Bipolar Neutrosophic Projection Based Models for Solving Multi-attribute Decision Making Problems

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Abstract. Bipolar neutrosophic sets are the extension of neutrosophic sets and are based on the idea of positive and negative preferences of information. Projection measure is a useful apparatus for modelling real life decision making problems. In the paper, we define projection, bidirectional projection and hybrid projection measures between bipolar neutrosophic sets. Three new methods based on the proposed projection measures are developed for solving multi-attribute decision making problems. In the solution process, the ratings of performance values of the alternatives with respect to the attributes are expressed in terms of bipolar neutrosophic values. We calculate projection, bidirectional projection, and hybrid projection measures between each alternative and ideal alternative with bipolar neutrosophic information. All the alternatives are ranked to identify the best alternative. Finally, a numerical example is provided to demonstrate the applicability and effectiveness of the developed methods. Comparison analysis with the existing methods in the literature in bipolar neutrosophic environment is also performed.

Keywords: Bipolar neutrosophic sets; projection measure; bidirectional projection measure; hybrid projection measure; multi-attribute decision making.

1 Introduction

For describing and managing indeterminate and inconsistent information, Smarandache [1] introduced neutrosophic set which has three independent components namely truth membership degree (T), indeterminacy membership degree (I) and falsity membership degree (F) where T, I, and F lie in \([0, 1]\). Later, Wang et al. [2] proposed single valued neutrosophic set (SVNS) to deal real decision making problems where \(T, I, F\) lie in \([0, 1]\).

Zhang [3] grounded the notion of bipolar fuzzy sets by extending the concept of fuzzy sets [4]. The value of membership degree of an element of bipolar fuzzy set belongs to \([-1, 1]\). With reference to a bipolar fuzzy set, the membership degree zero of an element reflects that the element is irrelevant to the corresponding property, the membership degree belongs to \([0, 1]\) of an element reflects that the element somewhat satisfies the property, and the membership degree belongs to \([-1,0]\) of an element reflects that the element somewhat satisfies the implicit counter-property.

Deli et al. [5] extended the concept of bipolar fuzzy set to bipolar neutrosophic set (BNS). With reference to a bipolar neutrosophic set \(Q\), the positive membership degrees \(T^+_Q(x)\), \(I^+_Q(x)\), and \(F^+_Q(x)\) represent respectively the truth membership, indeterminate membership and falsity membership of an element \(x \in X\) corresponding to the bipolar neutrosophic set \(Q\) and the negative membership degrees \(T^-_Q(x)\), \(I^-_Q(x)\), and \(F^-_Q(x)\) denote respectively the truth membership, indeterminate membership and false membership degree of an element \(x \in X\) to some implicit counter-property corresponding to the bipolar neutrosophic set \(Q\).

Projection measure is a useful decision making device as it takes into account the distance as well as the included angle for measuring the closeness degree between two objects [6, 7]. Yue [6] and Zhang et al. [7] studied projection based multi-attribute decision making (MADM) in crisp environment i.e. projections are defined by ordinary numbers or crisp numbers. Yue [8] further investigated a new multi-attribute group decision making (MAGDM) method based on determining the weights of the decision makers by employing projection technique with interval data. Yue and Jia [9] established a methodology for MAGDM based on a new normalized projection measure, in which the attribute values are provided by decision makers in hybrid form with crisp values and interval data.


In neutrosophic environment, Chen and Ye [18] developed projection based model of neutrosophic numbers and presented MADM method to select clay-bricks in construction field. Bidirectional projection measure [19, 20] considers the distance and included angle between two vectors \( x, y \). Ye [19] defined bidirectional projection measure as an improvement of the general projection measure of SVNSs to overcome the drawback of the general projection measure. In the same study, Ye [19] developed MADM method for selecting problems of mechanical design schemes under a single-valued neutrosophic environment. Ye [20] also presented bidirectional projection method for MAGDM with neutrosophic numbers.

Ye [21] defined credibility – induced interval neutrosophic weighted arithmetic averaging operator and credibility – induced interval neutrosophic weighted geometric averaging operator and developed the projection measure based ranking method for MADM problems with interval neutrosophic information and credibility information. Dey et al. [22] proposed a new approach to neutrosophic soft MADM using grey relational projection method. Dey et al. [23] defined weighted projection measure with interval neutrosophic assessments and applied the proposed concept to solve MADM problems with interval valued neutrosophic information. Pramanik et al. [24] defined projection and bidirectional projection measures between rough neutrosophic sets and proposed two new multi-criteria decision making (MCDM) methods based on projection and bidirectional projection measures in rough neutrosophic set environment.

In the field of bipolar neutrosophic environment, Deli et al. [5] defined score, accuracy, and certainty functions in order to compare BNSs and developed bipolar neutrosophic weighted average (BNWA) and bipolar neutrosophic weighted geometric (BNWG) operators to obtain collective bipolar neutrosophic information. In the same study, Deli et al. [5] also proposed a MCDM approach on the basis of score, accuracy, and certainty functions and BNWA, BNWG operators. Deli and Subas [25] presented a single valued bipolar neutrosophic MCDM through correlation coefficient similarity measure. Şahin et al. [26] provided a MCDM method based on Jaccard similarity measure of BNS. Üluçay et al. [27] defined Dice similarity, weighted Dice similarity, hybrid vector similarity, weighted hybrid vector similarity measures under BNSs and developed MCDM methods based on the proposed similarity measures. Dey et al. [28] defined Hamming and Euclidean distance measures to compute the distance between BNSs and investigated a TOPSIS approach to derive the most desirable alternative.

In this study, we define projection, bidirectional projection and hybrid projection measures under bipolar neutrosophic information. Then, we develop three methods for solving MADM problems with bipolar neutrosophic assessments. We organize the rest of the paper in the following way. In Section 2, we recall several useful definitions concerning SVNSs and BNSs. Section 3 defines projection, bidirectional projection and hybrid projection measures between BNSs. Section 4 is devoted to present three models for solving MADM under bipolar neutrosophic environment. In Section 5, we solve a decision making problem with bipolar neutrosophic information on the basis of the proposed measures. Comparison analysis is provided to demonstrate the feasibility and flexibility of the proposed methods in Section 6. Finally, Section 7 provides conclusions and future scope of research.

2 Basic Concepts Regarding SVNSs and BNSs

In this Section, we provide some basic definitions regarding SVNSs, BNSs which are useful for the construction of the paper.

2.1 Single valued neutrosophic sets [2]

Let \( X \) be a universal space of points with a generic element of \( X \) denoted by \( x \), then a SVNS \( P \) is characterized by a truth membership function \( T_p(x) \), an indeterminate membership function \( I_p(x) \) and a falsity membership function \( F_p(x) \). A SVNS \( P \) is expressed in the following way:

\[
P = \{x, \{T_p(x), I_p(x), F_p(x)\} \mid x \in X\}
\]

where, \( T_p(x), I_p(x), F_p(x) : X \rightarrow [0, 1] \) and \( 0 \leq T_p(x) + I_p(x) + F_p(x) \leq 3 \) for each point \( x \in X \).

2.2 Bipolar neutrosophic set [5]

Consider \( X \) be a universal space of objects, then a BNS \( Q \) in \( X \) is presented as follows:

\[
Q = \{x, \{T^+_Q(x), I^+_Q(x), F^+_Q(x), T^-_Q(x), I^-_Q(x), F^-_Q(x)\} \mid x \in X\},
\]
where 
\[ T_{q_0}^+(x), I_{q_0}^+(x), F_{q_0}^+(x) : X \to [0, 1] \text{ and } T_{q_0}^-(x), I_{q_0}^-(x), F_{q_0}^-(x) : X \to [-1, 0] \] 
The positive membership degrees \( T_{q_0}^+(x) \), \( I_{q_0}^+(x) \), \( F_{q_0}^+(x) \) denote the truth membership, indeterminate membership, and falsity membership functions of an element \( x \in X \) corresponding to a BNS \( Q \), and the negative membership degrees \( T_{q_0}^-(x) \), \( I_{q_0}^-(x) \), \( F_{q_0}^-(x) \) denote the truth membership, indeterminate membership, and falsity membership of an element \( x \in X \) to several implicit counter property associated with a BNS \( Q \). For convenience, a bipolar neutrosophic value (BNV) is presented as \( \vec{q} = < T_{q_0}^+, I_{q_0}^+, F_{q_0}^+, T_{q_0}^-, I_{q_0}^-, F_{q_0}^- > \).

**Definition 1** [5]

Let, \( Q_1 = \{ x, \{ T_{q_0}^+(x), I_{q_0}^+(x), F_{q_0}^+(x), T_{q_0}^-(x), I_{q_0}^-(x), F_{q_0}^-(x) \} | x \in X \} \) and \( Q_2 = \{ x, \{ T_{q_0}^+(x), I_{q_0}^+(x), F_{q_0}^+(x), T_{q_0}^-(x), I_{q_0}^-(x), F_{q_0}^-(x) \} | x \in X \} \) be any two BNSs. Then \( Q_1 \subseteq Q_2 \) if and only if

\( T_{q_0}^+(x) \leq T_{q_0}^+(x), I_{q_0}^+(x) \leq I_{q_0}^+(x), F_{q_0}^+(x) \geq F_{q_0}^+(x) \text{ ; } T_{q_0}^-(x) \geq T_{q_0}^-(x), I_{q_0}^-(x) \geq I_{q_0}^-(x), F_{q_0}^-(x) \leq F_{q_0}^-(x) \) for all \( x \in X \).

**Definition 2** [5]

Let, \( Q_1 = \{ x, \{ T_{q_0}^+(x), I_{q_0}^+(x), F_{q_0}^+(x), T_{q_0}^-(x), I_{q_0}^-(x), F_{q_0}^-(x) \} | x \in X \} \) and \( Q_2 = \{ x, \{ T_{q_0}^+(x), I_{q_0}^+(x), F_{q_0}^+(x), T_{q_0}^-(x), I_{q_0}^-(x), F_{q_0}^-(x) \} | x \in X \} \) be any two BNSs. Then \( Q_1 = Q_2 \) if and only if

\( T_{q_0}^+(x) = T_{q_0}^+(x), I_{q_0}^+(x) = I_{q_0}^+(x), F_{q_0}^+(x) = F_{q_0}^+(x) ; T_{q_0}^-(x) = T_{q_0}^-(x), I_{q_0}^-(x) = I_{q_0}^-(x), F_{q_0}^-(x) = F_{q_0}^-(x) \) for all \( x \in X \).

**Definition 3** [5]

Let, \( Q = \{ x, \{ T_{q_0}^+(x), I_{q_0}^+(x), F_{q_0}^+(x), T_{q_0}^-(x), I_{q_0}^-(x), F_{q_0}^-(x) \} | x \in X \} \) be a BNS. The complement of \( Q \) is represented by \( Q^c \) and is defined as follows:

\( T_{q_0}^c(x) = \{ 1 \} - T_{q_0}^+(x), I_{q_0}^c(x) = \{ 1 \} - I_{q_0}^+(x), F_{q_0}^c(x) = \{ 1 \} - F_{q_0}^+(x) \); 
\( T_{q_0}^c(x) = \{ 1 \} - T_{q_0}^-(x), I_{q_0}^c(x) = \{ 1 \} - I_{q_0}^-(x), F_{q_0}^c(x) = \{ 1 \} - F_{q_0}^-(x) \).

**Definition 4**

Let, \( Q_1 = \{ x, \{ T_{q_0}^+(x), I_{q_0}^+(x), F_{q_0}^+(x), T_{q_0}^-(x), I_{q_0}^-(x), F_{q_0}^-(x) \} | x \in X \} \) and \( Q_2 = \{ x, \{ T_{q_0}^+(x), I_{q_0}^+(x), F_{q_0}^+(x), T_{q_0}^-(x), I_{q_0}^-(x), F_{q_0}^-(x) \} | x \in X \} \) be any two BNSs. Their union \( Q_1 \cup Q_2 \) is defined as follows:

\( Q_1 \cup Q_2 = \{ \max (T_{q_0}^+(x), T_{q_0}^-(x)), \min (I_{q_0}^+(x), I_{q_0}^-(x)), \max (F_{q_0}^+(x), F_{q_0}^-(x)), \min (F_{q_0}^+(x), F_{q_0}^-(x)), \max (I_{q_0}^+(x), I_{q_0}^-(x)), \max (F_{q_0}^+(x), F_{q_0}^-(x)) \} \forall x \in X.

Their intersection \( Q_1 \cap Q_2 \) is defined as follows:

\( Q_1 \cap Q_2 = \{ \min (T_{q_0}^+(x), T_{q_0}^-(x)), \max (I_{q_0}^+(x), I_{q_0}^-(x)), \max (F_{q_0}^+(x), F_{q_0}^-(x)), \min (I_{q_0}^+(x), \min (F_{q_0}^+(x), F_{q_0}^-(x)) \} \forall x \in X.

**Definition 5** [5]

Let \( \vec{q}_1 = < T_{q_0}^+, I_{q_0}^+, F_{q_0}^+, T_{q_0}^-, I_{q_0}^-, F_{q_0}^- > \) and \( \vec{q}_2 = < T_{q_0}^+, I_{q_0}^+, F_{q_0}^+, T_{q_0}^-, I_{q_0}^-, F_{q_0}^- > \) be any two BNVs, then

i. \( \beta \cdot \vec{q}_1 = < 1 - (1 - T_{q_0}^+)^{1, \beta}, (I_{q_0}^+)^{1, \beta}, (F_{q_0}^+)^{1, \beta}, - (T_{q_0}^-)^{1, \beta} >;
\)

ii. \( \vec{q}_1 \beta = < (T_{q_0}^+)^{1, \beta}, 1 - (1 - I_{q_0}^+)^{1, \beta}, 1 - (1 - F_{q_0}^+)^{1, \beta}, - (1 - (-T_{q_0}^-))^\beta >;
\)

iii. \( \vec{q}_1 + \vec{q}_2 = < T_{q_0}^+ + T_{q_0}^-, T_{q_0}^+, T_{q_0}^-, I_{q_0}^+, I_{q_0}^-, F_{q_0}^+, F_{q_0}^- > ;
\)

iv. \( \vec{q}_1 - \vec{q}_2 = < -T_{q_0}^+ - T_{q_0}^-, I_{q_0}^+ - I_{q_0}^-, F_{q_0}^+ - F_{q_0}^- > ;
\)

3 Projection, bidirectional projection and hybrid projection measures of BNSs

This Section proposes a general projection, a bidirectional projection and a hybrid projection measures for BNSs.

**Definition 6**

Assume that \( X = (x_1, x_2, ..., x_m) \) be a finite universe of discourse and \( Q \) be a BNS in \( X \), then modulus of \( Q \) is defined as follows:

\[ \|Q\| = \left( \sum_{j=1}^{m} \alpha_j \right)^2 = \sqrt{\sum_{j=1}^{m} (T_{q_0}^+)^2 + (I_{q_0}^+)^2 + (F_{q_0}^+)^2 + (T_{q_0}^-)^2 + (I_{q_0}^-)^2 + (F_{q_0}^-)^2 \} \]

where \( \alpha_j = \{ T_{q_0}^+(x), I_{q_0}^+(x), F_{q_0}^+(x), T_{q_0}^-(x), I_{q_0}^-(x), F_{q_0}^-(x) \} \), \( j = 1, 2, ..., m \).
Definition 7 [10, 29]
Assume that \( u = (u_1, u_2, \ldots, u_m) \) and \( v = (v_1, v_2, \ldots, v_n) \) be two vectors, then the projection of vector \( u \) onto vector \( v \) can be defined as follows:

\[
\text{Proj} (u) = \| u \| \cos (u, v) = \frac{\sum_{i=1}^{n} u_i^2}{\sqrt{\sum_{i=1}^{n} v_i^2}} \cdot \frac{\sum_{i=1}^{n} (u_i v_j)}{\sqrt{\sum_{i=1}^{n} u_i^2} \cdot \sqrt{\sum_{i=1}^{n} v_i^2}} = \frac{\sum_{i=1}^{n} (u_i v_j)}{\sqrt{\sum_{i=1}^{n} v_i^2}}
\]

where, \( \text{Proj} (u) \) represents the closeness of \( u \) and \( v \) in magnitude.

Definition 8
Assume that \( X = (x_1, x_2, \ldots, x_m) \) be a finite universe of discourse and \( R, S \) be any two BNSs in \( X \), then

\[
\text{Proj} (R_S) = \| R \| \cos (R, S) = \frac{1}{\| S \|} (R_S)
\]

is called the projection of \( R \) on \( S \), where

\[
\| R \| = \sqrt{\sum_{i=1}^{n} (T_{i+}^2(x_i) + (I_{i+}^2(x_i)) + (F_{i+}^2(x_i)) + (T_{i-}^2(x_i)) + (I_{i-}^2(x_i)) + (F_{i-}^2(x_i)))}
\]

\[
\| S \| = \sqrt{\sum_{i=1}^{n} (T_{i+}^2(x_i)) + (I_{i+}^2(x_i)) + (F_{i+}^2(x_i)) + (T_{i-}^2(x_i)) + (I_{i-}^2(x_i)) + (F_{i-}^2(x_i))),}
\]

and

\[
R.S = \frac{\sum_{i=1}^{n} (T_{i+}(x_i) + I_{i+}(x_i) + F_{i+}(x_i) + T_{i-}(x_i) + I_{i-}(x_i) + F_{i-}(x_i))}{\| R \| \cdot \| S \|},
\]

Example 1. Suppose that \( R = (0.5, 0.3, 0.2, -0.2, -0.1, 0.05) \), \( S = (0.7, 0.3, 0.1, -0.4, -0.2, -0.3) \) be the two BNSs in \( X \), then the projection of \( R \) on \( S \) is obtained as follows:

\[
\text{Proj} (R) = \| R \| = \frac{0.5(0.7) + 0.3(0.3) + (0.2)(-0.1) + (-0.2)(-0.4) + (-0.1)(-0.2) + (-0.05)(-0.3)}{\sqrt{0.7^2 + 0.3^2 + (0.2)^2 + (-0.4)^2 + (0.1)^2 + (-0.2)^2}} = 0.612952
\]

The bigger value of \( \text{Proj} (R) \) reflects that \( R \) and \( S \) are closer to each other.

However, in single valued neutrosophic environment, Ye [20] observed that the general projection measure cannot describe accurately the degree of \( \alpha \) close to \( \beta \). We also notice that the general projection incorporated by Xu [11] is not reasonable in several cases under bipolar neutrosophic setting, for example let, \( \alpha = \beta = \alpha = \beta = \alpha = \beta > \alpha > \gamma > 2, a, 2a, -2a, -2a, -2a > \), then \( \text{Proj} (\alpha) = \text{Proj} (\gamma) = 4.8989979 \| \alpha \| \) and \( \text{Proj} (\gamma) = 2.449449 \| \alpha \| \). This shows that \( \beta \) is much closer to \( \gamma \) than \( \alpha \) which is not true because \( \alpha = \beta \). Ye [20] opined that \( \alpha \) is equal to \( \beta \) whenever \( \text{Proj} (\alpha) = \text{Proj} (\beta) \) should be equal to 1. Therefore, Ye [20] proposed an alternative method called bidirectional projection measure to overcome the limitation of general projection measure as given below.

Definition 9 [20]
Consider \( x \) and \( y \) be any two vectors, then the bidirectional projection between \( x \) and \( y \) is defined as follows:

\[
\text{B-proj} (x, y) = \frac{1}{1+\| x \| \cdot \| y \|} \| x \| \cdot \| y \| - \| x \| - \| y \| + \| x \cdot y \|
\]

where \( \| x \| \), \( \| y \| \) denote the moduli of \( x \) and \( y \) respectively, and \( x \cdot y \) is the inner product between \( x \) and \( y \).

Here, \( \text{B-proj} (x, y) = 1 \) if and only if \( x = y \) or \( 0 \leq \text{B-proj} (x, y) \leq 1 \), i.e. bidirectional projection is a normalized measure.

Definition 10
Consider \( R = \left\{ T_{R+}(x_i), I_{R+}(x_i), F_{R+}(x_i), T_{R-}(x_i), I_{R-}(x_i), F_{R-}(x_i) \right\} \) and \( S = \left\{ T_{S+}(x_i), I_{S+}(x_i), F_{S+}(x_i), T_{S-}(x_i), I_{S-}(x_i), F_{S-}(x_i) \right\} \) be any two BNSs in \( X = (x_1, x_2, \ldots, x_n) \), then the bidirectional projection measure between \( R \) and \( S \) is defined as follows:

\[
\text{B-proj} (R, S) = \frac{1}{1+\| R \| \cdot \| S \|} \| R \| \cdot \| S \| - \| R \| - \| S \| + \| R \cdot S \|
\]

where

\[
\| R \| = \sqrt{\sum_{i=1}^{n} (T_{i+}^2(x_i) + (I_{i+}^2(x_i)) + (F_{i+}^2(x_i)) + (T_{i-}^2(x_i)) + (I_{i-}^2(x_i)) + (F_{i-}^2(x_i)))}
\]

\[
\| S \| = \sqrt{\sum_{i=1}^{n} (T_{i+}^2(x_i)) + (I_{i+}^2(x_i)) + (F_{i+}^2(x_i)) + (T_{i-}^2(x_i)) + (I_{i-}^2(x_i)) + (F_{i-}^2(x_i)))}
\]

and \( R.S = \frac{\sum_{i=1}^{n} (T_{i+}(x_i) + I_{i+}(x_i) + F_{i+}(x_i) + T_{i-}(x_i) + I_{i-}(x_i) + F_{i-}(x_i))}{\| R \| \cdot \| S \|} \).

Proposition 1. Let \( \text{B-proj} (R) \) be a bidirectional projection measure between any two BNSs \( R \) and \( S \), then

1. \( 0 \leq \text{B-proj} (R, S) \leq 1 \);
2. \( \text{B-proj} (R, S) = \text{B-proj} (S, R) \);
3. \( \text{B-proj} (R, S) = 1 \) for \( R = S \).

Proof.
1. For any two non-zero vectors \( R \) and \( S \),

\[
\frac{1}{1+\| R \| \cdot \| S \|} \| R \| \cdot \| S \| - \| R \| - \| S \| + \| R \cdot S \| > 0, \quad \frac{1}{1+x} > 0, \quad \text{when} \quad x > 0
\]
\( \therefore B-\text{Proj} (R, S) > 0 \), for any two non-zero vectors \( R \) and \( S \).

\( B-\text{Proj} (R, S) = 0 \) if and only if either \( \| R \| = 0 \) or \( \| S \| = 0 \)

i.e. when either \( R = (0, 0, 0, 0, 0) \) or \( S = (0, 0, 0, 0, 0) \)

which is trivial case.

\( \therefore B-\text{Proj} (R, S) \geq 0 \).

For two non-zero vectors \( R \) and \( S \),

\[
\| R \| \| S \| + \| R \| - \| S \| \leq \| R \| + \| S \| \leq \| R \| + \| S \| = R.S
\]

\( \therefore \| R \| \| S \| \leq \| R \| + \| S \| - \| R \| \leq R.S \|

\( \therefore \| R \| \| S \| \leq \| R \| + \| S \| - \| R \| \| S \| \leq R.S \|

\( \therefore B-\text{Proj} (R, S) \leq 1.

\( \therefore 0 \leq B-\text{Proj} (R, S) \leq 1; \)

2. From definition, \( R.S = \mathcal{S}.R \), therefore,

\[
B-\text{Proj} (R, S) = \frac{\| R \| \| S \| - \text{Proj} (R, S)}{\| R \| \| S \| - \| R \| \| S \|} = \frac{\| R \| \| S \| - \| R \| \| S \|}{\| R \| \| S \| - \| R \| \| S \|} = B-\text{Proj} (S, R).
\]

Obviously, \( B-\text{Proj} (R, S) = 1 \), only when \( \| R \| = \| S \| \) i.e. when \( T_R^p (x_i) = T_S^p (x_i) \), \( I_R^p (x_i) = I_S^p (x_i) \), \( F_R^p (x_i) = F_S^p (x_i) \), \( T_R^- (x_i) = T_S^- (x_i) \), \( I_R^- (x_i) = I_S^- (x_i) \), \( F_R^- (x_i) = F_S^- (x_i) \).

This completes the proof.

Example 2. Assume that \( R = (0.5, 0.3, 0.2, -0.2, -0.1, -0.05) \), \( S = (0.7, 0.3, 0.1, -0.4, -0.2, -0.3) \) be the BNSs in \( X \) then the bidirectional projection measure between \( R \) and \( S \) is computed as given below.

\[
B-\text{Proj} (R, S) = \frac{(0.6576473)(0.9380832)}{(0.6576473)(0.9380832) + |(0.9380832 - 0.6576473)|} = 0.7927845
\]

Definition 11

Let \( R = \{ T_R^p (x_i), I_R^p (x_i), F_R^p (x_i), T_R^- (x_i), I_R^- (x_i), F_R^- (x_i) \} \) and \( S = \{ T_S^p (x_i), I_S^p (x_i), F_S^p (x_i), T_S^- (x_i), I_S^- (x_i), F_S^- (x_i) \} \) be any two BNSs in \( X = (x_1, x_2, ..., x_m) \), then the hybrid projection measure is defined as the combination of projection measure and bidirectional projection measure. The hybrid projection measure between \( R \) and \( S \) is represented as follows:

\[
\text{Hyb-Proj} (R, S) = \rho \text{Proj} (R, S) + (1 - \rho) \cdot B-\text{Proj} (R, S)
\]

\[
= \rho \frac{R.S}{\| R \| \| S \|} + (1 - \rho) \frac{\| R \| \| S \| - \| R \| \| S \|}{R.S} \quad (6)
\]

where

\[
\| R \| = \sqrt{\sum_{i=1}^{n} (T_R^p (x_i) + T_R^- (x_i))^2 + (F_R^p (x_i) + F_R^- (x_i))^2 + (I_R^p (x_i) + I_R^- (x_i))^2 + (T_S^p (x_i) + T_S^- (x_i))^2 + (F_S^p (x_i) + F_S^- (x_i))^2 + (I_S^p (x_i) + I_S^- (x_i))^2},
\]

\[
\| S \| = \sqrt{\sum_{i=1}^{n} (T_S^p (x_i) + T_S^- (x_i))^2 + (F_S^p (x_i) + F_S^- (x_i))^2 + (I_S^p (x_i) + I_S^- (x_i))^2 + (T_R^p (x_i) + T_R^- (x_i))^2 + (F_R^p (x_i) + F_R^- (x_i))^2 + (I_R^p (x_i) + I_R^- (x_i))^2},
\]

and

\[
R.S = \sum_{i=1}^{n} (T_R^p (x_i) + T_R^- (x_i))(T_S^p (x_i) + T_S^- (x_i)) + (F_R^p (x_i) + F_R^- (x_i))(F_S^p (x_i) + F_S^- (x_i)) + (I_R^p (x_i) + I_R^- (x_i))(I_S^p (x_i) + I_S^- (x_i))
\]

where \( 0 \leq \rho \leq 1 \).

Proposition 2

Let \( \text{Hyb-Proj} (R, S) \) be a hybrid projection measure between any two BNSs \( R \) and \( S \), then

1. \( 1.0 \leq \text{Hyb-Proj} (R, S) \leq 1; \)
2. \( \text{Hyb-Proj} (R, S) = B-\text{Proj} (S, R); \)
3. \( \text{Hyb-Proj} (R, S) = 1 \) for \( R = S \).

Proof. The proofs of the properties under Proposition 2 are similar as Proposition 1.

Example 3. Assume that \( R = (0.5, 0.3, 0.2, -0.2, -0.1, -0.05), S = (0.7, 0.3, 0.1, -0.4, -0.2, -0.3) \) be the two BNSs, then the hybrid projection measure between \( R \) and \( S \) with \( \rho = 0.7 \) is calculated as given below.

\( \text{Hyb- Proj} (R, S) = (0.7)(0.612952) + (1 - 0.7)\) (0.7927845) = 0.6669018

4 Projection, bidirectional projection and hybrid projection based decision making methods for MADM problems with bipolar neutrosophic information

In this section, we develop projection based decision making models to MADM problems with bipolar neutrosophic assessments. Consider \( E = \{ E_1, E_2, ..., E_m \} \), \( m \geq 2 \) a discrete set of \( m \) feasible alternatives, \( F = \{ F_1, F_2, ..., F_n \} \), \( n \geq 2 \) be a set of attributes under consideration and \( w = (w_1, w_2, ..., w_m) \) be the weight vector of the attributes such that \( 0 \leq w_i \leq 1 \) and \( \sum_{j=1}^{m} w_j = 1 \). Now, we present three algorithms for MADM problems involving bipolar neutrosophic information.

4.1 Method 1

Step 1. The rating of evaluation value of alternative \( E_i \) (\( i = 1, 2, ..., m \)) for the predefined attribute \( F_j \) (\( j = 1, 2, ..., n \)) is presented by the decision maker in terms of bipolar neutrosophic values and the bipolar neutrosophic decision matrix is constructed as given below.

\[
\{ q_{ij} \}_{m \times n} = \begin{bmatrix}
q_{11} & q_{12} & \cdots & q_{1n} \\
q_{21} & q_{22} & \cdots & q_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
q_{m1} & q_{m2} & \cdots & q_{mn}
\end{bmatrix}
\]

Surapati Pramanik, Partha Pratim Dey, Bibhas C. Giri, Florentin Smarandache, Bipolar Neutrosophic Projection Based Models for Solving Multi-attribute Decision Making Problems
where \( q_j = < (T^+_j, I^+_j, F^+_j), (T^-_j, I^-_j, F^-_j) > \) with \( T^+_j, I^+_j, F^+_j, T^-_j, I^-_j, F^-_j \) in \([0, 1] \) and \( 0 \leq T^+_j + I^+_j + F^+_j - T^-_j - I^-_j - F^-_j \leq 6 \) for \( i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \).

Step 2. We formulate the bipolar weighted decision matrix \( W_{ij} \) of the attributes as follows:

\[
W_{ij} = \{ [ z_{i1}, z_{i2}, \ldots, z_{in}], [ z_{21}, z_{22}, \ldots, z_{2n}], \ldots, [ z_{m1}, z_{m2}, \ldots, z_{mn}] \}
\]

where \( z_{ij} = w_i \cdot q_{ij} < 1 - (1 - T^+_j)^{n_i}, (I^+_j)^{n_i}, (F^+_j)^{n_i}, -(T^-_j)^{n_i}, -(I^-_j)^{n_i}, -(F^-_j)^{n_i} > = \mu^{x_i} \cdot \nu^{x_i} \cdot \omega^{x_i}, \mu^{x_i} \cdot \nu^{x_i} \cdot \omega^{x_i} > \) with \( \mu^{x_i}, \nu^{x_i}, \omega^{x_i} > 0 \) and \( 0 \leq \mu^{x_i} + \nu^{x_i} + \omega^{x_i} - \omega^{x_i} \leq 6 \) for \( i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \).

Step 3. We identify the bipolar neutrosophic positive ideal solution (BNPIS)

\[
\begin{align*}
& \text{Step 3. Identify the bipolar neutrosophic positive ideal solution (BNPIS):} \\
& z_{BPNIS} = \left\{ (e^+_1, f^+_1, g^+_1), (e^-_1, f^-_1, g^-_1), \ldots, (e^+_m, f^+_m, g^+_m), (e^-_m, f^-_m, g^-_m) \right\}
\end{align*}
\]

Step 4. Compute the bidirectional projection measure between \( z_{BPNIS} \) and \( Z^i = \left\{ (e^+_j, f^+_j, g^+_j), (e^-_j, f^-_j, g^-_j), \ldots, (e^+_m, f^+_m, g^+_m), (e^-_m, f^-_m, g^-_m) \right\} \) for all \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \) using the Eq. as given below.

\[
\text{B-Proj} (Z^i, z_{BPNIS}) = \frac{\| Z^i \| - \| z_{BPNIS} \|}{\| Z^i \| - \| z_{BPNIS} \| + \| Z^i \| - \| z_{BPNIS} \|} (8)
\]

where \( \| Z^i \| = \sum_{j=1}^{n} ([e^+_j]^2 + [f^+_j]^2 + [g^+_j]^2 + [e^-_j]^2 + [f^-_j]^2 + [g^-_j]^2] \) and \( \| Z^i \cdot z_{BPNIS} \| = \sum_{j=1}^{n} ([e^+_j]^2 + [f^+_j]^2 + [g^+_j]^2 + [e^-_j]^2 + [f^-_j]^2 + [g^-_j]^2] \) respectively.

Step 5. According to the bidirectional projection measure B-Proj \( (Z^i, z_{BPNIS}) \) for \( i = 1, 2, \ldots, m \) the alternatives are ranked and the highest value of B-Proj \( (Z^i, z_{BPNIS}) \) reflects the best option.

4.3. Method 3

Step 1. Construct the bipolar neutrosophic decision matrix

\[
\langle q_{ij} \rangle_{m \times n}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n.
\]

Step 2. Formulate the weighted bipolar neutrosophic decision matrix

\[
\langle z_{ij} \rangle_{m \times n}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n.
\]

Step 3. Identify \( z_{BPNIS} = \left\{ (e^+_1, f^+_1, g^+_1), (e^-_1, f^-_1, g^-_1), \ldots, (e^+_m, f^+_m, g^+_m), (e^-_m, f^-_m, g^-_m) \right\} \) for \( j = 1, 2, \ldots, n \).

Step 4. By combining projection measure \( \text{Proj} (Z^i, z_{BPNIS}) \) and bidirectional projection measure B-Proj \( (Z^i, z_{BPNIS}) \), we calculate the hybrid projection measure between \( z_{BPNIS} \) and \( Z^i \)

\[
\langle z_{ij} \rangle_{m \times n}, \text{for all i = 1, 2, \ldots, m, j = 1, 2, \ldots, n as follows.}
\]
Hyb-Proj \((Z^*, z^{\text{PIS}}) = \rho \text{ Proj} \((Z^c) - \rho \text{ B-Proj} \((Z^c), z^{\text{PIS}}) = \\
\rho \left( \frac{Z^* z^{\text{PIS}}}{z^{\text{PIS}}} \right) + (1 - \rho) \left( \frac{Z^* z^{\text{PIS}}}{z^{\text{PIS}}} \right)
\]
where \(\|Z^i\| = \sum_{i=1}^{m} \left( (\mu_i)^2 + (v_i)^2 + (\alpha_i)^2 \right), i = 1, 2, \ldots, m, \)
\(\|z^{\text{PIS}}\| = \sum_{j=1}^{n} \left( (\mu_j)^2 + (v_j)^2 + (\alpha_j)^2 \right), \)
\(Z^* z^{\text{PIS}} = \sum_{i=1}^{m} \left( \mu_i e_i + v_i f_i + \alpha_i g_i \right), i = 1, 2, \ldots, m, \) with \(0 \leq \rho \leq 1.\)

Step 5. We rank all the alternatives in accordance with the hybrid projection measure \(\text{Hyb-Proj} \((Z^c), z^{\text{PIS}})\) and greater value of \(\text{Hyb-Proj} \((Z^c), z^{\text{PIS}})\) indicates the better alternative.

5 A numerical example

We solve the MADM studied in [5, 28] where a customer desires to purchase a car. Suppose four types of car (alternatives) \(E_i, (i = 1, 2, 3, 4)\) are taken into consideration in the decision making situation. Four attributes namely Fuel economy \((F_1)\), Aerod \((F_2)\), Comfort \((F_3)\) and Safety \((F_4)\) are considered to evaluate the alternatives. Assume the weight vector \([5]\) of the attribute is given by \(w = (w_1, w_2, w_3, w_4) = (0.5, 0.25, 0.125, 0.125).\)

Method 1: The proposed projection measure based decision making with bipolar neutrosophic information for car selection is presented in the following steps:

Step 1: Construct the bipolar neutrosophic decision matrix

The bipolar neutrosophic decision matrix \(\{d_{ij}\}_{m \times n}\) presented by the decision maker as given below (see Table 1)

<table>
<thead>
<tr>
<th>(F_1)</th>
<th>(F_2)</th>
<th>(F_3)</th>
<th>(F_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E_1)</td>
<td>&lt;0.293, 0.837, 0.447, -0.837, -0.028&gt;</td>
<td>&lt;0.120, 0.795, 0.841, 0.915, 0.028&gt;</td>
<td>&lt;0.140, 0.956, 0.917, 0.972, 0.028&gt;</td>
</tr>
<tr>
<td>(E_2)</td>
<td>&lt;0.084, 0.837, 0.707, -0.837, -0.028&gt;</td>
<td>&lt;0.250, 0.899, 0.946, 0.915, 0.028&gt;</td>
<td>&lt;0.230, 0.892, 0.917, 0.956, 0.028&gt;</td>
</tr>
<tr>
<td>(E_3)</td>
<td>&lt;0.163, 0.632, 0.447, -0.774, -0.548, -0.452&gt;</td>
<td>&lt;0.054, 0.669, 0.669, 0.795, 0.915, 0.028&gt;</td>
<td>&lt;0.054, 0.917, 0.938, 0.917, 0.028&gt;</td>
</tr>
<tr>
<td>(E_4)</td>
<td>&lt;0.648, 0.837, 0.447, -0.894, -0.548, -0.051&gt;</td>
<td>&lt;0.084, 0.841, 0.669, -0.841, -0.054&gt;</td>
<td>&lt;0.084, 0.917, 0.750, 0.956, -0.028&gt;</td>
</tr>
</tbody>
</table>

Step 2. Construction of weighted bipolar neutrosophic decision matrix

The weighted decision matrix \(\{z_{ij}\}_{m \times n}\) is obtained by multiplying weights of the attributes to the bipolar neutrosophic decision matrix as follows (see Table 2).

$$
\{z_{ij}\}_{m \times n} = \{w_i \times \{d_{ij}\}_{m \times n}\}
$$

Step 3. Selection of BNPSIS

The BNPSIS \((z^{\text{PIS}}) = \{e_i^+, f_i^+, g_i^+, e_i^-, f_i^-, g_i^-\}_j, j = 1, 2, 3, 4)\) is computed from the weighted decision matrix as follows:

\[\{e_i^+, f_i^+, g_i^+, e_i^-, f_i^-, g_i^-\}_j = \{0.684, 0.632, 0.447, -0.894, -0.548, -0.051\}^j;\]
\[\{e_i^+, f_i^+, g_i^+, e_i^-, f_i^-, g_i^-\}_j = \{0.26, 0.669, 0.669, -0.915, -0.841, -0.026\}^j;\]
\[\{e_i^+, f_i^+, g_i^+, e_i^-, f_i^-, g_i^-\}_j = \{0.25, 0.892, 0.917, -0.972, -0.917, -0.028\}^j;\]
\[\{e_i^+, f_i^+, g_i^+, e_i^-, f_i^-, g_i^-\}_j = \{0.14, 0.818, 0.86, -0.917, -0.75, -0.028\}^j.\]
Step 4. Determination of weighted projection measure
The projection measure between positive ideal bipolar neutrosophic solution \( \mathbf{z}_{\text{PIS}} \) and each weighted decision matrix \( \{z_i\}_{m \times n} \) can be obtained as follows:

\[
\text{Proj} (Z^1)_{mn} = 3.4214, \quad \text{Proj} (Z^2)_{mn} = 3.4972, \quad \text{Proj} (Z^3)_{mn} = 3.1821, \quad \text{Proj} (Z^4)_{mn} = 3.3904.
\]

Step 5. Rank the alternatives
We observe that \( \text{Proj} (Z^3)_{mn} > \text{Proj} (Z^1)_{mn} > \text{Proj} (Z^4)_{mn} > \text{Proj} (Z^2)_{mn} \). Therefore, the ranking order of the cars is \( E_2 > E_1 > E_4 > E_3 \). Hence, \( E_2 \) is the best alternative for the customer.

Method 2: The proposed bidirectional projection measure based decision making for car selection is presented as follows:
Step 1. Same as Method 1
Step 2. Same as Method 1
Step 3. Same as Method 1
Step 4. Calculation of bidirectional projection measure
The bidirectional projection measure between positive ideal bipolar neutrosophic solution \( \mathbf{z}_{\text{PIS}} \) and each weighted decision matrix \( \{z_i\}_{m \times n} \) can be determined as given below.

\[
\text{B-Proj} (Z^1, \mathbf{z}_{\text{PIS}}) = 0.8556, \quad \text{B-Proj} (Z^2, \mathbf{z}_{\text{PIS}}) = 0.8101, \quad \text{B-Proj} (Z^3, \mathbf{z}_{\text{PIS}}) = 0.9503, \quad \text{B-Proj} (Z^4, \mathbf{z}_{\text{PIS}}) = 0.8969.
\]

Step 5. Ranking the alternatives
Here, we notice that \( \text{B-Proj} (Z^3, \mathbf{z}_{\text{PIS}}) > \text{B-Proj} (Z^1, \mathbf{z}_{\text{PIS}}) > \text{B-Proj} (Z^4, \mathbf{z}_{\text{PIS}}) > \text{B-Proj} (Z^2, \mathbf{z}_{\text{PIS}}) \) and therefore, the ranking order of the alternatives is obtained as \( E_3 > E_1 > E_4 > E_2 \). Hence, \( E_3 \) is the best choice among the alternatives.

Method 3: The proposed hybrid projection measure based MADM with bipolar neutrosophic information is provided as follows:
Step 1. Same as Method 1
Step 2. Same as Method 1
Step 3. Same as Method 1
Step 4. Computation of hybrid projection measure
The hybrid projection measures for different values of \( \rho \in [0, 1] \) and the ranking order are shown in the Table 3.

<table>
<thead>
<tr>
<th>Similarity measure</th>
<th>( \rho )</th>
<th>Measure values</th>
<th>Ranking order</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Hyb-Proj</strong> ( Z, \mathbf{z}_{\text{PIS}} )</td>
<td>0.25</td>
<td>( \text{Hyb-Proj} (Z^1, \mathbf{z}_{\text{PIS}}) = 1.4573 )</td>
<td>( E_1 &gt; E_2 &gt; E_3 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \text{Hyb-Proj} (Z^2, \mathbf{z}_{\text{PIS}}) = 1.4551 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \text{Hyb-Proj} (Z^3, \mathbf{z}_{\text{PIS}}) = 1.5297 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \text{Hyb-Proj} (Z^4, \mathbf{z}_{\text{PIS}}) = 1.5622 )</td>
<td></td>
</tr>
<tr>
<td><strong>Hyb-Proj</strong> ( Z, \mathbf{z}_{\text{PIS}} )</td>
<td>0.50</td>
<td>( \text{Hyb-Proj} (Z^1, \mathbf{z}_{\text{PIS}}) = 2.0334 )</td>
<td>( E_1 &gt; E_2 &gt; E_3 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \text{Hyb-Proj} (Z^2, \mathbf{z}_{\text{PIS}}) = 2.0991 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \text{Hyb-Proj} (Z^3, \mathbf{z}_{\text{PIS}}) = 2.0740 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \text{Hyb-Proj} (Z^4, \mathbf{z}_{\text{PIS}}) = 2.0740 )</td>
<td></td>
</tr>
<tr>
<td><strong>Hyb-Proj</strong> ( Z, \mathbf{z}_{\text{PIS}} )</td>
<td>0.75</td>
<td>( \text{Hyb-Proj} (Z^1, \mathbf{z}_{\text{PIS}}) = 2.9490 )</td>
<td>( E_1 &gt; E_2 &gt; E_3 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \text{Hyb-Proj} (Z^2, \mathbf{z}_{\text{PIS}}) = 2.7432 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \text{Hyb-Proj} (Z^3, \mathbf{z}_{\text{PIS}}) = 2.6182 )</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>( \text{Hyb-Proj} (Z^4, \mathbf{z}_{\text{PIS}}) = 2.6919 )</td>
<td></td>
</tr>
<tr>
<td><strong>Hyb-Proj</strong> ( Z, \mathbf{z}_{\text{PIS}} )</td>
<td>0.90</td>
<td>( \text{Hyb-Proj} (Z^1, \mathbf{z}_{\text{PIS}}) = 3.1370 )</td>
<td>( E_1 &gt; E_2 &gt; E_3 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \text{Hyb-Proj} (Z^2, \mathbf{z}_{\text{PIS}}) = 3.1296 )</td>
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<tr>
<td></td>
<td></td>
<td>( \text{Hyb-Proj} (Z^3, \mathbf{z}_{\text{PIS}}) = 2.9448 )</td>
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<tr>
<td></td>
<td></td>
<td>( \text{Hyb-Proj} (Z^4, \mathbf{z}_{\text{PIS}}) = 3.0308 )</td>
<td></td>
</tr>
</tbody>
</table>

6 Comparative analysis

In the Section, we compare the results obtained from the proposed methods with the results derived from other existing methods under bipolar neutrosophic environment to show the effectiveness of the developed methods.

Dey et al. [28] assume that the weights of the attributes are not identical and weights are fully unknown to the decision maker. Dey et al. [28] formulated maximizing deviation model under bipolar neutrosophic assessment to compute unknown weights of the attributes as \( w = (0.2585, 0.2552, 0.2278, 0.2585) \). By considering \( w = (0.2585, 0.2552, 0.2278, 0.2585) \), the proposed projection measures are shown as follows:

\[
\text{Proj} (Z^1)_{mn} = 3.3954, \quad \text{Proj} (Z^2)_{mn} = 3.3872, \quad \text{Proj} (Z^3)_{mn} = 3.1625, \quad \text{Proj} (Z^4)_{mn} = 3.2567.
\]

Since, \( \text{Proj} (Z^1)_{mn} > \text{Proj} (Z^2)_{mn} > \text{Proj} (Z^3)_{mn} > \text{Proj} (Z^4)_{mn} \), therefore the ranking order of the four alternatives is given by \( E_1 > E_2 > E_4 > E_3 \). Thus, \( E_1 \) is the best choice for the customer.

Now, by taking \( w = (0.2585, 0.2552, 0.2278, 0.2585) \), the bidirectional projection measures are calculated as given below,

\[
\text{B-Proj} (Z^1, \mathbf{z}_{\text{PIS}}) = 0.8113, \quad \text{B-Proj} (Z^2, \mathbf{z}_{\text{PIS}}) = 0.8111, \quad \text{B-Proj} (Z^3, \mathbf{z}_{\text{PIS}}) = 0.9854, \quad \text{B-Proj} (Z^4, \mathbf{z}_{\text{PIS}}) = 0.9974.
\]

Since, \( \text{B-Proj} (Z^1, \mathbf{z}_{\text{PIS}}) > \text{B-Proj} (Z^2, \mathbf{z}_{\text{PIS}}) > \text{B-Proj} (Z^3, \mathbf{z}_{\text{PIS}}) > \text{B-Proj} (Z^4, \mathbf{z}_{\text{PIS}}) \), consequently the ranking of the alternatives is \( E_1 > E_2 > E_3 > E_4 \).
order of the four alternatives is given by $E_4 > E_3 > E_2 > E_1$. Hence, $E_4$ is the best option for the customer.

Also, by taking $w = (0.2585, 0.2552, 0.2278, 0.2585)$, the proposed hybrid projection measures for different values of $\rho \in [0, 1]$ and the ranking order are revealed in the Table 4.

Deli et al. [5] assume the weight vector of the attributes as $w = (0.5, 0.25, 0.125, 0.125)$ and the ranking order based on score values is presented as follows:

$$E_3 > E_2 > E_1 > E_4$$

Thus, $E_3$ was the most desirable alternative.

Dey et al. [28] employed maximizing deviation method to find unknown attribute weights as $w = (0.2585, 0.2552, 0.2278, 0.2585)$. The ranking order of the alternatives is presented based on the relative closeness coefficient as given below.

$$E_3 > E_2 > E_4 > E_1$$

Obviously, $E_3$ is the most suitable option for the customer.

Dey et al. [28] also consider the weight vector of the attributes as $w = (0.5, 0.25, 0.125, 0.125)$, then using TOPSIS method, the ranking order of the cars is represented as follows:

$$E_4 > E_2 > E_1 > E_3$$

So, $E_4$ is the most preferable alternative for the buyer. We observe that different projection measure provides different ranking order and the projection measure is weight sensitive. Therefore, decision maker should choose the projection measure and weights of the attributes in the decision making context according to his/her needs, desires and practical situation.

Conclusion

In this paper, we have defined projection, bidirectional projection measures between bipolar neutrosophic sets. Further, we have defined a hybrid projection measure by combining projection and bidirectional projection measures. Through these projection measures we have developed three methods for multi-attribute decision making models under bipolar neutrosophic environment. Finally, a car selection problem has been solved to show the flexibility and applicability of the proposed methods. Furthermore, comparison analysis of the proposed methods with the other existing methods has also been demonstrated.

The proposed methods can be extended to interval bipolar neutrosophic set environment. In future, we shall apply projection, bidirectional projection, and hybrid projection measures of interval bipolar neutrosophic sets for group decision making, medical diagnosis, weaver selection, pattern recognition problems, etc.

| Table 4. Results of hybrid projection measure for different values of $\rho$ |
|-----------------|----------------|----------------|
| Similarity measure | Measure values | Ranking order |
| Proj z (Z, PIS) | Proj z (Z, PIS) | Proj z (Z, PIS) | Proj z (Z, PIS) |
| $\rho$ | $E_1 > E_2 > E_3 > E_4$ | $E_3 > E_1 > E_2 > E_4$ | $E_2 > E_3 > E_4 > E_1$ | $E_3 > E_1 > E_2 > E_4$ |
| 0.25 | $1.4970$ | $2.1385$ | $2.7670$ | $3.1410$ |
| 0.50 | $1.4819$ | $2.5156$ | $2.6241$ | $2.9589$ |
| 0.75 | $1.5082$ | $2.0662$ | $2.9589$ | $3.1410$ |
| 0.90 | $1.5203$ | $2.1385$ | $2.7670$ | $3.1410$ |

References


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