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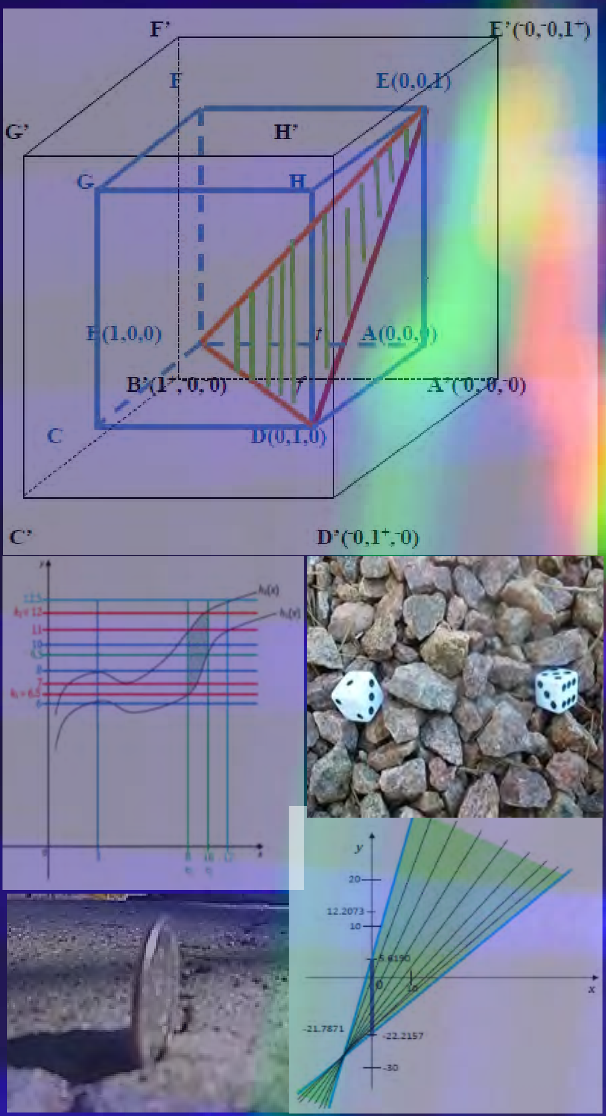

<A> <neutA> <antiA>

Florentin Smarandache , Mohamed Abdel-Basset
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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
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March 20, 2019

Prof. Florentin Smarandache
Univ New Mexico, Gallup Campus

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Thank you very much.

Sincerely,

Marian Hollingsworth
Director, Publisher Relations

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Contents

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Contents

Neutrosophic Sets and Systems, Vol. 28, 2019
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Reduction of indeterminacy of gray-scale image in bipolar neutrosophic domain

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Abstract: Neutrosophy is a branch of philosophy introduced by Florentin Smarandache. Neutrosophic set (NS) is the derivative of neutrosophy; it is a powerful tool to handle uncertainty. Here we applied neutrosophic set to gray scale image domain for image analysis. Several authors contributed in neutrosophic image analysis and image processing. We propose a novel approach on representation of grayscale images in bipolar neutrosophic domain (BNS). The reduction of noise in images is one of the challenging task in every field. While we transform a grayscale image into bipolar neutrosophic domain, the indeterminacy degree of both positive and negative memberships are reduced significantly. Indeed, we extract some useful information from indeterminacy domain; it leads to perform image analysis and processing in noisy images in a better manner. We discuss the representation of medical images in bipolar neutrosophic domain with examples.

Keywords: Bipolar neutrosophic set, Image analysis, Neutrosophy, Digital image processing.

1. Introduction

Neutrosophy is one of the useful tool to handle uncertainty in real world problems. It is the extension of fuzzy theory. Neutrosophy is a branch of philosophy which was introduced by Florentin Smarandache [1-3]. Neutrosophy deals with origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophy is the basis of neutrosophic sets (derivative of neutrosophy).

Neutrosophic set contains three parameters as true-membership degree, indeterminacy-membership degree and falsity-membership degree. These three membership degrees are independent and has range, a non-standard interval $]^-0,1^+[$. But for real life problems, non-standard interval is not applicable. Wang et al. [5] introduced single valued neutrosophic sets which is a neutrosophic set defined in the range [0, 1]. Later, Pinaki Majumdar et al. and Ali Aydogdu [6, 4] proposed some similarity and entropy measurements of single valued neutrosophic sets. In 2015, Deli et al. [7] introduced the concepts of bipolar neutrosophic sets (BNS) as an extension of neutrosophic sets. In 2016, Uluçay et al. [8] proposed some measures of similarities of bipolar neutrosophic sets.

Nowadays reduction of noise in images is difficult task in every field. Cheng and Guo [10] introduced the representation of image in neutrosophic domain and proposed image thresholding technique using neutrosophic domain. Guo and Cheng [11] proposed some concepts about image denoising through neutrosophic domain. Yanhui Guo and H.D. Cheng [9] introduced a new neutrosophic approach on image segmentation. A. A. Salama et al. [12, 14] proposed some image processing techniques using neutrosophic sets. G. Xu et al.[18] proposed image segmentation using TOPSIS method. Mohammed Abdel Basset et al. proposed some concepts of TOPSIS method for decision making problems in medical field [15, 19, 20, 22]. In 2017, Mumtaz Ali et al. introduced the concepts of bipolar neutrosophic soft sets which is a combined version of bipolar neutrosophic set and neutrosophic soft set. Arulpandy et al. [17] proposed some similarity and entropy measurements of bipolar neutrosophic soft sets. Several authors contributed to decision making and performance analysis using neutrosophic field [21, 23, 24].

In this paper, we proposed a novel approach on representation of any gray scale image in bipolar neutrosophic domain. In section 4, we applied our approach to MRI (Magnetic Resonance Image) medical images and discuss their nature with histogram representation. We analyze transformed images with some of the popular metrics Peak Signal-to-Noise Ratio (PSNR) and Mean Squared Error (MSE). In this transformation, the indeterminacy of both positive and negative membership degree is reduced significantly. This is the main advantage of this bipolar neutrosophic domain. Indeed, we extract useful information from original image through BNS domain; it is not available in neutrosophic domain.

2. Preliminaries

Definition 1. [1, 2, 3] Let X be the universe of discourse contains x . A Neutrosophic set $NS(A)$ is defined by $NS(A) = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$. Where $T_A(x), I_A(x), F_A(x)$ represents truth-membership degree, indeterminacy-membership degree and falsity-membership degree respectively. Here $T_A(x), I_A(x), F_A(x) : X \to]^-0, 1^+[$ along with the following condition

$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.

Example. Let $X = \{x_1, x_2, x_3\}$ be the universal set. Here, x_1, x_2, x_3 represents capacity, trustworthiness and price of a machine, respectively. Then $T_A(x), I_A(x), F_A(x)$ gives the degree of 'good service', degree of indeterminacy, degree of 'poor service' respectively. The neutrosophic set is defined

by $NS(A) = \{\langle x_1, 0.3, 0.4, 0.5\rangle, \langle x_2, 0.5, 0.2, 0.3\rangle, \langle x_3, 0.7, 0.2, 0.2\rangle\}$ where $^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.

Definition 2. [4,5,6] Single valued neutrosophic set(SVNS) is the immediate result of neutrosophic set if it is defined over standard unit interval [0,1] instead of the non-standard unit interval $]^-0, 1^+[$. A single valued neutrosophic set SVNS (A) is defined by $SVNS(A) = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$ where $T_A(x), I_A(x), F_A(x) : X \to [0,1]$ such that $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 3. [7, 8] Let X be the universal set which contains arbitrary points x. A bipolar neutrosophic set (BNS) $BNS(A)$ is defined by

$$BNS(A) = \{\langle x, T_A^+(x), I_A^+(x), F_A^+(x), T_A^-(x), I_A^-(x), F_A^-(x)\rangle : x \in X\}$$

Where

$T_A^+, I_A^+, F_A^+ : E \to [0,1]$ (Positive membership-degrees)

$T_A^-, I_A^-, F_A^- : E \to [-1,0]$ (Negative membership-degrees)

Such that

$0 \leq T_A^+(x) + I_A^+(x) + F_A^+(x) \leq 3$, $-3 \leq T_A^-(x) + I_A^-(x) + F_A^-(x) \leq 0$.

Example. Let $X = \{x_1, x_2, x_3\}$ be the universal set. A bipolar neutrosophic set (BNS) is defined by

$$A = \{\langle x_1, 0.3, 0.4, 0.5, -0.2, -0.4, -0.1\rangle, \langle x_2, 0.5, 0.2, 0.3, -0.2, -0.7, -0.5\rangle, \langle x_3, 0.7, 0.2, 0.2, -0.5, -0.4, -0.5\rangle\}$$

Where $0 \le T_A^+(x) + I_A^+(x) + F_A^+(x) \le 3$ and $-3 \le T_A^-(x) + I_A^-(x) + F_A^-(x) \le 0$. Also $T_A^+(x), I_A^+(x), F_A^+(x) \rightarrow [0,1]$ and $T_A^-(x), I_A^-(x), F_A^-(x) \rightarrow [-1,0]$.

3. Grayscale image in bipolar neutrosophic domain

Neutrosophy has wide range of applications in science and engineering. In particular, it is very useful in fields such as Data analytics, financial market, Social network analysis, Quantum theory, robotics in terms of decision making problems. In this section, we discuss about the applications of neutrosophic sets in image analysis. In 2008, H.D Cheng and Yanhui guo[10] introduced the representation of grayscale image in neutrosophic domain. After that, so many papers have been published about neutrosophic image such as image denoising, image thresholding, image segmentation etc.

3. 1. Image in neutrosophic domain

Let X be a universe of discourse, W be the set contained in X, which is composed by bright pixels. A neutrosophic image P_{NS} is characterized by three subset T, I and F. A pixel P in an image is described as P(T, I, F) and belongs to W in the following way: it is *t*% true , *i* % indeterminate and *f* % false in the bright pixel set, where *t* varies in T, *i* varies in I and *f* varies in F. Each component has a value in [0, 1].

Pixel P(i,j) in the image domain is transformed into neutrosophic domain $P_{NS}(i,j) = T(i,j), I(i,j), F(i,j)$, where $T(i,j), I(i,j), F(i,j)$ represents probabilities belonging to white set, indeterminate set and non-white set, respectively, which are defined as:

$$T(i,j) = \frac{\bar{g}(i,j) - \bar{g}_{min}}{\bar{g}_{max} - \bar{g}_{min}}, \quad I(i,j) = \frac{\delta(i,j) - \delta_{min}}{\delta_{max} - \delta_{min}}, \quad F(i,j) = 1 - T(i,j) = \frac{\bar{g}_{max} - \bar{g}(i,j)}{\bar{g}_{max} - \bar{g}_{min}}$$

Where $\bar{g}(i,j)$ represents mean intensity of pixel in some neighborhoods in W. Here,

$$g(i,j) = \frac{1}{W \times W} \sum_{m=i-w/2}^{i+w/2} \sum_{n=j-w/2}^{j+w/2} g(m,n)$$

$$\delta(i,j) = |g(i,j) - \bar{g}(i,j)|$$

$$\delta_{max} = max\delta(i,j) \qquad \delta_{min} = min\delta(i,j).$$

Example 1. We consider the original Lena image and represent it in neutrosophic domain as follows:

Figure 1. Original Lena image.

(a) (b) (c)

Figure 2. Neutrosophic domain images of Original Lena image.

(a) T domain (b) I domain (c) F domain

Above images represents truth-membership domain, indeterminacy domain and false-membership domain of original Lena image respectively. We mainly focus on truth-membership domain for image analysis along with indeterminacy domain. Truth-membership domain is correlated with indeterminacy domain.

3. 2. Image in bipolar neutrosophic domain

Now we introduce grayscale image representation in bipolar neutrosophic domain. Main advantage of this representation is, when we transform image into bipolar neutrosophic domain, the indeterminacy degree get reduced. Indeed, we extract some useful information from indeterminacy degree in bipolar neutrosophic domain which is not available in neutrosophic domain. We used MATLAB 2010 version for this transformation. The following steps are involved in this representation:

1. Load the original image. Convert this into grayscale if it is RGB color image.
2. Represent image in pixel domain.
3. Find the median pixel value of entire image.
4. Consider pixels above the median value as foreground image and below the median value as background image.
5. Set the window size (size of neighborhood) to find local mean value. In our case, we take 3x3-neighborhood.
6. Transform image into bipolar neutrosophic domain by taking positive memberships for foreground pixels and negative memberships for background pixels.

We use the following membership values to transform any grayscale image to bipolar neutrosophic domain. Since the elements are pixels of an image, we use only unsigned integer to represent the membership functions. A pixel in bipolar neutrosophic domain is represented by

$$P_{BNS}(i,j) = \{T^+(i,j), I^+(i,j), F^+(i,j), T^-(i,j), I^-(i,j), F^-(i,j)\}.$$

Here

$$T^+(i,j) = \frac{\bar{g}(i,j) - \bar{g}_{min}}{\bar{g}_{max} - \bar{g}_{min}} \qquad I^+(i,j) = \frac{\delta(i,j) - \delta_{min}}{\delta_{max} - \delta_{min}}$$

$$F^+(i,j) = 1 - T^+(i,j) = \frac{\bar{g}_{max} - \bar{g}(i,j)}{\bar{g}_{max} - \bar{g}_{min}}$$

$$T^-(i,j) = \frac{\hat{g}(i,j) - \hat{g}_{min}}{\hat{g}_{max} - \hat{g}_{min}} \qquad I^-(i,j) = \frac{\delta(i,j) - \delta_{min}}{\delta_{max} - \delta_{min}}$$

$$F^-(i,j) = 1 - T^+(i,j) = \frac{\hat{g}_{max} - \hat{g}(i,j)}{\hat{g}_{max} - \hat{g}_{min}}$$

Where $\bar{g}(i,j)$ represents mean intensity of foreground pixel in some neighborhood W and $\hat{g}(i,j)$ represents the mean intensity of background pixel in some neighborhood in W^*.

Here

$$g(i,j) = \frac{1}{W \times W} \sum_{m=i-w/2}^{i+w/2} \sum_{n=j-w/2}^{j+w/2} g(m,n)$$

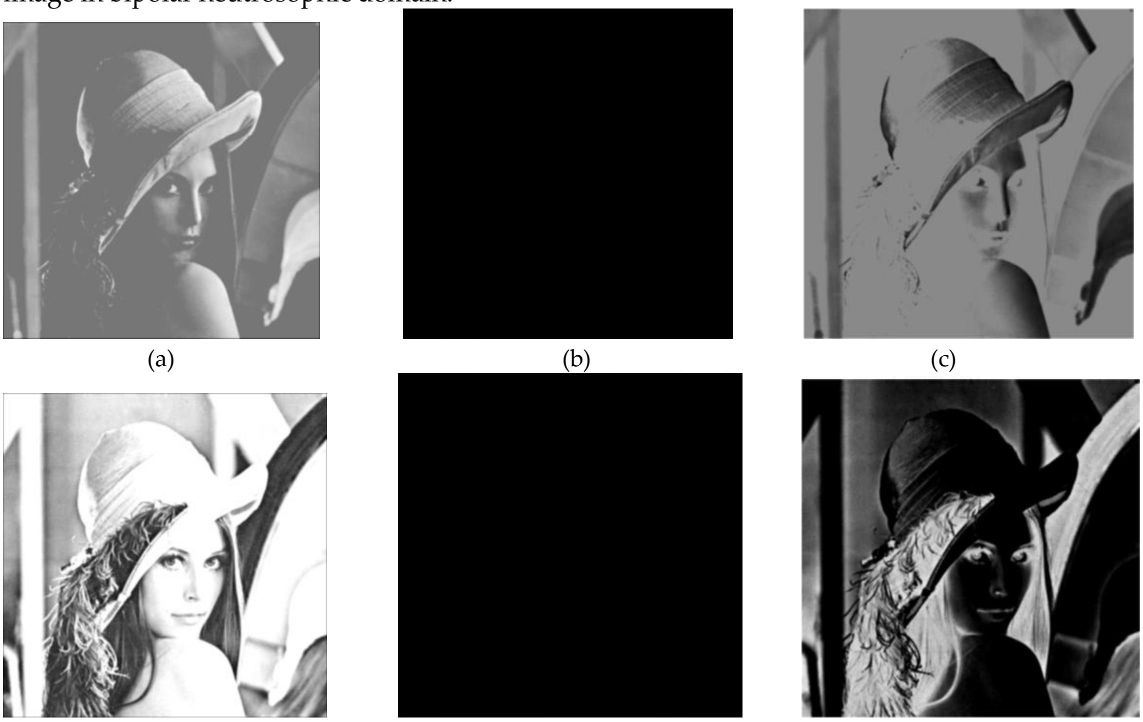
$$g(i,j) = \frac{1}{W^* \times W^*} \sum_{m=i-w^*/2}^{i+w^*/2} \sum_{n=j-w^*/2}^{j+w^*/2} g(m,n)$$

$$\delta(i,j) = |g(i,j) - \bar{g}(i,j)|$$

$$\delta(i,j) = |g(i,j) - \hat{g}(i,j)|$$

$$\delta_{max} = max\delta(i,j) \qquad \delta_{min} = min\delta(i,j).$$

Example 2. Consider the original Lena image in the previous example. The following image shows the image in bipolar neutrosophic domain.



(a) (b) (c)

(d) (e) (f)

Figure 3. Bipolar neutrosophic representation of lena image (fig 1.)

(a) T+ domain, (b) I+ domain, (c) F+ domain, (d) T- domain, (e) I- domain, (f) F- domain

Note that in the above images, I- domain and I+ domain images looks identical and black in color. It means both images contained only black pixels (pixels which has value zero). So from this we eliminate the indeterminacy of both positive and negative membership domains. The following histogram images shows that the gray level distribution of each images in BNS domain.

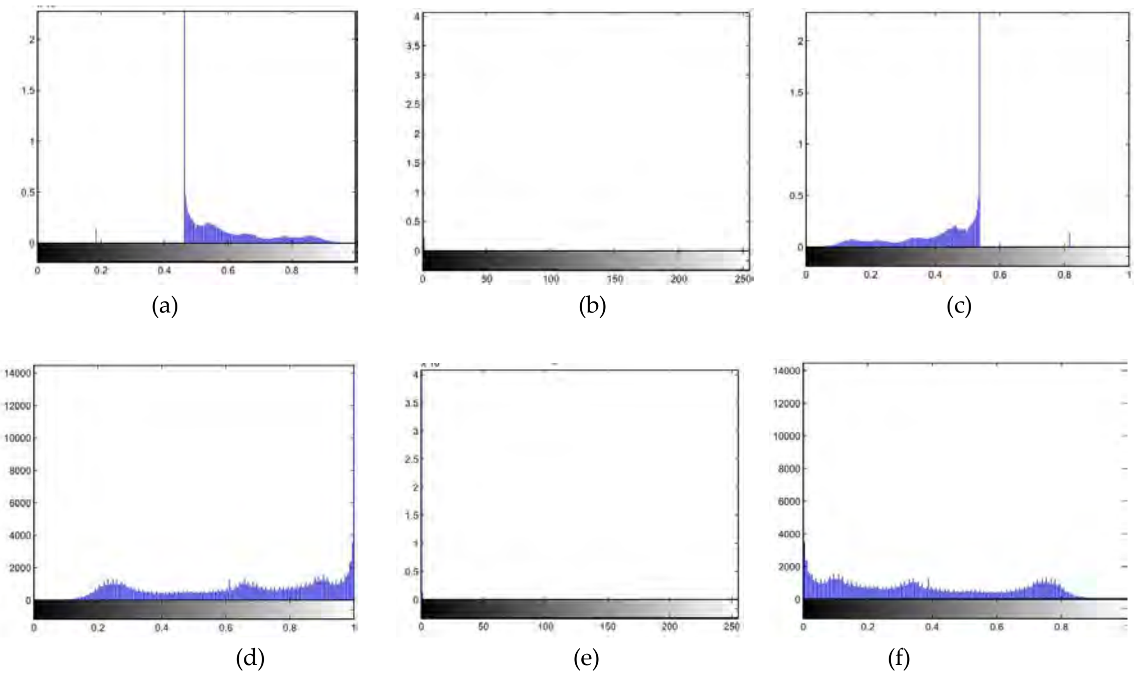


Figure 4. Histogram of transformed images (Fig 3.)
(a) Histogram of T+ , (b) Histogram of I+ , (c) Histogram of F+ , (d) Histogram of T- , (e) Histogram of I- , (f) Histogram of F- .

3. 3. Entropy of image in bipolar neutrosophic domain

Bipolar neutrosophic image entropy is defined as sum of entropies of all subsets $T^+, I^+, F^+, T^-, I^-, F^-$, which is used to evaluate the distribution of pixels in bipolar neutrosophic domain. $En_{BNS} = En_{T^+} + En_{I^+} + En_{F^+} + En_{T^-} + En_{I^-} + En_{F^-}$.

Here

$$En_{T^+} = -\sum p_{T^+}(i) \ln p_{T^+}(i)$$
$$En_{I^+} = -\sum p_{I^+}(i) \ln p_{I^+}(i)$$
$$En_{F^+} = -\sum p_{F^+}(i) \ln p_{F^+}(i)$$
$$En_{T^-} = -\sum p_{T^-}(i) \ln p_{T^-}(i)$$
$$En_{I^-} = -\sum p_{I^-}(i) \ln p_{I^-}(i)$$
$$En_{F^-} = -\sum p_{F^-}(i) \ln p_{F^-}(i).$$

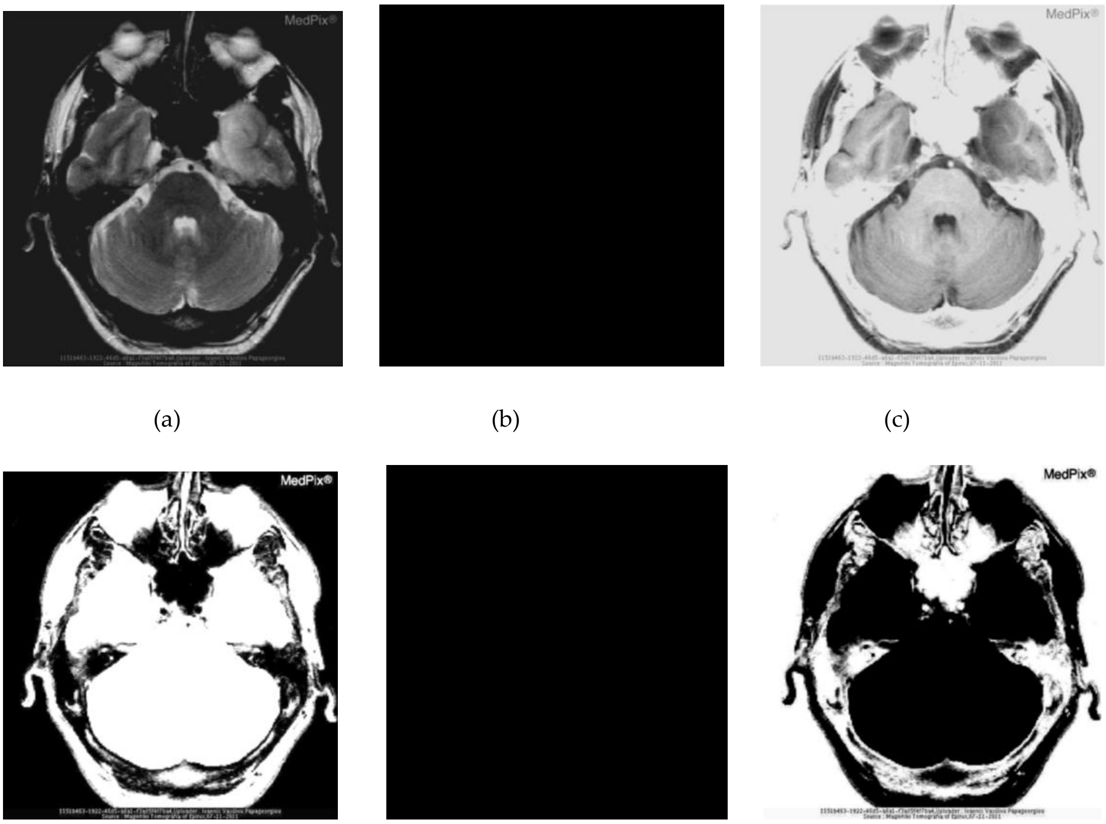
4. Bipolar neutrosophic representation of medical image

Nowadays image denoising is the challenging task in every field. Especially, in medical field, it is very useful for X-ray images, MRI images, CT images, Ultra sound images etc. In this section, we take MRI scan brain image and transform it to BNS domain and analyze various parameters.

Consider the following brain MRI image.

Figure 5. MRI Brain image

The following image shows the brain image in bipolar neutrosophic domain.



(a) (b) (c)

(d) (e) (f)

Figure 6. Bipolar neutrosophic representation of MRI brain images as

(a) T+ domain, (b) I+ domain, (c) F+ domain, (d) T- domain, (e) I- domain, (f) F- domain

From the above images, we can clearly see that the variations between each images. Every image has some useful information. We may neglect indeterminate images I+ and I-, since it has only black pixels. Peak Signal-to-noise Ratio (PSNR) values mostly used to find the noise level in the transformed image and we can check similarity level between original image and transformed image. PSNR value is calculated using the following formula:

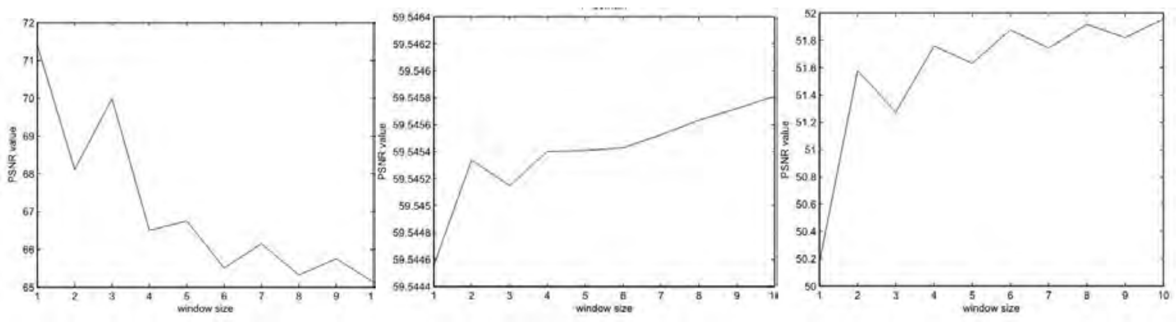
$$PSNR = -10log\left[\frac{\sum_{i=0}^{i=H-1}\sum_{j=0}^{j=W-1}[A(i,j) - A'(i,j)]^2}{M \times N \times 255^2}\right].$$

Here, the local mean average determines the variations in the transformed image. Local mean average of an image is depend on the window size (neighborhood size) which is used in the local mean average. Here, we analyze the PSNR value of the original image and images in the transformed domain for different neighborhood sizes.

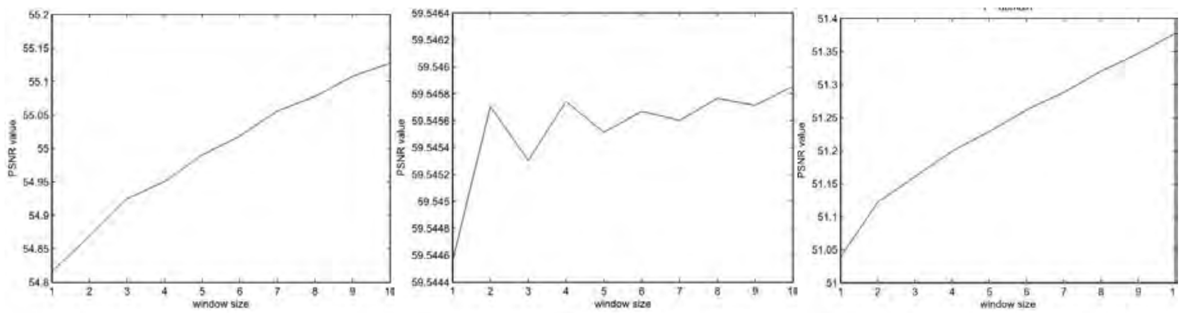
Window Size	T+ domain	I+ domain	F+ domain	T- domain	I- domain	F- domain
1x1	71.393	59.544	50.173	54.816	59.544	51.039
2x2	68.115	59.545	51.579	54.868	59.545	51.121
3x3	69.987	59.545	51.275	54.924	59.545	51.160
4x4	66.502	59.545	51.758	54.950	59.545	51.199
5x5	66.755	59.545	51.632	54.990	59.545	51.228
6x6	65.509	59.545	51.877	55.018	59.545	51.261
7x7	66.151	59.545	51.744	55.055	59.545	51.287
8x8	65.323	59.545	51.916	55.077	59.545	51.320
9x9	65.752	59.545	51.819	55.107	59.545	51.346
10x10	65.128	59.545	51.955	55.127	59.545	51.377

Table 1. PSNR values of brain image in BNS domain associated with different neighborhood windows.

Following plots shows the variations in PSNR values when we increase the size of the window in local mean average.



(a) (b) (c)



(d) (e) (f)

Figure 7. Comparison of PSNR values and neighborhood window size in (a) T+ domain, (b) I+ domain, (c) F+ domain, (d) T- domain, (e) I- domain, (f) F- domain

Mean Square Error (MSE) is another parameter to check the quality of transformed image. MSE is calculated using the following formula:

$$MSE = \frac{\sum_{i=0}^{i=H-1}\sum_{j=0}^{j=W-1}[A(i,j) - A'(i,j)]^2}{M \times N}.$$

Following table shows that the mean square error between original image and transformed images with different window size.

Window Size	T+ domain	I+ domain	F+ domain	T- domain	I- domain	F- domain
1x1	0.00472	0.07221	0.62480	0.21449	0.07221	0.51189
2x2	0.01003	0.07220	0.45200	0.21194	0.07220	0.50223
3x3	0.00652	0.07220	0.48482	0.20922	0.07220	0.49778
4x4	0.01455	0.07220	0.43370	0.20797	0.07219	0.49336
5x5	0.01373	0.07220	0.44652	0.20609	0.07220	0.49000
6x6	0.01829	0.07220	0.42206	0.20475	0.07220	0.48633
7x7	0.01577	0.07220	0.43519	0.20300	0.07220	0.48337
8x8	0.01909	0.07220	0.41824	0.20198	0.07219	0.47981
9x9	0.01729	0.07220	0.42767	0.20059	0.07220	0.47688
10x10	0.01996	0.07219	0.41447	0.19968	0.07219	0.47353

Table 2. MSE values of brain image in BNS domain associated with different neighborhood windows.

Following plots shows the variations in MSE when we increase the window size.

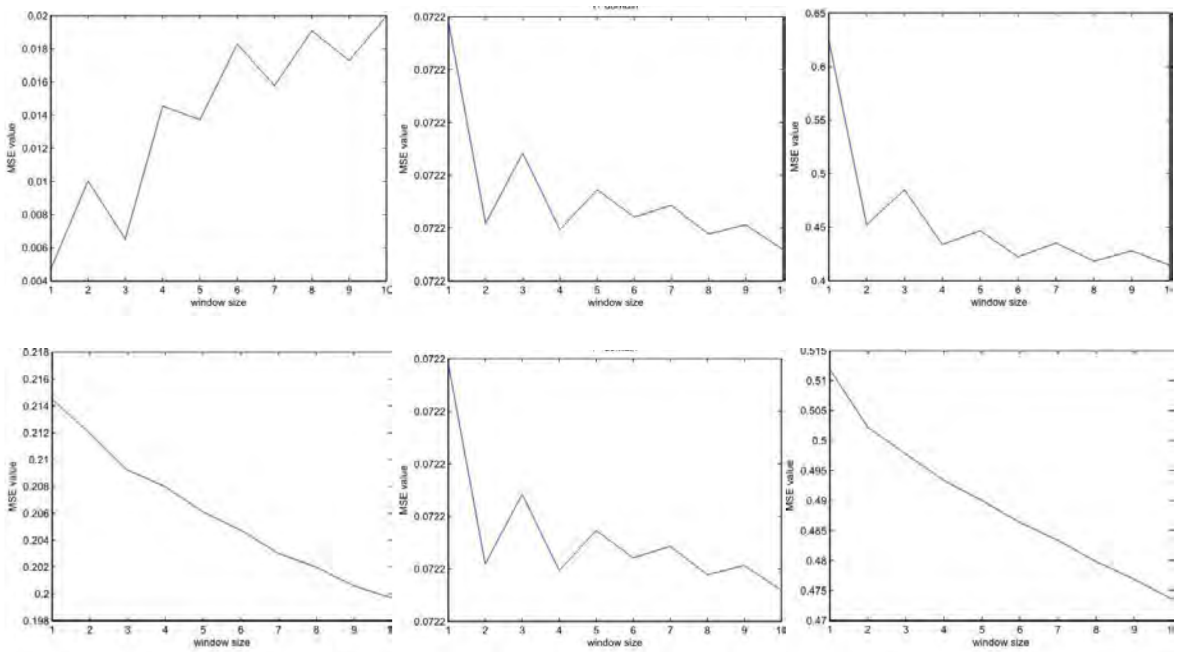


Figure 8. Comparison of PSNR values and neighborhood window size in (a) T+ domain, (b) I+ domain, (c) F+ domain, (d) T- domain, (e) I- domain, (f) F- domain.

The following table shows the entropies of each images in bipolar neutrosophic domain. It represents the uncertainty level of a gray-scale image. Particularly, higher entropy value means, it gives more detailed information about the image; likewise, lower entropy value means, it gives less information. Roughly speaking, higher entropy represents distribution level high intensity pixels and lower entropy represents distribution level of low intensity pixels.

	T+ domain	**I+ domain**	**F+ domain**	**T- domain**	**I- domain**	**F- domain**
Entropy Value	4.5872	0.0258	4.5872	3.9005	0.0361	3.9005

Table 3. Entropy values of brain image in BNS domain

	T domain	**I domain**	**F domain**
Entropy Value	6.0492	3.9579	6.0492

Table 4. Entropy values of brain image in NS domain

From the above Table 3 and Table 4. we can clearly see that the variations of entropy values between neutrosophic domain and bipolar neutrosophic domain. Entropy values of indeterminacy domain in bipolar neutrosophic domain is significantly reduced when compared to neutrosophic domain. So we conclude that our bipolar neutrosophic domain of gray scale image performed well.

5. Conclusions

A new technique to represent gray scale image in bipolar neutrosophic domain is proposed. While the image is transformed into bipolar neutrosophic domain, the indeterminacy degree of both positive and negative membership domain is reduced significantly. So this transformation gives more useful information compared to neutrosophic domain. Further, we discussed about the gray level distribution of images in bipolar neutrosophic domain through histogram. Selection of neighborhood window is important in this transformation. Large window gives best transformation, but we lose essential information of original image. We compared most popular metrics PSNR and MSE for our transformed images associated with different neighborhood sizes. PSNR and MSE both are useful parameters to determine the quality of gray-scale images by analyzing distribution of gray levels. Our future work will include image analysis and image processing through bipolar neutrosophic domain.

Conflicts of Interest: The authors declare no conflict of interest.

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Single Valued Neutrosophic Coloring

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Abstract: Neutrosophic set was introduced by Smarandache in 1998. Due to some real time situation, decision makers deal with uncertainty and inconsistency to identify the best result. Neutrosophic concept helps to investigate the vague or indeterminacy values. Graph structures used to reduce the complications in solving the system of equations for finding the decision of some real-life problems. In this research study, we introduced the single-valued neutrosophic coloring concept. We introduce various notions, single valued neutrosophic vertex coloring, single valued neutrosophic edge coloring, and single valued neutrosophic total coloring and support those definitions with some examples.

Keywords: single-valued neutrosophic graphs; single-valued neutrosophic vertex coloring; single-valued neutrosophic edge coloring; single-valued neutrosophic total coloring.

1. Introduction

Graph theory plays a vital role in real time problems Graph represents the connection among the points by lines and is the useful tool to solve the network problems. It is applicable in many fields such as computer science, physical science, electrical communication engineering, economics and Operation Research etc. In 1852, Francis Guthrie's four-color conjecture gave the sparkle for the new branch, graph coloring in graph theory. Graph coloring is assigning the color to the vertices or edges or both vertices and edges of the graph based on some conditions. After three decades got the solution to Guthrie's conjecture. Graph coloring technique used in many areas like telecommunication, scheduling, computer networks etc. Sometime in real-life have to deal with imprecise data and uncertain relation between points, in that case fuzzy technique where came. In 1965, Fuzzy set theory was introduced by Zadeh [39] and further work on fuzzy graph theory developed by A. Rosenfeld [33] in 1975. The fuzzy chromatic number was introduced by Munoz et al. [36] in 2004 and extended by C.Eslahchi and B.N.Onagh [23] in 2006. In 2009, S.Lavanya and R.Sattanathan [30] introduced the concept fuzzy total coloring. In 2014, Anjaly Kishore, M.S.Sunitha [7] discussed the strong chromatic number of fuzzy graphs in their research paper.

Intuitionistic fuzzy sets are dealing membership and non-membership data. Kassimir T.Atanassov [13] introduced the concept of intuitionistic fuzzy sets in 1986 and intuitionistic fuzzy graph in 1999. Ismail and Rifayathali [28] discussed the coloring of intuitionaistic fuzzy graphs using (α, β) cuts in 2015, Rifayathali et al. [32] discussed intuitionistic fuzzy coloring and strong intuitionistic fuzzy coloring in 2017 and 2018.

Vague set concept introduced by Gau and Buehrer [26] in 1993 and in 2014, Akram et al. [11] discussed vague graphs and further work extended by Borzooei et al. [14, 15], Vertex and Edge coloring of Vague graphs were introduced by Arindam Dey et al [12] in 2018.

In all real-time cases, the membership and non-membership values are not enough to find the result. Sometimes the vague or indeterminacy qualities need to be considered for the decision making, in that case intuitionistic fuzzy logic insufficient to give the solution. This situation reasoned for to move the new concept, F.Smarandache came with a solution"Neutrosophic logic". Neutrosophic logic play a vital role in several of the real valued problems like law, medicine, industry, finance, engineering, IT, etc.

Neutrosophic set was introduced by F.Smarandache [35] in 1998, Neutrosophic set a generalisation of the intuitionistic fuzzy set. It consists truth value, indeterminacy value and false values.Wang et al. [38] worked on Single valued neutrosophic sets in 2010. Strong Neutrosophic graph and its properties were introduced and discussed by Dhavaseelan et al. [25] in 2015 and Single valued neutrosophic concept introduced in 2016 by Akram and Shahzadi [8, 9, 10]. Broumi et al. [16, 17, 18, 19, 20, 21, 22] extended their works in Single valued neutrosophic graphs, Isolated single valued graphs, Uniform single valued graphs, Interval valued neutrosophic graphs (IVNG) and Bipolar neutrosophic graphs. Dhavaseelan et al. [24] in 2018, discussed Single valued co-neutrosophic graphs in their paper. Sinha et al. [34] extended the single valued work for signed digraphs in 2018 and Vasile [37] proposed five penta-valued refined neutrosophic indexes representation in his work. In 2019, jan et al. in their paper [29] have reviewed the following definitions: Interval-Valued Fuzzy Graphs (IVFG), Interval-Valued Intuitionistic Fuzzy Graphs (IVIFG), Complement of IVFG, SVNG, IVNG and the complement of SVNG and IVNG. They have modified those definitions, supported with some examples. Neutrosophic graphs happen to play a vital role in the building of neutrosophic models. Also, these graphs can be used in networking, Computer technology, Communication, Genetics, Economics, Sociology, Linguistics, etc., when the concept of indeterminacy is present.

Abdel-Basset et al. used Neutrosophic concept in their papers [1, 2, 3, 4, 5, 6, 31] to find the decisions for some real-life operation research and IoT-based enterprises in 2019. The above papers given the idea to interlink the graph coloring concept in SVNG when deal with vague or indeterminacy qualities.

In this research paper, we introduced the concept of single valued neutrosophic vertex coloring, single valued neutrosophic edge coloring and single valued neutrosophic total coloring of single valued neutrosophic graph and also Strong and Complete Single valued neutrosophic graph coloring are discussed with examples.

Definition 1.1. [35]

Let X be a space of points(objects). A neutrosophic set A in X is characterized by truth-membership function $t_A(x)$, an indeterminacy-membership function $i_A(x)$ and a falsity-membership function $f_A(x)$. The functions $t_A(x)$, $i_A(x)$, and $f_A(x)$, are real standard or non-standard subsets of $]0^-, 1^+[$. That is, $t_A(x): X \to]0^-, 1^+[$, $i_A(x): X \to]0^-, 1^+[$ and $f_A(x): X \to]0^-, 1^+[$ and $0^- \leq t_A(x) + i_A(x) + f_A(x) \leq 3^+$.

Definition 1.2. [9]

A single-valued neutrosophic graphs (SVNG) G = (X, Y) is a pair where X: N → [0,1] is a single-valued neutrosophic set on N and Y: N × N → [0,1] is a single-valued neutrosophic relation on N such that

$$t_Y(xy) \leq \min\{t_X(x), t_X(y)\},$$

$$i_Y(xy) \leq \min\{i_X(x), i_X(y)\},$$

$$f_Y(xy) \leq \max\{f_X(x), f_X(y)\},$$

for all x,y ∈ N. X and Y are called the single-valued neutrosophic vertex set of G and the single-valued neutrosophic edge set of G, respectively. A single-valued neutrosophic relation Y is said to be symmetric if $t_Y(xy) = t_Y(yx)$, $i_Y(xy) = i_Y(yx)$ and $f_Y(xy) = f_Y(yx)$, for all x,y ∈ N. Single-valued neutrosophic be abbreviated here as SVN.

Definition 1.3. [10]

The complement of a SVNG G = (X, Y) is a SVNG $\bar{G} = (\bar{X}, \bar{Y})$, where

1. $\bar{X} = X$
2. $\bar{t_X}(x) = t_X(x), \bar{i_X}(x) = i_X(x), \bar{f_X}(x) = f_X(x)\ for\ all\ x \in X$
3. $\bar{t_X}(xy) = \begin{cases} \min\{t_X(x), t_X(y)\}\ if\ \ t_Y(xy) = 0 \\ \min\{t_X(x), t_X(y)\} - t_Y(xy)\ if\ \ t_Y(xy) > 0 \end{cases}$

$$\bar{i_X}(xy) = \begin{cases} \min\{i_X(x), i_X(y)\}\ if\ \ i_Y(xy) = 0 \\ \min\{i_X(x), i_X(y)\} - i_Y(xy)\ if\ \ i_Y(xy) > 0 \end{cases}$$

$$\bar{f_X}(xy) = \begin{cases} \max\{f_X(x), f_X(y)\}\ if\ \ f_Y(xy) = 0 \\ \max\{f_X(x), f_X(y)\} - f_Y(xy)\ if\ \ f_Y(xy) > 0 \end{cases}$$

for all $x, y \in X$.

2. Single-Valued Neutrosophic Vertex Coloring (SVNVC)

In this section, we have developed SVNVC and this coloring has verified through some examples of SVNG, CSVNG and SSVNG. Also discussed some theorems.

Definition 2.1.

A family $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ of SVN fuzzy set is called a k-SVNVC of a SVNG G = (X, Y) if

1. $\vee\, \gamma_i(x) = X,\ \forall x \in X$
2. $\gamma_i \wedge \gamma_j = 0$
3. For every incident vertices of edge xy of G, $\min\{\gamma_i(m_1(x)), \gamma_i(m_1(y))\} = 0$, $\min\{\gamma_i(i_1(x)), \gamma_i(i_1(y))\} = 0\ and\ \max\{\gamma_i(n_1(x)), \gamma_i(n_1(y))\} = 1, (1 \leq i \leq k)$.

This k-SVNVC of G is denoted by $\chi_v(G)$, is called the SVN chromatic number of the SVNG G.

Example 2.2.

Consider the SVNG G = (X,E) with SVN vertex set $X = \{x_1, x_2, x_3, x_4, x_5\}$ and SVN edge set $E = \{X_iX_j | ij = 12,14,15,23,24,25,34,35,45\}$ the membership functions defined as,

$$\left(m_1(x_i), i_1(x_i), n_1(x_i)\right) = \begin{cases} (0.3,0.2,0.6)\ for\ i = 1,2 \\ (0.7,0.1,0.2)\ for\ i = 3 \\ (0.2,0.1,0.7)\ for\ i = 4 \\ (0.5,0.1,0.7)\ for\ i = 5 \end{cases}$$

$$\left(m_2(x_ix_j), i_2(x_ix_j), n_2(x_ix_j)\right) = \begin{cases} (0.3,0.2,0.6)\ for\ ij = 12 \\ (0.2,0.1,0.7)\ for\ ij = 14,24,34,45 \\ (0.3,0.1,0.6)\ for\ ij = 15,23,25 \\ (0.5,0.1,0.7)\ for\ ij = 35 \end{cases}$$

Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ be a family of SVN fuzzy sets defined on X as follows:

$$\gamma_1(x_i) = \begin{cases} (0.3,0.2,0.6)\ for\ i = 1,3 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_2(x_i) = \begin{cases} (0.7,0.1,0.2)\ for\ i = 2 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_3(x_i) = \begin{cases} (0.5,0.1,0.7)\ for\ i = 4 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_4(x_i) = \begin{cases} (0.2,0.1,0.7)\ for\ i = 5 \\ (0,0,1)\ for\ others \end{cases}$$

Hence the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ fulfilled the conditions of SVNVC of the graph G. Any families below four points could not satisfy our definition. Hence the SVN chromatic number $\chi_v(G)$ of the above example is 4.

Definition 2.3.

A SVNG G = (X, Y) is called complete single-valued neutrosophic graph (CSVNG) if the following conditions are satisfied:

$$t_Y(xy) = \min\{t_X(x), t_X(y)\},$$

$$i_Y(xy) = \min\{i_X(x), i_X(y)\},$$

$$f_Y(xy) = \max\{f_X(x), f_X(y)\},$$

for all x,y ∈ X.

Definition 2.4.

A SVNG G = (X, Y) is called strong single-valued neutrosophic graph (SSVNG) if the following conditions are satisfied:

$$t_Y(xy) = \min\{t_X(x), t_X(y)\},$$

$$i_Y(xy) = \min\{i_X(x), i_X(y)\},$$

$$f_Y(xy) = \max\{f_X(x), f_X(y)\},$$

for all (x,y) ∈ Y .

Example 2.5.

Consider the SSVNG G = (X,Y) with SVN vertex set $X = \{x_1, x_2, x_3, x_4, x_5\}$ and SVN edge set $Y = \{x_i x_j | ij = 12,15,23,34,45\}$ the membership functions defined as,

$$\left(m_1(x_i), i_1(x_i), n_1(x_i)\right) = \begin{cases} (0.1,0.2,0.9)\ for\ i = 1 \\ (0.6,0.7,0.4)\ for\ i = 2 \\ (0.3,0.3,0.7)\ for\ i = 3 \\ (0.7,0.8,0.2)\ for\ i = 4 \\ (0.5,0.5,0.6)\ for\ i = 5 \end{cases}$$

$$\left(m_2(x_i x_j), i_2(x_i x_j), n_2(x_i x_j)\right) = \begin{cases} (0.1,0.2,0.9)\ for\ ij = 12,15 \\ (0.3,0.3,0.7)\ for\ ij = 23,34 \\ (0.5,0.5,0.6)\ for\ ij = 45 \end{cases}$$

Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ be a family of SVN fuzzy sets defined on X as follows:

$$\gamma_1(x_i) = \begin{cases} (0.1,0.2,0.9)\ for\ i = 1 \\ (0.3,0.3,0.7)\ for\ i = 3 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_2(x_i) = \begin{cases} (0.6,0.7,0.4)\ for\ i = 2 \\ (0.7,0.8,0.2)\ for\ i = 4 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_3(x_i) = \begin{cases} (0.5,0.5,0.6)\ for\ i = 5 \\ (0,0,1)\ for\ others \end{cases}$$

Hence the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ fulfilled the conditions of Strong SVNVC of the graph G. Any families below three points could not satisfy our definition. Hence the SSVN chromatic number $\chi_v(G)$ of the above example is 3.

Example 2.6.

Consider the CSVNG G = (X,Y) with SVN vertex set $X = \{x_1, x_2, x_3, x_4, x_5\}$ and SVN edge set $Y = \{x_i x_j | ij = 12,13,14,23,24,34\}$ the membership functions defined as,

$$\left(m_1(x_i), i_1(x_i), n_1(x_i)\right) = \begin{cases} (0.7,0.7,0.1)\ for\ i = 1 \\ (0.6,0.7,0.3)\ for\ i = 2 \\ (0.3,0.3,0.7)\ for\ i = 3 \\ (0.1,0.1,0.8)\ for\ i = 4 \end{cases}$$

$$\left(m_2(x_i x_j), i_2(x_i x_j), n_2(x_i x_j)\right) = \begin{cases} (0.6,0.7,0.3)\ for\ ij = 12 \\ (0.3,0.3,0.7)\ for\ ij = 13,23 \\ (0.1,0.1,0.8)\ for\ ij = 14,24,34 \end{cases}$$

Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ be a family of SVN fuzzy sets defined on X as follows:

$$\gamma_1(x_i) = \begin{cases} (0.7,0.7,0.1)\ for\ i = 1 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_2(x_i) = \begin{cases} (0.6,0.7,0.3)\ for\ i = 2 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_3(x_i) = \begin{cases} (0.3,0.3,0.7)\ for\ i = 3 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_4(x_i) = \begin{cases} (0.1,0.1,0.8)\ for\ i = 4 \\ (0,0,1)\ for\ others \end{cases}$$

Hence the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ fulfilled the conditions of complete SVNVC of the graph G. Any families below four points could not satisfy our definition. Hence the SVN chromatic number $\chi_v(G)$ of the above example is 4.

Theorem 2.7.

For any graph CSVNG with n vertices, $\chi_v(G) = n$.

Proof:

By the definition of CSVNG, all the vertices are adjacent to each other. Each color class contains exactly one vertex with the value (t_X(x), t_X(x),t_X(x))> 0, thus remaining vertices are with the value (t_X(x), t_X(x),t_X(x)) = 0. Hence $\chi_v(G) = n$.

Theorem 2.8.

For any SSVNG G, then $\overline{\overline{\chi_v}}(G) = \chi_v(G)$.

Proof. It is obvious.

3. Single-Valued Neutrosophic Edge Coloring (SVNEC)

In this section, we introduced and discussed SVNEC with an example and theorems.

Definition 3.1.

A family $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ of SVN fuzzy set is called a k-SVNEC of a SVNG G = (X,Y) if

1. $\vee\, \gamma_i(xy) = Y,\ \forall xy \in Y$
2. $\gamma_i \wedge \gamma_j = 0$
3. For every strong edge xy of G, $\min\{\gamma_i(m_2(xy))\} = 0,\ \min\{\gamma_i(i_2(xy))\} = 0\ and\ \max\{\gamma_i(n_2(xy))\} = 1, (1 \le i \le k)$.

This k-SVNEC of G is denoted by$\chi_e(G)$, is called the SVN chromatic number of the SVNG G.

Example 3.2.

Consider the SVNG G = (X,Y) with SVN vertex set $X = \{x_1, x_2, x_3, x_4\}$ and SVN edge set $Y = \{x_i x_j | ij = 12,13,14,23,24,34\}$ the membership functions defined as,

$$\left(m_1(x_i), i_1(x_i), n_1(x_i)\right) = \begin{cases} (0.3,0.1,0.6)\ for\ i = 1 \\ (0.2,0.1,0.4)\ for\ i = 2 \\ (0.5,0.2,0.4)\ for\ i = 3 \\ (0.4,0.1,0.4)\ for\ i = 4 \end{cases}$$

$$\left(m_2(x_ix_j), i_2(x_ix_j), n_2(x_ix_j)\right) = \begin{cases} (0.2,0.1,0.4)\ for\ ij = 12,23,24 \\ (0.3,0.1,0.6)\ for\ ij = 13,14 \\ (0.4,0.1,0.4)\ for\ ij = 24 \end{cases}$$

Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ be a family of SVN fuzzy sets defined on Y as follows:

$$\gamma_1(x_ix_j) = \begin{cases} (0.2,0.1,0.4)\ for\ i = 12,34 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_2(x_ix_j) = \begin{cases} ((0.3,0.1,0.6))\ for\ i = 14,23 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_3(x_ix_j) = \begin{cases} (0.4,0.1,0.4)\ for\ i = 13,24 \\ (0,0,1)\ for\ others \end{cases}$$

Hence the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ fulfills the conditions of SVNEC of SVNG. Any families below three members could not satisfy our definition. Hence, the SVN chromatic number $\chi_e(G)$ of the above example is 3.

4. Single-Valued Neutrosophic Total Coloring (SVNTC)

In this section, we defined SVNTC supported by an example.

Definition 4.1.

A family $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ of SVN fuzzy sets on the SVN vertex set X is called a k-SVNTC of SVNG G = (X, Y) if

1. $\vee\, \gamma_i(x) = X,\ \forall x \in X$ and $\vee\, \gamma_i(xy) = Y,\ \forall xy \in Y$
2. $\gamma_i \wedge \gamma_j = 0$
3. For every incident vertices of edge xy of G, $\min\{\gamma_i(m_1(x)), \gamma_i(m_1(y))\} = 0$, $\min\{\gamma_i(i_1(x)), \gamma_i(i_1(y))\} = 0\ and\ \max\{\gamma_i(n_1(x)), \gamma_i(n_1(y))\} = 1, (1 \le i \le k)$. For every strong edge xy of G, $\min\{\gamma_i(m_2(xy))\} = 0$, $\min\{\gamma_i(i_2(xy))\} = 0\ and\ \max\{\gamma_i(n_2(xy))\} = 1, (1 \le i \le k)$.

This k-SVNTC of G is denoted by$\chi_t(G)$, is called the SVN chromatic number of the SVNG G.

Example 4.2.

Consider the SVNG G = (X,Y) with SVN vertex set $X = \{x_1, x_2, x_3, x_4, x_5\}$ and SVN edge set $Y = \{x_ix_j | ij = 12,13,14,15,23,24,25,34,35,45\}$ the membership functions defined as,

$$\left(m_1(x_i), i_1(x_i), n_1(x_i)\right) = \begin{cases} (0.3,0.1,0.7)\ for\ i = 1 \\ (0.5,0.3,0.5)\ for\ i = 2 \\ (0.4,0.2,0.6)\ for\ i = 3 \\ (0.8,0.6,0.2)\ for\ i = 4 \\ (0.7,0.5,0.3)\ for\ i = 5 \end{cases}$$

$$\left(m_2(x_ix_j), i_2(x_ix_j), n_2(x_ix_j)\right) = \begin{cases} (0.3,0.1,0.7)\ for\ ij = 12,13,14,15 \\ (0.8,0.6,0.2)\ for\ ij = 45 \\ (0.4,0.2,0.6)\ for\ ij = 23,24,25 \\ (0.5,0.3,0.5)\ for\ ij = 34,35 \end{cases}$$

Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ be a family of SVN fuzzy sets defined on Y as follows:

$$\gamma_1(x_i) = \begin{cases} (0.3,0.1,0.7)\ for\ i = 1 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_2(x_i) = \begin{cases} (0.5,0.3,0.5)\ for\ i = 2 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_3(x_i) = \begin{cases} (0.4,0.2,0.6)\ for\ i = 3 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_4(x_i) = \begin{cases} (0.8,0.6,0.2)\ for\ i = 4 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_5(x_i) = \begin{cases} (0.7,0.5,0.3)\ for\ i = 5 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_1(x_i x_j) = \begin{cases} (0.3,0.1,0.7)\ for\ i = 12 \\ (0.5,0.3,0.5)\ for\ i = 35 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_2(x_i x_j) = \begin{cases} (0.3,0.1,0.7)\ for\ i = 13 \\ (0.4,0.2,0.6)\ for\ i = 24 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_3(x_i x_j) = \begin{cases} (0.3,0.1,0.7)\ for\ i = 14 \\ (0.4,0.2,0.6)\ for\ i = 25 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_4(x_i x_j) = \begin{cases} (0.8,0.6,0.2)\ for\ i = 45 \\ (0.4,0.2,0.6)\ for\ i = 23 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_5(x_i x_j) = \begin{cases} (0.3,0.1,0.7)\ for\ i = 15 \\ (0.5,0.3,0.5)\ for\ i = 34 \\ (0,0,1)\ for\ others \end{cases}$$

Hence the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ fulfills the conditions of SVNTC of SVNG. Any families below five members could not satisfy our definition. Hence the SVN chromatic number $\chi_t(G)$ of the above example is 5.

5. **Conclusions**

Single Valued Neutrosophic Coloring concept introduced in this paper. Single valued neutrosophic vertex coloring, single valued neutrosophic edge coloring and single valued neutrosophic total coloring are defined. All thus definitions are developed and supported by some of the examples. In future, it will be extended to examine the theory of SVNC with the irregular colorings of graphs.

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An Integrated Neutrosophic and MOORA for Selecting Machine Tool

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Abstract: The selection of suitable machine tools for a manufacturing company is one of the significant points to achieving high competitiveness in the market. Besides, an appropriate choice of machine tools is very significant as it helps to realize full production quickly. Today's market offers many more choices for machine tool alternatives. There are also many factors one should consider as part of the appropriate machine tool selection process, including productivity, flexibility, compatibility, safety, cost, etc. Consequently, evaluation procedures involve several objectives, and it is often necessary to compromise among possibly conflicting tangible and intangible factors. For these reasons, multiple criteria decision making (MCDM) is a useful approach to solve this kind of problem. Most of the MCDM models are mathematical and ignore qualitative and often subjective considerations. The use of neutrosophic set theory allows incorporating qualitative and partially known information into the decision model. This paper describes a neutrosophic Multi-Objective Optimization on the basis of Ratio Analysis (MOORA) based methodology for evaluation and selection of vertical CNC machining centers for a manufacturing company in Tenth of Ramadan, Egypt.

Keywords: Machine Tool; Neutrosophic MOORA; MCDM

1. Introduction

Selecting an appropriate machine tool is one of the most complicated and time-consuming problems for manufacturing companies due to many feasible alternatives and conflicting objectives. The determination and evaluation of positive and negative characteristics of one alternative relative to others is a difficult task. The selection process of suitable machine tools has to begin with a critical evaluation of the procedures on the shop floor by considering an array of quantitative, qualitative, and economic concerns. Hence the decision-maker (engineer or manager) needs a lot of criteria to be found and a large amount of data to be analyzed for a proper and sufficient evaluation. Consequently using proper machine tools in a manufacturing facility can improve the production process, provide effective utilization of resources, increase productivity, and enhance system flexibility, repeatability, and reliability. Many potential criteria, such as flexibility, compatibility, safety, maintainability, cost, etc. must be considered in the selection procedure of a machine tool. Therefore machine tool selection can be viewed as a multiple criteria decision making (MCDM) problem in the presence of many quantitative and qualitative criteria. The MCDM methods deal with the process of making decisions in the presence of multiple criteria or objectives. A decision-maker (DM) is required to choose among quantifiable or non-quantifiable and various criteria. The DM's evaluations on qualitative criteria are always subjective and thus imprecise. The objectives are usually conflicting, and therefore, the solution is highly dependent on the preferences of the DM. Besides, it is complicated to develop a selection criterion that can precisely describe the choice of one alternative over another. The evaluation data of machine tool alternatives suitability for various subjective criteria and the weights of the criteria are usually expressed in linguistic terms. This makes neutrosophic logic a more natural approach to this kind of problems.

Many researchers have attempted to use fuzzy MCDM methods for selection problems. The purpose of this paper is to present a hybrid method between MOORA and Neutrosophic in the framework of neutrosophic for the selection of machine tool with a focus on multi-criteria and multi-group environment. These days, Companies, organizations, factories seek to provide a fast and a good service to meet the requirements of peoples or customers. The selecting of the best supplier increasing the efficiency of any organization whether company, factory according to [1]. Hence, for selecting the best supplier selection there are much of methodologies we presented some of them such as fuzzy sets (FS), Analytic network process (ANP), Analytic hierarchy process(AHP), (TOPSIS) technique for order of preference by similarity to ideal solution, (DSS) Decision support system, (MOORA)multi-objective optimization by ratio analysis.

1.1 Supplier selection

A Supplier choice is viewed as one of the most significant parts of creation and indecency the board for some, association's administration. The primary objective of provider choice is to recognize providers with the most outstanding ability for gathering an association needs reliably and with the base expense. They are utilizing a lot of standard criteria and measures for abroad examination of providers. Be that as it may, the degree of detail used for inspecting potential providers may differ contingent upon an association's needs.

The fundamental reason and target objective of determination are to recognize high-potential providers. To pick providers, the present association judge of every provider as per the capacity of gathering the association reliably and financially savvy its needs utilizing choice criteria and proper measure. Criteria and standards are created to be material to every one of the providers being considered and to mirror the company's needs and its supply and innovation technique. We show supplier evaluation and selection process in Fig.1 and in Fig.2.

Figure 1. Supplier evaluation and selection process.

Figure 2. Supplier evaluation and selection process.

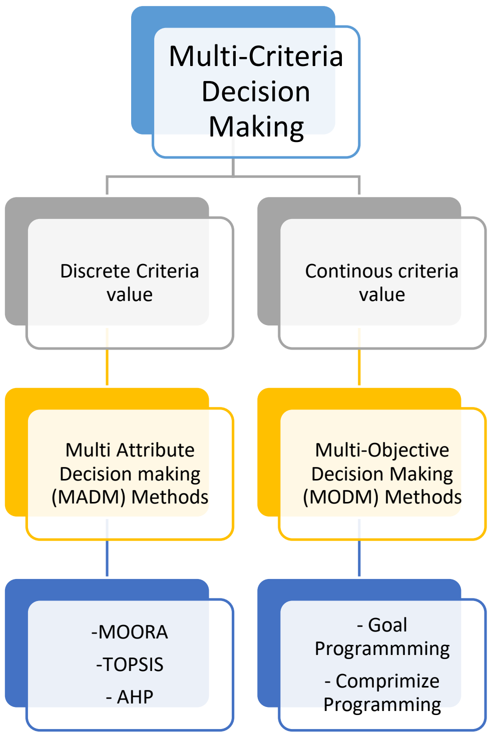

Figure 3. MOORA method belongs to MADM

1.2 MOORA

Multi-Objective Optimization based on Ratio Analysis (MOORA), otherwise called multi-criteria or multi-property advancement. MOORA the technique looks to rank or chooses the best elective from accessible choice was presented by Brauers and Zavadskas in 2006. The MOORA technique has a considerable scope of utilizations to settle on choices in the clashing and troublesome region of production network condition.

MOORA can be connected in the task determination, process structure choice, area choice, item choice and so on the way toward characterizing the choice objectives, gathering essential data and choosing the best ideal option is known as necessary leadership process. The fundamental thought of the MOORA technique is to ascertain the general execution of every opportunity as the contrast between the wholes of its standardized exhibitions, which has a place with expense and advantage criteria. This strategy connected in different fields effectively, for example, venture the executives. Fig.3 shows to which category belongs the method of MOORA.

1.3 Neutrosophic

There are numerous vulnerabilities in everyday life. The rationale of old-style science regularly lacks to clarify these vulnerabilities. Since it isn't always conceivable to call a circumstance or occasion right or wrong, for instance, we can't generally call the climate cold or hot. It very well may be heated for a few, frozen for a few and cool for other people.

Comparable circumstances in which we stay ambivalent may show up in the expert capability appraisal. It is frequently hard to decide if work is done or an item delivered is consistently definite great or unmistakable awful. Such a circumstance lessens the unwavering quality of assessing proficient proficiencies. To adapt to these vulnerabilities, Smarandache characterized the idea of the neutrosophic rationale and neutrosophic set [2] in 1998. In the concept of the neutrosophic explanation and neutrosophic bunches, there is a T level of participation, and I level of indeterminacy and F level of non-enrollment. These degrees are characterized autonomously of one another. It has a neutrosophic esteem (T, I, F) structure. A condition is dealt with as indicated by the two its precision and its error and its vulnerability. In this way, neutrosophic rationale and neutrosophic set assistance us to clarify numerous vulnerabilities in our lives. Furthermore, various scientists have made examinations on this hypothesis [3 - 7].

We present some of the methodologies that are used in the multi-criteria decision making and presenting the illustration between supplier selection, MOORA, and Neutrosophic. Hence the goal of this paper to present the hybrid of the MOORA method with neutrosophic as a methodology for MCDM.

This is ordered as follows: Section 2 gives an insight into some basic definitions on neutrosophic sets and MOORA. Section 3 explains the proposed methodology of neutrosophic MOORA model. In Section 4 a numerical example is presented in order to explain the proposed methodology. Finally, the conclusions

2. Preliminaries

In this Section, the fundamental definitions including neutrosophic set, single-esteemed neutrosophic sets, trapezoidal neutrosophic numbers and tasks on trapezoidal neutrosophic numbers are characterized.

Definition 2.1 Let X be a space of points and $x \in X$. A neutrosophic set A in X is definite by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$, $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or real nonstandard subsets of $]{-0}, 1{+}[$. That is $T_A(x){:}X \to]{-0}, 1{+}[$, $I_A(x){:}X \to]{-0}, 1{+}[$ and $F_A(x){:}X \to]{-0}, 1{+}[$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0- \leq \sup(x) + \sup x + \sup x \leq 3+$.

Definition 2.2: Let X be a universe of discourse. A single valued neutrosophic set A over X is an object taking the form $A = \{\langle x, T_A(x), I_A(x), F_A(x), \rangle : x \in X\}$, where $T_A(x){:}X \to [0,1]$, $I_A(x){:}X \to [0,1]$ and $F_A(x){:}X \to [0,1]$ with $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ for all $x \in X$. The intervals $T_A(x)$, $I_A(x)$ and $F_A(x)$ represent the truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively. For convenience, a SVN number is represented by $A = (a$, b, c), where a, b, $c \in [0, 1]$ and $a+b+c \leq 3$.

Definition 2.3: Suppose that $\alpha_{\tilde{a}}$, $\theta_{\tilde{a}}$, $\beta_{\tilde{a}} \in [0,1]$ and $a_1, a_2, a_3, a_4 \in$ R where $a_1 \leq a_2 \leq a_3 \leq a_4$. Then a single valued trapezoidal neutrosophic number, $\tilde{a} = \langle (a_1, a_2, a_3, a_4); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle$ is a special neutrosophic set on the real line set R whose truth-membership, indeterminacy-membership and falsity-membership functions are defined as:

$$T_{\tilde{a}}(x) = \begin{cases} \alpha_{\tilde{a}} \left(\frac{x-a_1}{a_2-a_1}\right) & (a_1 \leq x \leq a_2) \\ \alpha_{\tilde{a}} & (a_2 \leq x \leq a_3) \\ \alpha_{\tilde{a}} \left(\frac{a_4-x}{a_4-a_3}\right) & (a_3 \leq x \leq a_4) \\ 0 & otherwise \end{cases} \quad (1)$$

$$I_{\tilde{a}}(x) = \begin{cases} \frac{(a_2-x+\theta_{\tilde{a}}(x-a_1))}{(a_2-a_1)} & (a_1 \leq x \leq a_2) \\ \alpha_{\tilde{a}} & (a_2 \leq x \leq a_3) \\ \frac{(x-a_3+\theta_{\tilde{a}}(a4-x))}{(a_4-a_3)} & (a_3 \leq x \leq a_4) \\ 1 & otherwise \end{cases}, \quad (2)$$

$$F_{\tilde{a}}(x) = \begin{cases} \frac{(a_2-x+\beta_{\tilde{a}}(x-a_1))}{(a_2-a_1)} & (a_1 \leq x \leq a_2) \\ \alpha_{\tilde{a}} & (a_2 \leq x \leq a_3) \\ \frac{(x-a_3+\beta_{\tilde{a}}(a4-x))}{(a_4-a_3)} & (a_3 \leq x \leq a_4) \\ 1 & otherwise \end{cases}, \quad (3)$$

Where $\alpha_{\tilde{a}}$, $\theta_{\tilde{a}}$ and $\beta_{\tilde{a}}$ and represent the maximum truth-membership degree, minimum indeterminacy-membership degree and minimum falsity-membership degree respectively. A single valued trapezoidal neutrosophic number $\tilde{a}$=$\langle(a_1 , a_2 , a_3 , a_4); \alpha_{\tilde{a}} , \theta_{\tilde{a}} , \beta_{\tilde{a}}\rangle$ may express an ill-defined quantity of the range, which is approximately equal to the interval $[a_2 , a_3]$.

Definition 2.4: Let $\tilde{a}$=$\langle(a_1 , a_2 , a_3 , a_4); \alpha_{\tilde{a}} , \theta_{\tilde{a}} , \beta_{\tilde{a}}\rangle$ and $\tilde{b}$=$\langle(b_1 , b_2 , b_3 , b_4); \alpha_{\tilde{b}} , \theta_{\tilde{b}} , \beta_{\tilde{b}}\rangle$ be two single valued trapezoidal neutrosophic numbers and ϒ≠ 0 be any real number. Then,

1. Addition of two trapezoidal neutrosophic numbers

   $\tilde{a}+\tilde{b}$ =$\langle(a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle$
2. Subtraction of two trapezoidal neutrosophic numbers

   $\tilde{a}-\tilde{b}$ =$\langle(a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle$
3. Inverse of trapezoidal neutrosophic number

   $\tilde{a}^{-1}$ =$\langle(\frac{1}{a_4} , \frac{1}{a_3} , \frac{1}{a_2} , \frac{1}{a_1}) ; \alpha_{\tilde{a}} , \theta_{\tilde{a}} , \beta_{\tilde{a}}\rangle$ where ($\tilde{a}\neq 0$)
4. Multiplication of trapezoidal neutrosophic number by constant value

   $$\Upsilon\tilde{a} = \begin{cases} \langle(\Upsilon a_1 , \Upsilon a_2 , \Upsilon a_3 , \Upsilon a_4); \alpha_{\tilde{a}} , \theta_{\tilde{a}} , \beta_{\tilde{a}}\rangle & \text{if } (\Upsilon > 0) \\ \langle(\Upsilon a_4 , \Upsilon a_3 , \Upsilon a_2 , \Upsilon a_1); \alpha_{\tilde{a}} , \theta_{\tilde{a}} , \beta_{\tilde{a}}\rangle & \text{if } (\Upsilon < 0) \end{cases}$$
5. Division of two trapezoidal neutrosophic numbers

   $$\frac{\tilde{a}}{\tilde{b}} = \begin{cases} \langle(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1}); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle & \text{if } (a_4 > 0 , b_4 > 0) \\ \langle(\frac{a_4}{b_4}, \frac{a_3}{b_3}, \frac{a_2}{b_2}, \frac{a_1}{b_1}); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle & \text{if } (a_4 < 0 , b_4 > 0) \\ \langle(\frac{a_4}{b_1}, \frac{a_3}{b_2}, \frac{a_2}{b_3}, \frac{a_1}{b_4}); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle & \text{if } (a_4 < 0 , b_4 < 0) \end{cases}$$
6. Multiplication of trapezoidal neutrosophic numbers

$$\tilde{a}\tilde{b} = \begin{cases} \langle(a_1b_1 , a_2b_2 , a_3b_3 , a_4b_4); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle & \text{if } (a_4 > 0 , b_4 > 0) \\ \langle(a_1b_4 , a_2b_3 , a_3b_2 , a_4b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle & \text{if } (a_4 < 0 , b_4 > 0) \\ \langle(a_4b_4 , a_3b_3 , a_2b_2 , a_1b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle & \text{if } (a_4 < 0 , b_4 < 0) \end{cases}$$

3. Methodology

The functionality of linguistic variables, words have more extent to describe the semantic and sentimental expressions compared with numbers. This research chooses trapezoidal neutrosophic numbers, which includes nine parameters to model linguistic variables. The trapezoidal neutrosophic scales used in this proposed research exhibited in Table 1.

Table 1. Semantic expressions for the significance weights of criteria

Linguistic expressions	Trapezoidal neutrosophic numbers (T, I, I, F; $\alpha_{\tilde{a}}$, $\theta_{\tilde{a}}$, $\beta_{\tilde{a}}$)
Just Equal (JE)	(0.1, 0.2, 0.3, 0.4; 0.5, 0.1, 0.3) (0.2, 0.3, 0.3, 0.4; 0.8, 0.2, 0.3)
Equal importance (EI)	(0.3, 0.4, 0.4, 0.5; 1.0, 0.1, 0.1)
Weak importance of one over another (WIO)	(0.4, 0.5, 0.5, 0.6; 0.7, 0.3, 0.2)
Essential or strong importance (VRS)	(0.5, 0.6, 0.6, 0.7; 0.9, 0.2, 0.1)
Very Strong Importance (AS)	(0.6, 0.7, 0.7, 0.8; 0.8, 0.3, 0.5) (0.7, 0.8, 0.8, 0.9; 0.8, 0.3, 0.5) (0.9, 1.0, 1.0,1.0; 0.1, 0.2, 0.2)

In this section, the steps of the suggested neutrosophic MOORA framework are presented with detail. The suggested framework consists of such steps as follows:

Step 1. Constructing model and problem structuring.

a. Constitute a group of decision-makers.
b. Formulate the problem based on the opinions of decision-makers

Step 2. Making the pairwise comparisons matrix and determining the weight based on opinions of (DMs).

a. Identify the criteria and sub criteria C = {C_1, C_2, $C_3 \ldots C_m$}.
b. Making matrix among criteria n × m based on opinions of decision-makers.

$$W = \begin{array}{c|cccc} & C_1 & C_2 & \ldots & C_m \\ \hline \begin{matrix} C_1 \\ C_2 \\ C_3 \\ Cn \end{matrix} & \multicolumn{4}{l}{\begin{bmatrix} (l_{11}, m_{11l}, m_{11u}, u_{11}) & (l_{11}, m_{11l}, m_{11u}, u_{11}) & \cdots & (l_{1n}, m_{1nl}, m_{1nu}, u_{1n}) \\ (l_{21}, m_{21l}, m_{21u}, u_{21}) & (l_{22}, m_{22l}, m_{22u}, u_{22}) & \cdots & (l_{2n}, m_{2nl}, m_{2nu}, u_{2n}) \\ \cdots & \cdots & \cdots & \cdots \\ (l_{n1}, m_{n1l}, m_{n1u}, u_{n1}) & (l_{n2}, m_{n2l}, m_{n2u}, u_{n2}) & \cdots & (l_{nn}, m_{nnl}, m_{nnu}, u_{nn}) \end{bmatrix}} \end{array} \quad (4)$$

c. Decision-makers make pairwise comparisons matrix between criteria compared to each criterion.
d. According to, the opinion of decision-makers should be among from 0 to 1 not negative. Then, we transform neutrosophic matrix to pairwise comparisons deterministic matrix by adding (α, θ, β) and using the following equation to calculate the accuracy and score.

i. $$S(\tilde{a}_{ij}) = \frac{1}{16}[a_1 + b_1 + c_1 + d_1] \times (2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} - \beta_{\tilde{a}}) \quad (5)$$

ii. $$A(\tilde{a}_{ij}) = \frac{1}{16}[a_1 + b_1 + c_1 + d_1] \times (2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} + \beta_{\tilde{a}}) \quad (6)$$

e. We obtain the deterministic matrix by using S ($\tilde{a}_{ij}$).

f. From the deterministic matrix we obtain the weighting matrix by dividing each entry on the sum of the column.

Step 3. Determine the decision-making matrix (DMM). The method begin with define the available alternatives and criteria.

$$R = \begin{array}{c c} & \begin{array}{cccc} C_1 & C_2 & \dots & C_m \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ An \end{array} & \begin{bmatrix} (l_{11}, m_{11l}, m_{11u}, u_{11}) & (l_{11}, m_{11l}, m_{11u}, u_{11}) & \cdots & (l_{1n}, m_{1nl}, m_{1nu}, u_{1n}) \\ (l_{21}, m_{21l}, m_{21u}, u_{21}) & (l_{22}, m_{22l}, m_{22u}, u_{22}) & \cdots & (l_{2n}, m_{2nl}, m_{2nu}, u_{2n}) \\ \dots & \dots & \cdots & \dots \\ (l_{n1}, m_{n1l}, m_{n1u}, u_{n1}) & (l_{n2}, m_{n2l}, m_{n2u}, u_{n2}) & \cdots & (l_{nn}, m_{nnl}, m_{nnu}, u_{nn}) \end{bmatrix} \end{array} \quad (7)$$

Where A_i represents the available alternatives where i = 1… n and the C_j represents criteria

a. Decision makers (DMs) make pairwise comparisons matrix between criteria compared to each criterion. Using the Eqs. (5, 6) to calculate the accuracy and score.

b. We obtain the deterministic matrix by using S ($\tilde{a}_{ij}$).

Step 4. Calculate the normalized decision-making matrix from previous matrix (DMM).

a. Thereby, normalization is carried out, where the Euclidean norm is obtained according to Eq. (8) to the criterionE_j.

i. $$|Ey_j| = \sqrt{\sum_1^n E_i^2} \quad (8)$$

The normalization of each entry is undertaken according to Eq. (9)

ii. $$NE_{ij} = \frac{E_{ij}}{|E_j|} \quad (9)$$

Step 5. Compute the aggregated weighted neutrosophic decision matrix (AWNDM) as the following:

i. $$\acute{X} = X \times W \quad (10)$$

Step 6. Compute the contribution of each alternative Ny_i the contribution of each alternative

i. $$Ny_i = \sum_{i=1}^{g} Ny_i - \sum_{j=g+1}^{m} Nx_j \quad (11)$$

Step 7. Rank the alternatives.

4. Practical example

4.1 Case study

A real-world case issue is chosen to represent the utilization of the proposed methodology. The picked organization is a medium-sized assembling endeavor, which utilizes around 75 individuals and situated in the Tenth of Ramadan, Egypt. It makes a wide assortment of extra parts for the car business. In particular, the organization concentrated on sizeable measured gathering and assembling organizations working for the car business. Its creation fan is full including motor mountings, encasings, front suspension arms, fan sharp edges, indoor regulator lodgings, numerous sorts of riggings, entryway rollers, entryway handles, and so forth. The organization likewise delivers molds which are utilized to fabricate the elastic, metal, and aluminum parts. While different kinds of CNC and manual machine devices are utilized for normal generation, once in a while manual machine apparatuses are for the most part utilized as reinforcements. The organization is a metal machining activity venture demonstrating qualities of both occupation shop and clump creation. Thus client request sizes go in a wide edge. Truly, the organization has gotten an abnormal state of benefits, which began to decay as a result of a decrease in the interest level because of an innovative change and economic situations.

For instance, once in a while an essential client's requests require the expansion of the new CNC machining focuses. In addition, in some cases existing client requests require improved machining abilities including the buy of the specific CNC machining focuses. Therefore, the organization the board chose to pull in new clients by offering new aptitudes which incorporate growing machining limit and ability, lessening creation costs, expanding item quality, and shortening conveyance time. This is a basic inspiration for the first venture. First, a project team, including three engineers and two managers working for the company, was constructed. Then a detailed interview was conducted to determine the most suitable type of equipment for the company's competitiveness. At this point, new vertical CNC machining centers for the company' immediate needs were decided to purchase. The company considered four different alternative models of the three different manufacturers, which are denoted as A1, A2, A3, and A4, respectively. Furthermore, a detailed questionnaire related to the data regarding the qualitative and quantitative criteria for the machine tool selection model was prepared. Then a lot of face-to-face interviews were held to develop reliable information on the selected criteria and alternatives. After a set of interviews, four criteria were determined to perform the analysis. The four criteria are cost, operative flexibility, installation easiness, maintainability, and serviceability, which are denoted as C1, C2, C3, and C4, respectively. Cost is the purchasing cost of the machine tool. Operative flexibility means the possibility of using the machine tool as desired. It must be utilized when needed. Installation easiness means having the positive effects of the convenience of installation. Simple installation is practical and fast, along with installation time savings without requiring any particular technical ability.

Maintainability imparts to a machine tool an inherent ability to be maintained with reduced person-hours and skill levels, and fewer tools and support equipment. It is also the probability that a machine can be kept in an operational condition. Serviceability is defined as the ease with which all maintenance activities can be performed on a system. It is also defined as the ease with which all services, including implementation services, post-implementation professional services, and managed services can be performed.

4.2 Results

The aim of using Neutrosophic MOORA is to determine the importance weight of the criteria, then used to the ranking of the alternatives.

Step 1. Constitute a group of decision-makers.

Step 2. We determine the importance of each criteria based on opinion of all decision-makers as in Table 2, using the Eq.4.

Table 2. The comparison matrix between criteria for calculating weights

weights	C_1	C_2	C_3	C_4	W
C_1	(0.5, 0.5,0.5,0.5)	(0.2, 0.3, 0.3, 0.4; 0.8, 0.2, 0.3)	(0.5, 0.6, 0.6, 0.7; 0.9, 0.2, 0.1)	(0.9, 1.0, 1.0,1.0; 0.1, 0.2, 0.2)	0.17
C_2	(0.2, 0.3, 0.3, 0.4; 0.8, 0.2, 0.3)	(0.5, 0.5,0.5,0.5)	(0.7, 0.8, 0.8, 0.9; 0.8, 0.3, 0.5)	(0.2, 0.3, 0.3, 0.4; 0.8, 0.2, 0.3)	0.23
C_3	(0.7, 0.8, 0.8, 0.9; 0.8, 0.3, 0.5)	(0.4, 0.5, 0.5, 0.6; 0.7, 0.3, 0.2)	(0.5, 0.5,0.5,0.5)	(0.9, 1.0, 1.0,1.0; 0.1, 0.2, 0.2)	0.33
C_4	(0.9, 1.0, 1.0,1.0; 0.1, 0.2, 0.2)	(0.5, 0.6, 0.6, 0.7; 0.9, 0.2, 0.1)	(0.2, 0.3, 0.3, 0.4; 0.8, 0.2, 0.3)	(0.5, 0.5,0.5,0.5)	0.27

We show the weights of criteria in Fig.4.

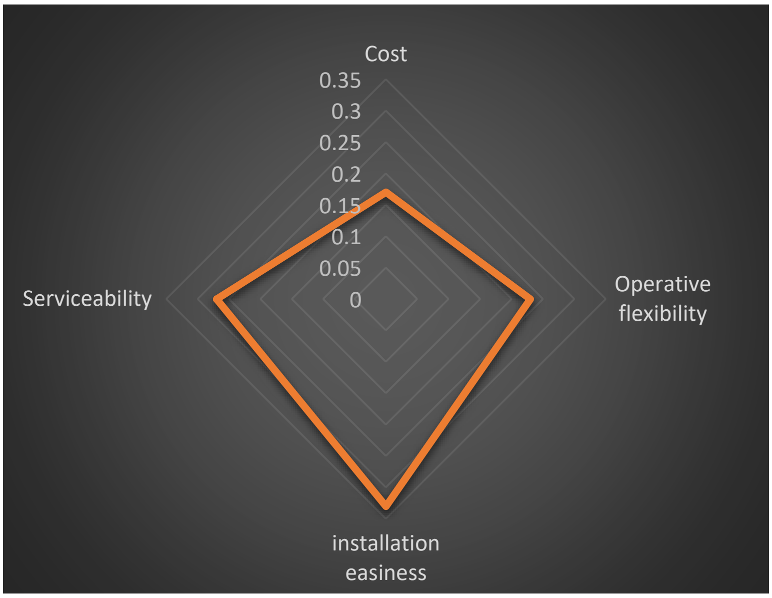

Figure 4. Weights of criteria.

Step 3. Construct the matrix that representing the ratings given by every DM between the criteria and alternatives, by using the Eq.7.

Every decision maker makes the evaluation matrix via comparing the four alternatives relative to each criteria by using the trapezoidal neutrosophic numbers scale in Table 1 as shown in Table 3.

Table 3. The comparison matrix between criteria for calculating weights

	C_1	C_2	C_3	C_4
A_1	(0.4, 0.5, 0.5, 0.6; 0.7, 0.3, 0.2)	(0.2, 0.3, 0.3, 0.4; 0.8, 0.2, 0.3)	(0.5, 0.6, 0.6, 0.7; 0.9, 0.2, 0.1)	(0.5, 0.6, 0.6, 0.7; 0.9, 0.2, 0.1)
A_2	(0.2, 0.3, 0.3, 0.4; 0.8, 0.2, 0.3)	(0.5, 0.6, 0.6, 0.7; 0.9, 0.2, 0.1)	(0.9, 1.0, 1.0,1.0; 0.1, 0.2, 0.2)	(0.9, 1.0, 1.0,1.0; 0.1, 0.2, 0.2)
A_3	(0.7, 0.8, 0.8, 0.9; 0.8, 0.3, 0.5)	(0.9, 1.0, 1.0,1.0; 0.1, 0.2, 0.2)	(0.9, 1.0, 1.0,1.0; 0.1, 0.2, 0.2)	(0.9, 1.0, 1.0,1.0; 0.1, 0.2, 0.2)
A_4	(0.2, 0.3, 0.3, 0.4; 0.8, 0.2, 0.3)	(0.5, 0.6, 0.6, 0.7; 0.9, 0.2, 0.1)	(0.2, 0.3, 0.3, 0.4; 0.8, 0.2, 0.3)	(0.2, 0.3, 0.3, 0.4; 0.8, 0.2, 0.3)

From previous Table 3 we can determine the weight of each criteria by using Eq.5 or Eq.6 in the similarity case.

Step 4. Calculate the normalized decision-making matrix from Table 3, by using Eq. (8, 9).then calculating the weights using Eq.9.

a. Sum of squares and their square roots in Table 4.

Table 4. Sum of squares and their square roots

	C_1	C_2	C_3	C_4
A_1	0.11	0.20	0.32	0.27
A_2	0.11	0.23	0.26	0.20
A_3	0.10	0.16	0.08	0.18
A_4	0.25	0.19	0.11	0.07
SS	0.17	0.14	0.20	0.14
SR	0.35	0.39	0.44	0.38

b. Objectives divided by their square roots in Table 5.

Table 5. Objectives divided by their square roots

	C_1	C_2	C_3	C_4
A_1	0.28	0.55	0.65	0.47
A_2	0.32	0.38	0.50	0.55
A_3	0.25	0.44	0.12	0.16
A_4	0.64	0.21	0.25	0.17

Step 5. Compute the contribution of each alternative by using Eq.11 as presented in Table 6

Table 6. Ranking of the alternatives.

	C_1	C_2	C_3	C_4	Y_i	Rank
A_1	0.43	0.19	0.47	0.46	0.65	2
A_2	0.45	0.56	0.24	0.33	0.85	1
A_3	0.23	0.43	0.35	0.32	0.60	3
A_4	0.65	0.32	0.33	0.28	0.45	4

Step 6. Rank the alternatives.

The higher the closeness means the better the rank, so the relative closeness to the ideal solution of the alternatives can be substituted as follows: A2 > A1 > A3 > A4 as shown in Fig.5. A2 is defined as the best alternative for this company. The obtained result is discussed in the company just as to investigate the meaningfulness of the selected alternative.

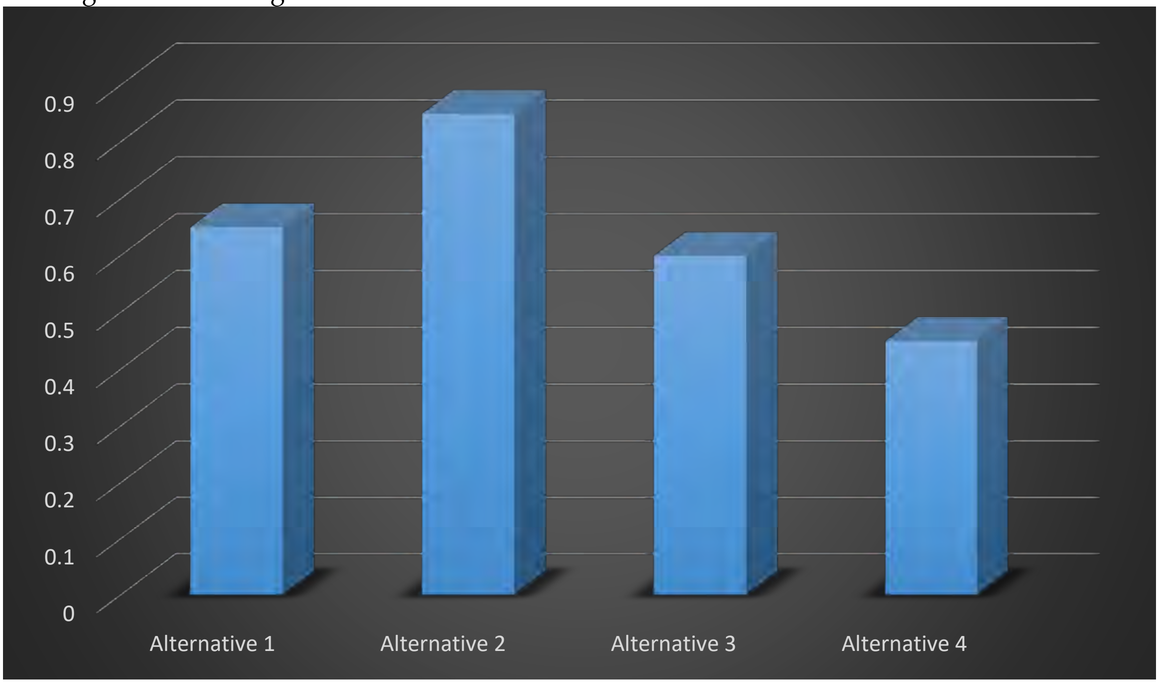

Figure 5. Ranking of the alternatives.

5. Conclusions

In this paper, a methodology based on neutrosophic and MOORA for selecting the most suitable machine tools is suggested. Also, the ranking scores are the outcomes of the methodology, and by using ranking scores, DM can obtain not only a ranking of the alternatives but also the degree of superiority among the alternatives. For dealing uncertainty and improving lack of precision in evaluating criteria and machine tool alternatives, neutrosophic methods are used. Our approach applies trapezoidal numbers into traditional MOORA method. By applying for neutrosophic numbers, DM enables to get better results in the overall importance of criteria and real alternatives.

As a result of the study, we find that the proposed method is practical for ranking machine tool alternatives concerning multiple conflicting criteria.

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Plithogenic Fuzzy Whole Hypersoft Set, Construction of Operators and their Application in Frequency Matrix Multi Attribute Decision Making Technique

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Abstract: In this paper, initially a matrix representation of Plithogenic Hypersoft Set (PHSS) is introduced and then with the help of this matrix some local operators for Plithogenic Fuzzy Hypersoft set (PFHSS) are developed. These local operators are used to generalize PFHSS to Plithogenic Fuzzy Whole Hypersoft set (PFWHSS). The generalized PFWHSS set is hybridization of Fuzzy Hypersoft set (which represent multiattributes and their subattributes as a combined whole membership i.e. case of having an exterior view of the event) and the Plithogenic Fuzzy Hypersoft set (in which multi attributes and their subattributes are represented with individual memberships case of having interior view). Thus, the speciality of PFWHSS is its presentation of an exterior and interior view of a situation simultaneously. Later, the PFWHSS is employed in development of multi attributes decision making scheme named as Frequency Matrix Multi Attributes Decision making scheme (FMMADMS). This innovative technique is not only simpler than any of the former MADM techniques, but also has a unique capability of dealing mathematically a variety of human mind psychologies at every level that are working in different environments (fuzzy, intuitionistic, neutrosophic, plithogenic). Besides, FMMADMS also provides the percentage authenticity of the final ranking which in itself is a new idea providing a transparent and unbiased ranking. Moreover, the new introduced idea of frequency matrix handles the ranking ties in the best possible way and has an ability to provide the authenticity comparative analysis of previously developed schemes. Lastly, application of this FMMADMS is described as a numerical example for a case of ranking and selecting the best alternative.

Keywords: Plithogenic Hypersoft set, Exterior view, Plithogenic Whole Hypersoft set, Interior view, Frequency Matrix, Multi Attribute Decision making Scheme, Percentage authenticity.

1. Introduction

The theory of uncertainty in mathematics was initially introduced by Zadeh [26] in 1965 named as fuzzy set theory (FST). A fuzzy set is a set where each element of the universe of discourse X has some degree of belongingness in unit closed interval $[0,1]$ in given set A, where A is subset of universal set X with respect to some attribute say M with imposing condition that the sum of membership and non membership is one unlike crisp set where element from the universe either belong to given set A or does not belong to A. In Fuzzy set theory, elements of set are expressed with one quantity i.e. degree of membership. To represent this degree of membership a notation $\mu_A(x) \in [0,1] \forall x \in X$ was used and to represent the degree of nonmembership a notation $\upsilon_A(x) \in [0,1] \forall x \in X$ was used. The members of fuzzy set are represented by using one quantity i.e. the degree of membership $\{x: \mu_A(x)\}$.

Due to the condition $\mu_A(x) + \upsilon_A(x) = 1 \forall x \in X$ imposed by Zadeh the degree of non membership $\upsilon_A(x)$ to A will be $1 - \mu_A(x)$, where $\upsilon_A(x) \in [0,1] \forall x \in X$.

Further generalization of fuzzy set was made by Atanassov [1] in 1986 which are known as Intuitionistic fuzzy set (IFS). In IFS the natural concept of hesitation in human mind was used in assigning a degree of membership in unit closed interval such that sum of degree of membership, degree of non membership and degree of hesitation should be one. The degree of hesitation or indeterminacy was represented by the notation "ι" now the improved condition is $\mu_A(x) + \upsilon_A(x) + \iota_A(x) = 1 \forall x \in X$. The members of IFS are represented by using two quantities $\mu_A(x)$and $\upsilon_A(x) \{x: (\mu_A(x), \upsilon_A(x))\}$. Later, IFS were further generalized by Smarandache [15]. He considered membership $\mu_A(x)$, nonmembership $\upsilon_A(x)$ and indeterminacy $\iota_A(x)$ as independent quantities or functions in the unit cube, representing three axis of the unit cube in non standard unit interval $]0^-, 1^+[$. Smarandache represented the elements of Neutrosophic set (NS) by using three independent quantities and introduced "Neutrosophy"[16-17] as a new branch of philosophy which studies the origin nature,by considering neutrality and opposite and their interactions with different ideational spectra. Mathematically, a NS is represented by $\{x: (\mu_A(x), \upsilon_A(x), \iota_A(x))\}$ with condition $0 \le \mu_A(x) + \upsilon_A(x) + \iota_A(x) \le 3$. The new defined approach of dealing with human mind consciousness in form of Neutrosophic Set is utilized in MCDM and MADM techniques ([2-7],[9], ,[12], [18] ,[25]).

Furthermore, Smarandache[13] has generalized the Soft set to Hypersoft set by transforming the function F of one attribute into a multi attribute function where $a_1, a_2, \ldots, a_n$ for $n \ge 1$ be n distinct attributes, whose corresponding attributes values are respectively the set $A_1, A_2, \ldots, A_n$ with $A_i \cap A_j = \varphi$ for $i \ne j$ and $i, j \in \{1, 2, \ldots, n\}$ and assigning a combine membership $\mu_{A_1 \times A_2 \times \ldots \times A_N}(x)$, non membership $\upsilon_{A_1 \times A_2 \times \ldots \times A_N}(x)$ and Indeterminacy $\iota_{A_1 \times A_2 \times \ldots \times A_N}(x)$ $\forall x \in X$ with condition and introduced a hybrids of Crisp/ Fuzzy/ Intuitonistic Fuzzy and Neutrosophic Hypersoft set and then generalized Hypersoft set to Plithogenic Hypersoft set (PHSS) by assigning a separate degree of membership, nonmembership and indeterminacy

$\mu_{A_i}(x), \upsilon_{A_i}(x), \iota_{A_i}(x)$ respectively to each attribute value A_i. Thus a Plithogenic Set, as the generalization of Crisp, Fuzzy, Intuitionistic Fuzzy, Picture Fuzzy and Neutrosophic Set was introduced by F.Smarandache in 2017 [14].

In this paper, we have firstly presented to our reader an entirely new concept of looking at a Plithogenic Hypersoft set in a form of a matrix. This matrix representation is further utilized in the emergence of some new local operators such as disjunction, conjunction and averaging operators for Plithogenic Fuzzy Hyper soft sets (PFHSS). In the second stage, we have utilized these local operators to the define a new idea of a Plithogenic Fuzzy Whole Hypersoft Set (PFWHSS). This new PWHSS not only present a deep insight into a Plithogenic decision making environment but also a broader outlook of a situation which clearly is more generalized and precise approach of modelling human mind capabilities. Moreover, the new PWHSS are employed in development of a multi attribute decision making scheme named as Frequency Matrix Multi Attributes Decision Making Scheme (FMMADMS).

In most MADM techniques, ranking is achieved by generating a comparison of alternatives with ideal and non ideal solution ([8], [19], [20]) etc. Mostly, comparison are made on the basis of distance, inclusion, and similarity measurements etc. These scheme when studied analytically are actually representing fuzzy behavior of human mind. The ideal solution represents membership and the non ideal solution represents nonmembership behavior of fuzzy environment. Besides, the selection of any input information taken from any background (fuzzy, intuitionistic fuzzy, neutrosophic or any other) the use of ideal and non ideal solution in modelling of different MADM schemes actually drives the entire scheme to a fuzzy environment. So the ranking is based on optimist and pessimist human behavior. In this new FMMADMS, the ranking includes the three behavior of human mind, optimist behavior (represented mathematically by using Max operator employed in construction of local operators which are involved in ranking procedure), pessimist behavior (represented mathematically by using Min operator used in designing local disjunction also used in ranking process) and the neutral behavior (represented mathematically by using averaging operator). The final decision is made by combining the three human mind behaviors in a matrix called Frequency Matrix which gives the ultimate ranking of alternatives. The major advantage of the new scheme is its capacity of indulging many human mind behaviors by introducing variety of operators between Min, Max and averaging operators. Thus, generalizing the scheme from neutrosophic to plithogenic modelling environment [14]. Also, in our scheme at its final stage a ratios authenticity of the ranking operators is provided to guarantee the rightfulness of the final decision.

With a brief introduction of our work in Section 1, we have organized the rest of the paper in following sections: Section 2, is a collection of all the necessary preliminaries required for understanding of this work while in Section 3, we have presented the new concept of representing a

Plithogenic Fuzzy Hyper Soft Set in form of a matrix. Moreover, have introduced some new local operators on this set and constructed a whole membership using these local operators. This whole membership over a PFHSS set gives a birds eye view of the entire situation thus driving to new idea of Plithogenic Fuzzy Whole Hyper Soft Set. Furthermore, the newly defined PFWHSS is used in constructing a new MADM technique called Frequency Matrix Multi Attributes Decision making scheme (FMMADMS). In Section 4, a numerical example is presented to elaborate the new scheme while in Section 5 we give the final Conclusion of this work along with some open problems related to this field.

2. Preliminaries

In this section, we will present some basic definitions of soft set, fuzzy soft set, hypersoft set, crisp hypersoft set, fuzzy hypersoft set, plithogenic hypersoft set, plithogenic crisp hypersoft set and plithogenic fuzzy hypersoft set which are useful in development of our literature.

Definition 2.1 [21] **(Soft Set)**

Let U be the initial universe of discourse, and E is a set of parameters or attributes with respect to U. Let $P(U)$ denote the power set of U, and $A \subseteq E$ is a set of attributes. Then pair (F, A), where $F: A \longrightarrow P(U)$ is called **Soft Set** over U. In other words, a soft set (F, A) over U is parameterized family of subset of U. For $e \in A$, $F(e)$ may be considered as set of e elements or e approximate elements

$(F, A) = \{(F(e) \in P(U): e \in E, F(e) = \varphi \text{if} e \notin A\}$.

Definition 2.2 [24] **(Soft subset)**

For two soft set (F, A) and (G, B) over a universe U, we say that (F, A) is a soft subset of (G, B) if

(i) $A \subseteq B$, and

(ii) $\forall e \in A, F(e) \subseteq G(e)$

The set of all soft set over U will be denoted by $S(U)$.

Definition 2.3 [26] **(Fuzzy set)**

Let U be the universe . A fuzzy set X over U is a set defined by a membership function μ_X representing a mapping $\mu_X : U \longrightarrow [0,1]$

The vale of $\mu_X(x)$ for the fuzzy set X is called the membership value of the grade of membership of $x \in U$.The membership value represent the degree of belonging to fuzzy set X. Then a fuzzy set X on U can be represented as follows.

$X = \{(\mu_X(x)/x): x \in U, \mu_X(x) \in [0,1]\}$.

Definition 2.4 [9] **(Fuzzy soft set)**

Let U be the initial universe of discourse, $F(U)$ be all fuzzy set over U. E be the set of all parameters or attributes with respect to U and $A \subseteq E$ is a set of attributes. A fuzzy soft set Γ_A on the universe U is defined by the set of ordered pairs as follows, $\Gamma_A = \{x, \gamma_A(x): x \in E, \gamma_A(x) \in F(U)\}$
where $\gamma_A: E \longrightarrow F(U)$ such that $\gamma_A(x) = \varphi$ if $x \notin A$

$$\gamma_A(x) = \{\mu_{\gamma_A(x)}(u)/u: u \in U, \mu_{\gamma_A(x)}(u) \in [0,1]\}.$$

Definition 2.5 [13] **(Hypersoft set)**

Let U be the initial universe of discourse $P(U)$ the power set of U and $a_1, a_2, \ldots, a_n$ for $n \geq 1$ be n distinct attributes, whose corresponding attributes values are respectively the set $A_1, A_2, \ldots, A_n$ with $A_i \cap A_j = \varphi$ for $i \neq j$ and $i, j \in \{1,2,\ldots,n\}$.
Then the pair $(F, A_1 \times A_2 \times \ldots \times A_n)$ where, $F: A_1 \times A_2 \times \ldots \times A_n \rightarrow P(U)$.
is called a Hypersoft set over U

Definition 2.6 [13] (Crisp Universe of Discourse)

A Universe of Discourse U_C is called Crisp if $\forall x \in U_C$, $x \in 100\%$ to U_C or membership of x $T(x)$ with respect to A in M is 1 denoted as $x(1)$.

Definition 2.7 [13] (Fuzzy Universe of Discourse)

A Universe of Discourse U_F is called Fuzzy if $\forall x \in U_C$ x partially belongs to U_F or membership of x $T(x) \subseteq [0,1]$ where $T(x)$ may be subset , an interval, a hesitant set, a single value set, etc. denoted as $x(T_x)$.

Definition 2.8 [13] (Plithogenic Universe of Discourse)

A Universe of Discourse U_p over a set V of attributes values, where $V = \{v_1, v_2, \ldots, v_n\}$, $n \geq 1$, is called Plithogenic if $\forall x \in U_p$ x belongs to U_p in the degree $d_x^0(v_i)$ with respect to the attribute value v_i, for all $i \in \{1,2,\ldots,n\}$. Since the degree of membership may be Crisp, Fuzzy, Intuitionistic Fuzzy, or Neutrosophic, the Plithogenic Universe of discourse may can be Crisp, fuzzy, Intuitionistic fuzzy, or Neutrosophic.

Definition 2.9 [13] (Crisp Hypersoft set)

Let U_C be the initial universe of discourse $P(U_C)$ the power set of U.

Let $a_1, a_2, \ldots, a_n$ for $n \geq 1$ be n distinct attributes, whose corresponding attributes values are respectively the set $A_1, A_2, \ldots, A_n$ with $A_i \cap A_j = \varphi$ for $i \neq j$ and $i, j \in \{1,2,\ldots,n\}$. Then the pair $(F_C, A_1 \times A_2 \times \ldots \times A_n)$ where $F_C: A_1 \times A_2 \times \ldots \times A_n \rightarrow P(U_C)$, is called Crisp Hypersoft set over U_C.

Definition 2.10 [13] (Fuzzy Hypersoft set)

Let U_F be the initial universe of discourse $P(U_F)$ the power set of U_F.

$a_1, a_2, \ldots, a_n$ for $n \geq 1$ be n distinct attributes whose corresponding attributes values are respectively the set $A_1, A_2, \ldots, A_n$ with $A_i \cap A_j = \varphi$ for $i \neq j$ and $i, j \in \{1,2,\ldots,n\}$. Then the pair

$(F_F, A_1 \times A_2 \times \ldots \times A_n)$ where $F_F: A_1 \times A_2 \times \ldots \times A_n \rightarrow P(U_F)$, is called Fuzzy Hypersoft set over U_F.

Definition 2.11 [13] (Plithogenic Hypersoft set)

Now instead of assigning combined membership $\mu_{A_1 \times A_2 \times \ldots \times A_N}(x)$ $\forall x \in U_c/\ U_F/U_{IF}/U_N$ for Hyper Soft set if each attribute A_j is assigned an individual membership $\mu_{A_j}(x)$, non membership $\upsilon_{A_j}(x)$ and Indeterminacy $\iota_{A_j}(x)$ $j = 1,2,\ldots,n$ in Crisp/Fuzzy/Intuitionistic Fuzzy and Neutrosophic Hypersoft set then these generalized Crisp/Fuzzy/Intuitionistic Fuzzy and neutrosophic Hypersoft set are called Plithogenic Crisp/ Fuzzy/Intuitionistic Fuzzy and Neutrosophic Hypersoft set.

3. Plithogenic Fuzzy Hyper Soft set, their representation in a Matrix form and generalization to Plithogenic Fuzzy Whole Hypersoft set

In this section, we define initially Crisp Whole Hypersoft set, Fuzzy Whole Hypersoft set, Intuitionistic Fuzzy Whole Hypersoft set, Neutrosophic Whole Hypersoft set.

Definition 3.1 (Plithogenic Crisp/ Fuzzy/ Intuitionistic Fuzzy and neutrosophic Whole Hypersoft set)

Let $U_{pl}(X)$ be the plithogenic universe of discourse and $F: A_1^k \times A_2^k \times \ldots \times A_N^k \rightarrow P(U_{pl})$ where $k = 1,2,3,\ldots,M$ represent Numeric values of attributes A_j for each j, k: and A^krepresent sub attributes of the given attributes, can attain different numeric values. Now if in Plithogenic Crisp/Fuzzy/Intuitionistic Fuzzy/Neutrosophic Hypersoft set all attributes A_j^k have both an individual membership $\mu_{A_j}(x)$, non membership $\upsilon_{A_j}(x)$and indeterminacy $\iota_{A_j}(x)$ where $j = 1,2,\ldots,N$ and a whole combined membership $\mu_{A_1 \times A_2 \times \ldots \times A_N}(x)$ denoted by $\Omega(x)$, non membership $\upsilon_{A_1 \times A_2 \times \ldots \times A_N}(x)$ denoted by $\Phi(x)$ and Indeterminacy $\iota_{A_1 \times A_2 \times \ldots \times A_N}(x)$ denoted by $\Psi(x)$ then these generalized Plithogenic Crisp/Fuzzy/Intuitionistic Fuzzy /Neutrosophic Hypersoft set are called *Plithogenic Crisp/ Fuzzy/Intuitionistic Fuzzy / Neutrosophic Whole Hypersoft set.*

The Plithogenic Whole Hypersoft set is hybridization of Plithogenic Hypersoft set and Hypersoft set. If we are representing our set only with fuzzy memberships say $\mu_{A_j}(x)$ for individual attributes and Fuzzy whole memberships $\mu_{A_1 \times A_2 \times \ldots \times A_N}(x)$ say $\Omega(x)$ for combined attributes then the set under consideration are *Plithogenic Fuzzy Whole Hypersoft set.* Initially the literature is developed only for Plithogenic Fuzzy Hypersoft set and Plithogenic Fuzzy Whole Hypersoft set.

3.1 Plithogenic Fuzzy Whole Hypersoft set and Frequency Matrix Multi Attributes Decision Making Scheme (FMMADMS)

For convenience in dealing with plithogenic hypersoft set the data or informations i.e. memberships will be represented in the form of matrix denoted by C^{α}_{ij} for some combination of numeric values of attributes where α represent the given combination of attributes, i represent rows of matrix with respect to objects x_i, j represents columns of matrix with respect to numeric values of attributes A_j. These matrices will be helpful in construction of local Disjunction, Conjunction and Averaging operators. Furthermore, local constructed operators are used for the development of whole memberships denoted by Ω and then these memberships are used to generalize the Plithogenic Hypersoft Set to Plithogenic Whole Hypersoft Set and in development of a multi attributes decision making scheme named as Frequency Matrix Multi Attributes Decision Making Scheme (FMMADMS). The speciality of these local operators is that they deal within the matrix constructed by using informations or one can say within one combination of attributes which gives interior view of the event. In this section, we shall be dealing with PFHSS only. Later the idea can be generalized to other environments (intuitionistic, neutrosophic, plithogenic) etc. Let us now formally introduce the steps of FMMADMS. In this scheme, the first four steps are related to the matrix construction of PFHSS and their local operators while in the next three steps PFWHSS are developed using these operators and are utilized in defining the local ranking. Moreover, a final ranking is obtained using a frequency matrix. Also, a percentage authenticity is calculated to guarantee the transparency of the process.

Step 1. Decision of universe: Consider universe of discourse $U_{pl} = \{x_i\}$ $i = 1,2,3,\ldots, M$ and then $T = \{x_i\} \subset U_{pl}$ where i could be chosen between 1 to M. Here x_i represent the objects under consideration.

Step 2. Defining attributes and mapping: Let $A^k_1, A^k_2, A^k_3, \ldots, A^k_N$ be the attributes. Choose some attributes represented by A_j, $j = 1,2,3,\ldots,N$ and then assign k some numeric values can be presented by A^k_j where k and j can take values $1,2,3,\ldots,N$. The data of the numerical values is based on the decision maker's opinion by using the linguistic scales [[10],[11],[23]]. Define $F: A^k_1 \times A^k_2 \times A^k_3 \times \ldots \times A^k_N \rightarrow P(U_{pl})$, where F is a mappings from combination of attributes to some subset of power set of U_{pl}.

Step 3. Matrix representation: Write the data or information (Memberships) in the form of a matrix. Let C^{α}_{ij}; $j = 1,2,3,\ldots,N$ and $i = 1,2,3,\ldots, M$: be the matrix and let α represent the given combination of attributes A^k_j for some j and k.

$$C_{ij}^{\alpha} = \begin{array}{c} \\ x_1 \\ x_2 \\ . \\ . \\ . \\ x_M \end{array} \begin{array}{c} \begin{array}{ccccccc} A_1^k & A_2^k & & & & & A_N^k \end{array} \\ \begin{bmatrix} \mu_{A_1}(x_1) & \mu_{A_2}(x_1) & . & . & . & \mu_{A_N}(x_1) \\ \mu_{A_1}(x_2) & \mu_{A_1}(x_2) & . & . & . & \mu_{A_N}(x_2) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \mu_{A_1}(x_M) & \mu_{A_2}(x_M) & . & . & . & \mu_{A_N}(x_M) \end{bmatrix} \end{array} \quad (3.1)$$

Step 4. Construction of Local operators and Global whole memberships: Now by using individual memberships $\mu_j(x_i)$, for $x_i \in T$ and varying j from 1 to N one can develop a combined whole membership, say $\Omega^t(x_i)$ to x_i in T with respect to given combination of attributes by using different operators on rows of matrices of representation C_{ij}^{α} for Construction of local operators. These operators can be represented by taking different integer values of t i.e. $t = 1$ represent local disjunction operator , $t = 2$ represent local conjunction operator and $t = 3$ represent local averaging operator. The following local operators are constructed. Here, we define some local operators for Plithogenic Fuzzy Hypersoft Set. It is observed that the same operators are applicable for Plithogenic Crisp Hypersoft set but as the results are trivial so we will consider here only the case of Plithogenic Fuzzy Hypersoft set

Local Disjunction Operator for Plithogenic Fuzzy Hypersoft Set :

$$\vee_{loc}(F) = \cup(C_{ij}^{\alpha}) = Max_j(C_{ij}^{\alpha}) = Max_j(\mu_j(x_i)) \quad (3.2)$$

(Choose maximum membership from i_{th} row)

Here $\vee_{loc}$, $\cup$ are representations for local disjunctions operators for F, $\mu_j(x_i)$ is the membership for j_{th} attribute with respect to i_{th} object.

Local Conjunction Operators for Plithogenic Fuzzy Hypersoft Set :

$$\wedge_{loc}(F) = \cap(C_{ij}^{\alpha}) = Min_j(C_{ij}^{\alpha}) = Min_j(\mu_j(x_i)) \quad (3.3)$$

(Choose minimum membership from i_{th} row amongst j columns) and the result will be a column matrix representation three entities. Here $\wedge_{loc}$ are representations for local conjunctions operators for F, $\mu_j(x_i)$ is the membership for j_{th} attribute with respect to i_{th} object.

Local Averaging Operator for Plithogenic Fuzzy Hypersoft Set :

$$\Gamma(F) = \Gamma(C_{ij}^{\alpha}) = \sum_{j=1}^{N} \frac{\mu_j(x_i)}{N} \quad (3.4)$$

Here Γ represent averaging operator for mapping F for α combination of attributes applied on the given matrix of representation C_{ij}^{α} by taking average of memberships for i_{th} row.

Local Compliment for Plithogenic Fuzzy Hypersoft Set :

$$C_{loc}(F) = C\left(C_{ij}^{\alpha}\right) = \begin{Bmatrix} Max_j\left(1-\mu_j(x_i)\right) \\ Min_j\left(1-\mu_j(x_i)\right) \\ \sum_{j=1}^{N} \frac{\left(1-\mu_j(x_i)\right)}{N} \end{Bmatrix} \quad (3.5)$$

Here C_{loc} represent the local compliment for F mapping forα combination of attributes applied over matrix of representation C_{ij}^{α} by taking compliment of memberships for i_{th} row and then choosing either maximum or minimum or taking average of them. By applying Local disjunction, Local conjunction and Local averaging operators (3.2, 3.3, 3.4) to (3.1) one can develop a combined whole membership, denoted by $\Omega_{\alpha}^{t}(x_i)$.

Note: Here we have not used the compliment operator to develop the whole membership. But the choice is open for reader to work with this operator or any other operator of their choice.

Here $\Omega_{\alpha}^{t}(x_i)$ is representation for whole combined membership for i_{th} object withe respect to α combination of attributes in subset of $P(U_{pi})$

$$\Omega_{\alpha}^{1}(x_i) = \cup_j\ (C_{ij}^{\alpha}) = Max_j\left(\mu_j(x_i)\right) \quad (3.6)$$

$\Omega_{\alpha}^{1}(x_i)$ represent the combined (whole) membership for i_{th} object obtained by using disjunction operator ($t = 1$) developed in (3.2).

$$\Omega_{\alpha}^{2}(x_i) = \cap_j\ (C_{ij}^{\alpha}) = Min_j\left(\mu_j(x_i)\right) \quad (3.7)$$

$\Omega_{\alpha}^{2}(x_i)$ represent the combined (whole) membership for i_{th} object obtained by using conjunction operator ($t = 2$) developed in (3.3).

$$\Omega_{\alpha}^{3}(x_i) = \Gamma(C_{ij}^{\alpha}) = \sum_{j=1}^{N} \frac{(\mu_j(x_i))}{N} \quad (3.8)$$

$\Omega_{\alpha}^{3}(x_i)$ represent the combined (whole) membership for i_{th} object obtained by using averaging operator Γ ($t = 3$) developed in (3.4).

We shall use $\Omega_\alpha^1(x_i), \Omega_\alpha^2(x_i)$ and $\Omega_\alpha^3(x_i)$ for three different whole memberships of Plithogenic Fuzzy Whole hypersoft set.

Step 5. Matrix representation of Plithogenic Fuzzy Whole Hypersoft set and initial ranking:

Write the data or information (local individual membership and global whole memberships) in the form of an other matrix denoted by $C_{ij}^{\alpha t}$; $j = 1,2,3,\dots,N$ and $i = 1,2,3,\dots,M$ and α represents the given combination of attributes and $t = 1,2,3$ represent the local operators used to get the whole combined memberships where $C_{ij}^{\alpha t}$ is the matrix representation for Plithogenic Fuzzy whole Hypersoft set.

$$C_{ij}^{\alpha t} = \begin{array}{c} \\ x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_M \end{array} \begin{array}{c} \begin{array}{cccccc} A_1^k & A_2^k & & A_N^k \dots\dots & & \Omega_\alpha^t \end{array} \\ \begin{bmatrix} \mu_{A_1}(x_1) & \mu_{A_2}(x_1) & \cdot & \cdot & \cdot & \mu_{A_N}(x_1) & \Omega_\alpha^t(x_1) \\ \mu_{A_1}(x_2) & \mu_{A_1}(x_2) & \cdot & \cdot & \cdot & \mu_{A_N}(x_2) & \Omega_\alpha^t(x_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_{A_1}(x_M) & \mu_{A_2}(x_M) & \cdot & \cdot & \cdot & \mu_{A_N}(x_M) & \Omega_\alpha^t(x_M) \end{bmatrix} \end{array}$$

Where in A_j^k, k takes values with respect to given some α combination and in Ω_α^t and in $C_{ij}^{\alpha t}$while i represent rows of matrix and j represent its columns and $C_{ij}^{\alpha t}$: Plithogenic Fuzzy Whole Hypersoft Matrix (PFWHSM). For $t = 1,2,3$ we shall get three PFWHSM's.

In particular, for a fixed t and for some α combination of attributes $A_j, j = 1,2,3,\dots,N$ we will get an initial ranking for alternatives $T = \{x_i\}$ under consideration in $C_{ij}^{\alpha t}$ from the last column of $C_{ij}^{\alpha t}$ which is the column of whole membership value Ω_α^t . The first position is assigned to an alternative having highest whole membership $\Omega_\alpha^t(x_i)$ [which is the highest numeric value in last column] and the second position to one having second largest membership and so on. If a tie occurs for the position of alternatives in this initial ranking, it will be removed in final ranking. In this step, by varying $t = 1,2,3$ we shall obtain the three types of initial ranking of our alternatives based on three operators see (3.6,3.7 and 3.8). All of these ranking will be utilized in next stage to get the final ranking of alternatives.

It is worth mentioning here the fact that these initial rankings presents three human mind behaviors for three different choices of operators. To be more specific for $t = 1$ the use of Max operator will provide the choice of optimist behavior of human mind. Similarly for $t = 2$ which represent the use of Min operator one can represent the pessimist behavior of human mind. Furthermore, the choice of $t = 3$ i.e., the use of averaging operator will represent the neutral behavior of human mind. Finally in the next step by using the frequency matrix we will combine the three human mind behaviors to provide the final results of the ranking procedure.

Step 6. Construction of frequency matrix F_{qp} for final ranking:

Finally, we have constructed the frequency matrix of positions F_{qp} from initial ranking where $q = 1,2,\ldots M$ is used to represent rows (alternatives) of frequency matrix F_{qp} and $p = 1,2,\ldots,M$ is used to represent columns (positions attained by these alternatives) of frequency matrix F_{qp}.

$$F_{qp} = \begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ x_M \end{array}\!\!\begin{array}{c} \begin{array}{cccc} p_1 & p_2 & \cdots & p_M \end{array} \\ \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1M} \\ f_{21} & f_{22} & \cdots & f_{2M} \\ \vdots & \vdots & & \vdots \\ f_{M1} & f_{M2} & \cdots & f_{MM} \end{bmatrix} \end{array}$$

The final frequency matrix F_{qp} of alternatives and positions is a square matrix of order $M \times M$ i.e. number of ordering positions will be equal to the number of alternatives, The selection of first position to any alternative will be made by looking into the first column corresponding to the position 1 i.e. p_1. The alternative having the largest frequency value in this column will be assigned first position. Once first position is decided, the entire row corresponding to this alternative and the first column will be excluded from the process of selection. Next, we shall look into the second column to select the candidate having the largest frequency value to be assigned the second position of ordering. Once done he shall be excluded from the process by excluding his row and the second column from the process. This procedure of selection will continue until all the positions are assigned to the rightful alternative.

In final frequency matrix if two alternatives have the same frequency of position 1 which is a very rare case, then we check their frequency of position 2, the one having higher frequency value in position 2 will be assigned the first position. After this selection the particular alternative and the position 1 will be excluded from selection procedure. Then other competitor will be assigned the second position. In this way all the ties can be fairly handled in this process.

Step 7. Percentage measure of authenticity of ranking: Finally the percentage measure of authenticity can be obtained by using the ratios formula:

Percentage authenticity of p_{th} position for q_{th} alternative $= \frac{\max_q (f_{qp})}{\sum_q f_{qp}} \times 100$, where f_{qp} is the obtained frequency of the p_{th} position for q_{th} alternative and $\sum_q f_{qp}$ is the total frequency of p_{th} position.

4. Numerical Example

Step 1. Decision of universe: let $U = \{x_1, x_2, x_3, x_4, x_5\}$ set of five members of Engineering department and $T = \{x_2, x_3, x_5\} \subset U$ set of three members who have applied for the post of Assistant professor.

Step 2. Defining Attributes and mapping:

Let the attributes be $A_j^k; j = 1,2,3,4$ and k may have any value from 1 to 3

$A_1^k =$ Subject skill area with numeric values, $k = 1,2,3$

$A_1^1 =$ Mathematics, $A_1^2 =$ Physics, $A_1^3 =$ Computer science

$A_2^k =$ Qualification with numeric values, $k = 1,2$

$A_2^1 =$ Higher qualification like Ph.D. or equivalent, $A_2^2 =$ lower qualification like MS or equivalent $A_3^k =$ Teaching experience with numeric values, $k = 1,2$

$A_3^1 =$ Three years or less, $A_3^2 =$ More than three years

$A_4^k =$ Age, with numeric values $k = 1,2,3$

$A_4^1 =$ Age is less than thirty years, $A_4^2 =$ Age is between thirty to forty years $A_4^3 =$Age is greater than forty years

We need to select faculty members.

Let the Function F be given by,

$F: A_1^k \times A_2^k \times A_3^k \times A_4^k \rightarrow P(U)$ for $k = 1,1,1,2$ respectively.

We are interested in ranking of these three candidates for the Engineering department with the following criteria.

1. Subject skill area: Mathematics: $k = 1$
2. Qualification: Higher qualification like Ph.D or Equivalent $k = 1$
3. Teaching experience: Three years or less $k = 1$
4. Age: Age required is between thirty to forty years $k = 2$

$F(A_1^1, A_2^1, A_3^1, A_4^2) = \{x_2, x_3, x_5\}$ let we name $A_1^1, A_2^1, A_3^1, A_4^2$ combination as α

With respect to T =$\{x_2, x_3, x_5\}$ have memberships in PFHSS. Consider the memberships of x_2, x_3, x_5 as $\mu_j(x_i)$ for $i = 2,3,5$ and $j = 1$ to 4 in T with respect to α combination of attributes.

$F(\alpha) = F(A_1^1, A_2^1, A_3^1, A_4^2) = \{x_2(0.3,0.6,0.4,0.5), x_3(0.4,0.5,0.3,0.1), x_5(0.6,0.3,0.3,0.7)\}$

Step 3. Matrix representation: Let C_{ij}^{α} represented in 3.1 is the matrix of representation for the combination of attributes α,in PFHSS. Here rows are representing x_2, x_3, x_5 and columns are representing $A_1^k, A_2^k, A_3^k, A_4^k$.

$$C_{ij}^{\alpha} = \begin{array}{c} \\ x_2 \\ x_3 \\ x_5 \end{array} \begin{array}{c} \begin{array}{cccc} A_1^1 & A_2^1 & A_3^1 & A_4^2 \end{array} \\ \begin{bmatrix} 0.3 & 0.6 & 0.4 & 0.5 \\ 0.4 & 0.5 & 0.3 & 0.1 \\ 0.6 & 0.3 & 0.3 & 0.7 \end{bmatrix} \end{array}$$

Step 4. Construction of Local operators and Global whole memberships for PFHSS: By using individual memberships $\mu_j^{\alpha}(x_i)$, for $x_i \in T \subset U$ now with respect to α combination of attributes by fixing $i = 2{,}3$ and 5 and varying j from 1 to 4 in 3.6, 3.7 and 3.8 one can assign a combined (whole) membership, $\Omega_{\alpha}^{t}(x_i)$ to $x_i \in U$ in T with respect to α combination of attributes by using operators developed in 3.6, 3.7 and 3.8 on rows of matrix of representation C_{ij}^{α}. Using (3.1)

$$\Omega_{\alpha}^{1}(x_i) = \cup\,(\mu_j^{\alpha}(x_i)) = Max_j\left(\mu_j^{\alpha}(x_i)\right)$$

$$\Omega_{\alpha}^{2}(x_i) = \cap\left(\mu_j^{\alpha}(x_i)\right) = Min_j\left(\mu_j^{\alpha}(x_i)\right)$$

$$\Omega_{\alpha}^{3}\ (x_i) = \Gamma\left(\mu_j^{\alpha}(x_i)\right) = \Sigma_{j=1}^{N}\frac{(\mu_j^{\alpha}(x_i))}{N}$$

This membership is used in Generalization of PFHSS to Plithogenic Fuzzy Whole Hyper Soft set.

$\Omega_{\alpha}^{1}(x_2) = \cup\left(\mu_j^{\alpha}(x_2)\right) = Max_j\left(\mu_j^{\alpha}(x_2)\right) = 0.6$ for $i = 2$ and varying j from 1 to 4

$\Omega_{\alpha}^{1}(x_3) = \cup\left(\mu_j^{\alpha}(x_3)\right) = Max_j\left(\mu_j^{\alpha}(x_3)\right) = 0.5$ for $i = 3$ and varying j from 1 to 4

$\Omega_{\alpha}^{1}(x_5) = \cup\left(\mu_j^{\alpha}(x_5)\right) = Max_j\left(\mu_j^{\alpha}(x_5)\right) = 0.7$ for $i = 5$ and varying j from 1 to 4

$\Omega_{\alpha}^{2}(x_2) = \cap\left(\mu_j^{\alpha}(x_2)\right) = Min_j\left(\mu_j^{\alpha}(x_2)\right) = 0.3$ for $i = 2$ and varying j from 1 to 4

$\Omega_{\alpha}^{2}(x_3) = \cap\left(\mu_j^{\alpha}(x_3)\right) = Min_j\left(\mu_j^{\alpha}(x_3)\right) = 0.1$ for $i = 3$ and varying j from 1 to 4

$\Omega_{\alpha}^{2}(x_5) = \cap\left(\mu_j^{\alpha}(x_5)\right) = Min_j\left(\mu_j^{\alpha}(x_5)\right) = 0.3$ for $i = 5$ and varying j from 1 to 4

$$\Omega_{\alpha}^{3}(x_2) = \Gamma\left(\mu_j^{\alpha}(x_2)\right) = \sum_{j=1}^{4} \frac{\left(\mu_j^{\alpha}(x_2)\right)}{4} = 0.45 \text{ for } i = 2, N = 4 \text{and varying } j \text{ from } 1 \text{ to } 4$$

$$\Omega_{\alpha}^{3}(x_3) = \Gamma\left(\mu_j^{\alpha}(x_3)\right) = \sum_{j=1}^{4} \frac{\left(\mu_j^{\alpha}(x_3)\right)}{4} = 0.325 \text{ for } i = 2, N = 4 \text{ and varying } j \text{ from } 1 \text{ to } 4$$

$$\Omega_{\alpha}^{3}(x_5) = \Gamma\left(\mu_j^{\alpha}(x_5)\right) = \sum_{j=1}^{4} \frac{\left(\mu_j^{\alpha}(x_5)\right)}{4} = 0.45 \text{ for } i = 2, N = 4 \text{ and varying } j \text{ from } 1 \text{ to } 4$$

Step 5 Matrix representation of Plithogenic Fuzzy Whole Hypersoft set and initial ranking:

$$C_{ij}^{\alpha 1} = \begin{array}{c} \\ x_2 \\ x_3 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1^1 & A_2^1 & A_3^1 & A_4^2 & \Omega_\alpha^1 \end{array} \\ \begin{bmatrix} 0.3 & 0.6 & 0.4 & 0.5 & 0.6 \\ 0.4 & 0.5 & 0.3 & 0.1 & 0.5 \\ 0.6 & 0.3 & 0.3 & 0.7 & 0.7 \end{bmatrix} \end{array}$$

For choosing the best one will select the largest value from last column i.e. $x_5 = 0.7$ The initial ranking for $t = 1$, is Position 1: for x_5,Position 2: for x_2 and Position 3: for x_3.

$$C_{ij}^{\alpha 2} = \begin{array}{c} \\ x_2 \\ x_3 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1^1 & A_2^1 & A_3^1 & A_4^2 & \Omega_\alpha^1 \end{array} \\ \begin{bmatrix} 0.3 & 0.6 & 0.4 & 0.5 & 0.3 \\ 0.4 & 0.5 & 0.3 & 0.1 & 0.1 \\ 0.6 & 0.3 & 0.3 & 0.7 & 0.3 \end{bmatrix} \end{array}$$

For choosing the best one will select the largest value from last column i.e. $x_2 = x_5 = 0.3$. The initial ranking for $t = 2$, is Position 1: could be assigned to both the candidates x_5 and x_2. This tie will be removed in final step of ranking.

$$C_{ij}^{\alpha 3} = \begin{array}{c} \\ x_2 \\ x_3 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1^1 & A_2^1 & A_3^1 & A_4^2 & \Omega_\alpha^1 \end{array} \\ \begin{bmatrix} 0.3 & 0.6 & 0.4 & 0.5 & 0.45 \\ 0.4 & 0.5 & 0.3 & 0.1 & 0.325 \\ 0.6 & 0.3 & 0.3 & 0.7 & 0.475 \end{bmatrix} \end{array}$$

For choosing the best one will select the largest value from last column i.e. $x_5 = 0.7$ The initial ranking for $t = 3$, is Position 1: for x_5,Position 2: for x_2 and Position 3: for x_3.

Step 6. Construction of frequency matrix F_{qp} for final ranking: Next we construct a frequency matrix to get the final ranking using the data of step 5.

$$F_{qp} = \begin{array}{c c} & \begin{array}{ccc} p_1 & p_2 & p_3 \end{array} \\ \begin{array}{c} x_2 \\ x_3 \\ x_5 \end{array} & \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \end{bmatrix} \end{array}$$

This frequency matrix shows the frequency of getting first position for $x_2$is 1, for $x_3$is 0 and for x_5 is 3, the frequency of getting second position for $x_2$is 2, for $x_3$is 0 and for x_5 is 3,and the frequency of getting third position for $x_2$is 3, for $x_3$is 0 and for x_5 is 0. We can see here the initial ranking for $t = 1$ is $x_5 \rhd x_2 \rhd x_3$, for $t = 2$ is $x_5 = x_2 \rhd x_3$ and ranking for $t = 3$ is $x_5 \rhd x_2 \rhd x_3$ and the final ranking from the frequency matrix F_{qp} is same i.e., $x_5 \rhd x_2 \rhd x_3$ which shows use of frequency matrix increases the authenticity of the ranking and selection of right candidate for the post.

Step 7. Percentage measure of authenticity of ranking:

Percentage authenticity of first position for $x_5 = \frac{\max_q(f_{qp})}{\sum_q f_{qp}} = \frac{f_{31}}{\sum_q f_{q1}} \times 100 = 75\%$

Percentage authenticity of second position for $x_2 = \frac{f_{12}}{\sum_q f_{q2}} \times 100 = 100\%$

Percentage authenticity of third position for $x_3 = \frac{f_{23}}{\sum_q f_{q3}} \times 100 = 100\%$

5. Conclusion

A novice idea of matrix representation of Plithogenic Fuzzy Hypersoft Set (PFHSS) is introduced along with construction of their local operators such as conjunction, disjunction and averaging operators. These local operators are utilized in defining a new concept of Plithogenic Fuzzy Whole Hyper Fuzzy Soft Set (PFWHSS). The PFWHSS deals fuzziness of the data or information as a combined vision (external view) in case of combined membership of a combination of attributes and individually (internal view) as a in case of considering individual memberships. Furthermore, an innovative yet simple MADM technique called Frequency Matrix Multi Attributes Decision Making Scheme (FMMADMS) is developed. In this technique, at first stage, we have employed three different PFWHSS to get three initial rankings of alternatives representing decisions made by three different human mind behaviors of being optimist (the case in which whole membership is obtained by using conjunction (Max) operator), pessimist (the case in which whole membership is obtained using disjunction (Min) operator) and the neutral behavior (the case in which whole membership is obtained using averaging operator). In the next stage, we have introduced a new concept of frequency matrix that combines all the three possibilities of human mind behavior to provide with a final ranking decision of alternatives. In many decision making schemes, there are possibilities of ties between ranking alternatives. The use frequency matrix in FMMADMS provides a unique way of handling these ties. It results into a final ranking free of ties. Lastly, the scheme works with a percentage measure to guarantee the authenticity and accuracy of the final ranking. This itself, is entirely a new idea to get to get an authenticity of different ranking schemes which shows that the final decision is transparent and unbiased.

Moreover, this technique is more generalized since it use PFWHSS which deals with not only attributes but also sub attributes at the same time. One of the beauty of this scheme is its simplicity as the user need not to handle with complicated long calculations based operators. Also this new technique have a flexible approach of using wide range of operators that can absorb changes according to the requirement of the provided environment. To be more specific, the selection of three operators represent a neutrosophic behavior which clearly is a special case of plithogenic attitude as mentioned in [14]. Now introducing more operators among these three neutrosophic elemental behaviors (membership, nonmembership, neutrality) we can generalize the model of this scheme in plithogenic environment which may handle more of human mind complexities.

Some of the open problems that could be addressed: This work have vast extensions by developments of some new literature on operators, their properties and applications in different environments like Crisp, Fuzzy, Intuitionistic Fuzzy and Neutrosophic etc. and development of multi attributes decision making techniques in different environments. Moreover, the matrix representation of plithogenic whole hypersoft set opens new dimensions towards development of many operators and MADM techniques.

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Pi-Distance of Rough Neutrosophic Sets for Medical Diagnosis

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Abstract. The objective of the study is to find out the relationship between the disease and the symptoms seen with the patient and diagnose the disease that impacted the patient using rough neutrosophic set. Neoteric method [PI-distance] is devised in rough neutrosophic set. Utilization of medical diagnosis is commenced with using prescribed procedures to identify a person suffering from the disease for a considerable period. The result showed that the proposed method is free from shortcomings that affect the existing methods and found to be more accurate in diagnosing the diseases.

Keywords: Neutrosophic set, rough neutrosophic set, Pi-distance, medical diagnosis.

1. Introduction

Mathematical principles play a vital role in solving the real life problems in engineering, medical sciences, social sciences, economics and so on. These problems are having no definite data and they are mostly imprecise in character. We are therefore employing probability theory, fuzzy set theory, rough set theory *etc.*, in Mathematics to find solutions to these problems. In the same way, fuzzy logic techniques have been integrated with conventional clinical decision in healthcare industry. As clinicians find it hard to have a fool proof diagnosis, they are initiating certain steps without any guidance from the experts. Neutrosophic set which is a generalized set possesses all attributes necessary to encode medical knowledge base and capture medical inputs.

The law of average has been applied in Medical diagnosis combining the information of which most of them are quantifiable derived through various sources and the inconsistent data derived through intuitive thought and the whole process offers low intra and inter personal consistency. So contradictions, inconsistency, indeterminacy and fuzziness should be accepted as unavoidable as they are integrated in the behavior of biological systems as well as in their characterization. To model an expert doctor it is imperative that it should not disallow uncertainty as it would be then inapt to capture fuzzy or incomplete knowledge that might lead to the danger of fallacies due to misplaced precision.

As medical diagnosis contains lots of uncertainties and increased volume of information available to physicians from new medical technologies, the process of classifying different sets of symptoms under a single name of disease becomes difficult. The main advantage of rough set theory is that it does not need any preliminary or additional information about data(like the probability in statistics, the value of possibility in fuzzy set theory *etc.*,).So, rough neutrosophic sets play a vital role in medical diagnosis.

In 1965, Fuzzy set theory was firstly given by Zadeh[1] which is applied in many real applications to handle uncertainty. Sometimes membership function itself is uncertain and hard to be defined by a crisp value. So the concept of interval valued fuzzy sets was proposed to capture the uncertainty of grade of membership. In 1986, Atanassov[3] introduced the intuitionistic fuzzy sets which consider both truth-membership and falsity-membership. Edward Samuel and Narmadhagnanam[4] proposed the tangent inverse distance and sine similarity measure of intuitionistic fuzzy sets and apply them in medical diagnosis.

Later on, intuitionistic fuzzy sets were extended to the interval valued intuitionistic fuzzy sets. Intuitionistic fuzzy sets and interval valued intuitionistic fuzzy sets can only handle incomplete information not the indeterminate information and inconsistent information which exists commonly in belief systems. So, Neutrosophic set (generalization of fuzzy sets, intuitionistic fuzzy sets and so on) defined by FlorentinSmarandache[5] has capability to deal with uncertainty, imprecise, incomplete and inconsistent information which exists in real world from philosophical point of view.

In 1982, Pawlak[2] introduced the concept of rough set (RS), as a formal tool for modeling and processing incomplete information in information systems. There are two basic elements in rough set theory, crisp set and equivalence relation, which constitute the mathematical basis of rough sets. The basic idea of rough set is based upon the approximation of sets by a pair of sets known as the lower approximation and the upper approximation of a set. Here, the lower and upper approximation operators are based on equivalence relation. Nanda and Majumdar [6] examined fuzzy rough sets. Broumi *et al* [7] introduced rough neutrosophic sets.

SurapatiPramanik and KalyanMondal [8,9] introduced cosine and cotangent similarity measures of rough neutrosophic sets. Pramanik *et al* [10] introduced correlation coefficient of rough neutrosophic sets. Edward Samuel and Narmadhagnanam [11] proposed order function among roughneutrosophic sets. Pramanik *et al* [12] introduced several trigonometric Hamming similarity measures under interval rough neutrosophic environment. Pramanik *et al* [13] introduced Multi attribute decision making strategy on projection and bidirectional projection measures of interval rough neutrosophic sets. Mondal *et al* [14] examined TOPSIS in rough neutrosophic environment.

Mondal *et al* [15] proposed variational coefficient similarity measure under rough neutrosophic environment. Mondal *et al* [16] proposed several trigonometric Hamming similarity measures of rough neutrosophic sets. Mondal and Pramanik [17] proposed grey relational analysis among rough neutrosophic sets. Pramanik and Mondal [18] proposed some similarity measures among rough neutrosophic sets. Mondal *et al* [19] proposed aggregation operators among rough neutrosophic sets. Pramanik *et al* [20] introduced Multi criteria decision making based on projection and bidirectional projection measures of rough neutrosophic sets. Neutrosophic set is applied to different areas including decision making by many researchers[21-27]. Mohana and Mohanasundari [28] proposed similarity measures of single valued neutrosophic rough sets. Tuhin Bera and Nirmal KumarMahapatra[29] applied generalised single valued neutrosophic number in neutrosophic linear programming. Ulucay et al [30] proposed a new approach for multi-attribute decision-making problems in bipolar neutrosophic sets.Broumi et al [31] proposed single valued (2N+1) sided polygonal neutrosophic numbers and single valued (2N) sided polygonal neutrosophic numbers. Li et al [32] proposed slope stability assessment method using the arctangent and tangent similarity measure of neutrosophic numbers.

Rest of the article is structured as follows. In Section 2, we briefly present the basic definitions.Section 3 deals with proposed definition (PI distance) and some of its properties. Sections 4, 5 and 6 deal with methodology,algorithm and case study related to medical diagnosis respectively.Conclusion is given in Section 7.

2. Preliminaries

2.1 Definition [33]

Let X be a Universe of discourse, with a generic element in X denoted by x, the neutrosophic set(NS) A is an object having the form $A = \{\langle x : T_A(x), I_A(x), F_A(x)\rangle, x \in X\}$ where the functions define $T, I, F : X \to \left]^{-}0,1^{+}\right[$ respectively the degree of membership (or Truth), the degree of indeterminacy and the degree of non-membership(or Falsehood) of the element $x \in X$ to the set A with the conditio $^{-}0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^{+}$

2.2 Definition [7]

Let U be a non-null set and R be an equivalence relation on U. Let P be neutrosophic set in U with the membership function T_P, indeterminacy function I_P and non-membership function F_P. The lower and the upper approximations of P in the approximation (U,R) denoted by $\underline{N}(P)$ & $\overline{N}(P)$ are respectively defined as follows:

$$\underline{N}(P) = \left\langle \left\langle x, T_{\underline{N}(P)}(x), I_{\underline{N}(P)}(x), F_{\underline{N}(P)}(x) \right\rangle / \, y \in [x]_R, x \in U \right\rangle$$

$$\overline{N}(P) = \left\langle \left\langle x, T_{\overline{N}(P)}(x), I_{\overline{N}(P)}(x), F_{\overline{N}(P)}(x) \right\rangle / \, y \in [x]_R, x \in U \right\rangle$$

where

$$T_{\underline{N}(P)}(x) = \bigwedge_{y \in [x]_R} T_P(y)$$

$$I_{\underline{N}(P)}(x) = \bigvee_{y \in [x]_R} I_P(y)$$

$$F_{\underline{N}(P)}(x) = \bigvee_{y \in [x]_R} F_P(y)$$

$$T_{\overline{N}(P)}(x) = \bigvee_{y \in [x]_R} T_P(y)$$

$$I_{\overline{N}(P)}(x) = \bigwedge_{y \in [x]_R} I_P(y)$$

$$F_{\overline{N}(P)}(x) = \bigwedge_{y \in [x]_R} F_P(y)$$

So, $0 \leq T_{\underline{N}(P)}(x) + I_{\underline{N}(P)}(x) + F_{\underline{N}(P)}(x) \leq 3$ and $0 \leq T_{\overline{N}(P)}(x) + I_{\overline{N}(P)}(x) + F_{\overline{N}(P)}(x) \leq 3$, where $\vee$ and $\wedge$ mean "max" and "min" operators respectively, $T_P(y), I_P(y)$ and $F_P(y)$ are the membership, indeterminacy and non-membership of with respect to P. It is easy to see that $\underline{N}(P)$ & $\overline{N}(P)$ are two neutrosophic sets in U, thus the NS mappings $\underline{N}, \overline{N} : N(U) \to N(U)$ are respectively, referred to as the lower and upper rough neutrosophic set approximation operators, and the pair $\left(\underline{N}(P), \overline{N}(P)\right)$ is called the rough neutrosophic set in (U,R).

3 Proposed definitions

3.1. Pi-distance

Let $A = \left\langle \left(\underline{T}_A(x_i), \underline{I}_A(x_i), \underline{F}_A(x_i)\right), \left(\overline{T}_A(x_i), \overline{I}_A(x_i), \overline{F}_A(x_i)\right) \right\rangle$ and $B = \left\langle \left(\underline{T}_B(x_i), \underline{I}_B(x_i), \underline{F}_B(x_i)\right), \left(\overline{T}_B(x_i), \overline{I}_B(x_i), \overline{F}_B(x_i)\right) \right\rangle$ be two rough neutrosophic sets, then the Pi-distance is defined as

$$PI_{RNS}(A,B) = \sum_{i=1}^{n} \frac{\left[\left|\underline{T}_A(x_i) - \underline{T}_B(x_i)\right| + \left|\underline{I}_A(x_i) - \underline{I}_B(x_i)\right| + \left|\underline{F}_A(x_i) - \underline{F}_B(x_i)\right| + \left|\overline{T}_A(x_i) - \overline{T}_B(x_i)\right| + \left|\overline{I}_A(x_i) - \overline{I}_B(x_i)\right| + \left|\overline{F}_A(x_i) - \overline{F}_B(x_i)\right|\right]^2}{\pi + \left[\sqrt{\underline{T}_A(x_i) + \underline{T}_B(x_i) + \underline{I}_A(x_i) + \underline{I}_B(x_i) + \underline{F}_A(x_i) + \underline{F}_B(x_i)}\sqrt{\overline{T}_A(x_i) + \overline{T}_B(x_i) + \overline{I}_A(x_i) + \overline{I}_B(x_i) + \overline{F}_A(x_i) + \overline{F}_B(x_i)}\right]}$$

3.1.1. Boundedness

Let $A_i = \left\langle \left(\underline{T}_A(x_i), \underline{I}_A(x_i), \underline{F}_A(x_i)\right), \left(\overline{T}_A(x_i), \overline{I}_A(x_i), \overline{F}_A(x_i)\right) \right\rangle$ $(i = 1,2,\ldots,n)$ be a collection of rough neutrosophic sets and

$$\overset{-}{A} = \left\langle \left[\min_i\left[\underline{T}_A(x_i)\right], \min_i\left[\overline{T}_A(x_i)\right]\right], \left[\max_i\left[\underline{I}_A(x_i)\right], \max_i\left[\overline{I}_A(x_i)\right]\right], \left[\max_i\left[\underline{F}_A(x_i)\right], \max_i\left[\overline{F}_A(x_i)\right]\right] \right\rangle \ \&$$

$$\overset{-}{A} = \left\langle \left[\min_i\left[\underline{T}_A(x_i)\right], \min_i\left[\overline{T}_A(x_i)\right]\right], \left[\max_i\left[\underline{I}_A(x_i)\right], \max_i\left[\overline{I}_A(x_i)\right]\right], \left[\max_i\left[\underline{F}_A(x_i)\right], \max_i\left[\overline{F}_A(x_i)\right]\right] \right\rangle$$

$$\overset{+}{A} = \left\langle \left[\max_i\left[\underline{T}_A(x_i)\right], \max_i\left[\overline{T}_A(x_i)\right]\right], \left[\min_i\left[\underline{I}_A(x_i)\right], \min_i\left[\overline{I}_A(x_i)\right]\right], \left[\min_i\left[\underline{F}_A(x_i)\right], \min_i\left[\overline{F}_A(x_i)\right]\right] \right\rangle$$

then $\overset{-}{A} \leq PI_{RNS}\left(A_1, A_2, \ldots, A_n\right) \leq \overset{+}{A}$

3.1.2. Proposition 1

(i) $PI_{RNS}(A,B)\geq 0$

(ii) $PI_{RNS}(A,B)=0$ if and only if $A=B$

(iii) $PI_{RNS}(A,B)=PI_{RNS}(B,A)$

(iv) If $A\subseteq B\subseteq C$ then $PI_{RNS}(A,C)\geq PI_{RNS}(A,B)$ & $PI_{RNS}(A,C)\geq PI_{RNS}(B,C)$

Proof

(i) We know that, the truth-membership function, indeterminacy –membership function and falsity–membership function in rough neutrosophic sets are within $[0,1]$. Hence $PI_{RNS}(A,B)\geq 0$

(ii) If $A=B$, then $\underline{T}_A(x_i)=\underline{T}_B(x_i),\underline{I}_A(x_i)=\underline{I}_B(x_i),\underline{F}_A(x_i)=\underline{F}_B(x_i),\overline{T}_A(x_i)=\overline{T}_B(x_i),\overline{I}_A(x_i)=\overline{I}_B(x_i)\&\overline{F}_A(x_i)=\overline{F}_B(x_i)$ for $i=1,2,\cdots,n\ \&\ x_i\in X$. Therefore, $PI_{RNS}(A,B)=0$. If $IHD_{RNS}(A,B)=0$, this implies

$$\left|\underline{T}_A(x_i)-\underline{T}_B(x_i)\right|=0;\quad \left|\underline{I}_A(x_i)-\underline{I}_B(x_i)\right|=0$$
$$\left|\underline{F}_A(x_i)-\underline{F}_B(x_i)\right|=0;\quad \left|\overline{T}_A(x_i)-\overline{T}_B(x_i)\right|=0$$
$$\left|\overline{I}_A(x_i)-\overline{I}_B(x_i)\right|=0;\quad \left|\overline{F}_A(x_i)-\overline{F}_B(x_i)\right|=0$$

Since its denominator is not equal to zero. Then,

$\underline{T}_A(x_i)=\underline{T}_B(x_i),\underline{I}_A(x_i)=\underline{I}_B(x_i),\underline{F}_A(x_i)=\underline{F}_B(x_i),\overline{T}_A(x_i)=\overline{T}_B(x_i),\overline{I}_A(x_i)=\overline{I}_B(x_i)\&\overline{F}_A(x_i)=\overline{F}_B(x_i)$ for $i=1,2,\cdots,n\ \&\ x_i\in X$. Hence $A=B$.

(iii) We know that,

$$\left|\underline{T}_A(x_i)-\underline{T}_B(x_i)\right|=\left|\underline{T}_B(x_i)-\underline{T}_A(x_i)\right|;\ \left|\underline{I}_A(x_i)-\underline{I}_B(x_i)\right|=\left|\underline{I}_B(x_i)-\underline{I}_A(x_i)\right|;\ \left|\underline{F}_A(x_i)-\underline{F}_B(x_i)\right|=\left|\underline{F}_B(x_i)-\underline{F}_A(x_i)\right|$$
$$\left|\overline{T}_A(x_i)-\overline{T}_B(x_i)\right|=\left|\overline{T}_B(x_i)-\overline{T}_A(x_i)\right|;\ \left|\overline{I}_A(x_i)-\overline{I}_B(x_i)\right|=\left|\overline{I}_B(x_i)-\overline{I}_A(x_i)\right|;\ \left|\overline{F}_A(x_i)-\overline{F}_B(x_i)\right|=\left|\overline{F}_B(x_i)-\overline{F}_A(x_i)\right|$$

Hence $PI_{RNS}(A,B)=PI_{RNS}(B,A)$.

(iv) We know that,

$$\underline{T}_A(x_i)\leq\underline{T}_B(x_i)\leq\underline{T}_C(x_i);\ \overline{T}_A(x_i)\leq\overline{T}_B(x_i)\leq\overline{T}_C(x_i)$$
$$\underline{I}_A(x_i)\geq\underline{I}_B(x_i)\geq\underline{I}_C(x_i);\ \overline{I}_A(x_i)\geq\overline{I}_B(x_i)\geq\overline{I}_C(x_i)$$
$$\underline{F}_A(x_i)\geq\underline{F}_B(x_i)\geq\underline{F}_C(x_i);\ \overline{F}_A(x_i)\geq\overline{F}_B(x_i)\geq\overline{F}_C(x_i)$$
$$[\because A\subseteq B\subseteq C]$$

Hence,

$$\left|\underline{T}_A(x_i)-\underline{T}_B(x_i)\right|\leq\left|\underline{T}_A(x_i)-\underline{T}_C(x_i)\right|;\ \left|\overline{T}_A(x_i)-\overline{T}_B(x_i)\right|\leq\left|\overline{T}_A(x_i)-\overline{T}_C(x_i)\right|$$
$$\left|\underline{I}_A(x_i)-\underline{I}_B(x_i)\right|\leq\left|\underline{I}_A(x_i)-\underline{I}_C(x_i)\right|;\ \left|\overline{I}_A(x_i)-\overline{I}_B(x_i)\right|\leq\left|\overline{I}_A(x_i)-\overline{I}_C(x_i)\right|$$
$$\left|\underline{F}_A(x_i)-\underline{F}_B(x_i)\right|\leq\left|\underline{F}_A(x_i)-\underline{F}_C(x_i)\right|;\ \left|\overline{F}_A(x_i)-\overline{F}_B(x_i)\right|\leq\left|\overline{F}_A(x_i)-\overline{F}_C(x_i)\right|$$
$$\left|\underline{T}_B(x_i)-\underline{T}_C(x_i)\right|\leq\left|\underline{T}_A(x_i)-\underline{T}_C(x_i)\right|;\ \left|\overline{T}_B(x_i)-\overline{T}_C(x_i)\right|\leq\left|\overline{T}_A(x_i)-\overline{T}_C(x_i)\right|$$
$$\left|\underline{I}_B(x_i)-\underline{I}_C(x_i)\right|\leq\left|\underline{I}_A(x_i)-\underline{I}_C(x_i)\right|;\ \left|\overline{I}_B(x_i)-\overline{I}_C(x_i)\right|\leq\left|\overline{I}_A(x_i)-\overline{I}_C(x_i)\right|$$
$$\left|\underline{F}_B(x_i)-\underline{F}_C(x_i)\right|\leq\left|\underline{F}_A(x_i)-\underline{F}_C(x_i)\right|;\ \left|\overline{F}_B(x_i)-\overline{F}_C(x_i)\right|\leq\left|\overline{F}_A(x_i)-\overline{F}_C(x_i)\right|$$

Here, the PI- distance is an increasing function

$\therefore\ PI_{RNS}(A,C)\geq PI_{RNS}(A,B)\ \&\ PI_{RNS}(A,C)\geq PI_{RNS}(B,C)$

4. Methodology

In this section, we present an application of rough neutrosophic set in medical diagnosis. In a given pathology, Suppose S is a set of symptoms, D is a set of diseases and P is a set of patients and let Q be a rough neutrosophic relation from the set of patients to the symptoms.i.e., $Q(P \to S)$ and R be a rough neutrosophic relation from the set of symptoms to the diseases i.e., $R(S \to D)$ and then the methodology involves three main jobs:

1. Determination of symptoms.
2. Formulation of medical knowledge based on rough neutrosophic sets.
3. Determination of diagnosis on the basis of new computation technique of rough neutrosophic sets.

5. Algorithm

Step 1 : The symptoms of the patients are given to obtain the patient symptom relation Q and are noted in Table 1.

Step 2 : The medical knowledge relating the symptoms with the set of diseases under consideration are given to obtain the symptom-disease relation *R* and are noted in Table 2.

Step 3 : The Computation *T* (relation between patients and diseases) is found using (3.1) between Table 1 & Table 2 and are noted in Table 3

Step 4: Finally, we select the minimum value from Table 3 of each row for possibility of the patient affected with the respective disease and then we conclude that the patient P_k is suffering from the disease D_r.

6. Case study [8]

In this section, an example adapted from Surapati Pramanik and Kalyan Mondal (Cosine Similarity Measure of Rough Neutrosophic Sets and its application in medical diagnosis) is used. Let there be three patients $P = \{P_1, P_2, P_3\}$ and the set of symptoms S={Temperature, Headache, Stomach pain,Cough, Chest pain}.The Rough Neutrosophic Relation $Q(P \to S)$ is given as in Table 1. Let the set of diseases D = {Viral fever, Malaria, Stomach problem, Chest problem}.The Rough Neutrosophic Relation $R(S \to D)$ is given as in Table 2.

Table 1: Patient-symptom relation (using step 1)

Q	Temperature	Headache	Stomach pain	Cough	Chest pain
P_1	$\langle(0.6,0.4,0.3),(0.8,0.2,0.1)\rangle$	$\langle(0.4,0.4,0.4),(0.6,0.2,0.2)\rangle$	$\langle(0.5,0.3,0.2),(0.7,0.1,0.2)\rangle$	$\langle(0.6,0.2,0.4),(0.8,0.0,0.2)\rangle$	$\langle(0.4,0.4,0.4),(0.6,0.2,0.2)\rangle$
P_2	$\langle(0.5,0.3,0.4),(0.7,0.3,0.2)\rangle$	$\langle(0.5,0.5,0.3),(0.7,0.3,0.3)\rangle$	$\langle(0.5,0.3,0.4),(0.7,0.1,0.4)\rangle$	$\langle(0.5,0.3,0.3),(0.9,0.1,0.3)\rangle$	$\langle(0.5,0.3,0.3),(0.7,0.1,0.3)\rangle$
P_3	$\langle(0.6,0.4,0.4),(0.8,0.2,0.2)\rangle$	$\langle(0.5,0.2,0.3),(0.7,0.0,0.1)\rangle$	$\langle(0.4,0.3,0.4),(0.8,0.1,0.2)\rangle$	$\langle(0.6,0.1,0.4),(0.8,0.1,0.2)\rangle$	$\langle(0.5,0.3,0.3),(0.7,0.1,0.1)\rangle$

Table 2: Symptom-Disease relation (Using step 2)

R	Viral fever	Malaria	Stomach problem	Chestproblem
Temperature	$\langle(0.6,0.5,0.4),(0.8,0.3,0.2)\rangle$	$\langle(0.1,0.4,0.4),(0.5,0.2,0.2)\rangle$	$\langle(0.3,0.4,0.4),(0.5,0.2,0.2)\rangle$	$\langle(0.2,0.4,0.6),(0.4,0.4,0.4)\rangle$
Headache	$\langle(0.5,0.3,0.4),(0.7,0.3,0.2)\rangle$	$\langle(0.2,0.3,0.4),(0.6,0.3,0.2)\rangle$	$\langle(0.2,0.3,0.3),(0.4,0.1,0.1)\rangle$	$\langle(0.1,0.5,0.5),(0.5,0.3,0.3)\rangle$
Stomach pain	$\langle(0.2,0.3,0.4),(0.4,0.3,0.2)\rangle$	$\langle(0.1,0.4,0.4),(0.3,0.2,0.2)\rangle$	$\langle(0.4,0.3,0.4),(0.6,0.1,0.2)\rangle$	$\langle(0.1,0.4,0.6),(0.3,0.2,0.4)\rangle$
Cough	$\langle(0.4,0.3,0.3),(0.6,0.1,0.1)\rangle$	$\langle(0.3,0.3,0.3),(0.5,0.1,0.3)\rangle$	$\langle(0.1,0.6,0.6),(0.3,0.4,0.4)\rangle$	$\langle(0.5,0.3,0.4),(0.7,0.1,0.2)\rangle$
Chest pain	$\langle(0.2,0.4,0.4),(0.4,0.2,0.2)\rangle$	$\langle(0.1,0.3,0.3),(0.3,0.1,0.1)\rangle$	$\langle(0.1,0.4,0.4),(0.3,0.2,0.2)\rangle$	$\langle(0.4,0.4,0.4),(0.6,0.2,0.2)\rangle$

Table 3: Pi-distance

T	Viral fever	Malaria	Stomach problem	Chest problem
P_1	**0.4115**	0.9147	1.2435	1.0821
P_2	**0.4963**	0.7953	1.3853	0.7419
P_3	**0.5233**	0.8466	1.3912	1.3189

7. Conclusion

This study discovers the relationship between the symptoms found with the patients and the set of diseases. This study will help the researcher to find out the diseases accurately that impacted the patients. This method is apt for handling the medical diagnosis problems and its efficiency and rationality have been proved without any doubt. The method employed is free from the limitations that are commonly found in other studies. Without such limitations, a new theory on image processing, cluster analysis etc., has been developed. In the same way it will grow and extend itself to other types of neutrosophic sets.

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Machine learning in Neutrosophic Environment: A Survey

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Abstract: Veracity in big data analytics is recognized as a complex issue in data preparation process, involving imperfection, imprecision and inconsistency. Single-valued Neutrosophic numbers (SVNs), have prodded a strong capacity to model such complex information. Many Data mining and big data techniques have been proposed to deal with these kind of dirty data in preprocessing stage. However, only few studies treat the imprecise and inconsistent information inherent in the modeling stage. However, this paper summarizes all works done about mapping machine learning algorithms from crisp number space to Neutrosophic environment. We discuss also contributions and hybridization of machine learning algorithms with Single-valued Neutrosophic numbers (SVNs) in modeling imperfect information, and then their impacts on resolving reel world problems. In addition, we identify new trends for future research, then we introduce, for the first time, a taxonomy of Neutrosophic learning algorithms, clarifying what algorithms are already processed or not, which makes it easier for domain researchers.

Keywords: Neutrosophic; Machine Learning; Single-valued Neutrosophic numbers; Neutrosophic simple linear regression; Neutrosophic-k-NN; Neutrosophic-SVM; Neutrosophic C-means; Neutrosophic Hierarchical Clustering.

1. Introduction

Although Machine learning algorithms have caught extensive attention in last decade, seen their abilities to solve a wide problems remained obscure for years. Most of these techniques work under the some hypotheses that data should be pure, perfect and complete information. As a result, formally if the learning problems are formulated under a set of indeterminate or inconsistent information, the machine learning system becomes unable to work and the data must treated in preparation phase, which is make data science process very long, and impracticable.

However, real learning problems are often involves imperfect information such as uncertainty, inconsistency, inaccuracy and incompleteness. If we can modeling the learning problem as it in real form, exploiting the information's imperfections, we can reduce the data science process which is in many times come back from modeling that is the last step to preparation step that is the first step in the process of data science.

Single-valued neutrosophic set (SVNs) aims to provide a framework to model imperfect information. In contrast to classical machine learning methods, single-valued neutrosophic learning algorithm manipulate information with imperfections to deal with learning problems modeling complex information. To improve the performance of existing learning algorithms and handle the imperfect information in real-world, many machine learning techniques has recently been mapped into Neutrosophic Sets (NSs) environment.

Hence, the main notions and concepts of Neutrosophic are defined, also some achievements and its extensions on the NSs are undertaken. Thus, to manipulate indeterminacy, uncertainty, or inconsistency in information, that often characterizes real situations, Smarandache [1 - 3], introduced Neutrosophic set (NS), which consists of three elements, truth-membership, an indeterminacy membership, and a falsity-membership degrees independently.

Every element of the NS's features has not only a certain degree of truth(T), but also a falsity degree (F) and indeterminacy degree(I). This concept is generated from many others such as crisp set, intuitionistic fuzzy set, fuzzy set, interval-valued fuzzy set, interval-valued intuitionistic fuzzy set, etc.

Nonetheless, the NS as a philosophical concept is hard to apply in real applications. In order to overcome this situation, Smarandache and al. [4] concretize this concept introducing single-valued neutrosophic set (SVNS). SVNS can be applied quite well in real scientific and engineering fields to handle the uncertainty, imprecise, incomplete, and inconsistent information. Broumi and Smarandache [5, 6] studied basic properties of similarity and distances applied in Neutrosophic environment using single valued neutrosophic set (SVN).

Hybridization between Neutrosophic and machine learning algorithms, have also been studied, several papers [7- 11] on Neutrosophic Machine Learning (NML) have been published in the last few years.

However, there is no survey papers summarize those new learning techniques and approaches, removing the barrier for researchers currently working in the area of Neutrosophic Machine Learning. This has the twofold advantage of making such techniques more readily reachable by researchers and, conversely, avoid wasting time for to have idea which Machine learning approaches to be mapped to Neutrosophic.

The rest of this paper is organized as follows. We discuss the origins of the connection between Neutrosophic and machine learning in Section 2. Next, in Section 3, we summarize a wide variety of hybrid Neutrosophic Machine Learning techniques. Research trends and outstanding issues are discussed in Section 4.1. Then, in section 4.2, we introduce, for the first time, a taxonomy of Neutrosophic learning algorithms, clarifying what algorithms are already processed or not, which makes it easier for domain researchers.

2. Origins of connection between Neutrosophic and Machine learning

We cannot understand this connection without understanding how the Neutrosophic community works. In recent years there has been an augmenting passion from this community of neutrosophic in working, in different directions, the use of Neutrosophic to treat imperfections information in many methods and domaines. This has led to the development of a new mathematic domaine called Neutrosophic, then the connections with many others areas, such as machine learning and artificial intelligence. In the early 1999s, the pioneer of the field Florentin Smarandache generalized the intuitionistic fuzzy set (IFS), paraconsistent set, and intuitionistic set to the neutrosophic set (NS), and he underlined the distinctions between NS and IFS by reel examples. With his biggest passion and faith, Florentin Smarandache, in a quiet small town in south U.S. called Gallup, start defend his theory of Neutrality and why the three elements truth-membership (T), indeterminacy (I), and falsehood-nonmembership (F) are over 1, reproducing the history of science by story as many concepts and theory that considered primitives, and then changed by new ones.

In addition to several papers of the Neutrosophic science international association (NSIA) members, gathered in Encyclopedia Neutrosophic Researchers [12], much advances has been done. Today there are several fields of Neutrosophic to tackle a variety of problems, including Neutrosophic Computing and Machine Learning. These efforts are valued by launching a science international journal of Neutrosophic Computing and Machine Learning [13], which issued its 7th volume in 2019. In which, all published papers have wrote by NSIA's researchers.

The international journal of Neutrosophic Computing and Machine Learning with its all volumes can be seen as broad overview of the field of machine learning in Neutrosophic provided by NSIA's researchers.

The main contributions of this paper: (1) summarizes research achievements on Neutrosophic Computing and Machine Learning from the point of view of non NSIA's researchers. In a different way, try to collect the different articles on Neutrosophic machine learning papers published on several journals around the world other than those published in Neutrosophic Computing and Machine Learning journal, among it, each volume is can be considered a state of art. In order to present to researchers, the global state of art of advances research on Neutrosophic Machine Learning approaches. (2) Try to taxonomy, cluster and identify differences Neutrosophic Machines learning approaches.

3. Literature review

There are several Machine learning in Neutrosophic algorithms and approaches surveyed in this article. Then, a natural questions arise: how we can categorize all hybrid methods?

Our view of the general relationship between the fields of machine learning and Neutrosophic is the re-searchers try to map the basic operations from crisp number to Neutrosophic environment, however they rewrite machine learning algorithm instead of using simple mathematical formulas, and they use Neutrosophic formulas. But the main question should the researchers in this hybrid field (Machine learning and Neutrosophic) respond is, does this hybridization make sense to tackle the real world issues or just a theoretical formulation?

Before trying to respond this question, we synthesis all hybrid methods according to commonly used categories, summary all surveyed papers in a table 1. There are four categories of machine learning algorithms, supervised learning with two subcategories classification and prediction, semi-supervised learning, unsupervised learning and reinforcement learning.

3.1. Neutrosophic supervised learning

3.1.1. Neutrosophic Classification

Neutrosophic-k-NN Classifier [14]: K-Nearest Neighbor (K-NN) method isn't a learning method, but based on saving the training examples (all training examples), at prediction time, it find the k training examples $(x_1, y_1), \cdots, (x_k, y_k)$ that are closest to the test example x, and then affect to the most frequent class among those y_i's. This initial version of K-NN suffers from slowness because to classify x, one need to loop over all training examples. Actually, some tricks to speed are introduced such as classes represented by medoid (Representative point), or centroid (central value), etc. The Neutrosophic K-NN method we present here is the mapping of method based on Centroid, in which we consider c_j the center of cluster or class j, a constant m, regularization parameter δ, and (T_{ij}, I_{ij}, F_{ij}), where T_{ij} denote truth, I_{ij} indeterminacy and N_{ij} falsity membership values of point i for class j.

$$T_{ij} = \frac{(x_i - c_j)^{-(\frac{2}{m-1})}}{\sum_{j=1}^{C}(x_i - c_j)^{-(\frac{2}{m-1})} + (x_i - c_{imax})^{-(\frac{2}{m-1})} + \delta^{-(\frac{2}{m-1})}}, \quad (1)$$

$$F_{ij} = \frac{\delta^{-(\frac{2}{m-1})}}{\sum_{j=1}^{C}(x_i - c_j)^{-(\frac{2}{m-1})} + (x_i - c_{imax})^{-(\frac{2}{m-1})} + \delta^{-(\frac{2}{m-1})}}, \quad (2)$$

$$I_{ij} = \frac{(x_i - c_{imax})^{-(\frac{2}{m-1}}}{\sum_{j=1}^{C}(x_i - c_j)^{-(\frac{2}{m-1})} + (x_i - c_{imax})^{-(\frac{2}{m-1})} + \delta^{-(\frac{2}{m-1})}}, \tag{3}$$

At the time of prediction, the membership value of unknown point x_u to class j is defined by as follow:

$$x_{uj} = \frac{\sum_{i=1}^{k} d_i (T_{ij} + F_{ij} - I_{ij})}{\sum_{i=1}^{k} d_i}, \tag{4}$$

With $d_i = \frac{1}{(x_u - x_i)^{\frac{2}{q-1}}}$

Then unknown point x_u get the label of class maximizing $max\{x_{uj}; j = 1,2 \cdots, C\}$.

The authors didn't show the usefulness of the proposed method but they proposed an interesting idea to apply it on imbalanced data-set problems.

Neutrosophic SVM (N-SVM) [15] : Let's assume that (x_i, y_i) a set of training data, in which eve

with $i = 1,2 \cdots, N$

$$t_i = 1 - \frac{\|(x_j - C_+)\|}{max_{x_k \in P}\|(x_j - C_+)\|}, \tag{5}$$

ry x_i belonging to class y_i with a triple t_i, f_i, and i_i as its Neutrosophic components.

$$i_i = 1 - \frac{\|(x_j - C_{all})\|}{max_{x_k \in P}\|(x_j - C_{all})\|}, \tag{6}$$

$$f_i = 1 - \frac{\|(x_j - C_-)\|}{max_{x_k \in P}\|(x_j - C_-)\|}, \tag{7}$$

Where P and N represent the positive and negative samples subsets respectively, $y_i = +1$ for all $x_i \in P$ and $y_i = -1$ for $x_i \in N$.

$$t_i = 1 - \frac{\|(x_j - C_-)\|}{max_{x_k \in N}\|(x_j - C_-)\|}, \tag{8}$$

$$i_i = 1 - \frac{\|(x_j - C_{all})\|}{max_{x_k \in N}\|(x_j - C_{all})\|}, \tag{9}$$

$$f_i = 1 - \frac{\|(x_j - C_+)\|}{max_{x_k \in N}\|(x_j - C_+)\|}, \tag{10}$$

with $C_+ = \frac{1}{n_+}\sum_{k=1}^{n_+} x_k$, $C_- = \frac{1}{n_-}\sum_{k=1}^{n_-} x_k$, and $C_{all} = \frac{1}{2}(C_+ + C_-)$

We define g_j as weighting function:

$$g_j = t_i + i_i - f_i, \tag{11}$$

The optimal hyper-plane problem in the reformulated SVM is the solution to:

$$\text{minimize } g_j = \frac{1}{2}\omega \cdot \omega \sum_{j=1}^{k} g_j \, \zeta_j, \tag{12}$$

Subject to

$$\boldsymbol{y_j(\omega_j + b) > 1 - \zeta_j \quad i = 1, 2 \cdots, n} \tag{13}$$

N-SVM (Neutrosophic-Support Vector Machine) improves performance over standard SVM and reduces the effects of outliers in learning samples.

3.1.2. Neutrosophic Regression

Neutrosophic simple linear Regression: Salama and al. [16] studied and introduced Neutrosophic simple linear regression model with its possible utility to predict value of a dependent variable y according to predictor variable x. Below a pseudo code of Neutrosophic Linear Regression algorithm.

Algorithm 1 Neutrosophic Simple Linear Regression

Require: Training data $(x_i, y_j), i, j = 1,2, \cdots, N$

A model define the relationship between input x and y, y = ax + b, where (a and b) represent estimated Neutrosophic (intercept and slope) coefficients, y estimated Neutrosophic output

Define degree of membership, non-membership, and indeterminacy :
$((\mu_A(x_1), \lambda_A(x_1), \nu_A(x_1)), (\mu_B(x_1), \lambda_B(x_1), \nu_B(x_1)), i, j = 1,2 \cdots, N$
Define cost function $J(a, b) = \sum (ax_i + b - y_i)^2$
Repeat
    Calculate the gradients of J
    Update the weights a
Repeat until the cost $J(a, b)$ stops reducing, or some other predefined termination criteria is met

3.2. Neutrosophic unsupervised learning

3.2.1. Neutrosophic Clustering

Neutrosophic C-means: In this method, authors [10] have given a meaning to the three basic Neutrosophic components T_{ij} as membership values belonging to the determinate clusters I_i as boundary regions, and N_i noisy data set.

$$\overline{c}_{imax} = \frac{c_{pi} + c_{qi}}{2}, \tag{14}$$

We define p_i and q_i are the cluster numbers with the biggest and second biggest value of T respectively, and m is a constant.

$$p_i = \lambda \cdot argmax_{j=1,2\cdots,C}(T_{ij}), \tag{15}$$

$$q_i = argmax_{j \neq p_i \cap 1,2\cdots,C}(T_{ij}), \tag{16}$$

Membership Neutrosophic values are defined by follow formulas:

$$T_{ij} = \frac{\varpi_2\varpi_3(x_i - c_j)^{-(\frac{2}{m-1})}}{\sum_{j=1}^{C}(x_i - c_j)^{-(\frac{2}{m-1})} + (x_i - c_{imax})^{-(\frac{2}{m-1})} + \delta^{-(\frac{2}{m-1})}}, \tag{17}$$

$$F_{ij} = \frac{\varpi_1\varpi_3\delta^{-(\frac{2}{m-1})}}{\sum_{j=1}^{C}(x_i - c_j)^{-(\frac{2}{m-1})} + (x_i - c_{imax})^{-(\frac{2}{m-1})} + \delta^{-(\frac{2}{m-1})}}, \tag{18}$$

$$I_{ij} = \frac{\varpi_1\varpi_2(x_i - c_{imax})^{-(\frac{2}{m-1}}}{\sum_{j=1}^{C}(x_i - c_j)^{-(\frac{2}{m-1})} + (x_i - c_{imax})^{-(\frac{2}{m-1})} + \delta^{-(\frac{2}{m-1})}}, \tag{19}$$

with $\boldsymbol{i = 1, 2 \cdots, N}$

$$c_j = \frac{\sum_{i=1}^{N}(\varpi_1 T_{ij})^m x_i}{\sum_{i=1}^{N}(\varpi_1 T_{ij})^m}, \tag{20}$$

$$J_{NCM}(T, F, I, c) = \sum_{i=1}^{N}(\varpi_1 T_{ij})^m (x_i - c_j)^2 + \sum_{i=1}^{N}(\varpi_2 F_i)^m (x_i - \overline{c}_{imax})^2 + \delta^2 \sum_{i=1}^{N}(\varpi_3 I_i)^m, \tag{21}$$

The separation between classes is performed by iteration optimizing objective function, that is based on updating the Neutrosophic membership values (T_{ij},F_i, I_i), the centers c_j , and $\overline{c}_{imax}$ according to the equations defined above. The loop stop when $\parallel T_{ij}^{(k+1)} - T_{ij}^{(k)} \parallel < \epsilon$ with ϵ is condition check and k is step.

For nonlinear clustering problem an extended Method have been proposed called Kernel NCMA in which we use a function kernel K, $K(x_i, z_j)$ instead of$(x_i - z_j)$, such as $K(x_i, \overline{c}_{imax})$ in place of $x_i - \overline{c}_{imax}$. The NCMA can be summarized as follow :

Algorithm 2 KNCM algorithm

Assign each data into the class with the largest TM
Choose kernel function and its parameters
  Initialize $T^{(0)}$, $F^{(0)}$, $I^{(0)}$, $C, m, \delta, \epsilon, \varpi_1, \varpi_2, \varpi_3$ parameters
  While $\parallel T_{ij}^{(k+1)} - T_{ij}^{(k)} \parallel < \epsilon$ **do**
    Calculate the centers vectors $c^{(k)}$ at k ste
    Compute the $\overline{c}_{imax}$ using the clusters centers with the largest and second largest value of T_{ij}
    Update $T_{ij}(k)$ to $T_{ij}(k+1)$, $F_{ij}(k)$ to $T_{ij}(k+1)$, and $I_{ij}(k)$ to $I_{ij}(k+1)$
  End while

NCM and KNCM as mentioned by authors may handle veracity in data such as outliers and noise using their new objective function. And then possibility to deal with raw data in modeling phase instead while data cleaning phase.

3.2.2. Neutrosophic Hierarchical Clustering

Agglomerative Hierarchical Clustering Algorithm [17]: First, every SVNSs A_k with $(k = 1, \cdots, n)$ considered as single cluster. In a loop, until we get a single cluster of size n, the SVNSs A_k the SVNS are then compared to each other and are merged into a single group based on the closest pair of groups (with the smallest distance), based on a weighted distance (Hamming distance or Euclidean distance). At each stage, only two clusters can be merged and they cannot be separated once merged. The center of each cluster is recalculated using the arithmetic mean of the SVNS offered to the cluster. The distance between the centers of each group is considered as the distance between two groups.

Algorithm 3 Agglomerative Hierarchical Clustering algorithm

Let us consider a collection of n SVNSs $A_k(k = 1, \cdots, n)$
Assign each of the n SVNSs $A_k(k = 1, \cdots, n)$ to a single cluster
While All A_k clustered into a single cluster of size n **do**
SVNSs $A_k(k = 1, \cdots, n)$ are then compared among themselves and are merged them into a single
Cluster according to the closest (with smaller distance) pair of clusters, based on a weighted distance
(Hamming distance or Euclidean distance)
End while

Table 1. List of major contributions on machine learning algorithms in Neutrosophic environment.

Authors	Title	Reference	Publisher
Salama, A. A., Eisa, M., ELhafeez, S. A., Lotfy, M. M. (2015)	Review of recommender systems algorithms utilized in social networks based e-Learning systems neutrosophic system	[18]	Neutrosophic Sets and Systems 8 : 32-40
Ansari, A. Q., Biswas, R., Aggarwal, S. (2013)	Neutrosophic classifier: An extension of fuzzy classifer	[19]	Applied Soft Computing, 13(1), 563-573
Zhang, M., Zhang, L., Cheng, H. D. (2010)	A neutrosophic approach to image segmentation based on watershed method	[20]	Signal Processing, 90(5), 1510-1517
Zhang, X., Bo, C., Smarandache, F., Dai, J. (2018)	New inclusion relation of neutrosophic sets with applications and related lattice structure	[21]	International Journal of Machine Learning and Cybernetics, 9, 1753-1763
Mondal, K. A. L. Y. A. N., Pramanik, S. U. R. A. P. A. T. I., Giri, B. C. (2016)	Role of neutrosophic logic in data mining. New Trends in Neutrosophic Theory and Application	[22]	Pons Editions, Brussels, 15-23.
Sengur, A., Guo, Y. (2011)	Color texture image segmentation based on neutrosophic set and wavelet transformation	[23]	Computer Vision and Image Understanding,115(8), 1134-1144
Akbulut, Y., engr, A., Guo, Y., Smarandache, F. (2017)	A novel neutrosophic weighted extreme learning machine for imbalanced data set	[24]	Symmetry, 9(8), 142
Kraipeerapun, P., Fung, C. C., Wong, K. W. (2007 August)	Ensemble neural networks using interval neutrosophic sets and bagging	[25]	In Third International Conference on Natural Computation (ICNC 2007) (Vol. 1, pp. 386-390). IEEE
Kavitha, B., Karthikeyan, S., Maybell, P. S(2012)	An ensemble design of intrusion detection system for handling uncertainty using Neutrosophic Logic Classifier	[26]	Knowledge-Based Systems, 28, 88-96

Ye, J. (2014).	Single-valued neutrosophic minimum spanning tree and its clustering method	[27]	Journal of intelligent Systems, 23(3), 311-324
Thanh, N. D., Ali, M. (2017, July)	Neutrosophic recommender system for medical diagnosis based on algebraic similarity measure and clustering	[28]	In 2017 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE) (pp. 1-6). IEEE
Akbulut, Y., engr, A., Guo, Y., Polat, K. (2017)	KNCM: Kernel neutrosophic c-means clustering	[10]	Applied Soft Computing, 52, 714-724
Kraipeerapun, P., Fung, C. C., Wong, K. W. (2006)	Multiclass classification using neural networks and interval neutrosophic sets	[29]	World Scientific and Engineering Academy and Society (WSEAS)
Ali, M., Khan, M., Tung, N. T. (2018)	Segmentation of dental X-ray images in medical imaging using neutrosophic orthogonal matrices	[30]	Expert Systems with Applications, 91, 434-441
Long, H. V., Ali, M., Khan, M., Tu, D. N. (2019)	A novel approach for fuzzy clustering based on neutrosophic association matrix	[31]	Computers and Industrial Engineering, 127, 687-697
Kraipeerapun, P., Fung, C. C. (2008, February)	Comparing performance of interval neutrosophic sets and neural networks with support vector machines for binary classification problems	[32]	In 2008 2nd IEEE International Conference on Digital Ecosystems and Technologies (pp. 34-37). IEEE
Thanh, N. D., Ali, M. (2017)	A novel clustering algorithm in a neutrosophic recommender system for medical diagnosis	[33]	Cognitive Computation, 9(4), 526-544
Gaber, T., Ismail, G., Anter, A., Soliman, M., Ali, M., Semary, N., Snasel, V. (2015, August)	Thermogram breast cancer prediction approach based on Neutrosophic sets and fuzzy c-means algorithm	[34]	In 2015 37th Annual International Conference of the IEEE Engineering in Medicine and Biology Society (EMBC) (pp. 4254-4257). IEEE
Ye, J. (2017)	Single-valued neutrosophic clustering algorithms based on similarity measures	[35]	Journal of Classification, 34(1), 148-162
Tuan, T. M., Chuan, P. M., Ali, M., Ngan, T. T., Mittal, M. (2018)	Fuzzy and neutrosophic modeling for link prediction in social networks	[36]	Evolving Systems, 1-6
Ju, W., Cheng, H. D. (2008, December)	Discrimination of outer membrane proteins using reformulated support vector machine based on neutrosophic set	[37]	In 11th Joint International Conference on Information Sciences. Atlantis Press
Shan, J., Cheng, H. D., Wang, Y. (2012)	A novel segmentation method for breast ultrasound images based on neutrosophic lmeans clustering	[38]	Medical physics, 39(9), 5669-5682
Basha, S. H., Abdalla, A. S., Hassanien, A. E. (2016, December)	GNRCS: hybrid classification system based on neutrosophic logic and genetic algorithm	[39]	In 2016 12th International Computer Engineering Conference (ICENCO) (pp. 53-58). IEEE
Kraipeerapun, P., Fung, C. C., Wong, K. W. (2007)	Uncertainty assessment using neural networks and interval neutrosophic sets for multiclass classification problems	[40]	WSEAS Transactions on Computers, 6(3)
Dhingra, G., Kumar, V., Joshi, H. D. (2019)	A novel computer vision based neutrosophic approach for leaf disease identification and classification	[41]	Measurement, 135, 782-794
Rashno, E., Akbari, A., Nasersharif, B. (2019)	A Convolutional Neural Network model based on Neutrosophy for Noisy Speech Recognition	[42]	arXiv preprint arXiv:1901.10629

4. Discussions

4.1. Research trends and open issues

Hybridization between Neutrosophic and machine learning algorithms, have also been studied. In supervision learning, Akbulut and al. [14] introduced intuitive supervised learning method called Neutrosophic-k-NN Classifier K-Nearest Neighbor (K-NN). Due to its results as a powerful machine learning methods, several tries to map SVM in Neutrosophic, Ju and al. [15] proposed Neutrosophic-support vector machines (N-SVM). In [32], authors Compared performance of interval neutrosophic sets and neural networks with support vector machines for binary classification problems. Ju and al [37] reformulated SVM, based on neutrosophic set, to discriminate outer membrane proteins using reformulated support vector machine based on neutrosophic set. In recent years, Artificial neural networks (ANN) has recognized huge advances, which explain many attempts of hybridization between ANN and Neutrosophic, Kraipeerapun and al. [40] demonstrated how to assess uncertainty using neural networks and interval neutrosophic sets for multi-class classification problems, then its application on multi-class classification problems [29], afterward, for more robustness ensemble neural networks using interval neutrosophic sets and bagging [25].

Likewise, in unsupervised learning, Alsmadi and al. [7] introduced a hybrid Fuzzy C-Means and Neutrosophic for jaw lesions segmentation. Inspired from fuzzy c-means and the neutrosophic set framework, Guo and al. [9] proposed a new clustering algorithm, neutrosophic c-means (NCM), for uncertain data cluster-ing. Akbulu and al. [10] developed KNCM: Kernel Neutrosophic c-Means Clustering, neutrosophic c-means (NCM), in order to alleviate the limitations of the popular fuzzy c-means (FCM) clustering algorithm by introducing a new objective function which contains two types of rejection. To deal with indeterminacy, Qureshi and al. [11] improved the Method for Image Segmentation Using K-Means Clustering with Neutrosophic Logic. Ye and al. [35] proposed Single-valued neutrosophic clustering algorithms based on similarity measures. Akhtar and al. [8] applied K-mean algorithm in Neutrosophics environment for Image Segmentation, Gaber and al. [34] to predict thermogram breast cancer, and Shan and al. [38] use neutrosophic l-means clustering to breast ultrasound images based.

Conversely, in reinforcement learning, we haven't find any resources about mixture between the both approaches, because this type of algorithms of reinforcement is under development, to be subject of hybridization.

4.2. Taxonomy of Neutrosophic Machine learning

The trends also involve the question of where machine learning areas to apply Neutrosophic, whether to it is more appropriate to employ instead of crisp number the SVN numbers. Hence, we have classified different Neutrosophic machine learning algorithms. Below a summarizing of all Neutrosophic Learning Methods and algorithms, according to standard taxonomy of machine learning.

- Supervised (inductive) learning (training data includes desired outputs)
  - Prediction : (Regression) to predict continuous values
    - Neutrosophic simple linear regression
  - Classification (discrete labels) : predict categorical values
    - Neutrosophic-k-NN [14]
    - Neutrosophic-Support Vector Machines (N-SVM)[15], [32],[37]
    - Neutrosophy-Artificial neural networks (N-ANN)[40], [29]
    - Neutrosophy-Ensemble neural networks, Bagging [25]
- Unsupervised learning (training data does not include desired outputs)
  - Clustering
    - Neutrosophic C-Means (NCM) [7], [9], [11], [35], [8], [38], [34]
    - Kernel Neutrosophic c-Means(KNCM) [10]
  - Neutrosophic Hierarchical Clustering
    - Neutrosophic Agglomerative Hierarchical Clustering [17]
    - Neutrosophic Divisive Hierarchical Clustering
  - Finding association (in features)
  - Dimension reduction
- Neutrosophic semi-supervised learning : Neutrosophic Semi-supervised learning (training data includes a few desired outputs
- Neutrosophic Reinforcement learning : Learning from sequential data
  - Q-Learning
  - State-Action-Reward-State-Action (SARSA)
  - Deep Q Network (DQN)
  - Deep Deterministic Policy Gradient (DDPG)

5. Conclusions

In this paper, we have explored how Neutrosophic contributes to enhance machine learning algorithms generally and how to modeling and exploit information's imperfection such as uncertainty as a source of information, not a kind of noises. We tried to cover hybrid approaches. However, it is still several machine learning algorithms to map to Neutrosophic environment, demonstrate the utility of Neutrosophic with machine learning to tackle real world challenges.

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Neutrosophic Bipolar Vague Set and its Application to Neutrosophic Bipolar Vague Graphs

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Abstract: A bipolar model is a significant model wherein positive data revels the liked object, while negative data speaks the disliked object. The principle reason for analysing the vague graphs is to demonstrate the stability of few properties in a graph, characterized or to be characterized in using vagueness. In this present research article, the new concept of neutrosophic bipolar vague sets are initiated. Further, its application to neutrosophic bipolar vague graphs are introduced. Moreover, some remarkable properties of strong neutrosophic bipolar vague graphs, complete neutrosophic bipolar vague graphs and complement neutrosophic bipolar vague graphs are explored and the proposed ideas are outlined with an appropriate example

Keywords: Neutrosophic bipolar vague set, Neutrosophic bipolar vague graphs, Complete neutrosophic bipolar vague graph, Strong neutrosophic bipolar vague graph.

1. Introduction

Fuzzy set theory richly contains progressive frameworks comprising of data with various degrees of accuracy. Vague sets are first investigated by Gau and Buehrer [30] which is an extension of fuzzy set theory. Various issues in real-life problems have fluctuations, one has to handle these vulnerabilities, vague set is introduced. Vague sets are regarded as a special case of context dependent fuzzy sets and it is applicable in real-time systems consisting of information with multiple levels of precision. So as to deal with the uncertain and conflicting data, the neutrosophic set is presented by the creator Smarandache and studied widely about it [13, 21, 28, 31, 41, 42, 4, 5, 43, 44, 22, 23, 45]. Neutrosophic sets are the more generalized sets, one can manage with uncertain informations in a more successful way with a progressive manner when appeared differently in relation to fuzzy sets. It have the greater adaptability, accuracy and similarity to the framework when contrasted with past existing fuzzy models. The neutrosophic set has three completely independent parts, which are truth-membership degree, indeterminacy-membership degree and falsity-membership degree with the sum of these values lies between **0** and **3**; therefore, it is applied to many different areas, such as algebra [32, 33] and decision-making problems (see [46] and references therein).

Bipolar fuzzy sets are extension of fuzzy sets whose membership degree ranges from $[-\mathbf{1},\mathbf{1}]$. Themembership degree $(\mathbf{0},\mathbf{1}]$ represents that an object satisfies a certain property whereas the membership degree $[-\mathbf{1},\mathbf{0})$ represents that the element satisfies the implicit counter-property. The positive information indicates that the consideration to be possible and negative information indicates that the consideration is granted to be impossible. Notable that bipolar fuzzy sets and vague sets appear to be comparative, but they are completely different sets. Even though both sets handle with incomplete data, they will not adapt the indeterminate or inconsistent information which appears in many domains like decision support systems. Many researchers pay attention to the development of neutrosophic and bipolar neutrosophic graphs [39, 40]. For example, in [17], the authors studied neutrosophic soft topological K-algebras. In [48], complex neutrosophic graphs are developed. Bipolar single valued neutrosophic graphs are established in [25]. Bipolar neutrosophic sets and its application to incidence graphs are discussed in [15]. In [16], bipolar neutrosophic graphs are established.

Recently, a variety of decision making problems are based on two-sided bipolar judgements on a positive side and a negative side. Nowadays bipolar fuzzy sets are playing a substantial role in chemistry, economics, computer science, engineering, medicine and decision making problems (for more details see [27, 28, 31, 34, 38, 46] and references therein). Akram [8] introduced bipolar fuzzy graphs and discuss its various properties and several new concepts on bipolar neutrosophic graphs and bipolar neutrosophic hypergraphs have been studied in [7] and references therein. In [4], he established the certain notions including strong neutrosophic soft graphs and complete neutrosophic soft graphs. The author Shawkat Alkhazaleh introduces the concept of neutrosophic vague set theory [6]. The authors [3] introduces the concept of neutrosophic vague soft expert set which is a combination of neutrosophic vague set and soft expert set to improve the reasonability of decision making in reality. It is remarkable that the Definition 2.6 in [37] has a flaw and it not defined in a proper manner. We focussed on to redefine that definition in a proper way and explained with an example and also we applied to neutrosophic bipolar vague graphs. Motivation of the mentioned works as earlier [10], we mainly contribute the definition of neutrosophic bipolar vague set is redefined. In addition, it is applied to neutrosophic bipolar vague graphs and strong neutrosophic bipolar vague graphs. The developed results will find an application in NBVGs and also in decision making. The objectives in this work as follows:

- Newly defined the neutrosophic bipolar vague set
- Introduce the operations like union and intersection with example in section 2.
- In section 3, neutrosophic bipolar vague graphs are developed with an example. Further, the concepts of neutrosophic bipolar vague subgraph, adjacency, path, connectedness and degree of neutrosophic bipolar vague graph are evolved.
- Further we presented some remarkable properties of strong neutrosophic bipolar vague graphs in section 5, followed by a remark by comparing other types of bipolar graphs. The obtained results will improve the existing result [37].

2. Preliminaries

Definition 2.1 *[18] A vague set A on a non empty set X is a pair (T_A, F_A), where $T_A: X \to [0,1]$ and $F_A: X \to [0,1]$are true membership and false membership functions, respectively, such that*

$$0 \leq T_A(x) + F_A(y) \leq 1 \text{ for any } x \in X.$$

Let X and Y be two non-empty sets. A vague relation R of X to Y is a vague set R on $X \times Y$ that is $R = (T_R, F_R)$, where $T_R: X \times Y \rightarrow [0,1], F_R: X \times Y \rightarrow [0,1]$ which satisfies the condition:

$$0 \leq T_R(x, y) + F_R(x, y) \leq 1 \text{ for any } x \in X.$$

Let $G = (V, E)$ be a graph. A pair $G = (J, K)$ is called a vague graph on G^* or a vague graph where $J = (T_J, F_J)$ is a vague set on V and $K = (T_K, F_K)$ is a vague set on $E \subseteq V \times V$ such that for each $xy \in E$,

$$T_K(xy) \leq (T_J(x) \wedge T_J(y)) \text{ and } F_K(xy) \geq (T_J(x) \vee F_J(y)).$$

Definition 2.2 *[4] A Neutrosophic set A is contained in another neutrosophic set B, (i.e) $A \subseteq B$ if $\forall x \in X, T_A(x) \leq T_B(x), I_A(x) \geq I_B(x)$and $F_A(x) \geq F_B(x)$.*

Definition 2.3 *[27, 30] Let X be a space of points (objects), with a generic elements in X denoted by x. A single valued neutrosophic set (SVNS) A in X is characterized by truth-membership function $T_A(x)$, indeterminacy-membership function $I_A(x)$ and falsity-membership-function $F_A(x)$.*

For each point x in X, $T_A(x), F_A(x), I_A(x) \in [0,1], A = \{\langle x, T_A(x), F_A(x), I_A(x)\rangle\}$ and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 2.4 *[9] A neutrosophic graph is defined as a pair $G^* = (V, E)$ where*

(i) $V = \{v_1, v_2, .., v_n\}$ such that $T_1 = V \rightarrow [0,1]$, $I_1 = V \rightarrow [0,1]$ and $F_1 = V \rightarrow [0,1]$ denote the degree of truth-membership function, indeterminacy function and falsity-membership function, respectively and

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$$

(ii) $E \subseteq V \times V$ where $T_2 = E \rightarrow [0,1]$, $I_2 = E \rightarrow [0,1]$ and $F_2 = E \rightarrow [0,1]$ are such that

$$T_2(uv) \leq \{T_1(u) \wedge T_1(v)\},$$
$$I_2(uv) \leq \{I_1(u) \wedge I_1(v)\},$$
$$F_2(uv) \leq \{F_1(u) \vee F_1(v)\},$$
$$\text{and } 0 \leq T_2(uv) + I_2(uv) + F_2(uv) \leq 3, \forall uv \in E.$$

Definition 2.5 *[46] A bipolar neutrosophic set A in X is defined as an object of the form*

$A = \{< x, T^P(x), I^P(x), F^P(x), T^N(x), I^N(x), F^N(x) >: x \in X\}$, where $T^P, I^P, F^P: X \rightarrow [0,1]$ and $T^N, I^N, F^N: X \rightarrow [-1,0]$ The Positive membership degree $T^P(x), I^P(x), F^P(x)$ denotes the truth membership, indeterminate membership and false membership of an element $x \in X$ corresponding to a bipolar neutrosophic set A and the negative membership degree $T^N(x), I^N(x), F^N(x)$ denotes the truth membership, indeterminate membership and false membership of an element $x \in X$ to some implicit counter-property corresponding to a bipolar neutrosophic set A.

Definition 2.6 *[46] Let X be a non-empty set. Then we call* $A = \{\langle x, T^P(x), I^P(x), F^P(x), T^N(x), I^N(x), F^N(x)\rangle, x \in X\}$a bipolar single valued neutrosophic relation on X such that $T_A^P(x, y) \in [0,1], I_A^P(x, y) \in [0,1], F_A^P(x, y) \in [0,1]$ and $T_A^N(x, y) \in [-1,0], I_A^N(x, y) \in [-1,0], F_A^N(x, y) \in [-1,0]$.

Definition 2.7 *[46] Let $A = (T_P^A, I_P^A, F_P^A, T_N^A, I_N^A, F_N^A)$ and $B = (T_P^B, I_P^B, F_P^B, T_N^B, I_N^B, F_N^B)$ be bipolar single valued neutrosophic set on X. If $B = (T_P^B, I_P^B, F_P^B, T_N^B, I_N^B, F_N^B)$ is a bipolar single valued neutrosophic relation on $A = (T_P^A, I_P^A, F_P^A, T_N^A, I_N^A, F_N^A)$ then*

$$T_B^P(xy) \leq (T_A^P(x) \wedge T_A^P(y)), T_B^N(xy) \geq (T_A^N(x) \vee T_A^N(y))$$

$I_B^P(xy) \geq (I_A^P(x) \vee I_A^P(y)), I_B^N(xy) \leq (I_A^N(x) \wedge I_A^N(y))$

$F_B^P(xy) \geq (F_A^P(x) \vee F_A^P(y)), F_B^N(xy) \leq (F_A^N(x) \wedge F_A^N(y))$

A bipolar single valued neutrosophic relation B on X is called symmetric if $T_B^P(xy) = T_B^P(yx), I_B^P(xy) = I_B^P(yx), F_B^P(xy) = F_B^P(yx)$ and $T_B^N(xy) = T_B^N(yx), I_B^N(xy) = I_B^N(yx), F_B^N(xy) = F_B^N(yx)$ for all $xy \in X$.

Definition 2.8 *[6] A neutrosophic vague set A_{NV} (NVS in short) on the universe of discourse X written as* $A_{NV} = \{\langle x, \hat{T}_{A_{NV}}(x), \hat{I}_{A_{NV}}(x), \hat{F}_{A_{NV}}(x)\rangle, x \in X\}$ whose truth-membership, indeterminacy membership and falsity-membership function is defined as $\hat{T}_{A_{NV}}(x) = [\hat{T}^-(x), \hat{T}^+(x)], [\hat{I}^-(x), \hat{I}^+(x)], [\hat{F}^-(x), \hat{F}^+(x)]$, where $T^+(x) = 1 - F^-(x), F^+(x) = 1 - T^-(x)$, and $0 \leq T^-(x) + I^-(x) + F^-(x) \leq 2$.

Definition 2.9 *[20] The complement of NVS A_{NV} is denoted by A_{NV}^c and it is defined by*

$$\hat{T}_{A_{NV}}^c(x) = [1 - T^+(x), 1 - T^-(x)],$$
$$\hat{I}_{A_{NV}}^c(x) = [1 - I^+(x), 1 - I^-(x)],$$
$$\hat{F}_{A_{NV}}^c(x) = [1 - F^+(x), 1 - F^-(x)],$$

Definition 2.10 *[6] Let A_{NV} and B_{NV} be two NVSs of the universe U. If for all $u_i \in U, \hat{T}_{A_{NV}}(u_i) =$* $\hat{T}_{B_{NV}}(u_i), \hat{I}_{A_{NV}}(u_i) = \hat{I}_{B_{NV}}(u_i), \hat{F}_{A_{NV}}(u_i) = \hat{F}_{B_{NV}}(u_i)$ then the NVS A_{NV} are included by B_{NV}, denoted by $A_{NV} \subseteq B_{NV}$ where $1 \leq i \leq n$.

Definition 2.11 *[6] The union of two NVSs A_{NV} and B_{NV} is a NVSs, C_{NV}, written as $C_{NV} = A_{NV} \cup B_{NV}$, whose truth membership function, indeterminacy-membership function and false-membership function are related to those of A_{NV} and B_{NV} by*

$$\hat{T}_{C_{NV}}(x) = [(\hat{T}_{A_{NV}}^-(x) \vee \hat{T}_{B_{NV}}^-(x)), (\hat{T}_{A_{NV}}^+(x) \vee \hat{T}_{B_{NV}}^+(x))]$$
$$\hat{I}_{C_{NV}}(x) = [(\hat{I}_{A_{NV}}^-(x) \wedge \hat{I}_{B_{NV}}^-(x)), (\hat{I}_{A_{NV}}^+(x) \wedge \hat{I}_{B_{NV}}^+(x))]$$
$$\hat{F}_{C_{NV}}(x) = [(\hat{F}_{A_{NV}}^-(x) \wedge \hat{F}_{B_{NV}}^-(x)), (\hat{F}_{A_{NV}}^+(x) \wedge \hat{F}_{B_{NV}}^+(x))]$$

Definition 2.12 *[6] The intersection of two NVSs A_{NV} and B_{NV} is a NVSs C_{NV}, written as $C_{NV} = A_{NV} \cap B_{NV}$, whose truth membership function, indeterminacy-membership function and false-membership function are related to those of A_{NV} and B_{NV} by*

$$\hat{T}_{C_{NV}}(x) = [(\hat{T}_{A_{NV}}^-(x) \wedge \hat{T}_{B_{NV}}^-(x)), (\hat{T}_{A_{NV}}^+(x) \wedge \hat{T}_{B_{NV}}^+(x))]$$
$$\hat{I}_{C_{NV}}(x) = [(\hat{I}_{A_{NV}}^-(x) \vee \hat{I}_{B_{NV}}^-(x)), (\hat{I}_{A_{NV}}^+(x) \vee \hat{I}_{B_{NV}}^+(x))]$$
$$\hat{F}_{C_{NV}}(x) = [(\hat{F}_{A_{NV}}^-(x) \vee \hat{F}_{B_{NV}}^-(x)), (\hat{F}_{A_{NV}}^+(x) \vee \hat{F}_{B_{NV}}^+(x))]$$

Definition 2.13 *[39] Let $G^* = (V, E)$ be a graph. A pair $G = (J, K)$ is called a neutrosophic vague graph (NVG) on G^* or a neutrosophic graph where $J = (\hat{T}_J, \hat{I}_J, \hat{F}_J)$ is a neutrosophic vague set on V and $K = (\hat{T}_K, \hat{I}_K, \hat{F}_K)$ is a neutrosophic vague set $E \subseteq V \times V$ where*

(1)$V = \{v_1, v_2, \ldots, v_n\}$ such that $T_J^-: V \to [0,1], I_J^-: V \to [0,1], F_J^-: V \to [0,1]$ which satisfies the condition $F_J^- = [1 - T_J^+]$

$T_J^+: V \to [0,1], I_J^+: V \to [0,1], F_J^+: V \to [0,1]$ which satisfies the condition $F_J^+ = [1 - T_1^-]$

denotes the degree of truth membership function, indeterminacy membership and falsity membership of the element $v_i \in V$, and

$$0 \leq T_J^-(v_i) + I_J^-(v_i) + F_J^-(v_i) \leq 2.$$
$$0 \leq T_J^+(v_i) + I_J^+(v_i) + F_J^+(v_i) \leq 2.$$

(2) $E \subseteq V \times V$ where

$$T_K^-: V \times V \to [0,1], I_K^-: V \times V \to [0,1], F_K^-: V \times V \to [0,1]$$
$$T_K^+: V \times V \to [0,1], I_K^+: V \times V \to [0,1], F_K^+: V \times V \to [0,1]$$

denotes the degree of truth membership function, indeterminacy membership and falsity membership of the element $v_i, v_j \in E$ respectively and such that

$$0 \leq T_K^-(v_i) + I_K^-(v_i) + F_K^-(v_i) \leq 2.$$
$$0 \leq T_K^+(v_i) + I_K^+(v_i) + F_K^+(v_i) \leq 2.$$

such that

$$T_K^-(xy) \leq \{T_J^-(x) \wedge T_J^-(y)\}$$
$$I_K^-(xy) \leq \{I_J^-(x) \wedge I_J^-(y)\}$$
$$F_K^-(xy) \leq \{F_J^-(x) \vee F_J^-(y)\},$$

similarly

$$T_K^+(xy) \leq \{T_J^+(x) \wedge T_J^+(y)\}$$
$$I_K^+(xy) \leq \{I_J^+(x) \wedge I_J^+(y)\}$$
$$F_K^+(xy) \leq \{F_J^+(x) \vee F_J^+(y)\}.$$

Example 2.14 *Consider a neutrosophic vague graph* $G = (J, K)$ *such that* $J = \{a, b, c\}$ *and* $K = \{ab, bc, ca\}$ *defined by*

$$\hat{a} = T[0.5,0.6], I[0.4,0.3], F[0.4,0.5], \hat{b} = T[0.4,0.6], I[0.7,0.3], F[0.4,0.6],$$
$$\hat{c} = T[0.4,0.4], I[0.5,0.3], F[0.6,0.6]$$
$$a^- = (0.5,0.4,0.4), b^- = (0.4,0.7,0.4), c^- = (0.4,0.5,0.6)$$
$$a^+ = (0.6,0.3,0.5), b^+ = (0.6,0.3,0.6), c^+ = (0.4,0.3,0.6)$$

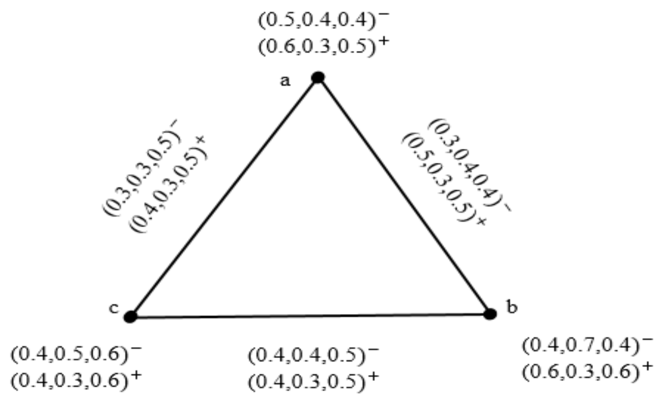


Figure 1neutrosophic vague graph

3. Neutrosophic Bipolar Vague Set

In this section, the definition of NBVS, complement of NBVS, operations like union, intersection are elaborated with an example.

Definition 3.1 *In a universe of discourse* X, *the neutrosophic bipolar vague set (NBVS), denoted as* A_{NBVS} *represented as,*

$$A_{NBV} = \{\langle x, \hat{T}_{A_{NBV}}^P(x), \hat{I}_{A_{NBV}}^P(x), \hat{F}_{A_{NBV}}^P(x), \hat{T}_{A_{NBV}}^N(x), \hat{I}_{A_{NBV}}^N(x), \hat{F}_{A_{NBV}}^N(x)\rangle, x \in X\}$$

whose truth-membership, indeterminacy membership and falssity-membership function is expanded as

$$\hat{T}^P_{A_{NBV}}(x) = [(T^-)^P(x), (T^+)^P(x)], \hat{I}^P_{A_{NBV}}(x) = [(I^-)^P(x), (I^+)^P(x)], \hat{F}^P_{A_{NBV}}(x) = [(F^-)^P(x), (F^+)^P(x)],$$

where $(T^+)^P(x) = 1 - (F^-)^P(x), (F^+)^P(x) = 1 - (T^-)^P(x)$, and provided that,

$$0 \leq (T^-)^P(x) + (I^-)^P(x) + (F^-)^P(x) \leq 2.$$

Also

$$\hat{T}^N_{A_{NBV}}(x) = [(T^-)^N(x), (T^+)^N(x)], \hat{I}^N_{A_{NBV}}(x) = [(I^-)^N(x), (I^+)^N(x)], \hat{F}^N_{A_{NBV}}(x) = [(F^-)^N(x), (F^+)^N(x)],$$

where $(T^+)^N(x) = -1 - (F^-)^N(x), (F^+)^N(x) = -1 - (T^-)^N(x)$,

and provided that,

$$0 \geq (T^-)^N(x) + (I^-)^N(x) + (F^-)^N(x) \geq -2.$$

Example 3.2 *Let* $U = \{x_1, x_2, x_3\}$ *be a set of universe we define the NBV set* A_{NBV} *as follows*

$$A_{NBV} = \{\frac{x_1}{[0.3,0.6]^P, [0.5,0.5]^P, [0.4,0.7]^P, [-0.3,-0.5]^N, [-0.4,-0.4]^N, [-0.5,-0.7]^N},$$

$$\frac{x_2}{[0.4,0.6]^P, [0.4,0.6]^P, [0.4,0.6]^P, [-0.4,-0.4]^N, [-0.5,-0.5]^N, [-0.6,-0.6]^N},$$

$$\frac{x_3}{[0.3,0.7]^P, [0.6,0.4]^P, [0.3,0.7]^P, [-0.4,-0.6]^N, [-0.5,-0.6]^N, [-0.4,-0.6]^N}\}$$

Definition 3.3 *IN NBVS, the complement of* A^c_{NBV} *be expanded as,*

$$(\hat{T}^c_{A_{NBV}}(x))^P = \{(1 - T^+(x))^P, (1 - T^-(x))^P\}, (\hat{T}^c_{A_{NBV}}(x))^N = \{(-1 - T^+(x))^N, (-1 - T^-(x))^N\}$$

$$(\hat{I}^c_{A_{NBV}}(x))^P = \{(1 - I^+(x))^P, (1 - I^-(x))^P\}, (\hat{I}^c_{A_{NBV}}(x))^N = \{(-1 - I^+(x))^N, (-1 - I^-(x))^N\}$$

$$(\hat{F}^c_{A_{NBV}}(x))^P = \{(1 - F^+(x))^P, (1 - F^-(x))^P\}, (\hat{F}^c_{A_{NBV}}(x))^N = \{(-1 - F^+(x))^N, (-1 - F^-(x))^N\}$$

Example 3.4 *Considering above example we have*

$$A_{NBV} = \{\frac{x_1}{[0.7,0.4]^P, [0.5,0.5]^P, [0.6,0.3]^P, [-0.7,-0.5]^N, [-0.6,-0.6]^N, [-0.5,-0.3]^N},$$

$$\frac{x_2}{[0.6,0.4]^P, [0.6,0.4]^P, [0.6,0.4]^P, [-0.6,-0.6]^N, [-0.5,-0.5]^N, [-0.4,-0.4]^N},$$

$$\frac{x_3}{[0.7,0.3]^P, [0.4,0.6]^P, [0.7,0.3]^P, [-0.6,-0.4]^N, [-0.5,-0.4]^N, [-0.6,-0.4]^N}\}$$

Definition 3.5 *Two NBVSs* A_{NBV} *and* B_{NBV} *of the universe* U *are said to be equal, if for all* $u_i \in U$,

$$(\hat{T}_{A_{NBV}})^P(u_i) = (\hat{T}_{B_{NBV}})^P(u_i), (\hat{I}_{A_{NBV}})^P(u_i) = (\hat{I}_{B_{NBV}})^P(u_i), (\hat{F}_{A_{NBV}})^P(u_i) = (\hat{F}_{B_{NBV}})^P(u_i)$$

and

$$(\hat{T}_{A_{NBV}})^N(u_i) = (\hat{T}_{B_{NBV}})^N(u_i), (\hat{I}_{A_{NBV}})^N(u_i) = (\hat{I}_{B_{NBV}})^N(u_i), (\hat{F}_{A_{NBV}})^N(u_i) = (\hat{F}_{B_{NBV}})^N(u_i).$$

Definition 3.6 *In the Universe* U, *two NBVSs,* A_{NBV}, B_{NBV} *be given as,*

$$(\hat{T}_{A_{NBV}})^P(u_i) \leq (\hat{T}_{B_{NBV}})^P(u_i), (\hat{I}_{A_{NBV}})^P(u_i) \geq (\hat{I}_{B_{NBV}})^P(u_i), (\hat{F}_{A_{NBV}})^P(u_i) \geq (\hat{F}_{B_{NBV}})^P(u_i)$$

and

$$(\hat{T}_{A_{NBV}})^N(u_i) \geq (\hat{T}_{B_{NBV}})^N(u_i), (\hat{I}_{A_{NBV}})^N(u_i) \leq (\hat{I}_{B_{NBV}})^N(u_i), (\hat{F}_{A_{NBV}})^N(u_i) \leq (\hat{F}_{B_{NBV}})^N(u_i)$$

then the NBVS $(A_{NBV})^P$ are included by $(B_{NBV})^P$, denoted by $(A_{NBV})^P \subseteq (B_{NBV})^P$ where $1 \leq i \leq n$ and $(A_{NBV})^N$ are included by $(B_{NBV})^N$, denoted by $(A_{NBV})^N \subseteq (B_{NBV})^N$ where $1 \leq i \leq n$.

Definition 3.7 *The union of two NVSs A_{NBV} and B_{NBV} is a NBVSs, C_{NBV}, written as $C_{NBV} = A_{NBV} \cup B_{NBV}$, whose truth membership function, indeterminacy-membership function and false-membership function are related to those of A_{NBV} and B_{NBV} by*

$$(\hat{T}_{C_{NBV}})^P(x) = [((T^-_{A_{NBV}})^P(x) \vee (T^-_{B_{NBV}})^P(x)), ((T^+_{A_{NBV}})^P(x) \vee (T^+_{B_{NBV}})^P(x))]$$
$$(\hat{I}_{C_{NBV}})^P(x) = [((I^-_{A_{NBV}})^P(x) \wedge (I^-_{B_{NBV}})^P(x)), ((I^+_{A_{NBV}})^P(x) \wedge (I^+_{B_{NBV}})^P(x))]$$
$$(\hat{F}_{C_{NBV}})^P(x) = [((F^-_{A_{NBV}})^P(x) \wedge (F^-_{B_{NBV}})^P(x)), ((F^+_{A_{NBV}})^P(x) \wedge (F^+_{B_{NBV}})^P(x))], \text{and}$$
$$(\hat{T}_{C_{NBV}})^N(x) = [((T^-_{A_{NBV}})^N(x) \wedge (T^-_{B_{NBV}})^N(x)), ((T^+_{A_{NBV}})^N(x) \wedge (T^+_{B_{NBV}})^N(x))]$$
$$(\hat{I}_{C_{NBV}})^N(x) = [((I^-_{A_{NBV}})^N(x) \vee (I^-_{B_{NBV}})^N(x)), ((I^+_{A_{NBV}})^N(x) \vee (I^+_{B_{NBV}})^N(x))]$$
$$(\hat{F}_{C_{NBV}})^N(x) = [((F^-_{A_{NBV}})^N(x) \vee (F^-_{B_{NBV}})^N(x)), ((F^+_{A_{NBV}})^N(x) \vee (F^+_{B_{NBV}})^N(x))]$$

Definition 3.8 *The intersection of two NVSs A_{NBV} and B_{NBV} is a NBVSs C_{NBV}, written as $C_{NBV} = A_{NBV} \cap B_{NBV}$, whose truth membership function, indeterminacy-membership function and false-membership function are related to those of A_{NBV} and B_{NBV} by*

$$(\hat{T}_{C_{NBV}})^P(x) = [((T^-_{A_{NBV}})^P(x) \wedge (T^-_{B_{NBV}})^P(x)), ((T^+_{A_{NBV}})^P(x) \wedge (T^+_{B_{NBV}})^P(x))]$$
$$(\hat{I}_{C_{NBV}})^P(x) = [((I^-_{A_{NBV}})^P(x) \vee (I^-_{B_{NBV}})^P(x)), ((I^+_{A_{NBV}})^P(x) \vee (I^+_{B_{NBV}})^P(x))]$$
$$(\hat{F}_{C_{NBV}})^P(x) = [((F^-_{A_{NBV}})^P(x) \vee (F^-_{B_{NBV}})^P(x)), ((F^+_{A_{NBV}})^P(x) \vee (F^+_{B_{NBV}})^P(x))], \text{and}$$
$$(\hat{T}_{C_{NBV}})^N(x) = [((T^-_{A_{NBV}})^N(x) \vee (T^-_{B_{NBV}})^N(x)), ((T^+_{A_{NBV}})^N(x) \vee (T^+_{B_{NBV}})^N(x))]$$
$$(\hat{I}_{C_{NBV}})^N(x) = [((I^-_{A_{NBV}})^N(x) \wedge (I^-_{B_{NBV}})^N(x)), ((I^+_{A_{NBV}})^N(x) \wedge (I^+_{B_{NBV}})^N(x))]$$
$$(\hat{F}_{C_{NBV}})^N(x) = [((F^-_{A_{NBV}})^N(x) \wedge (F^-_{B_{NBV}})^N(x)), ((F^+_{A_{NBV}})^N(x) \wedge (F^+_{B_{NBV}})^N(x))]$$

Definition 3.9 *Let U be a set of universe and let A_{NBV} and B_{NBV} be NBVSs, then the union $A_{NBV} \cap B_{NBV}$ is defined as follows:*

$$A_{NBV} = \{\frac{x_1}{[0.3,0.6]^P, [0.6,0.6]^P, [0.4,0.7]^P, [-0.4,-0.7]^N, [-0.6,-0.6]^N, [-0.3,-0.6]^N},$$
$$\frac{x_2}{[0.4,0.6]^P, [0.6,0.4]^P, [0.4,0.6]^P, [-0.5,-0.5]^N, [-0.7,-0.3]^N, [-0.5,-0.5]^N},$$
$$\frac{x_3}{[0.7,0.8]^P, [0.6,0.6]^P, [0.2,0.3]^P, [-0.5,-0.4]^N, [-0.5,-0.5]^N, [-0.6,-0.5]^N}\}$$

$$B_{NBV} = \{\frac{x_1}{[0.2,0.8]^P, [0.5,0.4]^P, [0.2,0.8]^P, [-0.5,-0.7]^N, [-0.7,-0.7]^N, [-0.3,-0.5]^N},$$
$$\frac{x_2}{[0.3,0.8]^P, [0.6,0.5]^P, [0.2,0.7]^P, [-0.5,-0.6]^N, [-0.4,-0.3]^N, [-0.4,-0.5]^N},$$
$$\frac{x_3}{[0.2,0.5]^P, [0.5,0.2]^P, [0.5,0.8]^P, [-0.5,-0.5]^N, [-0.4,-0.3]^N, [-0.5,-0.5]^N}\}$$

$$A_{NBV} \cap B_{NBV} = H_{NBV}$$
$$= \{\frac{x_1}{[0.2,0.6]^P, [0.6,0.6]^P, [0.4,0.8]^P, [-0.4,-0.7]^N, [-0.7,-0.7]^N, [-0.3,-0.6]^N},$$
$$\frac{x_2}{[0.3,0.6]^P, [0.6,0.5]^P, [0.4,0.7]^P, [-0.5,-0.5]^N, [-0.7,-0.3]^N, [-0.5,-0.5]^N},$$

$$\frac{x_3}{[0.2,0.5]^P,[0.6,0.6]^P,[0.5,0.8]^P,[-0.5,-0.4]^N,[-0.5,-0.5]^N,[-0.6,-0.5]^N}\}$$

4 Neutrosophic Bipolar Vague graphs

In this section, neutrosophic bipolar vague graphs are defined. The concepts of neutrosophic bipolar vague subgraph, adjacency, path, connectedness and degree of neutrosophic bipolar vague graph are discussed.

Definition 4.1 *In a crisp graph* $G^* = (V,E)$*. A pair* $G = (J,K)$ *is called a neutrosophic bipolar vague graph (NBVG) on* G^* *or a neutrosophic bipolar vague graph where* J *is a neutrosophic bipolar vague set and* K *is a neutrosophic bipolar vague relation in* G^* *such that* $J^P = ((\hat{T}_J)^P,(\hat{I}_J)^P,(\hat{F}_J)^P), J^N = ((\hat{T}_J)^N,(\hat{I}_J)^N,(\hat{F}_J)^N)$ *is a neutrosophic bipolar vague set on* V *and* $K^P = ((\hat{T}_K)^P,(\hat{I}_K)^P,(\hat{F}_K)^P), K^N = ((\hat{T}_K)^N,(\hat{I}_K)^N,(\hat{F}_K^N))$ *is a neutrosophic Bipolar vague set* $E \subseteq V \times V$ *where*

(1) $V = \{v_1, v_2, \dots, v_n\}$ such that

$$(T_J^-)^P: V \to [0,1], (I_J^-)^P: V \to [0,1], (F_J^-)^P: V \to [0,1]$$

which satisfies the condition $(F_J^-)^P = [1 - (T_J^+)^P]$

$$(T_J^+)^P: V \to [0,1], (I_J^+)^P: V \to [0,1], (F_J^+)^P: V \to [0,1]$$

which satisfies the condition $(F_J^+)^P = [1 - (T_J^-)^P]$, and

$$(T_J^-)^N: V \to [-1,0], (I_J^-)^N: V \to [-1,0], (F_J^-)^N: V \to [-1,0]$$

which satisfies the condition $(F_J^-)^N = [-1 - (T_J^+)^N]$

$(T_J^+)^N: V \to [-1,0], (I_J^+)^N: V \to [-1,0], (F_J^+)^N: V \to [-1,0]$ which satisfies the condition $(F_J^+)^N = [-1 - (T_J^-)^N]$ denotes the degree of truth membership function, indeterminacy membership and falsity membership of the element $v_i \in V$, and

$$0 \le (T_J^-)^P(v_i) + (I_J^-)^P(v_i) + (F_J^-)^P(v_i) \le 2$$
$$0 \le (T_J^+)^P(v_i) + (I_J^+)^P(v_i) + (F_J^+)^P(v_i) \le 2$$
$$0 \ge (T_J^-)^N(v_i) + (I_J^-)^N(v_i) + (F_J^-)^N(v_i) \ge -2$$
$$0 \le (T_J^+)^N(v_i) + (I_J^+)^N(v_i) + (F_J^+)^N(v_i) \ge -2.$$

(2) $E \subseteq V \times V$ where

$$(T_K^-)^P: V \times V \to [0,1], (I_K^-)^P: V \times V \to [0,1], (F_K^-)^P: V \times V \to [0,1]$$
$$(T_K^+)^P: V \times V \to [0,1], (I_K^+)^P: V \times V \to [0,1], (F_K^+)P: V \times V \to [0,1]\text{and}$$
$$(T_K^-)^N: V \times V \to [-1,0], (I_K^-)^N: V \times V \to [-1,0], (F_K^-)^N: V \times V \to [-1,0]$$
$$(T_K^+)^N: V \times V \to [-1,0], (I_K^+)^N: V \times V \to [-1,0], (F_K^+)^N: V \times V \to [-1,0]$$

denotes the degree of truth membership function, indeterminacy membership and falsity membership of the element $v_i, v_j \in E$ respectively and such that

$$0 \le (T_K^-)^P(v_i,v_j) + (I_K^-)^P(v_i,v_j) + (F_K^-)^P(v_i,v_j) \le 2$$
$$0 \le (T_K^+)^P(v_i,v_j) + (I_K^+)^P(v_i,v_j) + (F_K^+)^P(v_i,v_j) \le 2$$
$$0 \ge (T_K^-)^N(v_i,v_j) + (I_K^-)^N(v_i,v_j) + (F_K^-)^N(v_i,v_j) \ge -2$$
$$0 \ge (T_K^+)^N(v_i,v_j) + (I_K^+)^N(v_i,v_j) + (F_K^+)^N(v_i,v_j) \ge -2,$$

such that

$$(T_K^-)^P(xy) \le \{(T_J^-)^P(x) \wedge (T_J^-)^P(y)\}$$
$$(I_K^-)^P(xy) \le \{(I_J^-)^P(x) \wedge (I_J^-)^P(y)\}$$

$(F_K^-)^P(xy) \leq \{(F_J^-)^P(x) \vee (F_J^-)^P(y)\}$

$(T_K^+)^P(xy) \leq \{(T_J^+)^P(x) \wedge (T_J^+)^P(y)\}$

$(I_K^+)^P(xy) \leq \{(I_J^+)^P(x) \wedge (I_J^+)^P(y)\}$

$(F_K^+)^P(xy) \leq \{(F_J^+)^P(x) \vee (F_J^+)^P(y)\}$,

and

$(T_K^-)^N(xy) \geq \{(T_J^-)^N(x) \vee (T_J^-)^N(y)\}$

$(I_K^-)^N(xy) \geq \{(I_J^-)^N(x) \vee (I_K^-)^N(y)\}$

$(F_K^-)^N(xy) \geq \{(F_J^-)^N(x) \wedge (F_J^-)^N(y)\}$,

$(T_K^+)^N(xy) \geq \{(T_J^+)^N(x) \vee (T_J^+)^N(y)\}$

$(I_K^+)^N(xy) \geq \{(I_J^+)^N(x) \vee (I_J^+)^N(y)\}$

$(F_K^+)^N(xy) \geq \{(F_J^+)^N(x) \wedge (F_J^+)^N(y)\}$.

Example 4.2 *Consider a neutrosophic bipolar vague graph* $G = (J, K)$ *such that* $J = \{a, b, c\}$ *and* $K = \{ab, bc, ca\}$ *defined by*

$(\hat{a})^P = T[0.5,0.6], I[0.4,0.3], F[0.4,0.5]$,

$(\hat{b})^P = T[0.4,0.6], I[0.7,0.3], F[0.4,0.6]$,

$(\hat{c})^P = T[0.4,0.4], I[0.5,0.3], F[0.6,0.6]$

$(a^-)^P = (0.5,0.4,0.4), (b^-)^P = (0.4,0.7,0.4), (c^-)^P = (0.4,0.5,0.6)$

$(a^+)^P = (0.6,0.3,0.5), (b^+)^P = (0.6,0.3,0.6), (c^+)^P = (0.4,0.3,0.6)$

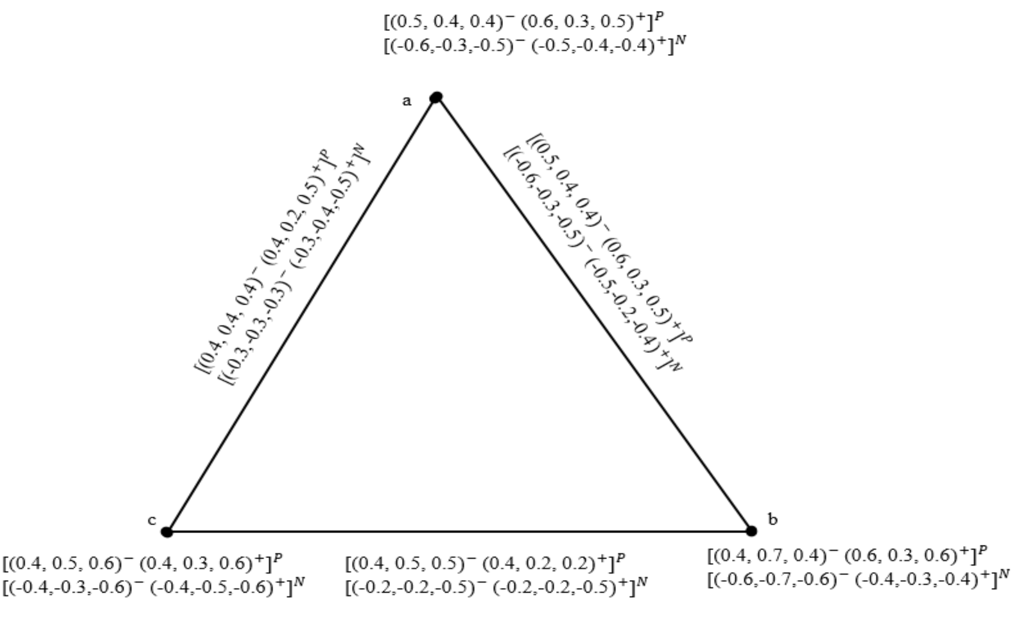
$(\hat{a})^N = T[-0.6, -0.5], I[-0.3, -0.4], F[-0.5, -0.4]$,

$(\hat{b})^N = T[-0.6, -0.4], I[-0.7, -0.3], F[-0.6, -0.4]$,

$(\hat{c})^N = T[-0.4, -0.4], I[-0.3, -0.5], F[-0.6, -0.6]$

$(a^-)^N = (-0.6, -0.3, -0.5), (b^-)^N = (-0.6, -0.7, -0.6), (c^-)^N = (-0.4, -0.3, -0.6)$

$(a^+)^N = (-0.5, -0.4, -0.4), (b^+)^P = (-0.4, -0.3, -0.4), (c^+)^P = (-0.4, -0.5, -0.6)$



$Figure$ 2 NEUTROSOPHIC BIPOLAR VAGUE GRAPH

Definition 4.3 *A neutrosophic bipolar vague graph* $H = (J'(x), K'(x))$ *is said to be a neutrosophic bipolar vague subgraph of the NVG* $G = (J, K)$ *if* $J'(x) \subseteq J(x)$ *and* $K'(xy) \subseteq K'(xy)$, *in other words, if*

$$(\hat{T}_J')^P(x) \leq (\hat{T}_J)^P(x)$$
$$(\hat{I}_J')^P(x) \leq (\hat{I}_J)^P(x)$$
$$(\hat{F}_J')^P(x) \leq (\hat{F}_J)^P(x)\ \forall x \in V$$
$$(\hat{T}_K')^P(xy) \leq (\hat{T}_K)^P(xy)$$
$$(\hat{I}_K')^P(xy) \leq (\hat{I}_K)^P(xy)$$
$$(\hat{F}_K')^P(xy) \leq (\hat{F}_K)^P(xy), \forall xy \in E.$$

Also,

$$(\hat{T}_J')^N(x) \geq (\hat{T}_J)^N(x)$$
$$(\hat{I}_J')^N(x) \geq (\hat{I}_J)^N(x)$$
$$(\hat{F}_J')^N(x) \geq (\hat{F}_J)^N(x), \forall x \in V$$

and

$$(\hat{T}_K')^N(xy) \geq (\hat{T}_K)^N(xy)$$
$$(\hat{I}_K')^N(xy) \geq (\hat{I}_K)^N(xy)$$
$$(\hat{F}_K')^N(xy) \geq (\hat{F}_K)^N(xy), \forall xy \in E.$$

Definition 4.4 *The two vertices are said to be adjacent in a neutrosophic bipolar vague graph* $G = (J, K)$ *if*

$$(T_K^-)^P(xy) = \{(T_J^-)^P(x) \wedge (T_J^-)^P(y)\}$$
$$(I_K^-)^P(xy) = \{(I_J^-)^P(x) \wedge (I_J^-)^P(y)\}$$
$$(F_K^-)^P(xy) = \{(F_J^-)^P(x) \vee (F_J^-)^P(y)\},$$
$$(T_K^+)^P(xy) = \{(T_J^+)^P(x) \wedge (T_J^+)^P(y)\}$$
$$(I_K^+)^P(xy) = \{(I_J^+)^P(x) \wedge (I_J^+)^P(y)\}$$
$$(F_K^+)^P(xy) = \{(F_J^+)^P(x) \vee (F_J^+)^P(y)\},$$
$$(T_K^-)^N(xy) = \{(T_J^-)^N(x) \vee (T_J^-)^N(y)\}$$
$$(I_K^-)^N(xy) = \{(I_J^-)^N(x) \vee (I_K^-)^N(y)\}$$
$$(F_K^-)^N(xy) = \{(F_J^-)^N(x) \wedge (F_J^-)^N(y)\},$$
$$(T_K^+)^N(xy) = \{(T_J^+)^N(x) \vee (T_J^+)^N(y)\}$$
$$(I_K^+)^N(xy) = \{(I_J^+)^N(x) \vee (I_J^+)^N(y)\}$$
$$(F_K^+)^N(xy) = \{(F_J^+)^N(x) \wedge (F_J^+)^N(y)\},$$

Here, x is the neighbour of y and vice versa, also (xy) is incident at x and y.

Definition 4.5 *In a neutrosophic bipolar vague graph* $G = (J, K)$, *a path* ρ *is meant to be a sequence of different points* $x_0, x_1, \ldots, x_n$ *such an extent that*

$$(T_K^-)^P(x_{i-1}, x_1) > 0, (I_K^-)^P(x_{i-1}, x_1) > 0, (F_K^-)^P(x_{i-1}, x_1) > 0,$$
$$(T_K^+)^P(x_{i-1}, x_1) > 0, (I_K^+)^P(x_{i-1}, x_1) > 0, (F_K^+)^P(x_{i-1}, x_1) > 0,$$

and

$$(T_K^-)^N(x_{i-1}, x_1) < 0, (I_K^-)^N(x_{i-1}, x_1) < 0, (F_K^-)^N(x_{i-1}, x_1) < 0,$$
$$(T_K^+)^N(x_{i-1}, x_1) < 0, (I_K^+)^N(x_{i-1}, x_1) < 0, (F_K^+)^N(x_{i-1}, x_1) < 0,$$

for every i lies between 0 and 1. $n \leq 1$ is known as the path length.. A single vertex x_i can represent as a path.

Definition 4.6 *A neutrosophic bipolar vague graph* $G = (J, K)$, *if every pair of vertices has at least one neutrosophic bipolar vague path between them is known as connected, otherwise it is disconnected.*

Definition 4.7 *A vertex* $x_i \in V$ *of neutrosophic bipolar vague graph* $G = (J, K)$ *is said to be isolatedvertex if there is no effective edge incident at* x_i.

Definition 4.8 *A vertex in a neutrosophic bipolar vague graph* $G = (J, K)$ *having exactly one neighbours is called a pendent vertex. Otherwise, it is called non-pendent vertex. An edge in a neutrosophic bipolar vague graph incident with a pendent vertex is called a pendent edge other words it is called non-pendent edge. A vertex in a neutrosophic bipolar vague graph adjacent to the pendent vertex is called an support of the pendent edge.*

Definition 4.9 *A neutrosophic bipolar vague graph* $G = (J, K)$ *that has neither self loops nor parallel edge is called simple neutrosophic bipolar vague graph.*

Definition 4.10 *Let* $G = (J, K)$ *be a neutrosophic bipolar vague graph. Then the degree of a vertex* $x \in G$ *is a sum of degree truth membership, sum of indeterminacy membership and sum of falsity membership of all those edges which are incident on vertex* x *denoted by*

$$(d(x))^P = ([(d_{T_J}^-)P(x), (d_{T_J}^+)^P(x)], [(d_{I_J}^-)^P(x), (d_{I_J}^+)^P(x)], [(d_{F_J}^-)^P(x), (d_{F_J}^+)^P(x)])$$

$$(d(x))^N = ([(d_{T_J}^-)N(x), (d_{T_J}^+)^N(x)], [(d_{I_J}^-)^N(x), (d_{I_J}^+)^N(x)], [(d_{F_J}^-)^N(x), (d_{F_J}^+)^N(x)])$$

where $(d_{T_J}^-)^P(x) = \sum_{x \neq y} (T_K^-)^P(xy)$, $(d_{T_J}^+)^P(x) = \sum_{x \neq y} (T_K^+)^P(xy)$ denotes the positive degree of truth membership vertex, $(d_{I_J}^-)^P(x) = \sum_{x \neq y} (I_K^-)^P(xy)$, $(d_{I_J}^+)^P(x) = \sum_{x \neq y} (I_K^+)^P(xy)$ denotes the positive degree of indeterminacy membership vertex, $(d_{F_J}^-)^P(x) = \sum_{x \neq y} (F_K^-)^P(xy)$, $(d_{F_J}^+)^P(x) = \sum_{x \neq y} (F_K^+)^P(xy)$ denotes the positive degree of falsity membership vertex for all $x, y \in J$.

Similarly, $(d_{T_J}^-)^N(x) = \sum_{x \neq y} (T_K^-)^N(xy)$, $(d_{T_J}^+)^N(x) = \sum_{x \neq y} (T_K^+)^N(xy)$ denotes the negative degree of truth membership vertex, $(d_{I_J}^-)^N(x) = \sum_{x \neq y} (I_K^-)^N(xy)$, $(d_{I_J}^+)^N(x) = \sum_{x \neq y} (I_K^+)^N(xy)$ denotes the negative degree of indeterminacy membership vertex, $(d_{F_J}^-)^N(x) = \sum_{x \neq y} (F_K^-)^N(xy)$, $(d_{F_J}^+)^N(x) = \sum_{x \neq y} (F_K^+)^N(xy)$ denotes the negative degree of falsity membership vertex for all $x, y \in J$.

Definition 4.11 *A neutrosophic bipolar vague graph* $G = (J, K)$ *is called constant if degree of each vertex is* $A = (A_1, A_2, A_3)$ *that is* $d(x) = (A_1, A_2, A_3)$ *for all* $x \in V$.

5 Strong Neutrosophic Bipolar Vague Graphs

In this section, we presented some remarkable properties of strong neutrosophic bipolar vague graphs and a remark is provided by comparing other types of bipolar graphs. Finally conclusion is given.

Definition 5.1 *A neutrosophic bipolar vague graph* $G = (J, K)$ *of* $G^* = (V, E)$ *is called strong neutrosophic bipolar vague graph if*

$$(T_K^-)^P(xy) = \{(T_J^-)^P(x) \wedge (T_J^-)^P(y)\}$$

$$(I_K^-)^P(xy) = \{(I_J^-)^P(x) \wedge (I_J^-)^P(y)\}$$

$$(F_K^-)^P(xy) = \{(F_J^-)^P(x) \vee (F_J^-)^P(y)\},$$

$$(T_K^+)^P(xy) = \{(T_J^+)^P(x) \wedge (T_J^+)^P(y)\}$$

$$(I_K^+)^P(xy) = \{(I_J^+)^P(x) \wedge (I_J^+)^P(y)\}$$

$$(F_K^+)^P(xy) = \{(F_J^+)^P(x) \vee (F_J^+)^P(y)\},$$

$$(T_K^-)^N(xy) = \{(T_J^-)^N(x) \vee (T_J^-)^N(y)\}$$
$$(I_K^-)^N(xy) = \{(I_J^-)^N(x) \vee (I_K^-)^N(y)\}$$
$$(F_K^-)^N(xy) = \{(F_J^-)^N(x) \wedge (F_J^-)^N(y)\},$$
$$(T_K^+)^N(xy) = \{(T_J^+)^N(x) \vee (T_J^+)^N(y)\}$$
$$(I_K^+)^N(xy) = \{(I_J^+)^N(x) \vee (I_J^+)^N(y)\}$$
$$(F_K^+)^N(xy) = \{(F_J^+)^N(x) \wedge (F_J^+)^N(y)\}, \forall((xy) \in K)$$

Definition 5.2 *The complement of neutrosophic bipolar vague graph $G = (J, K)$ on G^* is a neutrosophic bipolar vague graph G^c where*

- $(J^c)^P(x) = (J)^P(x)$
- $(T_J^{-c})^P(x) = (T_J^-)^P(x), (I_J^{-c})^P(x) = (I_J^-)^P(x), (F_J^{-c})^P(x) = (F_J^-)^P(x)$ for all $x \in V$.
- $(T_J^{+c})^P(x) = (T_J^+)^P(x), (I_J^{+c})^P(x) = (I_J^+)^P(x), (F_J^{+c})^P(x) = (F_J^+)^P(x)$ for all $x \in V$.
- $(T_K^{-c})^P(xy) = \{(T_J^-)^P(x) \wedge (T_J^-)^P(y)\} - (T_K^-)^P(xy)$, $(I_K^{-c})^P(xy) = \{(I_J^-)^P(x) \wedge (I_J^-)^P(y)\} - (I_K^-)^P(xy)$

  $(F_K^{-c})^P(xy) = \{(F_J^-)^P(x) \vee (F_J^-)^P(y)\} - (F_K^-)^P(xy)$ for all $(xy) \in E$
- $(T_K^{+c})^P(xy) = \{(T_J^+)^P(x) \wedge (T_J^+)^P(y)\} - (T_K^+)^P(xy)$, $(I_K^{+c})^P(xy) = \{(I_J^+)^P(x) \wedge (I_J^+)^P(y)\} - (I_K^+)^P(xy)$

  $(F_K^{+c})^P(xy) = \{(F_J^+)^P(x) \vee (F_J^+)^P(y)\} - (F_K^+)^P(xy)$ for all $(xy) \in E$
- $(J^c)^N(x) = (J)^N(x)$
- $(T_J^{-c})^N(x) = (T_J^-)^N(x), (I_J^{-c})^N(x) = (I_J^-)^N(x), (F_J^{-c})^N(x) = (F_J^-)^N(x)$ for all $x \in V$.
- $(T_J^{+c})^N(x) = (T_J^+)^N(x), (I_J^{+c})^N(x) = (I_J^+)^N(x), (F_J^{+c})^N(x) = (F_J^+)^N(x)$ for all $x \in V$.
- $(T_K^{-c})^N(xy) = \{(T_J^-)^N(x) \vee (T_J^-)^N(y)\} - (T_K^-)^N(xy)$

  $(I_K^{-c})^N(xy) = \{(I_J^-)^N(x) \vee (I_J^-)^N(y)\} - (I_K^-)^N(xy)$

  $(F_K^{-c})^N(xy) = \{(F_J^-)^N(x) \wedge (F_J^-)^N(y)\} - (F_K^-)^N(xy)$ for all $(xy) \in E$
- $(T_K^{+c})^N(xy) = \{(T_J^+)^N(x) \vee (T_J^+)^N(y)\} - (T_K^+)^N(xy)$

  $(I_K^{+c})^N(xy) = \{(I_J^+)^N(x) \vee (I_J^+)^N(y)\} - (I_K^+)^N(xy)$

  $(F_K^{+c})^N(xy) = \{(F_J^+)^N(x) \wedge (F_J^+)^N(y)\} - (F_K^+)^N(xy)$ for all $(xy) \in E$

Remark 5.3 *If $G = (J, K)$ is a neutrosophic bipolar vague graph on G^* then from above definition, it follows that G^{c^c} is given by the neutrosophic bipolar vague graph $G^{c^c} = (J^{c^c}, K^{c^c})$ on G^* where*

- $((J^c)^c)^P(x) = (J(x))^P$
- $((T_J^{-c})^c)^P(x) = (T_J^-)^P(x), ((I_J^{-c})^c)^P(x) = (I_J^-)^P(x), ((F_J^{-c})^c)^P(x) = (F_J^-)^P(x)$ for all $x \in V$.
- $((T_J^{+c})^c)^P(x) = (T_J^+)^P(x), ((I_J^{+c})^c)^P(x) = (I_J^+)^P(x), ((F_J^{+c})^c)^P(x) = (F_J^+)^P(x)$ for all $x \in V$.
- $((T_K^{-c})^c)^P(xy) = \{(T_J^-)^P(x) \wedge (T_J^-)^P(y)\} - (T_K^-)^P(xy)$

  $((I_K^{-c})^c)^P(xy) = \{(I_J^-)^P(x) \wedge (I_J^-)^P(y)\} - (I_K^-)^P(xy)$

  $((F_K^{-c})^c)^P(xy) = \{(F_J^-)^P(x) \vee (F_J^-)^P(y)\} - (F_K^-)^P(xy)$ for all $(xy) \in E$
- $((T_K^{+c})^c)^P(xy) = \{(T_J^+)^P(x) \wedge (T_J^+)^P(y)\} - (T_K^+)^P(xy)$

  $((I_K^{+c})^c)^P(xy) = \{(I_J^+)^P(x) \wedge (I_J^+)^P(y)\} - (I_K^+)^P(xy)$

  $((F_K^{+c})^c)^P(xy) = \{(F_J^+)^P(x) \vee (F_J^+)^P(y)\} - (F_K^+)^P(xy)$ for all $(xy) \in E$
- $((J^c)^c)^N(x) = (J(x))^N$

- $((T_J^{-c})^c)^N(x) = (T_J^-)^N(x), ((I_J^{-c})^c)^N(x) = (I_J^-)^N(x), ((F_J^{-c})^c)^N(x) = (F_J^-)^N(x)$ for all $x \in V$.
- $((T_J^{+c})^c)^N(x) = (T_J^+)^N(x), ((I_J^{+c})^c)^N(x) = (I_J^+)^N(x), ((F_J^{+c})^c)^N(x) = (F_J^+)^N(x)$ for all $x \in V$.
- $((T_K^{-c})^c)^N(xy) = \{(T_J^-)^N(x) \vee (T_J^-)^N(y)\} - (T_K^-)^N(xy)$

$$((I_K^{-c})^c)^N(xy) = \{(I_J^-)^N(x) \vee (I_J^-)^N(y)\} - (I_K^-)^N(xy)$$

$$((F_K^{-c})^c)^N(xy) = \{(F_J^-)^N(x) \wedge (F_J^-)^N(y)\} - (F_K^-)^N(xy) \text{ for all } (xy) \in E$$

- $((T_K^{+c})^c)^N(xy) = \{(T_J^+)^N(x) \vee (T_J^+)^N(y)\} - (T_K^+)^N(xy)$

$$((I_K^{+c})^N)^N(xy) = \{(I_J^+)^N(x) \vee (I_J^+)^N(y)\} - (I_K^+)^N(xy)$$

$$((F_K^{+c})^c)^N(xy) = \{(F_J^+)^N(x) \wedge (F_J^+)^N(y)\} - (F_K^+)^N(xy) \text{ for all } (xy) \in E.$$

for any neutrosophic bipolar vague graph G, G^c is strong neutrosophic bipolar vague graph and $G \subseteq G^c$.

Definition 5.4 *Suppose G^c is the complement of neutrosophic bipolar vague graph G. In a strong neutrosophic bipolar vague graph G, $G \cong G^c$ then it is called self-complementary.*

Proposition 5.5 *Let $G = (J, K)$ be a strong neutrosophic bipolar vague graph if*

$$(T_K^-)^P(xy) = \{(T_J^-)^P(x) \wedge (T_J^-)^P(y)\}$$
$$(I_K^-)^P(xy) = \{(I_J^-)^P(x) \wedge (I_J^-)^P(y)\}$$
$$(F_K^-)^P(xy) = \{(F_J^-)^P(x) \vee (F_J^-)^P(y)\},$$
$$(T_K^+)^P(xy) = \{(T_J^+)^P(x) \wedge (T_J^+)^P(y)\}$$
$$(I_K^+)^P(xy) = \{(I_J^+)^P(x) \wedge (I_J^+)^P(y)\}$$
$$(F_K^+)^P(xy) = \{(F_J^+)^P(x) \vee (F_J^+)^P(y)\},$$
$$(T_K^-)^N(xy) = \{(T_J^-)^N(x) \vee (T_J^-)^N(y)\}$$
$$(I_K^-)^N(xy) = \{(I_J^-)^N(x) \vee (I_K^-)^N(y)\}$$
$$(F_K^-)^N(xy) = \{(F_J^-)^N(x) \wedge (F_J^-)^N(y)\},$$
$$(T_K^+)^N(xy) = \{(T_J^+)^N(x) \vee (T_J^+)^N(y)\}$$
$$(I_K^+)^N(xy) = \{(I_J^+)^N(x) \vee (I_J^+)^N(y)\}$$
$$(F_K^+)^N(xy) = \{(F_J^+)^N(x) \wedge (F_J^+)^N(y)\}, \forall((xy) \in K)$$

Then G is self complementary.

Proof. Let $G = (J, K)$ be a strong neutrosophic bipolar vague graph such that

$$(\hat{T}_K)^P(xy) = \frac{1}{2}[(\hat{T}_J)^P(x) \wedge (\hat{T}_J)^P(y)]$$

$$(\hat{I}_K)^P(xy) = \frac{1}{2}[(\hat{I}_J)^P(x) \wedge (\hat{I}_J)^P(y)]$$

$$(\hat{F}_K)^P(xy) = \frac{1}{2}[(\hat{F}_J)^P(x) \vee (\hat{F}_J)^P(y)],$$

and

$$(\hat{T}_K)^N(xy) = \frac{1}{2}[(\hat{T}_J)^N(x) \vee (\hat{T}_J)^N(y)]$$

$$(\hat{I}_K)^N(xy) = \frac{1}{2}[(\hat{I}_J)^N(x) \vee (\hat{I}_J)^N(y)]$$

$$(\hat{F}_K)^N(xy) = \frac{1}{2}[(\hat{F}_J)^N(x) \wedge (\hat{F}_J)^N(y)]$$

for all $xy \in J$ then $G \approx G^{c^c}$, implies G is self complementary. Hence proved

Proposition 5.6 *Assume that, G is a self complementary neutrosophic bipolar vague graph then*

$$\sum_{x\neq y}(\hat{T}_K)^P(xy) = \frac{1}{2}\sum_{x\neq y}\{(\hat{T}_J)^P(x) \wedge (\hat{T}_J)^P(y)\}$$

$$\sum_{x\neq y}(\hat{I}_K)^P(xy) = \frac{1}{2}\sum_{x\neq y}\{(\hat{I}_J)^P(x) \wedge (\hat{I}_J)^P(y)\}$$

$$\sum_{x\neq y}(\hat{F}_K)^P(xy) = \frac{1}{2}\sum_{x\neq y}\{(\hat{F}_J)^P(x) \vee (\hat{F}_J)^P(y)\}$$

$$\sum_{x\neq y}(\hat{T}_K)^N(xy) = \frac{1}{2}\sum_{x\neq y}\{(\hat{T}_J)^N(x) \vee (\hat{T}_J)^N(y)\}$$

$$\sum_{x\neq y}(\hat{I}_K)^N(xy) = \frac{1}{2}\sum_{x\neq y}\{(\hat{I}_J)^N(x) \vee (\hat{I}_J)^N(y)\}$$

$$\sum_{x\neq y}(\hat{F}_K)^N(xy) = \frac{1}{2}\sum_{x\neq y}\{(\hat{F}_J)^N(x) \wedge (\hat{F}_J)^N(y)\}$$

Proof. Suppose that G be an self complementary neutrosophic bipolar vague graph, by its definition, we have isomorphism $f: J_1 \to J_2$ satisfy

$$(\hat{T}^c_{J_1})^P(f(x)) = (\hat{T}_{J_1})^P(f(x)) = (\hat{T}_{J_1})^P(x)$$
$$(\hat{I}^c_{J_1})^P(f(x)) = (\hat{I}_{J_1})^P(f(x)) = (\hat{I}_{J_1})^P(x)$$
$$(\hat{F}^c_{J_1})^P(f(x)) = (\hat{F}_{J_1})^P(f(x)) = (\hat{F}_{J_1})^P(x)$$

and

$$(\hat{T}^c_{K_1})^P(f(x),f(y)) = (\hat{T}_{K_1})^P(f(x),f(y)) = (\hat{T}_{K_1})^P(xy)$$
$$(\hat{I}^c_{K_1})^P(f(x),f(y)) = (\hat{I}_{K_1})^P(f(x),f(y)) = (\hat{I}_{K_1})^P(xy)$$
$$(\hat{F}^c_{K_1})^P(f(x),f(y)) = (\hat{F}_{K_1})^P(f(x),f(y)) = (\hat{F}_{K_1})^P(xy)$$

we have $(\hat{T}^c_{K_1})^P(f(x),f(y)) = ((\hat{T}^c_{J_1})^P(x) \wedge (\hat{T}^c_{J_1})^P(y)) - (\hat{T}_{K_1})^P(f(x),f(y))$.

i.e,$(\hat{T}_{K_1})^P(xy) = ((\hat{T}^c_{J_1})^P(x) \wedge (\hat{T}^c_{J_1})^P(y)) - (\hat{T}_{K_1})^P(f(x),f(y))$.

$(\hat{T}_{K_1})^P(xy) = ((\hat{T}^c_{J_1})^P(x) \wedge (\hat{T}^c_{J_1})^P(y)) - (\hat{T}_{K_1})^P(xy)$, hence

$$\textstyle\sum_{x\neq y}(\hat{T}_{K_1})^P(xy) + \sum_{x\neq y}(\hat{T}_{K_1})^P(xy) = \sum_{x\neq y}((\hat{T}_{J_1})^P(x) \wedge (\hat{T}_{J_1})^P(y)).$$

Similarly, $\sum_{x\neq y}(\hat{I}_{K_1})^P(xy) + \sum_{x\neq y}(\hat{I}_{K_1})^P(xy) = \sum_{x\neq y}((\hat{I}_{J_1})^P(x) \wedge (\hat{I}_{J_1})^P(y))$

$$\sum_{x\neq y}(\hat{F}_{K_1})^P(xy) + \sum_{x\neq y}(\hat{F}_{K_1})^P(xy) = \sum_{x\neq y}((\hat{F}_{J_1})^P(x) \vee (\hat{F}_{J_1})^P(y))$$

$$2\sum_{x\neq y}(\hat{T}_{K_1})^P(xy) = \sum_{x\neq y}((\hat{T}_{J_1})^P(x) \wedge (\hat{T}_{J_1})^P(y))$$

$$2\sum_{x\neq y}(\hat{I}_{K_1})^P(xy) = \sum_{x\neq y}((\hat{I}_{J_1})^P(x) \wedge (\hat{I}_{J_1})^P(y))$$

$$2\sum_{x\neq y}(\hat{F}_{K_1})^P(xy) = \sum_{x\neq y}((\hat{F}_{J_1})^P(x) \vee (\hat{F}_{J_1})^P(y))$$

Similarly one can prove for the negative condition, from the equation of the proposition (5.5) holds.

Proposition 5.7 *Suppose G_1 and G_2 is neutrosophic bipolar vague graph which is strong, $\overline{G_1} \approx \overline{G_2}$(isomorphism)*

Proof. Assume that G_1 and G_2 are isomorphic there exist a bijective map $f: J_1 \to J_2$ satisfying,

$$(\hat{T}_{J_1})^P(x) = (\hat{T}_{J_2})^P(f(x)),$$
$$(\hat{I}_{J_1})^P(x) = (\hat{I}_{J_2})^P(f(x)),$$
$$(\hat{F}_{J_1})^P(x) = (\hat{F}_{J_2})^P(f(x)), \text{for all} x \in J_1$$
$$(\hat{T}_{J_1})^N(x) = (\hat{T}_{J_2})^N(f(x)),$$
$$(\hat{I}_{J_1})^N(x) = (\hat{I}_{J_2})^N(f(x)),$$
$$(\hat{F}_{J_1})^N(x) = (\hat{F}_{J_2})^N(f(x)), \text{for all} x \in J_1$$

and

$$(\hat{T}_{K_1})^P(xy) = (\hat{T}_{K_2})^P(f(x), f(y))$$
$$(\hat{I}_{K_1})^P(xy) = (\hat{I}_{K_2})^P(f(x), f(y))$$
$$(\hat{F}_{K_1})^P(xy) = (\hat{F}_{K_2})^P(f(x), f(y)) \forall xy \in K_1$$
$$(\hat{T}_{K_1})^N(xy) = (\hat{T}_{K_2})^N(f(x), f(y))$$
$$(\hat{I}_{K_1})^N(xy) = (\hat{I}_{K_2})^N(f(x), f(y))$$
$$(\hat{F}_{K_1})^N(xy) = (\hat{F}_{K_2})^N(f(x), f(y)) \forall xy \in K_1$$

by definition (5.2) we have

$$(T^c_{K_1})^P(xy) = ((T_{J_1})^P(x) \wedge (T_{J_1})^P(y)) - (T_{K_1})^P(xy)$$
$$= ((T_{J_2})^P f(x) \wedge (T_{J_2})^P f(y)) - (T_{K_2})^P(f(x)f(y))$$
$$= (T^c_{K_2})^P(f(x)f(y))$$
$$(I^c_{K_1})^P(xy) = ((I_{J_1})^P(x) \wedge (I_{J_1})^P(y)) - (I_{K_1})^P(xy)$$
$$= ((I_{J_2})^P f(x) \wedge (I_{J_2})^P f(y)) - (I_{K_2})^P(f(x)f(y))$$
$$= (I^c_{K_2})^P(f(x)f(y))$$
$$(F^c_{K_1})^P(xy) = ((F_{J_1})^P(x) \vee (F_{J_1})^P(y)) - (F_{K_1})^P(xy)$$
$$= ((F_{J_2})^P f(x) \vee (F_{J_2})^P f(y)) - (F_{K_2})^P(f(x)f(y))$$
$$= (F^c_{K_2})^P(f(x)f(y))$$

Hence $G_1^c \approx G_2^c$ for all $(xy) \in K_1$

Definition 5.8 *A neutrosophic bipolar vague graph $G = (J, K)$ is complete if*

$$(T_K^-)^P(xy) = \{(T_J^-)^P(x) \wedge (T_J^-)^P(y)\}$$
$$(I_K^-)^P(xy) = \{(I_J^-)^P(x) \wedge (I_J^-)^P(y)\}$$
$$(F_K^-)^P(xy) = \{(F_J^-)^P(x) \vee (F_J^-)^P(y)\},$$
$$(T_K^+)^P(xy) = \{(T_J^+)^P(x) \wedge (T_J^+)^P(y)\}$$
$$(I_K^+)^P(xy) = \{(I_J^+)^P(x) \wedge (I_J^+)^P(y)\}$$
$$(F_K^+)^P(xy) = \{(F_J^+)^P(x) \vee (F_J^+)^P(y)\},$$
$$(T_K^-)^N(xy) = \{(T_J^-)^N(x) \vee (T_J^-)^N(y)\}$$
$$(I_K^-)^N(xy) = \{(I_J^-)^N(x) \vee (I_K^-)^N(y)\}$$
$$(F_K^-)^N(xy) = \{(F_J^-)^N(x) \wedge (F_J^-)^N(y)\},$$
$$(T_K^+)^N(xy) = \{(T_J^+)^N(x) \vee (T_J^+)^N(y)\}$$
$$(I_K^+)^N(xy) = \{(I_J^+)^N(x) \vee (I_J^+)^N(y)\}$$

$$(F_K^+)^N(xy) = \{(F_J^+)^N(x) \wedge (F_J^+)^N(y)\}, \forall((xy) \in J)$$

Remark 5.9 The complement of NBVGs are NBVGs provided the graph is strong. According to [9], the complement of Single-Valued Neutrosophic Graph (SVNG) is not a SVNG. By the same idea, we implement the definition for NBVGs to obtain the proposed concepts. For other type of bipolar graphs, the complement of Bipolar Fuzzy Graph (BFG) is BFG [6]. The complement of Bipolar Fuzzy Soft Graph (BFSG) and Bipolar Neutrosophic Graph (BNG) are BFSG and BNG, [14, 16] respectively, provided if the graph is strong. The complement of complete bipolar SVNG is bipolar SVFG [25].

Conclusion

This present work characterised the new concept of neutrosophic bipolar vague sets and its application to NBVGs are introduced. Moreover, some remarkable properties of strong NBVGs, complete NBVGs and complement NBVGs have been investigated and the proposed concepts are illustrated with the examples. The obtained results are extended to interval neutrosophic bipolar vague sets. Further we can extend to investigate the domination number, regular and isomorphic properties of the proposed graph.

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Neutrosophic $\mathcal{N}$-structures over UP-algebras

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Abstract: The notions of (special) neutrosophic $\mathcal{N}$-UP-subalgebras, (special) neutrosophic $\mathcal{N}$-near UP-filters, (special) neutrosophic $\mathcal{N}$-UP-filters, (special) neutrosophic $\mathcal{N}$-UP-ideals, and (special) neutrosophic $\mathcal{N}$-strongly UP-ideals of UP-algebras are introduced, and several properties are investigated. Conditions for neutrosophic $\mathcal{N}$-structures to be (special) neutrosophic $\mathcal{N}$-UP-subalgebras, (special) neutrosophic $\mathcal{N}$-near UP-filters, (special) neutrosophic $\mathcal{N}$-UP-filters, (special) neutrosophic $\mathcal{N}$-UP-ideals, and (special) neutrosophic $\mathcal{N}$-strongly UP-ideals of UP-algebras are provided. Relations between (special) neutrosophic $\mathcal{N}$-UP-subalgebras (resp., (special) neutrosophic $\mathcal{N}$-near UP-filters, (special) neutrosophic $\mathcal{N}$-UP-filters, (special) neutrosophic $\mathcal{N}$-UP-ideals, (special) neutrosophic $\mathcal{N}$-strongly UP-ideals) and their level subsets are considered.

Keywords: UP-algebra; (special) neutrosophic $\mathcal{N}$-UP-subalgebra; (special) neutrosophic $\mathcal{N}$-near UP-filter; (special) neutrosophic $\mathcal{N}$-UP-filter; (special) neutrosophic $\mathcal{N}$-UP-ideal; (special) neutrosophic $\mathcal{N}$-strongly UP-ideal

1. Introduction

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [14], BCI-algebras [15], BCH-algebras [11], KU-algebras [28], SU-algebras [21] and others. They are strongly connected with logic. For example, BCI-algebras were introduced by Iséki [15] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [14, 15] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

The branch of the logical algebra, UP-algebras was introduced by Iampan [12] in 2017, and it is known that the class of KU-algebras [28] is a proper subclass of the class of UP-algebras. It have been examined by several researchers, for example, Somjanta et al. [32] introduced the notion of fuzzy sets in UP-algebras, the notion of intuitionistic fuzzy sets in UP-algebras was introduced by Kesorn et al. [22], Kaijae et al. [20] introduced the notions of anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras of UP-algebras, the notion of Q-fuzzy sets in UP-algebras was introduced by Tanamoon et al. [37], etc.

Neutrosophy provides a foundation for a whole family of new mathematical theories with the generalization of both classical and fuzzy counterparts. In a neutrosophic set, an element has three associated defining functions such as truth membership function (T), indeterminate membership function (I) and false membership function (F) defined on a universe of discourse X. These three functions are independent completely. The concept of neutrosophic logics was first introduced by Smarandache [31] in 1999. Jun et al. [16] introduced a new function, called a negative-valued function, and constructed $\mathcal{N}$-structures in 2009. Khan et al. [23] discussed neutrosophic $\mathcal{N}$-structures and their applications in semigroups in 2017. Jun et al. [17, 33] considered neutrosophic $\mathcal{N}$-structures applied to BCK/BCI-algebras and neutrosophic commutative $\mathcal{N}$-ideals in BCK-algebras in 2017. Jun et al. [19] studied neutrosophic positive implicative $\mathcal{N}$-ideals in BCK-algebras in 2018. Abdel-Baset and his colleagues applied the notion of neutrosophic set theory in the new fields (see [1, 2, 3, 4, 5, 6, 27]). Jun and his colleagues applied the notion of neutrosophic set theory in BCK/BCI-algebras (see [8, 18, 24, 26, 35, 36]).

The remaining part of the paper is structured as follows: Section 2 gives some definitions and properties of UP-algebras. Section 3 introduces the notions of neutrosophic $\mathcal{N}$-UP-subalgebras, neutrosophic $\mathcal{N}$-near UP-filters, neutrosophic $\mathcal{N}$-UP-filters, neutrosophic $\mathcal{N}$-UP-ideals, and neutrosophic $\mathcal{N}$-strongly UP-ideals of UP-algebras, and a level subset of a neutrosophic $\mathcal{N}$-structure is proved in Section 4. Section 5 introduces the notions of special neutrosophic $\mathcal{N}$-UP-subalgebras, special neutrosophic $\mathcal{N}$-near UP-filters, special neutrosophic $\mathcal{N}$-UP-filters, special neutrosophic $\mathcal{N}$-UP-ideals, and special neutrosophic $\mathcal{N}$-strongly UP-ideals of UP-algebras, and a level subset of a neutrosophic $\mathcal{N}$-structure of special type is proved in Section 6. This paper has been finalized with that result.

2. Basic results on UP-algebras

Before we begin our study, we will give the definition of a UP-algebra.

Definition 2.1 [12] An algebra $X=(X,\cdot,0)$ of type $(2,0)$ is called a *UP-algebra* where X is a nonempty set, $\cdot$ is a binary operation on X, and 0 is a fixed element of X (i.e., a nullary operation) if it satisfies the following axioms:

(UP-1) $(\forall x,y,z\in X)((y\cdot z)\cdot((x\cdot y)\cdot(x\cdot z))=0)$,

(UP-2) $(\forall x\in X)(0\cdot x=x)$,

(UP-3) $(\forall x\in X)(x\cdot 0=0)$, and

(UP-4) $(\forall x,y\in X)(x\cdot y=0, y\cdot x=0\Rightarrow x=y)$.

From [12], we know that the notion of UP-algebras is a generalization of KU-algebras (see [28]).

Example 2.2 [30] Let X be a universal set and let $\Omega\in \mathcal{P}(X)$ where $\mathcal{P}(X)$ means the power set of X. Let $\mathcal{P}_{\Omega}(X)=\{A\in\mathcal{P}(X)\mid \Omega\subseteq A\}$. Define a binary operation $\cdot$ on $\mathcal{P}_{\Omega}(X)$ by putting $A\cdot B=B\cap(A^C\cup\Omega)$ for all $A,B\in\mathcal{P}_{\Omega}(X)$ where A^C means the complement of a subset A. Then $(\mathcal{P}_{\Omega}(X),\cdot,\Omega)$ is a UP-algebra and we shall call it the generalized power UP-algebra of type 1 with respect to Ω. Let $\mathcal{P}^{\Omega}(X)=\{A\in\mathcal{P}(X)\mid A\subseteq\Omega\}$. Define a binary operation $*$ on $\mathcal{P}^{\Omega}(X)$ by putting $A*B=B\cup(A^C\cap\Omega)$ for all $A,B\in\mathcal{P}^{\Omega}(X)$. Then $(\mathcal{P}^{\Omega}(X),*,\Omega)$ is a UP-algebra and we shall call it the generalized power UP-algebra of type 2 with respect to Ω. In particular, $(\mathcal{P}(X),\cdot,\varnothing)$ is a UP-algebra and we shall call it the power UP-algebra of type 1, and $(\mathcal{P}(X),*,X)$ is a UP-algebra and we shall call it the power UP-algebra of type 2.

Example 2.3 [9] Let $\mathbf{N}$ be the set of all natural numbers with two binary operations $\circ$ and $\bullet$ defined by

$$(\forall x,y\in\mathbf{N})\left(x\circ y=\begin{cases} y & \text{if } x<y,\\ 0 & \text{otherwise}\end{cases}\right) \text{ and } (\forall x,y\in\mathbf{N})\left(x\bullet y=\begin{cases} y & \text{if } x>y \text{ or } x=0,\\ 0 & \text{otherwise}\end{cases}\right).$$

Then $(\mathbf{N},\circ,0)$ and $(\mathbf{N},\bullet,0)$ are UP-algebras.

Example 2.4 [25] Let $X=\{0,1,2,3,4,5\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	0	0	2	3	2	5
2	0	1	0	3	1	5
3	0	1	2	0	4	5
4	0	0	0	3	0	5
5	0	0	2	0	2	0

Then $(X,\cdot,0)$ is a UP-algebra.

For more examples of UP-algebras, see [7, 13, 29, 30].

The following proposition is very important for the study of UP-algebras.

Proposition 2.5 [12, 13] In a UP-algebra $X=(X,\cdot,0)$, the following properties hold:

1. $(\forall x\in X)(x\cdot x=0)$,
2. $(\forall x,y,z\in X)(x\cdot y=0, y\cdot z=0\Rightarrow x\cdot z=0)$,
3. $(\forall x,y,z\in X)(x\cdot y=0\Rightarrow (z\cdot x)\cdot(z\cdot y)=0)$,
4. $(\forall x,y,z\in X)(x\cdot y=0\Rightarrow (y\cdot z)\cdot(x\cdot z)=0)$,
5. $(\forall x,y\in X)(x\cdot(y\cdot x)=0)$,
6. $(\forall x,y\in X)((y\cdot x)\cdot x=0\Leftrightarrow x=y\cdot x)$,
7. $(\forall x,y\in X)(x\cdot(y\cdot y)=0)$,
8. $(\forall a,x,y,z\in X)((x\cdot(y\cdot z))\cdot(x\cdot((a\cdot y)\cdot(a\cdot z)))=0)$,
9. $(\forall a,x,y,z\in X)((((a\cdot x)\cdot(a\cdot y))\cdot z)\cdot((x\cdot y)\cdot z)=0)$,
10. $(\forall x,y,z\in X)(((x\cdot y)\cdot z)\cdot(y\cdot z)=0)$,
11. $(\forall x,y,z\in X)(x\cdot y=0\Rightarrow x\cdot(z\cdot y)=0)$,
12. $(\forall x,y,z\in X)(((x\cdot y)\cdot z)\cdot(x\cdot(y\cdot z))=0)$, and
13. $(\forall a,x,y,z\in X)(((x\cdot y)\cdot z)\cdot(y\cdot(a\cdot z))=0)$.

On a UP-algebra $X=(X,\cdot,0)$, we define a binary relation $\leq$ on X [12] as follows:

$$(\forall x,y\in X)(x\leq y\Leftrightarrow x\cdot y=0).$$

Definition 2.6 [10, 12, 32] A nonempty subset S of a UP-algebra $(X,\cdot,0)$ is called

1. a *UP-subalgebra* of X if $(\forall x,y\in S)(x\cdot y\in S)$.
2. a *near UP-filter* of X if
   (a) the constant 0 of X is in S, and
   (b) $(\forall x,y\in X)(y\in S\Rightarrow x\cdot y\in S)$.
3. a *UP-filter* of X if
   (a) the constant 0 of X is in S, and
   (b) $(\forall x,y\in X)(x\cdot y\in S, x\in S\Rightarrow y\in S)$.
4. a *UP-ideal* of X if
   (a) the constant 0 of X is in S, and
   (b) $(\forall x,y,z\in X)(x\cdot(y\cdot z)\in S, y\in S\Rightarrow x\cdot z\in S)$.
5. a *strongly UP-ideal* of X if

(a) the constant 0 of X is in S, and
(b) $(\forall x, y, z \in X)((z \cdot y) \cdot (z \cdot x) \in S, y \in S \Rightarrow x \in S)$.

Guntasow et al. [10] proved that the notion of UP-subalgebras is a generalization of near UP-filters, near UP-filters is a generalization of UP-filters, UP-filters is a generalization of UP-ideals, and UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra X is the only one strongly UP-ideal of itself.

Theorem 2.7 *Let $\mathcal{N}$ be a nonempty family of near UP-filters of a UP-algebra $X = (X, \cdot, 0)$. Then $\cap\mathcal{N}$ and $\cup\mathcal{N}$ are near UP-filters of X.*

Proof. Clearly, $0 \in N$ for all $N \in \mathcal{N}$. Then $0 \in \cap\mathcal{N}$. Let $x \in X$ and $y \in \cap\mathcal{N}$. Then $y \in N$ for all $N \in \mathcal{N}$. Since N is a near UP-filter of X, we have $x \cdot y \in N$ for all $N \in \mathcal{N}$ and so $x \cdot y \in \cap\mathcal{N}$. Hence, $\cap\mathcal{N}$ is a near UP-filter of X. Since $\cap\mathcal{N} \subseteq \cup\mathcal{N}$, we have $0 \in \cup\mathcal{N}$. Let $x \in X$ and $y \in \cup\mathcal{N}$. Then $y \in N$ for some $N \in \mathcal{N}$. Since N is a near UP-filter of X, we have $x \cdot y \in N \subseteq \cup\mathcal{N}$,. Hence, $\cup\mathcal{N}$ is a near UP-filter of X.

3. Neutrosophic $\mathcal{N}$-structures

We denote the family of all functions from a nonempty set X to the closed interval $[-1,0]$ of the real line by $\mathrm{F}(X,[-1,0])$. An element of $\mathrm{F}(X,[-1,0])$ is called a *negative-valued function* from X to $[-1,0]$ (briefly, $\mathcal{N}$-function on X). An ordered pair (X, f) of X and an $\mathcal{N}$-function f on X is called an *$\mathcal{N}$-structure*.

A *neutrosophic $\mathcal{N}$-structure* over a nonempty universe of discourse X [23] is defined to be the structure

$$X_N = \{(x, T_N(x), I_N(x), F_N(x)) \mid x \in X\}$$

where T_N, I_N and F_N are $\mathcal{N}$-functions on X which are called the *negative truth membership function*, the *negative indeterminacy membership function* and the *negative falsity membership function* on X, respectively.

For the sake of simplicity, we will use the notation X_N or $X_N = (X, T_N, I_N, F_N)$ instead of the neutrosophic $\mathcal{N}$-structure [16].

Definition 3.1 Let X_N be a neutrosophic $\mathcal{N}$-structure over a nonempty set X. The neutrosophic $\mathcal{N}$-structure $\overline{X}_N = (X, \overline{T}_N, \overline{I}_N, \overline{F}_N)$ defined by

$$(\forall x \in X)\begin{pmatrix} \overline{T}_N(x) & = -1 - T_N(x) \\ \overline{I}_N(x) & = -1 - I_N(x) \\ \overline{F}_N(x) & = -1 - F_N(x) \end{pmatrix} \tag{3.1}$$

is called the *complement* of X_N in X.

Remark 3.2 *For all neutrosophic $\mathcal{N}$-structure X_N over a nonempty set X, we have $X_N = \overline{\overline{X}}_N$.*

Lemma 3.3 [33] *Let f be an $\mathcal{N}$-function on a nonempty set X. Then the following statements hold:*

1. $(\forall x, y \in X)(-1 - \max\{f(x), f(y)\} = \min\{-1 - f(x), -1 - f(y)\})$, and

2. $(\forall x, y \in X)(-1-\min\{f(x), f(y)\} = \max\{-1-f(x), -1-f(y)\})$.

The following lemmas are easily proved

Lemma 3.4 *Let* f *be an* $\mathcal{N}$*-function on a nonempty set* X. *Then the following statements hold:*

1. $(\forall x, y, z \in X)(\overline{f}(x) \geq \min\{\overline{f}(y), \overline{f}(z)\} \Leftrightarrow f(x) \leq \max\{f(y), f(z)\})$,
2. $(\forall x, y, z \in X)(\overline{f}(x) \leq \min\{\overline{f}(y), \overline{f}(z)\} \Leftrightarrow f(x) \geq \max\{f(y), f(z)\})$,
3. $(\forall x, y, z \in X)(\overline{f}(x) \geq \max\{\overline{f}(y), \overline{f}(z)\} \Leftrightarrow f(x) \leq \min\{f(y), f(z)\})$, and
4. $(\forall x, y, z \in X)(\overline{f}(x) \leq \max\{\overline{f}(y), \overline{f}(z)\} \Leftrightarrow f(x) \geq \min\{f(y), f(z)\})$.

In what follows, let X denote a UP-algebra $(X, \cdot, 0)$ unless otherwise specified.

Now, we introduce the notions of neutrosophic $\mathcal{N}$-UP-subalgebras, neutrosophic $\mathcal{N}$-near UP-filters, neutrosophic $\mathcal{N}$-UP-filters, neutrosophic $\mathcal{N}$-UP-ideals, and neutrosophic $\mathcal{N}$-strongly UP-ideals of UP-algebras, provide the necessary examples, investigate their properties, and prove their generalizations.

Definition 3.5 A neutrosophic $\mathcal{N}$-structure X_N over X is called a *neutrosophic* $\mathcal{N}$*-UP-subalgebra* of X if it satisfies the following conditions:

$$(\forall x, y \in X)(T_N(x \cdot y) \leq \max\{T_N(x), T_N(y)\}), \tag{3.2}$$
$$(\forall x, y \in X)(I_N(x \cdot y) \geq \min\{I_N(x), I_N(y)\}), \tag{3.3}$$
$$(\forall x, y \in X)(F_N(x \cdot y) \leq \max\{F_N(x), F_N(y)\}). \tag{3.4}$$

Example 3.6 Let $X = \{0,1,2,3,4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	4
3	0	0	2	0	4
4	0	0	0	0	0

Then $(X, \cdot, 0)$ is a UP-algebra. We define a neutrosophic $\mathcal{N}$-structure X_N over X as follows:

$$T_N(0) = -0.8,\ I_N(0) = -0.3,\ F_N(0) = -0.8,$$
$$T_N(1) = -0.6,\ I_N(1) = -0.7,\ F_N(1) = -0.8,$$
$$T_N(2) = -0.4,\ I_N(2) = -0.8,\ F_N(2) = -0.7,$$
$$T_N(3) = -0.1,\ I_N(3) = -0.5,\ F_N(3) = -0.5,$$
$$T_N(4) = -0.2,\ I_N(4) = -0.9,\ F_N(4) = -0.3.$$

Hence, X_N is a neutrosophic $\mathcal{N}$-UP-subalgebra of X.

Definition 3.7 A neutrosophic $\mathcal{N}$-structure X_N over X is called a *neutrosophic $\mathcal{N}$-near UP-filter* of X if it satisfies the following conditions:

$$(\forall x \in X)(T_N(0) \leq T_N(x)), \tag{3.5}$$

$$(\forall x \in X)(I_N(0) \geq I_N(x)), \tag{3.6}$$

$$(\forall x \in X)(F_N(0) \leq F_N(x)), \tag{3.7}$$

$$(\forall x, y \in X)(T_N(x \cdot y) \leq T_N(y)), \tag{3.8}$$

$$(\forall x, y \in X)(I_N(x \cdot y) \geq I_N(y)), \tag{3.9}$$

$$(\forall x, y \in X)(F_N(x \cdot y) \leq F_N(y)). \tag{3.10}$$

Example 3.8 Let $X = \{0,1,2,3,4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	2
2	0	1	0	3	1
3	0	1	2	0	4
4	0	0	0	3	0

Then $(X, \cdot, 0)$ is a UP-algebra. We define a neutrosophic $\mathcal{N}$-structure X_N over X as follows:

$$T_N(0) = -0.8,\ I_N(0) = -0.3,\ F_N(0) = -0.8,$$
$$T_N(1) = -0.6,\ I_N(1) = -0.7,\ F_N(1) = -0.6,$$
$$T_N(2) = -0.8,\ I_N(2) = -0.8,\ F_N(2) = -0.7,$$
$$T_N(3) = -0.1,\ I_N(3) = -0.5,\ F_N(3) = -0.5,$$
$$T_N(4) = -0.3,\ I_N(4) = -0.8,\ F_N(4) = -0.3.$$

Hence, X_N is a neutrosophic $\mathcal{N}$-near UP-filter of X.

Definition 3.9 A neutrosophic $\mathcal{N}$-structure X_N over X is called a *neutrosophic $\mathcal{N}$-UP-filter* of X if it satisfies the following conditions: (3.5), (3.6), (3.7), and

$$(\forall x, y \in X)(T_N(y) \leq \max\{T_N(x \cdot y), T_N(x)\}), \tag{3.11}$$

$$(\forall x, y \in X)(I_N(y) \geq \min\{I_N(x \cdot y), I_N(x)\}), \tag{3.12}$$

$$(\forall x, y \in X)(F_N(y) \leq \max\{F_N(x \cdot y), F_N(x)\}). \tag{3.13}$$

Example 3.10 Let $X = \{0,1,2,3,4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	0	0
2	0	1	0	0	4
3	0	1	2	0	4
4	0	1	2	3	0

Then $(X, \cdot, 0)$ is a UP-algebra. We define a neutrosophic $\mathcal{N}$-structure X_N over X as follows:

$$T_N(0) = -0.9,\ I_N(0) = -0.2,\ F_N(0) = -0.8,$$
$$T_N(1) = -0.5,\ I_N(1) = -0.8,\ F_N(1) = -0.6,$$
$$T_N(2) = -0.2,\ I_N(2) = -0.6,\ F_N(2) = -0.3,$$
$$T_N(3) = -0.6,\ I_N(3) = -0.3,\ F_N(3) = -0.7,$$

$$T_N(4) = -0.7,\ I_N(4) = -0.3,\ F_N(4) = -0.8.$$

Hence, X_N is a neutrosophic $\mathcal{N}$-UP-filter of X.

Definition 3.11 A neutrosophic $\mathcal{N}$ –structure X_N over X is called a *neutrosophic $\mathcal{N}$-UP-ideal* of X if it satisfies the following conditions: (3.5), (3.6), (3.7), and

$$(\forall x, y, z \in X)(T_N(x \cdot z) \leq \max\{T_N(x \cdot (y \cdot z)), T_N(y)\}), \tag{3.14}$$

$$(\forall x, y, z \in X)(I_N(x \cdot z) \geq \min\{I_N(x \cdot (y \cdot z)), I_N(y)\}), \tag{3.15}$$

$$(\forall x, y, z \in X)(F_N(x \cdot z) \leq \max\{F_N(x \cdot (y \cdot z)), F_N(y)\}). \tag{3.16}$$

Example 3.12 Let $X = \{0,1,2,3,4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	3	4
3	0	0	2	0	4
4	0	1	2	3	0

Then $(X, \cdot, 0)$ is a UP-algebra. We define a neutrosophic $\mathcal{N}$-structure X_N over X as follows:

$$T_N(0) = -0.8,\ I_N(0) = -0.3,\ F_N(0) = -0.8,$$
$$T_N(1) = -0.5,\ I_N(1) = -0.6,\ F_N(1) = -0.8,$$
$$T_N(2) = -0.4,\ I_N(2) = -0.8,\ F_N(2) = -0.7,$$
$$T_N(3) = -0.1,\ I_N(3) = -0.7,\ F_N(3) = -0.5,$$
$$T_N(4) = -0.2,\ I_N(4) = -0.8,\ F_N(4) = -0.3.$$

Hence, X_N is a neutrosophic $\mathcal{N}$-UP-ideal of X.

Definition 3.13 A neutrosophic $\mathcal{N}$-structure X_N over X is called a *neutrosophic $\mathcal{N}$-strongly UP-ideal* of X if it satisfies the following conditions: (3.5), (3.6), (3.7), and

$$(\forall x, y, z \in X)(T_N(x) \leq \max\{T_N((z \cdot y) \cdot (z \cdot x)), T_N(y)\}), \tag{3.17}$$

$$(\forall x, y, z \in X)(I_N(x) \geq \min\{I_N((z \cdot y) \cdot (z \cdot x)), I_N(y)\}), \tag{3.18}$$

$$(\forall x, y, z \in X)(F_N(x) \leq \max\{F_N((z \cdot y) \cdot (z \cdot x)), F_N(y)\}). \tag{3.19}$$

Example 3.14 Let $X = \{0,1,2,3,4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	3	2
2	0	1	0	3	1
3	0	1	2	0	4
4	0	0	0	3	0

Then $(X, \cdot, 0)$ is a UP-algebra. We define a neutrosophic $\mathcal{N}$-structure X_N over X as follows:

$$(\forall x \in X)\begin{pmatrix} T_N(x) & = -1 \\ I_N(x) & = -0.3 \\ F_N(x) & = -0.7 \end{pmatrix}.$$

Hence, X_N is neutrosophic $\mathcal{N}$-strongly UP-ideal of X.

Definition 3.15 A neutrosophic $\mathcal{N}$-structure X_N over X is said to be *constant* if X_N is a constant function from X to $[-1,0]^3$. That is, T_N, I_N, and F_N are constant functions from X to $[-1,0]$.

Theorem 3.16 Every neutrosophic $\mathcal{N}$-UP-subalgebra of X satisfies the conditions (3.5), (3.6), and (3.7).

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-UP-subalgebra of X. Then for all $x \in X$, by Proposition 2.5 (1), (3.2), (3.3), and (3.4), we have

$$T_N(0) = T_N(x \cdot x) \leq \max\{T_N(x), T_N(x)\} = T_N(x),$$
$$I_N(0) = I_N(x \cdot x) \geq \min\{I_N(x), I_N(x)\} = I_N(x),$$
$$F_N(0) = F_N(x \cdot x) \leq \max\{F_N(x), F_N(x)\} = F_N(x).$$

Hence, X_N satisfies the conditions (3.5), (3.6), and (3.7).

Theorem 3.17 A neutrosophic $\mathcal{N}$-structure X_N over X is constant if and only if it is a neutrosophic $\mathcal{N}$-strongly UP-ideal of X.

Proof. Assume that X_N is constant. Then for all $x \in X$, $T_N(x) = T_N(0), I_N(x) = I_N(0)$, and $F_N(x) = F_N(0)$ and so $T_N(0) \leq T_N(x), I_N(0) \geq I_N(x)$, and $F_N(0) \leq F_N(x)$. Next, for all $x, y, z \in X$,

$$T_N(x) = T_N(0) = \max\{T_N(0), T_N(0)\} = \max\{T_N((z \cdot y) \cdot (z \cdot x)), T_N(y)\},$$
$$I_N(x) = I_N(0) = \min\{I_N(0), I_N(0)\} = \min\{I_N((z \cdot y) \cdot (z \cdot x)), I_N(y)\},$$
$$F_N(x) = F_N(0) = \max\{F_N(0), F_N(0)\} = \max\{F_N((z \cdot y) \cdot (z \cdot x)), F_N(y)\}.$$

Hence, X_N is a neutrosophic $\mathcal{N}$-strongly UP-ideal of X.

Conversely, assume that X_N is a neutrosophic $\mathcal{N}$-strongly UP-ideal of X. For any $x \in X$, by Proposition 2.5 (1), (UP-2), (UP-3), (3.17), (3.18), and (3.19), we have

$$\begin{aligned} T_N(x) &\leq \max\{T_N((x \cdot 0) \cdot (x \cdot x)), T_N(0)\} = \max\{T_N(0 \cdot (x \cdot x)), T_N(0)\} = \max\{T_N(x \cdot x), T_N(0)\} \\ &= \max\{T_N(0), T_N(0)\} = T_N(0), \end{aligned}$$
$$\begin{aligned} I_N(x) &\geq \min\{I_N((x \cdot 0) \cdot (x \cdot x)), I_N(0)\} = \min\{I_N(0 \cdot (x \cdot x)), I_N(0)\} = \min\{I_N(x \cdot x), I_N(0)\} \\ &= \min\{I_N(0), I_N(0)\} = I_N(0), \end{aligned}$$
$$\begin{aligned} F_N(x) &\leq \max\{F_N((x \cdot 0) \cdot (x \cdot x)), F_N(0)\} = \max\{F_N(0 \cdot (x \cdot x)), F_N(0)\} = \max\{F_N(x \cdot x), F_N(0)\} \\ &= \max\{F_N(0), F_N(0)\} = F_N(0). \end{aligned}$$

Thus $T_N(x) = T_N(0), I_N(x) = I_N(0)$, and $F_N(x) = F_N(0)$ for all $x \in X$. Hence, X_N is constant.

Theorem 3.18 Every neutrosophic $\mathcal{N}$-strongly UP-ideal of X is a neutrosophic $\mathcal{N}$-UP-ideal.

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-strong UP-ideal of X. Then X_N satisfies the conditions (3.5), (3.6), and (3.7). By Theorem 3.17, we have X_N is constant. Then for all $x \in X$, $T_N(x) = T_N(0), I_N(x) = I_N(0)$, and $F_N(x) = F_N(0)$. By Proposition 2.5 (5), (UP-3), (3.5), (3.6), (3.7), (3.17), (3.18), and (3.19), we have

$$\begin{aligned} T_N(x \cdot z) &= \max\{T_N((z \cdot y) \cdot (z \cdot (x \cdot z))), T_N(y)\} = \max\{T_N((z \cdot y) \cdot 0), T_N(y)\} = \max\{T_N(0), T_N(y)\} = T_N(y) \\ &\leq \max\{T_N(x \cdot (y \cdot z)), T_N(y)\}, \end{aligned}$$
$$\begin{aligned} I_N(x \cdot z) &= \min\{I_N((z \cdot y) \cdot (z \cdot (x \cdot z))), I_N(y)\} = \min\{I_N((z \cdot y) \cdot 0), I_N(y)\} = \min\{I_N(0), I_N(y)\} = I_N(y) \\ &\geq \min\{I_N(x \cdot (y \cdot z)), I_N(y)\}, \end{aligned}$$
$$\begin{aligned} F_N(x \cdot z) &= \max\{F_N((z \cdot y) \cdot (z \cdot (x \cdot z))), F_N(y)\} = \max\{F_N((z \cdot y) \cdot 0), F_N(y)\} = \max\{F_N(0), F_N(y)\} = F_N(y) \\ &\leq \max F_N(x \cdot (y \cdot z)), F_N(y). \end{aligned}$$

Hence, X_N is a neutrosophic $\mathcal{N}$-UP-ideal of X.

The following example show that the converse of Theorem 3.18 is not true.

Example 3.19 Let $X = \{0,1,2,3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Then $(X,\cdot,0)$ is a UP-algebra. We define a neutrosophic $\mathcal{N}$-structure X_N over X as follows:

$$T_N(0) = -0.6,\ I_N(0) = -0.1,\ F_N(0) = -0.7,$$
$$T_N(1) = -0.4,\ I_N(1) = -0.5,\ F_N(1) = -0.5,$$
$$T_N(2) = -0.3,\ I_N(2) = -0.4,\ F_N(2) = -0.4,$$
$$T_N(3) = -0.2,\ I_N(3) = -0.4,\ F_N(3) = -0.3.$$

Hence, X_N is a neutrosophic $\mathcal{N}$-UP-ideal of X. Since X_N is not constant, it follows from Theorem 3.17 that it is not a neutrosophic $\mathcal{N}$-strongly UP-ideal of X.

Theorem 3.20 Every neutrosophic $\mathcal{N}$-UP-ideal of X is a neutrosophic $\mathcal{N}$-UP-filter.

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-UP-ideal of X. Then X_N satisfies the conditions (3.5), (3.6), and (3.7). Next, let $x, y \in X$. By (UP-2), (3.14), (3.15), and (3.16), we have

$$T_N(y) = T_N(0\cdot y) \le \max\{T_N(0\cdot(x\cdot y)), T_N(x)\} = \max\{T_N(x\cdot y), T_N(x)\},$$
$$I_N(y) = I_N(0\cdot y) \ge \min\{I_N(0\cdot(x\cdot y)), I_N(x)\} = \min\{I_N(x\cdot y), I_N(x)\},$$
$$F_N(y) = F_N(0\cdot y) \le \max\{F_N(0\cdot(x\cdot y)), F_N(x)\} = \max\{F_N(x\cdot y), F_N(x)\}.$$

Hence, X_N is a neutrosophic $\mathcal{N}$-UP-filter of X.

The following example show that the converse of Theorem 3.20 is not true.

Example 3.21 Let $X = \{0,1,2,3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	2
3	0	1	0	0

Then $(X,\cdot,0)$ is a UP-algebra. We define a neutrosophic $\mathcal{N}$-structure X_N over X as follows:

$$T_N(0) = -0.7,\ I_N(0) = -0.1,\ F_N(0) = -0.9,$$
$$T_N(1) = -0.6,\ I_N(1) = -0.5,\ F_N(1) = -0.8,$$
$$T_N(2) = -0.3,\ I_N(2) = -0.4,\ F_N(2) = -0.5,$$
$$T_N(3) = -0.3,\ I_N(3) = -0.4,\ F_N(3) = -0.5.$$

Hence, X_N is a neutrosophic $\mathcal{N}$-UP-filter of X. Since $F_N(2\cdot 3) = -0.3 > -0.8 = \max\{F_N(2\cdot(1\cdot 3)), F_N(1)\}$, we have X_N is not a neutrosophic $\mathcal{N}$-UP-ideal of X.

Theorem 3.22 Every neutrosophic $\mathcal{N}$-UP-filter of X is a neutrosophic $\mathcal{N}$-near UP-filter.

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-UP-filter. Then X_N satisfies the conditions (3.5), (3.6), and (3.7). Next, let $x, y \in X$. By Proposition 2.5 (5), (3.5), (3.6), (3.7), (3.11), (3.12), and (3.13), we have

$$T_N(x \cdot y) \leq \max\{T_N(y \cdot (x \cdot y)), T_N(y)\} = \max\{T_N(0), T_N(y)\} = T_N(y),$$
$$I_N(x \cdot y) \geq \min\{I_N(y \cdot (x \cdot y)), I_N(y)\} = \min\{I_N(0), I_N(y)\} = I_N(y),$$
$$F_N(x \cdot y) \leq \max\{F_N(y \cdot (x \cdot y)), F_N(y)\} = \max\{F_N(0), F_N(y)\} = F_N(y).$$

Hence, X_N is a neutrosophic $\mathcal{N}$-near UP-filter of X.

The following example show that the converse of Theorem 3.22 is not true.

Example 3.23 Let $X = \{0,1,2,3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	1	3
2	0	0	0	3
3	0	1	1	0

Then $(X, \cdot, 0)$ is a UP-algebra. We define a neutrosophic $\mathcal{N}$-structure X_N over X as follows:

$$T_N(0) = -0.9,\ I_N(0) = -0.3,\ F_N(0) = -0.8,$$
$$T_N(1) = -0.5,\ I_N(1) = -0.7,\ F_N(1) = -0.7,$$
$$T_N(2) = -0.2,\ I_N(2) = -0.8,\ F_N(2) = -0.6,$$
$$T_N(3) = -0.1,\ I_N(3) = -0.5,\ F_N(3) = -0.3.$$

Hence, X_N is a neutrosophic $\mathcal{N}$-near UP-filter of X. Since $I_N(2) = -0.8 < -0.7 = \min\{I_N(1 \cdot 2), I_N(1)\}$, we have X_N is not a neutrosophic $\mathcal{N}$-UP-filter of X.

Theorem 3.24 Every neutrosophic $\mathcal{N}$-near UP-filter of X is a neutrosophic $\mathcal{N}$-UP-subalgebra.

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-near UP-filter of X. Then for all $x, y \in X$, by (3.8), (3.9), and (3.10), we have

$$T_N(x \cdot y) \leq T_N(y) \leq \max\{T_N(x), T_N(y)\},$$
$$I_N(x \cdot y) \geq I_N(y) \geq \min\{I_N(x), I_N(y)\},$$
$$F_N(x \cdot y) \leq F_N(y) \leq \max\{F_N(x), F_N(y)\}.$$

Hence, X_N is a neutrosophic $\mathcal{N}$-UP-subalgebra of X.

The following example show that the converse of Theorem 3.24 is not true.

Example 3.25 Let $X = \{0,1,2,3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	1	2
2	0	0	0	2
3	0	0	0	0

Then $(X, \cdot, 0)$ is a UP-algebra. We define a neutrosophic $\mathcal{N}$-structure X_N over X as follows:

$$T_N(0) = -0.8,\ I_N(0) = -0.3,\ F_N(0) = -0.8,$$
$$T_N(1) = -0.6,\ I_N(1) = -0.6,\ F_N(1) = -0.8,$$
$$T_N(2) = -0.4,\ I_N(2) = -0.5,\ F_N(2) = -0.7,$$

$$T_N(3) = -0.1,\ I_N(3) = -0.7,\ F_N(3) = -0.5.$$

Hence, X_N is a neutrosophic $\mathcal{N}$-UP-subalgebra of X. Since $I_N(1 \cdot 2) = -0.6 < -0.5 = I_N(2)$, we have X_N is not a neutrosophic $\mathcal{N}$-near UP-filter of X.

By Theorems 3.18, 3.20, 3.22, and 3.24 and Examples 3.19, 3.21, 3.23, and 3.25, we have that the notion of neutrosophic $\mathcal{N}$-UP-subalgebras is a generalization of neutrosophic $\mathcal{N}$-near UP-filters, neutrosophic $\mathcal{N}$-near UP-filters is a generalization of neutrosophic $\mathcal{N}$-UP-filters, neutrosophic $\mathcal{N}$-UP-filters is a generalization of neutrosophic $\mathcal{N}$-UP-ideals, and neutrosophic $\mathcal{N}$-UP-ideals is a generalization of neutrosophic $\mathcal{N}$-strongly UP-ideals. Moreover, by Theorem 3.17, we obtain that neutrosophic $\mathcal{N}$-strongly UP-ideals and constant neutrosophic $\mathcal{N}$-structures coincide.

Theorem 3.26 If X_N is a neutrosophic $\mathcal{N}$-UP-subalgebra of X satisfying the following condition:

$$(\forall x, y \in X)\left(x \cdot y \neq 0 \Rightarrow \begin{cases} T_N(x) \leq T_N(y) \\ I_N(x) \geq I_N(y) \\ F_N(x) \leq F_N(y) \end{cases} \right), \tag{3.20}$$

then X_N is a neutrosophic $\mathcal{N}$-near UP-filter of X.

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-UP-subalgebra of X satisfying the condition (3.20). By Theorem 3.16, we have X_N satisfies the conditions (3.5), (3.6), and (3.7). Next, let $x, y \in X$.

Case 1: $x \cdot y = 0$. Then, by (3.5), (3.6), and (3.7), we have

$$T_N(x \cdot y) = T_N(0) \leq T_N(y),\ I_N(x \cdot y) = I_N(0) \geq I_N(y),\ F_N(x \cdot y) = F_N(0) \leq F_N(y).$$

Case 2: $x \cdot y \neq 0$. Then, by (3.2), (3.3), (3.4), and (3.20), we have

$$T_N(x \cdot y) \leq \max\{T_N(x), T_N(y)\} = T_N(y),\ I_N(x \cdot y) \geq \min\{I_N(x), I_N(y)\} = I_N(y),$$
$$F_N(x \cdot y) \leq \max\{F_N(x), F_N(y)\} = F_N(y).$$

Hence, X_N is a neutrosophic $\mathcal{N}$-near UP-filter of X.

Theorem 3.27 If X_N is a neutrosophic $\mathcal{N}$-near UP-filter of X satisfying the following condition:

$$T_N = I_N = F_N, \tag{3.21}$$

then X_N is a neutrosophic $\mathcal{N}$-UP-filter of X.

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-near UP-filter of X satisfying the condition (3.21). Then X_N satisfies the conditions (3.5), (3.6), and (3.7). Next, let $x, y \in X$. Then, by (3.8), (3.9), and (3.21), we have

$$\max\{T_N(x \cdot y), T_N(x)\} = \max\{I_N(x \cdot y), T_N(x)\} \geq \max\{I_N(y), T_N(x)\} = \max\{T_N(y), T_N(x)\} \geq T_N(y),$$
$$\min\{I_N(x \cdot y), I_N(x)\} = \min\{T_N(x \cdot y), I_N(x)\} \leq \min\{T_N(y), I_N(x)\} = \min\{I_N(y), I_N(x)\} \leq I_N(y),$$
$$\max\{F_N(x \cdot y), F_N(x)\} = \max\{I_N(x \cdot y), F_N(x)\} \geq \max\{I_N(y), F_N(x)\} = \max\{F_N(y), F_N(x)\} \geq F_N(y).$$

Hence, X_N is a neutrosophic $\mathcal{N}$-UP-filter of X.

Theorem 3.28 If X_N is a neutrosophic $\mathcal{N}$-UP-filter of X satisfying the following condition:

$$(\forall x, y, z \in X)\begin{pmatrix} T_N(y \cdot (x \cdot z)) & = T_N(x \cdot (y \cdot z)) \\ I_N(y \cdot (x \cdot z)) & = I_N(x \cdot (y \cdot z)) \\ F_N(y \cdot (x \cdot z)) & = F_N(x \cdot (y \cdot z)) \end{pmatrix}, \tag{3.22}$$

then X_N is a neutrosophic $\mathcal{N}$-UP-ideal of X.

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-UP-filter of X satisfying the condition (3.22). Then X_N satisfies the conditions (3.5), (3.6), and (3.7). Next, let $x,y,z \in X$. Then, by (3.11), (3.12), (3.13), and (3.22), we have

$$T_N(x\cdot z) \leq \max\{T_N(y\cdot(x\cdot z)), T_N(y)\} = \max\{T_N(x\cdot(y\cdot z)), T_N(y)\}, T_N$$
$$I_N(x\cdot z) \geq \min\{I_N(y\cdot(x\cdot z)), I_N(y)\} = \min\{I_N(x\cdot(y\cdot z)), I_N(y)\}, I_N$$
$$F_N(x\cdot z) \leq \max\{F_N(y\cdot(x\cdot z)), F_N(y)\} = \max\{F_N(x\cdot(y\cdot z)), F_N(y)\}.F_N$$

Hence, X_N is a neutrosophic $\mathcal{N}$-UP-ideal of X.

Theorem 3.29 If X_N is a neutrosophic $\mathcal{N}$-structure over X satisfying the following condition:

$$(\forall x,y,z \in X)\left(z \leq x\cdot y \Rightarrow \begin{cases} T_N(z) \leq \max\{T_N(x), T_N(y)\} \\ I_N(z) \geq \min\{I_N(x), I_N(y)\} \\ F_N(z) \leq \max\{F_N(x), F_N(y)\} \end{cases} \right), \tag{3.23}$$

then X_N is a neutrosophic $\mathcal{N}$-UP-subalgebra of X.

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-structure over X satisfying the condition (3.23). Let $x,y \in X$. By Proposition 2.5 (1), we have $(x\cdot y)\cdot(x\cdot y) = 0$, that is, $x\cdot y \leq x\cdot y$. It follows from (3.23) that

$$T_N(x\cdot y) \leq \max\{T_N(x), T_N(y)\},\ I_N(x\cdot y) \geq \min\{I_N(x), I_N(y)\},\ F_N(x\cdot y) \leq \max\{F_N(x), F_N(y)\}.$$

Hence, X_N is a neutrosophic $\mathcal{N}$-UP-subalgebra of X.

Theorem 3.30 If X_N is a neutrosophic $\mathcal{N}$-structure over X satisfying the following condition:

$$(\forall x,y,z \in X)\left(z \leq x\cdot y \Rightarrow \begin{cases} T_N(z) \leq T_N(y) \\ I_N(z) \geq I_N(y) \\ F_N(z) \leq F_N(y) \end{cases} \right), \tag{3.24}$$

then X_N is a neutrosophic $\mathcal{N}$-near UP-filter of X.

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-structure over X satisfying the condition (3.24). Let $x \in X$. By (UP-2) and Proposition 2.5 (1), we have $0\cdot(x\cdot x) = 0$, that is, $0 \leq x\cdot x$. It follows from (3.24) that $T_N(0) \leq T_N(x), I_N(0) \geq I_N(x)$, and $F_N(0) \leq F_N(x)$. Next, let $x,y \in X$. By Proposition 2.5 (1), we have $(x\cdot y)\cdot(x\cdot y) = 0$, that is, $x\cdot y \leq x\cdot y$. It follows from (3.24) that $T_N(x\cdot y) \leq T_N(y), I_N(x\cdot y) \geq I_N(y)$, and $F_N(x\cdot y) \leq F_N(y)$. Hence, X_N is a neutrosophic $\mathcal{N}$-near UP-filter of X.

Theorem 3.31 If X_N is a neutrosophic $\mathcal{N}$-structure over X satisfying the following condition:

$$(\forall x,y,z \in X)\left(z \leq x\cdot y \Rightarrow \begin{cases} T_N(y) \leq \max\{T_N(z), T_N(x)\} \\ I_N(y) \geq \min\{I_N(z), I_N(x)\} \\ F_N(y) \leq \max\{F_N(z), F_N(x)\} \end{cases} \right), \tag{3.25}$$

then X_N is a neutrosophic $\mathcal{N}$-UP-filter of X.

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-structure over X satisfying the condition (3.25). Let $x \in X$. By (UP-3), we have $x\cdot(x\cdot 0) = 0$, that is, $x \leq x\cdot 0$. It follows from (3.25) that

$$T_N(0) \leq \max\{T_N(x), T_N(x)\} = T_N(x),\ I_N(0) \geq \min\{I_N(x), I_N(x)\} = I_N(x),$$
$$F_N(0) \leq \max\{F_N(x), F_N(x)\} = F_N(x).$$

Next, let $x, y \in X$. By Proposition 2.5 (1), we have $(x \cdot y) \cdot (x \cdot y) = 0$, that is, $x \cdot y \leq x \cdot y$. It follows from (3.25) that

$$T_N(y) \leq \max\{T_N(x \cdot y), T_N(x)\},\ I_N(y) \geq \min\{I_N(x \cdot y), I_N(x)\},\ F_N(y) \leq \max\{F_N(x \cdot y), F_N(x)\}.$$

Hence, X_N is a neutrosophic $\mathcal{N}$-UP-filter of X.

Theorem 3.32 If X_N is a neutrosophic $\mathcal{N}$-structure over X satisfying the following condition:

$$(\forall a, x, y, z \in X)\left(a \leq x \cdot (y \cdot z) \Rightarrow \begin{cases} T_N(x \cdot z) \leq \max\{T_N(a), T_N(y)\} \\ I_N(x \cdot z) \geq \min\{I_N(a), I_N(y)\} \\ F_N(x \cdot z) \leq \max\{F_N(a), F_N(y)\} \end{cases} \right), \tag{3.26}$$

then X_N is a neutrosophic $\mathcal{N}$-UP-ideal of X.

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-structure over X satisfying the condition (3.26). Let $x \in X$. By (UP-3), we have $x \cdot (0 \cdot (x \cdot 0) = 0$, that is, $x \leq 0 \cdot (x \cdot 0)$. It follows from (3.26) and (UP-2) that

$$T_N(0) = T_N(0 \cdot 0) \leq \max\{T_N(x), T_N(x)\} = T_N(x),\ I_N(0) = I_N(0 \cdot 0) \geq \min\{I_N(x), I_N(x)\} = I_N(x),$$
$$F_N(0) = F_N(0 \cdot 0) \leq \max\{F_N(x), F_N(x)\} = F_N(x).$$

Next, let $x, y, z \in X$. By Proposition 2.5 (1), we have $(x \cdot (y \cdot z)) \cdot (x \cdot (y \cdot z)) = 0$, that is, $x \cdot (y \cdot z) \leq x \cdot (y \cdot z)$. It follows from (3.26) that

$$T_N(x \cdot z) \leq \max\{T_N(x \cdot (y \cdot z)), T_N(y)\},\ I_N(x \cdot z) \geq \min\{I_N(x \cdot (y \cdot z)), I_N(y)\},$$
$$F_N(x \cdot z) \leq \max\{F_N(x \cdot (y \cdot z)), F_N(y)\}.$$

Hence, X_N is a neutrosophic $\mathcal{N}$-UP-ideal of X.

For any fixed numbers $\alpha^-, \alpha^+, \beta^-, \beta^+, \gamma^-, \gamma^+ \in [-1, 0]$ such that $\alpha^- < \alpha^+, \beta^- < \beta^+, \gamma^- < \gamma^+$ and a nonempty subset G of X, a neutrosophic $\mathcal{N}$-structure $X_N^G[{}^{\alpha^-, \beta^+, \gamma^-}_{\alpha^+, \beta^-, \gamma^+}] = (X, T_N^G[{}^{\alpha^-}_{\alpha^+}], I_N^G[{}^{\beta^+}_{\beta^-}], F_N^G[{}^{\gamma^-}_{\gamma^+}])$ over X where $T_N^G[{}^{\alpha^-}_{\alpha^+}], I_N^G[{}^{\beta^+}_{\beta^-}]$, and $F_N^G[{}^{\gamma^-}_{\gamma^+}]$ are $\mathcal{N}$-functions on X which are given as follows:

$$T_N^G[{}^{\alpha^-}_{\alpha^+}](x) = \begin{cases} \alpha^- & \text{if } x \in G, \\ \alpha^+ & \text{otherwise,} \end{cases} \quad I_N^G[{}^{\beta^+}_{\beta^-}](x) = \begin{cases} \beta^+ & \text{if } x \in G, \\ \beta^- & \text{otherwise,} \end{cases} \quad F_N^G[{}^{\gamma^-}_{\gamma^+}](x) = \begin{cases} \gamma^- & \text{if } x \in G, \\ \gamma^+ & \text{otherwise.} \end{cases}$$

Lemma 3.33 If the constant 0 of X is in a nonempty subset G of X, then a neutrosophic $\mathcal{N}$-structure $X_N^G[{}^{\alpha^-, \beta^+, \gamma^-}_{\alpha^+, \beta^-, \gamma^+}]$ over X satisfies the conditions (3.5), (3.6), and (3.7).

Proof. If $0 \in G$, then $T_N^G[{}^{\alpha^-}_{\alpha^+}](0) = \alpha^-, I_N^G[{}^{\beta^+}_{\beta^-}](0) = \beta^+, F_N^G[{}^{\gamma^-}_{\gamma^+}](0) = \gamma^-$. Thus

$$(\forall x \in X)\begin{pmatrix} T_N^G[{}^{\alpha^-}_{\alpha^+}](0) = \alpha^- \leq T_N^G[{}^{\alpha^-}_{\alpha^+}](x) \\ I_N^G[{}^{\beta^+}_{\beta^-}](0) = \beta^+ \geq I_N^G[{}^{\beta^+}_{\beta^-}](x) \\ F_N^G[{}^{\gamma^-}_{\gamma^+}](0) = \gamma^- \leq F_N^G[{}^{\gamma^-}_{\gamma^+}](x) \end{pmatrix}.$$

Hence, $X_N^G[{}^{\alpha^-, \beta^+, \gamma^-}_{\alpha^+, \beta^-, \gamma^+}]$ satisfies the conditions (3.5), (3.6), and (3.7).

Lemma 3.34 If a neutrosophic $\mathcal{N}$-structure $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ over X satisfies the condition (3.5) (resp., (3.6), (3.7)), then the constant 0 of X is in a nonempty subset G of X.

Proof. Assume that the neutrosophic $\mathcal{N}$-structure $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ over X satisfies the condition (3.5). Then $T_N^G[^{\alpha^-}_{\alpha^+}](0) \leq T_N^G[^{\alpha^-}_{\alpha^+}](x)$ for all $x \in X$. Since G is nonempty, there exists $g \in G$. Thus $T_N^G[^{\alpha^-}_{\alpha^+}](g) = \alpha^-$, so $T_N^G[^{\alpha^-}_{\alpha^+}](0) \leq T_N^G[^{\alpha^-}_{\alpha^+}](g) = \alpha^- \leq T_N^G[^{\alpha^-}_{\alpha^+}](0)$, that is, $T_N^G[^{\alpha^-}_{\alpha^+}](0) = \alpha^-$. Hence, $0 \in G$.

Theorem 3.35 A neutrosophic $\mathcal{N}$-structure $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ over X is a neutrosophic $\mathcal{N}$-UP-subalgebra of X if and only if a nonempty subset G of X is a UP-subalgebra of X.

Proof. Assume that $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ is a neutrosophic $\mathcal{N}$-UP-subalgebra of X. Let $x, y \in G$. Then $T_N^G[^{\alpha^-}_{\alpha^+}](x) = \alpha^- = T_N^G[^{\alpha^-}_{\alpha^+}](y)$. Thus, by (3.2), we have

$$T_N^G[^{\alpha^-}_{\alpha^+}](x \cdot y) \leq \max\{T_N^G[^{\alpha^-}_{\alpha^+}](x), T_N^G[^{\alpha^-}_{\alpha^+}](y)\} = \alpha^- \leq T_N^G[^{\alpha^-}_{\alpha^+}](x \cdot y)$$

and so $T_N^G[^{\alpha^-}_{\alpha^+}](x \cdot y) = \alpha^-$. Thus $x \cdot y \in G$. Hence, G is a UP-subalgebra of X.

Conversely, assume that G is a UP-subalgebra of X. Let $x, y \in X$.

Case 1: $x, y \in G$. Then

$$T_N^G[^{\alpha^-}_{\alpha^+}](x) = \alpha^- = T_N^G[^{\alpha^-}_{\alpha^+}](y),\ I_N^G[^{\beta^+}_{\beta^-}](x) = \beta^+ = I_N^G[^{\beta^+}_{\beta^-}](y),\ F_N^G[^{\gamma^-}_{\gamma^+}](x) = \gamma^- = F_N^G[^{\gamma^-}_{\gamma^+}](y).$$

Thus

$$\max\{T_N^G[^{\alpha^-}_{\alpha^+}](x), T_N^G[^{\alpha^-}_{\alpha^+}](y)\} = \alpha^-,\ \min\{I_N^G[^{\beta^+}_{\beta^-}](x), I_N^G[^{\beta^+}_{\beta^-}](y)\} = \beta^+,\ \max\{F_N^G[^{\gamma^-}_{\gamma^+}](x), F_N^G[^{\gamma^-}_{\gamma^+}](y)\} = \gamma^-.$$

Since G is a UP-subalgebra of X, we have $x \cdot y \in G$ and so $T_N^G[^{\alpha^-}_{\alpha^+}](x \cdot y) = \alpha^-, I_N^G[^{\beta^+}_{\beta^-}](x \cdot y) = \beta^+$, and $F_N^G[^{\gamma^-}_{\gamma^+}](x \cdot y) = \gamma^-$. Hence,

$$T_N^G[^{\alpha^-}_{\alpha^+}](x \cdot y) = \alpha^- \leq \alpha^- = \max\{T_N^G[^{\alpha^-}_{\alpha^+}](x), T_N^G[^{\alpha^-}_{\alpha^+}](y)\},\ I_N^G[^{\beta^+}_{\beta^-}](x \cdot y) = \beta^+ \geq \beta^+ = \min\{I_N^G[^{\beta^+}_{\beta^-}](x), I_N^G[^{\beta^+}_{\beta^-}](y)\},$$

$$F_N^G[^{\gamma^-}_{\gamma^+}](x \cdot y) = \gamma^- \leq \gamma^- = \max\{F_N^G[^{\gamma^-}_{\gamma^+}](x), F_N^G[^{\gamma^-}_{\gamma^+}](y)\}.$$

Case 2: $x \notin G$ or $y \notin G$. Then

$$T_N^G[^{\alpha^-}_{\alpha^+}](x) = \alpha^+ \text{ or } T_N^G[^{\alpha^-}_{\alpha^+}](y) = \alpha^+,\ I_N^G[^{\beta^+}_{\beta^-}](x) = \beta^- \text{ or } I_N^G[^{\beta^+}_{\beta^-}](y) = \beta^-,\ F_N^G[^{\gamma^-}_{\gamma^+}](x) = \gamma^+ \text{ or } F_N^G[^{\gamma^-}_{\gamma^+}](y) = \gamma^+.$$

Thus

$$\max\{T_N^G[^{\alpha^-}_{\alpha^+}](x), T_N^G[^{\alpha^-}_{\alpha^+}](y)\} = \alpha^+,\ \min\{I_N^G[^{\beta^+}_{\beta^-}](x), I_N^G[^{\beta^+}_{\beta^-}](y)\} = \beta^-,\ \max\{F_N^G[^{\gamma^-}_{\gamma^+}](x), F_N^G[^{\gamma^-}_{\gamma^+}](y)\} = \gamma^+.$$

Therefore,

$$T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot y)\leq \alpha^+=\max\{T_N^G[^{\alpha^-}_{\alpha^+}](x),T_N^G[^{\alpha^-}_{\alpha^+}](y)\},\ I_N^G[^{\beta^+}_{\beta^-}](x\cdot y)\geq \beta^-=\min\{I_N^G[^{\beta^+}_{\beta^-}](x),I_N^G[^{\beta^+}_{\beta^-}](y)\},$$

$$F_N^G[^{\gamma^-}_{\gamma^+}](x\cdot y)\leq \gamma^+=\max\{F_N^G[^{\gamma^-}_{\gamma^+}](x),F_N^G[^{\gamma^-}_{\gamma^+}](y)\}.$$

Hence, $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ is a neutrosophic $\mathcal{N}$-UP-subalgebra of X.

Theorem 3.36 A neutrosophic $\mathcal{N}$-structure $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ over X is a neutrosophic $\mathcal{N}$-near UP-filter of X if and only if a nonempty subset G of X is a near UP-filter of X.

Proof. Assume that $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ is neutrosophic $\mathcal{N}$-near UP-filter of X. Since $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ satisfies the condition (3.5), it follows from Lemma 3.34 that $0\in G$. Next, let $x\in X$ and $y\in G$. Then $T_N^G[^{\alpha^-}_{\alpha^+}](y)=\alpha^-$. Thus, by (3.8), we have $T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot y)\leq T_N^G[^{\alpha^-}_{\alpha^+}](y)=\alpha^-\leq T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot y)$

and so $T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot y)=\alpha^-$. Thus $x\cdot y\in G.$ Hence, G is a near UP-filter of X.

Conversely, assume that G is a near UP-filter of X. Since $0\in G$, it follows from Lemma 3.33 that $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ satisfies the conditions (3.5), (3.6), and (3.7). Next, let $x,y\in X$.

Case 1: $y\in G$. Then $T_N^G[^{\alpha^-}_{\alpha^+}](y)=\alpha^-, I_N^G[^{\beta^+}_{\beta^-}](y)=\beta^+$, and $F_N^G[^{\gamma^-}_{\gamma^+}](y)=\gamma^-$. Since G is a near UP-filter of X, we have $x\cdot y\in G$ and so $T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot y)=\alpha^-, I_N^G[^{\beta^+}_{\beta^-}](x\cdot y)=\beta^+$, and $F_N^G[^{\gamma^-}_{\gamma^+}](x\cdot y)=\gamma^-$. Thus

$$T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot y)=\alpha^-\leq\alpha^-=T_N^G[^{\alpha^-}_{\alpha^+}](y),\ I_N^G[^{\beta^+}_{\beta^-}](x\cdot y)=\beta^+\geq\beta^+=I_N^G[^{\beta^+}_{\beta^-}](y),$$

$$F_N^G[^{\gamma^-}_{\gamma^+}](x\cdot y)=\gamma^-\leq\gamma^-=F_N^G[^{\gamma^-}_{\gamma^+}](y).$$

Case 2: $y\notin G$. Then $T_N^G[^{\alpha^-}_{\alpha^+}](y)=\alpha^+, I_N^G[^{\beta^+}_{\beta^-}](y)=\beta^-$, and $F_N^G[^{\gamma^-}_{\gamma^+}](y)=\gamma^+$. Thus

$$T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot y)\leq\alpha^+=T_N^G[^{\alpha^-}_{\alpha^+}](y),\ I_N^G[^{\beta^+}_{\beta^-}](x\cdot y)\geq\beta^-=I_N^G[^{\beta^+}_{\beta^-}](y),\ F_N^G[^{\gamma^-}_{\gamma^+}](x\cdot y)\leq\gamma^+=F_N^G[^{\gamma^-}_{\gamma^+}](y).$$

Hence, $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ is a neutrosophic $\mathcal{N}$-near UP-filter of X.

Theorem 3.37 A neutrosophic $\mathcal{N}$-structure $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ over X is a neutrosophic $\mathcal{N}$-UP-filter of X if and only if a nonempty subset G of X is a UP-filter of X.

Proof. Assume that $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ is a neutrosophic $\mathcal{N}$-UP-filter of X. Since $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ satisfies the condition (3.5), it follows from Lemma 3.34 that $0\in G$. Next, let $x,y\in X$ be such that $x\cdot y\in G$ and $x\in G$. Then $T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot y)=\alpha^-=T_N^G[^{\alpha^-}_{\alpha^+}](x)$. Thus, by (3.11), we have

$$T_N^G[^{\alpha^-}_{\alpha^+}](y) \leq \max\{T_N^G[^{\alpha^-}_{\alpha^+}](x \cdot y), T_N^G[^{\alpha^-}_{\alpha^+}](x)\} = \alpha^- \leq T_N^G[^{\alpha^-}_{\alpha^+}](y)$$

and so $T_N^G[^{\alpha^-}_{\alpha^+}](y) = \alpha^-$. Thus $y \in G$. Hence, G is a UP-filter of X.

Conversely, assume that G is a UP-filter of X. Since $0 \in G$, it follows from Lemma 3.33 that $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ satisfies the conditions (3.5), (3.6), and (3.7). Next, let $x, y \in X$.

Case 1: $x \cdot y \in G$ and $x \in G$. Then

$$T_N^G[^{\alpha^-}_{\alpha^+}](x \cdot y) = \alpha^- = T_N^G[^{\alpha^-}_{\alpha^+}](x),\ I_N^G[^{\beta^+}_{\beta^-}](x \cdot y) = \beta^+ = I_N^G[^{\beta^+}_{\beta^-}](x),\ F_N^G[^{\gamma^-}_{\gamma^+}](x \cdot y) = \gamma^- = F_N^G[^{\gamma^-}_{\gamma^+}](x).$$

Since G is a UP-filter of X, we have $y \in G$ and so $T_N^G[^{\alpha^-}_{\alpha^+}](y) = \alpha^-, I_N^G[^{\beta^+}_{\beta^-}](y) = \beta^+$, and $F_N^G[^{\gamma^-}_{\gamma^+}](y) = \gamma^-$. Thus

$$T_N^G[^{\alpha^-}_{\alpha^+}](y) = \alpha^- \leq \alpha^- = \max\{T_N^G[^{\alpha^-}_{\alpha^+}](x \cdot y), T_N^G[^{\alpha^-}_{\alpha^+}](x)\},\ I_N^G[^{\beta^+}_{\beta^-}](y) = \beta^+ \geq \beta^+ = \min\{I_N^G[^{\beta^+}_{\beta^-}](x \cdot y), I_N^G[^{\beta^+}_{\beta^-}](x)\},$$

$$F_N^G[^{\gamma^-}_{\gamma^+}](y) = \gamma^- \leq \gamma^- = \max\{F_N^G[^{\gamma^-}_{\gamma^+}](x \cdot y), F_N^G[^{\gamma^-}_{\gamma^+}](x)\}.$$

Case 2: $x \cdot y \notin G$ or $x \notin G$. Then

$$T_N^G[^{\alpha^-}_{\alpha^+}](x \cdot y) = \alpha^+ \text{ or } T_N^G[^{\alpha^-}_{\alpha^+}](x) = \alpha^+,\ I_N^G[^{\beta^+}_{\beta^-}](x \cdot y) = \beta^- \text{ or } I_N^G[^{\beta^+}_{\beta^-}](x) = \beta^-,$$

$$F_N^G[^{\gamma^-}_{\gamma^+}](x \cdot y) = \gamma^+ \text{ or } F_N^G[^{\gamma^-}_{\gamma^+}](x) = \gamma^+.$$

Thus

$$\max\{T_N^G[^{\alpha^-}_{\alpha^+}](x \cdot y), T_N^G[^{\alpha^-}_{\alpha^+}](x)\} = \alpha^+,\ \min\{I_N^G[^{\beta^+}_{\beta^-}](x \cdot y), I_N^G[^{\beta^+}_{\beta^-}](x)\} = \beta^-,\ \max\{F_N^G[^{\gamma^-}_{\gamma^+}](x \cdot y), F_N^G[^{\gamma^-}_{\gamma^+}](x)\} = \gamma^+.$$

Therefore,

$$T_N^G[^{\alpha^-}_{\alpha^+}](y) \leq \alpha^+ = \max\{T_N^G[^{\alpha^-}_{\alpha^+}](x \cdot y), T_N^G[^{\alpha^-}_{\alpha^+}](x)\},\ I_N^G[^{\beta^+}_{\beta^-}](y) \geq \beta^- = \min\{I_N^G[^{\beta^+}_{\beta^-}](x \cdot y), I_N^G[^{\beta^+}_{\beta^-}](x)\},$$

$$F_N^G[^{\gamma^-}_{\gamma^+}](y) \leq \gamma^+ = \max\{F_N^G[^{\gamma^-}_{\gamma^+}](x \cdot y), F_N^G[^{\gamma^-}_{\gamma^+}](x)\}.$$

Hence, $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ is a neutrosophic $\mathcal{N}$-UP-filter of X.

Theorem 3.38 A neutrosophic $\mathcal{N}$-structure $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ over X is a neutrosophic $\mathcal{N}$-UP-ideal of X if and only if a nonempty subset G of X is a UP-ideal of X.

Proof. Assume that $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ is a neutrosophic $\mathcal{N}$-UP-ideal of X. Since $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ satisfies the condition (3.5), it follows from Lemma 3.34 that $0 \in G$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in G$ and $y \in G$. Then $T_N^G[^{\alpha^-}_{\alpha^+}](x \cdot (y \cdot z)) = \alpha^- = T_N^G[^{\alpha^-}_{\alpha^+}](y)$. Thus, by (3.17), we have

$$T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot z)\leq \max\{T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot(y\cdot z)),T_N^G[^{\alpha^-}_{\alpha^+}](y)\}=\alpha^- \leq T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot z)$$

and so $T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot z)=\alpha^-$. Thus $x\cdot z\in G$. Hence, G is a UP-ideal of X.

Conversely, assume that G is a UP-ideal of X. Since $0\in G$, it follows from Lemma 3.33 that $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ satisfies the conditions (3.5), (3.6), and (3.7). Next, let $x,y,z\in X$.

Case 1: $x\cdot(y\cdot z)\in G$ and $y\in G$. Then

$$T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot(y\cdot z))=\alpha^-=T_N^G[^{\alpha^-}_{\alpha^+}](y),\ I_N^G[^{\beta^+}_{\beta^-}](x\cdot(y\cdot z))=\beta^+=I_N^G[^{\beta^+}_{\beta^-}](y),\ F_N^G[^{\gamma^-}_{\gamma^+}](x\cdot(y\cdot z))=\gamma^-=F_N^G[^{\gamma^-}_{\gamma^+}](y).$$

Thus

$$\max\{T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot(y\cdot z)),T_N^G[^{\alpha^-}_{\alpha^+}](y)\}=\alpha^-,\ \min\{I_N^G[^{\beta^+}_{\beta^-}](x\cdot(y\cdot z)),I_N^G[^{\beta^+}_{\beta^-}](y)\}=\beta^+,$$

$$\max\{F_N^G[^{\gamma^-}_{\gamma^+}](x\cdot(y\cdot z)),F_N^G[^{\gamma^-}_{\gamma^+}](y)\}=\gamma^-.$$

Since G is a UP-ideal of X, we have $x\cdot z\in G$ and so $T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot z)=\alpha^-, I_N^G[^{\beta^+}_{\beta^-}](x\cdot z)=\beta^+$, and $F_N^G[^{\gamma^-}_{\gamma^+}](x\cdot z)=\gamma^-$. Thus

$$T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot z)=\alpha^-\leq\alpha^-=\max\{T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot(y\cdot z)),T_N^G[^{\alpha^-}_{\alpha^+}](y)\},$$

$$I_N^G[^{\beta^+}_{\beta^-}](x\cdot z)=\beta^+\geq\beta^+=\min\{I_N^G[^{\beta^+}_{\beta^-}](x\cdot(y\cdot z)),I_N^G[^{\beta^+}_{\beta^-}](y)\},$$

$$F_N^G[^{\gamma^-}_{\gamma^+}](x\cdot z)=\gamma^-\leq\gamma^-=\max\{F_N^G[^{\gamma^-}_{\gamma^+}](x\cdot(y\cdot z)),F_N^G[^{\gamma^-}_{\gamma^+}](y)\}.$$

Case 2: $x\cdot(y\cdot z)\notin G$ or $y\notin G$. Then

$$T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot(y\cdot z))=\alpha^+ \text{ or } T_N^G[^{\alpha^-}_{\alpha^+}](y)=\alpha^+,\ I_N^G[^{\beta^+}_{\beta^-}](x\cdot(y\cdot z))=\beta^- \text{ or } I_N^G[^{\beta^+}_{\beta^-}](y)=\beta^-,$$

$$F_N^G[^{\gamma^-}_{\gamma^+}](x\cdot(y\cdot z))=\gamma^+ \text{ or } F_N^G[^{\gamma^-}_{\gamma^+}](y)=\gamma^+.$$

Thus

$$\max\{T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot(y\cdot z)),T_N^G[^{\alpha^-}_{\alpha^+}](y)\}=\alpha^+,\ \min\{I_N^G[^{\beta^+}_{\beta^-}](x\cdot(y\cdot z)),I_N^G[^{\beta^+}_{\beta^-}](y)\}=\beta^-,$$

$$\max\{F_N^G[^{\gamma^-}_{\gamma^+}](x\cdot(y\cdot z)),F_N^G[^{\gamma^-}_{\gamma^+}](y)\}=\gamma^+.$$

Therefore,

$$T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot z)\leq\alpha^+=\max\{T_N^G[^{\alpha^-}_{\alpha^+}](x\cdot(y\cdot z)),T_N^G[^{\alpha^-}_{\alpha^+}](y)\},$$

$$I_N^G[^{\beta^+}_{\beta^-}](x\cdot z)\geq\beta^-=\min\{I_N^G[^{\beta^+}_{\beta^-}](x\cdot(y\cdot z)),I_N^G[^{\beta^+}_{\beta^-}](y)\},$$

$$F_N^G[^{\gamma^-}_{\gamma^+}](x\cdot z)\leq\gamma^+=\max\{F_N^G[^{\gamma^-}_{\gamma^+}](x\cdot(y\cdot z)),F_N^G[^{\gamma^-}_{\gamma^+}](y)\}.$$

Hence, $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ is a neutrosophic $\mathcal{N}$-UP-ideal of X.

Theorem 3.39 A neutrosophic $\mathcal{N}$-structure $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ over X is a neutrosophic $\mathcal{N}$-strongly UP-ideal of X if and only if a nonempty subset G of X is a strongly UP-ideal of X.

Proof. Assume that $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ is a neutrosophic $\mathcal{N}$-strongly UP-ideal of X. By Theorem 3.17, we have $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ is constant, that is, $T_N^G[^{\alpha^-}_{\alpha^+}]$ is constant. Since G is nonempty, we have $T_N^G[^{\alpha^-}_{\alpha^+}](x)=\alpha^-$ for all $x\in X$. Thus $G=X$. Hence, G is a strongly UP-ideal of X.

Conversely, assume that G is a strongly UP-ideal of X. Then $G=X$, so

$$(\forall x\in X)\begin{pmatrix} T_N^G[^{\alpha^-}_{\alpha^+}](x) & =\alpha^- \\ I_N^G[^{\beta^+}_{\beta^-}](x) & =\beta^+ \\ F_N^G[^{\gamma^-}_{\gamma^+}](x) & =\gamma^- \end{pmatrix}.$$

Thus $T_N^G[^{\alpha^-}_{\alpha^+}], I_N^G[^{\beta^+}_{\beta^-}]$, and $F_N^G[^{\gamma^-}_{\gamma^+}]$ are constant, that is, $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ is constant. By Theorem 3.17, we have $X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]$ is a neutrosophic $\mathcal{N}$-strongly UP-ideal of X.

4. Level subsets of a neutrosophic $\mathcal{N}$-structure

In this section, we discuss the relationships among neutrosophic $\mathcal{N}$-UP-subalgebras (resp., neutrosophic $\mathcal{N}$-near UP-filters, neutrosophic $\mathcal{N}$-UP-filters, neutrosophic $\mathcal{N}$-UP-ideals, neutrosophic $\mathcal{N}$-strongly UP-ideals) of UP-algebras and their level subsets.

Definition 4.1 [34] Let f be an $\mathcal{N}$-function on a nonempty set X. For any $t\in[-1,0]$, the sets

$$U(f;t)=\{x\in X\mid f(x)\geq t\},\ L(f;t)=\{x\in X\mid f(x)\leq t\},\ E(f;t)=\{x\in X\mid f(x)=t\}$$

are called an *upper* t*-level subset*, a *lower* t*-level subset*, and an *equal* t*-level subset* of f, respectively.

Theorem 4.2 A neutrosophic $\mathcal{N}$-structure X_N over X is a neutrosophic $\mathcal{N}$-UP-subalgebra of X if and only if for all $\alpha,\beta,\gamma\in[-1,0]$, the sets $L(T_N;\alpha),U(I_N;\beta)$, and $L(F_N;\gamma)$ are UP-subalgebras of X if $L(T_N;\alpha),U(I_N;\beta)$, and $L(F_N;\gamma)$ are nonempty.

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-UP-subalgebra of X. Let $\alpha,\beta,\gamma\in[-1,0]$ be such that $L(T_N;\alpha),U(I_N;\beta)$, and $L(F_N;\gamma)$ are nonempty.

Let $x,y\in L(T_N;\alpha)$. Then $T_N(x)\leq\alpha$ and $T_N(y)\leq\alpha$, so α is an upper bound of $\{T_N(x),T_N(y)\}$. By (3.2), we have $T_N(x\cdot y)\leq\max\{T_N(x),T_N(y)\}\leq\alpha$. Thus $x\cdot y\in L(T_N;\alpha)$.

Let $x,y\in U(I_N;\beta)$. Then $I_N(x)\geq\beta$ and $I_N(y)\geq\beta$, so β is a lower bound of $\{I_N(x),I_N(y)\}$. By (3.3), we have $I_N(x\cdot y)\geq\min\{I_N(x),I_N(y)\}\geq\beta$. Thus $x\cdot y\in U(I_N;\beta)$.

Let $x,y\in L(F_N;\gamma)$. Then $F_N(x)\leq\gamma$ and $F_N(y)\leq\gamma$, so γ is an upper bound of $\{F_N(x),F_N(y)\}$. By (3.4), we have $F_N(x\cdot y)\leq\max\{F_N(x),F_N(y)\}\leq\gamma$. Thus $x\cdot y\in L(F_N;\gamma)$.

Hence, $L(T_N;\alpha),U(I_N;\beta)$, and $L(F_N;\gamma)$ are UP-subalgebras of X.

Conversely, assume that for all $\alpha,\beta,\gamma \in [-1,0]$, the sets $L(T_N;\alpha), U(I_N;\beta)$, and $L(F_N;\gamma)$ are UP-subalgebras of X if $L(T_N;\alpha), U(I_N;\beta)$, and $L(F_N;\gamma)$ are nonempty.

Let $x, y \in X$. Then $T_N(x), T_N(y) \in [-1,0]$. Choose $\alpha = \max\{T_N(x), T_N(y)\}$. Thus $T_N(x) \leq \alpha$ and $T_N(y) \leq \alpha$, so $x, y \in L(T_N;\alpha) \neq \emptyset$. By assumption, we have $L(T_N;\alpha)$ is a UP-subalgebra of X and so $x \cdot y \in L(T_N;\alpha)$. Thus $T_N(x \cdot y) \leq \alpha = \max\{T_N(x), T_N(y)\}$.

Let $x, y \in X$. Then $I_N(x), I_N(y) \in [-1,0]$. Choose $\beta = \min\{I_N(x), I_N(y)\}$. Thus $I_N(x) \geq \beta$ and $I_N(y) \geq \beta$, so $x, y \in U(I_N;\beta) \neq \emptyset$. By assumption, we have $U(I_N;\beta)$ is a UP-subalgebra of X and so $x \cdot y \in U(I_N;\beta)$. Thus $I_N(x \cdot y) \geq \beta = \min\{I_N(x), I_N(y)\}$.

Let $x, y \in X$. Then $F_N(x), F_N(y) \in [-1,0]$. Choose $\gamma = \max\{F_N(x), F_N(y)\}$. Thus $F_N(x) \leq \gamma$ and $F_N(y) \leq \gamma$, so $x, y \in L(F_N;\gamma) \neq \emptyset$. By assumption, we have $L(F_N;\gamma)$ is a UP-subalgebra of X and so $x \cdot y \in L(F_N;\gamma)$. Thus $F_N(x \cdot y) \leq \gamma = \max\{F_N(x), F_N(y)\}$.

Therefore, X_N is a neutrosophic $\mathcal{N}$-UP-subalgebra of X.

Theorem 4.3 A neutrosophic $\mathcal{N}$-structure X_N over X is a neutrosophic $\mathcal{N}$-near UP-filter of X if and only if for all $\alpha,\beta,\gamma \in [-1,0]$, the sets $L(T_N;\alpha), U(I_N;\beta)$, and $L(F_N;\gamma)$ are near UP-filters of X if $L(T_N;\alpha), U(I_N;\beta)$, and $L(F_N;\gamma)$ are nonempty.

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-near UP-filter of X. Let $\alpha,\beta,\gamma \in [-1,0]$ be such that $L(T_N;\alpha), U(I_N;\beta)$, and $L(F_N;\gamma)$ are nonempty.

Let $x \in L(T_N;\alpha)$. Then $T_N(x) \leq \alpha$. By (3.5), we have $T_N(0) \leq T_N(x) \leq \alpha$. Thus $0 \in L(T_N;\alpha)$. Next, let $x \in X$ and $y \in L(T_N;\alpha)$. Then $T_N(y) \leq \alpha$. By (3.8), we have $T_N(x \cdot y) \leq T_N(y) \leq \alpha$. Thus $x \cdot y \in L(T_N;\alpha)$.

Let $x \in U(I_N;\beta)$. Then $I_N(x) \geq \beta$. By (3.6), we have $I_N(0) \geq I_N(x) \geq \beta$. Thus $0 \in U(I_N;\beta)$. Next, let $x \in X$ and $y \in U(I_N;\beta)$. Then $I_N(y) \geq \beta$. By (3.9), we have $I_N(x \cdot y) \geq I_N(y) \geq \beta$. Thus $x \cdot y \in U(I_N;\beta)$.

Let $x \in L(F_N;\gamma)$. Then $F_N(x) \leq \gamma$. By (3.7), we have $F_N(0) \leq F_N(x) \leq \gamma$. Thus $0 \in L(F_N;\gamma)$. Next, let $x \in X$ and $y \in L(F_N;\gamma)$. Then $F_N(y) \leq \gamma$. By (3.10), we have $F_N(x \cdot y) \leq F_N(y) \leq \gamma$. Thus $x \cdot y \in L(F_N;\gamma)$.

Hence, $L(T_N;\alpha), U(I_N;\beta)$, and $L(F_N;\gamma)$ are near UP-filters of X.

Conversely, assume that for all $\alpha,\beta,\gamma \in [-1,0]$, the sets $L(T_N;\alpha), U(I_N;\beta)$, and $L(F_N;\gamma)$ are near UP-filters of X if $L(T_N;\alpha), U(I_N;\beta)$, and $L(F_N;\gamma)$ are nonempty.

Let $x \in X$. Then $T_N(x) \in [-1,0]$. Choose $\alpha = T_N(x)$. Thus $T_N(x) \leq \alpha$, so $x \in L(T_N;\alpha) \neq \emptyset$. By assumption, we have $L(T_N;\alpha)$ is a near UP-filter of X and so $0 \in L(T_N;\alpha)$. Thus $T_N(0) \leq \alpha = T_N(x)$. Next, let $x, y \in X$. Then $T_N(y) \in [-1,0]$. Choose $\alpha = T_N(y)$. Thus $T_N(y) \leq \alpha$, so $y \in L(T_N;\alpha) \neq \emptyset$. By assumption, we have $L(T_N;\alpha)$ is a near UP-filter of X and so $x \cdot y \in L(T_N;\alpha)$. Thus $T_N(x \cdot y) \leq \alpha = T_N(y)$.

Let $x \in X$. Then $I_N(x) \in [-1,0]$. Choose $\beta = I_N(x)$. Thus $I_N(x) \geq \beta$, so $x \in U(I_N;\beta) \neq \emptyset$. By assumption, we have $U(I_N;\beta)$ is a near UP-filter of X and so $0 \in U(I_N;\beta)$. Thus $I_N(0) \geq \beta = I_N(x)$. Next, let $x, y \in X$. Then $I_N(y) \in [-1,0]$. Choose $\beta = I_N(y)$. Thus $I_N(y) \geq \beta$, so $y \in U(I_N;\beta) \neq \emptyset$. By assumption, we have $U(I_N;\beta)$ is a near UP-filter of X and so $x \cdot y \in U(I_N;\beta)$. Thus $I_N(x \cdot y) \geq \beta = I_N(y)$.

Let $x \in X$. Then $F_N(x) \in [-1,0]$. Choose $\gamma = F_N(x)$. Thus $F_N(x) \leq \gamma$, so $x \in L(F_N;\gamma) \neq \emptyset$. By assumption, we have $L(F_N;\gamma)$ is a near UP-filter of X and so $0 \in L(F_N;\gamma)$. Thus

$F_N(0) \leq \gamma = F_N(x)$. Next, let $x, y \in X$. Then $F_N(y) \in [-1,0]$. Choose $\gamma = F_N(y)$. Thus $F_N(y) \leq \gamma$, so $y \in L(F_N;\gamma) \neq \varnothing$. By assumption, we have $L(F_N;\gamma)$ is a near UP-filter of X and so $x \cdot y \in L(F_N;\gamma)$. Thus $F_N(x \cdot y) \leq \gamma = F_N(y)$.

Therefore, X_N is a neutrosophic $\mathcal{N}$-near UP-filter of X .

Theorem 4.4 A neutrosophic $\mathcal{N}$-structure X_N over X is a neutrosophic $\mathcal{N}$-UP-filter of X if and only if for all $\alpha,\beta,\gamma \in [-1,0]$, the sets $L(T_N;\alpha), U(I_N;\beta)$, and $L(F_N;\gamma)$ are UP-filters of X if $L(T_N;\alpha), U(I_N;\beta)$, and $L(F_N;\gamma)$ are nonempty.

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-UP-filter of X . Let $\alpha,\beta,\gamma \in [-1,0]$ be such that $L(T_N;\alpha), U(I_N;\beta)$, and $L(F_N;\gamma)$ are nonempty.

Let $x \in L(T_N;\alpha)$. Then $T_N(x) \leq \alpha$. By (3.5), we have $T_N(0) \leq T_N(x) \leq \alpha$. Thus $0 \in L(T_N;\alpha)$. Next, let $x, y \in X$ be such that $x \cdot y \in L(T_N;\alpha)$ and $x \in L(T_N;\alpha)$. Then $T_N(x \cdot y) \leq \alpha$ and $T_N(x) \leq \alpha$, so α is an upper bound of $\{T_N(x \cdot y), T_N(x)\}$. By (3.11), we have $T_N(y) \leq \max\{T_N(x \cdot y), T_N(x)\} \leq \alpha$. Thus $y \in L(T_N;\alpha)$.

Let $x \in U(I_N;\beta)$. Then $I_N(x) \geq \beta$. By (3.5), we have $I_N(0) \geq I_N(x) \geq \beta$. Thus $0 \in U(I_N;\beta)$. Next, let $x, y \in X$ be such that $x \cdot y \in U(I_N;\beta)$ and $x \in U(I_N;\beta)$. Then $I_N(x \cdot y) \geq \beta$ and $I_N(x) \geq \beta$, so β is a lower bound of $\{I_N(x \cdot y), I_N(x)\}$. By (3.12), we have $I_N(y) \geq \min\{I_N(x \cdot y), I_N(x)\} \geq \beta$ Thus $y \in U(I_N;\beta)$.

Let $x \in L(F_N;\gamma)$. Then $F_N(x) \leq \gamma$. By (3.5), we have $F_N(0) \leq F_N(x) \leq \gamma$. Thus $0 \in L(F_N;\gamma)$. Next, let $x, y \in X$ be such that $x \cdot y \in L(F_N;\gamma)$ and $x \in L(F_N;\gamma)$. Next, let $x, y \in L(F_N;\gamma)$ and $x \in L(F_N;\gamma)$. Then $F_N(x \cdot y) \leq \gamma$ and $F_N(x) \leq \gamma$, so γ is an upper bound of $\{F_N(x \cdot y), F_N(x)\}$. By (3.13), we have $F_N(y) \leq \max\{F_N(x \cdot y), F_N(x)\} \leq \gamma$. Thus $y \in L(F_N;\gamma)$.

Hence, $L(T_N;\alpha), U(I_N;\beta)$, and $L(F_N;\gamma)$ are UP-filters of X .

Conversely, assume that for all $\alpha,\beta,\gamma \in [-1,0]$, the sets $L(T_N;\alpha), U(I_N;\beta)$, and $L(F_N;\gamma)$ are UP-filters of X if $L(T_N;\alpha), U(I_N;\beta)$, and $L(F_N;\gamma)$ are nonempty.

Let $x \in X$. Then $T_N(x) \in [-1,0]$. Choose $\alpha = T_N(x)$. Thus $T_N(x) \leq \alpha$, so $x \in L(T_N;\alpha) \neq \varnothing$. By assumption, we have $L(T_N;\alpha)$ is a UP-filter of X and so $0 \in L(T_N;\alpha)$. Thus $T_N(0) \leq \alpha = T_N(x)$. Next, let $x, y \in X$. Then $T_N(x \cdot y), T_N(x) \in [-1,0]$. Choose $\alpha = \max\{T_N(x \cdot y), T_N(x)\}$. Thus $T_N(x \cdot y) \leq \alpha$ and $T_N(x) \leq \alpha$, so $x \cdot y, x \in L(T_N;\alpha) \neq \varnothing$. By assumption, we have $L(T_N;\alpha)$ is a UP-filter of X and so $y \in L(T_N;\alpha)$. Thus $T_N(y) \leq \alpha = \max\{T_N(x \cdot y), T_N(x)\}$.

Let $x \in X$. Then $I_N(x) \in [-1,0]$. Choose $\beta = I_N(x)$. Thus $I_N(x) \geq \beta$, so $x \in U(I_N;\beta) \neq \varnothing$. By assumption, we have $U(I_N;\beta)$ is a UP-filter of X and so $0 \in U(I_N;\beta)$. Thus $I_N(0) \geq \beta = I_N(x)$. Next, let $x, y \in X$. Then $I_N(x \cdot y), I_N(x) \in [-1,0]$. Choose $\beta = \min\{I_N(x \cdot y), I_N(x)\}$. Thus $I_N(x \cdot y) \geq \beta$ and $I_N(x) \geq \beta$, so $x \cdot y, x \in U(I_N;\beta) \neq \varnothing$. By assumption, we have $U(I_N;\beta)$ is a UP-filter of X and so $y \in U(I_N;\beta)$. Thus $I_N(y) \geq \beta = \min\{I_N(x \cdot y), I_N(x)\}$.

Let $x \in X$. Then $F_N(x) \in [-1,0]$. Choose $\gamma = F_N(x)$. Thus $F_N(x) \leq \gamma$, so $x \in L(F_N;\gamma) \neq \varnothing$. By assumption, we have $L(F_N;\gamma)$ is a UP-filter of X and so $0 \in L(F_N;\gamma)$. Thus $F_N(0) \leq \gamma = F_N(x)$. Next, let $x, y \in X$. Then $F_N(x \cdot y), F_N(x) \in [-1,0]$. Choose $\gamma = \max\{F_N(x \cdot y), F_N(x)\}$. Thus $F_N(x \cdot y) \leq \gamma$ and $F_N(x) \leq \gamma$, so $x \cdot y, x \in L(F_N;\gamma) \neq \varnothing$. By assumption, we have $L(F_N;\gamma)$ is a UP-filter of X and so $y \in L(F_N;\gamma)$. Thus $F_N(y) \leq \gamma = \max\{F_N(x \cdot y), F_N(x)\}$.

Therefore, X_N is a neutrosophic $\mathcal{N}$-UP-filter of X .

Theorem 4.5 A neutrosophic $\mathcal{N}$-structure X_N over X is a neutrosophic $\mathcal{N}$-UP-ideal of X if and only if for all $\alpha,\beta,\gamma \in [-1,0]$, the sets $L(T_N;\alpha), U(I_N;\beta)$, and $L(F_N;\gamma)$ are UP-ideals of X if $L(T_N;\alpha), U(I_N;\beta)$, and $L(F_N;\gamma)$ are nonempty.

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-UP-ideal of X. Let $\alpha,\beta,\gamma \in [-1,0]$ be such that $L(T_N;\alpha), U(I_N;\beta)$, and $L(F_N;\gamma)$ are nonempty.

Let $x \in L(T_N;\alpha)$. Then $T_N(x) \leq \alpha$. By (3.5), we have $T_N(0) \leq T_N(x) \leq \alpha$. Thus $0 \in L(T_N;\alpha)$. Next, let $x,y,z \in X$ be such that $x \cdot (y \cdot z) \in L(T_N;\alpha)$ and $y \in L(T_N;\alpha)$. Then $T_N(x \cdot (y \cdot z)) \leq \alpha$ and $T_N(y) \leq \alpha$, so α is an upper bound of $\{T_N(x \cdot (y \cdot z)), T_N(y)\}$. By (3.14), we have $T_N(x \cdot z) \leq \max\{T_N(x \cdot (y \cdot z)), T_N(y)\} \leq \alpha$. Thus $x \cdot z \in L(T_N;\alpha)$.

Let $x \in U(I_N;\alpha)$. Then $I_N(x) \geq \beta$. By (3.5), we have $I_N(0) \geq I_N(x) \geq \beta$. Thus $0 \in U(I_N;\beta)$. Next, let $x,y,z \in X$ be such that $x \cdot (y \cdot z) \in U(I_N;\beta)$ and $y \in U(I_N;\beta)$. Then $I_N(x \cdot (y \cdot z)) \geq \beta$ and $I_N(y) \geq \beta$, so β is a lower bound of $\{I_N(x \cdot (y \cdot z)), I_N(y)\}$. By (3.15), we have $I_N(x \cdot z) \geq \min\{I_N(x \cdot (y \cdot z)), I_N(y)\} \geq \beta$. Thus $x \cdot z \in U(I_N;\beta)$.

Let $x \in L(F_N;\gamma)$. Then $F_N(x) \leq \gamma$. By (3.5), we have $F_N(0) \leq F_N(x) \leq \gamma$. Thus $0 \in L(F_N;\gamma)$. Next, let $x,y,z \in X$ be such that $x \cdot (y \cdot z) \in L(F_N;\gamma)$ and $y \in L(F_N;\gamma)$. Then $F_N(x \cdot (y \cdot z)) \leq \gamma$ and $F_N(y) \leq \gamma$, so γ is an upper bound of $\{F_N(x \cdot (y \cdot z)), F_N(y)\}$. By (3.16), we have $F_N(x \cdot z) \leq \max\{F_N(x \cdot (y \cdot z)), F_N(y)\} \leq \gamma$. Thus $x \cdot z \in L(F_N;\gamma)$.

Hence, $L(T_N;\alpha), U(I_N;\beta)$, and $L(F_N;\gamma)$ are UP-ideals of X.

Conversely, assume that for all $\alpha,\beta,\gamma \in [-1,0]$, the sets $L(T_N;\alpha), U(I_N;\beta)$, and $L(F_N;\gamma)$ are UP-ideals of X if $L(T_N;\alpha), U(I_N;\beta)$, and $L(F_N;\gamma)$ are nonempty.

Let $x \in X$. Then $T_N(x) \in [-1,0]$. Choose $\alpha = T_N(x)$. Thus $T_N(x) \leq \alpha$, so $x \in L(T_N;\alpha) \neq \emptyset$. By assumption, we have $L(T_N;\alpha)$ is a UP-ideal of X and so $0 \in L(T_N;\alpha)$. Thus $T_N(0) \leq \alpha = T_N(x)$. Next, let $x,y,z \in X$. Then $T_N(x \cdot (y \cdot z)), T_N(y) \in [-1,0]$. Choose $\alpha = \max\{T_N(x \cdot (y \cdot z)), T_N(y)\}$. Thus $T_N(x \cdot (y \cdot z)) \leq \alpha$ and $T_N(y) \leq \alpha$, so $x \cdot (y \cdot z), y \in L(T_N;\alpha) \neq \emptyset$. By assumption, we have $L(T_N;\alpha)$ is a UP-ideal of X and so $x \cdot z \in L(T_N;\alpha)$. Thus $T_N(x \cdot z) \leq \alpha = \max\{T_N(x \cdot (y \cdot z)), T_N(y)\}$.

Let $x \in X$. Then $I_N(x) \in [-1,0]$. Choose $\beta = I_N(x)$. Thus $I_N(x) \geq \beta$, so $x \in U(I_N;\beta) \neq \emptyset$. By assumption, we have $U(I_N;\beta)$ is a UP-ideal of X and so $0 \in U(I_N;\beta)$. Thus $I_N(0) \geq \beta = I_N(x)$. Next, let $x,y,z \in X$. Then $I_N(x \cdot (y \cdot z)), I_N(y) \in [-1,0]$. Choose $\beta = \min\{I_N(x \cdot (y \cdot z)), I_N(y)\}$. Thus $I_N(x \cdot (y \cdot z)) \geq \beta$ and $I_N(y) \geq \beta$, so $x \cdot (y \cdot z), y \in U(I_N;\beta) \neq \emptyset$. By assumption, we have $U(I_N;\beta)$ is a UP-ideal of X and so $x \cdot z \in U(I_N;\beta)$. Thus $I_N(x \cdot z) \geq \beta = \min\{I_N(x \cdot (y \cdot z)), I_N(y)\}$.

Let $x \in X$. Then $F_N(x) \in [-1,0]$. Choose $\gamma = F_N(x)$. Thus $F_N(x) \leq \gamma$, so $x \in L(F_N;\gamma) \neq \emptyset$. By assumption, we have $L(F_N;\gamma)$ is a UP-ideal of X and so $0 \in L(F_N;\gamma)$. Thus $F_N(0) \leq \gamma = F_N(x)$. Next, let $x,y,z \in X$. Then $F_N(x \cdot (y \cdot z)), F_N(y) \in [-1,0]$. Choose $\gamma = \max\{F_N(x \cdot (y \cdot z)), F_N(y)\}$. Thus $F_N(x \cdot (y \cdot z)) \leq \gamma$ and $F_N(y) \leq \gamma$, so $x \cdot (y \cdot z), y \in L(F_N;\gamma) \neq \emptyset$. By assumption, we have $L(F_N;\gamma)$ is a UP-ideal of X and so $x \cdot z \in L(F_N;\gamma)$. Thus $F_N(x \cdot z) \leq \gamma = \max\{F_N(x \cdot (y \cdot z)), F_N(y)\}$.

Therefore, X_N is a neutrosophic $\mathcal{N}$-UP-ideal of X.

Theorem 4.6 A neutrosophic $\mathcal{N}$-structure X_N over X is a neutrosophic $\mathcal{N}$-strongly UP-ideal of X if and only if the sets $E(T_N;T_N(0)), E(I_N;I_N(0))$, and $E(F_N;F_N(0))$ are strongly UP-ideals of X.

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-strongly UP-ideal of X. By Theorem 3.17, we have X_N is constant, that is, T_N, I_N, and F_N are constant. Thus

$$(\forall x \in X)\begin{pmatrix} T_N(x) & = T_N(0) \\ I_N(x) & = I_N(0) \\ F_N(x) & = F_N(0) \end{pmatrix}.$$

Hence, $E(T_N;T_N(0)) = X, E(I_N;I_N(0)) = X$, and $E(F_N;F_N(0)) = X$ and so $E(T_N;T_N(0)), E(I_N;I_N(0))$, and $E(F_N;F_N(0))$ are strongly UP-ideals of X.

Conversely, assume that $E(T_N;T_N(0)), E(I_N;I_N(0))$, and $E(F_N;F_N(0))$ are strongly UP-ideals of X. Then $E(T_N;T_N(0)) = X, E(I_N;I_N(0)) = X$, $E(F_N;F_N(0)) = X$ and so

$$(\forall x \in X)\begin{pmatrix} T_N(x) & = T_N(0) \\ I_N(x) & = I_N(0) \\ F_N(x) & = F_N(0) \end{pmatrix}.$$

Thus T_N, I_N, and F_N are constant, that is X_N is constant. By Theorem 3.17, we have X_N is a neutrosophic $\mathcal{N}$-strongly UP-ideal of X.

5. Neutrosophic $\mathcal{N}$-structures of special type

In this section, we introduce the notions of special neutrosophic $\mathcal{N}$-UP-subalgebras, special neutrosophic $\mathcal{N}$-near UP-filters, special neutrosophic $\mathcal{N}$-UP-filters, special neutrosophic $\mathcal{N}$-UP-ideals, and special neutrosophic $\mathcal{N}$-strongly UP-ideals of UP-algebras, provide the necessary examples, investigate their properties, and prove their generalizations.

Definition 5.1 A neutrosophic $\mathcal{N}$-structure X_N over X is called a *special neutrosophic $\mathcal{N}$-UP-subalgebra* of X if it satisfies the following conditions:

$$(\forall x, y \in X)(T_N(x \cdot y) \geq \min\{T_N(x), T_N(y)\}), \tag{5.1}$$

$$(\forall x, y \in X)(I_N(x \cdot y) \leq \max\{I_N(x), I_N(y)\}), \tag{5.2}$$

$$(\forall x, y \in X)(F_N(x \cdot y) \geq \min\{F_N(x), F_N(y)\}). \tag{5.3}$$

Example 5.2 Let $X = \{0,1,2,3,4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	0
2	0	1	0	0	0
3	0	1	2	0	0
4	0	1	2	3	0

Then $(X, \cdot, 0)$ is a UP-algebra. We define a neutrosophic $\mathcal{N}$-structure X_N over X as follows:

$$T_N(0) = -0.2,\ I_N(0) = -0.9,\ F_N(0) = -0.2,$$
$$T_N(1) = -0.4,\ I_N(1) = -0.8,\ F_N(1) = -0.4,$$
$$T_N(2) = -0.8,\ I_N(2) = -0.7,\ F_N(2) = -0.6,$$
$$T_N(3) = -0.3,\ I_N(3) = -0.5,\ F_N(3) = -0.7,$$
$$T_N(4) = -0.8,\ I_N(4) = -0.3,\ F_N(4) = -0.8.$$

Hence, X_N is a special neutrosophic $\mathcal{N}$-UP-subalgebra of X.

Definition 5.3 A neutrosophic $\mathcal{N}$-structure X_N over X is called a *special neutrosophic $\mathcal{N}$-near UP-filter* of X if it satisfies the following conditions:

$$(\forall x \in X)(T_N(0) \geq T_N(x)), \tag{5.4}$$

$$(\forall x \in X)(I_N(0) \leq I_N(x)), \tag{5.5}$$

$$(\forall x \in X)(F_N(0) \geq F_N(x)), \tag{5.6}$$

$$(\forall x, y \in X)(T_N(x \cdot y) \geq T_N(y)), \tag{5.7}$$

$$(\forall x, y \in X)(I_N(x \cdot y) \leq I_N(y)), \tag{5.8}$$

$$(\forall x, y \in X)(F_N(x \cdot y) \geq F_N(y)). \tag{5.9}$$

Example 5.4 Let $X = \{0,1,2,3,4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	3	0
2	0	2	0	3	0
3	0	2	2	0	0
4	0	2	2	3	0

Then $(X, \cdot, 0)$ is a UP-algebra. We define a neutrosophic $\mathcal{N}$-structure X_N over X as follows:

$$T_N(0) = -0.2,\ I_N(0) = -0.8,\ F_N(0) = -0.3,$$
$$T_N(1) = -0.5,\ I_N(1) = -0.5,\ F_N(1) = -0.7,$$
$$T_N(2) = -0.4,\ I_N(2) = -0.7,\ F_N(2) = -0.4,$$
$$T_N(3) = -0.3,\ I_N(3) = -0.4,\ F_N(3) = -0.6,$$
$$T_N(4) = -0.8,\ I_N(4) = -0.2,\ F_N(4) = -0.8.$$

Hence, X_N is a special neutrosophic $\mathcal{N}$-near UP-filter of X.

Definition 5.5 A neutrosophic $\mathcal{N}$-structure X_N over X is called a *special neutrosophic $\mathcal{N}$-UP-filter* of X if it satisfies the following conditions: (5.4), (5.5), (5.6), and

$$(\forall x, y \in X)(T_N(y) \geq \min\{T_N(x \cdot y), T_N(x)\}), \tag{5.10}$$

$$(\forall x, y \in X)(I_N(y) \leq \max\{I_N(x \cdot y), I_N(x)\}), \tag{5.11}$$

$$(\forall x, y \in X)(F_N(y) \geq \min\{F_N(x \cdot y), F_N(x)\}). \tag{5.12}$$

Example 5.6 Let $X = \{0,1,2,3,4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	0
2	0	1	0	3	0
3	0	1	2	0	0
4	0	1	2	3	0

Then $(X, \cdot, 0)$ is a UP-algebra. We define a neutrosophic $\mathcal{N}$-structure X_N over X as follows:

$$T_N(0) = -0.2,\ I_N(0) = -0.8,\ F_N(0) = -0.2,$$
$$T_N(1) = -0.8,\ I_N(1) = -0.5,\ F_N(1) = -0.8,$$
$$T_N(2) = -0.6,\ I_N(2) = -0.4,\ F_N(2) = -0.5,$$
$$T_N(3) = -0.7,\ I_N(3) = -0.6,\ F_N(3) = -0.7,$$

$$T_N(4) = -0.5,\ I_N(4) = -0.7,\ F_N(4) = -0.4.$$

Hence, X_N is a special neutrosophic $\mathcal{N}$-UP-filter of X.

Definition 5.7 A neutrosophic $\mathcal{N}$-structure X_N over X is called a *special neutrosophic $\mathcal{N}$-UP-ideal* of X if it satisfies the following conditions: (5.4), (5.5), (5.6), and

$$(\forall x, y, z \in X)(T_N(x \cdot z) \geq \min\{T_N(x \cdot (y \cdot z)), T_N(y)\}), \tag{5.13}$$

$$(\forall x, y, z \in X)(I_N(x \cdot z) \leq \max\{I_N(x \cdot (y \cdot z)), I_N(y)\}), \tag{5.14}$$

$$(\forall x, y, z \in X)(F_N(x \cdot z) \geq \min\{F_N(x \cdot (y \cdot z)), F_N(y)\}). \tag{5.15}$$

Example 5.8 Let $X = \{0,1,2,3,4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	0	4
2	0	0	0	0	0
3	0	3	2	0	4
4	0	3	2	0	0

Then $(X, \cdot, 0)$ is a UP-algebra. We define a neutrosophic $\mathcal{N}$-structure X_N over X as follows:

$$T_N(0) = -0.3,\ I_N(0) = -0.8,\ F_N(0) = -0.2,$$

$$T_N(1) = -0.6,\ I_N(1) = -0.6,\ F_N(1) = -0.3,$$

$$T_N(2) = -0.8,\ I_N(2) = -0.4,\ F_N(2) = -0.8,$$

$$T_N(3) = -0.6,\ I_N(3) = -0.6,\ F_N(3) = -0.3,$$

$$T_N(4) = -0.7,\ I_N(4) = -0.5,\ F_N(4) = -0.7.$$

Hence, X_N is a special neutrosophic $\mathcal{N}$-UP-ideal of X.

Definition 5.9 A neutrosophic $\mathcal{N}$-structure X_N over X is called a *special neutrosophic $\mathcal{N}$-strongly UP-ideal* of X if it satisfies the following conditions: (5.4), (5.5), (5.6), and

$$(\forall x, y, z \in X)(T_N(x) \geq \min\{T_N((z \cdot y) \cdot (z \cdot x)), T_N(y)\}), \tag{5.16}$$

$$(\forall x, y, z \in X)(I_N(x) \leq \max\{I_N((z \cdot y) \cdot (z \cdot x)), I_N(y)\}), \tag{5.17}$$

$$(\forall x, y, z \in X)(F_N(x) \geq \min\{F_N((z \cdot y) \cdot (z \cdot x)), F_N(y)\}). \tag{5.18}$$

Example 5.10 Let $X = \{0,1,2,3,4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	0
2	0	1	0	0	4
3	0	1	2	0	4
4	0	4	2	3	0

Then $(X, \cdot, 0)$ is a UP-algebra. We define a neutrosophic $\mathcal{N}$-structure X_N over X as follows:

$$(\forall x \in X)\begin{pmatrix} T_N(x) & = -0.5 \\ I_N(x) & = -1 \\ F_N(x) & = -0.3 \end{pmatrix}.$$

Hence, X_N is a special neutrosophic $\mathcal{N}$-strongly UP-ideal X.

Theorem 5.11 Every special neutrosophic $\mathcal{N}$-UP-subalgebra of X satisfies the conditions (5.4), (5.5), and (5.6).

Proof. Assume that X_N is a special neutrosophic $\mathcal{N}$-UP-subalgebra of X. Then for all $x \in X$, by Proposition 2.5 (1), (5.1), (5.2), and (5.3), we have

$$T_N(0) = T_N(x \cdot x) \geq \min\{T_N(x), T_N(x)\} = T_N(x), \; I_N(0) = I_N(x \cdot x) \leq \max\{I_N(x), I_N(x)\} = I_N(x),$$
$$F_N(0) = F_N(x \cdot x) \geq \min\{F_N(x), F_N(x)\} = F_N(x).$$

Hence, X_N satisfies the conditions (5.4), (5.5), and (5.6).

By Lemma 3.4 (1) and (4), we have the following five theorems.

Theorem 5.12 A neutrosophic $\mathcal{N}$-structure X_N over X is a neutrosophic $\mathcal{N}$-UP-subalgebra of X if and only if $\overline{X}_N$ is a special neutrosophic $\mathcal{N}$-UP-subalgebra of X.

Theorem 5.13 A neutrosophic $\mathcal{N}$-structure X_N over X is a neutrosophic $\mathcal{N}$-near UP-filter of X if and only if $\overline{X}_N$ is a special neutrosophic $\mathcal{N}$-near UP-filter of X.

Theorem 5.14 A neutrosophic $\mathcal{N}$-structure X_N over X is a neutrosophic $\mathcal{N}$-UP-filter of X if and only if $\overline{X}_N$ is a special neutrosophic $\mathcal{N}$-UP-filter of X.

Theorem 5.15 A neutrosophic $\mathcal{N}$-structure X_N over X is a neutrosophic $\mathcal{N}$-UP-ideal of X if and only if $\overline{X}_N$ is a special neutrosophic $\mathcal{N}$-UP-ideal of X.

Theorem 5.16 A neutrosophic $\mathcal{N}$-structure X_N over X is a neutrosophic $\mathcal{N}$-strongly UP-ideal of X if and only if $\overline{X}_N$ is a special neutrosophic $\mathcal{N}$-strongly UP-ideal of X.

Theorem 5.17 A neutrosophic $\mathcal{N}$-structure X_N over X is constant if and only if it is a special neutrosophic $\mathcal{N}$-strongly UP-ideal of X.

Proof. It is straightforward by Remark 3.2 and Theorems 3.17 and 5.16.

By Remark Remark 3.2 and Theorems 5.12, 5.13, 5.14, 5.15, and 5.16, we have that the notion of special neutrosophic $\mathcal{N}$-UP-subalgebras is a generalization of special neutrosophic $\mathcal{N}$-near UP-filters, special neutrosophic $\mathcal{N}$-near UP-filters is a generalization of special neutrosophic $\mathcal{N}$-UP-filters, special neutrosophic $\mathcal{N}$-UP-filters is a generalization of special neutrosophic $\mathcal{N}$-UP-ideals, and special neutrosophic $\mathcal{N}$-UP-ideals is a generalization of special neutrosophic $\mathcal{N}$-strongly UP-ideals. Moreover, by Theorem 5.17, we obtain that special neutrosophic $\mathcal{N}$-strongly UP-ideals and constant neutrosophic $\mathcal{N}$-structures coincide.

Theorem 5.18 If X_N is a special neutrosophic $\mathcal{N}$-UP-subalgebra of X satisfying the following condition:

$$(\forall x, y \in X)\left(x \cdot y \neq 0 \Rightarrow \begin{cases} T_N(x) \geq T_N(y) \\ I_N(x) \leq I_N(y) \\ F_N(x) \geq F_N(y) \end{cases} \right), \tag{5.19}$$

then X_N is a special neutrosophic $\mathcal{N}$-near UP-filter of X.

Proof. Assume that X_N is a special neutrosophic $\mathcal{N}$-UP-subalgebra of X satisfying the condition (5.19). By Theorem 5.11, we have X_N satisfies the conditions (5.4), (5.5), and (5.6). Next, let $x, y \in X$.

Case 1: $x \cdot y = 0$. Then, by (5.4), (5.5), and (5.6), we have

$$T_N(x \cdot y) = T_N(0) \geq T_N(y), \; I_N(x \cdot y) = I_N(0) \leq I_N(y), \; F_N(x \cdot y) = F_N(0) \geq F_N(y).$$

Case 2: $x \cdot y \neq 0$. Then, by (5.1), (5.2), (5.3), and (5.19), we have

$$T_N(x\cdot y) \geq \min\{T_N(x), T_N(y)\} = T_N(y),\ I_N(x\cdot y) \leq \max\{I_N(x), I_N(y)\} = I_N(y),$$
$$F_N(x\cdot y) \geq \min\{F_N(x), F_N(y)\} = F_N(y).$$

Hence, X_N is a special neutrosophic $\mathcal{N}$-near UP-filter of X.

Theorem 5.19 If X_N is a special neutrosophic $\mathcal{N}$-near UP-filter of X satisfying the condition (3.21), then X_N is a special neutrosophic $\mathcal{N}$-UP-filter of X.

Proof. Assume that X_N is a special neutrosophic $\mathcal{N}$-near UP-filter of X satisfying the condition (3.21). Then X_N satisfies the conditions (5.4), (5.5), and (5.6). Next, let $x, y, z \in X$. By (5.7), (5.8), and (3.21), we have

$$\min\{T_N(x\cdot y), T_N(x)\} = \min\{I_N(x\cdot y), T_N(x)\} \leq \min\{I_N(y), T_N(x)\} = \min\{T_N(y), T_N(x)\} \leq T_N(y),$$
$$\max\{I_N(x\cdot y), I_N(x)\} = \max\{T_N(x\cdot y), I_N(x)\} \geq \max\{T_N(y), I_N(x)\} = \max\{I_N(y), I_N(x)\} \geq I_N(y),$$
$$\min\{F_N(x\cdot y), F_N(x)\} = \min\{I_N(x\cdot y), F_N(x)\} \leq \min\{I_N(y), F_N(x)\} = \min\{F_N(y), F_N(x)\} \leq F_N(y).$$

Hence, X_N is a special neutrosophic $\mathcal{N}$-UP-filter of X.

Theorem 5.20 If X_N is a special neutrosophic $\mathcal{N}$-UP-filter of X satisfying the condition (3.22), then X_N is a special neutrosophic $\mathcal{N}$-UP-ideal of X.

Proof. Assume that X_N is a special neutrosophic $\mathcal{N}$-UP-filter of X satisfying the condition (3.22). Then X_N satisfies the conditions (5.4), (5.5), and (5.6). Next, let $x, y, z \in X$. By (5.10), (5.11), (5.12), and (3.22), we have

$$T_N(x\cdot z) \geq \min\{T_N(y\cdot(x\cdot z)), T_N(y)\} = \min\{T_N(x\cdot(y\cdot z)), T_N(y)\},$$
$$I_N(x\cdot z) \leq \max\{I_N(y\cdot(x\cdot z)), I_N(y)\} = \max\{I_N(x\cdot(y\cdot z)), I_N(y)\},$$
$$F_N(x\cdot z) \geq \min\{F_N(y\cdot(x\cdot z)), F_N(y)\} = \min\{F_N(x\cdot(y\cdot z)), F_N(y)\}.$$

Hence, X_N is a special neutrosophic $\mathcal{N}$-UP-ideal of X.

Theorem 5.21 If X_N is a neutrosophic $\mathcal{N}$-structure over X satisfying the following condition:

$$(\forall x, y, z \in X)\left(z \leq x\cdot y \Rightarrow \begin{cases} T_N(z) \geq \min\{T_N(x), T_N(y)\} \\ I_N(z) \leq \max\{I_N(x), I_N(y)\} \\ F_N(z) \geq \min\{F_N(x), F_N(y)\} \end{cases} \right), \tag{5.20}$$

then X_N is a special neutrosophic $\mathcal{N}$-UP-subalgebra of X.

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-structure over X satisfying the condition (5.20). Let $x, y \in X$. By Proposition 2.5 (1), we have $(x\cdot y)\cdot(x\cdot y) = 0$, that is, $x\cdot y \leq x\cdot y$. It follows from (5.20) that

$$T_N(x\cdot y) \geq \min\{T_N(x), T_N(y)\},\ I_N(x\cdot y) \leq \max\{I_N(x), I_N(y)\},\ F_N(x\cdot y) \geq \min\{F_N(x), F_N(y)\}.$$

Hence, X_N is a special neutrosophic $\mathcal{N}$-UP-subalgebra of X.

Theorem 5.22 If X_N is a neutrosophic $\mathcal{N}$-structure over X satisfying the following condition:

$$(\forall x, y, z \in X)\left(z \leq x\cdot y \Rightarrow \begin{cases} T_N(z) \geq T_N(y) \\ I_N(z) \leq I_N(y) \\ F_N(z) \geq F_N(y) \end{cases} \right), \tag{5.21}$$

then X_N is a special neutrosophic $\mathcal{N}$-near UP-filter of X.

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-structure over X satisfying the condition (5.21). Let $x \in X$. By (UP-2) and Proposition 2.5 (1), we have $0 \cdot (x \cdot x) = 0$, that is, $0 \le x \cdot x$. It follows from (5.21) that $T_N(0) \ge T_N(x), I_N(0) \le I_N(x)$, and $F_N(0) \ge F_N(x)$. Next, let $x, y \in X$. By Proposition 2.5 (1), we have $(x \cdot y) \cdot (x \cdot y) = 0$, that is, $x \cdot y \le x \cdot y$. It follows from (5.21) that $T_N(x \cdot y) \ge T_N(y), I_N(x \cdot y) \le I_N(y)$, and $F_N(x \cdot y) \ge F_N(y)$. Hence, X_N is a special neutrosophic $\mathcal{N}$-near UP-filter of X.

Theorem 5.23 If X_N is a neutrosophic $\mathcal{N}$-structure over X satisfying the following condition:

$$(\forall x, y, z \in X)\left(z \le x \cdot y \Rightarrow \begin{cases} T_N(y) \ge \min\{T_N(z), T_N(x)\} \\ I_N(y) \le \max\{I_N(z), I_N(x)\} \\ F_N(y) \ge \min\{F_N(z), F_N(x)\} \end{cases} \right), \tag{5.22}$$

then X_N is a special neutrosophic $\mathcal{N}$-UP-filter of X.

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-structure over X satisfying the condition (5.22). Let $x \in X$. By (UP-3), we have $x \cdot (x \cdot 0) = 0$, that is, $x \le x \cdot 0$. It follows from (5.22) that

$$T_N(0) \ge \min\{T_N(x), T_N(x)\} = T_N(x),\ I_N(0) \le \max\{I_N(x), I_N(x)\} = I_N(x),\ F_N(0) \ge \min\{F_N(x), F_N(x)\} = F_N(x).$$

Next, let $x, y \in X$. By Proposition 2.5 (1), we have $(x \cdot y) \cdot (x \cdot y) = 0$, that is, $x \cdot y \le x \cdot y$. It follows from (5.22) that

$$T_N(y) \ge \min\{T_N(x \cdot y), T_N(x)\},\ I_N(y) \le \max\{I_N(x \cdot y), I_N(x)\},\ F_N(y) \ge \min\{F_N(x \cdot y), F_N(x)\}.$$

Hence, X_N is a special neutrosophic $\mathcal{N}$-UP-filter of X.

Theorem 5.24 If X_N is a neutrosophic $\mathcal{N}$-structure over X satisfying the following condition:

$$(\forall a, x, y, z \in X)\left(a \le x \cdot (y \cdot z) \Rightarrow \begin{cases} T_N(x \cdot z) \ge \min\{T_N(a), T_N(y)\} \\ I_N(x \cdot z) \le \max\{I_N(a), I_N(y)\} \\ F_N(x \cdot z) \ge \min\{F_N(a), F_N(y)\} \end{cases} \right), \tag{5.23}$$

then X_N is a special neutrosophic $\mathcal{N}$-UP-ideal of X.

Proof. Assume that X_N is a neutrosophic N -structure over X satisfying the condition (5.23). Let $x \in X$. By (UP-3), we have $x \cdot (0 \cdot (x \cdot 0) = 0$, that is, $x \le 0 \cdot (x \cdot 0)$. It follows from (5.23) and (UP-2) that

$$T_N(0) = T_N(0 \cdot 0) \ge \min\{T_N(x), T_N(x)\} = T_N(x),\ I_N(0) = I_N(0 \cdot 0) \le \max\{I_N(x), I_N(x)\} = I_N(x),$$
$$F_N(0) = F_N(0 \cdot 0) \ge \min\{F_N(x), F_N(x)\} = F_N(x).$$

Next, let $x, y, z \in X$. By Proposition 2.5 (1), we have $(x \cdot (y \cdot z)) \cdot (x \cdot (y \cdot z)) = 0$, that is, $x \cdot (y \cdot z) \le x \cdot (y \cdot z)$. It follows from (5.23) that

$$T_N(x \cdot z) \ge \min\{T_N(x \cdot (y \cdot z)), T_N(y)\},\ I_N(x \cdot z) \le \max\{I_N(x \cdot (y \cdot z)), I_N(y)\},$$
$$F_N(x \cdot z) \ge \min\{F_N(x \cdot (y \cdot z)), F_N(y)\}.$$

Hence, X_N is a special neutrosophic $\mathcal{N}$-UP-ideal of X.

For any fixed numbers $\alpha^-, \alpha^+, \beta^-, \beta^+, \gamma^-, \gamma^+ \in [-1, 0]$ such that $\alpha^- < \alpha^+, \beta^- < \beta^+, \gamma^- < \gamma^+$ and a nonempty subset G of X, a neutrosophic $\mathcal{N}$-structure ${}^G X_N[^{\alpha^+, \beta^-, \gamma^+}_{\alpha^-, \beta^+, \gamma^-}] = (X, {}^G T_N[^{\alpha^+}_{\alpha^-}], {}^G I_N[^{\beta^-}_{\beta^+}], {}^G F_N[^{\gamma^+}_{\gamma^-}])$ over X where ${}^G T_N[^{\alpha^+}_{\alpha^-}], {}^G I_N[^{\beta^-}_{\beta^+}]$, and ${}^G F_N[^{\gamma^+}_{\gamma^-}]$ are $\mathcal{N}$-functions on X which are given as follows:

$$ {}^{G}T_N[^{\alpha^+}_{\alpha^-}](x) = \begin{cases} \alpha^+ & \text{if } x \in G, \\ \alpha^- & \text{otherwise,} \end{cases} \quad {}^{G}I_N[^{\beta^-}_{\beta^+}](x) = \begin{cases} \beta^- & \text{if } x \in G, \\ \beta^+ & \text{otherwise,} \end{cases} \quad {}^{G}F_N[^{\gamma^+}_{\gamma^-}](x) = \begin{cases} \gamma^+ & \text{if } x \in G, \\ \gamma^- & \text{otherwise.} \end{cases} $$

Lemma 5.25 Let $\alpha^-, \alpha^+, \beta^-, \beta^+, \gamma^-, \gamma^+ \in [-1,0]$. Then the following statements hold:

1. $\overline{X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]} = {}^{G}X_N[^{-1-\alpha^-,-1-\beta^+,-1-\gamma^-}_{-1-\alpha^+,-1-\beta^-,-1-\gamma^+}]$, and
2. $\overline{{}^{G}X_N[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]} = X_N^G[^{-1-\alpha^+,-1-\beta^-,-1-\gamma^+}_{-1-\alpha^-,-1-\beta^+,-1-\gamma^-}]$.

Proof. 1. Let $\overline{X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]}$ be a neutrosophic $\mathcal{N}$-structure over X. Then $\overline{X_N^G[^{\alpha^-,\beta^+,\gamma^-}_{\alpha^+,\beta^-,\gamma^+}]} = (X, \overline{T_N^G[^{\alpha^-}_{\alpha^+}]}, \overline{I_N^G[^{\beta^+}_{\beta^-}]}, \overline{F_N^G[^{\gamma^-}_{\gamma^+}]})$. Since

$$ T_N^G[^{\alpha^-}_{\alpha^+}](x) = \begin{cases} \alpha^- & \text{if } x \in G, \\ \alpha^+ & \text{otherwise,} \end{cases} \quad I_N^G[^{\beta^+}_{\beta^-}](x) = \begin{cases} \beta^+ & \text{if } x \in G, \\ \beta^- & \text{otherwise,} \end{cases} \quad F_N^G[^{\gamma^-}_{\gamma^+}](x) = \begin{cases} \gamma^- & \text{if } x \in G, \\ \gamma^+ & \text{otherwise} \end{cases}, $$

we have

$$ \overline{T_N^G[^{\alpha^-}_{\alpha^+}]}(x) = \begin{cases} -1-\alpha^- & \text{if } x \in G, \\ -1-\alpha^+ & \text{otherwise} \end{cases} = {}^{G}T_N[^{-1-\alpha^-}_{-1-\alpha^+}](x), \quad \overline{I_N^G[^{\beta^+}_{\beta^-}]}(x) = \begin{cases} -1-\beta^+ & \text{if } x \in G, \\ -1-\beta^- & \text{otherwise} \end{cases} = {}^{G}I_N[^{-1-\beta^+}_{-1-\beta^-}](x), $$

$$ \overline{F_N^G[^{\gamma^-}_{\gamma^+}]}(x) = \begin{cases} -1-\gamma^- & \text{if } x \in G, \\ -1-\gamma^+ & \text{otherwise} \end{cases} = {}^{G}F_N[^{-1-\gamma^-}_{-1-\gamma^+}](x). $$

Hence, $(X, {}^{G}T_N[^{-1-\alpha^-}_{-1-\alpha^+}], {}^{G}I_N[^{-1-\beta^+}_{-1-\beta^-}], {}^{G}F_N[^{-1-\gamma^-}_{-1-\gamma^+}]) = {}^{G}X_N[^{-1-\alpha^-,-1-\beta^+,-1-\gamma^-}_{-1-\alpha^+,-1-\beta^-,-1-\gamma^+}]$.

2. Let $\overline{{}^{G}X_N[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]}$ be a neutrosophic $\mathcal{N}$-structure over X. Then $\overline{{}^{G}X_N[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]} = (X, \overline{{}^{G}T_N[^{\alpha^+}_{\alpha^-}]}, \overline{{}^{G}I_N[^{\beta^-}_{\beta^+}]}, \overline{{}^{G}F_N[^{\gamma^+}_{\gamma^-}]})$. Since

$$ {}^{G}T_N[^{\alpha^+}_{\alpha^-}](x) = \begin{cases} \alpha^+ & \text{if } x \in G, \\ \alpha^- & \text{otherwise,} \end{cases} \quad {}^{G}I_N[^{\beta^-}_{\beta^+}](x) = \begin{cases} \beta^- & \text{if } x \in G, \\ \beta^+ & \text{otherwise,} \end{cases} \quad {}^{G}F_N[^{\gamma^+}_{\gamma^-}](x) = \begin{cases} \gamma^+ & \text{if } x \in G, \\ \gamma^- & \text{otherwise} \end{cases}, $$

we have

$$ \overline{{}^{G}T_N[^{\alpha^+}_{\alpha^-}]}(x) = \begin{cases} -1-\alpha^+ & \text{if } x \in G, \\ -1-\alpha^- & \text{otherwise} \end{cases} = T_N^G[^{-1-\alpha^+}_{-1-\alpha^-}](x), \quad \overline{{}^{G}I_N[^{\beta^-}_{\beta^+}]}(x) = \begin{cases} -1-\beta^- & \text{if } x \in G, \\ -1-\beta^+ & \text{otherwise} \end{cases} = I_N^G[^{-1-\beta^-}_{-1-\beta^+}](x), $$

$$ \overline{{}^{G}F_N[^{\gamma^+}_{\gamma^-}]}(x) = \begin{cases} -1-\gamma^+ & \text{if } x \in G, \\ -1-\gamma^- & \text{otherwise} \end{cases} = F_N^G[^{-1-\gamma^+}_{-1-\gamma^-}](x). $$

Hence, $(X, T_N^G[^{-1-\alpha^+}_{-1-\alpha^-}], I_N^G[^{-1-\beta^-}_{-1-\beta^+}], F_N^G[^{-1-\gamma^+}_{-1-\gamma^-}]) = X_N^G[^{-1-\alpha^+,-1-\beta^-,-1-\gamma^+}_{-1-\alpha^-,-1-\beta^+,-1-\gamma^-}]$.

Lemma 5.26 If the constant 0 of X is in a nonempty subset G of X, then a neutrosophic $\mathcal{N}$-structure ${}^{G}X_N[{}^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ over X satisfies the conditions (5.4), (5.5), and (5.6).

Proof. If $0 \in G$, then ${}^{G}T_N[{}^{\alpha^+}_{\alpha^-}](0) = \alpha^+, {}^{G}I_N[{}^{\beta^-}_{\beta^+}](0) = \beta^-$, and ${}^{G}F_N[{}^{\gamma^+}_{\gamma^-}](0) = \gamma^+$. Thus

$$(\forall x \in X)\begin{pmatrix} {}^{G}T_N[{}^{\alpha^+}_{\alpha^-}](0) = \alpha^+ \geq {}^{G}T_N[{}^{\alpha^+}_{\alpha^-}](x) \\ {}^{G}I_N[{}^{\beta^-}_{\beta^+}](0) = \beta^- \leq {}^{G}I_N[{}^{\beta^-}_{\beta^+}](x) \\ {}^{G}F_N[{}^{\gamma^+}_{\gamma^-}](0) = \gamma^+ \geq {}^{G}F_N[{}^{\gamma^+}_{\gamma^-}](x) \end{pmatrix}.$$

Hence, ${}^{G}X_N[{}^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ satisfies the conditions (5.4), (5.5), and (5.6).

Lemma 5.27 If a neutrosophic $\mathcal{N}$-structure ${}^{G}X_N[{}^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ over X satisfies the condition (5.4) (resp., (5.5), (5.6)), then the constant 0 of X is in a nonempty subset G of X

Proof. Assume that a neutrosophic $\mathcal{N}$-structure ${}^{G}X_N[{}^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ over X satisfies the condition (5.4). Then ${}^{G}T_N[{}^{\alpha^+}_{\alpha^-}](0) \geq {}^{G}T_N[{}^{\alpha^+}_{\alpha^-}](x)$ for all $x \in X$. Since G is nonempty, there exists $g \in G$. Thus ${}^{G}T_N[{}^{\alpha^+}_{\alpha^-}](g) = \alpha^+$, so ${}^{G}T_N[{}^{\alpha^+}_{\alpha^-}](0) \geq {}^{G}T_N[{}^{\alpha^+}_{\alpha^-}](g) = \alpha^+$, that is, ${}^{G}T_N[{}^{\alpha^+}_{\alpha^-}](0) = \alpha^+$. Hence, $0 \in G$.

Theorem 5.28 A neutrosophic $\mathcal{N}$-structure ${}^{G}X_N[{}^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ over X is a special neutrosophic $\mathcal{N}$-UP-subalgebra of X if and only if a nonempty subset G of X is a UP-subalgebra of X.

Proof. Assume that ${}^{G}X_N[{}^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a special neutrosophic $\mathcal{N}$-UP-subalgebra of X. Let $x, y \in G$. Then ${}^{G}T_N[{}^{\alpha^+}_{\alpha^-}](x) = \alpha^+ = {}^{G}T_N[{}^{\alpha^+}_{\alpha^-}](y)$. Thus

$${}^{G}T_N[{}^{\alpha^+}_{\alpha^-}](x \cdot y) \geq \min\{{}^{G}T_N[{}^{\alpha^+}_{\alpha^-}](x), {}^{G}T_N[{}^{\alpha^+}_{\alpha^-}](y)\} = \alpha^+ \geq {}^{G}T_N[{}^{\alpha^+}_{\alpha^-}](x \cdot y)$$

and so ${}^{G}T_N[{}^{\alpha^+}_{\alpha^-}](x \cdot y) = \alpha^+$. Thus $x \cdot y \in G$. Hence, G is a UP-subalgebra of X.

Conversely, assume that G is a UP-subalgebra of X. Let $x, y \in X$.

Case 1: $x, y \in G$. Then

$${}^{G}T_N[{}^{\alpha^+}_{\alpha^-}](x) = \alpha^+ = {}^{G}T_N[{}^{\alpha^+}_{\alpha^-}](y),\ {}^{G}I_N[{}^{\beta^-}_{\beta^+}](x) = \beta^- = {}^{G}I_N[{}^{\beta^-}_{\beta^+}](y),\ {}^{G}F_N[{}^{\gamma^+}_{\gamma^-}](x) = \gamma^+ = {}^{G}F_N[{}^{\gamma^+}_{\gamma^-}](y).$$

Thus

$$\min\{{}^{G}T_N[{}^{\alpha^+}_{\alpha^-}](x), {}^{G}T_N[{}^{\alpha^+}_{\alpha^-}](y)\} = \alpha^+,\ \max\{{}^{G}I_N[{}^{\beta^-}_{\beta^+}](x), {}^{G}I_N[{}^{\beta^-}_{\beta^+}](y)\} = \beta^-,\ \min\{{}^{G}F_N[{}^{\gamma^+}_{\gamma^-}](x), {}^{G}F_N[{}^{\gamma^+}_{\gamma^-}](y)\} = \gamma^+.$$

Since G is a UP-subalgebra of X, we have $x \cdot y \in G$ and so ${}^{G}T_{N}[^{\alpha^+}_{\alpha^-}](x \cdot y) = \alpha^+, {}^{G}I_{N}[^{\beta^-}_{\beta^+}](x \cdot y) = \beta^-$, and ${}^{G}F_{N}[^{\gamma^+}_{\gamma^-}](x \cdot y) = \gamma^+$. Hence,

$$ {}^{G}T_{N}[^{\alpha^+}_{\alpha^-}](x \cdot y) = \alpha^+ \geq \alpha^+ = \min\{{}^{G}T_{N}[^{\alpha^+}_{\alpha^-}](x), {}^{G}T_{N}[^{\alpha^+}_{\alpha^-}](y)\}, $$

$$ {}^{G}I_{N}[^{\beta^-}_{\beta^+}](x \cdot y) = \beta^- \leq \beta^- = \max\{{}^{G}I_{N}[^{\beta^-}_{\beta^+}](x), {}^{G}I_{N}[^{\beta^-}_{\beta^+}](y)\}, $$

$$ {}^{G}F_{N}[^{\gamma^+}_{\gamma^-}](x \cdot y) = \gamma^+ \geq \gamma^+ = \min\{{}^{G}F_{N}[^{\gamma^+}_{\gamma^-}](x), {}^{G}F_{N}[^{\gamma^+}_{\gamma^-}](y)\}. $$

Case 2: $x \notin G$ or $y \notin G$. Then

$$ {}^{G}T_{N}[^{\alpha^+}_{\alpha^-}](x) = \alpha^- \text{ or } {}^{G}T_{N}[^{\alpha^+}_{\alpha^-}](y) = \alpha^-, \ {}^{G}I_{N}[^{\beta^-}_{\beta^+}](x) = \beta^+ \text{ or } {}^{G}I_{N}[^{\beta^-}_{\beta^+}](y) = \beta^+, $$

$$ {}^{G}F_{N}[^{\gamma^+}_{\gamma^-}](x) = \gamma^- \text{ or } {}^{G}F_{N}[^{\gamma^+}_{\gamma^-}](y) = \gamma^-. $$

Thus

$$ \min\{{}^{G}T_{N}[^{\alpha^+}_{\alpha^-}](x), {}^{G}T_{N}[^{\alpha^+}_{\alpha^-}](y)\} = \alpha^-, \ \max\{{}^{G}I_{N}[^{\beta^-}_{\beta^+}](x), {}^{G}I_{N}[^{\beta^-}_{\beta^+}](y)\} = \beta^+, \ \min\{{}^{G}F_{N}[^{\gamma^+}_{\gamma^-}](x), {}^{G}F_{N}[^{\gamma^+}_{\gamma^-}](y)\} = \gamma^-. $$

Therefore,

$$ {}^{G}T_{N}[^{\alpha^+}_{\alpha^-}](x \cdot y) \geq \alpha^- = \min\{{}^{G}T_{N}[^{\alpha^+}_{\alpha^-}](x), {}^{G}T_{N}[^{\alpha^+}_{\alpha^-}](y)\}, $$

$$ {}^{G}I_{N}[^{\beta^-}_{\beta^+}](x \cdot y) \leq \beta^+ = \max\{{}^{G}I_{N}[^{\beta^-}_{\beta^+}](x), {}^{G}I_{N}[^{\beta^-}_{\beta^+}](y)\}, $$

$$ {}^{G}F_{N}[^{\gamma^+}_{\gamma^-}](x \cdot y) \geq \gamma^- = \min\{{}^{G}F_{N}[^{\gamma^+}_{\gamma^-}](x), {}^{G}F_{N}[^{\gamma^+}_{\gamma^-}](y)\}. $$

Hence, ${}^{G}X_{N}[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a special neutrosophic $\mathcal{N}$-UP-subalgebra of X.

Theorem 5.29 A neutrosophic $\mathcal{N}$-structure ${}^{G}X_{N}[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ over X is a special neutrosophic $\mathcal{N}$-near UP-filter of X if and only if a nonempty subset G of X is a near UP-filter of X.

Proof. Assume that ${}^{G}X_{N}[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a special neutrosophic $\mathcal{N}$-near UP-filter of X. Since ${}^{G}X_{N}[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ satisfies the condition (5.4), it follows from Lemma 5.27 that $0 \in G$. Next, let $x \in X$ and $y \in G$. Then ${}^{G}T_{N}[^{\alpha^+}_{\alpha^-}](y) = \alpha^+$. Thus, by (5.7), we have

$$ {}^{G}T_{N}[^{\alpha^+}_{\alpha^-}](x \cdot y) \geq {}^{G}T_{N}[^{\alpha^+}_{\alpha^-}](y) = \alpha^+ \geq {}^{G}T_{N}[^{\alpha^+}_{\alpha^-}](x \cdot y) $$

and so ${}^{G}T_{N}[^{\alpha^+}_{\alpha^-}](x \cdot y) = \alpha^+$. Thus $x \cdot y \in G$. Hence, G is a near UP-filter of X.

Conversely, assume that G is a near UP-filter of X. Since $0 \in G$, it follows from Lemma 5.26 that ${}^{G}X_{N}[^{\alpha^{+},\beta^{-},\gamma^{+}}_{\alpha^{-},\beta^{+},\gamma^{-}}]$ satisfies the conditions (5.4), (5.5), and (5.6). Next, let $x, y \in X$.

Case 1: $y \in G$. Then ${}^{G}T_{N}[^{\alpha^{+}}_{\alpha^{-}}](y) = \alpha^{+}, {}^{G}I_{N}[^{\beta^{-}}_{\beta^{+}}](y) = \beta^{-}$, and ${}^{G}F_{N}[^{\gamma^{+}}_{\gamma^{-}}](y) = \gamma^{+}$. Since G is a near UP-filter of X, we have $x \cdot y \in G$ and so ${}^{G}T_{N}[^{\alpha^{+}}_{\alpha^{-}}](x \cdot y) = \alpha^{+}, {}^{G}I_{N}[^{\beta^{-}}_{\beta^{+}}](x \cdot y) = \beta^{-}$, and ${}^{G}F_{N}[^{\gamma^{+}}_{\gamma^{-}}](x \cdot y) = \gamma^{+}$. Thus

$$ {}^{G}T_{N}[^{\alpha^{+}}_{\alpha^{-}}](x \cdot y) = \alpha^{+} \geq \alpha^{+} = {}^{G}T_{N}[^{\alpha^{+}}_{\alpha^{-}}](y),\ {}^{G}I_{N}[^{\beta^{-}}_{\beta^{+}}](x \cdot y) = \beta^{-} \leq \beta^{-} = {}^{G}I_{N}[^{\beta^{-}}_{\beta^{+}}](y), $$

$$ {}^{G}F_{N}[^{\gamma^{+}}_{\gamma^{-}}](x \cdot y) = \gamma^{+} \geq \gamma^{+} = {}^{G}F_{N}[^{\gamma^{+}}_{\gamma^{-}}](y). $$

Case 2: $y \notin G$. Then ${}^{G}T_{N}[^{\alpha^{+}}_{\alpha^{-}}](y) = \alpha^{-}, {}^{G}I_{N}[^{\beta^{-}}_{\beta^{+}}](y) = \beta^{+}$, and ${}^{G}F_{N}[^{\gamma^{+}}_{\gamma^{-}}](y) = \gamma^{-}$. Thus

$$ {}^{G}T_{N}[^{\alpha^{+}}_{\alpha^{-}}](x \cdot y) \geq \alpha^{-} = {}^{G}T_{N}[^{\alpha^{+}}_{\alpha^{-}}](y),\ {}^{G}I_{N}[^{\beta^{-}}_{\beta^{+}}](x \cdot y) \leq \beta^{+} = {}^{G}I_{N}[^{\beta^{-}}_{\beta^{+}}](y),\ {}^{G}F_{N}[^{\gamma^{+}}_{\gamma^{-}}](x \cdot y) \geq \gamma^{-} = {}^{G}F_{N}[^{\gamma^{+}}_{\gamma^{-}}](y). $$

Hence, ${}^{G}X_{N}[^{\alpha^{+},\beta^{-},\gamma^{+}}_{\alpha^{-},\beta^{+},\gamma^{-}}]$ is a special neutrosophic $\mathcal{N}$-near UP-filter of X.

Theorem 5.30 A neutrosophic $\mathcal{N}$-structure ${}^{G}X_{N}[^{\alpha^{+},\beta^{-},\gamma^{+}}_{\alpha^{-},\beta^{+},\gamma^{-}}]$ over X is a special neutrosophic $\mathcal{N}$-UP-filter of X if and only if a nonempty subset G of X is a UP-filter of X.

Proof. Assume that ${}^{G}X_{N}[^{\alpha^{+},\beta^{-},\gamma^{+}}_{\alpha^{-},\beta^{+},\gamma^{-}}]$ is a special neutrosophic $\mathcal{N}$-UP-filter of X. Since ${}^{G}X_{N}[^{\alpha^{+},\beta^{-},\gamma^{+}}_{\alpha^{-},\beta^{+},\gamma^{-}}]$ satisfies the condition (5.4), it follows from Lemma 5.27 that $0 \in G$. Next, let $x, y \in X$ be such that $x \cdot y \in G$ and $x \in G$. Then ${}^{G}T_{N}[^{\alpha^{+}}_{\alpha^{-}}](x \cdot y) = \alpha^{+} = {}^{G}T_{N}[^{\alpha^{+}}_{\alpha^{-}}](x)$. Thus, by (5.10), we have

$$ {}^{G}T_{N}[^{\alpha^{+}}_{\alpha^{-}}](y) \geq \min\{{}^{G}T_{N}[^{\alpha^{+}}_{\alpha^{-}}](x \cdot y), {}^{G}T_{N}[^{\alpha^{+}}_{\alpha^{-}}](x)\} = \alpha^{+} \geq {}^{G}T_{N}[^{\alpha^{+}}_{\alpha^{-}}](y) $$

and so ${}^{G}T_{N}[^{\alpha^{+}}_{\alpha^{-}}](y) = \alpha^{+}$. Thus $y \in G$. Hence, G is a UP-filter of X.

Conversely, assume that G is a UP-filter of X. Since $0 \in G$, it follows from Lemma 5.26 that ${}^{G}X_{N}[^{\alpha^{+},\beta^{-},\gamma^{+}}_{\alpha^{-},\beta^{+},\gamma^{-}}]$ satisfies the conditions (5.4), (5.5), and (5.6). Next, let $x, y \in X$.

Case 1: $x \cdot y \in G$ and $x \in G$. Then

$$ {}^{G}T_{N}[^{\alpha^{+}}_{\alpha^{-}}](x \cdot y) = \alpha^{+} = {}^{G}T_{N}[^{\alpha^{+}}_{\alpha^{-}}](x),\ {}^{G}I_{N}[^{\beta^{-}}_{\beta^{+}}](x \cdot y) = \beta^{-} = {}^{G}I_{N}[^{\beta^{-}}_{\beta^{+}}](x),\ {}^{G}F_{N}[^{\gamma^{+}}_{\gamma^{-}}](x \cdot y) = \gamma^{+} = {}^{G}F_{N}[^{\gamma^{+}}_{\gamma^{-}}](x). $$

Since G is a UP-filter of X, we have $y \in G$ and so ${}^{G}T_{N}[^{\alpha^{+}}_{\alpha^{-}}](y) = \alpha^{+}, {}^{G}I_{N}[^{\beta^{-}}_{\beta^{+}}](y) = \beta^{-}$, and ${}^{G}F_{N}[^{\gamma^{+}}_{\gamma^{-}}](y) = \gamma^{+}$. Thus

$$^{G}T_N[^{\alpha^+}_{\alpha^-}](y) = \alpha^+ \geq \alpha^+ = \min\{^{G}T_N[^{\alpha^+}_{\alpha^-}](x \cdot y), ^{G}T_N[^{\alpha^+}_{\alpha^-}](x)\},$$

$$^{G}I_N[^{\beta^-}_{\beta^+}](y) = \beta^- \leq \beta^- = \max\{^{G}I_N[^{\beta^-}_{\beta^+}](x \cdot y), ^{G}I_N[^{\beta^-}_{\beta^+}](x)\},$$

$$^{G}F_N[^{\gamma^+}_{\gamma^-}](y) = \gamma^+ \geq \gamma^+ = \min\{^{G}F_N[^{\gamma^+}_{\gamma^-}](x \cdot y), ^{G}F_N[^{\gamma^+}_{\gamma^-}](x)\}.$$

Case 2: $x \cdot y \notin G$ or $x \notin G$. Then

$$^{G}T_N[^{\alpha^+}_{\alpha^-}](x \cdot y) = \alpha^- \text{ or } ^{G}T_N[^{\alpha^+}_{\alpha^-}](x) = \alpha^-, \ ^{G}I_N[^{\beta^-}_{\beta^+}](x \cdot y) = \beta^+ \text{ or } ^{G}I_N[^{\beta^-}_{\beta^+}](x) = \beta^+,$$

$$^{G}F_N[^{\gamma^+}_{\gamma^-}](x \cdot y) = \gamma^- \text{ or } ^{G}F_N[^{\gamma^+}_{\gamma^-}](x) = \gamma^-.$$

Thus

$$\min\{^{G}T_N[^{\alpha^+}_{\alpha^-}](x \cdot y), ^{G}T_N[^{\alpha^+}_{\alpha^-}](x)\} = \alpha^-, \ \max\{^{G}I_N[^{\beta^-}_{\beta^+}](x \cdot y), ^{G}I_N[^{\beta^-}_{\beta^+}](x)\} = \beta^+,$$

$$\min\{^{G}F_N[^{\gamma^+}_{\gamma^-}](x \cdot y), ^{G}F_N[^{\gamma^+}_{\gamma^-}](x)\} = \gamma^-.$$

Therefore,

$$^{G}T_N[^{\alpha^+}_{\alpha^-}](x) \geq \alpha^- = \min\{^{G}T_N[^{\alpha^+}_{\alpha^-}](x \cdot y), ^{G}T_N[^{\alpha^+}_{\alpha^-}](x)\},$$

$$^{G}I_N[^{\beta^-}_{\beta^+}](x) \leq \beta^+ = \max\{^{G}I_N[^{\beta^-}_{\beta^+}](x \cdot y), ^{G}I_N[^{\beta^-}_{\beta^+}](x)\},$$

$$^{G}F_N[^{\gamma^+}_{\gamma^-}](x) \geq \gamma^- = \min\{^{G}F_N[^{\gamma^+}_{\gamma^-}](x \cdot y), ^{G}F_N[^{\gamma^+}_{\gamma^-}](x)\}.$$

Hence, $^{G}X_N[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a special neutrosophic $\mathcal{N}$-UP-filter of X.

Theorem 5.31 A neutrosophic $\mathcal{N}$-structure $^{G}X_N[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ over X is a special neutrosophic $\mathcal{N}$-UP-ideal of X if and only if a nonempty subset G of X is a UP-ideal of X.

Proof. Assume that $^{G}X_N[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a special neutrosophic $\mathcal{N}$-UP-ideal of X. Since $^{G}X_N[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ satisfies the condition (5.4), it follows from Lemma 5.27, that $0 \in G$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in G$ and $y \in G$. Then $^{G}T_N[^{\alpha^+}_{\alpha^-}](x \cdot (y \cdot z)) = \alpha^+ = ^{G}T_N[^{\alpha^+}_{\alpha^-}](y)$. Thus, by (5.13), we have

$$^{G}T_N[^{\alpha^+}_{\alpha^-}](x \cdot z) \geq \min\{^{G}T_N[^{\alpha^+}_{\alpha^-}](x \cdot (y \cdot z)), ^{G}T_N[^{\alpha^+}_{\alpha^-}](y)\} = \alpha^+ \geq ^{G}T_N[^{\alpha^+}_{\alpha^-}](x \cdot z)$$

and so $^{G}T_N[^{\alpha^+}_{\alpha^-}](x \cdot z) = \alpha^+$. Thus $x \cdot z \in G$. Hence, G is a UP-ideal of X.

Conversely, assume that G is a UP-ideal of X. Since $0 \in G$, it follows from Lemma 5.26 that $^{G}X_N[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ satisfies the conditions (5.4), (5.5), and (5.6). Next, let $x, y, z \in X$.

Case 1: $x \cdot (y \cdot z) \in G$ and $y \in G$. Then

$$^{G}T_N[^{\alpha^+}_{\alpha^-}](x\cdot(y\cdot z))=\alpha^+={}^{G}T_N[^{\alpha^+}_{\alpha^-}](y),\ {}^{G}I_N[^{\beta^-}_{\beta^+}](x\cdot(y\cdot z))=\beta^-={}^{G}I_N[^{\beta^-}_{\beta^+}](y),$$

$$^{G}F_N[^{\gamma^+}_{\gamma^-}](x\cdot(y\cdot z))=\gamma+={}^{G}F_N[^{\gamma^+}_{\gamma^-}](y).$$

Thus

$$\min\{{}^{G}T_N[^{\alpha^+}_{\alpha^-}](x\cdot(y\cdot z)),{}^{G}T_N[^{\alpha^+}_{\alpha^-}](y)\}=\alpha^+,\ \max\{{}^{G}I_N[^{\beta^-}_{\beta^+}](x\cdot(y\cdot z)),{}^{G}I_N[^{\beta^-}_{\beta^+}](y)\}=\beta^-,$$

$$\min\{{}^{G}F_N[^{\gamma^+}_{\gamma^-}](x\cdot(y\cdot z)),{}^{G}F_N[^{\gamma^+}_{\gamma^-}](y)\}=\gamma^+.$$

Since G is a UP-ideal of X, we have $x\cdot z\in G$ and so ${}^{G}T_N[^{\alpha^+}_{\alpha^-}](x\cdot z)=\alpha^+,{}^{G}I_N[^{\beta^-}_{\beta^+}](x\cdot z)=\beta^-$, and ${}^{G}F_N[^{\gamma^+}_{\gamma^-}](x\cdot z)=\gamma^+$. Thus

$$^{G}T_N[^{\alpha^+}_{\alpha^-}](x\cdot z)=\alpha^+\geq\alpha^+=\min\{{}^{G}T_N[^{\alpha^+}_{\alpha^-}](x\cdot(y\cdot z)),{}^{G}T_N[^{\alpha^+}_{\alpha^-}](y)\},$$

$$^{G}I_N[^{\beta^-}_{\beta^+}](x\cdot z)=\beta^-\leq\beta^-=\max\{{}^{G}I_N[^{\beta^-}_{\beta^+}](x\cdot(y\cdot z)),{}^{G}I_N[^{\beta^-}_{\beta^+}](y)\},$$

$$^{G}F_N[^{\gamma^+}_{\gamma^-}](x\cdot z)=\gamma^+\geq\gamma^+=\min\{{}^{G}F_N[^{\gamma^+}_{\gamma^-}](x\cdot(y\cdot z)),{}^{G}F_N[^{\gamma^+}_{\gamma^-}](y)\}.$$

Case 2: $x\cdot(y\cdot z)\notin G$ or $y\notin G$. Then

$$^{G}T_N[^{\alpha^+}_{\alpha^-}](x\cdot(y\cdot z))=\alpha^-\text{ or }{}^{G}T_N[^{\alpha^+}_{\alpha^-}](y)=\alpha^-,\ {}^{G}I_N[^{\beta^-}_{\beta^+}](x\cdot(y\cdot z))=\beta^+\text{ or }{}^{G}I_N[^{\beta^-}_{\beta^+}](y)=\beta^+,$$

$$^{G}F_N[^{\gamma^+}_{\gamma^-}](x\cdot(y\cdot z))=\gamma^-\text{ or }{}^{G}F_N[^{\gamma^+}_{\gamma^-}](y)=\gamma^-.$$

Thus

$$\min\{{}^{G}T_N[^{\alpha^+}_{\alpha^-}](x\cdot(y\cdot z)),{}^{G}T_N[^{\alpha^+}_{\alpha^-}](y)\}=\alpha^-,\ \max\{{}^{G}I_N[^{\beta^-}_{\beta^+}](x\cdot(y\cdot z)),{}^{G}I_N[^{\beta^-}_{\beta^+}](y)\}=\beta^+,$$

$$\min\{{}^{G}F_N[^{\gamma^+}_{\gamma^-}](x\cdot(y\cdot z)),{}^{G}F_N[^{\gamma^+}_{\gamma^-}](y)\}=\gamma^-.$$

Therefore,

$$^{G}T_N[^{\alpha^+}_{\alpha^-}](x\cdot z)\geq\alpha^-=\min\{{}^{G}T_N[^{\alpha^+}_{\alpha^-}](x\cdot(y\cdot z)),{}^{G}T_N[^{\alpha^+}_{\alpha^-}](y)\},$$

$$^{G}I_N[^{\beta^-}_{\beta^+}](x\cdot z)\leq\beta^+=\max\{{}^{G}I_N[^{\beta^-}_{\beta^+}](x\cdot(y\cdot z)),{}^{G}I_N[^{\beta^-}_{\beta^+}](y)\},$$

$$^{G}F_N[^{\gamma^+}_{\gamma^-}](x\cdot z)\geq\gamma^-=\min\{{}^{G}F_N[^{\gamma^+}_{\gamma^-}](x\cdot(y\cdot z)),{}^{G}F_N[^{\gamma^+}_{\gamma^-}](y)\}.$$

Hence, ${}^{G}X_N[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a special neutrosophic $\mathcal{N}$-UP-ideal of X.

Theorem 5.32 A neutrosophic $\mathcal{N}$-structure ${}^{G}X_N[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ over X is a special neutrosophic $\mathcal{N}$-strongly UP-ideal of X if and only if a nonempty subset G of X is a strongly UP-ideal of X.

Proof. Assume that ${}^{G}X_N[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a special neutrosophic $\mathcal{N}$-strongly UP-ideal of X. By Theorem 5.17, we have ${}^{G}T_N[^{\alpha^+}_{\alpha^-}]$ is constant, that is, ${}^{G}T_N[^{\alpha^+}_{\alpha^-}]$ is constant. Since G is nonempty, we have ${}^{G}T_N[^{\alpha^+}_{\alpha^-}](x)=\alpha^+$ for all $x\in X$. Thus $G=X$. Hence, G is a strongly UP-ideal of X.

Conversely, assume that G is a strongly UP-ideal of X. Then $G=X$, so

$$\left(\forall x\in X\right)\begin{pmatrix} {}^{G}T_N[^{\alpha^+}_{\alpha^-}](x) & =\alpha^+ \\ {}^{G}I_N[^{\beta^-}_{\beta^+}](x) & =\beta^- \\ {}^{G}F_N[^{\gamma^+}_{\gamma^-}](x) & =\gamma^+ \end{pmatrix}.$$

Thus ${}^{G}T_N[^{\alpha^+}_{\alpha^-}], {}^{G}I_N[^{\beta^-}_{\beta^+}]$, and ${}^{G}F_N[^{\gamma^+}_{\gamma^-}]$ are constant, that is, ${}^{G}X_N[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is constant. By Theorem 5.17, we have ${}^{G}X_N[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a special neutrosophic $\mathcal{N}$-strongly UP-ideal of X.

6. Level subset of a neutrosophic $\mathcal{N}$-structure of special type

In the last section of this paper, we discuss the relationships among special neutrosophic $\mathcal{N}$-UP-subalgebras (resp., special neutrosophic $\mathcal{N}$-near UP-filters, special neutrosophic $\mathcal{N}$-UP-filters, special neutrosophic $\mathcal{N}$-UP-ideals, special neutrosophic $\mathcal{N}$-strongly UP-ideals) of UP-algebras and their level subsets.

Theorem 6.1 A neutrosophic $\mathcal{N}$-structure X_N over X is a special neutrosophic $\mathcal{N}$-UP-subalgebra of X if and only if for all $\alpha,\beta,\gamma\in[-1,0]$, the sets $U(T_N;\alpha),L(I_N;\beta)$, and $U(F_N;\gamma)$ are UP-subalgebras of X if $U(T_N;\alpha),L(I_N;\beta)$, and $U(F_N;\gamma)$ are nonempty.

Proof. Assume that X_N is a special neutrosophic $\mathcal{N}$-UP-subalgebra of X. Let $\alpha,\beta,\gamma\in[-1,0]$ be such that $U(T_N;\alpha),L(I_N;\beta)$, and $U(F_N;\gamma)$ are nonempty.

Let $x,y\in U(T_N;\alpha)$. Then $T_N(x)\geq\alpha$ and $T_N(y)\geq\alpha$, so α is a lower bound of $\{T_N(x),T_N(y)\}$. By (5.1), we have $T_N(x\cdot y)\geq\min\{T_N(x),T_N(y)\}\geq\alpha$. Thus $x\cdot y\in U(T_N;\alpha)$.

Let $x,y\in L(I_N;\beta)$. Then $I_N(x)\leq\beta$ and $I_N(y)\leq\beta$, so β is an upper bound of $\{I_N(x),I_N(y)\}$. By (5.2), we have $I_N(x\cdot y)\leq\max\{I_N(x),I_N(y)\}\leq\beta$. Thus $x\cdot y\in L(I_N;\beta)$.

Let $x,y\in U(F_N;\gamma)$. Then $F_N(x)\geq\gamma$ and $F_N(y)\geq\gamma$, so γ is a lower bound of $\{F_N(x),F_N(y)\}$. By (5.3), we have $F_N(x\cdot y)\geq\min\{F_N(x),F_N(y)\}\geq\gamma$. Thus $x\cdot y\in U(F_N;\gamma)$.

Hence, $U(T_N;\alpha),L(I_N;\beta)$, and $U(F_N;\gamma)$ are UP-subalgebras of X.

Conversely, assume that for all $\alpha,\beta,\gamma\in[-1,0]$, the set $U(T_N;\alpha),L(I_N;\beta)$, and $U(F_N;\gamma)$ are UP-subalgebras if $U(T_N;\alpha),L(I_N;\beta)$, and $U(F_N;\gamma)$ are nonempty.

Let $x,y\in X$. Then $T_N(x),T_N(y)\in[-1,0]$ Choose $\alpha=\min\{T_N(x),T_N(y)\}$. Thus $T_N(x)\geq\alpha$ and $T_N(y)\geq\alpha$, so $x,y\in U(T_N;\alpha)\neq\varnothing$. By assumption, we have $U(T_N;\alpha)$ is a UP-subalgebra of X and so $x,y\in U(T_N;\alpha)$. Thus $T_N(x\cdot y)\geq\alpha=\min\{T_N(x),T_N(y)\}$.

Let $x, y \in X$. Then $I_N(x), I_N(y) \in [-1,0]$ Choose $\beta = \max\{I_N(x), I_N(y)\}$. Thus $I_N(x) \leq \beta$ and $I_N(y) \leq \beta$, so $x, y \in L(I_N;\beta) \neq \varnothing$. By assumption, we have $L(I_N;\beta)$ is a UP-subalgebra of X and so $x, y \in L(I_N;\beta)$. Thus $I_N(x \cdot y) \leq \beta = \max\{I_N(x), I_N(y)\}$.

Let $x, y \in X$. Then $F_N(x), F_N(y) \in [-1,0]$. Choose $\gamma = \min\{F_N(x), F_N(y)\}$. Thus $F_N(x) \geq \gamma$ and $F_N(y) \geq \gamma$, so $x, y \in U(F_N;\gamma) \neq \varnothing$. By assumption, we have $U(F_N;\gamma)$ is a UP-subalgebra of X and so $x, y \in U(F_N;\gamma)$. Thus $F_N(x \cdot y) \leq \gamma = \min\{F_N(x), F_N(y)\}$.

Therefore, X_N is a special neutrosophic $\mathcal{N}$-UP-subalgebra of X.

Theorem 6.2 A neutrosophic $\mathcal{N}$-structure X_N over X is a special neutrosophic $\mathcal{N}$-near UP-filter of X if and only if for all $\alpha,\beta,\gamma \in [-1,0]$, the sets $U(T_N;\alpha), L(I_N;\beta)$, and $U(F_N;\gamma)$ are near UP-filters of X if $U(T_N;\alpha), L(I_N;\beta)$, and $U(F_N;\gamma)$ are nonempty.

Proof. Assume that X_N is a special neutrosophic $\mathcal{N}$-near UP-filter of X. Let $\alpha,\beta,\gamma \in [-1,0]$ be such that $U(T_N;\alpha), L(I_N;\beta)$, and $U(F_N;\gamma)$ are nonempty.

Let $x \in U(T_N;\alpha)$. Then $T_N(x) \geq \alpha$. By (5.4), we have $T_N(0) \geq T_N(x) \geq \alpha$. Thus $0 \in U(T_N;\alpha)$. Next, let $y \in U(T_N;\alpha)$. Then $T_N(y) \geq \alpha$. By (5.7), we have $T_N(x \cdot y) \geq T_N(y) \geq \alpha$. Thus $x \cdot y \in U(T_N;\alpha)$.

Let $x \in L(I_N;\beta)$. Then $I_N(x) \leq \beta$. By (5.5), we have $I_N(0) \leq I_N(x) \leq \beta$. Thus $0 \in L(I_N;\beta)$. Next, let $y \in L(I_N;\beta)$. Then $I_N(y) \leq \beta$. By (5.8), we have $I_N(x \cdot y) \leq I_N(y) \leq \beta$. Thus $x \cdot y \in L(I_N;\beta)$.

Let $x \in U(F_N;\gamma)$. Then $F_N(x) \geq \gamma$. By (5.6), we have $F_N(0) \geq F_N(x) \geq \gamma$. Thus $0 \in U(F_N;\gamma)$. Next, $y \in U(F_N;\gamma)$. Then $F_N(y) \geq \gamma$. By (5.9), we have $F_N(x \cdot y) \geq F_N(y) \geq \gamma$. Thus $x \cdot y \in U(F_N;\gamma)$.

Hence, $U(T_N;\alpha)$, $L(I_N;\beta)$, and $U(F_N;\gamma)$ are near UP-filters of X.

Conversely, assume that for all $\alpha,\beta,\gamma \in [-1,0]$, the set $U(T_N;\alpha), L(I_N;\beta)$, and $U(F_N;\gamma)$ are near UP-filters if $U(T_N;\alpha), L(I_N;\beta)$, and $U(F_N;\gamma)$ are nonempty.

Let $x \in X$. Then $T_N(0) \in [-1,0]$. Choose $\alpha = T_N(x)$. Thus $T_N(x) \geq \alpha$, so $x \in L(T_N;\alpha) \neq \varnothing$. By assumption, we have $U(T_N;\alpha)$ is a near UP-filter of X and so $0 \in U(T_N;\alpha)$. Thus $T_N(0) \geq \alpha = T_N(x)$. Next, let $y \in X$. Then $T_N(y) \in [-1,0]$. Choose $\alpha = T_N(y)$. Thus $T_N(y) \geq \alpha$, so $y \in U(T_N;\alpha) \neq \varnothing$. By assumption, we have $U(T_N;\alpha)$ is a near UP-filter of X, and so $x \cdot y \in U(T_N;\alpha)$. Thus $T_N(x \cdot y) \geq \alpha = T_N(y)$.

Let $x \in X$. Then $I_N(0) \in [-1,0]$. Choose $\beta = I_N(x)$. Thus $I_N(x) \leq \beta$, so $x \in L(I_N;\beta) \neq \varnothing$. By assumption, we have $L(I_N;\beta)$ is a near UP-filter of X and so $0 \in L(I_N;\beta)$. Thus $I_N(0) \leq \beta = I_N(x)$. Next, let $y \in X$. Then $I_N(y) \in [-1,0]$. Choose $\beta = I_N(y)$. Thus $I_N(y) \leq \beta$, so $y \in L(I_N;\beta) \neq \varnothing$. By assumption, we have $L(I_N;\beta)$ is a near UP-filter of X, and so $x \cdot y \in L(I_N;\beta)$. Thus $I_N(x \cdot y) \leq \beta = I_N(y)$.

Let $x \in X$. Then $F_N(0) \in [-1,0]$. Choose $\gamma = F_N(x)$. Thus $F_N(x) \geq \gamma$, so $x \in U(F_N;\gamma) \neq \varnothing$. By assumption, we have $U(F_N;\gamma)$ is a near UP-filter of X and so $0 \in U(F_N;\gamma)$. Thus $F_N(0) \geq \gamma = F_N(x)$. Next, let $y \in X$. Then $F_N(y) \in [-1,0]$. Choose $\gamma = F_N(y)$. Thus $F_N(y) \geq \gamma$, so $y \in U(F_N;\gamma) \neq \varnothing$. By assumption, we have $U(F_N;\gamma)$ is a near UP-filter of X, and so $x \cdot y \in U(F_N;\gamma)$. Thus $F_N(x \cdot y) \geq \gamma = F_N(y)$.

Therefore, X_N is a special neutrosophic $\mathcal{N}$-near UP-filter of X.

Theorem 6.3 A neutrosophic $\mathcal{N}$-structure X_N over X is a special neutrosophic $\mathcal{N}$-UP-filter of X if and only if for all $\alpha,\beta,\gamma\in[-1,0]$, the sets $U(T_N;\alpha), L(I_N;\beta)$, and $U(F_N;\gamma)$ are UP-filters of X if $U(T_N;\alpha), L(I_N;\beta)$, and $U(F_N;\gamma)$ are nonempty.

Proof. Assume that X_N is a special neutrosophic $\mathcal{N}$-UP-filter of X. Let $\alpha,\beta,\gamma\in[-1,0]$ be such that $U(T_N;\alpha), L(I_N;\beta)$, and $U(F_N;\gamma)$ are nonempty.

Let $x\in U(T_N;\alpha)$. Then $T_N(x)\geq\alpha$. By (5.4), we have $T_N(0)\geq T_N(x)\geq\alpha$. Thus $0\in U(T_N;\alpha)$. Next, let $x\cdot y\in U(T_N;\alpha)$ and $x\in U(T_N;\alpha)$. Then $T_N(x\cdot y)\geq\alpha$ and $T_N(x)\leq\alpha$, so α is a lower bound of $\{T_N(x\cdot y),T_N(x)\}$. By (5.10), we have $T_N(y)\geq\min\{T_N(x\cdot y),T_N(x)\}\geq\alpha$. Thus $y\in U(T_N;\alpha)$.

Let $x\in L(I_N;\beta)$. Then $I_N(x)\leq\beta$. By (5.5), we have $I_N(0)\leq I_N(x)\leq\beta$. Thus $0\in L(I_N;\beta)$. Next, let $x\cdot y\in L(I_N;\beta)$ and $x\in L(I_N;\beta)$. Then $I_N(x\cdot y)\leq\beta$ and $I_N(x)\leq\beta$, so β is an upper bound of $\{I_N(x\cdot y),I_N(x)\}$. By (5.11), we have $I_N(y)\leq\max\{I_N(x\cdot y),I_N(x)\}\leq\beta$. Thus $y\in L(I_N;\beta)$.

Let $x\in U(F_N;\gamma)$. Then $F_N(x)\geq\gamma$. By (5.6), we have $F_N(0)\geq F_N(x)\geq\gamma$. Thus $0\in U(F_N;\gamma)$. Next, let $x\cdot y\in U(F_N;\gamma)$ and $x\in U(F_N;\gamma)$. Then $F_N(x\cdot y)\geq\gamma$ and $F_N(x)\geq\gamma$, so γ is a lower bound of $\{F_N(x\cdot y),F_N(x)\}$. By (5.12), we have $F_N(y)\geq\min\{F_N(x\cdot y),F_N(x)\}\geq\gamma$. Thus $y\in U(F_N;\gamma)$.

Hence, $U(T_N;\alpha), L(I_N;\beta)$, and $U(F_N;\gamma)$ are UP-filters of X.

Conversely, assume that for all $\alpha,\beta,\gamma\in[-1,0]$, the set $U(T_N;\alpha), L(I_N;\beta)$, and $U(F_N;\gamma)$ are UP-filters if $U(T_N;\alpha), L(I_N;\beta)$, and $U(F_N;\gamma)$ are nonempty.

Let $x\in X$. Then $T_N(x)\in[-1,0]$. Choose $\alpha=T_N(x)$. Thus $T_N(x)\geq\alpha$, so $x\in U(T_N;\alpha)\neq\varnothing$. By assumption, we have $U(T_N;\alpha)$ is a UP-filter of X and so $0\in U(T_N;\alpha)$. Thus $T_N(0)\geq\alpha=T_N(x)$. Next, let $x,y\in X$. Then $T_N(x\cdot y),T_N(x)\in[-1,0]$. Choose $\alpha=\min\{T_N(x\cdot y),T_N(x)\}$. Thus $T_N(x\cdot y)\geq\alpha$ and $T_N(x)\geq\alpha$, so $x\cdot y,x\in U(T_N;\alpha)\neq\varnothing$. By assumption, we have $U(T_N;\alpha)$ is a UP-filter of X and so $y\in U(T_N;\alpha)$. Thus $T_N(y)\geq\alpha=\min\{T_N(x\cdot y),T_N(x)\}$.

Let $x\in X$. Then $I_N(x)\in[-1,0]$. Choose $\beta=I_N(x)$. Thus $I_N(x)\leq\beta$, so $x\in L(I_N;\beta)\neq\varnothing$. By assumption, we have $L(I_N;\beta)$ is a UP-filter of X and so $0\in L(I_N;\beta)$. Thus $I_N(0)\leq\beta=I_N(x)$. Next, let $x,y\in X$. Then $I_N(x\cdot y),I_N(x)\in[-1,0]$. Choose $\beta=\max\{I_N(x\cdot y),I_N(x)\}$. Thus $I_N(x\cdot y)\leq\beta$ and $I_N(x)\leq\beta$, so $x\cdot y,x\in L(I_N;\beta)\neq\varnothing$. By assumption, we have $L(I_N;\beta)$ is a UP-filter of X and so $y\in L(I_N;\beta)$. Thus $I_N(y)\leq\beta=\max\{I_N(x\cdot y),I_N(x)\}$.

Let $x\in X$. Then $F_N(x)\in[-1,0]$. Choose $\gamma=F_N(x)$. Thus $F_N(x)\leq\gamma$, so $x\in U(F_N;\gamma)\neq\varnothing$. By assumption, we have $U(F_N;\gamma)$ is a UP-filter of X and so $0\in U(F_N;\gamma)$. Thus $F_N(0)\geq\gamma=F_N(x)$. Next, let $x,y\in X$. Then $F_N(x\cdot y),F_N(x)\in[-1,0]$. Choose $\gamma=\min\{F_N(x\cdot y),F_N(x)\}$. Thus $F_N(x\cdot y)\geq\gamma$ and $F_N(x)\geq\gamma$, so $x\cdot y,x\in U(F_N;\gamma)\neq\varnothing$. By assumption, we have $U(F_N;\gamma)$ is a UP-filter of X and so $y\in U(F_N;\gamma)$. Thus $F_N(y)\geq\gamma=\min\{F_N(x\cdot y),F_N(x)\}$.

Therefore, X_N is a special neutrosophic $\mathcal{N}$-UP-filter of X.

Theorem 6.4 A neutrosophic $\mathcal{N}$-structure X_N over X is a special neutrosophic $\mathcal{N}$-UP-ideals of X if and only if for all $\alpha,\beta,\gamma\in[-1,0]$, the sets $U(T_N;\alpha), L(I_N;\beta)$, and $U(F_N;\gamma)$ are UP-ideals of X if $U(T_N;\alpha), L(I_N;\beta)$, and $U(F_N;\gamma)$ are nonempty.

Proof. Assume that X_N is a special neutrosophic $\mathcal{N}$-UP-ideal of X. Let $\alpha,\beta,\gamma\in[-1,0]$ be such that $U(T_N;\alpha), L(I_N;\beta)$, and $U(F_N;\gamma)$ are nonempty.

Let $x\in U(T_N;\alpha)$. Then $T_N(x)\geq\alpha$. By (5.4), we have $T_N(0)\geq T_N(x)\geq\alpha$. Thus $0\in U(T_N;\alpha)$. Next, let $x\cdot(y\cdot z)\in U(T_N;\alpha)$ and $y\in U(T_N;\alpha)$. Then $T_N(x\cdot(y\cdot z))\geq\alpha$ and $T_N(y)\geq\alpha$, so α is a lower bound of $\{T_N(x\cdot(y\cdot z)),T_N(y)\}$. By (5.13), we have $T_N(x\cdot z)\ge\min\{T_N(x\cdot(y\cdot z)),T_N(y)\}\ge\alpha$. Thus $x\cdot z\in U(T_N;\alpha)$.

Let $x\in L(I_N;\beta)$. Then $I_N(x)\le\beta$. By (5.5), we have $I_N(0)\le I_N(x)\le\beta$. Thus $0\in L(I_N;\beta)$. Next, let $x\cdot(y\cdot z)\in L(I_N;\beta)$ and $y\in L(I_N;\beta)$. Then $I_N(x\cdot(y\cdot z))\le\beta$ and $I_N(y)\le\beta$, so β is an upper bound of $\{I_N(x\cdot(y\cdot z)),I_N(y)\}$. By (5.14), we have $I_N(x\cdot z)\le\max\{I_N(x\cdot(y\cdot z)),I_N(y)\}\le\beta$. Thus $x\cdot z\in L(I_N;\beta)$.

Let $x\in U(F_N;\gamma)$. Then $F_N(x)\ge\gamma$. By (5.6), we have $F_N(0)\ge F_N(x)\ge\gamma$. Thus $0\in U(F_N;\gamma)$. Next, let $x\cdot(y\cdot z)\in U(F_N;\gamma)$ and $y\in U(F_N;\gamma)$. Then $F_N(x\cdot(y\cdot z))\ge\gamma$ and $F_N(y)\ge\gamma$, so γ is a lower bound of $\{F_N(x\cdot(y\cdot z)),F_N(y)\}$. By (5.15), we have $F_N(x\cdot z)\ge\min\{F_N(x\cdot(y\cdot z)),F_N(y)\}\ge\gamma$. Thus $x\cdot z\in U(F_N;\gamma)$.

Hence, $U(T_N;\alpha),L(I_N;\beta)$, and $U(F_N;\gamma)$ are UP-ideals of X.

Conversely, assume that for all $\alpha,\beta,\gamma\in[-1,0]$, the set $U(T_N;\alpha),L(I_N;\beta)$, and $U(F_N;\gamma)$ are UP-ideals if $U(T_N;\alpha),L(I_N;\beta)$, and $U(F_N;\gamma)$ are nonempty.

Let $x\in X$. Then $T_N(x)\in[-1,0]$. Choose $\alpha=T_N(x)$. Thus $T_N(x)\ge\alpha$, so $x\in U(T_N;\alpha)\ne\varnothing$. By assumption, we have $U(T_N;\alpha)$ is a UP-ideal of X and so $0\in U(T_N;\alpha)$. Thus $T_N(0)\ge\alpha=T_N(x)$. Next, let $x,y,z\in X$. Then $T_N(x\cdot(y\cdot z)),T_N(y)\in[-1,0]$. Choose $\alpha=\min\{T_N(x\cdot(y\cdot z)),T_N(y)\}$. Thus $T_N(x\cdot(y\cdot z))\ge\alpha$ and $T_N(y)\ge\alpha$, so $x\cdot(y\cdot z),y\in U(T_N;\alpha)\ne\varnothing$. By assumption, we have $U(T_N;\alpha)$ is a UP-ideal of X and so $x\cdot z\in U(T_N;\alpha)$. Thus $T_N(x\cdot z)\ge\alpha=\min\{T_N(x\cdot(y\cdot z)),T_N(y)\}$.

Let $x\in X$. Then $I_N(x)\in[-1,0]$. Choose $\beta=I_N(x)$. Thus $I_N(x)\le\beta$, so $x\in L(I_N;\beta)\ne\varnothing$. By assumption, we have $L(I_N;\beta)$ is a UP-ideal of X and so $0\in L(I_N;\beta)$. Thus $I_N(0)\le\beta=I_N(x)$. Next, let $x,y,z\in X$. Then $I_N(x\cdot(y\cdot z)),I_N(y)\in[-1,0]$. Choose $\beta=\max\{I_N(x\cdot(y\cdot z)),I_N(y)\}$. Thus $I_N(x\cdot(y\cdot z))\le\beta$ and $I_N(y)\le\beta$, so $x\cdot(y\cdot z),y\in L(I_N;\beta)\ne\varnothing$. By assumption, we have $L(I_N;\beta)$ is a UP-ideal of X and so $x\cdot z\in L(I_N;\beta)$. Thus $I_N(x\cdot z)\le\beta=\max\{I_N(x\cdot(y\cdot z)),I_N(y)\}$.

Let $x\in X$. Then $F_N(x)\in[-1,0]$. Choose $\gamma=F_N(x)$. Thus $F_N(x)\ge\gamma$, so $x\in U(F_N;\gamma)\ne\varnothing$. By assumption, we have $U(F_N;\gamma)$ is a UP-ideal of X and so $0\in U(F_N;\gamma)$. Thus $F_N(0)\ge\gamma=F_N(x)$. Next, let $x,y,z\in X$. Then $F_N(x\cdot(y\cdot z)),F_N(y)\in[-1,0]$. Choose $\gamma=\min\{F_N(x\cdot(y\cdot z)),F_N(y)\}$. Thus $F_N(x\cdot(y\cdot z))\ge\gamma$ and $F_N(y)\ge\gamma$, so $x\cdot(y\cdot z),y\in U(F_N;\gamma)\ne\varnothing$. By assumption, we have $U(F_N;\gamma)$ is a UP-ideal of X and so $x\cdot z\in U(F_N;\gamma)$. Thus $F_N(x\cdot z)\ge\gamma=\min\{F_N(x\cdot(y\cdot z)),F_N(y)\}$.

Therefore, X_N is a special neutrosophic $\mathcal{N}$-UP-ideal of X.

Definition 6.5 Let X_N be a neutrosophic $\mathcal{N}$-structure over X. For $\alpha,\beta,\gamma\in[-1,0]$, the sets

$$ULU_{X_N}(\alpha,\beta,\gamma)=\{x\in X\mid T_N\ge\alpha,I_N\le\beta,F_N\ge\gamma\},$$

$$LUL_{X_N}(\alpha,\beta,\gamma)=\{x\in X\mid T_N\le\alpha,I_N\ge\beta,F_N\le\gamma\},$$

$$E_{X_N}(\alpha,\beta,\gamma)=\{x\in X\mid T_N=\alpha,I_N=\beta,F_N=\gamma\}$$

are called a ULU-(α,β,γ)*-level subset*, an LUL-(α,β,γ)*-level subset*, and an E-(α,β,γ)*-level subset* of X_N, respectively. Then we see that

$$ULU_{X_N}(\alpha,\beta,\gamma)=U(T_N;\alpha)\cap L(I_N;\beta)\cap U(F_N;\gamma),$$

$$LUL_{X_N}(\alpha,\beta,\gamma)=L(T_N;\alpha)\cap U(I_N;\beta)\cap L(F_N;\gamma),$$

$$E_{X_N}(\alpha,\beta,\gamma) = E(T_N;\alpha) \cap E(I_N;\beta) \cap E(F_N;\gamma).$$

Corollary 6.6 A neutrosophic $\mathcal{N}$-structure X_N over X is a neutrosophic $\mathcal{N}$-UP-subalgebra of X if and only if for all $\alpha,\beta,\gamma \in [-1,0]$, $LUL_{X_N}(\alpha,\beta,\gamma)$ is a UP-subalgebra of X where $LUL_{X_N}(\alpha,\beta,\gamma)$ is nonempty.

Proof. It is straightforward by Theorem 4.2.

Corollary 6.7 A neutrosophic $\mathcal{N}$-structure X_N over X is a neutrosophic $\mathcal{N}$-near UP-filter of X if and only if for all $\alpha,\beta,\gamma \in [-1,0]$, $LUL_{X_N}(\alpha,\beta,\gamma)$ is a near UP-filter of X where $LUL_{X_N}(\alpha,\beta,\gamma)$ is nonempty.

Proof. It is straightforward by Theorem 4.3.

Corollary 6.8 A neutrosophic $\mathcal{N}$-structure X_N over X is a neutrosophic $\mathcal{N}$-UP-filter of X if and only if for all $\alpha,\beta,\gamma \in [-1,0]$, $LUL_{X_N}(\alpha,\beta,\gamma)$ is a UP-filter of X where $LUL_{X_N}(\alpha,\beta,\gamma)$ is nonempty.

Proof. It is straightforward by Theorem 4.4.

Corollary 6.9 A neutrosophic $\mathcal{N}$-structure X_N over X is a neutrosophic $\mathcal{N}$-UP-ideal of X if and only if for all $\alpha,\beta,\gamma \in [-1,0]$, $LUL_{X_N}(\alpha,\beta,\gamma)$ is a UP-ideal of X where $LUL_{X_N}(\alpha,\beta,\gamma)$ is nonempty.

Proof. It is straightforward by Theorem 4.5.

Corollary 6.10 A neutrosophic $\mathcal{N}$-structure X_N over X is a neutrosophic $\mathcal{N}$-strongly UP-ideal of X if and only if $E(T_N, T_N(0)) = X, E(I_N, I_N(0)) = X$, and $E(F_N, F_N(0)) = X$.

Proof. It is straightforward by Theorem 4.6.

Corollary 6.11 A neutrosophic $\mathcal{N}$-structure X_N over X is a special neutrosophic $\mathcal{N}$-UP-subalgebra of X if and only if for all $\alpha,\beta,\gamma \in [-1,0]$, $ULU_{X_N}(\alpha,\beta,\gamma)$ is a UP-subalgebra of X where $ULU_{X_N}(\alpha,\beta,\gamma)$ is nonempty.

Proof. It is straightforward by Theorem 6.1.

Corollary 6.12 A neutrosophic $\mathcal{N}$-structure X_N over X is a special neutrosophic $\mathcal{N}$-near UP-filter of X if and only if for all $\alpha,\beta,\gamma \in [-1,0]$, $ULU_{X_N}(\alpha,\beta,\gamma)$ is a near UP-filter of X where $ULU_{X_N}(\alpha,\beta,\gamma)$ is nonempty.

Proof. It is straightforward by Theorem 6.2.

Corollary 6.13 A neutrosophic $\mathcal{N}$-structure X_N over X is a special neutrosophic $\mathcal{N}$-UP-filter of X if and only if for all $\alpha,\beta,\gamma \in [-1,0]$, $ULU_{X_N}(\alpha,\beta,\gamma)$ is a UP-filter of X where $ULU_{X_N}(\alpha,\beta,\gamma)$ is nonempty.

Proof. It is straightforward by Theorem 6.3.

Corollary 6.14 A neutrosophic $\mathcal{N}$-structure X_N over X is a special neutrosophic $\mathcal{N}$-UP-ideal of X if and only if for all $\alpha,\beta,\gamma \in [-1,0]$, $ULU_{X_N}(\alpha,\beta,\gamma)$ is a UP-ideal of X where $ULU_{X_N}(\alpha,\beta,\gamma)$ is nonempty.

Proof. It is straightforward by Theorem 6.4.

7. Conclusions

In this paper, we have introduced the notions of (special) neutrosophic N -UP-subalgebras, (special) neutrosophic N -near UP-filters, (special) neutrosophic N -UP-filters, (special) neutrosophic N -UP-ideals, and (special) neutrosophic N -strongly UP-ideals of UP-algebras and investigated some of their important properties. Then we have that the notion of (special) neutrosophic N -UP-subalgebras is a generalization of (special) neutrosophic N -near UP-filters, (special) neutrosophic N -near UP-filters is a generalization of (special) neutrosophic N -UP-filters, (special) neutrosophic N -UP-filters is a generalization of (special) neutrosophic N -UP-ideals, and (special) neutrosophic N -UP-ideals is a generalization of (special) neutrosophic N -strongly UP-ideals. Moreover, we obtain that (special) neutrosophic N -strongly UP-ideals and constant neutrosophic N -structures coincide.

In our future study, we will apply these notion/results to other type of neutrosophic N -structures in UP-algebras. Also, we will study the soft set theory/cubic set theory of such neutrosophic N -structures.

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New Operators on Interval Valued Neutrosophic Sets

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Abstract. As a generalization of fuzzy sets and intuitionistic fuzzy sets, neutrosophic sets have been developed by F. Smarandache to represent imprecise, incomplete and inconsistent information existing in the real world. A neutrosophic set is characterized by a truth-membership function, an indeterminacy-membership function, and a falsity-membership function. An interval neutrosophic set is an instance of a neutrosophic set, which can be used in real scientific and engineering applications. In this paper we have defined some new operators on interval valued neutrosophic sets and studied their properties. In addition, we give numerical examples to illustrate the defined operations.

Keywords: Neutrosophic set, new operators on interval valued neutrosophic sets.

1 **Introduction**

In 1999, a Russian scientist Molodstov [1] initiated the concept of soft set theory as a fundamental mathematical tool for modelling uncertainty, vague concepts and not clearly defined objects. Although various traditional tools, including but not limited to rough set theory [2], fuzzy set theory [3], intuitionistic fuzzy set theory [4] etc. have been used by many researchers to extract useful information hidden in the uncertain data, but there are inherent complications connected with each of these theories. Additionally, all these approachess lack in parameterizations of the tools and hence they couldn't be applied effectively in real life problems, especially in areas like environmental, economic and social problems. Soft set theory is standing uniquely in the sense that it is free from the above mentioned impediments and obliges approximate illustration of an object from the beginning, which makes this theory a natural mathematical formalism for approximate reasoning.

The notion of intuitionistic fuzzy set (IFS) was initiated by Atanassov as a significant generalization of fuzzy set. Intuitionistic fuzzy sets are very useful in situations when description of a problem by a linguistic variable, given in terms of a membership function only, seems too complicated. Recently intuitionistic fuzzy sets have been applied to many fields such as logic programming, medical diagnosis, decision making problems etc. The intuitionistic fuzzy sets can only handle the incomplete information considering both the truth membership (or simply membership) and falsity membership (or non-membership) values. But it doesn't handle the indeterminate and inconsistent information which exists in belief system. In 1995, F. Smarandache [05, 06] introduced the concept of neutrosphic set which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data. This concept has been successfully applied to many fields such as databases [7, 8], medical diagnosis problem [9], decision making problem [10], topology [11], control theory [12] etc.

Presently works on the neutrosophic set theory is progressing rapidly. Bhowmik and Pal [13, 14] defined intuitionistic neutrosophic set. Later on Salam and Alblowi [15] introduced another concept called Generalized neutrosophic set. Wang et al. [16] proposed another extension of neutrosophic set which is single valued neutrosophic. Also Wang et al. [17] introduced the notion of interval valued neutrosophic set which is an instance of neutrosophic set. It is characterized by an interval membership degree, interval indeterminacy degree and interval non-membership degree. Ye [18, 19] defined similarity measures between interval neutrosophic sets and their multicriteria decision-making method. Majumdar and Samanta [20] proposed some types of similarity and entropy of neutrosophic sets. Broumi and Smarandache [21, 22, 23] proposed several similarity measures of neutrosophic sets. S. Broumi and F. Smarandache defined four new operations on interval-valued intuitionistic hesitant fuzzy sets and studied their important properties. F.G. Lupianez [24] defined the notion of neutrosophic topology on the non-standard interval. Majumder [25] discussed the distance and similarity between two neutrosophic sets . He also introduced the notion of entropy to measure the amount of uncertainty expressed by a neutrosophic set. H. Zhang et al. [26] defined operations for interval neutrosophic sets and a comparison approach was put forward based on the related research of interval valued intuitionistic fuzzy sets. He also developed two interval neutrosophic number aggregation operators and using these, a multi-criteria decision making problem was explored. H.Wang et al. [27] presented various properties of interval neutrosophic sets based on set theoretic operators. In 2017, Bera and Mahapatra [28] initiated the concept of neutrosophic soft matrix and they successfully applied it to solve decision making problems. Song et al. [29] applied neutrosophic sets to ideals in BCK/BCI algebras. Shahzadi et al [30] applied single valued neutrosophic sets in medical diagnosis. Recently, Thao and Smaran [31] proposed the concept of divergence measure on neutrosophic sets with an application to medical problem. Some recent applications of neutrosophic sets can be found in [32-39].

This paper is an attempt to define some new operators on interval valued neutrosophic sets and to study their properties. In addition to that, we have given numerical examples to illustrate the defined operations. The organization of this paper is as follow: In section 2, we briefly present some basic definitions which will be used in the rest of the paper. In section 3, we define some new operations on interval valued neutrosophic sets and discuss their properties. In section 5, conclusion is given. Lastly all the related references are given.

2 Preliminaries

2.1 Definition [3]:

Let U be a non empty set. Then a fuzzy set τ on U is a set having the form $\tau=\{(\mathrm{x}, \mu_\tau(\mathrm{x})) : \mathrm{x}\in \mathrm{U}\}$

where the function $\mu_\tau : \mathrm{U}\to[0, 1]$ is called the membership function and $\mu_\tau(\mathrm{x})$ represents the degree of membership of each element $\mathrm{x}\in\mathrm{U}$.

2.2 Definition [4]:

Let U be a non empty set. Then an intuitionistic fuzzy set (IFS for short) τ is an object having the form $\tau=\{\langle \mathrm{x}, \mu_\tau(\mathrm{x}), \gamma_\tau(\mathrm{x})\rangle : \mathrm{x}\in\mathrm{U}\}$ where the functions $\mu_\tau:\mathrm{U}\to[0, 1]$ and $\gamma_\tau:\mathrm{U}\to[0, 1]$ are called membership function and non-membership function respectively. $\mu_\tau(\mathrm{x})$ and $\gamma_\tau(\mathrm{x})$ represent the degree of membership and the degree of non-membership respectively of each element $\mathrm{x}\in \mathrm{U}$ and $0\le\mu_\tau(\mathrm{x})+\gamma_\tau(\mathrm{x})\le 1$ for each $\mathrm{x}\in\mathrm{U}$.

We denote the class of all intuitionistic fuzzy sets on U by $\mathrm{IFS}^{\mathrm{U}}$.

2.3 Definition [5, 6]:

Let U be a non empty set. Then a neutrosophic set (NS for short) Γ is an object having the form $\Gamma = \{\langle x, \mu_\Gamma(x), \gamma_\Gamma(x), \delta_\Gamma(x)\rangle : x \in U\}$ where the functions $\mu_\Gamma, \gamma_\Gamma, \delta_\Gamma : U \to]^-0, 1^+[$ and $^-0 \le \mu_\Gamma(x) + \gamma_\Gamma(x) + \delta_\Gamma(x) \le 3^+$. From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. But in real life applications in scientific and engineering problems it is difficult to use neutrosophic sets with value from real standard or non-standard subsets of $]^-0, 1^+[$. Hence we consider the neutrophic set which takes the value from the subset of [0, 1] i.e; $0 \le \mu_\Gamma(x) + \gamma_\Gamma(x) + \delta_\Gamma(x) \le 3$ where $\mu_\Gamma, \gamma_\Gamma$ and δ_Γ are called truth membership function, indeterminacy membership function and falsity function respectively.

We denote the class of all neutrosophic sets on U by NS^U.

2.4 Definition [17]:

Let U be a non empty set. Then an interval valued neutrosophic set (IVNS for short) Γ is an object having the form

$$\Gamma = \left\{\left\langle x, \left[\inf\mu_\Gamma(x), \sup\mu_\Gamma(x)\right], \left[\inf\gamma_\Gamma(x), \sup\gamma_\Gamma(x)\right], \left[\inf\delta_\Gamma(x), \sup\delta_\Gamma(x)\right]\right\rangle : x \in U\right\}$$

where the functions $\mu_\Gamma, \gamma_\Gamma, \delta_\Gamma : U \to Int([0, 1])$ and $0 \le \sup\mu_\Gamma(x) + \sup\gamma_\Gamma(x) + \sup\delta_\Gamma(x) \le 3$.

We denote the class of all interval valued neutrosophic sets on U by $IVNS^U$.

2.5 Definition [17]:

Let Γ, Ω be two interval neutrosophic sets on U. Then

(a) Γ is called a subset of Ω, denoted by $\Gamma \subseteq \Omega$ if

$\inf\mu_\Gamma(x) \le \inf\mu_\Omega(x)$, $\sup\mu_\Gamma(x) \le \sup\mu_\Omega(x)$, $\inf\gamma_\Gamma(x) \ge \inf\gamma_\Omega(x)$, $\sup\gamma_\Gamma(x) \ge \sup\gamma_\Omega(x)$,
$\inf\delta_\Gamma(x) \le \inf\delta_\Omega(x)$, $\sup\delta_\Gamma(x) \le \sup\delta_\Omega(x)$ $\forall x \in U$.

(b) The intersection of Γ and Ω is denoted by $\Gamma \cap \Omega$ and is defined by

$$\begin{aligned}\Gamma \cap \Omega = \Big\{\Big\langle &\left[\min\left(\inf\mu_\Gamma(x), \inf\mu_\Omega(x)\right), \min\left(\sup\mu_\Gamma(x), \sup\mu_\Omega(x)\right)\right], \\ &\left[\max\left(\inf\gamma_\Gamma(x), \inf\gamma_\Omega(x)\right), \max\left(\sup\gamma_\Gamma(x), \sup\gamma_\Omega(x)\right)\right], \\ &\left[\max\left(\inf\delta_\Gamma(x), \inf\delta_\Omega(x)\right), \max\left(\sup\delta_\Gamma(x), \sup\delta_\Omega(x)\right)\right]\Big\rangle : x \in U\Big\}.\end{aligned}$$

(c) The union of Γ and Ω is denoted by $\Gamma \cup \Omega$ and is defined by

$$\begin{aligned}\Gamma \cup \Omega = \Big\{\Big\langle &\left[\max\left(\inf\mu_\Gamma(x), \inf\mu_\Omega(x)\right), \max\left(\sup\mu_\Gamma(x), \sup\mu_\Omega(x)\right)\right], \\ &\left[\min\left(\inf\gamma_\Gamma(x), \inf\gamma_\Omega(x)\right), \min\left(\sup\gamma_\Gamma(x), \sup\gamma_\Omega(x)\right)\right], \\ &\left[\min\left(\inf\delta_\Gamma(x), \inf\delta_\Omega(x)\right), \min\left(\sup\delta_\Gamma(x), \sup\delta_\Omega(x)\right)\right]\Big\rangle : x \in U\Big\}.\end{aligned}$$

(d) The complement of Γ is denoted by Γ^c and is defined by

$$\Gamma^c = \left\{\left\langle x, \left[\inf\delta_\Gamma(x), \sup\delta_\Gamma(x)\right], \left[1\text{-}\sup\gamma_\Gamma(x), 1 - \inf\gamma_\Gamma(x)\right], \left[\inf\mu_\Gamma(x), \sup\mu_\Gamma(x)\right]\right\rangle : x \in U\right\}$$

3. New Operators on Interval Valued Neutrosophic Sets

In this section we have proposed two new operators defined on interval valued neutrosophic sets. We also present their basic properties.

3.1 Definition:

The operator $\Box : IVNS^U \rightarrow IVNS^U$ is defined by

$$\Box\Gamma = \left\{ \left\langle \mathrm{x},\ \left[\inf \mu_\Gamma(\mathrm{x}), \sup \mu_\Gamma(\mathrm{x})\right],\ \left[\inf \gamma_\Gamma(\mathrm{x}), \sup \gamma_\Gamma(\mathrm{x})\right], \left[\inf \delta_\Gamma(\mathrm{x}), 1-\sup \mu_\Gamma(\mathrm{x})\right] \right\rangle :\ \mathrm{x} \in \mathrm{U} \right\},$$

for $\Gamma \in IVNS^U$.

3.2 Example:

Let us consider an interval valued neutrosophic set Γ on U given by

$$\Gamma = \left\{ \left\langle \mathrm{a},\ [0.2, 0.4],\ [0.6, 0.3], [0.3, 0.5] \right\rangle, \left\langle \mathrm{b},\ [0.6, 0.8],\ [0.5, 0.6], [0.1, 0.4] \right\rangle \right\}.$$

Then we have $\Box\Gamma = \left\{ \left\langle \mathrm{a},\ [0.2, 0.4],\ [0.6, 0.3], [0.3, 0.5] \right\rangle, \left\langle \mathrm{b},\ [0.6, 0.8],\ [0.5, 0.6], [0.1, 0.6] \right\rangle \right\}$.

3.3 Definition:

The operator $\Diamond : IVNS^U \rightarrow IVNS^U$ is defined by

$$\Diamond\Gamma = \left\{ \left\langle \mathrm{x},\ \left[\inf \mu_\Gamma(\mathrm{x}), 1-\sup \delta_\Gamma(\mathrm{x})\right],\ \left[\inf \gamma_\Gamma(\mathrm{x}), \sup \gamma_\Gamma(\mathrm{x})\right], \left[\inf \delta_\Gamma(\mathrm{x}), \sup \delta_\Gamma(\mathrm{x})\right] \right\rangle :\ \mathrm{x} \in \mathrm{U} \right\},$$

for $\Gamma \in IVNS^U$.

3.4 Example:

Let us consider an interval valued neutrosophic set Γ on U given by

$$\Gamma = \left\{ \left\langle \mathrm{a},\ [0.2, 0.4],\ [0.6, 0.3], [0.3, 0.5] \right\rangle, \left\langle \mathrm{b},\ [0.3, 0.8],\ [0.5, 0.6], [0.1, 0.4] \right\rangle \right\}.$$

Then we have $\Diamond\Gamma = \left\{ \left\langle \mathrm{a},\ [0.2, 0.5],\ [0.6, 0.3], [0.3, 0.5] \right\rangle, \left\langle \mathrm{b},\ [0.3, 0.6],\ [0.5, 0.6], [0.1, 0.4] \right\rangle \right\}$.

3.5 Theorem:

For $\Gamma \in IVNS^U$, we have the followings

(a) $\left(\Box\Gamma^c\right)^c = \Diamond\Gamma$

(b) $\left(\Diamond\Gamma^c\right)^c = \Box\Gamma$

(c) $\Box\Gamma \subseteq \Gamma \subseteq \Diamond\Gamma$

(d) $\Box\left(\Box\Gamma\right) = \Box\Gamma$

(e) $\Diamond\left(\Diamond\Gamma\right) = \Diamond\Gamma$

(f) $\Box\left(\Diamond\Gamma\right) = \Diamond\Gamma$

(g) $\Diamond\left(\Box\Gamma\right) = \Box\Gamma$

<u>Proof:</u>

(a) $\Gamma = \left\{ \left\langle \mathrm{x},\ \left[\inf \mu_\Gamma(\mathrm{x}), \sup \mu_\Gamma(\mathrm{x})\right],\ \left[\inf \gamma_\Gamma(\mathrm{x}), \sup \gamma_\Gamma(\mathrm{x})\right], \left[\inf \delta_\Gamma(\mathrm{x}), \sup \delta_\Gamma(\mathrm{x})\right] \right\rangle :\ \mathrm{x} \in \mathrm{U} \right\}$

$\Rightarrow \Gamma^c = \left\{ \left\langle \mathrm{x},\ \left[\inf \delta_\Gamma(\mathrm{x}), \sup \delta_\Gamma(\mathrm{x})\right],\ \left[1\text{-}\sup \gamma_\Gamma(\mathrm{x}), 1-\inf \gamma_\Gamma(\mathrm{x})\right], \left[\inf \mu_\Gamma(\mathrm{x}), \sup \mu_\Gamma(\mathrm{x})\right] \right\rangle :\ \mathrm{x} \in \mathrm{U} \right\}$

$$\therefore \Box\Gamma^{c}=\left\{\left\langle \text{x},\ \left[\inf\delta_{\Gamma}(\text{x}),\sup\delta_{\Gamma}(\text{x})\right],\ \left[1\text{-}\sup\gamma_{\Gamma}(\text{x}),1-\inf\gamma_{\Gamma}(\text{x})\right],\left[\inf\mu_{\Gamma}(\text{x}),1-\sup\delta_{\Gamma}(\text{x})\right]\right\rangle :\ \text{x}\in \text{U}\right\}.$$

Hence $\left(\Box\Gamma^{c}\right)^{c}$

$$=\left\{\left\langle \text{x},\left[\inf\mu_{\Gamma}(\text{x}),1-\sup\delta_{\Gamma}(\text{x})\right],\ \left[\inf\gamma_{\Gamma}(\text{x}),\sup\gamma_{\Gamma}(\text{x})\right],\left[\inf\delta_{\Gamma}(\text{x}),\sup\delta_{\Gamma}(\text{x})\right]\right\rangle :\ \text{x}\in \text{U}\right\}=\Diamond\Gamma.$$

(b) $\Gamma=\left\{\left\langle \text{x},\ \left[\inf\mu_{\Gamma}(\text{x}),\sup\mu_{\Gamma}(\text{x})\right],\ \left[\inf\gamma_{\Gamma}(\text{x}),\sup\gamma_{\Gamma}(\text{x})\right],\left[\inf\delta_{\Gamma}(\text{x}),\sup\delta_{\Gamma}(\text{x})\right]\right\rangle :\ \text{x}\in \text{U}\right\}$

$$\Rightarrow \Gamma^{c}=\left\{\left\langle \text{x},\ \left[\inf\delta_{\Gamma}(\text{x}),\sup\delta_{\Gamma}(\text{x})\right],\ \left[1\text{-}\sup\gamma_{\Gamma}(\text{x}),1-\inf\gamma_{\Gamma}(\text{x})\right],\left[\inf\mu_{\Gamma}(\text{x}),\sup\mu_{\Gamma}(\text{x})\right]\right\rangle :\ \text{x}\in \text{U}\right\}$$

$$\therefore \Diamond\Gamma^{c}=\left\{\left\langle \text{x},\ \left[\inf\delta_{\Gamma}(\text{x}),1-\sup\mu_{\Gamma}(\text{x})\right],\ \left[1\text{-}\sup\gamma_{\Gamma}(\text{x}),1-\inf\gamma_{\Gamma}(\text{x})\right],\left[\inf\mu_{\Gamma}(\text{x}),\sup\mu_{\Gamma}(\text{x})\right]\right\rangle :\ \text{x}\in \text{U}\right\}.$$

Hence $\left(\Diamond\Gamma^{c}\right)^{c}$

$$=\left\{\left\langle \text{x},\left[\inf\mu_{\Gamma}(\text{x}),\sup\mu_{\Gamma}(\text{x})\right],\ \left[\inf\gamma_{\Gamma}(\text{x}),\sup\gamma_{\Gamma}(\text{x})\right],\left[\inf\delta_{\Gamma}(\text{x}),1-\sup\mu_{\Gamma}(\text{x})\right]\right\rangle :\ \text{x}\in \text{U}\right\}=\Box\Gamma.$$

(c) Proof is straight forward.

(d) $\Gamma=\left\{\left\langle \text{x},\ \left[\inf\mu_{\Gamma}(\text{x}),\sup\mu_{\Gamma}(\text{x})\right],\ \left[\inf\gamma_{\Gamma}(\text{x}),\sup\gamma_{\Gamma}(\text{x})\right],\left[\inf\delta_{\Gamma}(\text{x}),\sup\delta_{\Gamma}(\text{x})\right]\right\rangle :\ \text{x}\in \text{U}\right\}$

$$\Rightarrow\Box\Gamma=\left\{\left\langle \text{x},\ \left[\inf\mu_{\Gamma}(\text{x}),\sup\mu_{\Gamma}(\text{x})\right],\ \left[\inf\gamma_{\Gamma}(\text{x}),\sup\gamma_{\Gamma}(\text{x})\right],\left[\inf\delta_{\Gamma}(\text{x}),1-\sup\mu_{\Gamma}(\text{x})\right]\right\rangle :\ \text{x}\in \text{U}\right\}$$

$$\Rightarrow\Box(\Box\Gamma)=\left\{\left\langle \text{x},\ \left[\inf\mu_{\Gamma}(\text{x}),\sup\mu_{\Gamma}(\text{x})\right],\ \left[\inf\gamma_{\Gamma}(\text{x}),\sup\gamma_{\Gamma}(\text{x})\right],\left[\inf\delta_{\Gamma}(\text{x}),1-\sup\mu_{\Gamma}(\text{x})\right]\right\rangle :\ \text{x}\in \text{U}\right\}$$

$$=\Box\Gamma.$$

(e) Proof is similar to (d).

(f) $\Gamma=\left\{\left\langle \text{x},\ \left[\inf\mu_{\Gamma}(\text{x}),\sup\mu_{\Gamma}(\text{x})\right],\ \left[\inf\gamma_{\Gamma}(\text{x}),\sup\gamma_{\Gamma}(\text{x})\right],\left[\inf\delta_{\Gamma}(\text{x}),\sup\delta_{\Gamma}(\text{x})\right]\right\rangle :\ \text{x}\in \text{U}\right\}$

$$\Rightarrow \Diamond\Gamma=\left\{\left\langle \text{x},\ \left[\inf\mu_{\Gamma}(\text{x}),1-\sup\delta_{\Gamma}(\text{x})\right],\ \left[\inf\gamma_{\Gamma}(\text{x}),\sup\gamma_{\Gamma}(\text{x})\right],\left[\inf\delta_{\Gamma}(\text{x}),\sup\delta_{\Gamma}(\text{x})\right]\right\rangle :\ \text{x}\in \text{U}\right\}$$

$$\Rightarrow\Box(\Diamond\Gamma)=\left\{\left\langle \text{x},\ \left[\inf\mu_{\Gamma}(\text{x}),1-\sup\delta_{\Gamma}(\text{x})\right],\ \left[\inf\gamma_{\Gamma}(\text{x}),\sup\gamma_{\Gamma}(\text{x})\right],\left[\inf\delta_{\Gamma}(\text{x}),\sup\delta_{\Gamma}(\text{x})\right]\right\rangle :\ \text{x}\in \text{U}\right\}$$

$$=\Diamond\Gamma.$$

(g) Proof is similar to (f).

3.6 Theorem:

For $\Gamma,\Omega\in IVNS^{U}$, we have the followings

(a) $\Box(\Gamma\cup\Omega)=\Box\Gamma\cup\Box\Omega$

(b) $\Box(\Gamma\cap\Omega)=\Box\Gamma\cap\Box\Omega$

(c) $\Diamond(\Gamma\cup\Omega)=\Diamond\Gamma\cup\Diamond\Omega$

(d) $\Diamond(\Gamma\cap\Omega)=\Diamond\Gamma\cap\Diamond\Omega$

Proof:

We have,

$\Gamma=\left\{\left\langle \text{x},\ \left[\inf\mu_{\Gamma}(\text{x}),\sup\mu_{\Gamma}(\text{x})\right],\ \left[\inf\gamma_{\Gamma}(\text{x}),\sup\gamma_{\Gamma}(\text{x})\right],\left[\inf\delta_{\Gamma}(\text{x}),\sup\delta_{\Gamma}(\text{x})\right]\right\rangle :\ \text{x}\in \text{U}\right\}$ and

$\Omega=\left\{\left\langle \text{x},\ \left[\inf\mu_{\Omega}(\text{x}),\sup\mu_{\Omega}(\text{x})\right],\ \left[\inf\gamma_{\Omega}(\text{x}),\sup\gamma_{\Omega}(\text{x})\right],\left[\inf\delta_{\Omega}(\text{x}),\sup\delta_{\Omega}(\text{x})\right]\right\rangle :\ \text{x}\in \text{U}\right\}$.

(a) $\Gamma\cup\Omega$

$$=\Big\{\Big\langle\Big[\max\big(\inf\mu_{\Gamma}(x),\inf\mu_{\Omega}(x)\big),\ \max\big(\sup\mu_{\Gamma}(x),\sup\mu_{\Omega}(x)\big)\Big],$$
$$\Big[\min\big(\inf\gamma_{\Gamma}(x),\inf\gamma_{\Omega}(x)\big),\min\big(\sup\gamma_{\Gamma}(x),\sup\gamma_{\Omega}(x)\big)\Big],$$
$$\Big[\min\big(\inf\delta_{\Gamma}(x),\inf\delta_{\Omega}(x)\big),\min\big(\sup\delta_{\Gamma}(x),\sup\delta_{\Omega}(x)\big)\Big]\Big\rangle : x\in U\Big\}.$$

Hence $\Box(\Gamma\cup\Omega)$

$$=\Big\{\Big\langle\Big[\max\big(\inf\mu_{\Gamma}(x),\inf\mu_{\Omega}(x)\big),\ \max\big(\sup\mu_{\Gamma}(x),\sup\mu_{\Omega}(x)\big)\Big],$$
$$\Big[\min\big(\inf\gamma_{\Gamma}(x),\inf\gamma_{\Omega}(x)\big),\min\big(\sup\gamma_{\Gamma}(x),\sup\gamma_{\Omega}(x)\big)\Big],$$
$$\Big[\min\big(\inf\delta_{\Gamma}(x),\inf\delta_{\Omega}(x)\big),1-\sup\Big(\max\big(\sup\mu_{\Gamma}(x),\sup\mu_{\Omega}(x)\big)\Big)\Big]\Big\rangle : x\in U\Big\}$$
$$=\Big\{\Big\langle\Big[\max\big(\inf\mu_{\Gamma}(x),\inf\mu_{\Omega}(x)\big),\ \max\big(\sup\mu_{\Gamma}(x),\sup\mu_{\Omega}(x)\big)\Big],$$
$$\Big[\min\big(\inf\gamma_{\Gamma}(x),\inf\gamma_{\Omega}(x)\big),\min\big(\sup\gamma_{\Gamma}(x),\sup\gamma_{\Omega}(x)\big)\Big],$$
$$\Big[\min\big(\inf\delta_{\Gamma}(x),\inf\delta_{\Omega}(x)\big),\min\big(1-\sup\mu_{\Gamma}(x),1-\sup\mu_{\Omega}(x)\big)\Big]\Big\rangle : x\in U\Big\}.$$

Again

$$\Box\Gamma=\Big\{\Big\langle x,\ \big[\inf\mu_{\Gamma}(x),\sup\mu_{\Gamma}(x)\big],\ \big[\inf\gamma_{\Gamma}(x),\sup\gamma_{\Gamma}(x)\big],\big[\inf\delta_{\Gamma}(x),1-\sup\mu_{\Gamma}(x)\big]\Big\rangle :\ x\in U\Big\}$$

and

$$\Box\Omega=\Big\{\Big\langle x,\ \big[\inf\mu_{\Omega}(x),\sup\mu_{\Omega}(x)\big],\ \big[\inf\gamma_{\Omega}(x),\sup\gamma_{\Omega}(x)\big],\big[\inf\delta_{\Omega}(x),1-\sup\mu_{\Omega}(x)\big]\Big\rangle :\ x\in U\Big\}$$

.

Hence $\Box\Gamma\cup\Box\Omega$

$$=\Big\{\Big\langle\Big[\max\big(\inf\mu_{\Gamma}(x),\inf\mu_{\Omega}(x)\big),\ \max\big(\sup\mu_{\Gamma}(x),\sup\mu_{\Omega}(x)\big)\Big],$$
$$\Big[\min\big(\inf\gamma_{\Gamma}(x),\inf\gamma_{\Omega}(x)\big),\min\big(\sup\gamma_{\Gamma}(x),\sup\gamma_{\Omega}(x)\big)\Big],$$
$$\Big[\min\big(\inf\delta_{\Gamma}(x),\inf\delta_{\Omega}(x)\big),\min\big(1-\sup\mu_{\Gamma}(x),1-\sup\mu_{\Omega}(x)\big)\Big]\Big\rangle : x\in U\Big\}.$$

Consequently, $\Box(\Gamma\cup\Omega)=\Box\Gamma\cup\Box\Omega$.

(b) Proof is similar to (a).

(c) $\Gamma\cup\Omega$

$$=\Big\{\Big\langle\Big[\max\big(\inf\mu_{\Gamma}(x),\inf\mu_{\Omega}(x)\big),\ \max\big(\sup\mu_{\Gamma}(x),\sup\mu_{\Omega}(x)\big)\Big],$$
$$\Big[\min\big(\inf\gamma_{\Gamma}(x),\inf\gamma_{\Omega}(x)\big),\min\big(\sup\gamma_{\Gamma}(x),\sup\gamma_{\Omega}(x)\big)\Big],$$
$$\Big[\min\big(\inf\delta_{\Gamma}(x),\inf\delta_{\Omega}(x)\big),\min\big(\sup\delta_{\Gamma}(x),\sup\delta_{\Omega}(x)\big)\Big]\Big\rangle : x\in U\Big\}.$$

Hence $\Diamond(\Gamma\cup\Omega)$

$$=\Big\{\Big\langle\Big[\max\big(\inf\mu_{\Gamma}(x),\inf\mu_{\Omega}(x)\big),\ 1-\sup\Big(\min\big(\sup\delta_{\Gamma}(x),\sup\delta_{\Omega}(x)\big)\Big)\Big],$$
$$\Big[\min\big(\inf\gamma_{\Gamma}(x),\inf\gamma_{\Omega}(x)\big),\min\big(\sup\gamma_{\Gamma}(x),\sup\gamma_{\Omega}(x)\big)\Big],$$
$$\Big[\min\big(\inf\delta_{\Gamma}(x),\inf\delta_{\Omega}(x)\big),\min\big(\sup\delta_{\Gamma}(x),\sup\delta_{\Omega}(x)\big)\Big]\Big\rangle : x\in U\Big\}$$

$$
\begin{aligned}
=\Big\{\Big\langle & \big[\max\big(\inf\mu_\Gamma(x),\inf\mu_\Omega(x)\big),\ \max\big(1-\sup\mu_\Gamma(x),1-\sup\mu_\Omega(x)\big)\big],\\
& \big[\min\big(\inf\gamma_\Gamma(x),\inf\gamma_\Omega(x)\big),\min\big(\sup\gamma_\Gamma(x),\sup\gamma_\Omega(x)\big)\big],\\
& \big[\min\big(\inf\delta_\Gamma(x),\inf\delta_\Omega(x)\big),\min\big(\sup\delta_\Gamma(x),\sup\delta_\Omega(x)\big)\big]\Big\rangle : x\in U\Big\}.
\end{aligned}
$$

Again

$$\Diamond\Gamma=\Big\{\Big\langle x,\ \big[\inf\mu_\Gamma(x),1-\sup\delta_\Gamma(x)\big],\ \big[\inf\gamma_\Gamma(x),\sup\gamma_\Gamma(x)\big],\big[\inf\delta_\Gamma(x),\sup\delta_\Gamma(x)\big]\Big\rangle :\ x\in U\Big\}$$

and

$$\Diamond\Omega=\Big\{\Big\langle x,\ \big[\inf\mu_\Omega(x),1-\sup\delta_\Omega(x)\big],\ \big[\inf\gamma_\Omega(x),\sup\gamma_\Omega(x)\big],\big[\inf\delta_\Omega(x),\sup\delta_\Omega(x)\big]\Big\rangle :\ x\in U\Big\}$$

Hence $\Box\Gamma\cup\Box\Omega$

$$
\begin{aligned}
=\Big\{\Big\langle & \big[\max\big(\inf\mu_\Gamma(x),\inf\mu_\Omega(x)\big),\ \max\big(1-\sup\delta_\Gamma(x),1-\sup\delta_\Omega(x)\big)\big],\\
& \big[\min\big(\inf\gamma_\Gamma(x),\inf\gamma_\Omega(x)\big),\min\big(\sup\gamma_\Gamma(x),\sup\gamma_\Omega(x)\big)\big],\\
& \big[\min\big(\inf\delta_\Gamma(x),\inf\delta_\Omega(x)\big),\min\big(\sup\delta_\Gamma(x),\sup\delta_\Omega(x)\big)\big]\Big\rangle : x\in U\Big\}.
\end{aligned}
$$

Consequently, $\Box(\Gamma\cup\Omega)=\Box\Gamma\cup\Box\Omega$.

(d) Proof is similar to (c).

3.7 Definition:

The operator $\circ : IVNS^U \rightarrow IFS^U$ is defined by

$$\circ\Gamma=\Big\{\Big\langle x,\ \inf\mu_\Gamma(x),\ \inf\gamma_\Gamma(x),\inf\delta_\Gamma(x)\Big\rangle :\ x\in U\Big\},\ \Gamma\in IVNS^U.$$

3.8 Example:

Let us consider an interval valued neutrosophic set Γ on U given by

$$\Gamma=\Big\{\Big\langle a,\ [0.2,0.4],\ [0.6,0.3],[0.3,0.5]\Big\rangle,\Big\langle b,\ [0.6,0.8],\ [0.5,0.6],[0.1,0.4]\Big\rangle\Big\}.$$

Then we have $\circ\Gamma=\Big\{\Big\langle a,\ 0.2,\ 0.6,0.3\Big\rangle,\Big\langle b,\ 0.6,\ 0.5,0.1\Big\rangle\Big\}$.

3.9 Theorem:

For $\Gamma\in IVNS^U$, we have

(a) $\circ(\Box\Gamma)=\circ\Gamma$

(b) $\circ(\Diamond\Gamma)=\circ\Gamma$

Proof:

We have,

$$\Gamma=\Big\{\Big\langle x,\ \big[\inf\mu_\Gamma(x),\sup\mu_\Gamma(x)\big],\ \big[\inf\gamma_\Gamma(x),\sup\gamma_\Gamma(x)\big],\big[\inf\delta_\Gamma(x),\sup\delta_\Gamma(x)\big]\Big\rangle :\ x\in U\Big\}.$$

Then

(a) $\Box\Gamma=\Big\{\Big\langle x,\ \big[\inf\mu_\Gamma(x),\sup\mu_\Gamma(x)\big],\ \big[\inf\gamma_\Gamma(x),\sup\gamma_\Gamma(x)\big],\big[\inf\delta_\Gamma(x),1-\sup\mu_\Gamma(x)\big]\Big\rangle :\ x\in U\Big\}$

and so $\circ(\Box\Gamma)=\Big\{\Big\langle x,\ \inf\mu_\Gamma(x),\ \inf\gamma_\Gamma(x),\inf\delta_\Gamma(x)\Big\rangle :\ x\in U\Big\}=\circ\Gamma.$

(b) Proof is similar to (a).

3.10 Theorem:

For $\Gamma, \Omega \in IVNS^U$, we have the followings

(a) $\circ(\Gamma \cup \Omega) = \circ\Gamma \cup \circ\Omega$

(b) $\circ(\Gamma \cap \Omega) = \circ\Gamma \cap \circ\Omega$

Proof:

We have,

$\Gamma = \{\langle \mathrm{x}, [\inf\mu_\Gamma(\mathrm{x}), \sup\mu_\Gamma(\mathrm{x})], [\inf\gamma_\Gamma(\mathrm{x}), \sup\gamma_\Gamma(\mathrm{x})], [\inf\delta_\Gamma(\mathrm{x}), \sup\delta_\Gamma(\mathrm{x})] \rangle : \mathrm{x} \in \mathrm{U}\}$ and

$\Omega = \{\langle \mathrm{x}, [\inf\mu_\Omega(\mathrm{x}), \sup\mu_\Omega(\mathrm{x})], [\inf\gamma_\Omega(\mathrm{x}), \sup\gamma_\Omega(\mathrm{x})], [\inf\delta_\Omega(\mathrm{x}), \sup\delta_\Omega(\mathrm{x})] \rangle : \mathrm{x} \in \mathrm{U}\}$.

(a) $\Gamma \cup \Omega$

$$= \{\langle x, [\max(\inf\mu_\Gamma(\mathrm{x}), \inf\mu_\Omega(\mathrm{x})), \max(\sup\mu_\Gamma(\mathrm{x}), \sup\mu_\Omega(\mathrm{x}))],$$
$$[\min(\inf\gamma_\Gamma(\mathrm{x}), \inf\gamma_\Omega(\mathrm{x})), \min(\sup\gamma_\Gamma(\mathrm{x}), \sup\gamma_\Omega(\mathrm{x}))],$$
$$[\min(\inf\delta_\Gamma(\mathrm{x}), \inf\delta_\Omega(\mathrm{x})), \min(\sup\delta_\Gamma(\mathrm{x}), \sup\delta_\Omega(\mathrm{x}))]\rangle : \mathrm{x} \in \mathrm{U}\}.$$

$\therefore \circ(\Gamma \cup \Omega)$

$$= \{\langle x, \max(\inf\mu_\Gamma(\mathrm{x}), \inf\mu_\Omega(\mathrm{x})), \min(\inf\gamma_\Gamma(\mathrm{x}), \inf\gamma_\Omega(\mathrm{x})), \min(\inf\delta_\Gamma(\mathrm{x}), \inf\delta_\Omega(\mathrm{x}))\rangle : \mathrm{x} \in \mathrm{U}\}.$$

Again we have, $\circ\Gamma \cup \circ\Omega$

$$= \{\langle \mathrm{x}, \inf\mu_\Gamma(\mathrm{x}), \inf\gamma_\Gamma(\mathrm{x}), \inf\delta_\Gamma(\mathrm{x}) \rangle : \mathrm{x} \in \mathrm{U}\} \cup \{\langle \mathrm{x}, \inf\mu_\Omega(\mathrm{x}), \inf\gamma_\Omega(\mathrm{x}), \inf\delta_\Omega(\mathrm{x}) \rangle : \mathrm{x} \in \mathrm{U}\}$$
$$= \{\langle x, \max(\inf\mu_\Gamma(\mathrm{x}), \inf\mu_\Omega(\mathrm{x})), \min(\inf\gamma_\Gamma(\mathrm{x}), \inf\gamma_\Omega(\mathrm{x})), \min(\inf\delta_\Gamma(\mathrm{x}), \inf\delta_\Omega(\mathrm{x}))\rangle : \mathrm{x} \in \mathrm{U}\}.$$

Consequently $\circ(\Gamma \cup \Omega) = \circ\Gamma \cup \circ\Omega$.

(b) Proof is similar to (a).

4. Conclusions

Neutrosophic set is a part of neutrosophy which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. In this paper we have defined the set-theoretic operators on interval valued neutrosophic sets and studied some properties. We hope that this paper will promote the future study on interval valued neutrosophic sets to carry out a general framework for their application in practical life. Moreover, with the motivations of ideas presented in the paper, one can think of similar operations on interval valued neutrosophic sets of type-2, hesitant interval valued neutrosophic sets, interval valued neutrosophic soft sets and interval valued hesitant neutrosophic soft sets.

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A Python Tool for Implementations on Bipolar Neutrosophic Matrices

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Abstract: Bipolar neutrosophic matrices (BNM) are obtained by bipolar neutrosophic sets. Each bipolar neutrosophic number represents an element of the matrix. The matrices are representable multi-dimensional arrays (3D arrays). The arrays have nested list data type. Some operations, especially the composition is a challenging algorithm in terms of coding because there are so many nested lists to manipulate. This paper presents a Python tool for bipolar neutrosophic matrices. The advantage of this work, is that the proposed Python tool can be used also for fuzzy matrices, bipolar fuzzy matrices, intuitionistic fuzzy matrices, bipolar intuitionistic fuzzy matrices and single valued neutrosophic matrices.

Keywords: Python; Neutrosophic sets; bipolar neutrosophic sets; matrix; composition operation

1. Introduction

Smarandache [1] gave the concept of neutrosophic set (NS) by considering the triplets independent components whose values belong to real standard or nonstandard unit interval] - 0, 1$^+$[. Later on, Smarandache [1] gave single valued neutrosophic set (SVNS) to apply into the various engineering applications. The various properties of SVNS is being studied by Wang et al. [2]. Further, Zhang et al. [3] presented a concept of interval-valued NS (IVNS) where the different membership degrees are represented by interval. In [4] Deli et al. introduced the concept of bipolar neutrosophic sets and their applications based on multicriteria decision making problems. The same author [5] proposed the bipolar neutrosophic refined sets and their applications in medical diagnosis for more details about the applications and its sets, we refer to [6]. Since the existence of NS, various scholars have presented the approaches related to SVNS and bipolar neutrosophic sets into the different fields. For instance, Mumtaz et al. [7] developed the concept of bipolar neutrosophic soft sets that combines soft sets and bipolar neutrosophic sets. In [8, 9] Broumi et al. introduced the notion of bipolar single valued neutrosophic graph theory and its shortest path problem. Dey et al. [10] considered TOPSIS method for solving the decision making problem under bipolar neutrosophic environment. Akram et al. [11] described bipolar neutrosophic TOPSIS method and bipolar neutrosophic ELECTRE-I method. Akram and Sarwar [12] studied the novel multiple criteria decision making methods based on bipolar neutrosophic sets and bipolar neutrosophic graphs. Akram and Sitara [13] introduced the concept of bipolar single-valued neutrosophic graph structures and discussed certain notions of bipolar single-valued neutrosophic graph structures with examples. Singh [14] introduced bipolar neutrosophic graph representation of concept lattice and it's processing using granular computing. Mullai and Broumi [15] presented shortest path problem by minimal spanning tree algorithm using bipolar neutrosophic numbers. Uluçay et al. [16] defined similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision making. Based on literal neutrosophic numbers, Mamouni et al. [17] defined the addition and multiplication of two neutrosophic fuzzy matrices. in the light of Fuzzy Neutrosophic soft sets, Arockiarani [18] present a new technique for handling decision making problems and proposed some new notions on matrix representation. Karaaslan and Hayat [19] introduced some novel operations on neutrosophic matrices. Uma et al. [20] introduced two types of fuzzy neutrosophic soft Matrices. The same authors in [21] decomposed fuzzy neutrosophic soft matrix by means of its section of fuzzy neutrosophic soft matrix of Type-I. Hassan et al. [22] defined some special types of bipolar single valued neutrosophic graphs. Akram and Siddique [23] discussed certain types of edge irregular bipolar neutrosophic graphs. Pramanik [24] developed cross entropy measures of bipolar neutrosophic sets and interval bipolar neutrosophic sets. Wang *et al*. [25] defined Frank operations of bipolar neutrosophic numbers (BNNs) and proposed Frank bipolar neutrosophic Choquet Bonferroni mean operators by combining Choquet integral operators and Bonferroni mean operators based on Frank operations of BNNs. In the same study, Akram and Nasir [26] introduced the concept of p-competition bipolar neutrosophic graphs. then they defined generalization of bipolar neutrosophic competition graphs called m-step bipolar neutrosophic competition graphs. AKRAM and SHUM [27] defined Bipolar Neutrosophic Planar Graphs. Hashim et al. [28] provide an application of neutrosophic bipolar fuzzy sets in daily life's problem related with HOPE foundation that is planning to build a children hospital. Akram, and Luqman [29] generalized the concept of bipolar neutrosophic sets to hypergraphs. Das et al. [30] proposes an algorithmic approach for group decision making (GDM) problems using neutrosophic soft matrix (NSM) and relative weights of experts.

Broumi et al. [31-34] applied the concept of IVNS on graph theory and studied some interesting results. Broumi et al. [35] developed a Matlab toolbox for computing operational matrices under the SVNS environments. Pramanik et al [36] developed a hybrid structure termed "rough bipolar neutrosophic set". In [37] Pramanik et al. presented Bipolar neutrosophic projection based models for solving multi-attribute decision making problems. Broumi et al [38] developed the concept of bipolar complex neutrosophic sets and its application in decision making problem. Akram, et al.[39] applied the concept of bipolar neutrosophic sets to incidence graphs and studied some properties. For more details on the application of neutrosophic set theory, we refer the readers to [46-52].

Among all the above, matrices play a vital job in the expansion region of science and engineering. However, the classical matrix theory neglects the role of uncertainties during the analysis. Therefore, the decision process may contain a lot of uncertainties. Thus, the role of the fuzzy matrices and their extension including triangular fuzzy matrices, type-2 triangular fuzzy matrices, interval valued fuzzy matrices, intuitionistic fuzzy matrices, interval valued intuitionistic fuzzy matrices are studied deeply by several scholars. In [40] Zahariev, developed a Matlab software package to the fuzzy algebras. In

[41], authors solved intuitionistic fuzzy relational rational calculus problems using a fuzzy toolbox. Later on, in [42] Karunambigai and Kalaivani proposed some computing procedures in Matlab for intuitionistic fuzzy operational matrices with suitable examples. Uma et al. [43] studied determinant theory for fuzzy neutrosophic soft square matrices. Also, in [44] Uma et al. introduced the determinant and adjoint of a square Fuzzy Neutrosophic Soft Matrices (FNSMs) a defined the circular FNSM and study some relations on square FNSM such as reflexivity, transitivity and circularity.

Recently few researchers [45] developed a Python programs for computing operations on neutrosophic numbers, but all these programs cannot deal with neutrosophic matrices, to do best of our knowledge, there is no work conducted on developing python codes to compute the operations on single valued neutrosophic matrices and bipolar neutrosophic matrices. Thus, there is a need to develop the work in that direction. For it, the presented paper discusses various operations of bipolar neutrosophic sets and their corresponding Python code for different metrics. To achieve it, rest of the manuscript is summarized as. In section 2, some concepts related to SVNS, BNS are presented. Section 3 deals with the generations of Python programs for bipolar neutrosophic matrices with a numerical example and lastly, conclusion is summarized in section 4.

2.BACKGROUND AND BIPOLAR NEUTROSOPHIC SETS

In this section, some basic concepts on SVNS, BNS are briefly presented over the universal set ξ [1, 2, 4].

Definition 2.1 [1] A set A is said to be A neutrosophic set 'A' consists of three components namely truth, indeterminate and falsity denoted by T_A, I_A(x) and F_A(x) such that

$T_A(x), I_A(x),\ F_A(x) \in]\ {}^-0, 1^+[$ and $-0 \leq \sup T_A(x) + \sup I_A(x) + \sup \mathrm{F_A}(\mathrm{x}) \leq 3+$ (1)

Definition 2.2 [2] A SVNS 'A' on X is given as

A = $\{< x: \mathrm{T_A(x)}, \mathrm{I_A}(x),\ \mathrm{F_A}(x) > x \in \xi\}$ (2)

where the functions $\mathrm{T_A(x)}$, $\mathrm{I_A(x)}$, $\mathrm{F_A(x)} \in$ [0. 1] are named "degree of truth, indeterminacy and falsity membership of x in A", such that

$0 \leq T_A(\mathrm{x}) + I_A(\mathrm{x}) + F_A(\mathrm{x}) \leq 3$ (3)

Definition 2.3[4]. A bipolar neutrosophic set A in ξ is defined as an object of the form

A={<x, $(T_A^P(x), I_A^P(x), F_A^P(x), T_A^N(x), I_A^N(x), F_A^N(x))$>: x $\in \xi$ }, where $T_A^P(x), I_A^P(x), F_A^P(x): \xi \rightarrow$ [1, 0] and $T_A^N(x), I_A^N(x), F_A^N(x): \xi \rightarrow$ [-1, 0]. The positive membership degree $T_A^P(x), I_A^P(x), F_A^P(x)$enotes the truth membership, indeterminate membership and false membership of an element $\in \xi$ corresponding to a bipolar neutrosophic set whereas the negative membership degree $T_A^N(x), I_A^N(x), F_A^N(x)$denotes the truth membership, indeterminate membership and false membership of an element $x \in \xi$ to some implicit counter-property corresponding to a bipolar neutrosophic set A. For convenience a bipolar neutrosophic number is represented by

A= <$(T_A^P, I_A^P, F_A^P, T_A^N, I_A^-, F_A^-)$> (4)

Definition 2.4 [4]. In order to make a comparison between two BNN. The score function is applied to compare the grades of BNS. This function shows that greater is the value, the greater is the bipolar neutrosophic sets and by using this concept paths can be ranked. Suppose

$\tilde{A} =< T^P, \mathrm{I}^P, \mathrm{F}^P, T^N, \mathrm{I}^N, \mathrm{F}^N >$ be a bipolar neutrosophic number. Then, the score function $s(\tilde{A})$, accuracy function $a(\tilde{A})$ and certainty function $c(\tilde{A})$ of a BNN are defined as follows:

(i) $s(\tilde{A}) = \left(\frac{1}{6}\right) \times \left[T^P + 1 - I^P + 1 - F^P + 1 + T^N - I^N - F^N \right]$ (5)

(ii) $a(\tilde{A}) = T^P - F^P + T^N - F^N$ (6)

(iii) $c(\tilde{A}) = T^P - F^N$ (7)

Comparison of bipolar neutrosophic numbers

Let $\tilde{A}_1 =< T_1^p, \mathrm{I}_1^p, \mathrm{F}_1^p, T_1^n, \mathrm{I}_1^n, \mathrm{F}_1^n >$ and $\tilde{A}_2 =< T_2^p, \mathrm{I}_2^p, \mathrm{F}_2^p, T_2^n, \mathrm{I}_2^n, \mathrm{F}_2^n >$ be two bipolar neutrosophic numbers then

i. If $s(\tilde{A}_1) \succ s(\tilde{A}_2)$, then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$.

ii. If $s(\tilde{A}_1) = s(\tilde{A}_2)$, and $a(\tilde{A}_1) \succ a(\tilde{A}_2)$ then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$.

iii. If $s(\tilde{A}_1) = s(\tilde{A}_2)$, $a(\tilde{A}_1) = a(\tilde{A}_2)$, and $\mathrm{c}(\tilde{A}_1) \succ c(\tilde{A}_2)$ then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$.

iv. If $s(\tilde{A}_1) = s(\tilde{A}_2)$, $a(\tilde{A}_1) = a(\tilde{A}_2)$, and $\mathrm{c}(\tilde{A}_1) = c(\tilde{A}_2)$ then $\tilde{A}_1$ is equal to $\tilde{A}_2$, that is, $\tilde{A}_1$ is indifferent to $\tilde{A}_2$, denoted by $\tilde{A}_1 = \tilde{A}_2$.

Definition 2.5 [4]: A bipolar neutrosophic matrix (BNM) of order m× n is defined as
A_{BNM}=$[< a_{ij}, a^P_{ij_T}, a^P_{ij_I}, a^P_{ij_F}, a^N_{ij_T}, a^N_{ij_I}, a^N_{ij_F} >]_{\mathrm{m\times n}}$ where
$a^P_{ij_T}$ is the positive membership value of element a_{ij} in A.
$a^N_{ij_T}$ is the negative membership value of element a_{ij} in A.
$a^P_{ij_T}$ is the positive indeterminate-membership value of element a_{ij} in A.
$a^N_{ij_T}$ is the negative indeterminate-membership value of element a_{ij} in A.
$a^P_{ij_T}$ is the positive non- membership value of element a_{ij} in A.
$a^N_{ij_T}$ is the negative non-membership value of element a_{ij} in A.

For simplicity, we write A as A_{BNM}= $[< a^P_{ij_T}, a^P_{ij_I}, a^P_{ij_F}, a^N_{ij_T}, a^N_{ij_I}, a^N_{ij_F} >]_{\mathrm{m\times n}}$.

3.COMPUTING THE BIPOLAR NEUTROSOPHIC MATRIX OPERATIONS USING PYTHON LANGUAGE

To generate the Python program for inputting the single valued neutrosophic matrices. The procedure is described as follows:

3.1 Checking the matrix is BNM or not

To generate the Python program for deciding for a given the matrix is bipolar neutrosophic matrix or, simple call of the function **BNMChecking ( )** is defined as follow:

```
# BNM is represented by 3D Numpy Array => row, column and bipolar number with 6 tuples for
BNM Checking
#A1.shape and A2.shape returns (3, 3, 6) the dimension of A. (row, column, numbers of element
(Bipolar Neutrosophic Number, 6 elements) )
# A.shape[0] = 3 rows
# A.shape[1] = 3 columns
# A.shape[2] = Each bipolar neutrosophic number has 6 tuple as usual
#One can use any matrices having arbitrary dimension
import numpy as np
#A1 is a BNM
A1= np.array([     [[0.000, 0.001, 0.002, -0.003, -0.004, -0.005],    [0.010, 0.011, 0.012, -0.013, -0.014, -
0.015] , [0.020, 0.021, 0.022, -0.023, -0.024, -0.025]    ],
[[0.100,0.101,0.102,-0.103,-0.104, -0.105], [0.110,0.111,0.112,-0.113,-0.114,-0.115], [0.120,0.121,0.122,-
0.123,-0.124,-0.125]     ],
 [[0.200,0.201,0.202,-0.203,-0.204,-0.205], [0.210, 0.211,0.212,-0.213,-0.214,-0.215], [0.220,0.221,0.222,-
0.223,-0.224,-0.225]     ] ])
#A2 is not BNM
A2= np.array([     [[0.000, 0.001, 0.002, -0.003, -0.004, -0.005],    [0.010, 0.011, 0.012, -0.013, -0.014, -
0.015] , [0.020, 0.021, 0.022, -0.023, -0.024, -0.025]    ],
[[0.100,0.101,0.102,-0.103,-0.104, -0.105],      [0.110,0.111,0.112,-0.113,-0.114,-0.115],
[0.120,0.121,0.122,-0.123,-0.124,-0.125]     ],
[[0.200,0.201,0.202,-0.203, 0.204,-0.205],       [0.210, 0.211,0.212,-0.213,-0.214,-0.215],
[0.220,0.221,0.222,-0.223,-0.224,-0.225]     ] ])
def BNMChecking (A):
    dimA=A.shape
    control=0
    counter = 0
    for i in range (0,dimA[0]):
        if counter == 1:
            break
        for j in range (0,dimA[0]):
            if counter == 1:
                break
            for  d in range (0, dimA[2]):
                if  counter ==0:
```

```
                    if (d==0 or d==1 or d==2) :
                        if   not (0 <=   A[i][j][d] <= 1):
                            counter=1
                            print (A[i][j], ' is not a bipolar neutrosophic number, so the matrix
is not a BNM')
                            control=1
                            break
                    if   (d==3 or d==4 or d==5) :
                        if not (-1 <=   A[i][j][d] <= 0) :
                            counter=1
                            print (A[i][j], ' is not a bipolar neutrosophic number, so the matrix
is not a BNM')
                            control=1
                            break
                else:
                    print (A[i][j], ' is not a bipolar neutrosophic number, so the matrix is not a
BNM')
                    break
    if control==0:
        print ('The matrix is a BNM')
```

Example 1. In this example we evaluate the checking the matrix C is BNM or not of order 4X4:

C=

$$\begin{pmatrix} <.5,.7,.2,-.7,-.3,-.6> & <.4,.4,.5,-.7,-.8,-.4> & <.7,.7,.5,-.8,-.7,-.6> & <.1,.5,.7,-.5,-.2,-.8> \\ <.9,.7,.5,-.7,-.7,-.1> & <.7,.6,.8,-.7,-.5,-.1> & <.9,.4,.6,-.1,-.7,-.5> & <.5,.2,.7,-.5,-.1,-.9> \\ <.9,.4,.2,-.6,-.3,-.7> & <.2,.2,.2,-.4,-.7,-.4> & <.9,.5,.5,-.6,-.5,-.2> & <.7,.5,.3,-.4,-.2,-.2> \\ <.9,.7,.2,-.8,-.6,-.1> & <.3,.5,.2,-.5,-.5,-.2> & <.5,.4,.5,-.1,-.7,-.2> & <.2,.4,.8,-.5,-.5,-.6> \end{pmatrix}$$

The bipolar neutrosophic matrix C can be inputted in Python environment like this:

```
Shell  AST
>>> %Run 'bnm checking.py'
>>> C= np.array([ [ [0.5,0.7,0.2,-0.7,-0.3,-0.6], [0.4,0.4,0.5,-0.7,-0.8,-0.4], [0.7,0.7,0.5,-0.8,-0.7,-0.6], [0.1,0.5,0.7,-0.5,-0.2
    ,-0.8]],
                  [[0.9,0.7,0.5,-0.7,-0.7,-0.1], [0.7,0.6,0.8,-0.7,-0.5,-0.1], [0.9,0.4,0.6,-0.1,-0.7,-0.5], [0.5,0.2,0.7,-0.
    5,-0.1,-0.9]],
                   [[0.9,0.4,0.2,-0.6,-0.3,-0.7], [0.2,0.2,0.2,-0.4,-0.7,-0.4], [0.9,0.5,0.5,-0.6,-0.5,-0.2], [0.7,0.5,0.3,-0
    .4,-0.2,-0.2]],
                   [[0.9,0.7,0.2,-0.8,-0.6,-0.1], [0.3,0.5,0.2,-0.5,-0.5,-0.2], [0.5,0.4,0.5,-0.1,-0.7,-0.2], [0.2,0.4,0.8,-0
    .5,-0.5,-0.6]] ])

>>> BNMChecking (C)
  The matrix is a BNM
...
```

3.2. Determining complement of bipolar neutrosophic matrix

For a given BNM A= $[< T_{ij}^P, I_{ij}^P, F_{ij}^P, T_{ij}^N, I_{ij}^N, F_{ij}^N >]_{m\times n}$, the complement of A is defined as follow:

$$A^c = [< \{1\} - T_{ij}^P, \{1\} - I_{ij}^P, \{-1\} - F_{ij}^P, \{1\} - T_{ij}^N, \{-1\} - I_{ij}^N, \{-1\} - F_{ij}^N >]_{m\times n} \quad (8)$$

$$A^c = [< F_{ij}^P, \{1\} - I_{ij}^P, T_{ij}^P, F_{ij}^N, \{-1\} - I_{ij}^N, T_{ij}^N >]_{m\times n} \quad (9)$$

To generate the Python program for finding complement of bipolar neutrosophic matrix, simple call of the function **BNMCompelementOf()** is defined as follow:

```
# BNM is represented by 3D Numpy Array => row, column and bipolar number with 6 tuples for (8)
import numpy as np
A= np.array([   [ [0.3,0.6,1,-0.2,-0.54,-0.4],   [0.1,0.2,0.8,-0.5,-0.34,-0.7]],
                [ [0.1,0.12,0,-0.27,-0.44,-0.92], [0.5,0.33,0.58,-0.33,-0.24,-0.22]],
                [ [0.11,0.22,0.6,-0.29,-0.24,-0.52],[0.22,0.63,0.88,-0.28,-0.54,-0.32] ]
                ])
#A.shape gives (3, 2, 6) the dimension of A. (row, column, numbers of element (Bipolar Neutrosophic Number, 6 elements) )
# A.shape[0] = 3 rows
# A.shape[1] = 2 columns
# A.shape[2] = each bipolar neutrosophic number with 6 tuple as usual
def BNMCompelementOf( A ):
    global Ac
    dimA=A.shape                                # Dimension of the matrix
    Ac= []      # Empty matrix with dimension of A to create complement of A
        for i in range (0,dimA[0]):         # for rows, here 3
        H=[]
        for j in range (0,dimA[1]):   # for columns, here 2
            H.extend([ [ 1-A[i][j][0], 1-A[i][j][1], 1-A[i][j][2], -1-(-A[i][j][3]), -1-(-A[i][j][4]), -1-(-A[i][j][5]) ] ])
            Ac.append(H)
    print   ('A= ', A)
    print ('*****************************************************************')
    print('Ac= ', np.array(Ac))
```

The function BNMCompelementOf (A) the below returns the complement matrix of a given bipolar neutrosophic matrix A for (9).

```
# BNM is representable by 3D Numpy Array ====> row, column and bipolar neutrosophic numbers having 6 tuples for (9)
import numpy as np
A= np.array([   [ [0.3,0.6,1,-0.2,-0.54,-0.4], [0.1,0.2,0.8,-0.5,-0.34,-0.7]    ],
                [ [0.1,0.12,0,-0.27,-0.44,-0.92], [0.5,0.33,0.58,-0.33,-0.24,-0.22] ],
                [ [0.11,0.22,0.6,-0.29,-0.24,-0.52],[0.22,0.63,0.88,-0.28,-0.54,-0.32]     ]])
#A.shape gives (3, 2, 6) the dimension of A. (row, column, numbers of element (Bipolar Neutrosophic Number, 6 elements) )
# A.shape[0] = 3 rows
# A.shape[1] = 2 columns
# A.shape[2] = Each bipolar neutrosophic number with 6 tuple as usual
def BNMCompelementOf( A ):
    global Ac
    dimA=A.shape                                # Dimension of the matrix
    Ac= []
```

```
    for i in range (0,dimA[0]):          # for rows, here 3
        H=[]
        for j in range (0,dimA[1]):   # for columns, here 2
            H.extend([[ A[i][j][2], 1-A[i][j][1], A[i][j][0], A[i][j][5], -1-(-A[i][j][4]), A[i][j][3] ] ])
        Ac.append(H)
  print   ('A= ', A)
  print ('*******************************************************************')
  print ('*******************************************************************')
  print('Ac= ', np.array(Ac))
```

The bipolar neutrosophic matrix A is a simple example, one can create his/her BNM and try it into the function **BNMCompelementOf** ( ):

3.3. Determining the score, accuracy and certainty matrices of bipolar neutrosophic matrix

To generate the python program for obtaining the score matrix, accuracy of bipolar neutrosophic matrix, simple call of the functions **ScoreMatrix( )**, **AccuracyMatrix( )** and **CertaintyMatrix( )** are defined as follow:

```
# BNM is represented by 3D Numpy Array => row, column and bipolar number with 6 tuples for (5,
6 and 7)
  import numpy as np
  A= np.array([    [        [0.3,0.6,1,-0.2,-0.54,-0.4], [0.1,0.2,0.8,-0.5,-0.34,-0.7]              ],
                   [        [0.1,0.12,0,-0.27,-0.44,-0.92], [0.5,0.33,0.58,-0.33,-0.24,-0.22]       ],
                   [        [0.11,0.22,0.6,-0.29,-0.24,-0.52],[0.22,0.63,0.88,-0.28,-0.54,-0.32]    ]])
  def ScoreMatrix( A ):
      score=[]
      dimA=A.shape                              # Dimension of the matrix
      for i in range (0,dimA[0]):               # for rows, here 3
          H=[]
          for j in range (0,dimA[1]):           # for columns, here 2
              H.extend([ [ ( A[i][j][0] + 1 - A[i][j][1] + 1 - A[i][j][2] + 1 + A[i][j][3] - A[i][j][4] -
A[i][j][5] )/6 ] ])
          score.append(H)
      print('Score Matrix= ', np.array(score))
  def AccuracyMatrix ( A ):
      accuracy=[]
      dimA=A.shape                              # Dimension of the matrix
      for i in range (0,dimA[0]):               # for rows, here 3
          H=[]
          for j in range (0,dimA[1]):           # for columns, here 2
              H.extend([ [   A[i][j][0] - A[i][j][2] + A[i][j][3] - A[i][j][5]   ] ])
          accuracy.append(H)
      print('Accuracy Matrix= ', np.array(accuracy))
```

```
def CertaintyMatrix ( A ):
    certainty = []
    dimA=A.shape                              # Dimension of the matrix
    for i in range (0,dimA[0]):              # for rows, here 3
        H=[]
        for j in range (0,dimA[1]):          # for columns, here 2
            H.extend([ [   A[i][j][0] - A[i][j][5]   ] ])
        certainty.append(H)
    print('Certainty Matrix= ', np.array(certainty))
```

3.4. Computing union of two bipolar neutrosophic matrices

The union of two bipolar neutrosophic matrices A and B is defined as follow:

$$A \cup B = C = \left[< c^P_{ij_T}, c^P_{ij_I}, c^P_{ij_F}, c^N_{ij_T}, c^N_{ij_I}, c^N_{ij_F} >\right]_{m\times n} \tag{10}$$

where

$c^P_{ij_T} = a^P_{ij_T} \vee b^P_{ij_T}$, $c^N_{ij_T} = a^N_{ij_T} \wedge b^N_{ij_T}$

$c^P_{ij_I} = a^P_{ij_I} \wedge b^P_{ij_I}$, $c^N_{ij_I} = a^N_{ij_I} \vee b^N_{ij_I}$

$c^P_{ij_F} = a^P_{ij_F} \wedge b^P_{ij_F}$, $c^N_{ij_F} = a^N_{ij_F} \vee b^N_{ij_F}$

To generate the python program for finding the union of two bipolar neutrosophic matrices, simple call of the following function **Union(A, B)** is defined as follow:

```
# BNM is represented by 3D Numpy Array => row, column and bipolar number with 6 tuples for (10)
import numpy as np
A= np.array([   [      [0.3,0.6,1,-0.2,-0.54,-0.4], [0.1,0.2,0.8,-0.5,-0.34,-0.7]    ],
                [        [0.1,0.12,0,-0.27,-0.44,-0.92], [0.5,0.33,0.58,-0.33,-0.24,-0.22] ],
                [   [0.11,0.22,0.6,-0.29,-0.24,-0.52],[0.22,0.63,0.88,-0.28,-0.54,-0.32]] ])
B= np.array([   [        [0.32,0.4,0.1,-0.25,-0.54,-0.4], [0.13,0.2,0.11,-0.55,-0.35,-0.72]      ],
                [[0.17,0.19,0.66,-0.87,-0.64,-0.92], [0.25,0.36,0.88,-0.33,-0.54,-0.22] ],
                [ [0.15,0.28,0.67,-0.39,-0.27,-0.55],[0.24,0.73,0.28,-0.26,-0.53,-0.52] ]
                ])
#A.shape gives (3, 2, 6) the dimension of A. (row, column, numbers of element (Bipolar Neutrosophic Number, 6 elements) )
# A.shape[0] = 3 rows
# A.shape[1] = 2 columns
# A.shape[2] = each bipolar neutrosophic number with 6 tuple as usual
union=[]
def Union( A, B ):
    if A.shape == B.shape:
        dimA=A.shape
        for i in range (0,dimA[0]):          # for rows, here 3
            H=[]
            for j in range (0,dimA[1]):    # for columns, here 2
```

```
                H.extend([[  max(A[i][j][0],B[i][j][0])  ,  min(A[i][j][1],  B[i][j][1]),  min(A[i][j][2],
B[i][j][2]), max(A[i][j][3],B[i][j][3]), min(A[i][j][4], B[i][j][4]), min(A[i][j][5], B[i][j][5]) ] ])
                union.append(H)
    print('union= ', np.array(union)
```

Example 2. In this example we Evaluate the union of the two bipolar neutrosophic matrices C and D of order 4X4:

C=

$$\begin{pmatrix} <.5,.7,.2,-.7,-.3,-.6> & <.4,.4,.5,-.7,-.8,-.4> & <.7,.7,.5,-.8,-.7,-.6> & <.1,.5,.7,-.5,-.2,-.8> \\ <.9,.7,.5,-.7,-.7,-.1> & <.7,.6,.8,-.7,-.5,-.1> & <.9,.4,.6,-.1,-.7,-.5> & <.5,.2,.7,-.5,-.1,-.9> \\ <.9,.4,.2,-.6,-.3,-.7> & <.2,.2,.2,-.4,-.7,-.4> & <.9,.5,.5,-.6,-.5,-.2> & <.7,.5,.3,-.4,-.2,-.2> \\ <.9,.7,.2,-.8,-.6,-.1> & <.3,.5,.2,-.5,-.5,-.2> & <.5,.4,.5,-.1,-.7,-.2> & <.2,.4,.8,-.5,-.5,-.6> \end{pmatrix}$$

The bipolar neutrosophic matrix C can be inputted in Python code like this:

```
C= np.array([ [ [0.5,0.7,0.2,-0.7,-0.3,-0.6], [0.4,0.4,0.5,-0.7,-0.8,-0.4], [0.7,0.7,0.5,-0.8,-0.7,-0.6], [0.1,0.5,0.7,-0.5,-0.2,-
0.8]],[[0.9,0.7,0.5,-0.7,-0.7,-0.1], [0.7,0.6,0.8,-0.7,-0.5,-0.1], [0.9,0.4,0.6,-0.1,-0.7,-0.5], [0.5,0.2,0.7,-0.5,-0.1,-0.9]],
[[0.9,0.4,0.2,-0.6,-0.3,-0.7], [0.2,0.2,0.2,-0.4,-0.7,-0.4], [0.9,0.5,0.5,-0.6,-0.5,-0.2], [0.7,0.5,0.3,-0.4,-0.2,-0.2]],
[[0.9,0.7,0.2,-0.8,-0.6,-0.1], [0.3,0.5,0.2,-0.5,-0.5,-0.2], [0.5,0.4,0.5,-0.1,-0.7,-0.2], [0.2,0.4,0.8,-0.5,-0.5,-0.6]] ])
```

D=

$$\begin{pmatrix} <.3,.4,.3,-.5,-.4,-.2> & <.1,.2,.7,-.5,-.2,-.3> & <.3,.2,.6,-.4,-.8,-.7> & <.2,.1,.3,-.2,-.4,-.4> \\ <.2,.2,.7,-.3,-.3,-.5> & <.3,.5,.6,-.6,-.7,-.4> & <.6,.5,.4,-.3,-.6,-.8> & <.3,.4,.4,-.3,-.5,-.3> \\ <.5,.3,.1,-.4,-.2,-.4> & <.5,.4,.3,-.3,-.8,-.2> & <.5,.8,.6,-.2,-.2,-.4> & <.4,.6,.5,-.1,-.6,-.5> \\ <.6,.1,.7,-.7,-.4,-.8> & <.4,.6,.4,-.4,-.2,-.5> & <.4,.9,.3,-.5,-.5,-.3> & <.4,.5,.4,-.3,-.7,-.4> \end{pmatrix}$$

The bipolar neutrosophic matrix D can be inputted in Python code like this:

```
D= np.array([[[0.3,0.4, 0.3,-0.5,-0.4,-0.2], [0.1,0.2,0.7,-0.5,-0.2,-0.3], [0.3,0.2,0.6,-0.4,-0.8,-0.7], [0.2,0.1,0.3,-0.2,-0.4,-
0.4]], [[0.2,0.2,0.7,-0.3,-0.3,-0.5], [0.3,0.5,0.6,-0.6,-0.7,-0.4], [0.6,0.5,0.4,-0.3,-0.6,-0.8], [0.3,0.4,0.4,-0.3,-0.5,-0.3]],
[[0.5,0.3,0.1,-0.4,-0.2,-0.4], [0.5,0.4,0.3,-0.3,-0.8,-0.2], [0.5,0.8,0.6,-0.2,-0.2,-0.4], [0.4,0.6,0.5,-0.1,-0.6,-0.5]],
[[0.6,0.1,0.7,-0.7,-0.4,-0.8], [0.4,0.6,0.4,-0.4,-0.2,-0.5], [0.4,0.9,0.3,-0.5,-0.5,-0.3], [0.4,0.5,0.4,-0.3,-0.7,-0.4]]])
```

So, the union matrix of two bipolar neutrosophic matrices is portrayed as follow

$$C_{BNS} \cup D_{BNS} = \begin{pmatrix} <.5,.4,.2,-.7,-.3,-.2> & <.4,.2,.5,-.7,-.2,-.3> & <.7,.2,.5,-.8,-.7,-.6> & <.2,.1,.3,-.5,-.2,-.4> \\ <.9,.2,.5,-.7,-.3,-.1> & <.7,.5,.6,-.7,-.5,-.1> & <.9,.4,.4,-.3,-.6,-.5> & <.5,.2,.4,-.5,-.1,-.3> \\ <.9,.3,.1,-.6,-.2,-.4> & <.5,.2,.2,-.4,-.7,-.2> & <.9,.5,.5,-.6,-.2,-.2> & <.7,.5,.3,-.4,-.2,-.2> \\ <.9,.1,.2,-.8,-.4,-.1> & <.4,.5,.2,-.5,-.2,-.2> & <.5,.4,.3,-.5,-.5,-.2> & <.4,.4,.4,-.5,-.5,-.4> \end{pmatrix}$$

The result of union matrix of two bipolar neutrosophic matrices C and D can be obtained by the call of the command Union (C, D):

```
>>> Union(C, D)
Union =
 [[[ 0.5   0.4   0.2 -0.7 -0.3 -0.2]  [ 0.4   0.2   0.5 -0.7 -0.2 -0.3] [ 0.7   0.2   0.5 -0.8 -0.7 -0.6]    [ 0.2   0.1   0.3 -0.5 -0.2 -0.4]]
 [[ 0.9   0.2   0.5 -0.7 -0.3 -0.1]  [ 0.7   0.5   0.6 -0.7 -0.5 -0.1]    [ 0.9   0.4   0.4 -0.3 -0.6 -0.5]    [ 0.5   0.2   0.4 -0.5 -0.1 -0.3]]
 [[ 0.9   0.3   0.1 -0.6 -0.2 -0.4]    [ 0.5   0.2   0.2 -0.4 -0.7 -0.2]    [ 0.9   0.5   0.5 -0.6 -0.2 -0.2]    [ 0.7   0.5   0.3 -0.4 -0.2 -0.2]]
 [[ 0.9   0.1   0.2 -0.8 -0.4 -0.1]    [ 0.4   0.5   0.2 -0.5 -0.2 -0.2]    [ 0.5   0.4   0.3 -0.5 -0.5 -0.2]    [ 0.4   0.4   0.4 -0.5 -0.5 -0.4]]]
```

3.5. Computing intersection of two bipolar neutrosophic matrices

The union of two bipolar neutrosophic matrices A and B is defined as follow:

$$A \cap B = D = \left[< d^{P}_{ij_T}, d^{P}_{ij_I}, d^{P}_{ij_F}, d^{N}_{ij_T}, d^{N}_{ij_I}, d^{N}_{ij_F} >\right]_{m\times n} \quad (11)$$

Where

$d^{P}_{ij_T} = a^{P}_{ij_T} \wedge b^{P}_{ij_T}, \quad d^{N}_{ij_T} = a^{N}_{ij_T} \vee b^{N}_{ij_T}$

$d^{P}_{ij_I} = a^{P}_{ij_I} \vee b^{P}_{ij_I}, \quad d^{N}_{ij_I} = a^{N}_{ij_I} \wedge b^{N}_{ij_I}$

$d^{P}_{ij_F} = a^{P}_{ij_F} \vee b^{P}_{ij_F}, \quad d^{N}_{ij_F} = a^{N}_{ij_F} \wedge b^{N}_{ij_F}$

To generate the python program for finding the intersection of two bipolar neutrosophic matrices, simple call of the function Intersection (A, B) is defined as follow:

```
# BNM is represented by 3D Numpy Array => row, column and bipolar number with 6 tuples for (11)
import numpy as np
A= np.array([  [      [0.3,0.6,1,-0.2,-0.54,-0.4], [0.1,0.2,0.8,-0.5,-0.34,-0.7]         ],
               [      [0.1,0.12,0,-0.27,-0.44,-0.92], [0.5,0.33,0.58,-0.33,-0.24,-0.22]   ],
               [      [0.11,0.22,0.6,-0.29,-0.24,-0.52],[0.22,0.63,0.88,-0.28,-0.54,-0.32] ]
               ])
B= np.array([  [      [0.32,0.4,0.1,-0.25,-0.54,-0.4], [0.13,0.2,0.11,-0.55,-0.35,-0.72]  ],
               [      [0.17,0.19,0.66,-0.87,-0.64,-0.92], [0.25,0.36,0.88,-0.33,-0.54,-0.22] ],
               [      [0.15,0.28,0.67,-0.39,-0.27,-0.55],[0.24,0.73,0.28,-0.26,-0.53,-0.52]   ]])
#A.shape gives (3, 2, 6) the dimension of A. (row, column, numbers of element (Bipolar Neutrosophic Number, 6 elements) )
# A.shape[0] = 3 rows
# A.shape[1] = 2 columns
# A.shape[2] = each bipolar neutrosophic number with 6 tuple as usual
intersection=[]
def Intersection( A, B ):
    if A.shape == B.shape:
        dimA=A.shape
        for i in range (0,dimA[0]):        # for rows, here 3
            H=[]
            for j in range (0,dimA[1]):    # for columns, here 2
                H.extend([[ min(A[i][j][0],B[i][j][0]) , max(A[i][j][1], B[i][j][1]), max(A[i][j][2], B[i][j][2]), min(A[i][j][3],B[i][j][3]), max(A[i][j][4], B[i][j][4]), max(A[i][j][5], B[i][j][5]) ] ])
                intersection.append(H)
    print('Intersection= ', np.array(intersection))
```

Example 3. In this example we evaluate the intersection of the two bipolar neutrosophic matrices C and D of order 4X4:

C=

$$\begin{pmatrix} <.5,.7,.2,-.7,-.3,-.6> & <.4,.4,.5,-.7,-.8,-.4> & <.7,.7,.5,-.8,-.7,-.6> & <.1,.5,.7,-.5,-.2,-.8> \\ <.9,.7,.5,-.7,-.7,-.1> & <.7,.6,.8,-.7,-.5,-.1> & <.9,.4,.6,-.1,-.7,-.5> & <.5,.2,.7,-.5,-.1,-.9> \\ <.9,.4,.2,-.6,-.3,-.7> & <.2,.2,.2,-.4,-.7,-.4> & <.9,.5,.5,-.6,-.5,-.2> & <.7,.5,.3,-.4,-.2,-.2> \\ <.9,.7,.2,-.8,-.6,-.1> & <.3,.5,.2,-.5,-.5,-.2> & <.5,.4,.5,-.1,-.7,-.2> & <.2,.4,.8,-.5,-.5,-.6> \end{pmatrix}$$

The bipolar neutrosophic matrix C can be inputted in Python code like this:

C= np.array([[[0.5,0.7,0.2,-0.7,-0.3,-0.6], [0.4,0.4,0.5,-0.7,-0.8,-0.4], [0.7,0.7,0.5,-0.8,-0.7,-0.6], [0.1,0.5,0.7,-0.5,-0.2,-0.8]], [[0.9,0.7,0.5,-0.7,-0.7,-0.1], [0.7,0.6,0.8,-0.7,-0.5,-0.1], [0.9,0.4,0.6,-0.1,-0.7,-0.5], [0.5,0.2,0.7,-0.5,-0.1,-0.9]], [[0.9,0.4,0.2,-0.6,-0.3,-0.7], [0.2,0.2,0.2,-0.4,-0.7,-0.4], [0.9,0.5,0.5,-0.6,-0.5,-0.2], [0.7,0.5,0.3,-0.4,-0.2,-0.2]], [[0.9,0.7,0.2,-0.8,-0.6,-0.1], [0.3,0.5,0.2,-0.5,-0.5,-0.2], [0.5,0.4,0.5,-0.1,-0.7,-0.2], [0.2,0.4,0.8,-0.5,-0.5,-0.6]])

D=

$$\begin{pmatrix} <.3,.4,.3,-.5,-.4,-.2> & <.1,.2,.7,-.5,-.2,-.3> & <.3,.2,.6,-.4,-.8,-.7> & <.2,.1,.3,-.2,-.4,-.4> \\ <.2,.2,.7,-.3,-.3,-.5> & <.3,.5,.6,-.6,-.7,-.4> & <.6,.5,.4,-.3,-.6,-.8> & <.3,.4,.4,-.3,-.5,-.3> \\ <.5,.3,.1,-.4,-.2,-.4> & <.5,.4,.3,-.3,-.8,-.2> & <.5,.8,.6,-.2,-.2,-.4> & <.4,.6,.5,-.1,-.6,-.5> \\ <.6,.1,.7,-.7,-.4,-.8> & <.4,.6,.4,-.4,-.2,-.5> & <.4,.9,.3,-.5,-.5,-.3> & <.4,.5,.4,-.3,-.7,-.4> \end{pmatrix}$$

The bipolar neutrosophic matrix D can be inputted in Python code like this:

D= np.array([[[0.3, 0.4, 0.3,-0.5,-0.4,-0.2], [0.1,0.2,0.7,-0.5,-0.2,-0.3], [0.3,0.2,0.6,-0.4,-0.8,-0.7], [0.2,0.1,0.3,-0.2,-0.4,-0.4]], [[0.2,0.2,0.7,-0.3,-0.3,-0.5], [0.3,0.5,0.6,-0.6,-0.7,-0.4], [0.6,0.5,0.4,-0.3,-0.6,-0.8], [0.3,0.4,0.4,-0.3,-0.5,-0.3]],

[[0.5,0.3,0.1,-0.4,-0.2,-0.4], [0.5,0.4,0.3,-0.3,-0.8,-0.2], [0.5,0.8,0.6,-0.2,-0.2,-0.4], [0.4,0.6,0.5,-0.1,-0.6,-0.5]],

[[0.6,0.1,0.7,-0.7,-0.4,-0.8], [0.4,0.6,0.4,-0.4,-0.2,-0.5], [0.4,0.9,0.3,-0.5,-0.5,-0.3], [0.4,0.5,0.4,-0.3,-0.7,-0.4]]])

So, the intersection matrix of two bipolar neutrosophic matrices is portrayed as follow

$$C_{BNS} \cap D_{BNS}$$
$$= \begin{pmatrix} <.3,.7,.3,-.5,-.4,-.6> & <.1,.4,.7,-.5,-.8,-.4> & <.3,.7,.6,-.4,-.8,-.7> & <.1,.5,.7,-.2,-.4,-.8> \\ <.2,.7,.7,-.3,-.7,-.5> & <.3,.6,.8,-.6,-.7,-.4> & <.6,.5,.6,-.1,-.7,-.8> & <.3,.4,.7,-.3,-.5,-.9> \\ <.5,.4,.2,-.4,-.3,-.7> & <.2,.4,.3,-.3,-.7,-.4> & <.5,.8,.6,-.2,-.5,-.4> & <.4,.6,.5,-.1,-.6,-.5> \\ <.6,.7,.7,-.7,-.6,-.8> & <.3,.6,.4,-.4,-.5,-.5> & <.4,.9,.5,-.1,-.7,-.3> & <.2,.5,.8,-.3,-.7,-.6> \end{pmatrix}$$

The result of intersection matrix of two bipolar neutrosophic matrices C and D can be obtained by the call of the command Intersection (C, D):

```
>>> Intersection (C, D)
Intersection =
[[[ 0.3   0.7   0.3 -0.5 -0.4 -0.6]   [ 0.1   0.4   0.7 -0.5 -0.8 -0.4]   [ 0.3   0.7   0.6 -0.4 -0.8 -0.7]   [ 0.1   0.5   0.7 -0.2 -0.4 -0.8]]
 [[ 0.2   0.7   0.7 -0.3 -0.7 -0.5]   [ 0.3   0.6   0.8 -0.6 -0.7 -0.4]   [ 0.6   0.5   0.6 -0.1 -0.7 -0.8]   [ 0.3   0.4   0.7 -0.3 -0.5 -0.9]]
 [[ 0.5   0.4   0.2 -0.4 -0.3 -0.7]   [ 0.2   0.4   0.3 -0.3 -0.8 -0.4]   [ 0.5   0.8   0.6 -0.2 -0.5 -0.4]   [ 0.4   0.6   0.5 -0.1 -0.6 -0.5]]
 [[ 0.6   0.7   0.7 -0.7 -0.6 -0.8]   [ 0.3   0.6   0.4 -0.4 -0.5 -0.5]   [ 0.4   0.9   0.5 -0.1 -0.7 -0.3]   [ 0.2   0.5   0.8 -0.3 -0.7 -0.6]]]
```

3.6. Computing addition operation of two bipolar neutrosophic matrices.

The addition of two bipolar neutrosophic matrices A and B is defined as follow:

$$A \oplus B = S = \left[< s^P_{ij_T}, s^P_{ij_I}, s^P_{ij_F}, s^N_{ij_T}, s^N_{ij_I}, s^N_{ij_F} >\right]_{m\times n} \quad (12)$$

Where

$$s^P_{ij_T} = a^P_{ij_T} + b^P_{ij_T} - a^P_{ij_T}.b^P_{ij_T}, \qquad s^N_{ij_T} = -\left(a^N_{ij_T}.b^N_{ij_T}\right)$$
$$s^P_{ij_I} = a^P_{ij_I}.b^P_{ij_I}, \qquad s^N_{ij_I} = -\left(-a^N_{ij_I} - b^N_{ij_I} - a^N_{ij_I}.b^N_{ij_I}\right)$$
$$s^P_{ij_F} = a^P_{ij_F}.b^P_{ij_F}, \qquad s^N_{ij_F} = -\left(-a^N_{ij_F} - b^N_{ij_F} - a^N_{ij_F}.b^N_{ij_F}\right)$$

To generate the python program for obtaining the addition of two bipolar neutrosophic matrices, simple call of the function **Addition (A, B)** is defined as follow:

```
# BNM is represented by 3D Numpy Array => row, column and bipolar number with 6 tuples for
(12)
```

```
import numpy as np
A= np.array([   [[0.3,0.6,1,-0.2,-0.54,-0.4], [0.1,0.2,0.8,-0.5,-0.34,-0.7] ],
                [[0.1,0.12,0,-0.27,-0.44,-0.92], [0.5,0.33,0.58,-0.33,-0.24,-0.22]],
                [ [0.11,0.22,0.6,-0.29,-0.24,-0.52],[0.22,0.63,0.88,-0.28,-0.54,-0.32] ]])
B= np.array([   [[0.32,0.4,0.1,-0.25,-0.54,-0.4], [0.13,0.2,0.11,-0.55,-0.35,-0.72] ],
                [[0.17,0.19,0.66,-0.87,-0.64,-0.92], [0.25,0.36,0.88,-0.33,-0.54,-0.22] ],
                [[0.15,0.28,0.67,-0.39,-0.27,-0.55],[0.24,0.73,0.28,-0.26,-0.53,-0.52]     ]])
#A.shape gives (3, 2, 6) the dimension of A. (row, column, numbers of element (Bipolar Neutrosophic Number, 6 elements) )
# A.shape[0] = 3 rows
# A.shape[1] = 2 columns
# A.shape[2] = each bipolar neutrosophic number with 6 tuples as usual
addition=[]
def Addition( A, B ):
    if A.shape == B.shape:
        dimA=A.shape
        for i in range (0,dimA[0]):         # for rows, here 3
            H=[]
            for j in range (0,dimA[1]):   # for columns, here 2
                H.extend([[A[i][j][0]+B[i][j][0]-A[i][j][0]*B[i][j][0],     A[i][j][1]* B[i][j][1],
A[i][j][2]* B[i][j][2] -(-A[i][j][3]*B[i][j][3]), -(-A[i][j][4]-B[i][j][4] -A[i][j][4]*B[i][j][4] ), -(-A[i][j][5]-
B[i][j][5]-A[i][j][5]*B[i][j][5]) ]])
              addition.append(H)
    print('Addition= ', np.array(addition))
```

Example 4. In this example we evaluate the addition of the two bipolar neutrosophic matrices C and D of order 4X4:

C=

$$\begin{pmatrix} <.5,.7,.2,-.7,-.3,-.6> & <.4,.4,.5,-.7,-.8,-.4> & <.7,.7,.5,-.8,-.7,-.6> & <.1,.5,.7,-.5,-.2,-.8> \\ <.9,.7,.5,-.7,-.7,-.1> & <.7,.6,.8,-.7,-.5,-.1> & <.9,.4,.6,-.1,-.7,-.5> & <.5,.2,.7,-.5,-.1,-.9> \\ <.9,.4,.2,-.6,-.3,-.7> & <.2,.2,.2,-.4,-.7,-.4> & <.9,.5,.5,-.6,-.5,-.2> & <.7,.5,.3,-.4,-.2,-.2> \\ <.9,.7,.2,-.8,-.6,-.1> & <.3,.5,.2,-.5,-.5,-.2> & <.5,.4,.5,-.1,-.7,-.2> & <.2,.4,.8,-.5,-.5,-.6> \end{pmatrix}$$

The bipolar neutrosophic matrix C can be inputted in Python code like this:

```
C= np.array([ [ [0.5,0.7,0.2,-0.7,-0.3,-0.6], [0.4,0.4,0.5,-0.7,-0.8,-0.4], [0.7,0.7,0.5,-0.8,-0.7,-0.6], [0.1,0.5,0.7,-0.5,-0.2,-
0.8]],[[0.9,0.7,0.5,-0.7,-0.7,-0.1], [0.7,0.6,0.8,-0.7,-0.5,-0.1], [0.9,0.4,0.6,-0.1,-0.7,-0.5], [0.5,0.2,0.7,-0.5,-0.1,-0.9]],

[[0.9,0.4,0.2,-0.6,-0.3,-0.7], [0.2,0.2,0.2,-0.4,-0.7,-0.4], [0.9,0.5,0.5,-0.6,-0.5,-0.2], [0.7,0.5,0.3,-0.4,-0.2,-0.2]],

[[0.9,0.7,0.2,-0.8,-0.6,-0.1], [0.3,0.5,0.2,-0.5,-0.5,-0.2], [0.5,0.4,0.5,-0.1,-0.7,-0.2], [0.2,0.4,0.8,-0.5,-0.5,-0.6]]])
```

D=

$$\begin{pmatrix} <.3,.4,.3,-.5,-.4,-.2> & <.1,.2,.7,-.5,-.2,-.3> & <.3,.2,.6,-.4,-.8,-.7> & <.2,.1,.3,-.2,-.4,-.4> \\ <.2,.2,.7,-.3,-.3,-.5> & <.3,.5,.6,-.6,-.7,-.4> & <.6,.5,.4,-.3,-.6,-.8> & <.3,.4,.4,-.3,-.5,-.3> \\ <.5,.3,.1,-.4,-.2,-.4> & <.5,.4,.3,-.3,-.8,-.2> & <.5,.8,.6,-.2,-.2,-.4> & <.4,.6,.5,-.1,-.6,-.5> \\ <.6,.1,.7,-.7,-.4,-.8> & <.4,.6,.4,-.4,-.2,-.5> & <.4,.9,.3,-.5,-.5,-.3> & <.4,.5,.4,-.3,-.7,-.4> \end{pmatrix}$$

The bipolar neutrosophic matrix D can be inputted in Python code like this:

D= np.array([[[0.3,0.4,0.3,-0.5,-0.4,-0.2], [0.1,0.2,0.7,-0.5,-0.2,-0.3], [0.3,0.2,0.6,-0.4,-0.8,-0.7], [0.2,0.1,0.3,-0.2,-0.4,-0.4]], [[0.2,0.2,0.7,-0.3,-0.3,-0.5], [0.3,0.5,0.6,-0.6,-0.7,-0.4], [0.6,0.5,0.4,-0.3,-0.6,-0.8], [0.3,0.4,0.4,-0.3,-0.5,-0.3]],

[[0.5,0.3,0.1,-0.4,-0.2,-0.4], [0.5,0.4,0.3,-0.3,-0.8,-0.2], [0.5,0.8,0.6,-0.2,-0.2,-0.4], [0.4,0.6,0.5,-0.1,-0.6,-0.5]], [[0.6,0.1,0.7,-0.7,-0.4,-0.8], [0.4,0.6,0.4,-0.4,-0.2,-0.5], [0.4,0.9,0.3,-0.5,-0.5,-0.3], [0.4,0.5,0.4,-0.3,-0.7,-0.4]]])

So, the addition matrix of two bipolar neutrosophic matrices is portrayed as follow

$C_{BNS} \oplus D_{BNS} =$

$$\begin{pmatrix} <.65,.28,.06,-.35,-.58,-.68> & <.46,.08,.35,-.35,-.84,-.58> & <.79,.14,.30,-.32,-.94,-.88> & <.28,.05,.21,-.10,-.52,-.88> \\ <.92,.14,.35,-.21,-.79,-.55> & <.79,.30,.48,-.42,-.85,-.46> & <.96,.20,.24,-.03,-.88,-.90> & <.65,.08,.28,-.15,-.55,-.93> \\ <.65,.12,.02,-.24,-.44,-.82> & <.60,.08,.06,-.12,-.94,-.52> & <.95,.40,.30,-.12,-.60,-.52> & <.82,.30,.15,-.04,-.68,-.60> \\ <.96,.07,.14,-.56,-.76,-.82> & <.58,.30,.08,-.20,-.60,-.60> & <.70,.36,.15,-.05,-.85,-.44> & <.52,.20,.32,-.15,-.85,-.76> \end{pmatrix}$$

The result of addition matrix of two bipolar neutrosophic matrices C and D can be obtained by the call of the command addition (C, D):

```
>>> Addition(C, D)

Addition=

[[[ 0.65  0.28  0.06  0.35 -0.58 -0.68] [ 0.46  0.08  0.35  0.35 -0.84 -0.58] [ 0.79  0.14  0.3   0.32 -0.94 -0.88] [ 0.28  0.05  0.21  0.1  -0.52 -0.88]]

[[ 0.92  0.14  0.35  0.21 -0.79 -0.55]  [ 0.79  0.3   0.48  0.42 -0.85 -0.46]  [ 0.96  0.2   0.24  0.03 -0.88 -0.9 ]  [ 0.65  0.08  0.28  0.15 -0.55 -0.93]]

[[ 0.95  0.12  0.02  0.24 -0.44 -0.82]  [ 0.6   0.08  0.06  0.12 -0.94 -0.52]  [ 0.95  0.4   0.3   0.12 -0.6  -0.52]  [ 0.82  0.3   0.15  0.04 -0.68 -0.6 ]]

[[ 0.96  0.07  0.14  0.56 -0.76 -0.82]  [ 0.58  0.3   0.08  0.2  -0.6  -0.6 ]  [ 0.7   0.36  0.15  0.05 -0.85 -0.44]  [ 0.52  0.2   0.32  0.15 -0.85 -0.76]]]
```

3.7. Computing product of two bipolar neutrosophic matrices

The product of two bipolar neutrosophic matrices A and B is defined as follow:

$$A \odot B = R = \left[< r_{ij_T}^P, r_{ij_I}^P, r_{ij_F}^P, r_{ij_T}^N, r_{ij_I}^N, r_{ij_F}^N >\right]_{m\times n} \tag{13}$$

Where

$$r_{ij_T}^P = a_{ij_T}^P . b_{ij_T}^P, \quad r_{ij_T}^N = -\left(-a_{ij_T}^N - b_{ij_T}^N - a_{ij_T}^N . b_{ij_T}^N\right)$$
$$r_{ij_I}^P = a_{ij_I}^P + b_{ij_I}^P - a_{ij_I}^P . b_{ij_I}^P, \quad r_{ij_I}^N = -\left(a_{ij_I}^N . b_{ij_I}^N\right)$$
$$r_{ij_F}^P = a_{ij_F}^P + b_{ij_F}^P - a_{ij_F}^P . b_{ij_F}^P, \quad r_{ij_F}^N = -\left(a_{ij_F}^N . b_{ij_F}^N\right)$$

To generate the python program for finding the product operation of two bipolar neutrosophic matrices, simple call of the function **Product (A, B)** is defined as follow:

```
# BNM is represented by 3D Numpy Array => row, column and bipolar number with 6 tuples for (13)
import numpy as np
A= np.array([ [     [0.3,0.6,1,-0.2,-0.54,-0.4], [0.1,0.2,0.8,-0.5,-0.34,-0.7]          ],
              [     [0.1,0.12,0,-0.27,-0.44,-0.92], [0.5,0.33,0.58,-0.33,-0.24,-0.22]    ],
              [     [0.11,0.22,0.6,-0.29,-0.24,-0.52],[0.22,0.63,0.88,-0.28,-0.54,-0.32] ]])
B= np.array([ [     [0.32,0.4,0.1,-0.25,-0.54,-0.4], [0.13,0.2,0.11,-0.55,-0.35,-0.72]        ],
```

```
                 [     [0.17,0.19,0.66,-0.87,-0.64,-0.92], [0.25,0.36,0.88,-0.33,-0.54,-0.22]     ],
                 [     [0.15,0.28,0.67,-0.39,-0.27,-0.55],[0.24,0.73,0.28,-0.26,-0.53,-0.52]     ]])
#A.shape gives (3, 2, 6) the dimension of A. (row, column, numbers of element (Bipolar Neutrosophic Number, 6 elements) )
# A.shape[0] = 3 rows
# A.shape[1] = 2 columns
# A.shape[2] = each bipolar neutrosophic number with 6 tuple as usual
product=[]
def Product( A, B ):
    if A.shape == B.shape:
        dimA=A.shape
        for i in range (0,dimA[0]):          # for rows, here 3
            H=[]
            for j in range (0,dimA[1]):    # for columns, here 2
                H.extend([[ A[i][j][0]*B[i][j][0]) , A[i][j][1]+ B[i][j][1]- (A[i][j][1]*B[i][j][1]),
A[i][j][2]+ B[i][j][2]- (A[i][j][2]*B[i][j][2]), -(-A[i][j][3]-B[i][j][3]-A[i][j][3]*B[i][j][3]), -(A[i][j][4]*
B[i][j][4]), -(A[i][j][5]* B[i][j][5]) ] ])
            product.append(H)
    print(' Product = ', np.array(product))
```

Example 5. In this example we evaluate the product of the two bipolar neutrosophic matrices C and D of order 4X4:

C=

$$\begin{pmatrix} <.5,.7,.2,-.7,-.3,-.6> & <.4,.4,.5,-.7,-.8,-.4> & <.7,.7,.5,-.8,-.7,-.6> & <.1,.5,.7,-.5,-.2,-.8> \\ <.9,.7,.5,-.7,-.7,-.1> & <.7,.6,.8,-.7,-.5,-.1> & <.9,.4,.6,-.1,-.7,-.5> & <.5,.2,.7,-.5,-.1,-.9> \\ <.9,.4,.2,-.6,-.3,-.7> & <.2,.2,.2,-.4,-.7,-.4> & <.9,.5,.5,-.6,-.5,-.2> & <.7,.5,.3,-.4,-.2,-.2> \\ <.9,.7,.2,-.8,-.6,-.1> & <.3,.5,.2,-.5,-.5,-.2> & <.5,.4,.5,-.1,-.7,-.2> & <.2,.4,.8,-.5,-.5,-.6> \end{pmatrix}$$

The bipolar neutrosophic matrix C can be inputted in Python code like this:

```
C= np.array([ [ [0.5,0.7,0.2,-0.7,-0.3,-0.6], [0.4,0.4,0.5,-0.7,-0.8,-0.4], [0.7,0.7,0.5,-0.8,-0.7,-0.6], [0.1,0.5,0.7,-0.5,-0.2,-
0.8]], [[0.9,0.7,0.5,-0.7,-0.7,-0.1], [0.7,0.6,0.8,-0.7,-0.5,-0.1], [0.9,0.4,0.6,-0.1,-0.7,-0.5], [0.5,0.2,0.7,-0.5,-0.1,-0.9]],
[[0.9,0.4,0.2,-0.6,-0.3,-0.7], [0.2,0.2,0.2,-0.4,-0.7,-0.4], [0.9,0.5,0.5,-0.6,-0.5,-0.2], [0.7,0.5,0.3,-0.4,-0.2,-0.2]],
[[0.9,0.7,0.2,-0.8,-0.6,-0.1], [0.3,0.5,0.2,-0.5,-0.5,-0.2], [0.5,0.4,0.5,-0.1,-0.7,-0.2], [0.2,0.4,0.8,-0.5,-0.5,-0.6]] ])
```

$$\mathbf{D}=\begin{pmatrix} <.3,.4,.3,-.5,-.4,-.2> & <.1,.2,.7,-.5,-.2,-.3> & <.3,.2,.6,-.4,-.8,-.7> & <.2,.1,.3,-.2,-.4,-.4> \\ <.2,.2,.7,-.3,-.3,-.5> & <.3,.5,.6,-.6,-.7,-.4> & <.6,.5,.4,-.3,-.6,-.8> & <.3,.4,.4,-.3,-.5,-.3> \\ <.5,.3,.1,-.4,-.2,-.4> & <.5,.4,.3,-.3,-.8,-.2> & <.5,.8,.6,-.2,-.2,-.4> & <.4,.6,.5,-.1,-.6,-.5> \\ <.6,.1,.7,-.7,-.4,-.8> & <.4,.6,.4,-.4,-.2,-.5> & <.4,.9,.3,-.5,-.5,-.3> & <.4,.5,.4,-.3,-.7,-.4> \end{pmatrix}$$

The bipolar neutrosophic matrix D can be inputted in Python code like this:

```
D= np.array([[[0.3,0.4,0.3,-0.5,-0.4,-0.2], [0.1,0.2,0.7,-0.5,-0.2,-0.3], [0.3,0.2,0.6,-0.4,-0.8,-0.7], [0.2,0.1,0.3,-0.2,-0.4,-
0.4]], [[0.2,0.2,0.7,-0.3,-0.3,-0.5], [0.3,0.5,0.6,-0.6,-0.7,-0.4], [0.6,0.5,0.4,-0.3,-0.6,-0.8], [0.3,0.4,0.4,-0.3,-0.5,-0.3]],

 [[0.5,0.3,0.1,-0.4,-0.2,-0.4], [0.5,0.4,0.3,-0.3,-0.8,-0.2], [0.5,0.8,0.6,-0.2,-0.2,-0.4], [0.4,0.6,0.5,-0.1,-0.6,-0.5]],

 [[0.6,0.1,0.7,-0.7,-0.4,-0.8], [0.4,0.6,0.4,-0.4,-0.2,-0.5], [0.4,0.9,0.3,-0.5,-0.5,-0.3], [0.4,0.5,0.4,-0.3,-0.7,-0.4]]])
```

So, the product matrix of two bipolar neutrosophic matrices is portrayed as follow

$$C_{BNS} \odot D_{BNS} = \begin{pmatrix} <.15,.82,.44,-.85,-.12,-.12> & <.04,.52,.85,-.85,-1.16,-.12> & <.21,.76,.80,-.88,-.56,-.42> & <.02,.55,.79,-.60,-.008,-.32> \\ <.18,.76,.85,-.79,-.21,-.05> & <.21,.80,.92,-.88,-.35,-.04> & <.54,.70,.76,-.37,-.42,-0.40> & <.15,.52,.82,-.65,-.05,-.27> \\ <.45,.58,.28,-.76,-.06,-.28> & <.10,.52,.44,-.58,-.56,-.08> & <.45,.90,.80,-.68,-.10,-.08> & <.28,.80,.65,-.46,-.12,-.10> \\ <.54,.73,.76,-.94,-.24,-.08> & <.12,.80,.52,-.70,-.10,-.10> & <.20,.94,.65,-.55,-.35,-.06> & <.08,.70,.88,-.65,-.35,-.24> \end{pmatrix}$$

The result of product matrix of two bipolar neutrosophic matrices C and D can be obtained by the call of the command Product (C, D):

```
>>> Product(C, D)
Product=
[[[ 0.15   0.82   0.44 -0.85 -0.12 -0.12]    [ 0.04   0.52   0.85 -0.85 -0.16 -0.12]    [ 0.21   0.76   0.8   -0.88 -
0.56 -0.42] [ 0.02   0.55   0.79 -0.6   -0.08 -0.32]]
 [[ 0.18   0.76   0.85 -0.79 -0.21 -0.05]    [ 0.21   0.8    0.92 -0.88 -0.35 -0.04]    [ 0.54   0.7    0.76 -0.37 -
0.42 -0.4 ]    [ 0.15   0.52   0.82 -0.65 -0.05 -0.27]]
 [[ 0.45   0.58   0.28 -0.76 -0.06 -0.28]    [ 0.1    0.52   0.44 -0.58 -0.56 -0.08]    [ 0.45   0.9    0.8   -0.68 -
0.1   -0.08]    [ 0.28   0.8    0.65 -0.46 -0.12 -0.1 ]]
 [[ 0.54   0.73   0.76 -0.94 -0.24 -0.08] [ 0.12   0.8    0.52 -0.7   -0.1   -0.1 ] [ 0.2    0.94   0.65 -0.55 -0.35 -
0.06] [ 0.08   0.7    0.88 -0.65 -0.35 -0.24]]]
```

3.8. Computing transpose of bipolar neutrosophic matrix

To generate the python program for finding the transpose of bipolar neutrosophic matrix, simple call of the function **Transpose (A)** is defined as follow:

```
# BNM is represented by 3D Numpy Array => row, column and bipolar number with 6 tuples for transpose
import numpy as np
A=np.array([[ [0.3,0.6,1,-0.2,-0.54,-0.4], [0.1,0.2,0.8,-0.5,-0.34,-0.7]   ],
              [ [0.1,0.12,0,-0.27,-0.44,-0.92],[0.5,0.33,0.58,-0.33,-0.24,-0.22]],
             [ [0.11,0.22,0.6,-0.29,-0.24,-0.52],[0.22,0.63,0.88,-0.28,-0.54,-0.32]] ])
#A.shape gives (3, 2, 6) the dimension of A. (row, column, numbers of element (Bipolar Neutrosophic Number, 6 elements) )
# A.shape[0] = 3 rows
# A.shape[1] = 2 columns
# A.shape[2] = each bipolar neutrosophic number with 6 tuple as usual
def Transpose( A ):
    DimA= A. shape
    print (' the matrix ', DimA[0],' x ', DimA[1], ' dimension')
    trA = A.transpose()
    DimtrA= trA. shape
    print ('\n')
    print (' its transpose ', DimtrA[1],' x ', DimtrA[2], ' dimension')
    print ('\n' )
    print(' Transpose = ', trA)
```

Example 6. In this example we evaluate the transpose of the bipolar neutrosophic matrix C of order 4X4:

C=

$$\begin{pmatrix} <.5,.7,.2,-.7,-.3,-.6> & <.4,.4,.5,-.7,-.8,-.4> & <.7,.7,.5,-.8,-.7,-.6> & <.1,.5,.7,-.5,-.2,-.8> \\ <.9,.7,.5,-.7,-.7,-.1> & <.7,.6,.8,-.7,-.5,-.1> & <.9,.4,.6,-.1,-.7,-.5> & <.5,.2,.7,-.5,-.1,-.9> \\ <.9,.4,.2,-.6,-.3,-.7> & <.2,.2,.2,-.4,-.7,-.4> & <.9,.5,.5,-.6,-.5,-.2> & <.7,.5,.3,-.4,-.2,-.2> \\ <.9,.7,.2,-.8,-.6,-.1> & <.3,.5,.2,-.5,-.5,-.2> & <.5,.4,.5,-.1,-.7,-.2> & <.2,.4,.8,-.5,-.5,-.6> \end{pmatrix}$$

The bipolar neutrosophic matrix C can be inputted in Python code like this:

```
C= np.array([ [ [0.5,0.7,0.2,-0.7,-0.3,-0.6], [0.4,0.4,0.5,-0.7,-0.8,-0.4], [0.7,0.7,0.5,-0.8,-0.7,-0.6], [0.1,0.5,0.7,-0.5,-0.2,-0.8]], [[0.9,0.7,0.5,-0.7,-0.7,-0.1], [0.7,0.6,0.8,-0.7,-0.5,-0.1], [0.9,0.4,0.6,-0.1,-0.7,-0.5], [0.5,0.2,0.7,-0.5,-0.1,-0.9]],

[[0.9,0.4,0.2,-0.6,-0.3,-0.7], [0.2,0.2,0.2,-0.4,-0.7,-0.4], [0.9,0.5,0.5,-0.6,-0.5,-0.2], [0.7,0.5,0.3,-0.4,-0.2,-0.2]],

[[0.9,0.7,0.2,-0.8,-0.6,-0.1], [0.3,0.5,0.2,-0.5,-0.5,-0.2], [0.5,0.4,0.5,-0.1,-0.7,-0.2], [0.2,0.4,0.8,-0.5,-0.5,-0.6]] ])
```

So, the transpose matrix of bipolar neutrosophic matrices is portrayed as follow

<0.50, 0.70, 0.20,-0.70, -0.30, -0.60>	**<0.90, 0.70, 0.50,-0.70, -0.70, -0.10>**	**<0.30, 0.40, 0.20,-0.60, -0.30, -0.70>**	**<0.90, 0.70, 0.20,-0.80, -0.60, -0.10>**
<0.40, 0.40, 0.50,-0.70, -0.80, -0.40>	**<0.70, 0.60, 0.80,-0.70, -0.50, -0.10>**	**<0.20, 0.20, 0.20,-0.40, -0.70, -0.40>**	**<0.30, 0.50, 0.20,-0.50, -0.50, -0.20>**
<0.70, 0.70, 0.50,-0.80, -0.70, -0.60>	**<0.90, 0.40, 0.60,-0.10, -0.70, -0.50>**	**<0.90, 0.50, 0.50,-0.60, -0.50, -0.20>**	**<0.50, 0.40, 0.50,-0.10, -0.70, -0.20>**
<0.10, 0.50, 0.70,-0.50, -0.20, -0.80>	**<0.50, 0.20, 0.70,-0.50, -0.10, -0.90>**	**<0.70, 0.50, 0.30,-0.40, -0.20, -0.20>**	**<0.20, 0.40, 0.80,-0.50, -0.50, -0.60>**

```
>>> Transpose(C)
  The matrix 4 x4 dimension
  Its transpose 4 x 4 dimension
Transpose =
[[[ 0.5   0.9   0.9   0.9] [ 0.4   0.7   0.2   0.3]   [ 0.7   0.9   0.9   0.5] [ 0.1   0.5   0.7   0.2]]
[[ 0.7   0.7   0.4   0.7] [ 0.4   0.6   0.2   0.5] [ 0.7   0.4   0.5   0.4] [ 0.5   0.2   0.5   0.4]]
[[ 0.2   0.5   0.2   0.2] [ 0.5   0.8   0.2   0.2] [ 0.5   0.6   0.5   0.5] [ 0.7   0.7   0.3   0.8]]
[[-0.7 -0.7 -0.6 -0.8]    [-0.7 -0.7 -0.4 -0.5]      [-0.8 -0.1 -0.6 -0.1]      [-0.5 -0.5 -0.4 -0.5]]
[[-0.3 -0.7 -0.3 -0.6]      [-0.8 -0.5 -0.7 -0.5]      [-0.7 -0.7 -0.5 -0.7]      [-0.2 -0.1 -0.2 -0.5]]
[[-0.6 -0.1 -0.7 -0.1]    [-0.4 -0.1 -0.4 -0.2]    [-0.6 -0.5 -0.2 -0.2]    [-0.8 -0.9 -0.2 -0.6]]]
```

3.9 Computing composition of two bipolar neutrosophic matrices

To generate the python program for finding the composition of two bipolar neutrosophic matrices, simple call of the function **Composition ()** is defined as follow:

```
# BNM is represented by 3D Numpy Array => row, column and bipolar number with 6 tuples for Composition
#A.shape and B.shape returns (3, 3, 6) the dimension of A. (row, column, numbers of element (Bipolar Neutrosophic Number, 6 elements) )
# A.shape[0] = 3 rows
# A.shape[1] = 3 columns
# A.shape[2] = Each bipolar neutrosophic number has 6 tuple as usual
#One can use matrices with any dimensions but dimensions of two matrices must be the same and nxn
```

```
import math
import numpy as np
A= np.array( [ [ [0.3, 0.6, 1, -0.2, -0.54, -0.4], [0.1, 0.2, 0.8, -0.5, -0.34, -0.7], [0.020,0.021,0.022,-0.023,-
0.024,-0.025]    ],
[ [0.17,0.19,0.66,-0.87,-0.64,-0.92], [0.25,0.36,0.88,-0.33,-0.54,-0.22], [0.120,0.121,0.122,-0.123,-0.124,-
0.125]    ],
[ [0.15,0.28,0.67,-0.39,-0.27,-0.55],[0.24,0.73,0.28,-0.26,-0.53,-0.52], [0.220,0.221,0.222,-0.223,-0.224,-
0.225]    ]    ] )
B=np.array([ [0.11,0.22,0.6,-0.29,-0.24,-0.52], [0.32,0.4,0.1,-0.25,-0.54,-0.4], [0.13,0.2,0.11,-0.55,-0.35,-
0.72] ],
[ [0.100,0.101,0.102,-0.103,-0.104,-0.105], [1,0.111,0.112,-0.113,-0.114,-0.115], [0.720,0.821,0.152,-
0.143,-0.194,-0.1]    ],
[ [0,0.73,0.202,-0.203,-0.204,-0.205],    [0.22,0.63,0.88,-0.28,-0.54,-0.32], [0.3,0,0.47,-0.223,-0.254,-0.295]
] ] )
def Composition( A, B ):
    global composition
    composition=[]
    dimA = A.shape
    H=[ ]
    if A.shape == B.shape and dimA[0] == dimA[1]:
        for i in range (0,dimA[0]):
            for j in range (0,dimA[0]):
                counter0=0
                for d in range (0, dimA[0]):
                    if counter0 ==0:
                        maxtt    =       [ A[i][d][0],B[d][j][0] ]
                        maxT = min(maxtt)
                        minii =    [A[i][d][1],B[d][j][1] ]
                        minI =    max(minii)
                        minff = [ A[i][d][2],B[d][j][2]]
                        minF = max( minff)
                        minntt= [      A[i][d][3],B[d][j][3] ]
                        minNT = max (minntt)
                        maxnii = [ A[i][d][4],B[d][j][4]    ]
                        maxNI =    min( maxnii    )
                        maxnff= [      A[i][d][5],B[d][j][5]    ]
                        maxNF = min (maxnff)
                        counter0    = 1
                    else:
                        maxT1              = [    A[i][d][0],B[d][j][0]    ]
                        maxT11           = min(maxT1)
                        maxT112       = [    maxT11 ,    maxT    ]
```

```
                    maxT          =   max(maxT112)
                    minI1         =   [ A[i][d][1],B[d][j][1]   ]
                    minI11        =   max(minI1)
                    minI112   = [ minI11, minI      ]
                    minI          = min(   minI112)
                    minF1         =     [ A[i][d][2],B[d][j][2]   ]
                    minF11        =   max(minF1)
                    minF112       = [ minF11, minF]
                    minF          = min(minF112 )
                    minNT1        =    [   A[i][d][3],B[d][j][3]        ]
                    minNT11       =   max( minNT1     )
                    minNT112  =    [ minNT11, minNT ]
                    minNT         =   min( minNT112   )
                    maxNI1        = [ A[i][d][4],B[d][j][4] ]
                    maxNI11       = min(   maxNI1     )
                    maxNI112      = [   maxNI11, maxNI   ]
                    maxNI         = max(maxNI112)
                    maxNF1        =   [ A[i][d][5],B[d][j][5]   ]
                    maxNF11       =   min ( maxNF1 )
                    maxNF112 =   [ maxNF11, maxNF ]
                    maxNF         =   max (   maxNF112   )
               H.append( [maxT,   minI, minF, minNT, maxNI, maxNF] )
          composition.extend(H)
     global nested
     nested = [   ]
     for k in range( int(math.sqrt(len(composition))) ):
               nested.append(composition[k:k+int(math.sqrt(len(composition))) ] )
     print('Composition= ', np.array(nested))
```

Example 7. In this example we evaluate the composition of the two bipolar neutrosophic matrices C and D of order 4X4:

C=

$$\begin{pmatrix} \langle .5,.7,.2,-.7,-.3,-.6\rangle & \langle .4,.4,.5,-.7,-.8,-.4\rangle & \langle .7,.7,.5,-.8,-.7,-.6\rangle & \langle .1,.5,.7,-.5,-.2,-.8\rangle \\ \langle .9,.7,.5,-.7,-.7,-.1\rangle & \langle .7,.6,.8,-.7,-.5,-.1\rangle & \langle .9,.4,.6,-.1,-.7,-.5\rangle & \langle .5,.2,.7,-.5,-.1,-.9\rangle \\ \langle .9,.4,.2,-.6,-.3,-.7\rangle & \langle .2,.2,.2,-.4,-.7,-.4\rangle & \langle .9,.5,.5,-.6,-.5,-.2\rangle & \langle .7,.5,.3,-.4,-.2,-.2\rangle \\ \langle .9,.7,.2,-.8,-.6,-.1\rangle & \langle .3,.5,.2,-.5,-.5,-.2\rangle & \langle .5,.4,.5,-.1,-.7,-.2\rangle & \langle .2,.4,.8,-.5,-.5,-.6\rangle \end{pmatrix}$$

The bipolar neutrosophic matrix C can be inputted in Python code like this:

C= np.array([[[0.5,0.7,0.2,-0.7,-0.3,-0.6], [0.4,0.4,0.5,-0.7,-0.8,-0.4], [0.7,0.7,0.5,-0.8,-0.7,-0.6], [0.1,0.5,0.7,-0.5,-0.2,-0.8]], [[0.9,0.7,0.5,-0.7,-0.7,-0.1], [0.7,0.6,0.8,-0.7,-0.5,-0.1], [0.9,0.4,0.6,-0.1,-0.7,-0.5], [0.5,0.2,0.7,-0.5,-0.1,-0.9]], [[0.9,0.4,0.2,-0.6,-0.3,-0.7], [0.2,0.2,0.2,-0.4,-0.7,-0.4], [0.9,0.5,0.5,-0.6,-0.5,-0.2], [0.7,0.5,0.3,-0.4,-0.2,-0.2]], [[0.9,0.7,0.2,-0.8,-0.6,-0.1], [0.3,0.5,0.2,-0.5,-0.5,-0.2], [0.5,0.4,0.5,-0.1,-0.7,-0.2], [0.2,0.4,0.8,-0.5,-0.5,-0.6]]])

D=

$$\begin{pmatrix} <.3,.4,.3,-.5,-.4,-.2> & <.1,.2,.7,-.5,-.2,-.3> & <.3,.2,.6,-.4,-.8,-.7> & <.2,.1,.3,-.2,-.4,-.4> \\ <.2,.2,.7,-.3,-.3,-.5> & <.3,.5,.6,-.6,-.7,-.4> & <.6,.5,.4,-.3,-.6,-.8> & <.3,.4,.4,-.3,-.5,-.3> \\ <.5,.3,.1,-.4,-.2,-.4> & <.5,.4,.3,-.3,-.8,-.2> & <.5,.8,.6,-.2,-.2,-.4> & <.4,.6,.5,-.1,-.6,-.5> \\ <.6,.1,.7,-.7,-.4,-.8> & <.4,.6,.4,-.4,-.2,-.5> & <.4,.9,.3,-.5,-.5,-.3> & <.4,.5,.4,-.3,-.7,-.4> \end{pmatrix}$$

The bipolar neutrosophic matrix D can be inputted in Python code like this:

```
D= np.array([[[0.3,0.4,0.3,-0.5,-0.4,-0.2], [0.1,0.2,0.7,-0.5,-0.2,-0.3], [0.3,0.2,0.6,-0.4,-0.8,-0.7], [0.2,0.1,0.3,-0.2,-0.4,-0.4]], [[0.2,0.2,0.7,-0.3,-0.3,-0.5], [0.3,0.5,0.6,-0.6,-0.7,-0.4], [0.6,0.5,0.4,-0.3,-0.6,-0.8], [0.3,0.4,0.4,-0.3,-0.5,-0.3]],
[[0.5,0.3,0.1,-0.4,-0.2,-0.4], [0.5,0.4,0.3,-0.3,-0.8,-0.2], [0.5,0.8,0.6,-0.2,-0.2,-0.4], [0.4,0.6,0.5,-0.1,-0.6,-0.5]],
[[0.6,0.1,0.7,-0.7,-0.4,-0.8], [0.4,0.6,0.4,-0.4,-0.2,-0.5], [0.4,0.9,0.3,-0.5,-0.5,-0.3], [0.4,0.5,0.4,-0.3,-0.7,-0.4]]])
```

So, the composition matrix of two bipolar neutrosophic matrices is portrayed as follow

$C_{BNS} \odot D_{BNS}$=

$$\begin{pmatrix} <.5,.4,.3,-.5,-.4,-.5> & <.5,.5,.5,-.6,-.2,-.4> & <.5,.5,.5,-.5,-.5,-.6> & <.4,.4,.3,-.3,-.4,-.4> \\ <.5,.5,.5,-.6,-.2,-.4> & <.5,.5\ .5,-.5,-.5,-.6> & <.4,.4,.3,-.3,-.4,-.4> & <.5,.2,.5,-.5,-.4,-.2> \\ <.5\ .5,.5,-.5,-.5,-.6> & <.4,.4,.4,-.3,-.4,-.4> & <.5,.2,.5,-.5,-.4,-.2> & <.5,.4,.6,-.6,-.2,-.3> \\ <.4,.4,.3,-.3,-.4,-.1> & <.5,.2,.5,-.5,-.4\ -.2> & <.5,.4,.6,-.6,-.2,-.3> & <.6,.6,.6,-.5,-.5,-.5> \end{pmatrix}$$

The result of composition t matrix of two bipolar neutrosophic matrices C and D can be obtained by the call of the command Composition (C, D):

```
>>> Composition(C, D)

Composition=

[[[ 0.5  0.4  0.3 -0.5 -0.4 -0.5] [ 0.5  0.5  0.5 -0.6 -0.2 -0.4] [ 0.5  0.5  0.5 -0.5 -0.5 -0.6] [ 0.4  0.4  0.3 -0.3 -0.4 -0.4]]

[[ 0.5  0.5  0.5 -0.6 -0.2 -0.4] [ 0.5  0.5  0.5 -0.5 -0.5 -0.6] [ 0.4  0.4  0.3 -0.3 -0.4 -0.4] [ 0.5  0.2  0.5 -0.5 -0.4 -0.2]]

[[ 0.5  0.5  0.5 -0.5 -0.5 -0.6] [ 0.4  0.4  0.3 -0.3 -0.4 -0.4] [ 0.5  0.2  0.5 -0.5 -0.4 -0.2] [ 0.5  0.4  0.6 -0.6 -0.2 -0.3]]

[[ 0.4  0.4  0.3 -0.3 -0.4 -0.4] [ 0.5  0.2  0.5 -0.5 -0.4 -0.2] [ 0.5  0.4  0.6 -0.6 -0.2 -0.3] [ 0.6  0.6  0.6 -0.5 -0.5 -0.5]]]
```

4. Conclusion

In this paper, we have presented a useful Python tool for the calculations of matrices obtained by bipolar neutrosophic sets. The matrices have nested list data type, in other words, multi-dimensional arrays in the Python Programming Language. The importance of this work, is that the proposed Python tool can be used also for fuzzy matrices, bipolar fuzzy matrices, intuitionistic fuzzy matrices, bipolar intuitionistic fuzzy matrices and single valued neutrosophic matrices. This work will be extending with the implementation of Bipolar Complex Neutrosophic Matrices in the future. We have used Python Numpy module in order to provide convenience for possible users. We hope that the tool might be useful in data science, physics, scientific computing, decision making, engineering studies and other fields.

Author Contributions

S.T. implemented codes of the bipolar neutrosophic matrices and their operations and created the scripts on Python 3.7 by using Numpy module. S.B. offered the project paper and reviewed the implementations. Conceptualization, S.B. and S.T.; Methodology, S.T.; Validation, S.B., S.T., A.B., M.T and F.S.; Investigation, S.B. and S.T.; Resources, S.B., S.T., A.B., M.T and F.S; Writing-Original Draft

Preparation, S.B..; Writing—Review and Editing, S.B., S.T., A.B., M.T and F.S.; Supervision, S.B. and F.S.

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Neutrosophic αgs Continuity And Neutrosophic αgs Irresolute Maps

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Abstract. Neutrosophic Continuity functions very first introduced by A.A.Salama et.al.Aim of this present paper is, we introduce and investigate new kind of Neutrosophic continuity is called Neutrosophic αgs Continuity maps in Neutrosophic topological spaces and also discussed about some properties and characterization of Neutrosophic αgs Irresolute Map.

Keywords: Neutrosophic α-closed sets, Neutrosophic semi-closed sets, Neutrosophic αgs-closed sets Neutrosophic αgs Continuity maps, Neutrosophic αgs irresolute maps

1. Introduction

Neutrosophic set theory concepts first initiated by F.Smarandache[11] which is Based on K. Atanassov's intuitionistic[6]fuzzy sets & L.A.Zadeh's [20]fuzzy sets. Also it defined by three parameters truth(T), indeterminacy (I),and falsity(F)-membership function. Smarandache's neutrosophic concept have wide range of real time applications for the fields of [1,2,3,4&5] Information Systems, Computer Science, Artificial Intelligence, Applied Mathematics, decision making. Mechanics, Electrical & Electronic, Medicine and Management Science etc,.

A.A.Salama[16] introduced Neutrosophic topological spaces by using Smarandache's Neutrosophic sets. I.Arokiarani.[7] et.al., introduced Neutrosophic α-closed sets.P. Ishwarya, [13]et.al., introduced and studied Neutrosophic semi-open sets in Neutrosophic topological spaces. Neutrosophic continuity functions introduced by A.A.Salama[15]. Neutrosophic αgs-closed set[8] introduced by V.Banu priya&S.Chandrasekar. Aim of this present paper is, we introduce and investigate new kind of Neutrosophic continuity is called Neutrosophic αgs Continuity maps in Neutrosophic topological spaces and also we discussed about properties and characterization Neutrosophic αgs Irresolute Maps

2. Preliminaries

In this section, we introduce the basic definition for Neutrosophic sets and its operations.

Definition 2.1 [11]

Let E be a non-empty fixed set. A Neutrosophic set λ writing the format is

$$\lambda = \{<e, \eta_\lambda(e), \sigma_\lambda(e), \gamma_\lambda(e)>: e\in E\}$$

Where $\eta_\lambda(e)$, $\sigma_\lambda(e)$ and $\gamma_\lambda(e)$ which represents Neutrosophic topological spaces the degree of membership function, indeterminacy and non-membership function respectively of each element $e \in E$ to the set λ.

Remark 2.2 [11]

A Neutrosophic set $\lambda=\{<e, \eta_\lambda(e), \sigma_\lambda(e), \gamma_\lambda(e)>: e\in E\}$ can be identified to an ordered triple $<\eta_\lambda, \sigma_\lambda, \gamma_\lambda>$ in $]^{-}0,1^{+}[$ on E.

Remark 2.3[11]

Neutrosophic set λ={<e, η_λ(e),σ_λ(e),γ_λ(e) >:e$\in$E}our convenient we can write λ=<e, η_λ, σ_λ, γ_λ>.

Example 2.4 [11]

we must introduce the Neutrosophic set 0_N and 1_N in E as follows:

0_N may be defined as:

(0_1) 0_N={<e, 0, 0, 1>: e$\in$E}

(0_2) 0_N={<e, 0, 1, 1>: e$\in$E}

(0_3) 0_N ={<e, 0, 1, 0 >:e$\in$E}

(0_4) 0_N={<e, 0, 0, 0>: e$\in$E}

1_N may be defined as:

(1_1) 1_N = {<e, 1, 0, 0>: e$\in$E}

(1_2) 1_N = {<e, 1, 0, 1 >: e$\in$E}

(1_3) 1_N ={<e, 1, 1, 0 >: e$\in$E}

(1_4) 1_N ={<e, 1, 1, 1 >: e$\in$E}

Definition 2.5 [11]

Let λ=<η_λ, σ_λ,γ_λ> be a Neutrosophic set on E, then λ^C defined as λ^C={<e , γ_λ(e) ,1- σ_λ(e), η_λ(e) >: e $\in$E}

Definition 2.6 [11]

Let E be a non-empty set, and Neutrosophic sets λ and μ in the form

λ ={<e, η_λ(e), $\sigma\lambda$(e), $\gamma\lambda$(e)>:e$\in$E} and

μ ={<e, η_μ(e), σ_μ(e), γ_μ(e)>: e$\in$E}.

Then we consider definition for subsets ($\lambda\subseteq\mu$).

$\lambda\subseteq\mu$ defined as: $\lambda\subseteq\mu \Leftrightarrow \eta_\lambda(e) \leq \eta_\mu(e)$, $\sigma_\lambda(e) \leq \sigma_\mu(e)$ and $\gamma_\lambda(e) \geq \gamma_\mu(e)$ for all e$\in$E

Proposition 2.7 [11]

For any Neutrosophic set λ, then the following condition are holds:

(i) $0_N\subseteq\lambda$, $0_N\subseteq 0_N$

(ii) $\lambda\subseteq 1_N$, $1_N\subseteq 1_N$

Definition 2.8 [11]

Let E be a non-empty set, and λ=<e, η_μ(e),σ_λ(e), γ_λ(e)> , μ =<e, η_μ(e), σ_μ(e), γ_μ(e)> be two Neutrosophic sets. Then

(i) $\lambda\cap\mu$ defined as :$\lambda\cap\mu$ =<e, $\eta_\lambda(e)\wedge\eta_\mu(e)$, $\sigma_\lambda(e)\wedge\sigma_\mu(e)$,$\gamma_\lambda(e)\vee\gamma_\mu(e)$>

(ii) $\lambda\cup\mu$ defined as :$\lambda\cup\mu$ =<e, $\eta_\lambda(e)\vee\eta_\mu(e)$, $\sigma_\lambda(e)\vee\sigma_\mu(e)$, $\gamma_\lambda(e)\wedge\gamma_\mu(e)$>

Proposition 2.9 [11]

For all λ and μ are two Neutrosophic sets then the following condition are true:

(i) $(\lambda\cap\mu)^C=\lambda^C\cup\mu^C$

(ii) $(\lambda\cup\mu)^C=\lambda^C\cap\mu^C$.

Definition 2.10 [16]

A Neutrosophic topology is a non-empty set E is a family τ_N of Neutrosophic subsets in E satisfying the following axioms:

(i) 0_N, $1_N \in\tau_N$,

(ii) $G_1\cap G_2\in\tau_N$ for any G_1, $G_2\in\tau_N$,

(iii) $\cup G_i\in\tau_N$ for any family $\{G_i \mid i\in J\}\subseteq\tau_N$.

the pair (E, τ_N) is called a Neutrosophic topological space.

The element Neutrosophic topological spaces of τ_N are called Neutrosophic open sets.

A Neutrosophic set λ is closed if and only if λ^C is Neutrosophic open.

Example 2.11[16]

Let E={e} and

A_1= {<e, .6, .6, .5>:e$\in$E}

A_2= {<e, .5, .7, .9>:e$\in$E}

A_3= {<e, .6, .7, .5>:e$\in$E}

A_4= {<e, .5, .6, .9>:e$\in$E}

Then the family τ_N={0_N, 1_N,A_1, A_2, A_3, A_4}is called a Neutrosophic topological space on E.

Definition 2.12[16]

Let (E, τ_N) be Neutrosophic topological spaces and λ={<e, η_λ(e), σ_λ(e), γ_λ(e)>:e∈E} be a Neutrosophic set in E. Then the Neutrosophic closure and Neutrosophic interior of λ are defined by

Neu-cl(λ)=∩{D:D is a Neutrosophic closed set in E and λ⊆D}

Neu-int(λ)=∪{C:C is a Neutrosophic open set in E and C⊆λ}.

Definition 2.13

Let (E, τ_N) be a Neutrosophic topological space. Then λ is called

(i) Neutrosophic regular Closed set [7] (Neu-RCS in short) if λ=Neu-Cl(Neu-Int(λ)),

(ii) Neutrosophic α-Closed set[7] (Neu-αCS in short) if Neu-Cl(Neu-Int(Neu-Cl(λ)))⊆λ,

(iii) Neutrosophic semi Closed set [13] (Neu-SCS in short) if Neu-Int(Neu-Cl(λ))⊆λ ,

(iv) Neutrosophic pre Closed set [18] (Neu-PCS in short) if Neu-Cl(Neu-Int(λ))⊆λ,

Definition 2.14

Let (E, τ_N) be a Neutrosophic topological space. Then λ is called

(i). Neutrosophic regular open set [7](Neu-ROS in short) if λ=Neu-Int(Neu-Cl(λ)),

(ii). Neutrosophic α-open set [7](Neu-αOS in short) if λ⊆Neu-Int(Neu-Cl(Neu-Int(λ))),

(iii). Neutrosophic semi open set [13](Neu-SOS in short) if λ⊆Neu-Cl(Neu-Int(λ)),

(iv).Neutrosophic pre open set [18] (Neu-POS in short) if λ⊆Neu-Int(Neu-Cl(λ)),

Definition 2.15

Let (E, τ_N) be a Neutrosophic topological space. Then λ is called

(i).Neutrosophic generalized closed set[9](Neu-GCS in short) if Neu-cl(λ)⊆U whenever λ⊆U and U is a Neu-
OS in E ,

(ii).Neutrosophic generalized semi closed set[17] (Neu-GSCS in short) if Neu-scl(λ)⊆U Whenever λ⊆U and U
is a Neu-OS in E,

(iii).Neutrosophic α generalized closed set [14](Neu-αGCS in short) if Neu-αcl(λ)⊆U whenever λ⊆U and U is a
Neu-OS in E ,

(iv).Neutrosophic generalized alpha closed set [10] (Neu-GαCS in short) if Neu-αcl(λ)⊆U whenever λ⊆U and U
is a Neu-αOS in E .

The complements of the above mentioned Neutrosophic closed sets are called their respective Neutrosophic open sets.

Definition 2.16 [8]

Let (E, τ_N) be a Neutrosophic topological space.Then λ is called Neutrosophic α generalized Semi closed set (Neu-αGSCS in short) if Neu-αcl(λ)⊆U whenever λ⊆U and U is a Neu-SOS in E

The complements of Neutrosophic αGS closed sets is called Neutrosophic αGS open sets.

3. Neutrosophic αgs-Continuity maps

In this section we Introduce Neutrosophic α-generalized semi continuity maps and study some of its properties.

Definition 3.1.

A maps f:(E_1, τ_N)→(E_2, σ_N) is called a Neutrosophic α-generalized semi continuity(Neu-αGS continuity in short) $f^{-1}(\mu)$ is a Neu-αGSCS in (E_1, τ_N) for every Neu-CS μ of (E_2, σ_N)

Example 3.2.

Let E_1={a_1,a_2}, E_2={b_1,b_2}, U=<e_1,(.7,.5,.8),(.5,.5,.4)> andV=<e_2,(1,.5,.9),(.2,.5,.3)>.Then τ_N={0_N,U,1_N} and σ_N={0_N,V,1_N} are Neutrosophic Topologies on $E_1$and E_2 respectively.

Define a maps f:(E_1, τ_N)→(E_2, σ_N)by f(a_1)=b_1 and f(a_2)=b_2. Then f is a Neu-αGS continuity maps.

Theorem 3.3.

Every Neu-continuity maps is a Neu-αGS continuity maps.

Proof.

Let f:(E_1, τ_N)→(E_2, σ_N) be a Neu-continuity maps. Let λ be a Neu-CS in E_2. Since f is a Neu-continuity maps, $f^{-1}(\lambda)$ is a Neu-CS in E_1. Since every Neu-CS is a Neu-αGSCS,$f^{-1}(\lambda)$ is a Neu-αGSCS in E_1. Hence f is a Neu-αGS continuity maps.

Example 3.4.

Neu-αGS continuity maps is not Neu-continuity maps

Let E_1={a_1, a_2}, E_2={b_1, b_2},U=< e_1, (.5,.5,.3), (.7,.5, .8)> and V=< e_2,(.4,.5,.3), (.8,.5, .9)>. Then τ_N={0_N,U,1_N} and σ_N={0_N,V,1_N} are Neutrosophic sets on E_1 and E_2 respectively. Define a maps f:(E_1, τ_N)→(E_2, σ_N) by f(a_1)=b_1 and f(a_2)=b_2 . Since the Neutrosophic set λ=<y ,(.3,.5, .4),(.9,.5, .8)> is Neu-CS in E_2, $f^{-1}(\lambda)$ is a Neu-αGSCS but not Neu-CS in E_1. Therefore f is a Neu-αGS continuity maps but not a Neu-continuity maps.

Theorem 3.5.

Every Neu-α continuity maps is a Neu-αGS continuity maps.

Proof.

Let f:(E_1, τ_N)→(E_2, σ_N)be a Neu- α continuity maps. Let λ be a Neu-CS in E_2. Then by hypothesis $f^{-1}(\lambda)$ is a Neu-αCS in E_1. Since every Neu-αCS is a Neu-αGSCS,$f^{-1}(\lambda)$is a Neu-αGSCS in E_1. Hence f is a Neu-αGS continuity maps.

Example 3.6.

Neu-αGS continuity maps is not Neu-α continuity maps

Let E_1={a_1,a_2}, E_2={b_1,b_2},U=< e_1,(.5,.5, .6), (.7,.5,.6)> and V=< e_2 ,(.3,.5,.9), (.5,.5, .7)>. Then τ_N={0_N,U,1_N} and σ_N={0_N, V, 1_N} are Neutrosophic Topologies on E_1 and E_2 respectively. Define a maps f:(E_1, τ_N)→(E_2, σ_N) by f(a_1)=b_1 and f(a_2)=b_2. Since the Neutrosophic set λ=< e_2 , (.9,.5, .3), (.7,.5, .5)> is Neu-CS in E_2, $f^{-1}(\lambda)$ is a Neu-αGSCS continuity maps.

Remark 3.7.

Neu-G continuity maps and Neu-αGS continuity maps are independent of each other.

Example 3.8.

Neu-αGS continuity maps is not Neu-G continuity maps.

Let E_1={a_1, a_2}, E_2={b_1, b_2}, U=< e_1,(.5,.5, .6), (.8,.5,.4)> and V=< e_2 ,(.7,.5,.4), (.9,.5, .3)>. Then τ_N={0_N,U,1_N} and σ_N={0_N,V,1_N} are Neutrosophic Topologies on E_1 and E_2 respectively. Define a maps f:(E_1, τ_N)→(E_2, σ_N) by f(a_1)=b_1 and f(a_2)=b_2 .Then f is Neu-αGS continuity maps but not Neu-G continuity maps.

Since λ=< e_1,(.4,.5, .7), (.3,.5, .9)> is Neu-CS in E_2, $f^{-1}(\lambda)$=< e_2, (.4,.5, .7), (.7,.5, .3)> is not Neu-GCS in E_1.

Example 3.9.

Neu-G continuity maps is not Neu-αGS continuity maps.

Let E_1={a_1, a_2}, E_2={b_1,b_2}, U=<e_1,(.6,.5,.4), (.8,.5,.2)> and V=<e_2,(.3,.5,.7), (.1,.5, .9)>. Then τ_N={0_N,U,1_N} and σ_N={0_N,V,1_N} are Neutrosophic Topologies on E_1 and E_2 respectively. Define a maps f:(E_1, τ_N) →(E_2, σ_N) by f(a_1)=b_1 and f(a_2)=b_2 . Then f is Neu-G continuity maps but not a Neu-αGS continuity maps.

Since λ=< e_2 ,(.7,.5, .3), (.9,.5, .1)> is Neu-CS in E_2, $f^{-1}(\lambda)$=< e_1, (.7,.5, .3), (.9,.5,.1)> is not Neu-αGSCS in E_1.

Theorem 3.10.

Every Neu-αGS continuity maps is a Neu-GS continuity maps.

Proof.

Let f f:(E_1, τ_N)→(E_2, σ_N) be a Neu-αGS continuity maps. Let λ be a Neu-CS in E_2.Then by hypothesis $f^{-1}(\lambda)$ Neu-αGSCS in E_1. Since every Neu-αGSCS is a Neu-GSCS, $f^{-1}(\lambda)$ is a Neu-GSCS in E_1. Hence f is a Neu-GS continuity maps.

Example 3.11.

Neu-GS continuity maps is not Neu-αGS continuity maps.

Let E_1={a_1, a_2}, E_2={b_1, b_2}, U=< e_1,(.8,.5,.4), (.9,.5,.2)> and V=< e_2,(.3,.5,.9), (0.1,.5, .9)>. Then τ_N={0_N,U,1_N} and σ_N={0_N,V,1_N} are Neutrosophic Topologies on E_1 and E_2 respectively. Define a maps f:(E_1, τ_N)→(E_2, σ_N)by f(a_1)=b_1 and f(a_2)=b_2.Since the Neutrosophic set λ=< e_2,(.9,.5,.3),(.9, .5,.1)> is Neu-CS in E_2, $f^{-1}(\lambda)$ is Neu-GSCS in E_1 but not Neu-αGSCS in E_1. Therefore f is a Neu-GS continuity maps but not a Neu-αGS continuity maps.

Remark 3.12.

Neu-P continuity maps and Neu-αGS continuity maps are independent of each other.

Example 3.13.

Neu-P continuity maps is not Neu-αGS continuity maps Let $E_1=\{a_1, a_2\}$, $E_2=\{b_1, b_2\}$,U= < e_1, (.3,.5,.7),(.4,.5,.6)> and V=< e_2,(.8,.5,.3), (.9,.5, .2)>. Then $\tau_N=\{0_N,U,1_N\}$ and $\sigma_N=\{0_N, V, 1_N\}$ are Neutrosophic Topologies on E_1 and E_2 respectively. Define a maps f:(E_1, τ_N)→(E_2, σ_N)by f(a_1)=b_1 and f(a_2)=b_2.Since the Neutrosophic set λ=< e_2,(.3,.5, .8), (.2,.5, .9)> is Neu-CS in E_2, $f^{-1}(\lambda)$ is Neu-PCS in E_1 but not Neu-αGSCS in E_1. Therefore f is a Neu-P continuity maps but not Neu-αGS continuity maps.

Example 3.14.

Neu-αGS continuity maps is not Neu-P continuity maps

Let $E_1=\{a_1, a_2\}$, $E_2=\{b_1, b_2\}$, U=< e_1,(.4,.5,.8),(.5,.5,.7)> and V=< e_1,(.5,.5,.7), (.6,.5, .6)> and W=< e_2,(.8,.5,.4), (.5,.5,.7)>. Then $\tau_N=\{0_N,U,V,1_N\}$ and $\sigma_N=\{0_N,W,1_N\}$ are Neutrosophic Topologies on E_1 and E_2 respectively. Define a maps f:(E_1, τ_N)→(E_2, σ_N)by f(a_1) = b_1 and f(a_2)=b_2. Since the Neutrosophic set λ=<y ,(.4,.5, .8), (.7,.5, .5)> is Neu-αGSCS but not Neu-PCS in E_2, $f^{-1}(\lambda)$is Neu-αGSCS in E_1 but not Neu-PCS in E_1. Therefore f is a Neu-αGS continuity maps but not Neu-P continuity maps.

Theorem 3.15.

Every Neu-αGS continuity maps is a Neu-αG continuity maps.

Proof.

Let f:(E_1, τ_N)→(E_2, σ_N)be a Neu-αGS continuity maps. Let λ be a Neu-CS in E_2. Since f is Neu-αGS continuity maps, $f^{-1}(\lambda)$ is a Neu-αGSCS in E_1. Since every Neu-αGSCS is a Neu- αGCS, $f^{-1}(\lambda)$ is a Neu-αGCS in E_1. Hence f is a Neu- αG continuity maps.

Example 3.16.

Neu- αG continuity maps is not Neu-αGS continuity maps

Let $E_1=\{a_1, a_2\}$, $E_2=\{b_1, b_2\}$, U=< e_1,(.1,.5,.7),(.3,.5, .6)> and V=< e_2,(.7,.5,.4), (.6,.5, .5)>.Then $\tau_N=\{0_N,U,1_N\}$ and $\sigma_N=\{0_N,V,1_N\}$ are Neutrosophic Topologies on E_1 and E_2 respectively. Define a maps f:(E_1, τ_N)→(E_2, σ_N)by f(a_1)=b_1 and f(a_2)=b_2.Since the Neutrosophic set λ=< e_2,(.4,.5,.7),(.5,.5, .6)> is Neu-CS in E_2, $f^{-1}(\lambda)$ is Neu-αGCS in E_1 but not Neu-αGSCS in E_1. Therefore f is a Neu-αG continuity maps but not a Neu-αGS continuity maps.

Theorem 3.17.

Every Neu-αGS continuity maps is a Neu-Gα continuity maps.

Proof.

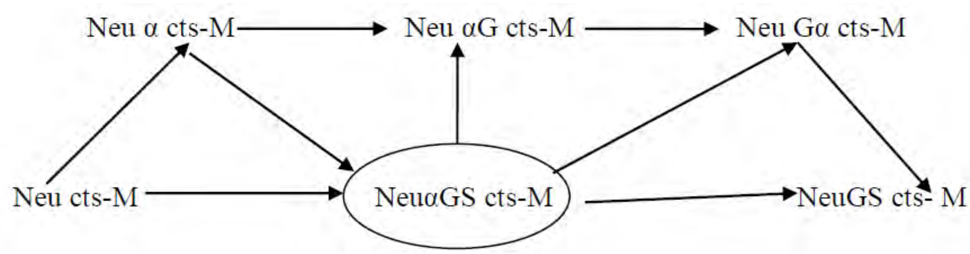
Let f:(E_1, τ_N)→(E_2, σ_N)be a Neu-αGS continuity maps. Let λ be a Neu-CS in E_2. Since f is Neu-αGS continuity maps, $f^{-1}(\lambda)$is a Neu-αGSCS in E_1. Since every Neu-αGSCS is a Neu-GαCS, $f^{-1}(\lambda)$ is a Neu-GαCS in E_1. Hence f is a Neu-Gα continuity maps.

Example 3.18.

Neu-Gα continuity maps is not Neu-αGS continuity maps Let $E_1=\{a_1, a_2\}$, $E_2=\{b_1, b_2\}$, U=< e_1, (.5,.5,.7), (.3,.5, .9)> and V=< e_2 ,(.6,.5,.6), (.5,.5,.7)>.Then $\tau_N=\{0_N,U,1_N\}$ and $\sigma_N=\{0_N,V,1_N\}$ are Neutrosophic Topologies on E_1 and E_2 respectively. Define a maps f:(E_1, τ_N)→(E_2, σ_N)by f(a_1)=b_1 and f(a_2)=b_2. Since the Neutrosophic set λ=<y,(.6,.5,.6), (.7,.5, .5)> is Neu-CS in E_2, $f^{-1}(\lambda)$is Neu-Gα CS in E_1 but not Neu-αGSCS in E_1. Therefore f is a Neu-Gα continuity maps but not a Neu-αGS continuity maps.

Remark 3.19.

We obtain the following diagram from the results we discussed above.



Theorem 3.20.

A maps f:(E_1,τ_N)→(E_2,σ_N)is Neu-αGS continuity if and only if the inverse image of each Neutrosophic set in E_2 is a Neu-αGSOS in E_1.

Proof.

first part Let λ be a Neutrosophic set in E_2. This implies λ^C is Neu-CS in E_2. Since f is Neu-αGS continuity, $f^{-1}(\lambda^C)$ is Neu-αGSCS in E_1. Since $f^{-1}(\lambda^C)=(f^{-1}(\lambda))^C$, $f^{-1}(\lambda)$ is a Neu-αGSOS in E_1.

Converse part Let λ be a Neu-CS in E_2. Then λ^C is a Neutrosophic set in E_2. By hypothesis $f^{-1}(\lambda^C)$ is Neu-αGSOS in E_1. Since $f^{-1}(\lambda^C)=(f^{-1}(\lambda))^C$, $(f^{-1}(\lambda))^C$ is a Neu-αGSOS in E_1. Therefore $f^{-1}(\lambda)$ is a Neu-αGSCS in E_1. Hence f is Neu-αGS continuity.

Theorem 3.21.

Let f:(E_1, τ_N)→(E_2, σ_N)be a maps and $f^{-1}(\lambda)$ be a Neu-RCS in $E_1$for every Neu-CS λ in E_2. Then f is a Neu-αGS continuity maps.

Proof.

Let λ be a Neu-CS in E_2 and $f^{-1}(\lambda)$ be a Neu-RCS in E_1. Since every Neu-RCS is a Neu-αGSCS, $f^{-1}(\lambda)$ is a Neu-αGSCS in E_1. Hence f is a Neu-αGS continuity maps.

Definition 3.22.

A Neutrosophic Topology (E, τ_N) is said to be an

(i)Neu-${}_{\alpha ga}U_{1/2}$(in short Neu- ${}_{\alpha ga}U_{1/2}$) space ,if every Neu-αGSCS in E is a Neu-CS in E,

(ii)Neu-${}_{\alpha gb}U_{1/2}$(in short Neu- ${}_{\alpha gb}U_{1/2}$) space ,if every Neu-αGSCS in E is a Neu-GCS in E,

(iii)Neu-${}_{\alpha gc}U_{1/2}$(in short Neu- ${}_{\alpha gc}U_{1/2}$) space, if every Neu-αGSCS in E is a Neu-GSCS in E.

Theorem 3.23.

Let f:(E_1, τ_N)→(E_2, σ_N)be a Neu-αGS continuity maps, then f is a Neu-continuity maps if E_1 is a Neu-${}_{\alpha ga}U_{1/2}$ space.

Proof.

Let λ be a Neu-CS in E_2. Then $f^{-1}(\lambda)$is a Neu-αGSCS in E_1, by hypothesis.Since E_1 is a Neu-${}_{\alpha ga}U_{1/2}$,$f^{-1}(\lambda)$ is a Neu-CS in E_1. Hence f is a Neu-continuity maps.

Theorem 3.24.

Let f:(E_1, τ_N)→(E_2, σ_N)be a Neu-αGS continuity maps, then f is a Neu-G continuity maps if E_1 is a Neu-${}_{\alpha gb}U_{1/2}$ space.

Proof.

Let λ be a Neu-CS in E_2. Then $f^{-1}(\lambda)$ is a Neu-αGSCS in E_1, by hypothesis. Since E_1 is a Neu- ${}_{\alpha gb}U_{1/2}$,$f^{-1}(\lambda)$ is a Neu-GCS in E_1. Hence f is a Neu-G continuity maps.

Theorem 3.25.

Let f:(E_1, τ_N)→(E_2, σ_N)be a Neu-αGS continuity maps, then f is a Neu-GS continuity maps if E_1 is a Neu-${}_{\alpha gc}U_{1/2}$ space.

Proof.

Let λ be a Neu-CS in E_2. Then $f^{-1}(\lambda)$ is a Neu-αGSCS in E_1, by hypothesis. Since E_1 is a Neu- ${}_{\alpha gc}U_{1/2}$,$f^{-1}(\lambda)$ is a Neu-GSCS in E_1. Hence f is a Neu-GS continuity maps.

Theorem 3.26.

Let f:(E_1, τ_N)→(E_2, σ_N)be a Neu-αGS continuity maps and g:(E_2, σ_N)→(E_3, ϱ_N) be an Neutrosophic continuity, then g∘f :(E_1, τ_N)→(E_3, ϱ_N) is a Neu-αGS continuity.

Proof.

Let λ be a Neu-CS in E_3. Then $g^{-1}(\lambda)$is a Neu-CS in E_2, by hypothesis. Since f is a Neu-αGS continuity maps, $f^{-1}(g^{-1}(\lambda))$ is a Neu-αGSCS in E_1. Hence g∘f is a Neu-αGS continuity maps.

Theorem 3.27.

Let f:(E_1, τ_N)→(E_2, σ_N)be a maps from Neutrosophic Topology in E_1 in to a Neutrosophic Topology E_2. Then the following conditions set are equivalent if E_1 is a Neu-${}_{\alpha ga}U_{1/2}$ space.

(i) f is a Neu-αGS continuity maps.

(ii) if μ is a Neutrosophic set in E_2 then $f^{-1}(\mu)$ is a Neu-αGSOS in E_1.

(iii) f^{-1}(Neu-int(μ))⊆Neu-int(Neu-Cl(Neu-int($f^{-1}(\mu)$))) for every Neutrosophic set μ in E_2.

Proof.

(i)→ (ii): is obviously true.

(ii)→ (iii): Let μ be any Neutrosophic set in E_2. Then Neu-int(μ) is a Neutrosophic set in E_2. Then f^{-1}(Neu-int(μ)) is a Neu-αGSOS in E_1. Since E_1 is a Neu-$_{\alpha ga}U_{1/2}$ space, f^{-1}(Neu-int(μ))is a Neutrosophic set in E_1.Therefore f^{-1}(Neu-int(μ))=Neu-int(f^{-1}(Neu-int(μ)))⊆Neu-int(Neu-Cl(Neu-int(f^{-1}(μ)))).
(iii)→(i) Let μ be a Neu-CS in E_2. Then its complement μ^C is a Neutrosophic set in E_2. By Hypothesis f^{-1}(Neu-int(μ^C))⊆Neu-int(Neu-Cl(Neu-int(f^{-1}(Neu-int(μ^C))))).This implies that $f^{-1}(\mu^C)$⊆Neu-int(Neu-Cl(Neu-int(f^{-1}(Neu-int(μ^C))))).Hence $f^{-1}(\mu^C)$ is a Neu-αOS in E_1. Since every Neu-αOS is a Neu-αGSOS, $f^{-1}(\mu^C)$is a Neu-αGSOS in E_1. Therefore f^{-1}(μ) is a Neu-αGSCS in E_1. Hence f is a Neu-αGS continuity maps.

Theorem 3.28.

Let f:(E_1, τ_N)→(E_2, σ_N)be a maps. Then the following conditions set are equivalent if E_1 is a Neu- $_{\alpha ga}U_{1/2}$ space.
(i) f is a Neu-αGS continuity maps.
(ii) $f^{-1}(\lambda)$ is a Neu-αGSCS in E_1 for every Neu-CS λ in E_2.
(iii) Neu-Cl(Neu-int(Neu-Cl($f^{-1}(\lambda)$)))⊆f^{-1}(Neu-Cl(λ)) for every Neutrosophic set λ in E_2.

Proof.

(i)→ (ii): is obviously true.
(ii)→ (iii): Let λ be a Neutrosophic set in E_2.Then Neu-Cl(λ) is a Neu-CS in E_2. By hypothesis,f^{-1}(Neu-Cl(λ))is a Neu-αGSCS in E_1. Since E_1 is a Neu-$_{\alpha ga}U_{1/2}$ space, f^{-1}(Neu- Cl(λ)) is a Neu-CS in E_1. Therefore Neu-Cl(f^{-1}(Neu-Cl(λ)))=f^{-1}(Neu-Cl(λ)).NowNeu-Cl(Neu-int(Neu-Cl($f^{-1}(\lambda)$)))⊆Neu-Cl(Neu-int(Neu-Cl(f^{-1}(Neu-Cl(λ))))) ⊆f^{-1}(Neu-Cl(λ)).
(iii)→(i): Let λ be a Neu-CS in E_2. By hypothesis Neu-Cl(Neu-int(Neu-Cl($f^{-1}(\lambda)$)))⊆f^{-1}(Neu-Cl(λ))=$f^{-1}(\lambda)$.This implies $f^{-1}(\lambda)$ is a Neu-αCS in E_1 and hence it is a Neu-αGSCS in E_1. Therefore f is a Neu-αGS continuity maps.

Definition 3.29.

Let (E, τ_N) be a Neutrospohic topology.The Neutrospohic alpha generalized semi closure (Neu-αGSCl(λ)in short) for any Neutrosophic set λ is Defined as follows. Neu-αGSCl(λ)=∩{ K|$K$is a Neu-αGSCS in $E_1$and λ ⊆K}. If λ is Neu-αGSCS, then Neu-αGSCl(λ)=λ.

Theorem 3.30.

Let f:(E_1, τ_N)→(E_2, σ_N)be a Neu-αGS continuity maps. Then the following conditions set are hold.
(i) f(Neu-αGSCl(λ))⊆Neu-Cl(f(λ)), for every Neutrosophic set λ in E_1.
(ii) Neu-αGSCl(f^{-1}(μ))⊆f^{-1}(Neu-Cl(μ)),for every Neutrosophic set μ in E_2.

Proof.

(i) Since Neu-Cl(f(λ))is a Neu-CS in E_2 and f is a Neu-αGS continuity maps, f^{-1}(Neu-Cl(f(λ)))is Neu-αGSCS in
E_1. That is Neu-αGSCl(λ)⊆f^{-1}(Neu-Cl(f(λ))). Therefore f(Neu-αGSCl(λ)) ⊆Neu-Cl(f(λ)),for every Neutrosophic set λ in E_1.
(ii) Replacing λ by f^{-1}(μ) in (i) we get f(Neu-αGSCl(f^{-1}(μ)))⊆Neu-Cl(f(f^{-1}(μ)))⊆Neu-Cl(μ).Hence Neu-αGSCl(
f^{-1}(μ))⊆f^{-1}(Neu-Cl(μ)), for every Neutrosophic set μ in E_2.

4. Neutrosophic α-Generalized Semi Irresolute Maps

In this section we Introduce Neutrosophic α-generalized semi irresolute maps and study some of its characterizations.

Definition 4.1.

A maps f:(E_1, τ_N)→(E_2, σ_N)is called a Neutrosophic alpha-generalized semi irresolute (Neu-αGS irresolute) maps if $f^{-1}(\lambda)$ is a Neu-αGSCSin (E_1, τ_N) for every Neu-αGSCS λ of (E_2, σ_N)

Theorem 4.2.

Let f:(E_1, τ_N)→(E_2, σ_N)be a Neu-αGS irresolute, then f is a Neu-αGS continuity maps.

Proof.

Let f be a Neu-αGS irresolute maps. Let λ be any Neu-CS in E_2. Since every Neu-CS is a Neu-αGSCS, λ is a Neu-αGSCS in E_2. By hypothesis $f^{-1}(\lambda)$ is a Neu-αGSCS in E_2. Hence f is a Neu-αGS continuity maps.

Example 4.3.

Neu-αGS continuity maps is not Neu-αGS irresolute maps.

Let E_1={a_1, a_2}, E_2={b_1, b_2}, U=< e_1,(.4,.5, .7), (.5,.5,.6)> and V=< e_2 ,(.8,.5,.3), (.4,.6, .7)>. Then τ_N={0_N,U,1_N} and σ_N = {0_N,V,1_N} are Neutrosophic Topologies on E_1 and E_2 respectively. Define a maps f:(E_1, τ_N)→(E_2, σ_N)by f(a_1)=b_1 and f(a_2)=b_2. Then f is a Neu-αGS continuity. We have μ=< e_2,(.2,.5, .9), (.6,.5, .5)> is a Neu-αGSCS in E_2 but $f^{-1}(\mu)$ is not a Neu-αGSCS in E_1. Therefore f is not a Neu-αGS irresolute maps.

Theorem 4.4.

Let f:(E_1, τ_N)→(E_2, σ_N)be a Neu-αGS irresolute, then f is a Neutrosophic irresolute maps if E_1 is a Neu-${}_{\alpha ga}U_{1/2}$ space.

Proof.

Let λ be a Neu-CS in E_2. Then λ is a Neu-αGSCS in E_2. Therefore $f^{-1}(\lambda)$ is a Neu-αGSCS in E_1, by hypothesis. Since E_1 is a Neu-${}_{\alpha ga}U_{1/2}$ space, $f^{-1}(\lambda)$ is a Neu-CS in E_1. Hence f is a Neutrosophic irresolute maps.

Theorem 4.5.

Let f:(E_1, τ_N)→(E_2, σ_N)and g:(E_2, σ_N)→(E_3, ϱ_N) be Neu-αGS irresolute maps, then g∘f:(E_1, τ_N)→(E_3, ϱ_N)is a Neu-αGS irresolute maps.

Proof.

Let λ be a Neu-αGSCS in E_3. Then $g^{-1}(\lambda)$ is a Neu-αGSCS in E_2. Since f is a Neu-αGS irresolute maps. $f^{-1}((g^{-1}(\lambda)))$ is a Neu-αGSCS in E_1. Hence g∘f is a Neu-αGS irresolute maps.

Theorem 4.6.

Let f:(E_1, τ_N)→(E_2, σ_N)be a Neu-αGS irresolute and g:(E_2, σ_N) →(E_3, ϱ_N) be Neu-αGS continuity maps, then g∘f:(E_1, τ_N)→(E_3, ϱ_N) is a Neu-αGS continuity maps.

Proof.

Let λ be a Neu-CS in E_3. Then $g^{-1}(\lambda)$ is a Neu-αGSCS in E_2. Since f is a Neu-αGS irresolute,
$f^{-1}((g^{-1}(\lambda))$ is a Neu-αGSCS in E_1. Hence g∘f is a Neu-αGS continuity maps.

Theorem 4.7.

Let f:(E_1, τ_N)→(E_2, σ_N)be a Neu-αGS irresolute, then f is a Neu-G irresolute maps if E_1 is a Neu-${}_{\alpha gb}U_{1/2}$ space.

Proof.

Let λ be a Neu-αGSCS in E_2. By hypothesis, $f^{-1}(\lambda)$ is a Neu-αGSCS in E_1. Since E_1 is a Neu- ${}_{\alpha gb}U_{1/2}$ space, $f^{-1}(\lambda)$ is a Neu-GCS in E_1. Hence f is a Neu-G irresolute maps.

Theorem 4.8.

Let f:(E_1, τ_N)→(E_2, σ_N)be a maps from a Neutrosophic Topology E_1 Into a Neutrosophic Topology E_2 . Then the following conditions set are equivalent if E_1 and E_2 are Neu- ${}_{\alpha ga}U_{1/2}$ spaces.

(i) f is a Neu-αGS irresolute maps.

(ii) $f^{-1}(\mu)$ is a Neu-αGSOS in E_1 for each Neu-αGSOS μ in E_2.

(iii) Neu-Cl($f^{-1}(\mu)$)⊆f^{-1}(Neu-Cl(μ)) for each Neutrosophic set μ of E_2.

Proof.

(i) →(ii) : Let μ be any Neu-αGSOS in E_2. Then μ^C is a Neu-αGSCS in E_2.Since f is Neu-αGS irresolute, $f^{-1}(\mu^C)$ is a Neu-αGSCS in E_1. But $f^{-1}(\mu^C)=(f^{-1}(\mu))^C$.Therefore $f^{-1}(\mu)$ is a Neu-αGSOS in E_1.

(ii)→(iii) : Let μ be any Neutrosophic set in $E_2$and μ⊆Neu-Cl(μ). Then $f^{-1}(\mu)$⊆f^{-1}(Neu-Cl(μ)). Since Neu-Cl(μ) is a Neu-CS in E_2, Neu-Cl(μ) is a Neu-αGSCS in E_2. Therefore $(\text{Neu-Cl}(\mu))^C$ is a Neu-αGSOS in E_2. By hypothesis, $f^{-1}((\text{Neu-Cl}(\mu))^C)$ is a Neu-αGSOS in E_1. Since $f^{-1}((\text{Neu-Cl}(\mu))^C)=(f^{-1}(\text{Neu-Cl}(\mu)))^C$,$f^{-1}$(Neu-Cl(μ)) is a Neu-$\alpha$GSCS in E_1. Since E_1 is Neu- ${}_{\alpha ga}U_{1/2}$ space,f^{-1}(Neu-Cl(μ)) is a Neu-CS in E_1. Hence Neu-Cl($f^{-1}(\mu)$)⊆Neu-Cl(f^{-1}(Neu-Cl(μ)))=f^{-1}(Neu-Cl(μ)). That is Neu-Cl($f^{-1}(\mu)$)⊆f^{-1}(Neu-Cl(μ)).

(iii)→(i) : Let μ be any Neu-αGSCS in E_2. Since E_2 is Neu-${}_{\alpha ga}U_{1/2}$ space, μ is a Neu-CS in E_2 and Neu-Cl(μ)=μ.Hence $f^{-1}(\mu)$=f^{-1}(Neu-Cl(μ)⊇Neu-Cl($f^{-1}(\mu)$)). But clearly $f^{-1}(\mu)$⊆Neu-Cl($f^{-1}(\mu)$). Therefore Neu-Cl($f^{-1}(\mu)$)=$f^{-1}(\mu)$. This implies $f^{-1}(\mu)$ is a Neu-CS and hence it is a Neu-αGSCS in E_1. Thus f is a Neu-αGS irresolute maps.

Conclusion

In this research paper using Neu-αGSCS(Neutrosophic αgs-closed sets) we are defined Neu-αGS continuity maps and analyzed its properties.after that we were compared already existing Neutrosophic continuity maps to Neu-αGSCS continuity maps. Furthermore we were extended to this maps to Neu-αGS irresolute maps , Finally This concepts can be extended to future Research for some mathematical applications.

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On NGSR Closed Sets in Neutrosophic Topological Spaces

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Abstract: The intention of this paper is to introduce the concept of GSR-closed sets in terms of neutrosophic topological spaces. Some of the properties of NGSR-closed sets are obtained. In addition, we inspect NGSR-continuity and NGSR-contra continuity in neutrosophic topological spaces.

Keywords: neutrosophic topology, NGSR-closed set, NGSR-continuous, NGSR-contra continuous mappings.

1. Introduction

In 1965, fuzzy concept was proposed by Zadeh [43] and he studied membership function. Chang [14] developed the theory of fuzzy topology in 1967. The notions of inclusion, union, intersection, complement, relation, convexity, and so forth, are expanded to such sets and several properties of these notions are established by various authors.

Atanassov [10, 11, 12] generalized the idea of fuzzy set to intuitionistic fuzzy set by adding the degree of non-membership. The intuitionistic fuzzy topology was advanced by Coker [16] using the notion of intuitionistic fuzzy sets. Intuitionistic fuzzy point was given by Coker et.al [15]. These approaches gave a wide field for exploration in the area of intuitionistic fuzzy topology and its application. Burillo et al.[13]studied the intuitionistic fuzzy relation and their properties. Thakur et.al [44] introduced generalized closed set in intuitionistic fuzzy topology. Various researchers [8, 24, 26, 33, 37, 38] extended the results of generalization of various Intuitionistic fuzzy closed sets in many directions.

The concepts of neutrosophy was introduced by Florentin Smarandache [18, 19, 20] in which he developed the degree of indeterminacy. In comparing with more uncertain ideology, the neutrosophic set can accord with indeterminacy situation. Salama et.al [34,35,36] transformed the idea of neutrosophic crisp set into neutrosophic topological spaces and introduced generalized neutrosophic set and generalized neutrosophic topological Spaces. Ishwarya et.al [22] studied Neutrosophic semi open sets in Neutrosophic topological spaces. Abdel-Basset et.al [1,2,3,4,5,6] gave a novel neutrosophic approach. Many researchers [28, 30, 31, 41, 42] added and studied semi open sets, α open sets, pre-open sets, semi alpha open sets etc., and developed several interesting properties and applications in Neutrosophic Topology. Several authors [7, 25, 27, 32, 39, 44] have contributed in topological spaces.

Mohana K et.al [29] introduced gsr -closed sets in soft topology in 2017. In this article we tend to provide the idea of NGSR-closed sets and NGSR-open sets. Also, we presented NGSR continuous and NGSR-contra continuous mappings.

2 Preliminaries

Definition 2.1. [20] Let X be a non-empty fixed set. A neutrosophic set (NS) A is an object having the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle : x \in X\}$ where $\mu_A(x), \sigma_A(x)$ and $\nu_A(x)$ represent the degree of membership, degree of indeterminacy and the degree of nonmembership respectively of each element $x \in X$ to the set A.

A Neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle : x \in X\}$ can be identified as an ordered triple $\langle \mu_A(x), \sigma_A(x), \nu_A(x)\rangle$ in $]^-0, 1^+[$ on X.

Definition 2.2. [20] Let $A = \langle \mu_A(x), \sigma_A(x), \nu_A(x)\rangle$ be a NS on X, then the complement C(A) may be defined as

1. $C(A) = \{\langle x, 1 - \mu_A(x), 1 - \nu_A(x)\rangle : x \in X\}$
2. $C(A) = \{\langle x, \nu_A(x), \sigma_A(x), \mu_A(x)\rangle : x \in X\}$
3. $C(A) = \{\langle x, \nu_A(x), 1 - \sigma_A(x), \mu_A(x)\rangle : x \in X\}$

Note that for any two neutrosophic sets A and B,

4. $C(A \cup B) = C(A) \cap C(B)$
5. $C(A \cap B) = C(A) \cup C(B)$.

Definition 2.3. [20] For any two neutrosophic sets $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \sigma_B(x), \nu_B(x)\rangle : x \in X\}$ we may have

1. $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \sigma_A(x) \leq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x) \forall\, x \in X$
2. $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \sigma_A(x) \geq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x) \forall\, x \in X$
3. $A \cap B = \langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x)$ and $\nu_A(x) \vee \nu_B(x)\rangle$
4. $A \cap B = \langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \vee \sigma_B(x)$ and $\nu_A(x) \vee \nu_B(x)\rangle$
5. $A \cup B = \langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x)$ and $\nu_A(x) \wedge \nu_B(x)\rangle$
6. $A \cup B = \langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \wedge \sigma_B(x)$ and $\nu_A(x) \wedge \nu_B(x)\rangle$

Definition 2.4. [34] A neutrosophic topology (NT) on a non-empty set X is a family τ of neutrosophic subsets in X satisfies the following axioms:

(NT_1) $0_N, 1_N \in \tau$
(NT_1) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$
(NT_1) $\cup G_i \in \tau \; \forall \{G_i : i \in J\} \subseteq \tau$

Definition 2.5. [34] Let A be an NS in NTS X. Then

Nint(A) = $\cup \{G : G$ is an NOS in X and $G \subseteq A\}$ is called a neutrosophic interior of A
Ncl(A) = $\cap \{K : K$ is an NCS in X and $A \subseteq K\}$ is called a neutrosophic closure of A

Definition 2.6. [18] A NS A of a NTS X is said to be

(1) a neutrosophic pre-open set (NPOS) if $A \subseteq NInt(NCl(A))$ and a neutrosophic pre-closed(NPCS) if $NCl(NInt(A)) \subseteq A$.

(2) a neutrosophic semi-open set (NSOS) if $A \subseteq NCl(NInt(A))$ and a neutrosophic semi-closed set (NSCS) if $NInt(NCl(A)) \subseteq A$.

(3) a neutrosophic α-open set (NαOS) if $A \subseteq NInt(NCl(NInt(A)))$ and a neutrosophic α-closed set (NαCS) if $NCl(NInt(NCl(A))) \subseteq A$.

(4) a neutrosophic regular open set (NROS) if $A = Nint(Ncl(A))$ and a neutrosophic regular closed set (NRCS) if $Ncl(Nint(A)) = A$.

Definition 2.7. [22] Consider a NS A in a NTS (X, τ).Then the neutrosophic semi interior and the neutrosophic

semi closure are defined as

Nsint(A) = ∪{G: G is a N Semi open set in X and G ⊆ A}

Nscl(A) = ∩ {K: K is a N Semi closed set in X and A ⊆ K }

Definition 2.8. [38] A subset A of a neutrosophic topological space (X, τ) is called a neutrosophic α generalized closed (Nαg-closed) set if Nαcl(A) ⊆ U whenever A ⊆ U and U is neutrosophic α-open in (X, τ).

3. NGSR closed sets

Definition 3.1. A NS A in a NTS X is stated to be a neutrosophic gsr closed set (NGSR-Closed set) if Nscl (A) ⊆ U for every A ⊆ U and U is a NROS (Neutrosophic Regular Open set) in X.

The complement C(A) of a NGSR-closed set A is a NGSR-open set in X.

Example 3.2. Let X = {a, b} and $\tau = \{0_1, G, 1_N\}$ be NT in which G_1 ⟨x, (0.4, 0.1), (0.3,0.2), (0.5, 0.5)⟩ and G_2 = ⟨x, (0.4,0,4), (0.4, 0.3), (0.5,0.4)⟩. Here A = ⟨x, (0.4,0.4), (0.3,0.2), (0.4,0.5)⟩ is an NGSR-closed set.

Theorem 3.3. Each NCS is a NGSR-closed set in X.

Proof. Let A ⊆ U wherein U is a NROS in X. Let A be an NCS in X.

We got Nscl(A) ⊆ Ncl(A) ⊆ U. Consequently A is a NGSR-closed set in X.

Example 3.4. Let X = {a, b} and $\tau = \{0_1, G, 1_N\}$ be an NT having G_1 = ⟨x, (0.4, 0.1), (0.3, 0.2), (0.5, 0.5)⟩ and G_2 = ⟨x, (0.4, 0.4), (0.4; 0.3), (0.5, 0.4)⟩ .Here A = ⟨x, (0.4,0.4), (0.3,0.2), (0.4,0.5)⟩ is an NGSR-closed set, however not NCS.

Theorem 3.5. Each Nα − closed set is a NGSR-closed set in X.

Proof. Let A ⊆ U inwhich U is a NROS in X. Let A be an Nα − closed set in X.

Now Nscl(A) ⊆ N ⊆ cl(A) ⊆ U . Consequently A is a NGSR-closed set in X.

Example 3.6. Let X ={a, b} and $\tau = \{0_1, G, 1_N\}$ be an NT in which

G_1 = ⟨x, (0.6,0.2), (0.1,0.5), (0.5, 0.4)⟩ and G_2 = ⟨ x, (0.5, 03), (0.3, 0.2), (0.6,0.4) ⟩

Here A = ⟨x, (0.6,0.3), (0.1,0.6), (0.5,0.4) ⟩ is an NGSR-closed set, but not Nα-closed set as

$$Ncl(Nint(Ncl(A))) = C(A) \nsubseteq A.$$

Theorem 3.7. Each Nsemi-closed set is a NGSR-closed set in X.

Proof. Suppose A is an Nsemi-closed set and $A \subseteq U$ wherein U is a NROS in X. Now $(A) = A \cup Nint(Ncl(A)) \subseteq A \cup A = A$. Therefore A is a NGSR-closed set in X.

Example 3.8. Let X = {a, b} and $\tau = \{0_1, G, 1_N\}$ be an NT in which

$G_1 = \langle x, (0.4,0.5), (0.3,0.2), (0.5,0.5)\rangle$ and $G_2 = \langle x, (0.4, 0.4), (0.4,0.3), (0.5, 0.4)\rangle$

Then $A = \langle x, (0.4,0.4), (0.3,0.2), (0.4,0.5)\rangle$ is an NGSR-closed set, however not Nsemi-closed set as $Nint(Ncl(A)) = G_1 \nsubseteq A$.

Theorem 3.9. Each $N\alpha G - closed$ set is a NGSR-closed set in X.

Proof. Let $A \subseteq U$ where U is a NROS in X. Let A be an $N\alpha G - closed$ set in X. Now $Nscl(A) \subseteq N\alpha cl(A) \subseteq U$. Therefore A is a NGSR-closed set in X.

Example 3.10. Let X = {a, b} and $\tau = \{0_1, G, 1_N\}$ be an NT where

$G_1 = \langle x, (0.6, 0.2), (0.1,0.5), (0.5, 0.4) \rangle$ and $G_2 = \langle x, (0.5, 0.3), (0.3,0.2), (0.6,0.4) \rangle$

Then $A = \langle x, (0.6,0.3), (0.1,0.6), (0.5,0.4) \rangle$ is an NGSR-closed set but not NαG-closed set.

Remark 3.11. The counter examples shows that NGSR-closed set is independent of NPCS.

Example 3.12. Let X = {a, b} and $\tau = \{0_1, G, 1_N\}$ be an NT where

$G_1 = \langle x, (0.6, 0.2), (01, 0.5), (0.5,0.4) \rangle$ and $G_2 = \langle x, (0.5,03.), (0.3, 0.2), (0.6,0.4) \rangle$

Here $A = \langle x, (0.6,0.3), (0.1,0.6), (0.5,0.4) \rangle$ be an NGSR-closed set, but not NPCS as Ncl(Nint(A)) = C(B) $\nsubseteq$ A .

Example 3.13. Let X ={a, b} and $\tau = \{0_1, G, 1_N\}$ be an NT where

$G_1 = \langle x, (0{:}5;\ 0{:}4), (0{:}3;\ 0{:}2), (0{:}5;\ 0{:}6) \rangle$,$G_2 = \langle x, (0{:}8;\ 0{:}7), (0{:}4;\ 0{:}3), (0{:}2;\ 0{:}3) \rangle$ and

$G_3 = \langle x, (0{:}2;\ 0{:}1), (0{:}3;\ 0{:}2), (0{:}8;\ 0{:}9) \rangle$

Then $A = \langle x, (0.5,0.3), (0.3,0.2), (0.5,0.7) \rangle$ is an NPCS, but not NGSR-closed set.

Theorem 3.14. Consider a NTS (X, τ). Then for each A ∈ NGSR-closed set and for each B ∈ NS in X, $A \subseteq B \subseteq Nscl(A)$ implies B ∈ NGSR-closed in (X, τ) .

Proof. Assume that $B \subseteq U$ and U is a NROS in (X, τ) which shows that $A \subseteq B, A \subseteq U$. Via speculation, B ⊆ Nscl(A). Consequently $Nscl(B) \subseteq Nscl(Nscl(A)) = Nscl(A) \subseteq U$, given that A is an NGSR-closed set in (X, τ). As a result B ∈ NGSR-closed in (X, τ).

Theorem 3.15. Consider a NROS A and a NGSR-closed set in (X, τ), then A is a NSemi-closed set in (X, τ).

Proof. Due to the fact A ⊆ A and A is a NROS in (X, τ),Via speculation, Nscl(A) ⊆ A.

However A ⊆ Nscl(A). Therefore Nscl(A) = A. Consequently A is a Nsemi-closed set in (X, τ).

Theorem 3.16. Let (X, τ) be a NTS. Then for each A ∈ NGSR-open X and for every B ∈ NS(X), Nsint (A) ⊆B⊆A implies B ∈ NGSR-open set in X.

Proof. Let A be any NGSR-open set of X and B be any NS of X. By means of speculation Nsint ⊆ B ⊆A. Then C(A) is a NGSR-closed in X and C(A) ⊆ C(B) ⊆Nscl(C(A)). By using Theorem 3.5, C(B) is a NGSRclosed in (X, τ). Thus B is a NGSR-Open in (X, τ). Hence B ∈ NGSR-open in X.

Theorem 3.17. A NS A is a NGSR-open in (X, τ) if and only if F _ Nsint(A) everytime F is a NRCS in (X, τ) and F ⊆ A.

Proof. **Necessity:** Assume that A is a NGSR-open in (X, τ) and F is a NRCS in (X, τ) such that F ⊆ A. Then C(F) is a NROS and C(A) ⊆ C(F). Via speculation C(A) is a NGSR-closed set in (X, τ), we've Nscl(C(A)) ⊆ C(F). Therefore F ⊆ Nsint(A).

Sufficiency: Let U be a NROS in (X, τ) such that C(A) ⊆ U. By hypothesis, C(U) ⊆ Nsint(A). Consequently Nscl(C(A)) ⊆ U and C(A) is an NGSR-closed set in (X, τ). Thus A is a NGSR-open set in (X, τ).

Theorem 3.18. A is Nsemi-closed if it is both Nsemi-open and NGSR-closed.
Proof. Considering A is each Nsemi-open and NGSR-closed set in X, then Nscl(A) ⊆ A. We additionally have A ⊆Nscl(A). Accordingly, Nscl(A) = A. Therefore, A is an Nsemi-closed set in X.

4 On NGSR-Continuity and NGSR-Contra Continuity

Definition 4.1. Let f be a mapping from a neutrosophic topological space (X, τ) to a neutrosophic topological space (Y, σ). Then f is referred to as a neutrosophic gsr-continuous(NGSR-continuous) mapping if $f^{-1}(B)$ is a NGSR-open set in X, for each neutrosophic-open set B in Y .

Theorem 4.2. Consider a mapping $f : (X,\tau) \rightarrow (Y,\sigma)$. Then (1) and (2) are equal.
(1) f is NGSR-continuous
(2) The inverse image of each N-closed set B in Y is NGSR-closed set in X.
Proof. This can be proved with the aid of using the complement and Definition 4.1.

Theorem 4.3. Consider an NGSR-continuous mapping $f : (X,\tau) \rightarrow (Y,\sigma)$ then the subsequent assertions hold:
(1) for all neutrosophic sets A in X, $f(NGSRNcl(A)) \subseteq Ncl(f(A))$
(2) for all neutrosophic sets B in Y, $NGSRNcl\ (f^{-1}(B)) \subseteq f^{-1}(Ncl(B))$.
Proof. (1) Let Ncl(f(A)) be a neutrosophic closed set in Y and f be NGSR-continuous, then it follows that f^{-1} (Ncl(f(A))) is NGSR-closed in X. In view that $A \subseteq f^{-1}\ (Ncl(f(A))), NGSRcl(A) \subseteq f^{-1}\ (Ncl(f(A)))$. Hence, $f(NGSRNcl(A)) \subseteq Ncl(f(A))$.
(2) We get $f(NGSRcl(f^{-1}\ (B))) \subseteq Nclf(f^{-1}\ (B))) \subseteq Ncl(B)$.
Hence, $NGSRcl(f^{-1}\ (B)) \subseteq f^{-1}\ (Ncl(B))$ by way of changing A with B in (1).

Definition 4.4. Let f be a mapping from a neutrosophic topological space (X, τ) to a neutrosophic topological space (Y, σ). Then f is known as neutrosophic gsr-contra continuous(NGSR-contra continuous) mapping if f^{-1} (B) is a NGSR-closed set in X for each neutrosophic-open set B in Y .

Theorem 4.5. Consider a mapping $f : (X,\tau) \rightarrow (Y,\sigma)$. Then the subsequent assertions are equivalent:
(1) f is a NGSR-contra continuous mapping
(2) f^{-1} (B) is an NGSR-closed set in X, for each NOS B in Y.
Proof. (1) $\Longrightarrow$ (2) Assume that f is NGSR-contra continuous mapping and B is NOS in Y. Then B_c is an NCS in Y. It follows that, $f^{-1}(B^c)$ is an NGSR-open set in X. For this reason, f^{-1} (B) is an NGSR-closed set in X.
(2) $\Longrightarrow$ (1) The converse is similar.

Theorem 4.6. Consider a bijective mapping $f : (X,\tau) \rightarrow (Y,\sigma)$. from an

NTS(X, τ)into an NTS(Y, σ).If $Ncl(f(A)) \subseteq f(NGSRint(A))$, for each NS B in X, then the mapping f is NGSR-contra continuous.

Proof. Consider a NCS B in Y. Then $Ncl(B) = B$ and f is onto, by way of assumption, $f(NGSRint(f^{-1}(B))) \subseteq Ncl(f(f^{-1}(B))) = Ncl(B) = B$. Consequently, $f^{-1}(f(NGSRint(f^{-1}(B)))) \subseteq f^{-1}(B)$. Additionally due to the fact that f is an into mapping, we have $NGSRint(f^{-1}(B)) = f^{-1}(f(NGSRint(f^{-1}(B)))) \subseteq f^{-1}(B)$.Consequently, $NGSRint(f^{-1}(B)) = f^{-1}(B)$, so f◎1(B) is an NGSR-open set in X. Hence, f is a NGSR-contra continuous mapping.

5. Conclusion and Future work

Neutrosophic topological space concept is used to deal with vagueness. This paper introduced NGSR closed set and some of its properties were discussed and derived some contradicting examples. This idea can be developed and extended in the real life applications such as in medical field and so on.

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Neutrosophic Vague Topological Spaces

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Abstract: The term topology was introduced by Johann Beredict Listing in the 19th century. Closed sets are fundamental objects in a topological space. In this paper, we use neutrosophic vague sets and topological spaces and we construct and develop a new concept namely "neutrosophic vague topological spaces". By using the fundamental definition and necessary example we have defined the neutrosophic vague topological spaces and have also discussed some of its properties. Also we have defined the neutrosophic vague continuity and neutrosophic vague compact space in neutrosophic vague topological spaces and their properties are deliberated.

Keywords: Neutrosophic vague set, neutrosophic vague topology, neutrosophic vague topological spaces, neutrosophic vague continuity.

1. Introduction:

Zadeh [19] in 1965 introduced and defined the fuzzy set which deals with the degree of membership/truth. Topology has become a powerful instrument of mathematical research. Topology is the modern version of geometry. It is commonly defined as the study of shapes and topological spaces. The topology is an area of mathematics, which is concerned with the properties of space that are preserved under continuous deformation including stretching and bending, but not tearing and gluing which include properties such as connectedness, continuity and boundary. The term topology was introduced by Johann Beredict Listing in the 19th century. Closed sets are fundamental objects in a topological space. In 1970, Levine [11] initiated the study of generalized closed sets.

The theory of fuzzy topology was introduced by Chang [8] in 1967; several researches were conducted on the generalizations of the notions of fuzzy sets and fuzzy topology. Atanassov [7] in1986 introduced the degree of non-membership/falsehood (F) and defined the intuitionistic fuzzy set as a generalization of fuzzy sets. Coker [9] in 1997 introduced the intuitionistic fuzzy topological spaces. As an extension of fuzzy set theory in 1993, the theory of vague sets was first proposed by Gau and Buehre[10]. Then, Smarandache [15] introduced the degree of indeterminacy/neutrality (I) as independent component in 1998 and defined the neutrosophic set. Various methods were proposed by Smarandache.et.al [13, 16, 17, 18] and Abdel-Basset.et.al [1, 2, 3] for neutrosophic sets.

Salama and Alblowi [12] in 2012 used this neutrosophic set and introduced neutrosophic topological spaces. Shawkat Alkhazaleh [14] in 2015 introduced the concept of neutrosophic vague set as a combination of neutrosophic set and vague set. Neutrosophic vague theory is an effective tool to process incomplete, indeterminate and inconsistent information. Al-Quran and Hassan [4, 5, 6] in 2017 introduced and gave more application on neutrosophic vague soft.

In this paper we define the notion of neutrosophic vague topological spaces and their properties are obtained. The purpose of this paper is to extend the classical topological spaces to neutrosophic vague topological spaces. Also we have defined the neutrosophic vague continuity and neutrosophic vague compact spaces which give the added advantage in neutrosophic vague topological spaces.

2. Preliminaries

Definition 2.1:[14] A neutrosophic vague set A_{NV} (NVS in short) on the universe of discourse X written as $A_{NV}=\left\{\left\langle x;\hat{T}_{A_{NV}}(x);\hat{I}_{A_{NV}}(x);\hat{F}_{A_{NV}}(x)\right\rangle;x\in X\right\}$, whose truth membership, indeterminacy membership and false membership functions is defined as:

$$\hat{T}_{A_{NV}}(x)=\left[T^-,T^+\right],\hat{I}_{A_{NV}}(x)=\left[I^-,I^+\right],\hat{F}_{A_{NV}}(x)=\left[F^-,F^+\right]$$

Where,

1) $T^+=1-F^-$
2) $F^+=1-T^-$ and
3) ${}^-0\le T^-+I^-+F^-\le 2^+$.

Definition 2.2:[14] Let A_{NV} and B_{NV} be two NVSs of the universe U. If $\forall u_i\in U$, $\hat{T}_{A_{NV}}(u_i)\le\hat{T}_{B_{NV}}(u_i)$; $\hat{I}_{A_{NV}}(u_i)\ge\hat{I}_{B_{NV}}(u_i)$; $\hat{F}_{A_{NV}}(u_i)\ge\hat{F}_{B_{NV}}(u_i)$, then the NVS A_{NV} is included by B_{NV}, denoted by $A_{NV}\subseteq B_{NV}$, where $1\le i\le n$.

Definition 2.3:[14] The complement of NVS A_{NV} is denoted by A^c_{NV} and is defined by

$$\hat{T}^c_{A_{NV}}(x)=\left[1-T^+,1-T^-\right],\hat{I}^c_{A_{NV}}(x)=\left[1-I^+,1-I^-\right],\hat{F}^c_{A_{NV}}(x)=\left[1-F^+,1-F^-\right].$$

Definition 2.4:[14] Let A_{NV} be NVS of the universe U where $\forall u_i\in U$, $\hat{T}_{A_{NV}}(x)=[1,1]$; $\hat{I}_{A_{NV}}(x)=[0,0]$; $\hat{F}_{A_{NV}}(x)=[0,0]$. Then A_{NV} is called unit NVS (1_{NV} in short), where $1\le i\le n$.

Definition 2.5:[14] Let A_{NV} be NVS of the universe U where $\forall u_i\in U$, $\hat{T}_{A_{NV}}(x)=[0,0]$; $\hat{I}_{A_{NV}}(x)=[1,1]$; $\hat{F}_{A_{NV}}(x)=[1,1]$. Then A_{NV} is called zero NVS (0_{NV} in short), where $1\le i\le n$.

Definition 2.6:[14] The union of two NVSs A_{NV} and B_{NV} is NVS C_{NV}, written as $C_{NV} = A_{NV} \cup B_{NV}$, whose truth-membership, indeterminacy-membership and false-membership functions are related to those of A_{NV} and B_{NV} given by,

$$\hat{T}_{C_{NV}}(x) = [\max(T^-_{A_{NV_x}}, T^-_{B_{NV_x}}), \max(T^+_{A_{NV_x}}, T^+_{B_{NV_x}})]$$

$$\hat{I}_{C_{NV}}(x) = [\min(I^-_{A_{NV_x}}, I^-_{B_{NV_x}}), \min(I^+_{A_{NV_x}}, I^+_{B_{NV_x}})]$$

$$\hat{F}_{C_{NV}}(x) = [\min(F^-_{A_{NV_x}}, F^-_{B_{NV_x}}), \min(F^+_{A_{NV_x}}, F^+_{B_{NV_x}})].$$

Definition 2.7:[14] The intersection of two NVSs A_{NV} and B_{NV} is NVS C_{NV}, written as $C_{NV} = A_{NV} \cap B_{NV}$, whose truth-membership, indeterminacy-membership and false-membership functions are related to those of A_{NV} and B_{NV} given by,

$$\hat{T}_{C_{NV}}(x) = [\min(T^-_{A_{NV_x}}, T^-_{B_{NV_x}}), \min(T^+_{A_{NV_x}}, T^+_{B_{NV_x}})]$$

$$\hat{I}_{C_{NV}}(x) = [\max(I^-_{A_{NV_x}}, I^-_{B_{NV_x}}), \max(I^+_{A_{NV_x}}, I^+_{B_{NV_x}})]$$

$$\hat{F}_{C_{NV}}(x) = [\max(F^-_{A_{NV_x}}, F^-_{B_{NV_x}}), \max(F^+_{A_{NV_x}}, F^+_{B_{NV_x}})].$$

Definition 2.8:[14] Let A_{NV} and B_{NV} be two NVSs of the universe U. If $\forall u_i \in U$, $\hat{T}_{A_{NV}}(u_i) = \hat{T}_{B_{NV}}(u_i)$; $\hat{I}_{A_{NV}}(u_i) = \hat{I}_{B_{NV}}(u_i)$; $\hat{F}_{A_{NV}}(u_i) = \hat{F}_{B_{NV}}(u_i)$, then the NVS A_{NV} and B_{NV}, are called equal, where $1 \le i \le n$.

Definition 2.9: Let $\{A_{i_{NV}} : i \in J\}$ be an arbitrary family of NVSs. Then

$$\cup A_{i_{NV}} = \left\{ \left\langle x; \left(\max_{i \in J}\left(T^-_{A_{i_{NV}}}\right), \max_{i \in J}\left(T^+_{A_{i_{NV}}}\right) \right), \left(\min_{i \in J}\left(I^-_{A_{i_{NV}}}\right), \min_{i \in J}\left(I^+_{A_{i_{NV}}}\right) \right), \left(\min_{i \in J}\left(F^-_{A_{i_{NV}}}\right), \min_{i \in J}\left(F^+_{A_{i_{NV}}}\right) \right) \right\rangle; x \in X \right\}$$

$$\cap A_{i_{NV}} = \left\{ \left\langle x; \left(\min_{i \in J}\left(T^-_{A_{i_{NV}}}\right), \min_{i \in J}\left(T^+_{A_{i_{NV}}}\right) \right), \left(\max_{i \in J}\left(I^-_{A_{i_{NV}}}\right), \max_{i \in J}\left(I^+_{A_{i_{NV}}}\right) \right), \left(\max_{i \in J}\left(F^-_{A_{i_{NV}}}\right), \max_{i \in J}\left(F^+_{A_{i_{NV}}}\right) \right) \right\rangle; x \in X \right\}$$

Corollary 2.10: Let A_{NV}, B_{NV} and C_{NV} be NVSs. Then

a) $A_{NV} \subseteq B_{NV}$ and $C_{NV} \subseteq D_{NV} \Rightarrow A_{NV} \cup C_{NV} \subseteq B_{NV} \cup D_{NV}$ and $A_{NV} \cap C_{NV} \subseteq B_{NV} \cap D_{NV}$

b) $A_{NV} \subseteq B_{NV}$ and $A_{NV} \subseteq C_{NV} \Rightarrow A_{NV} \subseteq B_{NV} \cap C_{NV}$

c) $A_{NV} \subseteq C_{NV}$ and $B_{NV} \subseteq C_{NV} \Rightarrow A_{NV} \cup B_{NV} \subseteq C_{NV}$

d) $A_{NV} \subseteq B_{NV}$ and $B_{NV} \subseteq C_{NV} \Rightarrow A_{NV} \subseteq C_{NV}$

e) $\overline{(A_{NV} \cup B_{NV})} = \overline{A_{NV}} \cap \overline{B_{NV}}$

f) $\overline{(A_{NV} \cap B_{NV})} = \overline{A_{NV}} \cup \overline{B_{NV}}$

g) $A_{NV} \subseteq B_{NV} \Rightarrow \overline{B_{NV}} \subseteq \overline{A_{NV}}$

h) $\overline{\left(\overline{A_{NV}}\right)} = A_{NV}$

i) $\bar{1}_{NV} = 0_{NV}$

j) $\bar{0}_{NV} = 1_{NV}$

Corollary 2.11: Let A_{NV}, B_{NV}, C_{NV} and $A_{i_{NV}} (i \in J)$ be NVSs. Then

a) $A_{i_{NV}} \subseteq B_{NV}$ for each $i \in J \Rightarrow \cup A_{i_{NV}} \subseteq B_{NV}$

b) $B_{NV} \subseteq A_{i_{NV}}$ for each $i \in J \Rightarrow B_{NV} \subseteq \cap A_{i_{NV}}$

c) $\overline{\cup A_{i_{NV}}} = \cap \overline{A_{i_{NV}}}$ and $\overline{\cap A_{i_{NV}}} = \cup \overline{A_{i_{NV}}}$

3. Neutrosophic Vague Topological Space:

Definition 3.1: A neutrosophic vague topology (NVT) on X_{NV} is a family τ_{NV} of neutrosophic vague sets (NVS) in X_{NV} satisfying the following axioms:

- $0_{NV}, 1_{NV} \in \tau_{NV}$
- $G_1 \cap G_2 \in \tau_{NV}$ for any $G_1, G_2 \in \tau_{NV}$
- $\cup G_i \in \tau_{NV}, \forall \{G_i : i \in J\} \subseteq \tau_{NV}$

In this case the pair (X_{NV}, τ_{NV}) is called neutrosophic vague topological space (NVTS) and any NVS in τ_{NV} is known as neutrosophic vague open set (NVOS) in X_{NV}.

The complement A_{NV}^{c} of NVOS in NVTS (X_{NV}, τ_{NV}) is called neutrosophic vague closed set (NVCS) in X_{NV}.

Example 3.2: Let $X_{NV} = \{e, f, g\}$ and

$$A_{NV} = \left\{ x, \frac{e}{\langle [0.1,0.5];[0.6,0.8];[0.5,0.9] \rangle}, \frac{f}{\langle [0.2,0.3];[0.4,0.5];[0.7,0.8] \rangle}, \frac{g}{\langle [0.2,0.6];[0.7,0.9];[0.4,0.8] \rangle} \right\},$$

$$B_{NV} = \left\{ x, \frac{e}{\langle [0.2,0.4];[0.3,0.7];[0.6,0.8] \rangle}, \frac{f}{\langle [0.5,0.8];[0.2,0.6];[0.2,0.5] \rangle}, \frac{g}{\langle [0.1,0.3];[0.1,0.7];[0.7,0.9] \rangle} \right\},$$

$$C_{NV} = \left\{ x, \frac{e}{\langle [0.2,0.5];[0.3,0.7];[0.5,0.8] \rangle}, \frac{f}{\langle [0.5,0.8];[0.2,0.5];[0.2,0.5] \rangle}, \frac{g}{\langle [0.2,0.6];[0.1,0.7];[0.4,0.8] \rangle} \right\},$$

$$D_{NV} = \left\{ x, \frac{e}{\langle [0.1,0.4];[0.6,0.8];[0.6,0.9] \rangle}, \frac{f}{\langle [0.2,0.3];[0.4,0.6];[0.7,0.8] \rangle}, \frac{g}{\langle [0.1,0.3];[0.7,0.9];[0.7,0.9] \rangle} \right\}.$$

Then the family $\tau_{NV} = \{0_{NV}, A_{NV}, B_{NV}, C_{NV}, D_{NV}, 1_{NV}\}$ of NVSs in X_{NV} is NVT on X_{NV}.

Definition 3.3: Let (X_{NV}, τ_{NV}) be NVTS and $A_{NV} = \left\{\left\langle x, \left[\hat{T}_A, \hat{I}_A, \hat{F}_A\right]\right\rangle\right\}$ be NVS in X_{NV}. Then the neutrosophic vague interior and neutrosophic vague closure are defined by

- $NV\,\mathrm{int}(A_{NV}) = \cup\{G_{NV} / G_{NV}$ is a NVOS in X_{NV} and $G_{NV} \subseteq A_{NV}\}$,
- $NVcl(A_{NV}) = \cap\{K_{NV} / K_{NV}$ is a NVCS in X_{NV} and $A_{NV} \subseteq K_{NV}\}$.

Note that for any NVS A_{NV} in (X_{NV}, τ_{NV}), we have $NVcl\left(A_{NV}^{c}\right) = \left(NV\,\mathrm{int}\left(A_{NV}\right)\right)^{c}$ and $NV\,\mathrm{int}\left(A_{NV}^{c}\right) = \left(NVcl\left(A_{NV}\right)\right)^{c}$.

It can be also shown that $NVcl(A_{NV})$ is NVCS and $NV\,\mathrm{int}(A_{NV})$ is NVOS in X_{NV}.

a) A_{NV} is NVCS in X_{NV} if and only if $NVcl(A_{NV}) = A_{NV}$.

b) A_{NV} is NVOS in X_{NV} if and only if $NV\,\mathrm{int}(A_{NV}) = A_{NV}$.

Example 3.4: Let $X_{NV} = \{e, f\}$ and let $\tau_{NV} = \{0_{NV}, G_1, G_2, 1_{NV}\}$ be NVT on X, where

$$G_1 = \left\{ x, \frac{e}{\langle [0.2,0.4];[0.7,0.9];[0.6,0.8] \rangle}, \frac{f}{\langle [0.3,0.5];[0.6,0.8];[0.5,0.7] \rangle} \right\} \text{ and}$$

$$G_2 = \left\{ x, \frac{e}{\langle [0.4,0.9];[0.1,0.3];[0.1,0.4] \rangle}, \frac{f}{\langle [0.5,0.7];[0.2,0.6];[0.3,0.5] \rangle} \right\}.$$

If $A_{NV} = \left\{ x, \frac{e}{\langle [0.3,0.5];[0.4,0.7];[0.5,0.7] \rangle}, \frac{f}{\langle [0.4,0.6];[0.5,0.8];[0.4,0.6] \rangle} \right\}$ then

$$NV\operatorname{int}(A_{NV}) = G_1 = \left\{ x, \frac{e}{\langle [0.2,0.4];[0.7,0.9];[0.6,0.8] \rangle}, \frac{f}{\langle [0.3,0.5];[0.6,0.8];[0.5,0.7] \rangle} \right\} \text{and}$$

$$NVcl(A_{NV}) = G_1^c = \left\{ x, \frac{e}{\langle [0.8,0.6];[0.1,0.3];[0.2,0.4] \rangle}, \frac{f}{\langle [0.5,0.7];[0.2,0.4];[0.3,0.5] \rangle} \right\}.$$

Proposition 3.5: Let A_{NV} be any NVS in X_{NV}. Then

i) $NV\operatorname{int}(1_{NV} - A_{NV}) = 1_{NV} - (NVcl(A_{NV}))$ and

ii) $NVcl(1_{NV} - A_{NV}) = 1_{NV} - (NV\operatorname{int}(A_{NV}))$

Proof: (i) By definition $NVcl(A_{NV}) = \cap\{K_{NV} / K_{NV} \text{ is a NVCS in } X_{NV} \text{ and } A_{NV} \subseteq K_{NV}\}$.

$$\begin{aligned} 1_{NV} - (NVcl(A_{NV})) &= 1_{NV} - \cap\{K_{NV} / K_{NV} \text{ is a NVCS in } X_{NV} \text{ and } A_{NV} \subseteq K_{NV}\} \\ &= \cup\{1_{NV} - K_{NV} / K_{NV} \text{ is a NVCS in } X_{NV} \text{ and } A_{NV} \subseteq K_{NV}\} \\ &= \cup\{G_{NV} / G_{NV} \text{ is an NVOS in } X_{NV} \text{ and } G_{NV} \subseteq 1_{NV} - A_{NV}\} \\ &= NV\operatorname{int}(1_{NV} - A_{NV}) \end{aligned}$$

(ii) The proof is similar to (i).

Proposition 3.6: Let (X_{NV}, τ_{NV}) be a NVTS and A_{NV}, B_{NV} be NVSs in X_{NV}. Then the following properties hold:

a) $NV\operatorname{int}(A_{NV}) \subseteq A_{NV}$,

a′) $A_{NV} \subseteq NVcl(A_{NV})$

b) $A_{NV} \subseteq B_{NV} \Rightarrow NV\operatorname{int}(A_{NV}) \subseteq NV\operatorname{int}(B_{NV})$,

b′) $A_{NV} \subseteq B_{NV} \Rightarrow NVcl(A_{NV}) \subseteq NVcl(B_{NV})$

c) $NV\operatorname{int}(NV\operatorname{int}(A_{NV})) = NV\operatorname{int}(A_{NV})$,

c′) $NVcl(NVcl(A_{NV})) = NVcl(A_{NV})$

d) $NV\operatorname{int}(A_{NV} \cap B_{NV}) = NV\operatorname{int}(A_{NV}) \cap NV\operatorname{int}(B_{NV})$,

d′) $NVcl(A_{NV} \cup B_{NV}) = NVcl(A_{NV}) \cup NVcl(B_{NV})$

e) $NV\operatorname{int}(1_{NV}) = 1_{NV}$,

e′) $NVcl(0_{NV}) = 0_{NV}$

Proof: (a), (b) and (e) are obvious, (c) follows from (a)

d) From $NV\operatorname{int}(A_{NV} \cap B_{NV}) \subseteq NV\operatorname{int}(A_{NV})$ and $NV\operatorname{int}(A_{NV} \cap B_{NV}) \subseteq NV\operatorname{int}(B_{NV})$ we obtain $NV\operatorname{int}(A_{NV} \cap B_{NV}) \subseteq NV\operatorname{int}(A_{NV}) \cap NV\operatorname{int}(B_{NV})$.

On the other hand, from the facts $NV\operatorname{int}(A_{NV}) \subseteq A_{NV}$ and $NV\operatorname{int}(B_{NV}) \subseteq B_{NV}$ $\Rightarrow NV\operatorname{int}(A_{NV}) \cap NV\operatorname{int}(B_{NV}) \subseteq A_{NV} \cap B_{NV}$ and $NV\operatorname{int}(A_{NV}) \cap NV\operatorname{int}(B_{NV}) \in \tau_{NV}$ we see that $NV\operatorname{int}(A_{NV}) \cap NV\operatorname{int}(B_{NV}) \subseteq NV\operatorname{int}(A_{NV} \cap B_{NV})$, for which we obtain the required result.

$(a') - (e')$ They can be easily deduced from $(a) - (e)$.

Definition 3.7: A NVS $A_{NV} = \langle x, [\hat{T}_A, \hat{I}_A, \hat{F}_A] \rangle$ in NVTS (X_{NV}, τ_{NV}) is said to be

i) *Neutrosophic Vague semi closed set* (NVSCS) if $NV\operatorname{int}(NVcl(A_{NV})) \subseteq A_{NV}$,

ii) *Neutrosophic Vague semi open set* (NVSOS) if $A_{NV} \subseteq NVcl(NV\operatorname{int}(A_{NV}))$,

iii) *Neutrosophic Vague pre- closed set* (NVPCS) if $NVcl(NV\operatorname{int}(A_{NV})) \subseteq A_{NV}$,

iv) *Neutrosophic Vague pre-open set* (NVPOS) if $A_{NV} \subseteq NV\operatorname{int}(NVcl(A_{NV}))$,

v) *Neutrosophic Vague* α *-closed set* (NV α CS) if $NVcl(NV\operatorname{int}(NVcl(A_{NV}))) \subseteq A_{NV}$,

vi) *Neutrosophic Vague* α *-open set* (NV α OS) if $A_{NV} \subseteq NV\operatorname{int}(NVcl(NV\operatorname{int}(A_{NV})))$,

vii) *Neutrosophic Vague semi pre- closed set* (NVSPCS) if $NV\operatorname{int}(NVcl(NV\operatorname{int}(A_{NV}))) \subseteq A_{NV}$,

viii) *Neutrosophic Vague semi pre-open set* (NVSPOS) if $A_{NV} \subseteq NVcl(NV\operatorname{int}(NVcl(A_{NV})))$,

ix) *Neutrosophic Vague regular open set* (NVROS) if $A_{NV} = NV\operatorname{int}(NVcl(A_{NV}))$,

x) *Neutrosophic Vague regular closed set* (NVRCS) if $A_{NV} = NVcl(NV\operatorname{int}(A_{NV}))$.

4. Neutrosophic Vague continuity:

Definition 4.1: We define the image and preimage of NVSs. Let X_{NV} and Y_{NV} be two nonempty sets and $f : X_{NV} \to Y_{NV}$ be a function, then the following statements hold:

a) If $B_{NV} = \left\{\left\langle x; \hat{T}_B(x); \hat{I}_B(x); \hat{F}_B(x)\right\rangle; x \in X\right\}$ is a NVS in Y_{NV}, then the preimage of B_{NV} under *f*, denoted by $f^{-1}(B_{NV})$, is the NVS in X_{NV} defined by

$$f^{-1}(B_{NV}) = \left\{\left\langle x; f^{-1}\left(\hat{T}_B\right)(x); f^{-1}\left(\hat{I}_B\right)(x); f^{-1}\left(\hat{F}_B\right)(x)\right\rangle; x \in X_{NV}\right\}.$$

b) If $A_{NV} = \left\{\left\langle x; \hat{T}_A(x); \hat{I}_A(x); \hat{F}_A(x)\right\rangle; x \in X_{NV}\right\}$ is a NVS in X_{NV}, then the image of A_{NV} under *f*, denoted by $f(A_{NV})$, is the NVS in Y_{NV} defined by

$$f(A_{NV}) = \left\{\left\langle y; f_{\sup}\left(\hat{T}_A\right)(y); f_{\inf}\left(\hat{I}_A\right)(y); f_{\inf}\left(\hat{F}_A\right)(y)\right\rangle; y \in Y_{NV}\right\}$$

where,

$$f_{\sup}\left(\hat{T}_A\right)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \hat{T}_A(x), & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

$$f_{\inf}\left(\hat{I}_A\right)(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \hat{I}_A(x), & \text{if } f^{-1}(y) \neq \phi \\ 1, & \text{otherwise} \end{cases}$$

$$f_{\inf}\left(\hat{F}_A\right)(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \hat{F}_A(x), & \text{if } f^{-1}(y) \neq \phi \\ 1, & \text{otherwise} \end{cases}$$

for each $y \in Y_{NV}$.

Corollary 4.2: Let $A_{NV}, A_{i_{NV}}\ (i \in J)$ be NVSs in X_{NV}, $B_{NV}, B_{j_{NV}}\ (j \in K)$ be NVSs in Y_{NV} and $f : X_{NV} \to Y_{NV}$ a function. Then

a) $A_{1_{NV}} \subseteq A_{2_{NV}} \Rightarrow f\left(A_{1_{NV}}\right) \subseteq f\left(A_{2_{NV}}\right)$, $B_{1_{NV}} \subseteq B_{2_{NV}} \Rightarrow f^{-1}\left(B_{1_{NV}}\right) \subseteq f^{-1}\left(B_{2_{NV}}\right)$,

b) $A_{NV} \subseteq f^{-1}(f(A_{NV}))$ (If f is injective, then $A_{NV} = f^{-1}(f(A_{NV}))$),

c) $f\left(f^{-1}(B_{NV})\right) \subseteq B_{NV}$ (If f is surjective, then $f\left(f^{-1}(B_{NV})\right) = B_{NV}$),

d) $f^{-1}\left(\cup B_{j_{NV}}\right) = \cup f^{-1}\left(B_{j_{NV}}\right)$,

e) $f^{-1}\left(\cap B_{j_{NV}}\right) = \cap f^{-1}\left(B_{j_{NV}}\right)$,

f) $f\left(\cup A_{i_{NV}}\right)=\cup f\left(A_{i_{NV}}\right)$,

g) $f\left(\cap A_{i_{NV}}\right)\subseteq \cap f\left(A_{i_{NV}}\right)$ (If f is injective, then $f\left(\cap A_{i_{NV}}\right)=\cap f\left(A_{i_{NV}}\right)$),

h) $f^{-1}\left(1_{NV}\right)=1_{NV}$,

i) $f^{-1}\left(0_{NV}\right)=0_{NV}$,

j) $f\left(1_{NV}\right)=1_{NV}$, if f is surjective,

k) $f\left(0_{NV}\right)=0_{NV}$,

l) $\overline{f\left(A_{NV}\right)}\subseteq f\left(\overline{A_{NV}}\right)$, if f is surjective,

m) $f^{-1}\left(\overline{B_{NV}}\right)=\overline{f^{-1}\left(B_{NV}\right)}$.

Definition 4.3: Let (X_{NV},τ_{NV}) and (Y_{NV},σ_{NV}) be two NVTSs and let $f:(X_{NV},\tau_{NV})\rightarrow(Y_{NV},\sigma_{NV})$ be a function. Then f is said to be neutrosophic vague continuous mapping iff the preimage of each neutrosophic vague closed set is in Y_{NV} is neutrosophic vague closed set in X_{NV}.

Definition 4.4: Let (X_{NV},τ_{NV}) and (Y_{NV},σ_{NV}) be two NVTSs and let $f:(X_{NV},\tau_{NV})\rightarrow(Y_{NV},\sigma_{NV})$ be a function. Then f is said to be neutrosophic vague open mapping iff the image of each neutrosophic vague open set is in X_{NV} is neutrosophic vague open set in Y_{NV}.

5. Neutrosophic Vague Compact Space:

Definition 5.1: Let (X_{NV},τ_{NV}) be NVTS.

i) If a family $\left\{\left\langle x,T_{A_i},I_{A_i},F_{A_i}\right\rangle : i\in J\right\}$ of NVOSs in X satisfies the condition $\bigcup\left\{\left\langle x,T_{A_i},I_{A_i},F_{A_i}\right\rangle : i\in J\right\}=1_{NV}$, then it is called neutrosophic vague open cover of X. A finite subfamily of neutrosophic vague open cover $\left\{\left\langle x,T_{A_i},I_{A_i},F_{A_i}\right\rangle : i\in J\right\}$ of X, which is also a neutrosophic vague cover of X, is called a neutrosophic vague finite subcover of $\left\{\left\langle x, T_{A_i}, I_{A_i}, F_{A_i}\right\rangle : i \in J\right\}$.

ii) A family $\left\{\left\langle x, T_{B_i}, I_{B_i}, F_{B_i}\right\rangle : i \in J\right\}$ of NVCSs in X satisfies the finite intersection property iff every finite subfamily $\left\{\left\langle x, T_{B_i}, I_{B_i}, F_{B_i}\right\rangle : i = 1, 2, ..., n\right\}$ of the family satisfies the condition $\bigcap_{i=1}^{n}\left\{\left\langle x, T_{B_i}, I_{B_i}, F_{B_i}\right\rangle\right\} \neq 0_{NV}$.

Definition 5.2: A NVTS (X_{NV}, τ_{NV}) is called neutrosophic vague compact iff every neutrosophic vague open cover of X has a neutrosophic vague finite subcover.

Corollary 5.3: A NVTS (X_{NV}, τ_{NV}) is neutrosophic vague compact iff every family $\left\{\left\langle x, T_{B_i}, I_{B_i}, F_{B_i}\right\rangle : i \in J\right\}$ of NVCSs in X having the FIP has a nonempty intersection.

Corollary 5.4: Let (X_{NV}, τ_{NV}), (Y_{NV}, σ_{NV}) be NVTSs and $f : (X_{NV}, \tau_{NV}) \to (Y_{NV}, \sigma_{NV})$ a neutrosophic vague continuous surjection. If (X_{NV}, τ_{NV}) is neutrosophic vague compact, then so is (Y_{NV}, σ_{NV}).

Definition 5.5: Let (X_{NV}, τ_{NV}) be NVTS and A_{NV} a NVS in X.

i) If a family $\left\{\left\langle x, T_{A_i}, I_{A_i}, F_{A_i}\right\rangle : i \in J\right\}$ of NVOSs in X satisfies the condition $A_{NV} \subseteq \bigcup\left\{\left\langle x, T_{A_i}, I_{A_i}, F_{A_i}\right\rangle : i \in J\right\}$, then it is called neutrosophic vague open cover of A_{NV}. A finite subfamily of neutrosophic vague open cover $\left\{\left\langle x, T_{A_i}, I_{A_i}, F_{A_i}\right\rangle : i \in J\right\}$ of A_{NV}, which is also a neutrosophic vague cover of A_{NV}, is called a neutrosophic vague finite subcover of $\left\{\left\langle x, T_{A_i}, I_{A_i}, F_{A_i}\right\rangle : i \in J\right\}$.

ii) A NVS in a NVTS (X_{NV}, τ_{NV}) is called neutrosophic vague compact iff every neutrosophic vague cover A_{NV} of has a neutrosophic vague finite subcover.

Corollary 5.6: Let (X_{NV}, τ_{NV}), (Y_{NV}, σ_{NV}) be NVTSs and $f:(X_{NV}, \tau_{NV}) \to (Y_{NV}, \sigma_{NV})$ a neutrosophic vague continuous function. If A_{NV} is neutrosophic vague compact in (X_{NV}, τ_{NV}), then so if $f(A_{NV})$ in (Y_{NV}, σ_{NV}).

Conclusion: Thus we have given the definition for neutrosophic vague topological spaces and suitable examples are also given. Along with those definition neutrosophic vague continuity and neutrosophic vague compact spaces where also discussed. Further, we can compare with all the neutrosophic vague sets and neutrosophic vague continuous functions in neutrosophic vague topological spaces.

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Neutrosophic Weibull distribution and Neutrosophic Family Weibull Distribution

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Abstract: Many problems in life are filled with ambiguity, uncertainty, impreciseness …etc., therefore we need to interpret these phenomena. In this paper, we will focus on studying neutrosophic Weibull distribution and its family, through explaining its special cases , and the functions' relationship with neutrosophic Weibull such as Neutrosophic Inverse Weibull, Neutrosophic Rayleigh, Neutrosophic three parameter Weibull, Neutrosophic Beta Weibull, Neutrosophic five Weibull, Neutrosophic six Weibull distributions (various parameters).This study will enable us to deal with indeterminate or inaccurate problems in a flexible manner. These problems will follow this family of distributions. In addition, these distributions are applied in various domains, such as reliability, electrical engineering, Quality Control ….. etc. Some properties and examples for these distributions are discussed.

Keywords: Weibull distribution, Neutrosophic logic, Neutrosophic number, Neutrosophic Weibull, Neutrosophic inverse Weibull, Neutrosophic Rayleigh, Neutrosophic Weibull with (three, four, five, six) parameters.

1. Introduction

The real world is overstuffed with vague, unclear, fuzzy (problems, situations, ideas). The classical probability ignores extreme, aberrant, unclear values, and therefore a new adequate tool had to emerge. Neutrosophic logic was introduced by Smarandache in 1995, as a generalization for the fuzzy logic and intuitionistic fuzzy logic [5, 6]. Smarandache [3, 7, 8] and Salamaa.et.al [3, 4] were presented the fundamental concepts of neutrosophic set. Smarandache extended the fuzzy set to the neutrosophic set [1, 3], introducing the neutrosophic components T, I, F which represent the membership, indeterminacy, and non-membership values respectively, where]-0, 1+[is the non-standard unit interval. Smarandache presented the neutrosophic statistics, which the data can be enigmatic, vague, imprecise, incomplete, even unknown.

The extension of classical distributions according to the neutrosophic logic means that the parameters of classical distribution take undetermined values[1,2,3,10], which allows dealing with all the situations that one may encounter while working with statistical data and especially when working with vague and inaccurate statistical data, such as the sample size may not be exactly known. The sample size could be between 50 and 70; the statistician is not sure about 20 sample persons if they belong or not to the population of interest; or because the 20 sample persons only partially belong to the population of interest, while partially they don't belong. This mean, in classical statistics all data are determined, while in neutrosophic statistic the data or a part of it are indeterminate in some degree. The neutrosophic researchers presented studies in objects different in neutrosophic statistic, such as Salama, Rafief [29], Abdel-Basset and others, see [20-28]. For more than a decade, Weibull distribution has been applied extensively in many areas and particularly used in the analysis of lifetime data for reliability engineering or biology (Rinne, 2008). However, the Weibull distribution has a weakness for modeling phenomenon with non-monotone failure rate. In this paper, we will define and study the Neutrosophic Weibull distribution, Neutrosophic family Weibull distribution for varies cases as Neutrosophic Weibull, Neutrosophic beta Weibull, Neutrosophic inverse Weibull, Neutrosophic Rayleigh, Neutrosophic with (three, four, five, six) parameters, and discuss some properties of these distributions, illustrated through examples and graphs.

2. Terminologies

In this section, we present some basic axioms of neutrosophic logic, and in particular, the work of Smarandache in [3, 7, 8] and Salama et al. [3, 4]. Smarandache introduced the neutrosophic components T, I, F which represent the membership, indeterminacy, and non-membership values respectively, where] 0-,1+[is nonstandard unit interval.

2.1 Some definitions

Definition 1 [1, 2, 3] "Neutrosophy is a new branch of philosophy which studies the origin, nature, and scope of neutralities, as well as their".

Definition 2 [1, 2, 3] Let T, I,F be real standard or nonstandard subsets of] 0-,1+[, with
Sup_T=t_sup, inf_T=t_inf
Sup_I=i_sup, inf_I=i_inf
Sup_F=f_sup, inf_F=f_inf
n-sup=t_sup+i_sup+f_sup
n-inf=t_inf+i_inf+f_inf,
T, I, F are called neutrosophic components.

Definition 3 [4, 5] Let X be a non-empty fixed set. A neutrosophic set (NS for short) A is an object having the form {x, $(\mu_A(x), \delta_A(x), \gamma_A(x)): x \in X\}$, where $\mu_A(x)$, $\delta_A(x)$ and $\gamma_A(x)$ which represent the degree of member ship function , the degree of indeterminacy , and the degree of non-member ship , respectively of each element x ∈ X to the set A .

Definition 4 [4, 5] The NSS 0_N and 1_N in X as follows:

0_N may be defined as:

$$0_1 = \{x \ \ 0,0,1: x \in X\}$$
$$0_2 = \{x \ \ 0,1,1: x \in X\}$$
$$0_3 = \{x \ \ 0,1,0: x \in X\}$$
$$0_4 = \{x \ \ 0,0,0: x \in X\}$$

1_N may be defined as:

$$1_1 = \{x \ \ 1,0,0: x \in X\}$$
$$1_2 = \{x \ \ 1,0,1: x \in X\}$$
$$1_3 = \{x \ \ 1,0,0: x \in X\}$$
$$1_4 = \{x \ \ 1,1,1: x \in X\}$$

2.2 *Neutrosophic probability*

Neutrosophic probability is a generalization of the classical probability in which the chance that event $A = \{X, A_1, A_2, A_3\}$ occurs is P(A1) true, P(A2) indeterminate , P(A3) false on a space X, then $NP(A) = \{X, P(A_1), P(A_2), P(A_3)\}$.

Definition 5 [3,4]

Let A and B be a neutrosophic events on a space X, then $NP(A) = \{X, P(A_1), P(A_2), P(A_3)\}$

And $NP(B) = \{X, P(B_1), P(B_2), P(B_3)\}$ their neutrosophic probabilities.

Definition 6 [3,4]

Let A and B be a neutrosophic events on a space X, and$NP(A) = \{X, P(A_1), P(A_2), P(A_3)\}$, and $NP(B) = \{X, P(B_1), P(B_2), P(B_3)\}$ are neutrosophic probabilities. Then we define

$NP(A \cap B) = \{X, P(A_1 \cap B_1), P(A_2 \cap B_2), P(A_3 \cap B_3)\}$

$NP(A \cup B) = \{X, P(A_1 \cup B_1), P(A_2 \cup B_2), P(A_3 \cup B_3)\}$

$\mathrm{NP}(\mathrm{A}^c) = \{\mathrm{X}, \mathrm{P}(\mathrm{A_1}^c), \mathrm{P}(\mathrm{A_2}^c), \mathrm{P}(\mathrm{A_3}^c)\}$

3 Weibull Distribution

Weibull distribution is one of most important distributions because it is widely used in reliability analysis, industrial and electrical engineering, in distribution of life time, in extreme value theory, … etc.; this distribution has various cases dependent on number of parameters such as two or three or five parameters α is the scale parameter, β is the shape parameter and γ is the location parameter. Also, it can be used to model a state where the failure function increases, decreases or remains constant with time.

4 Neutrosophic Weibull Distribution

A neutrosophic Weibull distribution (Neut-Weibull) of a continuous variable X is a classical Weibull distribution of x, but such that its mean α or β or γ are unclear or imprecise.

For example, α or β or γ can be an interval (open or closed or half open or half close) or can be set(s) with two or more elements. Then, the probability density function (p.d.f.) is:

$f_N(X) = \frac{\beta_N}{\alpha_N^{\beta_N}} X^{\beta_N - 1} e^{-(X/\alpha_N)^{\beta_N}}$, $X > 0$, Where β_N: is the shape parameter of distribution Net-Weibull, α_N: is the scale parameter of distribution Net- Weibull, such that N is a neutrosophic number.

4.1 Properties of Neutrosophic Weibull Distribution

- The distribution function (c.d.f.) is:

  $F_N(X) = 1 - e^{-(X/\alpha_N)^{\beta_N}}$,

  $E_N(X) = \alpha_N \Gamma\left(\frac{\beta_N + 1}{\beta_N}\right)$,

  $V_N(X) = \alpha^2{}_N \left[\Gamma\left(\frac{\beta_N + 2}{\beta_N}\right)\right] - \left[\Gamma\left(\frac{\beta_N + 1}{\beta_N}\right)\right]^2.$
- The hazard function is:

  $h_N = \beta_N X^{\beta_N - 1} X^{(\beta_N - 1/\alpha_N)^{\beta_N}}.$
- The moment rth about mean is:

  $\alpha_N{}^r \Gamma\left(\beta_N + \frac{r}{\beta_N}\right)$
- So, the reliability or survival function is:

$\overline{F_N}(X) = e^{-(X/\alpha_N)^{\beta_N}}$.

Now, we put β_N=1 in the formula (1), and we get the neutrosophic exponential distribution [13].

4.2 Example of Neutrosophic Weibull distribution

Let the product be an electric generator produced with high capacity of trademark that has a Weibull distribution with parameter α=1, β=[1.5,2]. Compute the probability of electric generator fails before the expiration of a five years warranty.

Solution :

In this example, we note that the shape parameter is indeterminate.

The electric generator can work through to one year:

$$f_N(X) = \frac{[1.5,2]}{\alpha_N^{[1.5,2]}} X^{[1.5,2]-1} e^{-(X/\alpha_N)^{[1.5,2]}}$$

If we take $\beta = 1.5$, and $\alpha = 1$

$$f_N(X = 1) = 0.5518$$

the probability of electric generator fails before the expiration of a five years warranty:

$$P(X \leq 5) = 1 - e^{-(5/1)^{1.5}}, = 0.999986$$

If we take $\beta = 2$, and $\alpha = 1$

$$f_N(X = 1) = 0.7357$$

$$P(X \leq 5) = 1 - e^{-(5/1)^{5}}, = 0.999999$$

Thus, the probability that the electric machine fails has the range between [0.5518, 0.7357].

Now, suppose $\beta = 2$ and $= [1,2]$, i.e the scale parameter α is indeterminate.

We take $\alpha = 1$ and $\beta = 2$

$$f_N(X = 1) = \frac{2}{e^1} = 0.7357$$

We take $\alpha = 2$ and $\beta = 2$

$$f_N(X = 1) = \frac{1}{2e^{1/4}} = 0.3894$$

In this case, the probability that the electric machine fails has the range between [0.7357, 0.3894]. Also, we can take more values of X, showed in Figure (1).

Now, we can compute

$F_N(X) = 1 - 1/e = 0.6321\,, \quad if\ \alpha = 1$

$F_N(X) = 1 - e^{1}/_{1.2840} = 0.2212\,, \quad if\ \alpha = 2.$

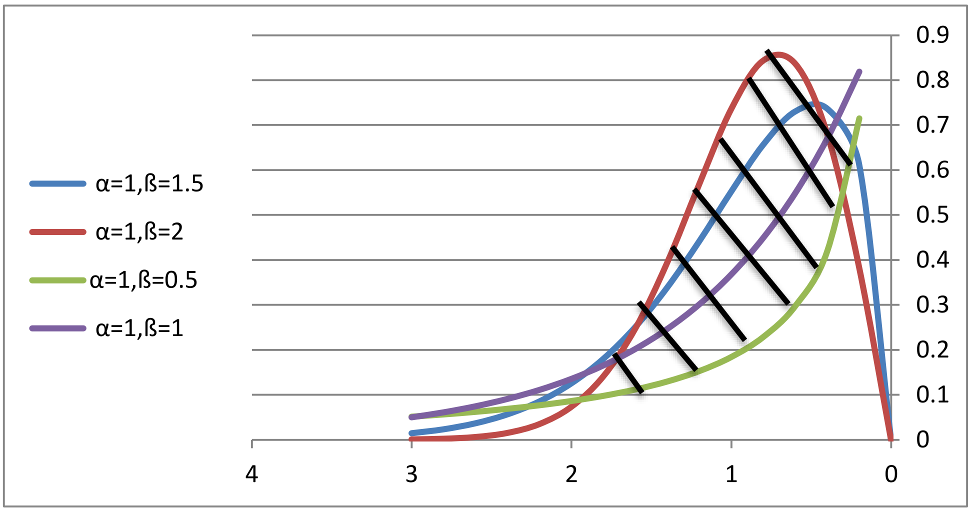

Figure 1: Neutrosophic Weibull distribution.

4.3 Comparison between Neutrosophic Weibull distribution and Weibull distribution

1- In classical Weibull, we noted that if the $\beta = 3.6$ *or more*, the probability distribution function (p.d.f) takes value error because it is greater than one, and this contradicts with law of probability,consedered Extreme values, while in neutrosophic Weibull this is applicable. See Figure (2).
2- In classical Weibull distribution, when X is increasing, the p.d.f. is decreasing, while in Neutrosophic Weibull distribution the p.d.f is unpredictable because of the aberrant values.
3- Many values that are larger than one are neglected in Weibull distribution, meanwhile in Neutrosophic Weibull these values are considered.
4- When $\alpha = \beta = 1$, the p.d.f. will equal zero when X=701,while in neutrosophic Weibull X can be of other values such as X={701,100} or [701,100] in this case p.d.f can be of different values.

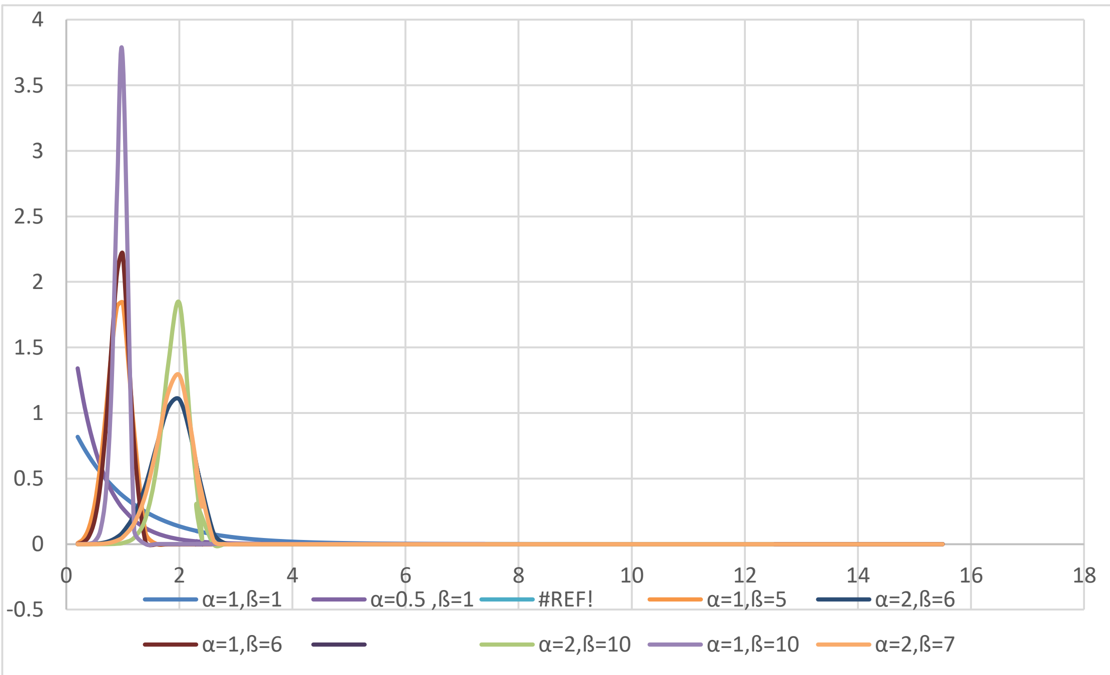

Figure 2: p.d.f of neutrosophic Weibull more than one.

5 The Family of Neutrosophic Weibull

In this section, we study the various types of Net-Weibull, such as neutrosophic Rayleigh distribution, neutrosophic inverse Weibull distribution, neutrosophic Beta-Weibull distribution and (three, four, five, six)-parameters Weibull distributions.

5.1 Neutrosophy Rayleigh Distribution

A Rayleigh distribution is often observed when the total size of a vector is linked to its directional components. Considering this distribution is important in the error analysis of various systems or individuals. It is also considered as a model for testing life failure/expiration. Rayleigh distribution is used in the study of the event of sea wave rise in the oceans and the study of wind speed, as well as in the information of the strength of signals and radiation at peak time of communications. The distribution is widely applied:

- In communications theory, to model multiple paths of dense scattered signals getting to a receiver;
- In the physics, to model wind speed, wave heights and sound/light radiation;
- In engineering, to measure the lifetime of an object, since the lifetime depends on the object's age (resistors, transformers, and capacitors in aircraft radar sets);
- In medical imaging examination, to study noise variance in magnetic resonance imaging.

Now, we define the probability density function of neutrosophic Rayleigh distribution as follows:

$R_N(X) = \frac{X}{{\delta_N}^2} e^{-X^2 \ /2{\delta_N}^2}$, $X > 0,\ \delta_N$ is the scale parameter.

this parameter δ_N can take the values of an interval or a set:
cumulative distribution is $F_N(X) = 1 - e^{-X^2 \ /2{\delta_N}^2}$,
the mean of Neutrosophic Rayleigh distribution is

$$\mathrm{E(X)} = \delta_N \sqrt{\frac{\pi}{2}},$$

the variance: var(x) = 2-π/2 ${\delta_N}^2$.

5.2 Neutrosophic Weibull with 3 Parameters

We can obtain the neutrosophic Weibull with 3-parameters by relaying on Weibull with 2-parameters and adding the third parameter, namely the location parameter (γ), this is in classical probability . Now, we define the neutrosophic Weibull with three parameters (an indeterminacy may exist in one parameter or in all parameters). Neutrosophic Weibull with 3-parameters is defined as follows:

$$f_N(X) = [\beta_N \frac{(X-\gamma_N)^{\beta_N - 1}}{\alpha_N^{\beta_N}}] e^{-((X-\gamma_N)/\alpha_N)^{\beta_N}} \quad , \ \gamma_N \le X$$

- The distribution function is:

  $F_N(X) = 1 - e^{-((X-\gamma_N)/\alpha_N)^{\beta_N}}$, $\gamma_N \le X$
- The hazard function is:

  $h_N(X) = \beta_N \ (X-\gamma_N)^{\beta_N}(1\ /\alpha_N)^{\beta_N}$, $\gamma_N \le X$
- The survival function is

  $\overline{F_N}(X) = e^{-((X-\gamma_N)/\alpha_N)^{\beta_N}}$
- The variance

$$V_N(X) = \alpha^2{}_N \left[\Gamma\left(\frac{\beta_N+2}{\beta_N}\right)\right] - \left[\Gamma\left(\frac{\beta_N+1}{\beta_N}\right)\right]^2,$$

- The expected value $E_N(X) = \gamma_N + \alpha_N \Gamma\left(\frac{\beta_N+1}{\beta_N}\right)$.

5.3 Four-Parameter Neutrosophic-Beta-Weibull

The Beta-Weibull was first proposed by Famoye et al. (2005) [11,12, 15]. We now define the new density function of neutrosophic-Beta-Weibull distribution (NBW) in neutrosophic logic with indeterminacy points for random variable or parameters as follows:

$$f(X) = \frac{\Gamma(c_N+\gamma_N)}{\Gamma(c_N)\Gamma(\gamma_N)} \frac{\alpha_N}{\beta_N}\left(\frac{X}{\beta_N}\right)^{\alpha_N-1} [1 - e^{-(X/\beta_N)^{\alpha_N}}]^{c_N-1} e^{-\gamma_N (X/\beta_N)^{\alpha_N}}$$

$$X > 0,\ \gamma_N, \beta_N, \alpha_N > 0$$

where these parameters $\gamma_N, \beta_N, \alpha_N$ can be set(s) or interval (closed or open or half):

$$\text{Because } \lim_{x\to 0} f(X) = \lim_{x\to 0} \frac{\Gamma(c_N+\gamma_N)}{\Gamma(c_N)\Gamma(\gamma_N)} \frac{\alpha_N}{\beta_N}\left(\frac{X}{\beta_N}\right)^{\alpha_N-1} \left[1 - e^{-(X/\beta_N)^{\alpha_N}}\right]^{c_N-1} e^{-\gamma_N (X/\beta_N)^{\alpha_N}}$$

$$= \frac{\Gamma(c_N+\gamma_N)}{\Gamma(c_N)\Gamma(\gamma_N)} \frac{\alpha_N}{\beta_N}\left(\frac{X}{\beta_N}\right)^{\alpha_N-1} e^{-\gamma_N (X/\beta_N)^{\alpha_N}} [1 - e^{-(X/\beta_N)^{\alpha_N}}]^{c_N-1}$$

$$= \frac{\Gamma(c_N+\gamma_N)}{\Gamma(c_N)\Gamma(\gamma_N)} \frac{\alpha_N}{\beta_N}\left(\frac{X}{\beta_N}\right)^{\alpha_N-1} e^{-\gamma_N (X/\beta_N)^{\alpha_N}} \left[1 - \frac{1}{2!}\left(\frac{X}{\beta_N}\right)^{\alpha_N} + \frac{1}{3!}\left(\frac{X}{\beta_N}\right)^{2\alpha_N} - \frac{1}{4!}\left(\frac{X}{\beta_N}\right)^{3\alpha_N} + \cdots\right]^{c_N-1}$$

Then the probability of density function is equal to

$$= \lim_{x\to 0} \frac{\alpha_N}{\beta_N} \frac{\Gamma(c_N+\gamma_N)}{\Gamma(c_N)\Gamma(\gamma_N)} \left(\frac{X}{\beta_N}\right)^{\alpha_N-1} = \begin{cases} \infty & \alpha_N\, c_N < 1 \\ \frac{\alpha_N}{\beta_N} \frac{\Gamma(c_N+\gamma_N)}{\Gamma(c_N)\Gamma(\gamma_N)} & \alpha_N\, c_N = 1 \\ 0 & \alpha_N\, c_N > 1 \end{cases}$$

where $\beta_N, c_N, \gamma_N, \alpha_{N,}$ are Neutrosophy numbers.

- When $c_N = \gamma_N = 1$, then the (NBW) is reduced to neutrosophic Weibull distribution.
- When $\beta_N = \alpha_N = 1, c_N = 2, \gamma_N = \delta\sqrt{2}$, the NBW is reduced to neutrosophy Rayleigh.
- In (1958) Kies defined the survival function to Weibull with four parameters in classical distribution.

Here we define Neutrosophic survival function in Neutrosophic distribution as follows:

$$\overline{F_N}(X) = e^{\left\{-\gamma_N \left(\frac{x-\alpha_N}{\beta_N - x}\right)^{k_N}\right\}},\ \gamma_N > 0, k_N > 0,\ 0 < \alpha_N < X < \beta_N < \infty.$$

5.4 Neutrosophic Weibull Distribution with 5 Parameters

Phani in (1987) [14] suggested model with survival function has five parameters. We define the neutrosophic Weibull with 5-parameters:

$$\overline{F_N}(X) = e^{\frac{-\gamma_N [X-\alpha_N]^{b_1}}{[\beta_N - X]^{b_2}}},\quad \gamma_N, b_1, b_2 > 0,\quad 0 < \alpha_N < X < \beta_N < \infty.$$

5.5 Neutrosophic Weibull Distribution with 6 Parameters

T, W, and Uraiwan in (2014) [15] proposed a mixed distribution is Beta exponential Weibull Poisson distribution. We define the neutrosophic Beta exponential Weibull Poisson distribution as follows:

Let x be the neutrosophic random variable with parameters $\gamma_N, k_N,\ \alpha_N,\ \beta_N, b_1, b_2$;

$$f(x) \ =\frac{\beta_N\alpha_N\,\gamma_N \quad {k_N}^{\beta}{}_N \quad X^{\beta_N-1}u(1-u)^{\alpha_N-1}e^{\gamma_N(1-u)^{\alpha_N}}}{\mathrm{B}(b_1,b_2)(e^{\,\gamma_N}-1)}.\left[\frac{e^{\gamma_N(1-u)^{\alpha_N}}-1}{(e^{\,\gamma_N}-1)}\right]^{\alpha_N-1}\left[1-\frac{e^{\gamma_N(1-u)^{\alpha_N}}-1}{(e^{\,\gamma_N}-1)}\right]^{b1_N-1}$$

where $u = e^{-(Xk_N)^{\beta_N}}$.

5.6 Neutrosophic Inverse Weibull Distribution

Keller et al. (1985) used the inverse Weibull distribution for reliability analysis of commercial vehicle engines. Here, we define Neutrosophic inverse Weibull distribution as follows:

$f_N(t) = \beta_N \alpha_N^{\beta_N}\, t^{-\beta_N-1} e^{-(\alpha_N/t)^{\beta_N}},\ \ t > 0$, So the Hazard function is $h_N(t) = \frac{\beta_N\alpha_N^{\beta_N}\,t^{-\beta_N-1}e^{-(\alpha_N/t)^{\beta_N}}}{1-e^{-(\alpha_N/t)^{\beta_N}}}$.

6 Applications

Many applications of Weibull families distributions are suitable for modeling and analysis of floods, rainfall, sea, electronic, manufacturing products, navigation and transportation control. The theories and tools of reliability engineering are applied into widespread fields, such as electronic and manufacturing products, aerospace equipment, earthquake and volcano forecasting, communication, navigation and transportation control, medical processor to the survival analysis of human being or biological species, and so on [14]. So the neutrosophic has the multi-applied in Decision-making, introduced by Abdel-Basset and others.

7 Conclusions

The study of neutrosophic probability distributions gives us a more comprehensive space in the applied field, as it takes into account more than the value of the distribution parameters and not only one value as in the classical distributions, and thus we will be able to solve and explain many of the issues that have been hindering us or we tended to ignore in classical logic. In this paper, we defined th new neutrosophic clasical distribution, the neutrosophic Weibull distribution and neutrosophic family Weibull (neutrosophic inverse Weibull, Neutrosophic Rayleigh distribution, Neutrosophic Weibull distribution with (3, 4, 5, 6)-parameters, and give clear examples. Because the weibull distribution has many applications in different fields.such as control system, relability and others. We also study some properties of these distributions (mean, variance, failure function and reliability function). In the future, we will apply these distributions to many problems and we will examine other distributions in neutrosophic logic.

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Some properties of Pentagonal Neutrosophic Numbers and its Applications in Transportation Problem Environment

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Abstract: In this research article we actually deals with the conception of pentagonal Neutrosophic number from a different frame of reference. Recently, neutrosophic set theory and its extensive properties have given different dimensions for researchers. This paper focuses on pentagonal neutrosophic numbers and its distinct properties. At the same time, we defined the disjunctive cases of this number whenever the truthiness, falsity and hesitation portion are dependent and independent to each other. Some basic properties of pentagonal neutrosophic numbers with its logical score and accuracy function is introduced in this paper with its application in real life operation research problem which is more reliable than the other methods.

Keywords: Neutrosophic set, neutrosophic number, Pentagonal Neutrosophic number; Score and Accuracy function.

1. Introduction

Recently, handling the uncertainty and vagueness is considered as one of the prominent research topic around the world. In this regard, mathematical algebra of Fuzzy set theory [1] has provided a well-established tool to deal with the same. Vagueness theory plays a key role to solve problems related with engineering and statistical computation. It is widely used in social science, networking, decision making problem or any kind of real life problem. Motivating from fuzzy sets the Atanassov [2] proposed the legerdemain idea of an intuitionistic fuzzy set in the field of Mathematics in which he considers the concept of membership function as well as non-membership function in case of intuitionistic fuzzy set. Afterwards, the invention of Liu F, Yuan XH in 2007 [3], ignited the concept of triangular intuitionistic fuzzy set, which in reality is the congenial mixture of triangular fuzzy set and intuitionistic fuzzy set. Later, Ye [4] introduced the elementary idea of trapezoidal intuitionistic fuzzy set where both truth function and falsity function are both trapezoidal number in nature instead of triangular. The uncertainty theory plays an influential role to create some interesting model in various fields of science and technological problem.

Smarandache in 1995 (published in 1998) [5] manifested the idea of neutrosophic set where there are three different components namely i) truthiness, ii) indeterminacies, iii) falseness. All the aspect of neutrosophic set is very much pertinent with our real-life system. Neutrosophic concept is a very effective & an exuberant idea in real life. Further, R. Helen [7] introduced the pentagonal fuzzy number and A.

Vigin [8] applied it in neural network. T.Pathinathan [9] gives the concept of reverse order triangular, trapezoidal, pentagonal fuzzy number. Later, Wang et al. [10] invented the perception of single typed neutrosophic set which so much useful to solve any complex problem. Later, Ye [11] presented the concept of trapezoidal neutrosophic fuzzy number and its application. A.Chakraborty [12] developed the conception of triangular neutrosophic number and its different form when the membership functions are dependent or independent. Recently, A.Chakraborty [13] also developed the perception of pentagonal fuzzy number and its different representation in research domain. Christi [14] applied the conception of pentagonal intuitionistic number for solving a transportation problem. Later, Chen [15, 16] solved MCDM problem with the help of FN-IOWA operator and using trapezoidal fuzzy number analyse fuzzy risk ranking problem respectively. Recently, S. Broumi [17-19] developed some important articles related with neutrosophic number in different branch of mathematics in various real life problems. Moreover, Prem [20-25] invented some useful results in neutrosophic arena, mainly associated with computer science engineering problem and networking field. Chakraborty A. [26, 27] applied the idea of vagueness in mathematical model for diabetes and inventory problem respectively. Recently, Abdel-Basset [28-34] introduced some interesting articles co-related with neutrosophic domain in disjunctive fields like MCDM problem; IoT based problem, Supply chain management problem, cloud computing problem etc. K . Mondal [35,36] apply the concept of neutrosophic number in teacher recruitment MCDM problem in education sector. Later, different types of developments in decision making problems, medical diagnoses problem and others in neutrosophic environment [37-49] are already published in this impreciseness arena. Recently, the conception of plithogenic set is being developed by Smarandache and it has a great impact in unceairty field in various domain of research.

1.1 Motivation

The perception of vagueness plays a crucial role in construction of mathematical modeling, engineering problem and medical diagnoses problem etc. Now there will be an important issue that if someone considers pentagonal neutrosophic number then what will be the linear form and what is the geometrical figure? How should we categorize the type-1, 2, 3 pentagonal neutrosophic numbers when the membership functions are related to each other? From this aspect we actually try to develop this research article. Later we invented some more interesting results on score and accuracy function and other application.

1.2 Contribution

In this paper, researchers mainly deal with the conception of pentagonal neutrosophic number in different aspect. We introduced the linear form of single typed pentagonal neutrosophic fuzzy number for distinctive categories. Basically, there are three categories of number will comes out when the three membership functions are dependent or independent among each other, namely Category-1, 2, 3 pentagonal neutrosophic numbers. All the disjunctive categories and their membership functions are addressed here simultaneously.

Researchers from all around the globe are very much interested to know that how a neutrosophic number is converted into a crisp number. Day by day, as research goes on they developed lots of techniques to solve the problem. We developed score and accuracy function and built up the conception of conversion of pentagonal neutrosophic fuzzy number in to a crisp number. In this current era, researchers are very much interested in doing transportation problem in neutrosophic domain. In this phenomenon, we consider a transportation problem in pentagonal neutrosophic domain where we utilize the idea of our developed score and accuracy function for solving the problem.

1.3 Novelties

There are a large number of works already published in this neutrosophic fuzzy set arena. Researchers already developed several formulations and application in various fields. However there will be many interesting results are still unknown. Our work is to try to develop the idea in the unknown points.

- Introduced the distinctive form of pentagonal neutrosophic fuzzy number and its definition for different cases.
- The graphical representation of pentagonal neutrosophic fuzzy number.
- Development of score and accuracy function.
- Application in transportation problem.

1.4 Verbal Phrase on Neutrosophic Arena

In case of daily life, an interesting question often arises: How can we connect the conception of vagueness and neutrosophic theory in real life domain and what are the verbal phrases in case of it?

Example: Let us consider a problem of vote casting. Suppose in an election we need to select some number of candidates among a finite number of candidates. People have different kind of emotions, feelings, demand, ethics, dream etc. So according to their viewpoint it can be any kind of fuzzy number like interval number, triangular fuzzy, intuitionistic, neutrosophic fuzzy number. Let us check the verbal phrases in each different case for the given problem.

Table 1.3.1: Verbal Phrases

Distinct parameter	Verbal Phrase	Information
Interval Number	[Low, High]	Voter will select according to their first priority within a certain range like [2^{nd}, 3rd] candidate.
Triangular Fuzzy Number	[Low, Median, High]	Voter will select according to their first priority containing an intermediate candidate like [1^{st},2^{nd},3^{rd}]
Intuitionistic (Triangular)	[Standard,Median,High; Very Low,Poor,Low]	Voters will select some candidate directly and reject others immediately according to their view.
Neutrosophic (Pentagonal)	[VeryLow,Low,Median,High, Very High; VeryLow,Low,Median,High, Very High; VeryLow,Low,Median,High, Very High]	Some Voters will select directly some candidates, some of them are in hesitation in casting vote and some of them directly reject voting according to their own viewpoints.

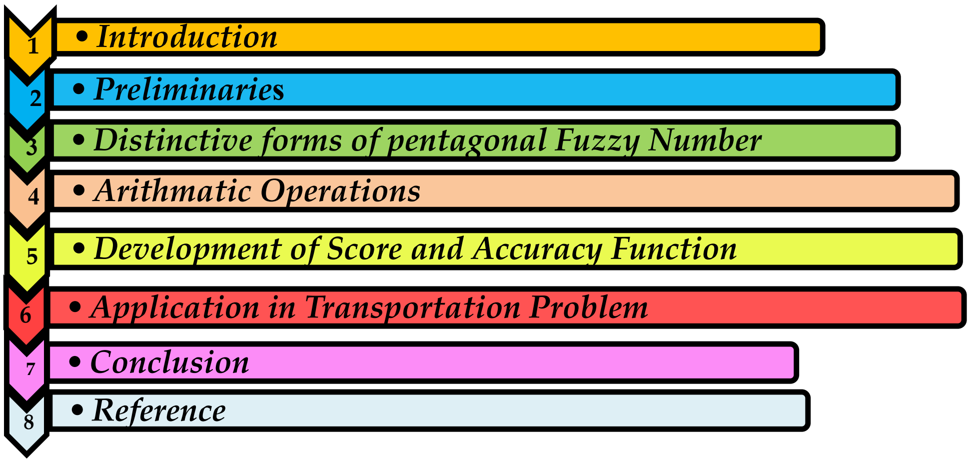
It can be observed that,in 1^{st} coloum of the above table which contains distinct parameters like interval number, triangular fuzzy number, triangular intuitionistic fuzzy number and neutrosophic number, obviously neutrosophic concept gives us a more reliable and logical result since it will contain truth,false as well as the hesitation information absent in the other parameters. Also it is a key question why we take pentagonal neutrosophic instead of triangular or trapezoidal? Now, if we observe the verbal phrase section we can observe that, in case of triangular it will contain only three phrase like low,median,high and trapezoidal contains four like low, semi median, quasi median,high. Suppose some voters choose truth part very strongly and reject the other two sections because these are very low or someone chooses truth part in an average way since he/she thought that rest of the portions are very low. That means we need to develop the verbal phrase such that it will contain much more distict categories. Pentagonal shapes give us atleast five disjunctive verbal categories like very low, low, median, high, very high which is much more logical, strong and it also contains more sensitive cases than the rest sections.

1.5 Need of Pentagonal Neutrosophic Fuzzy Number

The pentagonal neutrosophic fuzzy number stretches us superfluous opportunity to characterize flawed knowledge which leads to construct logical models in several realistic problems in a new way. Pentagonal neutrosophic represents the data and information in a complete way and the truth, hesitation and falsity can be characterized in more accurate and normal technique. The info is reserved throughout the operation and the full material can be utilized by the decision maker for further investigation. It can be finding its applications in different optimization complications, decision making problems and economic difficulties etc. which need fifteen components. In case of transportation problem, if the numbers of variable are five for each of the three components then it is problematic to signify by using Triangular or Trapezoidal neutrosophic Fuzzy numbers. Therefore, pentagonal neutrosophic fuzzy number can invention its dynamic applications in resolving the problem.

1.6 Structure of the paper

The article is developed as follows:



2. Mathematical Preliminaries

Definition 2.1: Fuzzy Set: [1] A set $\widetilde{B}$, defined as $\widetilde{B} = \{(x, \mu_{\widetilde{B}}(x)) : x \in X, \mu_{\widetilde{B}}(X) \in [0,1]\}$ and generally denoted by the pair $(x, \mu_{\widetilde{B}}(x))$, x belongs to the crisp set X and $\mu_{\widetilde{B}}(X)$ belongs to the interval $[0, 1]$, then set $\widetilde{B}$ is called a fuzzy set.

Definition 2.2: Intuitionistic Fuzzy Set (IFS): An Intuitionistic fuzzy set [2] $\tilde{S}$ in the universal discourse X which is denoted generically by x is said to be a Intuitionistic set if $\tilde{S} = \{\langle x; [\tau(x), \varphi(x)]\rangle : x \in X\}$, where $\tau(x): X \rightarrow [0,1]$ is named as the truth membership function which indicate the degree of assurance, $\varphi(x): X \rightarrow [0,1]$ is named the indeterminacy membership function which shows the degree of vagueness.

$\tau(x), \varphi(x)$ parades the following the relation $0 \le \tau(x) + \varphi(x) \le 1$.

2.3 Definition: Neutrosophic Set: Smarandache[5] A set $\widetilde{NeA}$ in the universal discourse X, symbolically denoted by x, it is called a neutrosophic set if $\widetilde{NeA} = \{\langle x; [\alpha_{\widetilde{NeA}}(x), \beta_{\widetilde{NeA}}(x), \gamma_{\widetilde{NeA}}(x)]\rangle : x \in X\}$, where $\alpha_{\widetilde{NeA}}(x): X \rightarrow [0,1]$ is said to be the truth membership function, which represents the degree of assurance, $\beta_{\widetilde{NeA}}(x): X \rightarrow [0,1]$ is said to be the indeterminacy membership, which denotes the degree of vagueness, and $\gamma_{\widetilde{NeA}}(x): X \rightarrow [0,1]$ is said to be the falsity membership, which indicates the degree of skepticism on the decision taken by the decision maker.

$\alpha_{\widetilde{NeA}}(x), \beta_{\widetilde{NeA}}(x) \& \gamma_{\widetilde{NeA}}(x)$exhibits the following relation: $-0 \leq \alpha_{\widetilde{NeA}}(x) + \beta_{\widetilde{NeA}}(x) + \gamma_{\widetilde{NeA}}(x) \leq 3+$.

2.4 Definition: Single-Valued Neutrosophic Set: A Neutrosophic set $\widetilde{NeA}$ in the definition 2.1 is said to be a single-Valued Neutrosophic Set $(\widetilde{SNeA})$ if x is a single-valued independent variable. $\widetilde{SNeA} = \{\langle x; [\alpha_{\widetilde{SNeA}}(x), \beta_{\widetilde{SNeA}}(x), \gamma_{\widetilde{SNeA}}(x)]\rangle : x \in X\}$, where $\alpha_{\widetilde{SNeA}}(x), \beta_{\widetilde{SNeA}}(x) \& \gamma_{\widetilde{SNeA}}(x)$ denoted the concept of accuracy, indeterminacy and falsity memberships function respectively.

If there exist three points $a_0, b_0 \& c_0$, for which $\alpha_{\widetilde{SNeA}}(a_0) = 1, \beta_{\widetilde{SNeA}}(b_0) = 1 \& \gamma_{\widetilde{SNeA}}(c_0) = 1$, then the $\widetilde{SNeA}$ is called neut-normal.

$\widetilde{SnS}$is called neut-convex, which implies that $\widetilde{SnS}$ is a subset of a real line by satisfying the following conditions:

i. $\alpha_{\widetilde{SNeA}}\langle \delta a_1 + (1-\delta) a_2 \rangle \geq min\langle \alpha_{\widetilde{SNeA}}(a_1), \alpha_{\widetilde{SNeA}}(a_2) \rangle$
ii. $\beta_{\widetilde{SNeA}}\langle \delta a_1 + (1-\delta) a_2 \rangle \leq max\langle \beta_{\widetilde{SNeA}}(a_1), \beta_{\widetilde{SNeA}}(a_2) \rangle$
iii. $\gamma_{\widetilde{SNeA}}\langle \delta a_1 + (1-\delta) a_2 \rangle \leq max\langle \gamma_{\widetilde{SNeA}}(a_1), \gamma_{\widetilde{SNeA}}(a_2) \rangle$

Where$a_1 \& a_2 \in \mathbb{R}$ and $\delta \in [0,1]$

2.5 Definition: Single-Valued Pentagonal Neutrosophic Number: A Single-Valued Pentagonal Neutrosophic Number $(\tilde{S})$ is defined as $\tilde{s} = \langle [(m^1, n^1, o^1, p^1, q^1); \pi], [(m^2, n^2, o^2, p^2, q^2); \rho], [(m^3, n^3, o^3, p^3, q^3); \sigma] \rangle$, where $\pi, \rho, \sigma \in [0,1]$. The accuracy membership function$(\tau_{\tilde{S}}): \mathbb{R} \to [0, \pi]$, the indeterminacy membership function $(\iota_{\tilde{S}}): \mathbb{R} \to [\rho, 1]$ and the falsity membership function $(\varepsilon_{\tilde{S}}): \mathbb{R} \to [\sigma, 1]$ are given as:

$$\tau_{\tilde{S}}(x) = \begin{cases} \tau_{\widetilde{Sl1}}(x) m^1 \leq x < n^1 \\ \tau_{\widetilde{Sl2}}(x)\, n^1 \leq x < o^1 \\ \mu \quad\quad x = o^1 \\ \tau_{\widetilde{Sr2}}(x)\, o^1 \leq x < p^1 \\ \tau_{\widetilde{Sr1}}(x) p^1 \leq x < q^1 \\ 0 \quad otherwise \end{cases}, \quad \iota_{\tilde{S}}(x) = \begin{cases} l_{\widetilde{Sl1}}(x) m^2 \leq x < n^2 \\ l_{\widetilde{Sl2}}(x)\, n^2 \leq x < o^2 \\ \vartheta \quad\quad x = o^2 \\ l_{\widetilde{Sr2}}(x)\, o^2 \leq x < p^2 \\ l_{\widetilde{Sr1}}(x) p^2 \leq x < q^2 \\ 1 \quad otherwise \end{cases}$$

$$\varepsilon_{\tilde{S}}(x) = \begin{cases} \varepsilon_{\widetilde{Sl1}}(x) m^3 \leq x < n^3 \\ \varepsilon_{\widetilde{Sl2}}(x)\, n^3 \leq x < o^3 \\ \vartheta \quad\quad x = o^3 \\ \varepsilon_{\widetilde{Sr2}}(x)\, o^3 \leq x < p^3 \\ \varepsilon_{\widetilde{Sr1}}(x) p^3 \leq x < q^3 \\ 1 \quad otherwise \end{cases}$$

3. Linear Generalized Pentagonal Neutrosophic number:

In this section, we introduce the linear and generalized neutrosophic number.

3.1 Pentagonal Single Typed Neutrosophic Number of Specification 1: When the quantity of the truth, hesitation and falsity are independent to each other.

A Pentagonal Single typed Neutrosophic Number (PTGNEU) of specification 1 is described as$\tilde{A}_{PtgNeu} = (p_1, p_2, p_3, p_4, p_5; q_1, q_2, q_3, q_4, q_5; r_1, r_2, r_3, r_4, r_5; \tau)$, whose truth membership; hesitation membership and falsity membership are specified as follows:

$$T_{\tilde{A}_{PtgNeu}}(x) = \begin{cases} \tau\dfrac{x-p_1}{p_2-p_1} if p_1 \le x \le p_2 \\ 1-(1-\tau)\dfrac{x-p_2}{p_3-p_2} if p_2 \le x \le p_3 \\ 1 \quad if x = p_3 \\ 1-(1-\tau)\dfrac{p_4-x}{p_4-p_3} if p_3 \le x \le p_4 \\ \tau\dfrac{p_5-x}{p_5-p_4} if p_4 \le x \le p_5 \\ 0 \quad Otherwise \end{cases}$$

$$I_{\tilde{A}_{PtgNeu}}(x) = \begin{cases} \tau\dfrac{q_2-x}{q_2-q_1} if q_1 \le x < q_2 \\ 1-(1-\tau)\dfrac{q_3-\mathrm{x}}{q_3-q_2} if q_2 \le x \le q_3 \\ 0 \quad if x = q_3 \\ 1-(1-\tau)\dfrac{x-q_3}{q_4-q_3}\ if q_3 \le x \le q_4 \\ \tau\dfrac{x-q_4}{q_5-q_4}\ if q_4 \le x \le q_5 \\ 1 \quad Otherwise \end{cases}$$

$$F_{\tilde{A}_{PtgNeu}}(x) = \begin{cases} \tau\dfrac{r_2-x}{r_2-r_1} if r_1 \le x < r_2 \\ 1-(1-\tau)\dfrac{r_3-\mathrm{x}}{r_3-r_2} if r_2 \le x \le r_3 \\ 0 \quad if x = r_3 \\ 1-(1-\tau)\dfrac{x-r_3}{r_4-r_3}\ if r_3 \le x \le r_4 \\ \tau\dfrac{x-r_4}{r_5-r_4}\ if r_4 \le x \le r_5 \\ 1 \quad Otherwise \end{cases}$$

Where $-0 \le T_{\tilde{A}_{PtgNeu}}(x) + I_{\tilde{A}_{PtgNeu}}(x) + F_{\tilde{A}_{PtgNeu}}(x) \le 3+, x \in \tilde{A}_{PtgNeu}$

The parametric form of the above type number is

$$\left(\tilde{A}_{PtgNeu}\right)_{\mu,\vartheta,\varphi} = \begin{bmatrix} T_{PtgNeu1L}(\mu), T_{PtgNeu2L}(\mu), T_{PtgNeu1R}(\mu), T_{PtgNeu2R}(\mu); \\ I_{PtgNeu1L}(\vartheta), I_{PtgNeu2L}(\vartheta), I_{PtgNeu1R}(\vartheta), I_{PtgNeu2R}(\vartheta); \\ F_{PtgNeu1L}(\varphi), F_{PtgNeu1L}(\varphi), F_{PtgNeu1L}(\varphi), F_{PtgNeu1L}(\varphi) \end{bmatrix}$$

where, $T_{PtgNeu1L}(\mu) = p_1 + \frac{\mu}{\tau}(p_2 - p_1)$ for $\mu \in [0,\tau]$, $T_{PtgNeu2L}(\mu) = p_2 + \frac{1-\mu}{1-\tau}(p_3 - p_2)for\mu \in [\tau, 1]$

$T_{PtgNeu2R}(\mu) = p_4 - \frac{1-\mu}{1-\tau}(p_4 - p_3)for\mu \in [\tau, 1]$, $T_{PtgNeu1R}(\mu) = p_5 - \frac{\mu}{\tau}(p_5 - p_4)for\mu \in [0,\tau]$

$I_{PtgNeu1L}(\vartheta) = q_2 - \frac{\vartheta}{\tau}(q_2 - q_1)for\vartheta \in [\tau, 1]$, $I_{PtgNeu2L}(\vartheta) = q_3 - \frac{1-\vartheta}{1-\tau}(q_3 - q_2)for\vartheta \in [0,\tau]$

$I_{PtgNeu2R}(\vartheta) = q_3 + \frac{1-\vartheta}{1-\tau}(q_4 - q_3)for\vartheta \in [0,\tau]$, $I_{PtgNeu1R}(\vartheta) = q_4 + \frac{\vartheta}{\tau}(q_5 - q_4)for\vartheta \in [\tau, 1]$

$F_{PtgNeu1L}(\varphi) = r_2 - \frac{\varphi}{\tau}(r_2 - r_1)for\varphi \in [\tau, 1]$, $F_{PtgNeu2L}(\varphi) = r_3 - \frac{1-\varphi}{1-\tau}(r_3 - r_2)for\varphi \in [0,\tau]$

$F_{PtgNeu2R}(\varphi) = r_3 + \frac{1-\varphi}{1-\tau}(r_4 - r_3)for\varphi \in [0,\tau]$, $F_{PtgNeu1R}(\varphi) = r_4 + \frac{\varphi}{\tau}(r_5 - r_4)for\varphi \in [\tau, 1]$

Here,$0 < \mu \le 1$, $0 < \vartheta \le 1$, $0 < \varphi \le 1$ and $-0 < \mu + \vartheta + \varphi \le 3+$

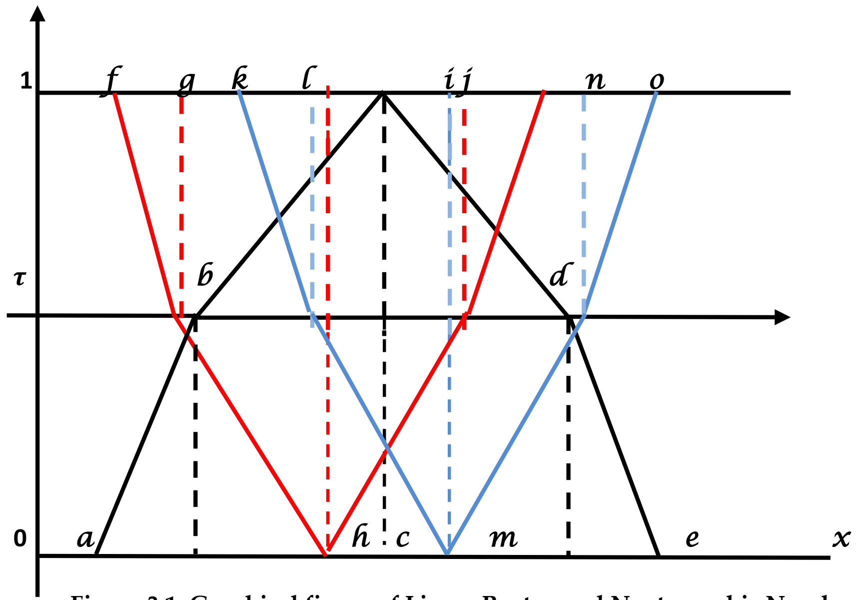


Figure 3.1: Graphical figure of Linear Pentagonal Neutrosophic Number.

Note 3.1 - Description of above figure: In this above figure we shall try to address the graphical representation of linear pentagonal neutrosophic number. The pentagonal shaped black marked line actually indicate the truthiness membership function, pentagonal shaped red marked line denotes the falseness membership function and pentagonal shaped blue marked line pointed the indeterminacy membership function of this corresponding number. Here, τ is a variable which follows the relation $0 \leq \tau \leq 1$. If $\tau = 0$ *or* 1 then the pentagonal number will be converted into triangular neutrosophic number.

3.2 Pentagonal Single Typed Neutrosophic Number of Specification 2: If the sections of Hesitation and Falsity functions are dependent to each other

A Pentagonal Single Typed Neutrosophic Number (PTGNEU) of specification 2 is described as $\tilde{A}_{PtgNeu} = (p_1, p_2, p_3, p_4, p_5; q_1, q_2, q_3, q_4, q_5; \theta_{PtgNeu}, \delta_{PtgNeu})$ whose truth membership; hesitation membership and falsity membership are specified as follows:

$$T_{\tilde{A}_{PtgNeu}}(x) = \begin{cases} \tau \dfrac{x - p_1}{p_2 - p_1} if p_1 \leq x \leq p_2 \\ 1 - (1 - \tau)\dfrac{x - p_2}{p_3 - p_2} if p_2 \leq x \leq p_3 \\ 1 \quad if x = p_3 \\ 1 - (1 - \tau)\dfrac{p_4 - x}{p_4 - p_3} if p_3 \leq x \leq p_4 \\ \tau \dfrac{p_5 - x}{p_5 - p_4} if p_4 \leq x \leq p_5 \\ 0 \quad Otherwise \end{cases}$$

$$I_{\tilde{A}_{PtgNeu}}(x)=\begin{cases}\dfrac{q_2-x+\theta_{PtgNeu}(x-q_1)}{q_2-q_1} & if\ q_1\le x<q_2\\ \dfrac{q_3-x+\theta_{PtgNeu}(x-q_2)}{q_3-q_2} & if q_2\le x<q_3\\ \theta_{PtgNeu} if x=q_3 & \\ \dfrac{x-q_3+\theta_{PtgNeu}(q_4-x)}{q_4-q_3} & if q_3\le x<q_4\\ \dfrac{x-q_4+\theta_{PtgNeu}(q_5-x)}{q_5-q_4} & if q_4\le x<q_5\\ 1 & Otherwise\end{cases}$$

and

$$F_{\tilde{A}_{PtgNeu}}(x)=\begin{cases}\dfrac{q_2-x+\delta_{PtgNeu}(x-q_1)}{q_2-q_1} & if\ q_1\le x<q_2\\ \dfrac{q_3-x+\delta_{PtgNeu}(x-q_2)}{q_3-q_2} & if q_2\le x<q_3\\ \delta_{PtgNeu} if x=q_3 & \\ \dfrac{x-q_3+\delta_{PtgNeu}(q_4-x)}{q_4-q_3} & if q_3\le x<q_4\\ \dfrac{x-q_4+\delta_{PtgNeu}(q_5-x)}{q_5-q_4} & if q_4\le x<q_5\\ 1 & Otherwise\end{cases}$$

where, $0\le\ T_{\tilde{A}_{PtgNeu}}(x)+I_{\tilde{A}_{PtgNeu}}(x)+F_{\tilde{A}_{PtgNeu}}(x)\le 2+, x\in\tilde{A}_{PtgNeu}$

The parametric form of the above type number is

$$\left(\tilde{A}_{PtgNeu}\right)_{\mu,\vartheta,\varphi}=\begin{bmatrix}T_{PtgNeu1L}(\mu),T_{PtgNeu2L}(\mu),T_{PtgNeu1R}(\mu),T_{PtgNeu2R}(\mu);\\ I_{PtgNeu1L}(\vartheta),I_{PtgNeu2L}(\vartheta),I_{PtgNeu1R}(\vartheta),I_{PtgNeu2R}(\vartheta);\\ F_{PtgNeu1L}(\varphi),F_{PtgNeu1L}(\varphi),F_{PtgNeu1L}(\varphi),F_{PtgNeu1L}(\varphi)\end{bmatrix}$$

$$T_{PtgNeu1L}(\mu)=p_1+\tfrac{\mu}{\tau}(p_2-p_1)\text{for}\mu\in[0,\tau],\ T_{PtgNeu2L}(\mu)=p_2+\tfrac{1-\mu}{1-\tau}(p_3-p_2)\text{for}\mu\in[\tau,1]$$

$$T_{PtgNeu2R}(\mu)=p_4-\tfrac{1-\mu}{1-\tau}(p_4-p_3)\text{for}\mu\in[\tau,1],T_{PtgNeu1R}(\mu)=p_5-\tfrac{\mu}{\tau}(p_5-p_4)\text{for}\mu\in[0,\tau]$$

$$I_{PtgNeu1L}(\vartheta)=\tfrac{q_2-\theta_{PtgNeu}q_1-\vartheta(q_2-q_1)}{1-\theta_{PtgNeu}}\text{for}\vartheta\in[\tau,1],I_{PtgNeu2L}(\vartheta)=\tfrac{q_3-\theta_{PtgNeu}q_2-\vartheta(q_3-q_2)}{1-\theta_{PtgNeu}}\text{for}\vartheta\in[0,\tau]$$

$$I_{PtgNeu2R}(\vartheta)=\tfrac{q_3-\theta_{PtgNeu}q_4+\vartheta(q_4-q_3)}{1-\theta_{PtgNeu}}\text{for}\vartheta\in[0,\tau],\ I_{PtgNeu2L}(\vartheta)=\tfrac{q_4-\theta_{PtgNeu}q_5+\vartheta(q_5-q_4)}{1-\theta_{PtgNeu}}\text{for}\vartheta\in[\tau,1]$$

$$F_{PtgNeu1L}(\varphi)=\tfrac{q_2-\delta_{PtgNeu}q_1-\varphi(q_2-q_1)}{1-\delta_{PtgNeu}}\text{for}\varphi\in[\tau,1],F_{PtgNeu2L}(\varphi)=\tfrac{q_3-\delta_{PtgNeu}q_2-\varphi(q_3-q_2)}{1-\delta_{PtgNeu}}\text{for}\varphi\in[0,\tau]$$

$$F_{PtgNeu2R}(\varphi)=\tfrac{q_3-\delta_{PtgNeu}q_4+\varphi(q_4-q_3)}{1-\delta_{PtgNeu}}\text{for}\varphi\in[0,\tau],\ F_{PtgNeu2L}(\varphi)=\tfrac{q_4-\delta_{PtgNeu}q_5+\varphi(q_5-q_4)}{1-\delta_{PtgNeu}}\text{for}\varphi\in[\tau,1]$$

Here, $0<\mu\le 1$, $\theta_{PtgNeu}<\vartheta\le 1$, $\delta_{PtgNeu}<\varphi\le 1$ and $-0<\vartheta+\varphi\le 1+$ and $-0<\mu+\vartheta+\varphi\le 2+$

4. Arithmetic Operations:

Suppose we consider two pentagonal neutrosophic fuzzy number as $\tilde{A}_{PtgNeu} = (p_1, p_2, p_3, p_4, p_5; \mu_a, \vartheta_a, \theta_a)$ and $\tilde{B}_{PtgNeu} = (q_1, q_2, q_3, q_4, q_5; \mu_b, \vartheta_b, \theta_b)$ then,

i) $\tilde{A}_{PtgNeu} + \tilde{B}_{PtgNeu} = [p_1 + q_1, p_2 + q_2, p_3 + q_3, p_4 + q_4, p_5 + q_5; \max\{\mu_a, \mu_b\}, \min\{\vartheta_a, \vartheta_b\}, \min\{\theta_a, \theta_b\}]$

ii) $\tilde{A}_{PtgNeu} - \tilde{B}_{PtgNeu} = [p_1 - q_5, p_2 - q_4, p_3 - q_3, p_4 - q_2, p_5 - q_1; \max\{\mu_a, \mu_b\}, \min\{\vartheta_a, \vartheta_b\}, \min\{\theta_a, \theta_b\}]$

iii) $k\tilde{A}_{PtgNeu} = [kp_1, kp_2, kp_3, kp_4, kp_5; \mu_a, \vartheta_a, \theta_a]\ if\ k > 0,\ = [kp_5, kp_4, kp_3, kp_2, kp_1; \mu_a, \vartheta_a, \theta_a]\ if\ k < 0$

iv)$\tilde{A}_{PtgNeu}{}^{-1} = (1/p_5, 1/p_4, 1/p_3, 1/p_2, 1/p_1; \mu_a, \vartheta_a, \theta_a)$

5. Proposed Score and Accuracy Function:

Score function and accuracy function of a pentagonal neutrosophic number is fully depend on the value of truth membership indicator degree, falsity membership indicator degree and hesitation membership indicator degree. The need of score and accuracy function is to compare or convert a pentagonal neutrosophic fuzzy number into a crisp number. In this section we will proposed a score function as follows.

For any Pentagonal Single typed Neutrosophic Number (PTGNEU)

$$\tilde{A}_{PtgNeu} = (p_1, p_2, p_3, p_4, p_5; q_1, q_2, q_3, q_4, q_5; r_1, r_2, r_3, r_4, r_5)$$

We consider the beneficiary degree of truth indicator part as = $\frac{p_1+p_2+p_3+p_4+p_5}{5}$

Non- beneficiary degree of falsity indicator part as =$\frac{r_1+r_2+r_3+r_4+r_5}{5}$.

And the hesitation degree of indeterminacy indicator as = $\frac{q_1+q_2+q_3+q_4+q_5}{5}$

Thus, we defined the score function as $SC_{PtgNeu} = \frac{1}{3}\left(2 + \frac{p_1+p_2+p_3+p_4+p_5}{5} - \frac{q_1+q_2+q_3+q_4+q_5}{5} - \frac{r_1+r_2+r_3+r_4+r_5}{5}\right)$,

Where, $SC_{PtgNeu} \in [0,1]$ and the Accuracy function is defined as $AC_{PtgNeu} = \left(\frac{p_1+p_2+p_3+p_4+p_5}{5} - \frac{r_1+r_2+r_3+r_4+r_5}{5}\right)$,

Where, $AC_{PtgNeu} \in [-1,1]$, Now we conclude that

If $A_{PtgNeu} =< (1,1,1,1,1; 0,0,0,0,0; 0,0,0,0,0) >$ then, $SC_{PtgNeu} = 1$ and $AC_{PtgNeu} = 1$

If $A_{PtgNeu} =< (0,0,0,0,0; 1,1,1,1,1; 1,1,1,1,1) >$ then, $SC_{PtgNeu} = 0$and $AC_{Sp_{neu}} = -1$

5.1 Relationship between any two pentagonal neutrosophic fuzzy numbers:

Let us consider any two pentagonal neutrosophic fuzzy number defined as follows

$A_{PtgNeu1} = (T_{PtgNeu1}, I_{PtgNeu1}, F_{PtgNeu1})$ and $A_{PtgNeu2} = (T_{PtgNeu2}, I_{PtgNeu2}, F_{PtgNeu2})$ if,

1) $SC_{A_{PtgNeu1}} > SC_{A_{PtgNeu2}}$, $then\ A_{PtgNeu1} > A_{PtgNeu2}$

2) $SC_{A_{PtgNeu1}} < SC_{A_{PtgNeu2}}$, $then\ A_{PtgNeu1} < A_{PtgNeu2}$

3) $SC_{A_{PtgNeu1}} = SC_{A_{PtgNeu2}}$, $then$

i)$AC_{A_{PtgNeu1}} > AC_{A_{PtgNeu2}}$, $then\ A_{PtgNeu1} > A_{PtgNeu2}$

$ii)$ $AC_{A_{PtgNeu1}} < AC_{A_{PtgNeu2}}$, $then\ A_{PtgNeu1} < A_{PtgNeu2}$

$iii)$ $AC_{A_{PtgNeu1}} = AC_{A_{PtgNeu2}}$, $then\ A_{PtgNeu1} \sim A_{PtgNeu2}$

6. Application in Neutrosophic Transportation Environment:

6.1 Mathematical Formulation of Model-I

In this section we consider a transportation problem in pentagonal neutrosophic environment where there are "p" sources and "q" destinations in which the decision makers need to choose a logical allotment such that it can start from "m"th source and it will went to "n"th section in such a way where the cost become the minimum once in presence of uncertainty, hesitation in transportation matrix. We also consider the available resources and required values are real number in nature.

Assumptions;

'm' is the source part for all m=1,2,3……p

'n' is destination part for all n=1,2,3…….q

$x_{mn}$is amount of portion product which can be transferred from m-th source to n-th destination.

$\widetilde{N_{mn}}$is the unit cost portion in neutrosophic nature which can be transferred from m-th source to n-th destination.

$u_m$is the total availability of the product at the source m.

$v_n$is the total requirement of the product at the source n.

Here, supply constraints $\sum_{n=0}^{q} x_{mn} = u_m$ for all m.

Demand constraints $\sum_{m=0}^{p} x_{mn} = \vartheta_n$ for all n.

Also, $\sum_{n=0}^{q} \vartheta_n = \sum_{m=0}^{p} u_m$,$x_{mn} \geq 0$

So, the mathematical formulation is, $Min\ Z = \sum_{n=0}^{q} \sum_{m=0}^{p} x_{mn} . \widetilde{N_{mn}}$, Subject to the constrain, $\sum_{n=0}^{q} x_{mn} = u_m$, $\sum_{m=0}^{p} x_{mn} = \vartheta_n$, Where, $x_{mn} \geq 0$ for all m, n.

Proposed Algorithm to find out the optimal solution of Model-I:

Step-1: Conversion of each pentagonal neutrosophic numbers into crisp using our proposed score functions and creates the transportation matrix into crisp system.

Step-2: Calculate the non-negative difference for each row and column between the smallest and next smallest elements row and column wise respectively.

Step-3: Take the highest difference and placed the availability or demand into the minimum allocated cell of the matrix. In case of tie in highest difference take any one arbitrarily.

Step-4: The process is going on unless and until the final optimal matrix is created. Lastly, check the number of allocated cells in the final matrix; it should be equal to row+column-1.

Step-5: Now, calculate the minimum total cost using the allocated cells.

Illustrative Example:

A company has three factories A, B, C which supplies some materials at D, E and F on monthly basis with pentagonal neutrosophic unit transportation cost whose capacities are 12,14,4 units respectively and the transportation matrix is defined as below and the requirements are 9, 10, 11 respectively. The problem is to find out the optimal solution and the minimum transportation cost.

	A	**B**	**C**	**Available**
D	<(10,15,20,25,30; 0,3,5,7,10; 0,1,2,3,4)>	<(1,1,1,1,1; 0,0,0,0,0; 0,0,0,0,0)>	<(10,20,30,40,50; 1,4,7,8,10; 0,1,1.5,2,2.5,3)>	12
E	<(2,3,4,7,9; 0,0.5,1,1.5,2; 0,0,0,0,0)>	<(5,10,15,20,25; 1,2,3,4,5; 1,1.5,2,2.5,3)>	<(0,0.5,1,1.5,2.5; 0,0.5,1,1.5,2; 0,0,0,0,0)>	14
F	<(5,9,11,12,13; 0,1,2,2.5,4.5;	<(10,15,20,25,30; 0,2,4,6,8;	<(15,20,25,30,50; 0,3,7,10,15;	4

	0,0.5,1,1.5,2)>	0,0,0,0,0)>	1,2,4,5,8)>	
Required	9	10	11	

Step-1

Table-1: We convert this pentagonal neutrosophic transportation problem in to a crisp model using the concept of score and accuracy function.

	A	B	C	u_m
D	**5**	**1**	**8**	**12**
E	**2**	**4**	**0**	**14**
F	**3**	**6**	**7**	**4**
v_n	**9**	**10**	**11**	

Step-2

Table-2:

	A	B	C	u_m(Penalty)
D	**5**	**1**	**8**	**12(4)**
E	**2**	**4**	**0**	**14(2)**
F	**3**	**6**	**7**	**4(3)**
v_n **(Pen-alty)**	**9 (1)**	**10 (3)**	**11 (7)**	

Step-3

Table-3: After processing the same operations finite number of times finally we get the final optimal solution matrix as, here $number\ of\ allocation = row + column - 1 = 5$

	A	B	C	u_i
D	$\boxed{2}$ **5**	$\boxed{10}$ **1**	**8**	**12**
E	$\boxed{3}$ **2**	**4**	$\boxed{11}$ **0**	**14**
F	$\boxed{4}$ **3**	**6**	**7**	**4**
v_j	**9**	**10**	**11**	

Thus, the total cost of this transportation problem is $Min\ Z = \sum_{n=0}^{3}\sum_{m=0}^{3} x_{mn} \cdot \widetilde{N_{mn}}$

=2*(10,15,20,25,30;0,3,5,7,10;0,1,2,3,4)+3*(1,2,3,6,8;0,0.5,1,1.5,2;-3,-2,1,0,1)+

4*(1,3,5,7,9;-5,-4,0,1,3;-5,-3,0,1,2)+10*(1,1,1,1,1;0,0,0,0,0;0,0,0,0,0)+11*(1,0,1,2,3;

0, 0.5, 1, 1.5, 2; 0, 0, 0, 0, 0)

$= <(26,58,90,128,163; -20,-3,24,39,60; -20,-16,1,10,19)>$

= 25.4 Units.

6.2 Mathematical Formulation of Model-II

Mathematical formulation is, $Min\ \check{Z} = \sum_{n=0}^{q} \sum_{m=0}^{p} \widetilde{x_{mn}} \cdot \widetilde{N_{mn}}$

Subject to the constrain, $\sum_{n=0}^{q} \widetilde{x_{mn}} = \widetilde{u_m}$

$$\sum_{m=0}^{p} \widetilde{x_{mn}} = \widetilde{\vartheta_n}$$

Also, $\sum_{n=0}^{q} \widetilde{\vartheta_n} = \sum_{m=0}^{p} \widetilde{u_m}$, Where, $\widetilde{x_{mn}} \geq 0$ for all m, n.

Here $\widetilde{N_{mn}}, \widetilde{u_m}, \widetilde{\vartheta_n}$ are all pentagonal neutrosophic numbers.

In formulation of Model-II with the help of pentagonal neutrosophic number cost, demand and supply formulated in the following table 1,First, we calculate the score value of individual neutrosophic cost to get crisp cost and consider the rest terms that is demand and supply as it is in neutrosophic nature. Now, for the allocation we first consider the score values of availability and demand and take the minimum value for the allocation. Then, we use the arithmetic operations in pentagonal neutrosophic domain to run the iteration process. Following the same above algorithm finally we get the optimal solution table where number of allocation=row+column-1 and finally we need to compute the final cost.

Table-1:

	A	B	C	Available
D	<(10,15,20,25,30; 0,3,5,7,10; 0,1,2,3,4)>	<(1,1,1,1,1; 0,0,0,0,0; 0,0,0,0,0)>	<(10,20,30,40,50; 1,4,7,8,10; 0,1,1.5,2,2.5,3)>	<(20,30,40,50,60; 3,5,6,10,12; 5,10,15,20,25)>
E	<(2,3,4,7,9; 0,0.5,1,1.5,2; 0,0,0,0,0)>	<(5,10,15,20,25; 1,2,3,4,5; 1,1.5,2,2.5,3)>	<(0,0.5,1,1.5,2.5; 0,0.5,1,1.5,2; 0,0,0,0,0)>	<(15,20,25,30,35; 5,10,15,20,30; 2,4,6,8,10)>
F	<(5,9,11,12,13; 0,1,2,2.5,4.5; 0,0.5,1,1.5,2)>	<(10,15,20,25,30; 0,2,4,6,8; 0,0,0,0,0)>	<(15,20,25,30,50; 0,3,7,10,15; 1,2,4,5,8)>	<(10,20,30,40,50; 4,6,8,10,12; 1,4,7,10,13)>
Required	<(30,40,50,60,70; 4,8,11,17,26; 4,8,12,16,20)>	<(10,20,30,40,50; 4,6,8,10,12; 3,6,9,12,15)>	<(5,10,15,20,25; 4,7,10,13,16; 1,4,7,10,13)>	

Step-1

Table-2:

	A	B	C	u_m
D	**5**	**1**	**8**	<(20,30,40,50,60; 3,5,6,10,12; 5,10,15,20,25)>
E	**2**	**4**	**0**	<(15,20,25,30,35; 5,10,15,20,30;

				2,4,6,8,10)>
F	**3**	**6**	**7**	<(10,20,30,40,50; 4,6,8,10,12; 1,4,7,10,13)>
v_n	<(30,40,50,60,70; 4,8,11,17,26; 4,8,12,16,20)>	<(10,20,30,40,50; 4,6,8,10,12; 3,6,9,12,15)>	<(5,10,15,20,25; 4,7,10,13,16; 1,4,7,10,13)>	

After the iteration process according to the proposed algorithm finally we get the allocations in the allocated cell as,

$$a_{11} =< (-30,-10,10,30,40;\ -9,-5,-2,4,8; -10,-2,6,14,22) >$$

$$a_{21} =< (-10,0,10,20,30;\ -11,-3,5,13,26; -11,-6,-1,4,9) >$$

$$a_{12} =< (10,20,30,40,50; 4,6,8,10,12; 3,6,9,12,15) >$$

$$a_{23} =< (5,10,15,20,25; 4,7,10,13,16; 1,4,7,10,13) >$$

$$a_{31} =< (-40,-10,30,70,110; -30,-9,8,25,46;\ -27,-10,7,24,41) >$$

Thus, the optimal solution of this model-II system is, $Min\ \check{Z} = \sum_{n=0}^{3}\sum_{m=0}^{3} \widetilde{x_{mn}} \cdot \widetilde{N_{mn}}$

$$= < (-30,-10,10,30,40;\ -9,-5,-2,4,8; -10,-2,6,14,22) > \times < (10,15,20,25,30; 0,3,5,7,10; 0,1,2,3,4) > +$$

$$<(-10,0,10,20,30; -11,-3,5,13,26;-11,-6,-1,4,9)>\times<(2,3,4,7,9;0,0.5,1,1.5,2;0,0,0,0,0)> +$$

$$< (10,20,30,40,50; 4,6,8,10,12; 3,6,9,12,15) >\times<(1,1,1,1,1;0,0,0,0,0;0,0,0,0,0)> +$$

$$< (5,10,15,20,25; 4,7,10,13,16; 1,4,7,10,13) >\times<(0,0.5,1,1.5,2.5;0,0.5,1,1.5,2;0,0,0,0,0)> +$$

$$<(-40,-10,30,70,110;-30,-9,8,25,46; -27,-10,7,24,41)>\times<(5,9,11,12,13;0,1,2,2.5,4.5;0,0.5,1,1.5,2)>$$

$$= < (-510,-215, 615,1800,3012.5;\ 0,-22,21,129.5,371;\ 0,-7,19,78,170) >$$

=263.53 units.

6.3 Discussion: In section 6.1, in model -I we observe that if we take pentagonal neutrosophic fuzzy number as a member of feasible solution then we get the $Min\ Z = 25.4$ units, whereas, if we take crisp number in this computation procedure then we get from table 3, Min Z= (2×5) + (3×2) + (4× 3) + $(10 \times 1) = (11 \times 0) = 38\ units$. Thus we can observe that pentagonal neutrosophic number give us better results. So, we follow the same technique in section 6.2 where we consider both availability and the demand as a pentagonal neutrosophic number.

The conception of pentagonal neutrosophic number is totally a new idea and till now, in this domain anyone doesn't considered the transportation problem so fur. Thus in future study, we can compare our work with the other established methods. Also, we can do comparative analysis in pentagonal neutrosophic arena whenever researchers from different section could develop some interesting and useful algorithm in this transportation domain.

7. Conclusion

In this current era, the conception of neutrosophic number plays a paramount role in different fields of research domain.There is a proliferating popularity for the conundrum concept of neutrosophic number presenting before the world a vibrant spice of logic and innovation to reach the zenith of excellence. The world is driven into a paradigm of brilliance as well as expertise with the formation of the corresponding number which assists the researcher dealing with uncertainty and also with the transportation problem.

Neutrosophic set conception is a generalization of intuitionistic fuzzy set which actually contains truthiness, falseness and indeterminacy concept. In this article, we developed a new concept of pentagonal neutrosophic fuzzy number, introduced its graphical representation and its properties. We also invented logical score and accuracy function which has a strong impact in conversion and ranking in this domain of research. Transportation problem is a very important application in operation research domain and we build up two different models in this article within neutrosophic environment. We also employed the arithmetic operations to find the solution which gives us better result than the general conception. Thus, it can be concluded that the approach for taking the pentagonal neutrosophic single-valued number is very helpful for the researchers who are involved in dealing the mathematical modelling with impreciseness in various fields of sciences and engineering. It reveals very realistic results in both mathematical points of view. There is still a massive amount of work in this field; hence much spectacular study can be explored with pentagonal neutrosophic parameters. Further, we can compare our research work with other established methods in pentagonal neutrosophic domain related with transportation problem.

In future, this article can be extended into multi criteria decision making problem. Also, researchers can apply this conception in various fields like engineering problem, pattern recognition problem, mathematical modeling etc.

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On MBJ – Neutrosophic β – Subalgebra

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Abstract: This paper studies about the definition of MBJ – Neutrosophic set in $\boldsymbol{\beta}$ – algebra, and introduce the concept of MBJ – Neutrosophic $\boldsymbol{\beta}$ – subalgebra. Homomorphic image and inverse image of MBJ – Neutrosophic $\boldsymbol{\beta}$ – subalgebra is provided. Also, Cartesian product of MBJ – Neutrosophic $\boldsymbol{\beta}$ – subalgebra is studied.

Keywords: MBJ–Neutrosophic set; MBJ–Neutrosophic β–subalgebra; MBJ-Neutrosophic Cartesian Product.

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1 Introduction

Zadeh [35, 36] introduced the fuzzy set to discuss uncertainty in many real requitals and as a generalization, the intuitionistic fuzzy set on an universe X was brought by Atanassov [8, 9]. The concept of Neutrosophic set is given by Smarandache [28, 29] with truth, indeterminate and false membership function and is explored to various dimensions by the authors of [10,16,17,18,32]. M. A. Basset et.al [1, 2, 3, 4, 5, 6] studies various topics in Neutrosophic set and its applications. As an extension the idea of MBJ – Neutrosophic structures was introduced in [34] where the BCK/BCI – algebra deals about a single binary operation (*).

The fuzzy sets have been connected in algebraic structure begins from Rosenfeld [27]. BCK – algebra is introduced by Iseki and Tanaka [8] and it has been analysed with several branches of fuzzy settings. As a generalization of BCK – algebra, Huang [11] and Iseki [14] discussed the notion of BCI – algebra. The structure of β – algebra was introduced by Neggers and Kim [25]. Also Jun and Kim [19] dealt some related topics on β – subalgebra. Later many researchers [7, 12, 33] developed to study β – algebra by relating with different fuzzy concepts. And as generalization of that, this paper applies the MBJ – Neutrosophic set in β–algebra and some results are given. The major difference of this work is handling an algebra with binary two operations (+ and −) whereas the existing other works involved single operation. This paper also provides a homomorphic image and pre-image of MBJ – Neutrosophic β – subalgebra and the cartesian product of MBJ – Neutrosophic β – subalgebra are also disputed.

2 Preliminaries

This part provides the essential definition and examples of β – algebra and some definitions of fuzzy sets.

2.1 Definition [7] A β – algebra is a non-empty set X with a constant 0 and binary operations + and – satisfying the following axioms:

i) $x - 0 = x$

ii) $(0 - x) + x = 0$

iii) $(x - y) - z = x - (y + z)$, for all $x, y, z \in X$.

2.2 Example Let X = {0, 1, 2, 3} be a set with constant 0 and two binary operations + and – are defined on X with the Cayley's table, then (X, +,-, 0) is a β – algebra.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

-	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

2.3 Definition [7] A non – empty subset S of a β – algebra (X, +, -, 0) is called a β – subalgebra of X, if
i) $x + y \in S$
ii) $x - y \in S, \forall\, x, y \in S$.

2.4 Example [33] Let X = {(0, 1, 2, 3),+,-,0} be a β – algebra with Cayley's table given above. Consider I_1 = {0, 2} and I_2 = {0, 1}. Then I_1 is a β – subalgebra of X, whereas I_2 does not satisfy the conditions to be an a β – subalgebra of X.

2.5 Definition [33] Let (X , +, -, 0_x) and (Y, +, -, 0_y) are β – algebras. A mapping $f : X \to Y$ is said to be a β – homomorphism if
i) $f(x + y) = f(x) + f(y)$
ii) $f(x - y) = f(x) - f(y), \forall\, x, y \in X$.

2.6 Definition A fuzzy set in a universal set X is defined as $\mu : X \to [0,1]$. For each $x \in X$, $\mu(x)$ is called the membership value of x.
2.7 Definition [9] An Intuitionistic fuzzy set in a non – empty set X is defined by $A = \{ < x, \mu(x), \nu(x) > / \ x \in X\}$ where $\mu_A : X \to [0,1]$ is a membership function of A and $\nu_A : X \to [0,1]$ is a non – membership function of A satisfying $0 \le \mu_A(x) + \nu_A(x) \le 1$, $\forall x \in X$.

2.8 Definition [12] An Interval valued fuzzy set on X is defined by $A = \{ (x, \bar{\mu}_A(x))\}, \forall x \in X$ where $\bar{\mu}_A : X \to D[0,1]$ and $D[0,1]$ denotes the family of all closed subintervals of [0,1]. Here $\bar{\mu}_A(x) = [\, {\mu_A}^L(x), {\mu_A}^U(x)], \forall x \in X$ and ${\mu_A}^L, {\mu_A}^U$ are fuzzy sets.

Remark: Let us define refined minimum (briefly, $rmin$) and refined maximum (briefly, $rmax$) of two elements in $D[\,0,1]$. We also define the symbols $\ge, \le, =$ in case of two elements in $D[\,0,1]$. Consider $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2] \in D[0,1]$ then $rmin\ (D_1, D_2) = [\min(a_1, a_2)\,, \min(b_1, b_2)]$, $rmax\ (D_1, D_2) = [\max(a_1, a_2)\,, \max(b_1, b_2)]$ $D_1 \ge D_2$ if and only if $a_1 \ge a_2$, $b_1 \ge b_2$. Likewise, $D_1 \le D_2$ and $D_1 = D_2$. For $D_i = [a_i, b_i] \in D[\,0,1]$, for i = 1,2,3…
We define $rsup_i(D_i) = [sup_i(a_i)\,, sup_i(b_i)]$ and $rinf_i(D_i) = [inf_i(a_i)\,, inf_i(b_i)]$.
Now, $D_1 \ge D_2$ if and only if $a_1 \ge a_2$, $b_1 \ge b_2$. Similarly, $D_1 \le D_2$ and $D_1 = D_2$.

2.9 Definition [8] An Interval valued Intuitionistic fuzzy set A on X is defined by $A = \{ < x, \bar{\mu}(x), \bar{\nu}(x) > / \ x \in X\}$. Here $\bar{\mu}_A : X \to D[0,1]$ and $\bar{\nu}_A : X \to D[0,1]$ and $D[0,1]$ is denoted as the set of all subintervals of [0,1].
Here $\bar{\mu}_A(x) = [\, {\mu_A}^L(x), {\mu_A}^U(x)]$, $\bar{\nu}_A(x) = [\, {\nu_A}^L(x), {\nu_A}^U(x)]$ with the condition $0 \le {\mu_A}^L(x) + {\nu_A}^L(x) \le 1$ and $0 \le {\mu_A}^U(x) + {\nu_A}^U(x) \le 1$.

2.10 Definition [28, 29] An Neutrosophic fuzzy set A on X is defined by
$A = \{ < x,\ A_T(x),\ A_I(x),\ A_F(x) > / \ x \in X\}$, where $A_T : X \to [\,0,1]$ is a truth membership function, A_I :

$X \to [0,1]$ is an indeterminate membership function and $A_F : X \to [0,1]$ is a false membership function.

2.11 Definition [34] Let X be a non – empty set. MBJ – Neutrosophic set in X, is a structure of the form $A = \{< x,\ M_A(x),\ \tilde{B}_A(x),\ J_A(x) > / \ x \in X\}$ where M_A and J_A are fuzzy sets in X and M_A is a truth membership function, J_A is a false membership function and $\tilde{B}_A$ is an interval valued fuzzy set in X and is an Indeterminate Interval Valued membership function.

2.12 Definition [12] the supremum property of the fuzzy set μ for the subset T in X is defined as $\mu(x_0) = \sup_{x \in T} \mu(x)$, if there exists $x, x_0 \in T$.

2.13 Definition [33] An Intuitionistic fuzzy set A with the degree membership $\mu_A : X \to [0,1]$ and the degree of non – membership function $\nu_A : X \to [0,1]$ is said to have $sup - inf$ property if for any subset T of X there exists $x_0 \in T$ such that $\mu_A(x_0) = \sup_{x \in T} \mu_A(x)$ and $\nu_A(x_0) = \inf_{x \in T} \nu_A(x)$

2.14 Definition

An Interval valued intuitionistic fuzzy set A in any set X is said to have the $rsup - rinf$ property if for subset T of X there exists $x_0 \in T$ such that $\bar{\mu}_A(x_0) = \operatorname{rsup}_{x \in T} \bar{\mu}_A(x)$ and $\bar{\nu}_A(x_0) = \operatorname{rinf}_{x \in T} \bar{\nu}_A(x)$ respectively.

In fuzzy theory, subsets are assumed to satisfy sup property, in intuitionistic fuzzy theory subsets are assumed to satisfy $sup - inf$ property and in interval valued intuitionistic fuzzy subsets are assumed to satisfy $rsup - rinf$ property. Analogously, in the following we define the notion of $sup - rsup - inf$ for an MBJ – Neutrosophic set.

2.15 Definition

An MBJ – Neutrosophic fuzzy set A in any set X is said to have the $sup - rsup - inf$ property if for subset T of X there exists $x_0 \in T$ such that $M_A(x_0) = \sup_{x \in T} M_A(x)$, $\tilde{B}_A(x_0) = \operatorname{rsup}_{x \in T} \bar{B}_A(x)$ and $J_A(x_0) = \inf_{x \in T} J_A(x)$ respectively.

3 MBJ – Neutrosophic Structures in β – Subalgebra

This division frames the structure of MBJ – Neutrosophic β – subalgebra of β – algebra and some relevant results are discussed.

3.1 Definition

Let X be a β – algebra. An MBJ – Neutrosophic set $A = (M_A, \tilde{B}_A, J_A)$ in X is called an MBJ – Neutrosophic β – subalgebra of X if it satisfies:

i) $M_A(x + y) \geq \min(M_A(x), M_A(y))$; and ii) $M_A(x - y) \geq \min(M_A(x), M_A(y))$;
$\tilde{B}_A(x + y) \geq \operatorname{rmin}(\tilde{B}_A(x), \tilde{B}_A(y))$; $\tilde{B}_A(x - y) \geq \operatorname{rmin}(\tilde{B}_A(x), \tilde{B}_A(y))$;
$J_A(x + y) \leq \max(J_A(x), J_A(y))$ $J_A(x - y) \leq \max(J_A(x), J_A(y))$

3.2 Example

1) Consider a β – algebra $X = (\{0,1,2\}, +, -)$ by the following cayley's table

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

-	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

and the MBJ – Neutrosophic set on X is defined by

$$M_A(x) = \begin{cases} 0.4 & , x = 0 \\ 0.3 & , otherwise \end{cases} \qquad \tilde{B}_A(x) = \begin{cases} [0.3, 0.8] & , x = 0 \\ [0.1, 0.5] & , otherwise \end{cases}$$

$J_A(x) = \begin{cases} 0.1 & , x = 0 \\ 0.3 & , otherwise \end{cases}$

Thus, A satisfy the terms to be an MBJ - Neutrosophic β - subalgebra of X.

2) Let $X = \{ (0, a, b, c), +, -\}$ be a β -algebra with the following cayley's table.

+	0	a	b	c
0	0	a	b	c
a	a	b	c	0
b	b	c	0	a
c	c	0	a	b

-	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Here, the MBJ – Neutrosophic set $A = \{ < x,\ M_A(x), \tilde{B}_A(x), J_A(x))>/\ x \in X\}$ on X is defined by

$M_A(x) = \begin{cases} 0.8, & x = 0 \\ 0.5, & x = b \\ 0.3, & x = a, c \end{cases}$ $\tilde{B}_A(x) = \begin{cases} [0.4,0.7], & x = 0 \\ [0.3,0.5], & x = b \\ [0.1,0.2], & x = a, c \end{cases}$

$J_A(x) = \begin{cases} 0.2, & x = 0 \\ 0.5, & x = b \\ 0.7, & x = a, c \end{cases}$ is an MBJ Neutrosophic β - subalgebra of X.

3.3 Theorem

If A_1 and A_2 are two MBJ Neutrosophic β - subalgebras of X, then $A_1 \cap A_2$ is an MBJ – Neutrosophic β - subalgebra of X.

Proof:

Let A_1 and A_2 be two MBJ – Neutrosophic β - subalgebra of X.

Now, $M_{A_1 \cap A_2}(x + y) = \min\{ M_{A_1}(x + y), M_{A_2}(x + y)\}$

$\geq \min\{\{M_{A_1}(x), M_{A_1}(y)\}, \min\{M_{A_2}(x), M_{A_2}(y)\}\}$

$= \min\{\{M_{A_1}(x), M_{A_2}(x)\}, \{M_{A_1}(y), M_{A_2}(y)\}\}$

$\geq \min\{M_{A_1 \cap A_2}(x), M_{A_1 \cap A_2}(y)\}$

$M_{A_1 \cap A_2}(x + y) \geq \min\{M_{A_1 \cap A_2}(x), M_{A_1 \cap A_2}(y)\}$.

Similarly, $M_{A_1 \cap A_2}(x - y) \geq \min\{M_{A_1 \cap A_2}(x), M_{A_1 \cap A_2}(y)\}$.

$\tilde{B}_{A_1 \cap A_2}(x + y) = [B^L_{A_1 \cap A_2}(x + y), B^U_{A_1 \cap A_2}(x + y)]$

$= [\min\left(B^L_{A_1}(x + y), B^L_{A_2}(x + y)\right), \min\left(B^U_{A_1}(x + y), B^U_{A_2}(x + y)\right)]$

$\geq [\min(B^L_{A_1 \cap A_2}(x), B^L_{A_1 \cap A_2}(y)), \min(B^U_{A_1 \cap A_2}(x), B^U_{A_1 \cap A_2}(y))]$

$= rmin\{\tilde{B}_{A_1 \cap A_2}(x), \tilde{B}_{A_1 \cap A_2}(y)\}$

$\tilde{B}_{A_1 \cap A_2}(x + y) \geq rmin\{\tilde{B}_{A_1 \cap A_2}(x), \tilde{B}_{A_1 \cap A_2}(y)\}$

Similarly, $\tilde{B}_{A_1 \cap A_2}(x - y) \geq rmin\{\tilde{B}_{A_1 \cap A_2}(x), \tilde{B}_{A_1 \cap A_2}(y)\}$

$J_{A_1 \cap A_2}(x + y) = \max\{ J_{A_1}(x + y), J_{A_2}(x + y)\}$

$\leq \max\{\{J_{A_1}(x), J_{A_1}(y)\}, \max\{J_{A_2}(x), J_{A_2}(y)\}\}$

$= \max\{\{J_{A_1}(x), J_{A_2}(x)\}, \{J_{A_1}(y), J_{A_2}(y)\}\}$

$\leq \max\{J_{A_1 \cap A_2}(x), J_{A_1 \cap A_2}(y)\}$

$J_{A_1 \cap A_2}(x + y) \leq \max\{J_{A_1 \cap A_2}(x), J_{A_1 \cap A_2}(y)\}$.

Similarly, $J_{A_1 \cap A_2}(x - y) \leq \max\{J_{A_1 \cap A_2}(x), J_{A_1 \cap A_2}(y)\}$.

Thus, $A_1 \cap A_2$ is an MBJ – Neutrosophic β - subalgebra of X.

3.4 Lemma

Let A be an MBJ – Neutrosophic β - subalgebra of X, then

$i)\ M_A(0) \geq M_A(x), \tilde{B}_A(0) \geq \tilde{B}_A(x)\ and\ J_A(0) \leq J_A(x),$

$ii) M_A(0) \geq M_A(x^*) \geq M_A(x), \tilde{B}_A(0) \geq \tilde{B}_A(x^*) \geq \tilde{B}_A(x)\ and\ J_A(0) \leq J_A(x^*) \leq J_A(x)$, where $x^* = 0 - x$, $\forall x \in X$.

Proof:

i) For any $x \in X$.

$M_A(0) = M_A(x-x) \geq \min(M_A(x), M_A(x))$

$= M_A(x)$

Therefore, $M_A(0) \geq M_A(x)$.

$\tilde{B}_A(0) = [B_A^L(0), B_A^U(0)]$

$\geq [B_A^L(x), B_A^U(x)]$

$= \tilde{B}_A(x)$

$J_A(0) = J_A(x-x) \leq \max(J_A(x), J_A(x)) = J_A(x)$

Thus, $J_A(0) \leq J_A(x)$.

ii) Also for $x \in X$,

$M_A(x^*) = M_A(0-x) \geq \min(M_A(0), M_A(x))$

$= M_A(x)$

$Hence, M_A(x^*) \geq M_A(x)$.

$\tilde{B}_A(x^*) = [B_A^L(x^*), B_A^U(x^*)]$

$= [B_A^L(0-x), B_A^U(0-x)]$

$= [\min(B_A^L(0), B_A^U(x)), \min(B_A^L(0), B_A^U(x))]$

$\geq [B_A^L(x), B_A^U(x)]$

$= \tilde{B}_A(x)$

$\therefore \tilde{B}_A(0) \geq \tilde{B}_A(x^*) \geq \tilde{B}_A(x)$

$J_A(x^*) = J_A(0-x) \leq \max(J_A(0), J_A(x)) = J_A(x)$

Thus, $J_A(0) \leq J_A(x^*) \leq J_A(x)$.

3.5 Theorem

If there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} M_A(x_n) = 1$, $\lim_{n\to\infty} \tilde{B}_A(x_n) = [1,1]$, $\lim_{n\to\infty} J_A(x_n) = 0$. And A be an MBJ – Neutrosophic β - subalgebra of X. Then

$M_A(0) = 1, \tilde{B}_A(0) = [1,1], and\ J_A(0) = 0$.

Proof:

Since, $M_A(0) \geq M_A(x), \forall x \in X$,

$M_A(0) \geq M_A(x_n)$.

Similarly, $\tilde{B}_A(0) \geq \tilde{B}_A(x_n)$ and $J_A(0) \leq J_A(x_n)$ for every positive integer n.

Note that, $1 \geq M_A(0) \geq \lim_{n\to\infty} M_A(x_n) = 1$,.

Hence $M_A(0) = 1$.

$[1,1] \geq \tilde{B}_A(0) \geq \lim_{n\to\infty} \tilde{B}_A(x_n) = [1,1]$

Implies $\tilde{B}_A(0) = [1,1]$

Also $0 \leq J_A(0) \leq \lim_{n\to\infty} J_A(x_n) = 0$.

Therefore, $J_A(0) = 0$.

3.6 Theorem

Given $A = (M_A, \tilde{B}_A, J_A)$ in X such that (M_A, J_A) is an intuitionistic fuzzy subalgebra of X and B_A^L, B_A^U are fuzzy subalgebra of X, then $A = (M_A, \tilde{B}_A, J_A)$ is an MBJ – Neutrosophic β - subalgebra of X.

Proof:

To prove this it's enough to verify that $\tilde{B}_A$ satisfies the conditions:

$\forall x, y \in X$.

$\tilde{B}_A(x+y) \geq rmin\{\tilde{B}_A(x), \tilde{B}_A(y)\}$

$\tilde{B}_A(x-y) \geq rmin\{\tilde{B}_A(x), \tilde{B}_A(y)\}$

For any $x, y \in X$, we get

$\tilde{B}_A(x+y) = [B_A^L(x+y), B_A^U(x+y)]$

$\geq [\min\{B_A^L(x), B_A^L(y)\}, \min\{B_A^U(x), B_A^U(y)\}]$

$= rmin\{[B_A^L(x), B_A^U(x)], [B_A^L(y), B_A^U(y)]\}$

$= rmin\{\tilde{B}_A(x), \tilde{B}_A(y)\}$

$\tilde{B}_A(x+y) \geq rmin\{\tilde{B}_A(x), \tilde{B}_A(y)\}$

Similarly, $\tilde{B}_A(x-y) \geq rmin\{\tilde{B}_A(x), \tilde{B}_A(y)\}$

$\tilde{B}_A$ satisfies the condition

$\therefore\ A = (M_A, \tilde{B}_A, J_A)$ is an MBJ – Neutrosophic β - subalgebra of X.

3.7 Theorem

If $A = (M_A, \tilde{B}_A, J_A)$ is an MBJ - Neutrosophic β - subalgebra of X. Then the sets $X_{M_A} = \{ x \in X / M_A(x) = M_A(0)\}$; $X_{\tilde{B}_A} = \{ x \in X/\tilde{B}_A(x) = \tilde{B}_A(0)\}$ and $X_{J_A} = \{ x \in X/J_A(x) = J_A(0)\}$ are subalgebra of X.

Proof:

For any $x, y \in X_{M_A}$.

Then $M_A(x) = M_A(0) = M_A(y)$

$M_A(x+y) \geq \min(M_A(x), M_A(y))$

$= \min(M_A(0), M_A(0)) = M_A(0)$

And $M_A(x-y) \geq \min(M_A(x), M_A(y))$

$= \min(M_A(0), M_A(0)) = M_A(0)$

$x + y \text{ and } x - y \in X_{M_A}$

Therefore, X_{M_A} is a subalgebra of X.

Let $x, y \in X_{\tilde{B}_A}$, then $\tilde{B}_A(x) = \tilde{B}_A(0) = \tilde{B}_A(y)$.

Now, $\tilde{B}_A(x+y) \geq rmin\{\tilde{B}_A(x), \tilde{B}_A(y)\}$

$= rmin\{\tilde{B}_A(0), \tilde{B}_A(0)\} = \tilde{B}_A(0)$

$\therefore \tilde{B}_A(x+y) \geq \tilde{B}_A(0)$

Similarly, $\tilde{B}_A(x-y) \geq \tilde{B}_A(0)$

$\therefore X_{\tilde{B}_A}$ is a subalgebra of X.

Let $x, y \in X_{J_A}$

$J_A(x) = J_A(0) = J_A(y)$

Now, $J_A(x+y) \leq \max(J_A(x), J_A(y))$

$= \max(J_A(0), J_A(0))$

$= J_A(0)$

$J_A(x-y) \leq \max(J_A(x), J_A(y))$

$= \max(J_A(0), J_A(0))$

$= J_A(0)$

$\therefore\ x + y \text{ and } x - y \in X_{J_A}$

X_{J_A} is a subalgebra of X.

3.8 Definition

$A = \{< x,\ M_A(x), \tilde{B}_A(x), J_A(x) > / \ x \in X\}$ be an MBJ – Neutrosophic set in X and f be mapping from X into Y then the image of A under f, $f(A)$ is defined as,

$f(A) = \{ < x, f_{sup}(M_A), f_{rsup}(\tilde{B}_A), f_{inf}(J_A) > / \ x \in Y \}$ where

$$f_{sup}(M_A)(y) = \begin{cases} \sup\limits_{x\in f^{-1}(y)} M_A(x), & if\ f^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$$

$$f_{rsup}(\tilde{B}_A)(y) = \begin{cases} \operatorname{rsup}\limits_{x\in f^{-1}(y)} \tilde{B}_A(x), & if\ f^{-1}(y) \neq \emptyset \\ [1,1] & otherwise \end{cases}$$

$$f_{inf}(J_A)(y) = \begin{cases} \inf\limits_{x\in f^{-1}(y)} J_A(x), & if\ f^{-1}(y) \neq \emptyset \\ 1 & otherwise \end{cases}$$

3.9 Definition [34]

Let $f : X \rightarrow Y$ be a function. Let A and B be the two MBJ – Neutrosophic β- subalgebra in X and Y respectively. Then inverse image of B under f is defined by

$f^{-1}(B) = \{x, f^{-1}(M_B(x)), f^{-1}\left(\tilde{B}_B(x)\right), f^{-1}(J_B(x)) \ / \ x \in X\}$ such that

$f^{-1}(M_B(x)) = M_B(f(x))$; $f^{-1}\left(\tilde{B}_B(x)\right) = \tilde{B}_B(f(x))$ and $f^{-1}(J_B(x)) = J_B(f(x))$.

3.10 Theorem

Let $(X,+,-,0)$ and $(Y,+,-,0)$ be two β – algebras and $f: X \to Y$ be an homomorphism. If A is an MBJ – Neutrosophic β – subalgebra of X, define
$f(A) = \{< x, M_f(x) = M(f(x)), \tilde{B}_f(x) = \tilde{B}(f(x)), J_f(x) = J(f(x)) >/ x \in X\}$. Then $f(A)$ is an MBJ – Neutrosophic β – subalgebra of Y.

Proof:

Let $x, y \in X$.

Now, $M_f(x+y) = M(f(x+y))$
$= M(f(x) + f(y))$
$\geq \min\{M(f(x)), M(f(y))\}$
$= \min\{M_f(x), M_f(y)\}$

$M_f(x+y) \geq \min\{M_f(x), M_f(y)\}$

Similarly, $M_f(x-y) \geq \min\{M_f(x), M_f(y)\}$

$\tilde{B}_f(x+y) = \tilde{B}(f(x+y))$
$= \tilde{B}(f(x) + f(y))$
$\geq \text{rmin}\{\tilde{B}(f(x)), \tilde{B}(f(y))\}$
$= \text{rmin}\{\tilde{B}_f(x), \tilde{B}_f(y)\}$

$\tilde{B}_f(x+y) \geq \text{rmin}\{\tilde{B}_f(x), \tilde{B}_f(y)\}$

Similarly, $\tilde{B}_f(x-y) \geq \text{rmin}\{\tilde{B}_f(x), \tilde{B}_f(y)\}$

$J_f(x+y) = J(f(x+y)) = J(f(x) + f(y))$
$\leq max\{J(f(x)), J(f(y))\}$
$= max\{J(f(x)), J(f(y))\}$

$J_f(x+y) \leq \max\{J_f(x), J_f(y)\}$

Similarly, $J_f(x-y) \leq \max\{J_f(x), J_f(y)\}$

Hence $f(A)$ is an MBJ – Neutrosophic β – subalgebra of Y.

3.11 Theorem

Let $f : X \to Y$ be a homomorphism of β – algebra X into a β – algebra Y. If
$A = \{< x, M_A(x), B_A(x), J_A(x) >/ x \in X\}$ is an MBJ – Neutrosophic β – subalgebra of X, then the image $f(A) = \{< x, f_{sup}(M_A), f_{rsup}(\tilde{B}_A), f_{inf}(J_A) >/ x \in X\}$ of A under f is an MBJ – Neutrosophic β – subalgebra of Y.

Proof:

$A = \{< x, M_A(x), B_A(x), J_A(x) >/ x \in X\}$ be an MBJ – Neutrosophic β – subalgebra of X.

Let $y_1, y_2 \in Y$

$\therefore \{x_1 + x_2 : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2), \} \subseteq \{x \in X : x \in f^{-1}(y_1 + y_2)\}$

Now,

$f_{sup}\{M_A(y_1 + y_2)\} = \sup\{M_A(x) / x \in f^{-1}(y_1 + y_2)\}$
$\geq \sup\{M_A(x_1 + x_2)/x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$
$\geq \sup\{\min\{M_A(x_1), M_A(x_2)\}/x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$
$= \min\{\sup\{M_A(x_1)/x_1 \in f^{-1}(y_1)\}, \sup\{M_A(x_2)/x_2 \in f^{-1}(y_2)\}\}$
$= \min\{f_{sup}(M_A(y_1)), f_{sup}(M_A(y_2))\}$

Similarly $f_{sup}\{M_A(y_1 - y_2)\} \geq \min\{f_{sup}(M_A(y_1)), f_{sup}(M_A(y_2))\}$

$f_{rsup}\{\tilde{B}_A(y_1 + y_2)\} = rsup\{\tilde{B}_A(x)/x \in f^{-1}(y_1 + y_2)\}$
$\geq \text{rsup}\{\tilde{B}_A(x_1 + x_2)/x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$
$\geq \text{rsup}\{\text{rmin}\{\tilde{B}_A(x_1), \tilde{B}_A(x_2)\}/x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$
$= r\text{min}\{r\text{sup}\{\tilde{B}_A(x_1)/x_1 \in f^{-1}(y_1)\}, r\sup\{\tilde{B}_A(x_2)/x_2 \in f^{-1}(y_2)\}\}$
$\geq \text{rmin}\{f_{rsup}(\tilde{B}_A(y_1)), f_{sup}(\tilde{B}_A(y_2))\}$

$f_{rsup}\{\tilde{B}_A(y_1 + y_2)\} \geq \text{rmin}\{f_{rsup}(\tilde{B}_A(y_1)), f_{sup}(\tilde{B}_A(y_2))\}$

Similarly, $f_{rsup}\{\tilde{B}_A(y_1 + y_2)\} \geq \text{rmin}\{f_{rsup}(\tilde{B}_A(y_1)), f_{sup}(\tilde{B}_A(y_2))\}$

$f_{inf}\{J_A(y_1 + y_2)\} = inf\{J_A(x) / x \in f^{-1}(y_1 + y_2)\}$
$\leq \inf\{J_A(x_1 + x_2)/x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$
$\leq \inf\{\max\{J_A(x_1), J_A(x_2)\}/x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$

$= \max\{\inf\{J_A(x_1)/x_1 \in f^{-1}(y_1)\}, \inf\{J_A(x_2)/x_2 \in f^{-1}(y_2)\}\}$

$= \max\{f_{inf}(J_A(y_1)), f_{inf}(J_A(y_2))\}$

Similarly, $f_{inf}\{J_A(y_1 + y_2)\} \leq \max\{f_{inf}(J_A(y_1)), f_{inf}(J_A(y_2))\}$.

3.12 Theorem

Let $f : X \rightarrow Y$ be a homomorphism of β – algebra. If $B = (M_B, \tilde{B}_B J_B)$ is an MBJ-Neutrosophic β – subalgebra of Y. Then $f^{-1}(B) = < (f^{-1}(M_B), f^{-1}(B_B), f^{-1}(J_B)) >$ is an MBJ – Neutrosophic β – subalgebra of X, where $f^{-1}(M_B(x)) = M_B(f(x))$; $f^{-1}\left(\tilde{B}_B(x)\right) = \tilde{B}_B(f(x))$ and $f^{-1}(J_B(x)) = J_B(f(x))$, for all $x \in X$.

Proof:

Let B be an MBJ – Neutrosophic β – subalgebra of Y and let $x, y \in X$

Then $f^{-1}(M_B)(x + y) = M_B(f(x + y))$

$= M_B(f(x) + f(y))$

$\geq \min\{M_B f(x)) + M_B f(y)\}$

$= \min\{f^{-1}(M_B(x)) + f^{-1}(M_B(y))\}$

$f^{-1}(M_B)(x + y) \geq \min\{f^{-1}(M_B(x)) + f^{-1}(M_B(y))\}$.

Similarly, $f^{-1}(M_B)(x - y) \geq \min\{f^{-1}(M_B(x)) + f^{-1}(M_B(y))\}$

$f^{-1}(\tilde{B}_B)(x + y) = \tilde{B}_B(f(x + y))$

$= \tilde{B}_B(f(x) + f(y))$

$\geq rmin\{\tilde{B}_B(f(x)), \tilde{B}_B(f(y))\}$

$= r\min\{f^{-1}\left(\tilde{B}_B(x)\right), f^{-1}\left(\tilde{B}_B(y)\right)\}$

$f^{-1}(\tilde{B}_B)(x + y) \geq r\min\{f^{-1}\left(\tilde{B}_B(x)\right), f^{-1}\left(\tilde{B}_B(y)\right)\}$

Similarly, $f^{-1}(\tilde{B}_B)(x - y) \geq r\min\{f^{-1}\left(\tilde{B}_B(x)\right), f^{-1}\left(\tilde{B}_B(y)\right)\}$

$f^{-1}(J_B)(x + y) = J_B(f(x + y))$

$= J_B(f(x) + f(y))$

$\leq \max\{J_B f(x) + J_B f(y)\}$

$= \max\{f^{-1}(J_B(x)) + f^{-1}(J_B(y))\}$

$f^{-1}(J_B)(x + y) \leq \min\{f^{-1}(J_B(x)) + f^{-1}(J_B(y))\}$.

Similarly, $f^{-1}(J_B)(x - y) \leq \min\{f^{-1}(J_B(x)) + f^{-1}(J_B(y))\}$.

Hence $f^{-1}(B) = (f^{-1}(M_B), f^{-1}(\tilde{B}_B), f^{-1}(J_B))$ is an MBJ – Neutrosophic β – subalgebra of X.

4 Product of MBJ – Neutrosophic Subalgebra

In this section the Cartesian product of the two MBJ – Neutrosophic β - subalgebra A and B of X and Y respectively is given.

4.1 Definition [12,33]

Let $A = \{< x,\ M_A(x), \tilde{B}_A(x), J_A(x) > / \ x \in X\}$ and $B = \{< y,\ M_A(y), \tilde{B}_A(y), J_A(y)) > / \ y \in Y\}$ be two MBJ – Neutrosophic sets of X and Y respectively. The Cartesian product of A and B is denoted by $A \times B$ is defined as $A \times B = \{< (x, y),\ M_{A\times B}(x, y), \tilde{B}_{A\times B}(x, y), J_{A\times B}(x, y) > / \ (x, y) \in X \times Y\}$ where

$M_{A\times B} : X \times Y \rightarrow [0,1]$, $\tilde{B}_{A\times B} : X \times Y \rightarrow D[0,1]$, $J_{A\times B} : X \times Y \rightarrow [0,1]$.

$M_{A\times B}(x, y) = \min\{M_A(x), M_A(y)\}$, $\tilde{B}_{A\times B}(x, y) = r\min\{\tilde{B}_A(x), \tilde{B}_A(y)\}$ and

$J_{A\times B}(x, y) = \max\{J_A(x), J_A(y)\}$.

4.2 Theorem

Let A *and* B be two MBJ – Neutrosophic β - subalgebra of X and Y respectively. Then $A \times B$ is also an

MBJ – Neutrosophic β - subalgebra of $X \times Y$.

Proof: Let A and B be an MBJ – Neutrosophic β - subalgebra of X and Y respectively.

Take $x = (x_1, x_2)$ and $y = (y_1, y_2) \in X \times Y$.

Now, $M_{A\times B}(x + y) = M_{A\times B}\big((x_1, x_2) + (y_1, y_2)\big)$

$= M_{A\times B}\big((x_1 + y_1), (y_1 + y_2)\big)$

$= \min\{M_A\big((x_1 + y_1)\big), M_B((y_1 + y_2))\}$

$\geq \min\{\min\big(M_A(x_1), M_B(y_1)\big), \min\big(M_A(x_2), M_B(y_2)\big)\}$

$= \min\{\min\big(M_A(x_1), M_B(x_2)\big), \min\big(M_A(y_1), M_B(y_2)\big)\}$

$= \min\{(M_{A\times B})(x_1, x_2), (M_{A\times B})(y_1, y_2)\}$

$= \min(M_{A\times B})(x), (M_{A\times B})(y)\}$

$M_{A\times B}(x + y) \geq \min\{(M_A \times M_B)(x), (M_A \times M_B)(y)\}$.

Similarly, $M_{A\times B}(x - y) \geq \min\{(M_A \times M_B)(x), (M_A \times M_B)(y)\}$

$\tilde{B}_{A\times B}(x + y) = \tilde{B}_{A\times B}\big((x_1, x_2) + (y_1, y_2)\big)$

$= \tilde{B}_{A\times B}\big((x_1 + y_1), (y_1 + y_2)\big)$

$= rmin\{\tilde{B}_A(x_1 + y_1), \tilde{B}_A(x_2 + y_2)\}$

$= rmin\{rmin(\tilde{B}_A(x_1), \tilde{B}_B(x_2)\}, rmin\ (\tilde{B}_A(y_1), \tilde{B}_B(y_2)\}$

$= rmin\{\tilde{B}_{A\times B}(x_1, x_2), \tilde{B}_{A\times B}(y_1, y_2)\}$

$\geq rmin\{\tilde{B}_{A\times B}(x), \tilde{B}_{A\times B}(y)\}$

$\tilde{B}_{A\times B}(x + y) \geq rmin\{\tilde{B}_{A\times B}(x), \tilde{B}_{A\times B}(y)\}$

Similarly, $\tilde{B}_{A\times B}(x - y) \geq rmin\{\tilde{B}_{A\times B}(x), \tilde{B}_{A\times B}(y)\}$

$J_{A\times B}(x + y) = J_{A\times B}\big((x_1, x_2) + (y_1, y_2)\big)$

$= J_{A\times B}\big((x_1 + y_1), (y_1 + y_2)\big)$

$= \max\{J\big((x_1 + y_1)\big), J_B((y_1 + y_2))\}$

$\geq \max\{\max\big(J(x_1), J(y_1)\big), \max\big(J_A(x_2), J(y_2)\big)\}$

$= \max\{\max\big(J_A(x_1), J_B(x_2)\big), \max\big(J_A(y_1), J_B(y_2)\big)\}$

$= \max\{(J_A \times J_B)(x_1, x_2), (J_A \times J_B)(y_1, y_2)\}$

$= \max(J_A \times J_B)(x), (J_A \times J_B)(y)\}$

$J_{A\times B}(x + y) \leq \max(J_A \times J_B)(x), (J_A \times J_B)(y)\}$

Similarly, $J_{A\times B}(x - y) \leq \max(J_A \times J_B)(x), (J_A \times J_B)(y)\}$.

Thus, $A \times B$ is also an MBJ – Neutrosophic β - subalgebra of $X \times Y$.

4.3 Theorem

Let $A_i = \{x \in X_i: M_{A_i}(x), \tilde{B}_{A_i}(x), J_{A_i}(x)\}$ be an MBJ – Neutrosophic β - subalgebra of X_i, i=1,2,…n. Then $\prod_{i=1}^{n} A_i$ is called direct product of finite MBJ – Neutrosophic β - subalgebra of $\prod_{i=1}^{n} X_i$ if

$i)$ $\prod_{i=1}^{n} M_{A_i}(x_i + y_i) \geq \min\{\prod_{i=1}^{n} M_{A_i}(x_i), \prod_{i=1}^{n} M_{A_i}(y_i)\}$

$\prod_{i=1}^{n} \tilde{B}_{A_i}(x_i + y_i) \geq r\min\{\prod_{i=1}^{n} \tilde{B}_{A_i}(x_i), \prod_{i=1}^{n} \tilde{B}_{A_i}(y_i)\}$

$\prod_{i=1}^{n} J_{A_i}(x_i + y_i) \leq \max\{\prod_{i=1}^{n} J_{A_i}(x_i), \prod_{i=1}^{n} J_{A_i}(y_i)\}$

ii) $\prod_{i=1}^{n} M_{A_i}(x_i - y_i) \geq \min\{\prod_{i=1}^{n} M_{A_i}(x_i), \prod_{i=1}^{n} M_{A_i}(y_i)\}$

$\prod_{i=1}^{n} \tilde{B}_{A_i}(x_i - y_i) \geq rmin\{\prod_{i=1}^{n} \tilde{B}_{A_i}(x_i), \prod_{i=1}^{n} \tilde{B}_{A_i}(y_i)\}$

$\prod_{i=1}^{n} J_{A_i}(x_i - y_i) \leq \max\{\prod_{i=1}^{n} J_{A_i}(x_i), \prod_{i=1}^{n} J_{A_i}(y_i)\}$.

Proof: The prove is clear by induction and using Theorem 4.2.

4.4 Theorem

Let $A_i = \{ x \in X_i: M_{A_i}(x), \tilde{B}_{A_i}(x), J_{A_i}(x)\}$ be an MBJ – Neutrosophic β - subalgebra of X_i, respectively for i=1,2,…n. Then $\prod_{i=1}^{n} A_i$ is an MBJ – Neutrosophic β - subalgebra of $\prod_{i=1}^{n} X_i$.

Proof: Let A be an MBJ – Neutrosophic β - subalgebra of X_i

Let $(x_1, x_2, \dots x_n)$ and $(y_1, y_2, \dots y_n) \in \prod_{i=1}^{n} X_i$

Take $a = (x_1, x_2, \dots x_n)$ and $b = (y_1, y_2, \dots y_n)$

Then

$\prod_{i=1}^{n} M_{A_i}(a + b) \geq \min\{M_{A_1}(a + b), \dots \dots . M_{A_n}(a + b)\}$

$= \min\{\min\{M_{A_1}(a), M_{A_1}(bn)\}, \dots \dots \dots . \min\{M_{A_n}(a), M_{A_n}(b)\}\}$

$= \min\{\min\{M_{A_1}(a), \dots \dots M_{A_n}(a)\}, \min\{M_{A_1}(b), \dots \dots M_{A_n}(b)\}\}$

$= \min\{\prod_{i=1}^{n} M_{A_i}(a), \prod_{i=1}^{n} M_{A_i}(b)\}$

$\prod_{i=1}^{n} M_{A_i}(a + b) \geq \min\{\prod_{i=1}^{n} M_{A_i}(a), \prod_{i=1}^{n} M_{A_i}(b)\}\}$

Similarly, $\prod_{i=1}^{n} M_{A_i}(a - b) \geq \min\{\prod_{i=1}^{n} M_{A_i}(a), \prod_{i=1}^{n} M_{A_i}(b)\}\}$

$\prod_{i=1}^{n} \tilde{B}_{A_i}(a + b) \geq \min\{\tilde{B}_{A_1}(a + b), \dots \dots . \tilde{B}_{A_n}(a + b)\}$

$= r\min\{r \min\{\tilde{B}_{A_1}(a), \tilde{B}_{A_1}(b)\}, \dots \dots \dots . \min\{\tilde{B}_{A_n}(a), \tilde{B}_{A_n}(b)\}\}$

$= r\min\{\mathrm{rmin}\{\tilde{B}_{A_1}(a), \dots \dots \tilde{B}_{A_n}(a)\}, \min\{\tilde{B}_{A_1}(b), \dots \dots \tilde{B}_{A_n}(b)\}\}$

$= r\min\{\prod_{i=1}^{n} \tilde{B}_{A_i}(a), \prod_{i=1}^{n} \tilde{B}_{A_i}(b)\}$

$\prod_{i=1}^{n} \tilde{B}_{A_i}(a + b) \geq \mathrm{rmin}\{\prod_{i=1}^{n} \tilde{B}_{A_i}(a), \prod_{i=1}^{n} \tilde{B}_{A_i}(b)\}\}$

Similarly, $\prod_{i=1}^{n} \tilde{B}_{A_i}(a - b) \geq \mathrm{rmin}\{\prod_{i=1}^{n} \tilde{B}_{A_i}(a), \prod_{i=1}^{n} \tilde{B}_{A_i}(b)\}\}$

$\prod_{i=1}^{n} J_{A_i}(a + b) \leq \max\{J_{A_1}(a + b), \dots \dots . J_{A_n}(a + b)\}$

$= \max\{\max\{J_{A_1}(a), J_{A_1}(b)\}, \dots \dots \dots . \max\{J_{A_n}(a), J_{A_n}(b)\}\}$

$= \max\{\max\{J_{A_1}(a), \dots \dots J_{A_n}(a)\}, \max\{J_{A_1}(b), \dots \dots J_{A_n}(b)\}\}$

$= \max\{\prod_{i=1}^{n} J_{A_i}(a), \prod_{i=1}^{n} J_{A_i}(b)\}$

$\prod_{i=1}^{n} J_{A_i}(a + b) \leq \max\{\prod_{i=1}^{n} J_{A_i}(a), \prod_{i=1}^{n} J_{A_i}(b)\}\}$

Similarly, $\prod_{i=1}^{n} J_{A_i}(a - b) \leq \max\{\prod_{i=1}^{n} J_{A_i}(a), \prod_{i=1}^{n} J_{A_i}(b)\}\}$

Thus, $\prod_{i=1}^{n} A_i$ is an MBJ – Neutrosophic β - subalgebra of $\prod_{i=1}^{n} X_i$.

Conclusion

Here, the MBJ – Neutrosophic substructure on β – algebra was introduced in double operations$+$ and $-$. Further, the study analysed the MBJ – Neutrosophic β – subalgebra using Homomorphic image, inverse image and Cartesian product. The same ideas can be extended to some other substructures like ideal, H- ideal and filters of a β – algebra for a future scope.

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Domination Number in Neutrosophic Soft Graphs

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Abstract: The soft set theory is a mathematical tool to represent uncertainty, imprecise, and vagueness is often employed in solving decision making problem. It has been widely used to identify irrelevant parameters and make reduction set of parameters for decision making in order to bring out the optimal choices. This manuscript is designed with the concept of neutrosophic soft graph structures. We introduce the domination number of neutrosophic soft graphs and elaborate them with suitable examples by using strength of path and strength of connectedness. Moreover, some remarkable properties of independent domination number, strong neighborhood domination, weights of a dominated graph and strong perfect domination of neutrosophic soft graph is investigated and the proposed concepts are described with suitable examples.

Keywords: Domination Number, Neutrosophic graphs, Strong neighborhood domination, Strong perfect domination, Soft graph.

1 Introduction

Fuzzy graph theory was introduced by Azriel Rosenfied in 1975. Still it is very young, it has been growing very fast and has crucial applications in various domain. Fuzzy set was introduced by Zadeh [8] whose basic components is only a membership function. The generalization of Zadeh's fuzzy set, called intuitionistic fuzzy set was introduced by atanassov [16] which is characterized by a membership function and a non membership function. According to Atanassov, the sum of membership degree and a non membership degree does not exceed one. A. Somasundaram and S. Somasundaram [33] presented more concept of independent domination, connected domination in fuzzy graphs, R. Parvathi and G. Thamilzhendhi [23] introduced domination in intuitionistic fuzzy graphs and discussed some of its properties.

The soft graphs represents need any addition information about the data such as the probability in statistic or possibility value in fuzzy graphs and give the accurate value. The theory use parameterization as its main vehicle in developing theory and its applications. The crucial model of parameter reduction and decision making is developing fascinating in dealing with uncertainties that making problems in soft set theory are interesting field. Molodtsov [25] introduced the concept of soft set theory as a new mathematical tool for dealing with uncertainties. Molodtsov's soft sets give us new technique for dealing with uncertainty from the view point of parameters. It has been revealed that soft sets have potential applications in several fields. In [7], author studied the fuzzy soft graphs. Operations of fuzzy soft graphs are studied in [8]. Recently, Akram M [9] introduced an idea about neutrosophic soft graphs and its application. Recently, the author Smarandache [29, 30, 13, 14, 31, 32, 17, 18, 19, 20, 35] introduced and studied extensively about neutrosophic set and it receives applications in many domains. The neutrosophic set has three completely independent parts, which are truth-membership degree, indeterminacy-membership degree and falsity-membership degree with the sum of these values lies between 0 and 3. Akram [9] established the certain notions including neutrosophic soft graphs, strong neutrosophic soft graphs, complete neutrosophic soft graphs. Motivation of the above, we introduced the concept of domination number in neutrosophic fuzzy soft graphs, strong neighborhood domination and strong perfect domination in neutrosophic fuzzy soft graphs. The major contribution of this work as follows:

- The domination set of neutrosophic soft graphs is established by using the concept of strength of a path, strength of connectedness and strong arc.
- The necessary and sufficient condition for the minimum domination set of neutrosophic soft graph is investigated.
- Some properties of independent domination number of neutrosophic soft graph are obtained and the proposed concepts are described with suitable examples.
- Further we presented a remarkable properties of independent domination number, strong neighborhood domination and strong perfect domination of neutrosophic soft graph.

2 Preliminaries

Definition 2.1 [30] A Neutrosophic set A is contained in another neutrosophic set B, (i.e) $A \subseteq C$ if $\forall x \in X, T_A(x) \leq T_B(x), I_A(x) \leq I_B(x)$and $F_A(x) \geq F_B(x)$.

Definition 2.2 [35] Let X be a space of points (objects), with a generic elements in X denoted by x. A single valued neutrosophic set (SVNS) A in X is characterized by truth-membership function $T_A(x)$, indeterminacy-membership function $I_A(x)$ and falsity-membership-function $F_A(x)$.
For each point x in X, $T_A(x), F_A(x), I_A(x) \in [0,1]$.

$$A = \{x, T_A(x), F_A(x), I_A(x)\} \text{ and } 0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$$

Definition 2.3 [17, 18] A neutrosophic graph is defined as a pair $G^* = (V, E)$ where

(i) $V = \{v_1, v_2, .., v_n\}$ such that $T_1 = V \to [0,1]$, $I_1 = V \to [0,1]$ and $F_1 = V \to [0,1]$ denote the degree of truth-membership function, indeterminacy function and falsity-membership function,respectively and

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$$

(ii) $E \subset V \times V$ where $T_2 = E \to [0,1]$,$I_2 = E \to [0,1]$ and $F_2 = E \to [0,1]$ are such that

$$T_2(uv) \leq \min\{T_1(u), T_1(v)\},$$
$$I_2(uv) \leq \min\{I_1(u), I_1(v)\},$$
$$F_2(uv) \leq \max\{F_1(u), F_1(v)\},$$
$$\text{and } 0 \leq T_2(uv) + I_2(uv) + F_2(uv) \leq 3, \forall uv \in E$$

Definition 2.4 Let (H, A) and (G, B) be two neutrosophic soft sets over the common universe U. (J, A) is said to be neutrosophic soft subset of (G, B) if $A \subset B$, if $T_{J(e)}(x) \leq T_{G(e)}(x), I_{J(e)}(x) \leq I_{G(e)}(x)$ and $F_{J(e)}(x) \geq F_{G(e)}(x)$ for all $e \in M, x \in U$.

Definition 2.5 Let (H, A) and (G, B) be two neutrosophic soft sets over the common universe U. The union of two neutrosophic soft sets (H, A) and (G, B) is neutrosophic soft set $(K, C) = (H, A) \cup (G, B)$, where $C = A \cup B$ and the truth-membership, indeterminacy-membership and falsity-membership of (K, C) are defined by $T_{K(e)}(x) = T_{H(e)}(x), \text{if} e \in A - B, T_{G(e)}(x), \text{if} e \in B - A, \max(T_{H(e)}(x), T_{G(e)}(x) \text{if} e \in A \cap B$.

Definition 2.6 Let U be an initial universe and P be the set of all parameters. $\rho(U)$ denotes the set of all neutrosophic sets of U. Let A be a subset of P. A pair (J, A) is called a neutrosophic soft set over U. Let $\rho(V)$ denotes the set of all neutrosophic sets of V and $\rho(E)$ denotes the set of all neutrosophic sets of E.

Definition 2.7 [9] A neutrosophic soft graph $G = (G^*, J, K, A)$ is an ordered four tuple, if it satisfies the following conditions:

(i)$G^* = (V, E)$ is a simple graph,

(ii)A is a non-empty set of parameters ,

(iii)(J, A) is a neutrosophic soft set over V,

(iv)(K, A) is a neutrosophic soft set over E,

(v)$(J(e), K(e))$ is a neutrosophic graph of G^*, then

$$T_{K(e)}(xy) \leq \{T_{J(e)}(x) \wedge T_{J(e)}(y)\},$$
$$I_{K(e)}(xy) \leq \{I_{J(e)}(x) \wedge I_{J(e)}(y)\},$$
$$F_{K(e)}(xy) \leq \{F_{J(e)}(x) \vee F_{J(e)}(y)\},$$

such that

$$0 \leq T_{K(e)}(xy) + I_{K(e)}(xy) + F_{K(e)}(xy) \leq 3 \text{ for all } e \in A \text{ and } x, y \in V.$$

The neutrosophic graph (J_e, K_e) is denoted by $H(e)$ for convenience. A neutrosophic soft graph is a parametrized family of neutrosophic graphs. The class of all neutrosophic soft graphs is denoted by $NS(G^*)$. Note that $T_{K(e)}(xy) = I_{K(e)}(xy) = 0$ and $F_{K(e)}(xy) = 1 \forall xy \in V \times V - E, e \notin A$.

Definition 2.8 [9] Let $G_1 = (F_1, K_1, A)$ and $G_2 = (F_2, K_2, B)$ be two neutrosophic soft graphs of G^*. Then G_1 is a neutrosophic subgraph of G_2 if

(i)$A \subseteq B$.

(ii)$H_1(e)$ is a partial subgraph of $H_2(e)$ for all $e \in A$.

3 MAIN RESULT

Definition 3.1 Let $G = (G^*, J, K, A)$ be a neutrosophic soft graph. Then the degree of a vertex $u \in G$ is a sum of degree truth membership, sum of indeterminacy membership and sum of falsity membership of all those edges which are incident on vertex u denoted by $d(u) = (d_{TJ(e)}(u), d_{IJ(e)}(u), d_{FJ(e)}(u))$ where

$d_{TJ(e)}(u) = \sum_{e \in A} (\sum_{u \notin v \in V} T_{K(e)}(u, v))$ called the degree of truth membership vertex

$d_{IJ(e)}(u) = \sum_{e \in A} (\sum_{u \notin v \in V} I_{K(e)}(u, v))$ called the degree of indeterminacy membership vertex

$d_{FJ(e)}(u) = \sum_{e \in A} (\sum_{u \notin v \in V} F_{K(e)}(u, v))$ called the degree of falsity membership vertex for all $e \in A, u, v \in V$.

Definition 3.2 Let $G = (G^*, J, K, A)$ be a neutrosophic soft graph. Then the total degree of a vertex $u \in G$ is defined by $td(u) = (td_{TJ(e)}(u), td_{IJ(e)}(u), td_{FJ(e)}(u))$ where

$td_{TJ(e)}(u) = \sum_{e \in A} (\sum_{u \notin v \in V} T_{K(e)}(u, v) + T_{J(e)}(u, v)$ called the degree of truth membership vertex

$td_{IJ(e)}(u) = \sum_{e \in A} (\sum_{u \notin v \in V} I_{K(e)}(u,v) + I_{J(e)}(u,v))$ called the degree of indeterminacy membership vertex

$td_{FJ(e)}(u) = \sum_{e \in A} (\sum_{u \notin v \in V} F_{K(e)}(u,v) + F_{J(e)}(u,v))$ called the degree of falsity membership vertex for all $e \in A, u, v \in V$.

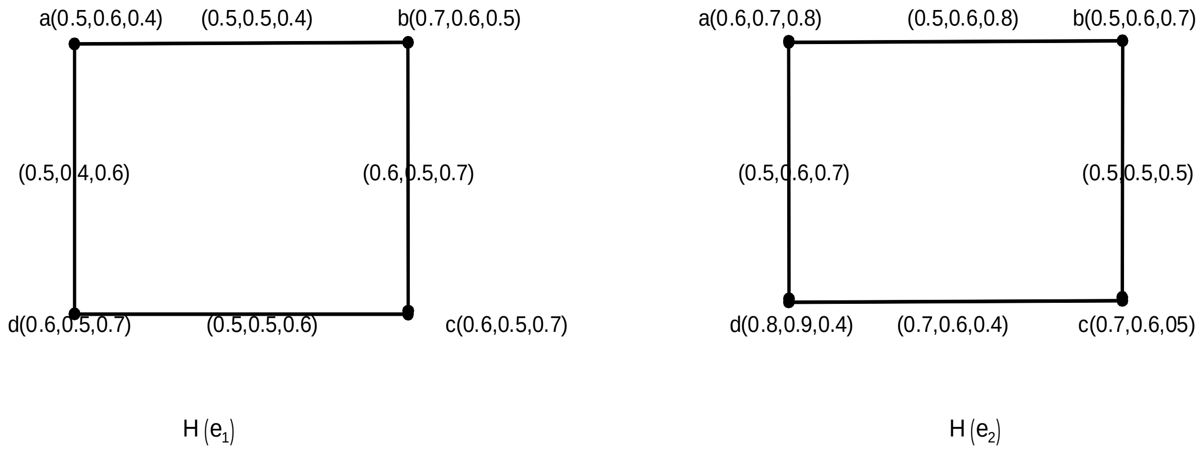

Figure 1

Example 3.3 Consider a simple graph $G^* = (V,E)$ such that $V = \{a,b,c,d\}$ and $E = \{(ab),(bc),(cd),(ad)\}$. Let $A = (J,A)$ be a neutrosophic soft over V with the approximation function $J: A \to \rho(V)$ defined by

$$J(e_1) = a(0.5,0.6,0.4), b(0.7,0.6,0.5), c(0.6,0.5,0.7), d(0.6,0.5,0.7)$$
$$J(e_2) = a(0.6,0.7,0.8), b(0.5,0.6,0.7), c(0.7,0.6,0.5), d(0.8,0.9,0.4)$$

Let (K,A) be a neutrosophic soft over E with neutrosophic approximation function $K: A \to \rho(E)$ defined by

$$K(e_1) = ab(0.5,0.5,0.4), bc(0.6,0.5,0.7), cd(0.5,0.5,0.6), ad(0.5,0.4,0.6)$$
$$K(e_2) = ab(0.5,0.6,0.8), bc(0.5,0.5,0.5), cd(0.7,0.6,0.4), ad(0.5,0.6,0.7)$$

Clearly, $H(e_1) = (J(e_1), K(e_1))$ and $H(e_2) = (J(e_2), K(e_2))$ are neutrosophic graphs corresponding to the parameters e_1 and e_2 respectively as shown in Figure 1.

For the graph $H(e_1)$ degree of vertices as follows, $deg(a) = (1,0.9,1.0), deg(b) = (1.1,1.0,1.1), deg(c) = (1.1,1.0,1.3), deg(d) = (1.0,0.9,1.2)$

For the graph $H(e_2)$ degree of vertices as follows, $deg(a) = (1.0,1.2,1.5)$, $deg(b) = (1.0,1.1,1.3), deg(c) = (1.2,1.1,0.9), deg(d) = (1.2,1.2,1.1)$

Definition 3.4 A simple graph G is said to be a regular if each vertices has a same degree for all $e \in A$, $x,y \in V$. Let $G^* = (V,E)$ be a neutrosophic graph then G is said to be a regular neutrosophic graph if $H(e)$ is a regular graph for all $e \in A$, if $H(e)$ is a regularr neutrosophic graph of degree r for all $e \in A$, then G is a $r-$ regular fuzzy graph. Let $G^* = (V,E)$ be a neutrosophic graph then G is said to be a totally regular neutrosophic graph if $H(e)$ is a totally regular graph for all $e \in A$, if $H(e)$ is a totally regular neutrosophic graph of degree r for all $e \in A$, then G is a $r-$totally regular neutrosophic fuzzy graph.

Example 3.5 Consider a simple graph $G^* = (V, E)$ such that $V = \{a, b, c, d\}$ and

$E = \{(ab), (bc), (cd), (ad)\}$. Let $A = \{e_1, e_2\}$. Let (J, A) be a neutrosophic soft over V with its approximation function $J = A \to \rho(V)$ defined by

$$J(e_1) = a(0.4,0.3,0.3), b(0.3,0.3,0.4), c(0.4,0.4,0.4), d(0.5,0.5,0.5)$$

$$J(e_2) = a(0.5,0.4,0.4), b(0.4,0.4,0.5), c(0.5,0.5,0.5), d(0.6,0.6,0.6).$$

Let (K, A) be a neutrosophic soft over E with neutrosophic approximation function $K: A \to \rho(E)$ defined by

$$K(e_1) = ab(0.2,0.2,0.2), bc(0.1,0.1,0.1), cd(0.2,0.2,0.2), ad(0.1,0.1,0.1)$$

$$K(e_2) = ab(0.2,0.2,0.2), bc(0.3,0.3,0.3), cd(0.2,0.2,0.2), ad(0.3,0.3,0.3).$$

Obviously, $H(e_1) = (F(e_1), K(e_1))$ and $H(e_2) = (F(e_2), K(e_2))$ are neutrosophic graphs corresponding to the parameters e_1 and e_2 respectively as shown in Figure 2

For the graph $H(e_1)$ degree of vertices as follows, $\deg(a) = (0.3,0.3,0.3)$, $\deg(b) = (0.3,0.3,0.3)$, $\deg(c) = (0.3,0.3,0.3)$, $\deg(d) = (0.3,0.3,0.3)$

For the graph $H(e_2)$ degree of vertices as follows, $\deg(a) = (0.5,0.5,0.5)$, $\deg(b) = (0.5,0.5,0.5)$, $\deg(c) = (0.5,0.5,0.5)$, $\deg(d) = (0.5,0.5,0.5)$

Here, $H(e_1)$ and $H(e_2)$ all the vertices degree are same so neutrosophic soft graph G is regular neutrosophic graph.

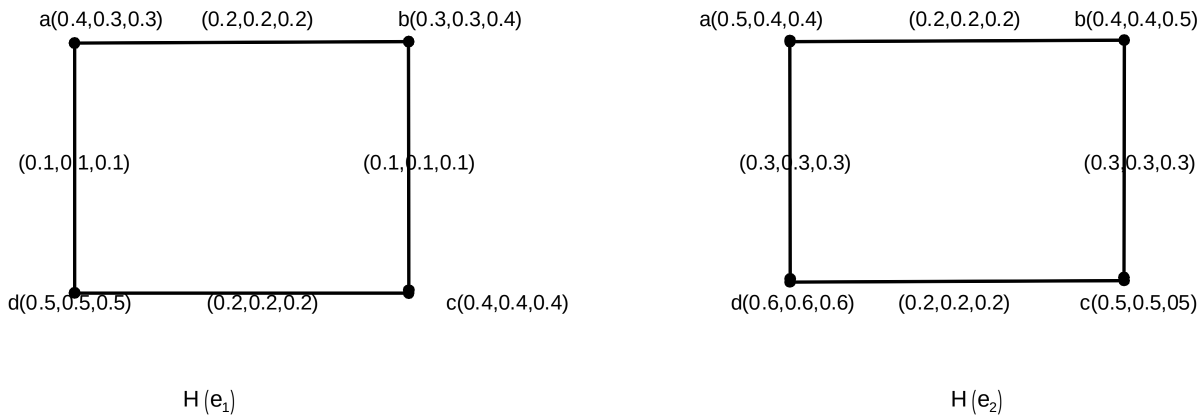

Figure 2

Definition 3.6 A graph $G^* = (V, E)$ is said to be a totally regular neutrosophic graph if each vertex has a same total degree for all $e \in A$, $u, v \in V$.

Example 3.7 Consider a simple graph $G^* = (V, E)$ such that $V = \{a, b, c, d, i, j, k\}$ and

$E = \{(ab), (bc), (cd), (ad), (ij), (jk), (kj)\}$. Let $A = \{e_1, e_2\}$ parameter set. Let (J, A) be a neutrosophic soft over V with its approximation function $J = A \to \rho(V)$ defined by

$$J(e_1) = \{a(0.5,0.6,0.4), b(0.4,0.7,0.6), c(0.4,0.6,0.7), d(0.5,0.5,0.5)\}$$

$$J(e_2) = \{i(0.6,0.7,0.5), j(0.5,0.7,0.9), k(0.6,0.6,0.7)\}$$

Let (K, A) be a neutrosophic soft over E with neutrosophic approximation function $K: A \to \rho(E)$ defined by

$$K(e_1) = ab(0.4,0.3,0.5), bc(0.4,0.3,0.3), cd(0.5,0.4,0.3), ad(0.3,0.4,0.5)$$

$$K(e_2) = ij(0.5,0.5,0.4), jk(0.6,0.5,0.4), ik(0.4,0.5,0.6),$$

clearly, $H(e_1) = (J(e_1), K(e_1))$ and $H(e_2) = (J(e_2), K(e_2))$ are neutrosophic graphs corresponding to the parameters e_1 and e_2 respectively as shown in Figure 3. For the graph $H(e_1)$ total degree of vertices as follows,

$tdeg(a) = (1.2,1.3,1.4)$, $tdeg(b) = (1.2,1.3,1.4)$, $tdeg(c) = (1.2,1.3,1.4)$, $tdeg(d) = (1.2,1.3,1.4)$

For the graph $H(e_2)$ degree of vertices as follow, $tdeg(i) = (1.5,1.6,1.5)$, $tdeg(j) = (1.5,1.6,1.5)$, $tdeg(k) = (1.5,1.6,1.5)$

Here $H(e_1)$ and $H(e_2)$ all the vertices total degrees are same so neutrosophic soft graph G is totally regular neutrosophic soft graph.

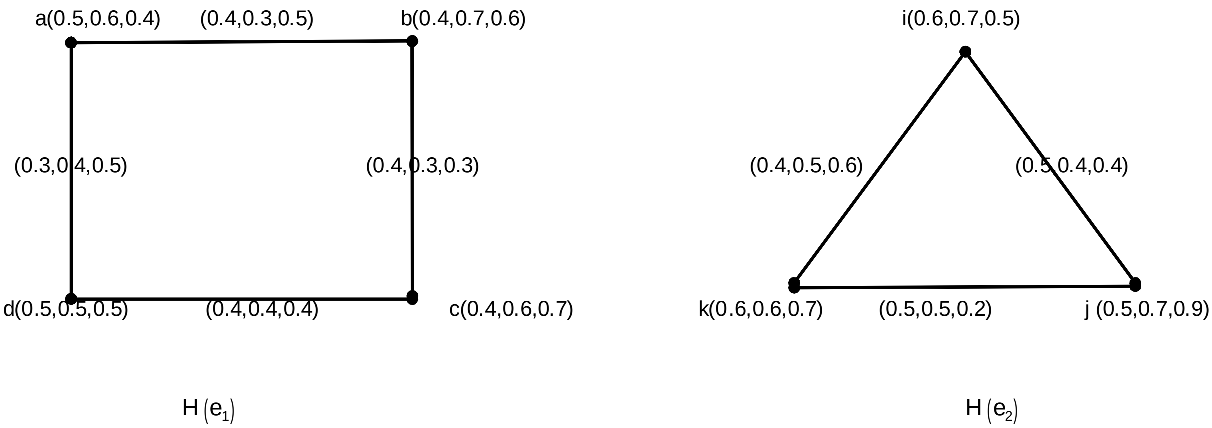

Figure 3

Definition 3.8 The order of a neutrosophic soft graph G is

$$Ord(G) = \sum_{e_i \in A} \left(\sum_{x \in V} T_{J(e_i)}(e_i)(x), \sum_{x \in V} I_{F(e_i)}(e_i)(x), \sum_{x \in V} F_{J(e_i)}(e_i)(x)\right).$$

Definition 3.9 The size of a neutrosophic soft graph G is

$$S(G) = \sum_{e_{i \in A}} \left(\sum_{xy \in V} T_{K_{ei}}(e_i)(xy), \sum_{xy \in V} I_{K_{ei}}(e_i)(xy), \sum_{xy \in V} F_{K_{ei}}(e_i)(xy), \right.$$

Example 3.10 In example Figure 1, we consider the order of neutrosophic soft graph is

$$Ord(G) = \sum_{e_i \in A} \left(\sum_{x \in V} T_{J(e_i)}(e_i)(x), \sum_{x \in V} I_{F(e_i)}(e_i)(x), \sum_{x \in V} F_{J(e_i)}(e_i)(x)\right).$$

$$Ord(G) = (5.0,5.0,4.7). \text{ Similarly } S(G) = (4.3,4.2,4.7)$$

Definition 3.11 Let $G = (G^*, J, K, A)$ be an neutrosophic soft graph. then cardinality of G is defined to be

$$|\mathbf{G}| = \sum_{\mathbf{e} \in \mathbf{A}} \left| \sum_{\mathbf{v_i} \in \mathbf{V}} \frac{\mathbf{1} + \mathbf{T}_{J(e)}(\mathbf{x}) + \mathbf{I}_{J(e)}(\mathbf{x}) - \mathbf{F}_{J(e)}(\mathbf{x})}{\mathbf{2}} + \right| \sum_{\mathbf{v_i}, \mathbf{v_j} \in \mathbf{V}} \frac{\mathbf{1} + \mathbf{T}_{J(e)}(\mathbf{xy}) + \mathbf{I}_{J(e)}(\mathbf{xy}) - \mathbf{F}_{J(e)}(\mathbf{xy})}{\mathbf{2}} \Bigg|$$

Example 3.12 Consider the above Figure 3, here $H(e_1)$ and $H(e_2)$ are neutrosophic soft graph of G corresponding to the parameter e_1, the cardinality is $G = 5.60$ and corresponding to the parameter e_2, the cardinality is $G = 4.60$

Definition 3.13 Let $GG = (G^*, J, K, A)$ be an neutrosophic soft graph, then vertex cardinality of G is defined to be

$$|V| = \sum_{e \in A} | \sum_{v_i \in V} \frac{1 + T_{J(e)}(x) + I_{J(e)}(x) - F_{J(e)}(x)}{2} |$$

Example 3.14 For the above Figure 3, $H(e_1)$ and $H(e_2)$ are neutrosophic soft graph of G corresponding to the parameter e_1 cardinality is $V = 0.85 + 0.75 + 0.65 + 0.75 = 3.0$ corresponding to the parameter e_2, the cardinality is $V = 2.30$. Then $G(V) = 5.30$

Definition 3.15 Let $G = (G^*, J, K, A)$ be an neutrosophic soft graph, Edge cardinality of E is defined to be

$$|E| = \sum_{e \in A} | \sum_{xy \in E} \frac{1 + T_{K(e)}(xy) + I_{K(e)}(xy) - F_{K(e)}(xy)}{2} |$$

Example 3.16 For the above Figure 3, $H(e_1)$ and $H(e_2)$ are neutrosophic soft graph of G corresponding to the parameter e_1 cardinality is $E = 2.6$ corresponding to the parameter e_2, the cardinality is $E = 2.30$ then $G(E) = 4.90$.

Definition 3.17 The sum of weight of the strong edges incident at v is means to be $d_G(v)$. in neutrosophic soft graph. The minimum $\deg(G)$ is $\delta(G) = \min\{d_g(v)/v \in V, e \in A.\}$

The maximum $\deg(G)$ is $\Delta(G) = \max\{d_g(v)/v \in V, e \in A.\}$

Definition 3.18 Two vertices x and y are said to be neighbors in neutrosophic soft graph if either one of the following conditions hold.

$$(1) T_{K(e)}(xy) > 0, I_{K(e)}(xy) > 0, F_{K(e)}(xy) > 0,$$
$$(2) T_{K(e)}(xy) > 0, I_{K(e)}(xy) = 0, F_{K(e)}(xy) > 0,$$
$$(3) T_{K(e)}(xy) > 0, I_{K(e)}(xy) > 0, F_{K(e)}(xy) = 0,$$
$$(4) T_{K(e)}(xy) = 0, I_{K(e)}(xy) > 0, F_{K(e)}(xy) > 0, \text{ for all } x, y \in V, e \in A.$$

Definition 3.19 A path in an neutrosophic is a sequence of distinct vertices $v_1, v_2, \ldots, v_n$, such that either one of the following conditions are satisfied.

$$(1) T_{K(e)}(xy) > 0, I_{K(e)}(xy) > 0, F_{K(e)}(xy) > 0,$$
$$(2) T_{K(e)}(xy) > 0, I_{K(e)}(xy) = 0, F_{K(e)}(xy) > 0,$$
$$(3) T_{K(e)}(xy) > 0, I_{K(e)}(xy) > 0, F_{K(e)}(xy) = 0,$$
$$(4) T_{K(e)}(xy) = 0, I_{K(e)}(xy) > 0, F_{K(e)}(xy) > 0, \text{ for all } x, y \in V, e \in A.$$

Definition 3.20 The length of a path $P = v_1, v_2, \ldots, v_{n+1} (n > 0)$ in Neutrosophic soft graph is n.

Definition 3.21 If v_i, v_j are vertices in G and if they are connected means of a path then the strength of that path is defined as $(\min_{i,j} T_{K(e)}(v_i, v_j), \min_{i,j} I_{K(e)}(v_i, v_j), \max_{i,j} F_{K(e)}(v_i, v_j))$ where $\min_{i,j} T_{K(e)}(v_i, v_j)$ is the $T_{K(e)}$- strength of weakest arc and $\min_{i,j} I_{K(e)}(v_i, v_j)$ is the $I_{K(e)}$- strength of weakest arc and $\max_{i,j} F_{K(e)}(v_i, v_j))$ is the $F_{K(e)}$- strength of strong arc.

Definition 3.22 If $v_i, v_j \in V \subseteq G$, the $T_{K(e)}$ -strength of connectedness between v_i and v_j is $T^{\infty}_{K(e)}(v_i, v_j) = \sup\{T^{K}_{K(e)}(v_i, v_j)/k = 1,2, \ldots n, e \in A\}$ and $I_{K(e)} -$ strength of connectedness between v_i and v_j is $I^{\infty}_{K(e)}(v_i, v_j) = \sup\{I^{k}_{K(e)}(v_i, v_j)/k = 1,2, \ldots n, e \in A\}$ and $F^{\infty}_{K(e)}(v_i, v_j) = \inf\{F^{k}_{K(e)}(v_i, v_j)/k = 1,2, \ldots n, e \in A\}$.

If u, v are connected by means of path of length k then $T^{k}_{K(e)}(v_i, v_j)$ is defined as

$$\sup\{T_{K(e)}(u, v_1) \wedge T_{K(e)}(v_1, v_2) \wedge T_{K(e)}(v_1, v_3) \ldots, T_{K(e)}(v_{k-1}, v_k)/u, v, v_1, \ldots v_{k-1}, v \in V\},$$

$I_{k(e)}^{k}(v_i, v_j)$ is defined as

$\sup\{I_{K(e)}(u, v_1) \wedge I_{K(e)}(v_1, v_2) \wedge I_{K(e)}(v_1, v_3) \ldots, I_{K(e)}(v_{k-1}, v_k)/u, v, v_1, \ldots v_{k-1}, v \in V\}$ and

$F_{K(e)}^{k}(v_i, v_j)$ is defined as

$\inf\{F_{K(e)}(u, v_1) \vee F_{K(e)}(v_1, v_2) \vee F_{K(e)}(v_1, v_3) \ldots, F_{K(e)}(v_{k-1}, v_k)/u, v, v_1, \ldots v_{k-1}, v \in V\}, e \in A.$

Definition 3.23 Two vertices that are joined by a path is called connected neutrosophic soft graph.

Definition 3.24 Let u be a vertex in an neutrosophic soft graph $G^* = (V, E)$, then $N(u) = \{v: v \in V\}$ and (u, v) is a strong arc is called neighborhood of u.

Definition 3.25 A vertex $u \in V$ of an neutrosophic soft graph $G = (V, E)$ is said to be an isolated vertex if $T_{K(e)}(u, v) = 0$, $I_{K(e)}(u, v)$ and $F_{K(e)}(u, v) = 0$, thus an isolated vertex does not dominated any other vertex in G.

Definition 3.26 An arc (u, v) is said to be strong arc, if $T_{K(e)}(u, v) \geq T_{K(e)}^{\infty}(u, v)$ and $I_{K(e)}(u, v) \geq I_{K(e)}^{\infty}(u, v)$ and $F_{K(e)}(u, v) \geq F_{K(e)}^{\infty}(u, v)$.

Definition 3.27 Let $G = (V, E)$ be an neutrosophic soft graph on V. Let $u, v \in V$, we say that u dominates v in G if there exists an strong arc between them.

Note:

1) For any $u, v \in V$, if u dominates v then v dominates u and hence domination is a symmetric relation on V.
2) For any $v \in V, N(v)$ is precisely the set of all vertices in V which are dominated by v.
3) If $T_{K(e)}(u, v) < T_{K(e)}^{\infty}(u, v)$ and $I_{K(e)}(u, v) < I_{K(e)}^{\infty}(u, v)$ and $F_{K(e)}(u, v) < F_{K(e)}^{\infty}(u, v)$, for all $u, v \in V$ and $e \in A$, then the only dominating set of G is V.

Definition 3.28 Given $S \subset V$ is called a dominating set in G if for every vertex $v \in V - S$ there exists a vertex $u \in S$ such that u dominates v. for all $e \in A, u, v \in V$.

Definition 3.29 A dominating set S of an Neutrosophic soft graph is said to be minimal domiating set if no proper subset of S is a dominating set. for all $e \in A, u, v \in V$.

Definition 3.30 Minimum cardinality among all minimal dominating set is called lower domination number of G, and is denoted by $\sum_{e \in A} (d_{NS}(G)) \forall e \in A, u, v \in V$.

Maximum cardinality among all minimal dominating set is called upper domination number of G, and is denoted by $\sum_{e \in A} (D_{NS}(G)) \forall e \in A, u, v \in V$.

Example 3.31 Consider an neutrosophic soft graph $G = (V, E)$, such that $V = \{a, b, c, d\}$ and $E = \{(ab), (bc), (cd), (da), (ac)\}$. Let $A = \{e_1, e_2\}$ be a set of parameters and let neutrosophic soft over V with neutrosophic approximation function $J: A \to \rho(v)$ defined by

$$J(e_1) = a(0.5,0.5,0.6), b(0.5,0.6,0.7), c(0.4,0.3,0.6), d(0.4,0.5,0.7)$$

$$J(e_2) = a(0.6,0.6,0.7), b(0.6,0.7,0.8), c(0.5,0.4,0.7), d(0.5,0.6,0.7)$$

Let (K, A) be a neutrosophic approximation function $K: A \to \rho(E)$ is defined by

$$K(e_1) = ab(0.4,0.5,0.6), bc(0.4,0.3,0.6), cd(0.4,0.3,0.6), ad(0.4,0.5,0.6), bd(0.4,0.5,07)$$

$$K(e_2) = ab(0.5,0.6,0.7), bc(0.5,0.4,0.7), cd(0.5,0.4,0.7), ad(0.5,0.6,0.7), ac(0.5,04,06)$$

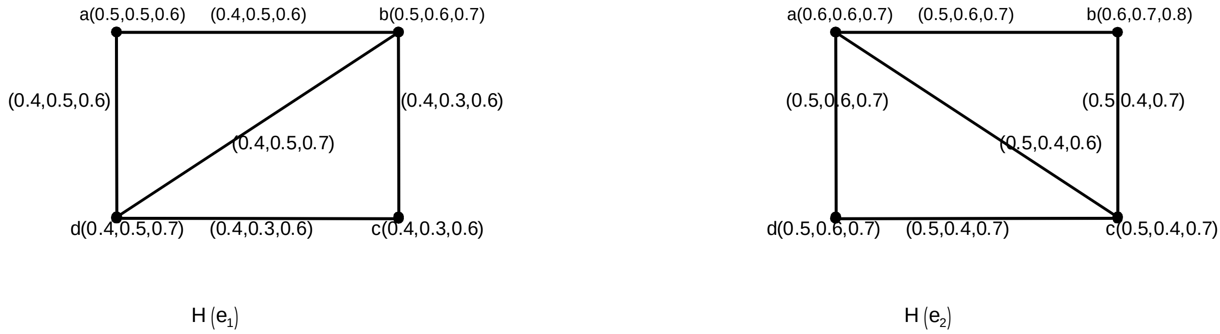

Figure 4

Here, corresponding to the parameter $H(e_1)$, the dominating set is $\{(a,b),(b,c),(c,d),(d,a),(a,b,c),(d,c,a),(b,d,a),(d)\}$

Corresponding to the parameter e_1, the minimum dominating set $\{d\}$.

Corresponding to the parameter e_1, the maximum dominating set $\{a,b\}$.

Corresponding to the parameter e_1, the minimum dominating number 0.6.

Corresponding to the parameter e_1, the maximum dominating number 1.4.

Here, corresponding to the parameter $H(e_2)$, the dominating set is $\{(a,b),(b,c),(c,d),(a,b,c),(d,c,a)\}$

Corresponding to the parameter e_2, the minimum dominating set $\{c,d\}$.

Corresponding to the parameter e_2, the maximum dominating set $\{a,b\}$.

Corresponding to the parameter e_2, the minimum dominating number 1.3.

Corresponding to the parameter e_2, the maximum dominating number 1.5.

For Figure 4, domination number is

$$\sum_{e\in A}(d_{NS}(G)) = 0.6 + 1.3 = 1.9$$

$$\sum_{e\in A}(D_{NS}(G)) = 1.4 + 1.5 = 2.9$$

Definition 3.32 Two vertices in an neutrosophic soft graph, $G = (V,E)$ are said to be independent if there is no strong arc between them.

Definition 3.33 Given $S \subset V$ is said to be independent set of G if $T_{K(e)}(u,v) < T^{\infty}_{K(e)}(u,v)$ and $I_{K(e)}(u,v) < I^{\infty}_{K(e)}(u,v)$ and $F_{K(e)}(u,v) < F^{\infty}_{K(e)}(u,v) \forall e \in A, u,v \in S$.

Definition 3.34 An indepentent set S of G in an neutrosophic soft graph is said to be maximal independent, if for every vertex $v \in V - S$, the set $S \cup \{v\}$ is not independent.

Definition 3.35 The minimum cardinality among all maximal independent set is called lower independence number of G, and it is denoted by $\sum_{e\in A}(i_{NS}(G))$. The maximum cardinality among all maximal independent set is called lower independence number of G, and it is denoted by $\sum_{e\in A}(I_{NS}(G))$.

Example 3.36 Consider an above example for an neutrosophic soft graph $G = (V,E)$, such that $V = \{a,b,c,d\}$ and $E = \{(a,b),(bc),(cd),(da),(ac)\}$. Let $A = \{e_1,e_2\}$ be a set of parameters and let neutrosophic soft over V with neutrosophic approximation function $J: A \to \rho(v)$ defined as follows:

we have corresponding to the parameter e_2 arc (ac) is weakest arc us does not dominated by $\{c\}$ and $\{a\}$.

$$J(e_1) = a(0.5,0.5,0.6), b(0.5,0.6,0.7), c(0.4,0.3,0.6), d(0.4,0.5,0.7)$$
$$J(e_2) = a(0.6,0.6,0.7), b(0.6,0.7,0.8), c(0.5,0.4,0.7), d(0.5,0.6,0.7)$$

Let (K, A) be a neutrosophic approximation function $K: A \to \rho(E)$ is defined by

$$K(e_1) = ab(0.4,0.5,0.6), bc(0.4,0.3,0.6), cd(0.4,0.3,0.6), ad(0.4,0.5,0.6), bd(0.4,0.5,07)$$
$$K(e_2) = ab(0.5,0.6,0.7), bc(0.5,0.4,0.7), cd(0.5,0.4,0.7), ad(0.5,0.6,0.7), ac(0.5,04,06).$$

For the Corresponding to the parameter e_1, the minimum Independent Dominating Det (IDS) is $\{a, c\}$.
For the Corresponding to the parameter e_1, the maximum (IDS) is $\{a, c\}$.
For the Corresponding to the parameter e_1, the minimum independent dominating number is 1.25.
For the Corresponding to the parameter e_1, the maximum independent dominating number is 1.25.
For the Corresponding to the parameter e_2, the minimum (IDS)is $\{c, a\}$.
For the Corresponding to the parameter e_2, the maximum (IDS)is $\{d, b\}$.
For the Corresponding to the parameter e_2, the minimum independent dominating number is 1.35.
For the Corresponding to the parameter e_2, the maximum independent dominating number is 1.45.
Independent domination number is $\sum_{e \in A} (i_{NS}(G)) = 2.60$ and $\sum_{e \in A} (I_{NS}(G)) = 2.70$

Theorem 3.37 A dominating set S of an NSG, $G = (G^*, J, K, A)$ is a minimal dominating set if and only if for each $d \in D$ one of the following conditions holds.
(i) d is not a strong neighbor of any vertex in D.
(ii) There is a vertex $v \in V - \{D\}$ such that $N(u) \cap D = d$.

Proof. Assume that D is a minimal dominating set of $G = (G^*, J, K, A)$. Then for every vertex $d \in D$, $D - \{d\}$ is not a dominating set and hence there exists $v \in V - (D - \{d\})$ which is not dominated by any vertex in $D - \{d\}$. If $v = d$, we get, v is not a strong neighbor of any vertex in D. If $v \neq d$,v is not dominated by $D - \{v\}$, but is dominated by D, then the vertex v is strong neighbor only to d in D. That is, $N(v) \cap D = d$. Conversely, assume that D is a dominating set and for each vertex $d \in D$, one of the two conditions holds, suppose D is not a minimal dominating set, then there exists a vertex $d \in D$, $D - \{d\}$ is a dominating set. Hence d is a strong neighbor to at least one vertex in $D - \{d\}$, the condition one does not hold. If $D - \{d\}$ is a dominating set then every vertex in $V - D$ is a strong neighbor at least one vertex in $D - \{d\}$, the second condition does not hold which contradicts our assumption that at least one of thse conditions holds. So D is a minimal dominating set.

Theorem 3.38 Let G be an NSG without isolated vertices and D is a minimal dominating set. Then $V - D$ is a dominating set of $G = (G^*, J, K, A)$.

Proof.D be a minimal dominating set. Let v be a any vertex of D. Since $G = (G^*, F, K, A)$ has no isolated vertices, there is a vertex $d \in N(v)$. v must be dominated by at least one vertex in $D - v$, that is $D - v$ is a dominating set. By above theorem, it follows that $d \in V - D$. Thus every vertex in D is dominated by at least one vertex in $V - D$, and $V - D$ is a dominating set.

Theorem 3.39 An independent set is a maximal independent set of NSG, $G = (G^*, J, K, A)$ if and only if it is independent and dominating set.

Proof. Let D be a maximal independent set in an NSG, and hence for every vertex $v \in V - D$, the set $D \cup v$ is not in dependent. For every vertex $v \in V - D$,there is a vertex $u \in D$ such that u is a strong neighbor to v. Thus D is a dominating set. Hence D is both dominating and independent set. Conversely, assume D is both independent and dominating. Suppose D is not maximal independent, then there exists a vertex $v \in V - D$, the set $D \cup v$ is independent. If $D \cup v$ is independent then no vertex in D is strong neighbor to v. Hence D cannot be a dominating set, which is contradiction, Hence D is a maximal independent set.

Theorem 3.40 Every maximal independent set in an NSG, $G = (G^*, J, K, A)$ is a minimal dominating set.

Proof. Let S be a maximal independent set in a NSG, by previous theorem, S is a dominating set. Suppose S is not a minimal dominating set, then there exists at least one vertex $v \in S$ for which $S - v$ is a dominating set, But if $S - v$ dominates $V - S - (v)$, then at least one vertex in $S - v$ must be strong neighbor to v. This contradicts the fact that S is an independent st of G. Therefore, must be a minimal dominating set.

4 STRONG NEIGHBORHOOD DOMINATION

Definition 4.1 Let $G = (V, E)$ be a neutrosophic soft graph and $u \in V$. Then $u \in V$ is called a strong neighbour of u if uv is a strong arc. the set of strong neighbor of u is called the strong neighborhood of u and is denoted by $N_s(u)$. The closed strong neighborhood of u is defined as $N_s[u] = N_s(u) \cup u$.for all $u \in V, e \in A$.

Definition 4.2 Let $G. = (V, E)$ be a strong neutrosophic soft graph and $v \in V$.

(i) The strong degree and the strong neighborhood degree of v are defined, respectively

$$d_s(v) = \sum_{e \in A} \Big(\sum_{u \in N_{(s)}(v)} T_{K(e)}(uv), \sum_{u \in N_{(s)}(v)} I_{K(e)}(uv) \sum_{u \in N_{(s)}(v)} F_{K(e)}(uv) \Big)$$

$$d_s N(v) = \sum_{e \in A} \Big(\sum_{u \in N_{(s)}(v)} T_{J(e)}(u), \sum_{u \in N_{(s)}(v)} I_{J(e)}(u) \sum_{u \in N_{(s)}(v)} F_{J(e)}(u) \Big)$$

Definition 4.3 The strong degree cardinality and the strong neighborhood degree cardinality of v are defined by

$$|d_s(v)| = \sum_{e \in A} \Big(\sum_{u \in N_s(v)} \frac{1 + T_{K(e)}(u,v) + I_{K(e)}(u,v) - F_{K(e)}(u,v)}{2} \Big)$$

$$|d_s N(v)| = \sum_{e \in A} \Big(\sum_{u \in N_s(v)} \frac{1 + T_{J(e)}(u) + I_{J(e)}(u) - F_{J(e)}(u)}{2} \Big)$$

The minimum and maximum strong degree of G are defined, respectively as

$\delta_s(G) = \wedge\, |d_s(v)|\ \forall v \in V$ and

$$\Delta_s(G) = \vee\, |d_s(v)|\ \forall u, v \in V, e \in A.$$

The minimum and maximum strong neighborhood degree of G are defined by

$\delta_s N(G) = \wedge\, |d_s N(v)| \forall v \in V$ and

$$\Delta_s N(G) = \vee\, |d_s N(v)| \forall u, v \in V, e \in A.$$

Example 4.4 Consider a neutrosophic soft graph $G = (V, E)$ in figure we see that

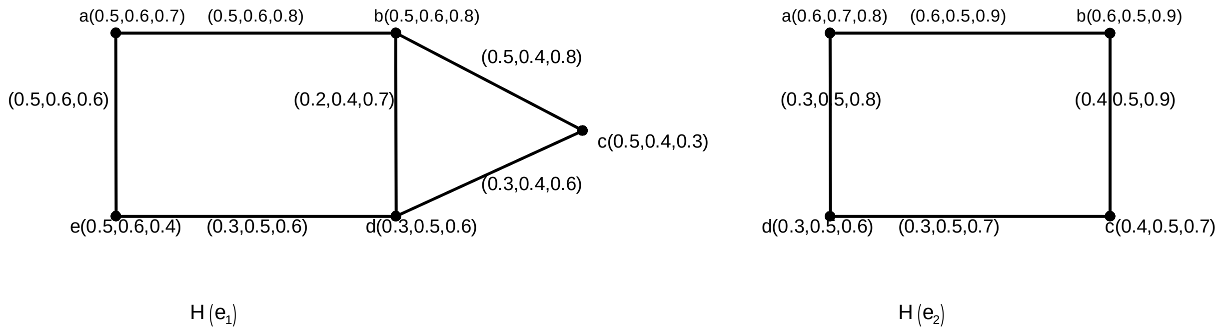

Figure 5

Corresponding to the parameter $H(e_1) = (ab), (bc), (cd), (de)$ are strong arc also for corresponding to the parameter $H(e_2)$ all arcs are strong.

Here for corresponding parameter $H(e_1)$, $d_s(a) = (0.5,0.6,0.8), d_s(b) = (1.0,1.0,1.6)$, $d_s(c) = (0.8,0.8,1.4), d_s(d) = (0.6,0.9,1.2), d_s(e) = (0.3,0.5,0.6)$

$$|d_s(a)| = (0.65), |d_s(b)| = (1.2), |d_s(c)| = (1.1), |d_s(d)| = (1.15), |d_s(e)| = (0.6)$$

Here $\delta_s(G) = 0.6$ and $\Delta_s(G) = 1.2$ and also corresponding to the parameter $H(e_2)$ we get, $d_s(a) = (0.9,1.0,1.7), d_s(b) = (1.0,1.0,1.8), d_s(c) = (0.7,1.0,1.6), d_s(d) = (0.6,1.0,1.5)$

$$|d_s(a)| = (1.1), |d_s(b)| = (1.1), |d_s(c)| = (1.05), |d_s(d)| = (1.05)$$

Here $\delta_s(G) = 1.05$ and $\Delta_s(G) = 1.1$ and also corresponding to the parameter $H(e_1)$ we get,

$$d_sN(a) = (0.5,0.6,0.8), d_sN(b) = (1.0,1.0,1.0), d_sN(c) = (0.8,1.1,1.4), d_sN(d) = (1.0,1.0,0.7), d_sN(e) = (0.3,0.5,0.6)$$

$$|d_sN(a)| = (0.65), |d_sN(b)| = (1.5), |d_sN(c)| = (1.25), |d_sN(d)| = (1.50), |d_sN(e)| = (0.6)$$

Here $\delta_sN(G) = 0.6$ and $\Delta_sN(G) = 1.50$ and also corresponding to the parameter $H(e_2)$, we get

$$d_sN(a) = (0.9,1.0,1.5), d_sN(b) = (1.0,1.2,1.5), d_sN(c) = (0.9,1.0,1.5), d_sN(d) = (1.0,1.2,1.5)$$

$$|d_sN(a)| = (1.2), |d_sN(b)| = (1.35), |d_sN(c)| = (1.2), |d_sN(d)| = (1.35),$$

Here $\delta_sN(G) = 1.2$and $\Delta_sN(G) = 1.35$.

Definition 4.5 The strong size and the strong order of neutrosophic soft graph of G are defined by

$$S_{NS}(G) = \{\sum_{e\in A} \sum_{uv\in E} \frac{1+T_{K(e)}(uv)+I_{K(e)}(uv)-F_{K(e)}(uv)}{2} / \text{ uv is a strongarc}\} \text{ and}$$

$$O_{NS}(G) = \{\sum_{e\in A} \sum_{u\in V} \frac{1 + T_{J(e)}(u) + I_{J(e)}(u) - F_{J(e)}(u)}{2} \text{ uv is a strongarc}\}$$

Example 4.6 Consider above Figure 5 neutrosophic soft graph G for a strong arc $H(e_1)$ is

$(ab), (bc), (cd), (de)$ in $H(e_1)$ we get for corresponding parameter $e_1 S_{NS}(e_1) = \frac{1.3+1.1+1.1+1.2}{2} = 2.35$.

Corresponding parameter e_2 all arcs are strong we get

$$S_{NS}(e_2) = \frac{1.2 + 1.0 + 1.1 + 1.0}{2} = 2.15$$

$S_{NS}(G) = 4.5$.

Corresponding parameter $e_1 O_{NS}(e_1) = \frac{1.4+1.3+1.6+1.2+1.7}{2} = 3.6.$

Corresponding parameter e_2 all arcs are strong we get $O_{NS}(e_2) = \frac{1.5+1.2+1.2+1.2}{2} = 2.55.$

$O_{NS}(G) = 6.15.$

Definition 4.7 Let D be a dominating set in a neutrosophic soft graph. The arc weight and the node weight of D are defined as follows, respectively,

$$W_e(D) = \{\sum_{e\in A}\sum_{u\in D, v\in N_S(u)} \frac{1 + \wedge T_{K(e)}(uv) + \wedge I_{K(e)}(uv) - \vee F_{K(e)}(uv)}{2}\}$$

$$W_v(D) = \sum_{e\in A}\sum_{u\in D, v\in N_S(u)} \frac{1 + \wedge T_{J(e)}(u) + \wedge I_{J(e)}(u) - \vee F_{J(e)}(u)}{2}$$

The strong domination number and the strong neighborhood domination number of G are defined as the minimum arc weight and the minimum node weight of dominating sets in G are denoted by $N\delta_S(G)$ and $N\delta_{SN}(G)$ respectively.

Example 4.8 Consider the neutrosophic soft graph G in Figure 5. The dominating set in G are, corresponding to the parameter e_1 sets are

$$D_1 = \{a, d\}, D_2 = \{b, d\}, D_3 = \{b, e\}, D_4 = \{a, b, d\}, D_5 = \{b, d, e\}$$

$$W_e(D_1) = 1.20, W_e(D_2) = 1.1, W_e(D_3) = 1.15, W_e(D_4) = 1.75, W_e(D_5) = 1.70$$

Here corresponding to the parameter e_1 minimum dominating set $N\delta_S(e_1) = \{b, d\}$ and domination number $N\delta_S(e_1) = 1.1$

Similarly, for corresponding to the parameter e_2, the domination sets are

$$D_1 = \{a, b\}, D_2 = \{b, c\}, D_3 = \{c, d\}, D_4 = \{a, c\}, D_5 = \{a, d\}, D_6 = \{b, d\}, D_7 = \{a, b, c\}, D_8 = \{b, d, c\}, D_9 = \{c, d, a\}$$

$$W_e(D_1) = 0.95, W_e(D_2) = 0.95, W_e(D_3) = 0.95, W_e(D_4) = 0.9, W_e(D_5) = 0.95, W_e(D_6) = 1.0, W_e(D_7) = 1.45, W_e(D_8) = 1.45, W_e(D_9) = 1.4$$

Here corresponding to the parameter e_2, the minimum dominating set $N\delta_S(e_2) = \{b, d\}$ and domination number $N\delta_S(e_1) = 0.95$

In addition ,we have Corresponding to the parameter e_1, the dominating set

$$D_1 = \{a, d\}, D_2 = \{b, d\}, D_3 = \{b, e\}, D_4 = \{a, b, d\}, D_5 = \{b, d, e\}$$

$$W_v(D_1) = 1.40, W_v(D_2) = 1.35, W_v(D_3) = 1.2, W_v(D_4) = 1.95, W_v(D_9) = 1.95$$

Here corresponding to the parameter e_1 minimum dominating set $N\delta_{SN}(e_1) = \{b, e\}$ and domination number $N\delta_{SN}(e_1) = 1.2$

similarly, corresponding to the parameter e_2 domination sets are

$$D_1 = \{a, b\}, D_2 = \{b, c\}, D_3 = \{c, d\}, D_4 = \{a, c\}, D_5 = \{a, d\}, D_6 = \{b, d\}, D_7 = \{a, b, c\}, D_8 = \{b, d, c\}, D_9 = \{c, d, a\}$$

$$W_v(D_1) = 1.15, W_v(D_2) = 1.0, W_v(D_3) = 1.0, W_v(D_4) = 1.05, W_v(D_5) = 1.15, W_v(D_6) = 1.1, W_v(D_7) = 1.60, W_v(D_8) = 1.55, W_v(D_9) = 1.60$$

Here corresponding to the parameter e_2 minimum dominating set $N\delta_{SN}(e_2) = D_2, D_3$ and domination number $N\delta_{SN}(e_2) = 1.0.$

5 STRONG PERFECT DOMINATION

In this section, we have define the perfect dominating set and strong perfect domination number of a neutrosophic soft graph using proper condition.

Definition 5.1 Let $G = (G^*, J, K, A)$ be a neutrosophic soft graph. A subset D of V is a perfect dominating set (or D^p) in G, if for every node $v \in V - D$, there exists a only one node $u \in D$ such that u dominates v. A set D^p is said to be minimal perfect dominating set if for each $v \in D^p$,$D^p - v$ is not a perfect dominating set in G.

Example 5.2 Consider the neutrosophic soft graph $G = (V, E)$ figure we see that all arcs are strong arc.

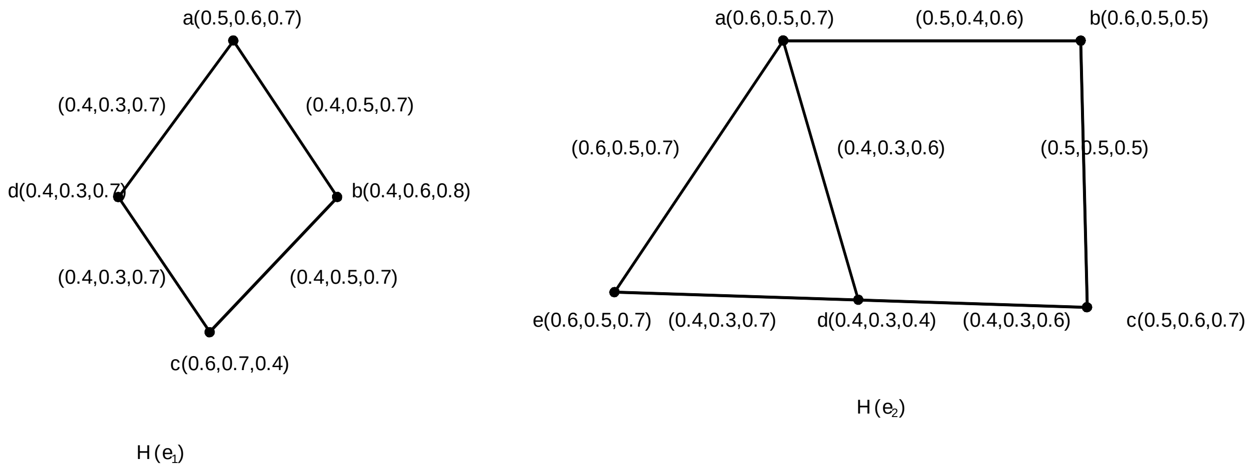

Figure 6

Here corresponding to the parameter e_1, the perfect dominating sets are

$$D_1^P = \{a, b\}, D_2^P = \{b, c\}, D_3^P = \{c, d\}, D_4^P = \{a, d\}$$

Then corresponding to the parameter e_2, the perfect dominating sets are

$$D_1^P = \{a, b\}, D_2^P = \{d, c\}, D_3^P = \{a, d, e\}$$

Proposition 5.3 Any perfect dominating set in neutrosophic soft graph G is a dominating set.

Remark 5.4 The converse of proposition in not correct in general cases. for this consider the neutrosophic soft graph G in figure 6, we see that in $D = \{a, c\}$ is a domination set in G, but it is not a a perfect domination set. Because b and d has two strong neighbors in D.

Definition 5.5 The strong perfect domination number of a neutrosophic soft graph G is defined as the minimum arc weights of perfect dominating sets of G which is denoted by $N\delta_{SP}(G)$.

Example 5.6 Consider the neutrosophic soft graph $G = (G^*, J, K, A)$ in Figure 6

Corresponding to the parameter e_1, the perfect domination sets are,

$D_1^P = \{a, b\}, D_2^P = \{b, c\}, D_3^P = \{c, d\}, D_4^P = \{a, d\}$ in $H(e_1)$ we get

$$W_e(D_1^P) = 1.1, W_e(D_2^P) = 1.1, W_e(D_3^P) = 1.0, W_e(D_4^P) = 1.0$$

Then $N\delta_{SP}(e_1) = 1.0$

Corresponding to the parameter e_2, the perfect domination sets are,

$D_1^P = \{a, b\}, D_2^P = \{d, c\}, D_3^P = \{a, d, e\}$ in $H(e_2)$ we get

$$W_e(D_1^P) = 1.15, W_e(D_2^P) = 1.05, W_e(D_3^P) = 1.50$$

Then $N_{SP}(e_2) = 1.05$, $N\delta_{SP}(G) = 1.05$.

Theorem 5.7 A perfect dominating set D^P of an NSG, $G = (G^*, J, K, A)$ is a minimal perfect dominating set if and only if for each $d \in D^P$ one of the following conditions holds.

(i) $N_s(v) \cap D^P = \{\emptyset\}$ or

(ii) There is a vertex $u \in V - \{D\}$ such that $N_s(u) \cap D^P = \{v\}$.

Proof. Let D^P be a minimal perfect dominating set and $v \in D^P$. Suppose that (i) and (ii) are not established. Then there exists a node $u \in D^P$ such that uv is strong and v has no strong neighbors in $V - D^P$. Therefore $D^P - \{v\}$ is a perfect dominating set in G, which is contradiction by the minimality of D^P.

Conversely, suppose that (i) or (ii) is established and D^P is not a minimal perfect dominating set in G. Then there exists $v \in V - D^P$ such that $D^P - \{v\}$ is a perfect dominating set. Hence v has a strong neighbor in D^P and so (i)is not established. Then there exists is a node $u \in V - D^P$ such that u isa strong neighbor of v and since $D^P - \{v\}$ is a dominating set, then u has a strong neighbor in $D^P - \{v\}$. Therefore $u \in V - D^P$ has two strong neighbors in D^P and so D^P is not a perfect dominating set, that is a contradiction. Then D^P is a minimal perfect dominating set in G.

Corollary 5.8 A dominating set D in a neutrosophic soft graph $G = (V, E)$ is a minimal dominating set if and only if for each node $v \in D$,either

(i) $N_s(v) \cap D^P = \{\emptyset\}$ or.

(ii) There is a vertex $u \in V - \{D\}$ such that $N_s(u) \cap D^P = \{v\}$.

Theorem 5.9 Let G be a neutrosophic soft graph which every its node has at least one strong neighbor. If D^P is a minimal perfect dominating set in G, then $V - D^P$ is a dominating set.

Proof. Suppose that D^P be a minimal perfect dominating set in G and $v \in V - (V - D^P)$. If there is no $u \in V - D^P$ such that $N_S(u) \cap D^P = \{v\}$. Then by above theorem, $N_S(v) \cap D^P = \{\emptyset\}$. Therefore there exists a node in G which has no strong neighbors that is contradiction. This implies that $V - D^P$ is a dominating set.

Corollary 5.10 Let G be a neutrosophic soft graph every node of which has at least one strong neighbor. If D is a minimal dominating set in G, then $V - D$ is a dominating set in G.

Theorem 5.11 Let G be a neutrosophic soft graph every node of which has exactly one strong neighbor. If D^P is a minimal perfect dominating set in G, then $V - D^P$ is a perfect dominating set in G.

Proof. Suppose that D^P is a minimal perfect dominating set in the neutrosophic soft graph G. Then by above theorem $V - D^P$ is a dominating set and since every node in G has exactly one strong neighbor, $V - D^P$ is a perfect dominating set in G.

6 Conclusion

In this work, we derived the domination number of neutrosophic soft graphs and elaborate them with suitable examples by using strength of path and strength of connectedness. Further, we investigate some remarkable properties of independent domination number, strong neighborhood domination and strong perfect domination of neutrosophic soft graph and the proposed concepts are described with suitable examples. Further we can extend to investigate the isomorphic properties of the proposed graph

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On Neutrosophic Vague Graphs

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Abstract: In this work, the new concept of neutrosophic vague graphs are introduced form the ideas of neutrosophic vague sets. Moreover, some remarkable properties of strong neutrosophic vague graphs, complete neutrosophic vague graphs and self-complementary neutrosophic vague graphs are investigated and the proposed concepts are described with suitable examples.

Keywords: Neutrosophic vague graphs, Complete neutrosophic vague graph, Strong neutrosophic vague graph.

1. Introduction

Initially, vague set theory was first investigated by Gau and Buehrer [30] which is an extension of fuzzy set theory. Vague sets are regarded as a special case of context-dependent fuzzy sets. In order to handle the indeterminate and inconsistent information, the neutrosophic set is introduced by the author Smarandache and studied extensively about neutrosophic set [14] - [37] and it receives applications in many fields. In neutrosophic set, the indeterminacy value is quantified explicitly and truth-membership, indeterminacy membership, and false-membership are defined completely independent, if the sum of these values lies between 0 and 3. The new developments of neutrosophic theory are extensively studied in [1] - [6]. Molodtsov [28] firstly introduced the soft set theory as a general mathematical tool to with uncertainty and vagueness. Akram [9] established the certain notions including strong neutrosophic soft graphs and complete neutrosophic soft graphs. The authors [7] first introduce the concept of neutrosophic vague soft expert set which is a combination of neutrosophic vague set and soft expert set to improve the reasonability of decision making in reality. Neutrosophic vague set theory are introduced in [8]. The operations on single valued neutrosophic graphs are studied in [11]. Further, intuitionistic neutrosophic soft set and graphs are developed in [13]. Now, the domination in vague graphs and its is application are discussed in [16]. Intuitionistic neutrosophic soft set are studied in [18]. Interval valued neutrosophic graphs are developed by the author Broumi [22,23,25]. Interval neutrosophic vague sets are intiated in [31]. Motivation of the aforementioned works, we introduced the concept of neutrosophic vague graphs and strong neutrosophic vague graphs. This is a new developed theory which is the combination of neutrsophic graphs and vague graphs. Here the sum of Truth, Intermediate and False membership value lies between 0 and 2 since the truth and false membership are dependent variables. Here the complement of neutrosophic vague graphs is again neutrosophic vague graphs. This development theory will be applied in Operation Research, Social network problems. Particularly, fake profile is one of the big problems of social networks. Now, it has become easier to create a fake profile. People often use fake profile to insult, harass someone, involve in unsocial activities, etc. This model can be reformulated in the abstract form to be applied in neutrosophic vague graphs. The major contribution of this work as follows:

- Newly introduced neutrosophic vague graphs, neutrosophic vague subgraphs, constant neutrosophic vague graphs with examples.
- Further we presented some remarkable properties of strong neutrosophic vague graphs with suitable examples.

2 Preliminaries

Definition 2.1 [10] A vague set A on a non empty set X is a pair (T_A, F_A), where $T_A: X \to [0,1]$ and $F_A: X \to [0,1]$are true membership and false membership functions, respectively, such that

$$0 \leq T_A(r) + F_A(r) \leq 1 \text{ for any } r \in X.$$

Let X and Y be two non-empty sets. A vague relation R of X to Y is a vague set R on $X \times Y$ that is $R = (T_R, F_R)$, where $T_R: X \times Y \to [0,1], F_R: X \times Y \to [0,1]$ which satisfies the condition:

$$0 \leq T_R(r,s) + F_R(r,s) \leq 1 \text{ for any } r \in X.$$

Let $G = (R, S)$ be a graph. A pair $G = (J, K)$ is named as a vague graph on G^* or a vague graph where $J = (T_J, F_J)$ is a vague set on R and $K = (T_K, F_K)$ is a vague set on $S \subseteq R \times R$ such that for each $rs \in S$,

$$T_K(rs) \leq (T_J(r) \wedge T_J(s)) \& F_K(rs) \geq (T_J(r) \vee F_J(s)).$$

Definition 2.2 [9] A Neutrosophic set $A \subset B$, (i.e) $A \subseteq C$ if $\forall r \in X, T_A(r) \leq T_B(r), I_A(r) \geq I_B(r)$and $F_A(r) \geq F_B(r)$.

Definition 2.3 [12, 26, 30] Let X be a space of points (objects), with a generic elements in X known by r. A single valued neutrosophic set (SVNS) A in X is characterized by truth-membership function $T_A(r)$, indeterminacy-membership function $I_A(r)$ and falsity-membership-function $F_A(r)$. For each point r in X, $T_A(r), F_A(r), I_A(r) \in [0,1]$.

$$A = \{r, T_A(r), F_A(r), I_A(r)\} \text{ and } 0 \leq T_A(r) + I_A(r) + F_A(r) \leq 3$$

Definition 2.4 [19, 20] A neutrosophic graph is represented as a pair $G^* = (V, E)$ where

(i) $R = \{r_1, r_2, .., r_n\}$ such that $T_1 = R \to [0,1]$, $I_1 = R \to [0,1]$ and $F_1 = R \to [0,1]$ denote the degree of truth-membership function, indeterminacy function and falsity-membership function, respectively and

$$0 \leq T_A(r) + I_A(r) + F_A(r) \leq 3$$

(ii) $S \subseteq R \times R$ where $T_2 = S \to [0,1]$, $I_2 = S \to [0,1]$ and $F_2 = S \to [0,1]$ are such that

$$T_2(rs) \leq \{T_1(r) \wedge T_1(s)\},$$
$$I_2(rs) \geq \{I_1(r) \vee I_1(s)\},$$
$$F_2(rs) \geq \{F_1(r) \vee F_1(s)\},$$
$$\text{and } 0 \leq T_2(rs) + I_2(rs) + F_2(rs) \leq 3, \forall rs \in R$$

Definition 2.5 [8] A neutrosophic vague set A_{NV} (NVS in short) on the universe of discourse X written as

$$A_{NV} = \{\langle r, \hat{T}_{A_{NV}}(r), \hat{I}_{A_{NV}}(r), \hat{F}_{A_{NV}}(r)\rangle, r \in X\}$$

whose truth-membership, indeterminacy membership and falsity-membership function is defined as

$$\hat{T}_{A_{NV}}(r) = [\hat{T}^-(r), \hat{T}^+(r)], [\hat{I}^-(r), \hat{I}^+(r)], [\hat{F}^-(r), \hat{F}^+(r)],$$

where $T^+(r) = 1 - F^-(r), F^+(r) = 1 - T^-(r)$, and $0 \le T^-(r) + I^-(r) + F^-(r) \le 2$.

Definition 2.6 [8] The complement of NVS A_{NV} is denoted by A^c_{NV} and it is represented by

$$\hat{T}^c_{A_{NV}}(r) = [1 - T^+(r), 1 - T^-(r)],$$
$$\hat{I}^c_{A_{NV}}(r) = [1 - I^+(r), 1 - I^-(r)],$$
$$\hat{F}^c_{A_{NV}}(r) = [1 - F^+(r), 1 - F^-(r)],$$

Definition 2.7 [8] Let A_{NV} and B_{NV} be two NVSs of the universe U. If for all $r_i \in U$,

$$\hat{T}_{A_{NV}}(r_i) = \hat{T}_{B_{NV}}(r_i), \hat{I}_{A_{NV}}(r_i) = \hat{I}_{B_{NV}}(r_i), \hat{F}_{A_{NV}}(r_i) = \hat{F}_{B_{NV}}(r_i)$$

then the NVS A_{NV} are included by B_{NV}, denoted by $A_{NV} \subseteq B_{NV}$ where $1 \le i \le n$.

Definition 2.8 [7] The union of two NVSs A_{NV} and B_{NV} is a NVSs, C_{NV}, written as $C_{NV} = A_{NV} \cup B_{NV}$, whose truth membership function, indeterminacy-membership function and false-membership function are related to those of A_{NV} and B_{NV} by

$$\hat{T}_{C_{NV}}(x) = [\max(\hat{T}^-_{A_{NV}}(r)\hat{T}^-_{B_{NV}}(r)), \max(\hat{T}^+_{A_{NV}}(r)\hat{T}^+_{B_{NV}}(r))]$$
$$\hat{I}_{C_{NV}}(x) = [\min(\hat{I}^-_{A_{NV}}(r)\hat{I}^-_{B_{NV}}(r)), \min(\hat{I}^+_{A_{NV}}(r)\hat{I}^+_{B_{NV}}(r))]$$
$$\hat{F}_{C_{NV}}(x) = [\min(\hat{F}^-_{A_{NV}}(r)\hat{F}^-_{B_{NV}}(r)), \min(\hat{F}^+_{A_{NV}}(r)\hat{F}^+_{B_{NV}}(r))]$$

Definition 2.9 [7] The intersection of two NVSs A_{NV} and B_{NV} is a NVSs C_{NV}, written as $C_{NV} = A_{NV} \cap B_{NV}$, whose truth membership function, indeterminacy-membership function and false-membership function are related to those of A_{NV} and B_{NV} by

$$\hat{T}_{C_{NV}}(x) = [\min(\hat{T}^-_{A_{NV}}(r)\hat{T}^-_{B_{NV}}(r)), \min(\hat{T}^+_{A_{NV}}(r)\hat{T}^+_{B_{NV}}(r))]$$
$$\hat{I}_{C_{NV}}(x) = [\max(\hat{I}^-_{A_{NV}}(r)\hat{I}^-_{B_{NV}}(r)), \max(\hat{I}^+_{A_{NV}}(r)\hat{I}^+_{B_{NV}}(r))]$$
$$\hat{F}_{C_{NV}}(x) = [\max(\hat{F}^-_{A_{NV}}(r)\hat{F}^-_{B_{NV}}(r)), \max(\hat{F}^+_{A_{NV}}(r)\hat{F}^+_{B_{NV}}(r))]$$

3 NEUTROSOPHIC VAGUE GRAPH

Definition 3.1 Let $G^* = (R, S)$ be a graph. A pair $G = (J, K)$ is named as a neutrosophic vague graph (NVG) on G^* or a neutrosophic graph where $J = (\hat{T}_J, \hat{I}_J, \hat{F}_J)$ is a neutrosophic vague set on R and $K = (\hat{T}_K, \hat{I}_K, \hat{F}_K)$ is a neutrosophic vague set $S \subseteq R \times R$ where

(1) $R = \{r_1, r_2, \ldots, r_n\}$ such that

$$T_J^-: R \to [0,1], I_J^-: R \to [0,1], F_J^-: R \to [0,1]$$

which satisfies the condition $F_J^- = [1 - T_J^+]$

$$T_J^+: R \to [0,1], I_J^+: R \to [0,1], F_J^+: R \to [0,1]$$

which satisfies the condition $F_J^+ = [1 - T_1^-]$ indicates the degree of truth membership function, indeterminacy membership and falsity membership of the element $r_i \in R.$, and

$$0 \le T_J^-(r_i) + I_J^-(r_i) + F_J^-(r_i) \le 2.$$
$$0 \le T_J^+(r_i) + I_J^+(r_i) + F_J^+(r_i) \le 2.$$

(2) $S \subseteq R \times R$ where

$$T_K^-: R \times R \to [0,1], I_K^-: R \times R \to [0,1], F_K^-: R \times R \to [0,1]$$
$$T_K^+: R \times R \to [0,1], I_K^+: R \times R \to [0,1], F_K^+: R \times R \to [0,1]$$

indicates the degree of truth membership function, indeterminacy membership and falsity membership of the element $r_i, r_j \in S.$ respectively and such that

$$0 \leq T_K^-(r_i) + I_K^-(r_i) + F_K^-(r_i) \leq 2.$$
$$0 \leq T_K^+(r_i) + I_K^+(r_i) + F_K^+(r_i) \leq 2.$$

such that

$$T_K^-(rs) \leq \{T_J^-(r) \wedge T_J^-(s)\}$$
$$I_K^-(rs) \leq \{I_J^-(r) \wedge I_J^-(s)\}$$
$$F_K^-(rs) \leq \{F_J^-(r) \vee F_J^-(s)\},$$

similarly

$$T_K^+(rs) \leq \{T_J^+(r) \wedge T_J^+(s)\}$$
$$I_K^+(rs) \leq \{I_J^+(r) \wedge I_J^+(s)\}$$
$$F_K^+(rs) \leq \{F_J^+(r) \vee F_J^+(s)\}.$$

Example 3.2 A neutrosophic vague graph $G = (J, K)$ such that $J = \{a, b, c\}$ and $K = \{ab, bc, ca\}$ indicated by

$$\hat{a} = T[0.5,0.6], I[0.4,0.3], F[0.4,0.5], \hat{b} = T[0.4,0.6], I[0.7,0.3], F[0.4,0.6],$$
$$\hat{c} = T[0.4,0.4], I[0.5,0.3], F[0.6,0.6]$$
$$a^- = (0.5,0.4,0.4), b^- = (0.4,0.7,0.4), c^- = (0.4,0.5,0.6)$$
$$a^+ = (0.6,0.3,0.5), b^+ = (0.6,0.3,0.6), c^+ = (0.4,0.3,0.6)$$

.

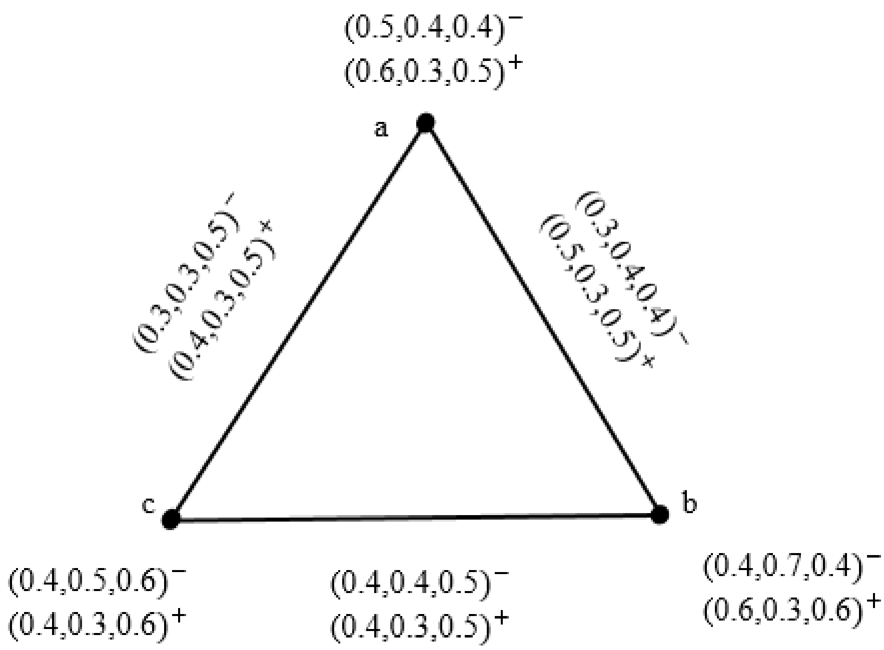


Figure 1

NEUTROSOPHIC VAGUE GRAPH

Definition 3.3 A neutrosophic vague graph $H = (J'(r), K'(r))$ is meant to be a neutrosophic vague subgraph of the NVG $G = (J, K)$ if $J'(r) \subseteq J(r)$ and $K'(rs) \subseteq K'(rs)$ in other words, if

$$\hat{T}_J^{'-}(r) \leq \hat{T}_J^-(r)$$
$$\hat{I}_J^{'-}(r) \leq \hat{I}_J^-(r)$$

$\hat{F}_J^{'-}(r) \geq \hat{F}_J^{-}(r) \forall r \in R$

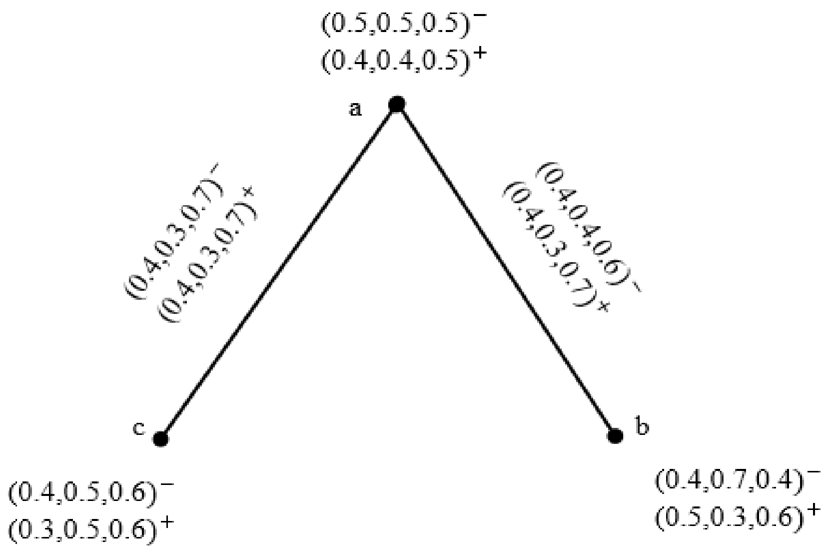
and

$\hat{T}_K^{'+}(rs) \leq \hat{T}_K^{-}(rs)$

$\hat{I}_K^{'+}(rs) \leq \hat{I}_K^{-}(rs)$

$\hat{F}_K^{'+}(rs) \geq \hat{F}_K^{-}(rs) \forall (rs) \in S.$

Example 3.4 A neutrosophic vague graph $G = (J, K)$ in Figure (1)



H_1 Figure 2

H_1 is a neutrosophic vague subgraph of G

Definition 3.5 The two vertices are said to be adjacent in a neutrosophic vague graph $G = (J, K)$ if

$$T_K^-(rs) = \{T_J^-(r) \wedge T_J^-(s)\}$$
$$I_K^-(rs) = \{I_J^-(r) \wedge I_J^-(s)\}$$
$$F_K^-(rs) = \{F_J^-(r) \vee F_J^-(s)\} \quad \text{and}$$
$$T_K^+(rs) = \{T_J^+(r) \wedge T_J^+(s)\}$$
$$I_K^+(rs) = \{I_J^+(r) \wedge I_J^+(s)\}$$
$$F_K^+(rs) = \{F_J^+(r) \vee F_J^+(s)\}$$

In this case, r and s are known to be neighbours and (rs) is incident at r and s also.

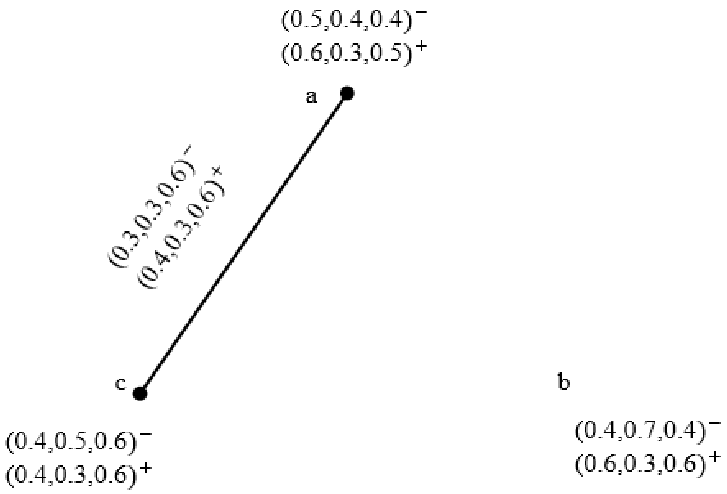
Definition 3.6 A path ρ in a NVG $G = (J, K)$ is a sequence of distinct vertices $r_0, r_1, \ldots, r_n$ such that $T_K^-(r_{i-1}, r_1) > 0$, $I_K^-(r_{i-1}, r_1) > 0$, $F_K^-(r_{i-1}, r_1) > 0$, $T_K^+(r_{i-1}, r_1) > 0$, $I_K^+(r_{i-1}, r_1) > 0$, $F_K^+(r_{i-1}, r_1) > 0$, for $0 \leq i \leq 1$, here $n \leq 1$ is called the length of the path ρ. A single node or single vertex r_i may all consider as a path.

Definition 3.7 A neutrosophic vague graph $G = (J, K)$ is said to be connected if every pair of vertices has at least on neutrosophic vague path between them otherwise it is disconnected.

Definition 3.8 A vertex $r_i \in R$ of neutrosophic vague graph $G = (J, K)$ called as a pendent vertex if there is no effective edge incident at x_i.

Definition 3.9 A vertex in a neutrosophic vague graph $G = (J, K)$ having exactly one neighbours is called a isolated vertex. otherwise,it is called non-isolated vertex. An edge in a neutrosophic vague graph incident with a isolated vertex is called a isolated edge other words it is called non-isolated edge. A vertex in a neutrosophic vague graph adjacent to the isolated vertex is called an support of the pendent edge.

Example 3.10 A neutrosophic vague graph $G = (J, K)$ in figure (3)



H_1 Figure 3

NEUTROSOPHIC VAGUE GRAPH

In figure (3), the neutrosophic vague vertex b is an pendent vertex.

Definition 3.11 Let $G = (J, K)$ be a neutrosophic vague graph. Then the degree of a vertex $r \in G$ is a sum of degree truth membership, sum of indeterminacy membership and sum of falsity membership of all those edges which are incident on vertex r represented by $d(r) = ([d^-_{T_J}(r), d^+_{T_J}(r)], [d^-_{I_J}(r), d^+_{I_J}(r)], [d^-_{F_J}(r), d^+_{F_J}(r)])$ where

$d^-_{T_J}(r) = \sum_{r\neq s} T^-_K(rs), d^+_{T_J}(r) = \sum_{r\neq s} T^+_K(rs)$ indicates the degree of truth membership vertex

$d^-_{I_J}(r) = \sum_{r\neq s} I^-_K(rs), d^+_{I_J}(r) = \sum_{r\neq s} I^+_K(rs)$ indicates the degree of indeterminacy membership vertex

$d^-_{F_J}(r) = \sum_{r\neq s} F^-_K(rs), d^+_{F_J}(r) = \sum_{r\neq s} F^+_K(rs)$ indicates the degree of falsity membership vertex for all $r, s \in J$.

Example 3.12 A neutrosophic vague graph $G = (J, K)$ in figure (1), we have the degree of each vertex as follows

$$d^-_T(a) = (0.6,0.7,0.9), d^-_F(b) = (0.7,0.8,1.3), d^-_F(c) = (0.7,0.7,1.0),$$
$$d^+_T(a) = (0.9,0.6,1.0), d^+_F(b) = (0.9,0.6,1.0), d^+_F(c) = (0.8,0.6,1.0)$$

Definition 3.13 A neutrosophic vague graph $G = (J, K)$ is called constant if degree of each vertex is $A = (A_1, A_2, A_3)$ that is $d(x) = (A_1, A_2, A_3)$ for all $x \in V$.

Example 3.14 Consider a neutrosophic vague graph $G = (J, K)$ in figure (4)defined by

$$\hat{a} = T[0.5,0.6], I[0.6,0.4], F[0.4,0.5], \hat{b} = T[0.4,0.4], I[0.4,0.6], F[0.6,0.6],$$

$$\hat{c} = T[0.4,0.6], I[0.7,0.3], F[0.4,0.6], \hat{d} = T[0.6,0.4], I[0.3,0.7], F[0.6,0.4]$$

$$a^- = (0.5,0.6,0.4), b^- = (0.4,0.4,0.6), c^- = (0.4,0.7,0.4), d^- = (0.6,0.3,0.6)$$

$$a^+ = (0.6,0.4,0.5), b^+ = (0.4,0.6,0.6), c^+ = (0.6,0.3,0.6), d^+ = (0.4,0.7,0.4).$$

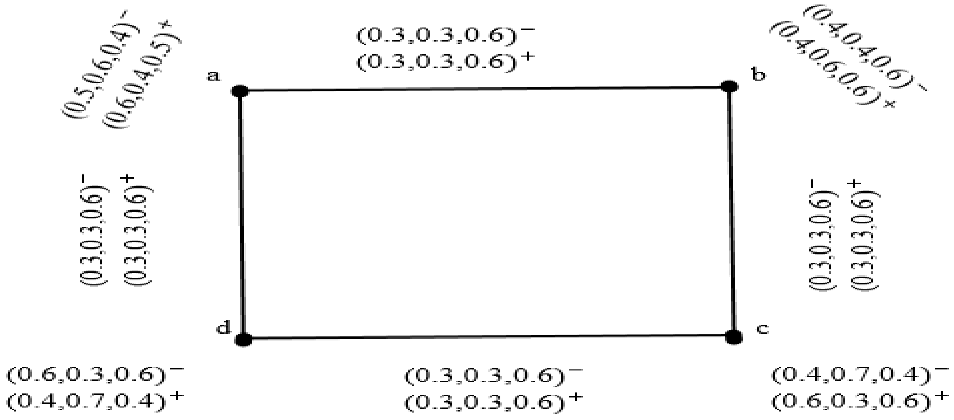


Figure 4

CONSTANT NEUTROSOPHIC VAGUE GRAPH

Clearly as it is seen in figure(4) G is constant neutrosophic vague graph since the degree of $(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ and $\hat{d} = (0.6,0.6,1.2)$.

Definition 3.15 The complement of neutrosophic vague graph $G = (J, K)$ on G^* is a neutrosophic vague graph G^c on G^* where

- $J^c(r) = J(r)$
- $T_J^{-c}(r) = T_J^-(r), I_J^{-c}(r) = I_J^-(r), F_J^{-c}(r) = F_J^-(r)$ for all $r \in R$.
- $T_J^{+c}(r) = T_J^+(r), I_J^{+c}(r) = I_J^+(r), F_J^{+c}(r) = F_J^+(r)$ for all $r \in R$.
- $T_K^{-c}(rs) = \{T_J^-(r) \wedge T_J^-(s)\} - T_K^-(rs)$

  $$I_K^{-c}(rs) = \{I_J^-(r) \wedge I_J^-(s)\} - I_K^-(rs)$$

  $$F_K^{-c}(rs) = \{F_J^-(r) \vee F_J^-(s)\} - F_K^-(rs) \text{ for all } (rs) \in S$$
- $T_K^{+c}(rs) = \{T_J^+(r) \wedge T_J^+(s)\} - T_K^+(rs)$

  $$I_K^{+c}(rs) = \{I_J^+(r) \wedge I_J^+(s)\} - I_K^+(rs)$$

  $$F_K^{+c}(rs) = \{F_J^+(r) \vee F_J^+(s)\} - F_K^+(rs) \text{ for all } (rs) \in S$$

4 Strong Neutrosophic Vague Graphs

Definition 4.1 A neutrosophic vague graph $G = (J, K)$ of $G^* = (R, S)$ is named as a strong neutrosophic vague graph if

$$T_K^-(rs) = \{T_J^-(r) \wedge T_J^-(s)\}$$

$$I_K^-(rs) = \{I_J^-(r) \wedge I_J^-(s)\}$$

$$F_K^-(rs) = \{F_J^-(r) \vee F_J^-(s)\} \quad \text{and}$$

$$T_K^+(rs) = \{T_J^+(r) \wedge T_J^+(s)\}$$

$$I_K^+(rs) = \{I_J^+(r) \wedge I_J^+(s)\}$$

$$F_K^+(rs) = \{F_J^+(r) \vee F_J^+(s)\} \text{ for all } (rs \in S)$$

Example 4.2 A neutrosophic vague graph $G = (J, K)$ such that $J = \{a, b, c\}$ and $K = \{ab, bc, ca\}$ defined by $\hat{a} = T[0.3,0.4], I[0.4,0.6], F[0.6,0.7], \hat{b} = T[0.6,0.4], I[0.6,0.7], F[0.6,0.4],$

$$\hat{c} = T[0.7,0.7], I[0.5,0.6], F[0.3,0.3]$$

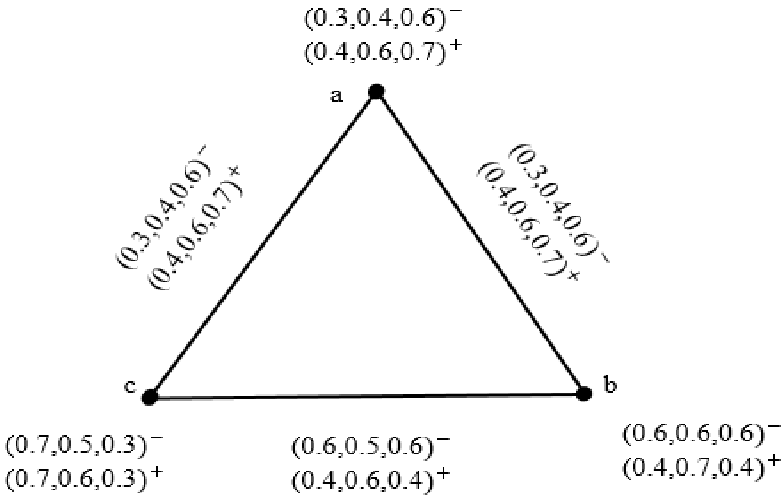


Figure 5

STRONG NEUTROSOPHIC VAGUE GRAPH

Remark 4.3 If $G = (J, K)$ is a neutrosophic vague graph on G^* then from above definition, it follow that G^{c^c} is given by the neutrosophic vague graph $G^{c^c} = J^{c^c}, K^{c^c}$ on G^* where

- $(J^c)^c(r) = J(r)$
- $(T_J^{-c})^c(r) = T_J^-(r), I_J^{-c}(r) = I_J^-(r), F_J^{-c}(r) = F_J^-(r)$ for all $r \in R$.
- $(T_J^{+c})^c(r) = T_J^+(r), I_J^{+c}(r) = I_J^+(r), F_J^{+c}(r) = F_J^+(r)$ for all $r \in R$.
- $(T_K^{-c})^c(rs) = \{T_J^-(r) \wedge T_J^-(s)\} - T_K^-(rs)$

$$(I_K^{-c})^c(rs) = \{I_J^-(r) \wedge I_J^-(s)\} - I_K^-(rs)$$

$$(F_K^{-c})^c(rs) = \{F_J^-(r) \vee F_J^-(s)\} - F_K^-(rs) \text{ for all } (rs) \in S$$

- $(T_K^{+c})^c(rs) = \{T_J^+(r) \wedge T_J^+(s)\} - T_K^+(rs)$

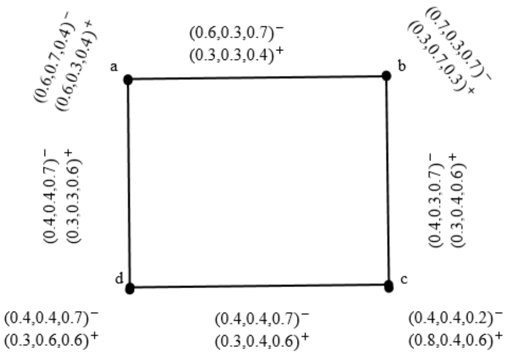
$$(I_K^{+c})^c(rs) = \{I_J^+(r) \wedge I_J^+(s)\} - I_K^+(rs)$$

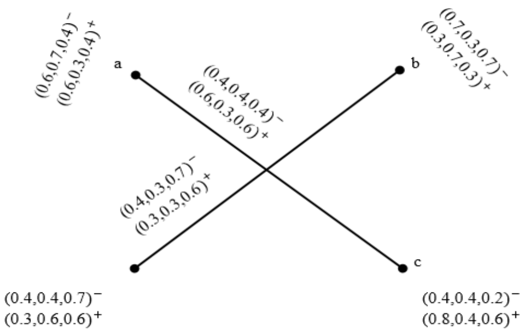
$$(F_K^{+c})^c(rs) = \{F_J^+(r) \vee F_J^+(s)\} - F_K^+(rs) \text{ for all } (rs) \in S$$

for any neutrosophic vague graph G, G^c is strong neutrosophic graph and $G \subseteq G^c$

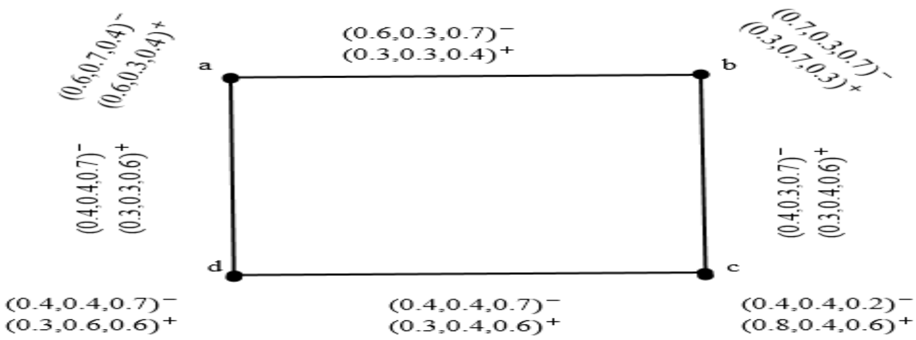
Definition 4.4 A strong neutrosophic graph G is called self-complementary if $G \cong G^c$where G^c is the complement of neutrosophic vague graph G.

Example 4.5 A neutrosophic vague graph $G = (J, K)$ such that $J = \{a, b, c, d\}$ and $K = \{ab, bc, cd, da\}$ defined as follows: consider a neutrosophic vague graph G as in figure(6)





G STRONG NVG **G^c STRONG NVG**



G^{c^c} STRONG NVG

Clearly, as it is seen in figure (6) $G \cong G^{c^c}$.

Hence G is self complementary.

Proposition 4.6 Let $G = (J, K)$ be a strong neutrosophic vague graph if

$$T_K^-(rs) = \{T_J^-(r) \wedge T_J^-(s)\}$$
$$I_K^-(rs) = \{I_J^-(r) \wedge I_J^-(s)\}$$
$$F_K^-(rs) = \{F_J^-(r) \vee F_J^-(s)\}$$
$$T_K^+(rs) = \{T_J^+(r) \wedge T_J^+(s)\}$$
$$I_K^+(rs) = \{I_J^+(r) \wedge I_J^+(s)\}$$
$$F_K^+(rs) = \{F_J^+(r) \vee F_J^+(s)\} \text{ for all } rs \in K$$

Then G is self complementary.

Proof. Let $G = (J, K)$ be a strong neutrosophic vague graph such that

$$\hat{T}_K(rs) = \frac{1}{2}\min[\hat{T}_J(r), \hat{T}_J(s)]$$

$$\hat{I}_K(rs) = \frac{1}{2}\min[\hat{I}_J(r), \hat{I}_J(s)]$$

$$\hat{F}_K(rs) = \frac{1}{2}\max\hat{F}_J(r), \hat{F}_J(s)$$

for all $rs \in J$ then $G \approx G^{c^c}$ under the identity map $I: J \to J$. Hence G is self complementary

Proposition 4.7 Let G be a self complementary neutrosophic vague graph then

$$\sum_{r\neq s} \widehat{T}_K(rs) = \frac{1}{2}\sum_{r\neq s} \min\{\widehat{T}_J(r), \widehat{T}_J(s)\}$$

$$\sum_{r\neq s} \hat{I}_K(rs) = \frac{1}{2}\sum_{r\neq s} \min\{\hat{I}_J(r), \hat{I}_J(s)\}$$

$$\sum_{r\neq s} \widehat{F}_K(rs) = \frac{1}{2}\sum_{r\neq s} \max\{\widehat{F}_J(r), \widehat{F}_J(s)\}$$

Proof. If G be an self complementary neutrosophic vague graph then there exist an isomorphism $f: J_1 \to J_2$ satisfy

$$\widehat{T}^c_{J_1}(f(r)) = \widehat{T}_{J_1}(f(r)) = \widehat{T}_{J_1}(r)$$
$$\hat{I}^c_{J_1}(f(r)) = \hat{I}_{J_1}(f(r)) = \hat{I}_{J_1}(r)$$
$$\widehat{F}^c_{J_1}(f(r)) = \widehat{F}_{J_1}(f(r)) = \widehat{F}_{J_1}(r)$$

and

$$\widehat{T}^c_{K_1}(f(r), f(s)) = \widehat{T}_{K_1}(f(r), f(s)) = \widehat{T}_{K_1}(rs)$$
$$\hat{I}^c_{K_1}(f(r), f(s)) = \hat{I}_{K_1}(f(r), f(s)) = \hat{I}_{K_1}(rs)$$
$$\widehat{F}^c_{K_1}(f(r), f(s)) = \widehat{F}_{K_1}(f(r), f(s)) = \widehat{F}_{K_1}(rs)$$

we have $\widehat{T}^c_{K_1}(f(r), f(s)) = \min(\widehat{T}^c_{J_1}(r), \widehat{T}^c_{J_1}(s)) - \widehat{T}_{K_1}(f(r), f(s))$. i.e, $\widehat{T}_{K_1}(rs) = \min(\widehat{T}^c_{J_1}(r), \widehat{T}^c_{J_1}(s)) - \widehat{T}_{K_1}(f(r), f(s))$. $\widehat{T}_{K_1}(rs) = \min(\widehat{T}^c_{J_1}(r), \widehat{T}^c_{J_1}(s)) - \widehat{T}_{K_1}(rs)$. that is

$$\textstyle\sum_{r\neq s} \widehat{T}_{K_1}(rs) + \sum_{r\neq s} \widehat{T}_{K_1}(rs) = \sum_{r\neq s} \min(\widehat{T}_{J_1}(r), \widehat{T}_{J_1}(s)).$$

Similarly, $\sum_{r\neq s} \hat{I}_{K_1}(rs) + \sum_{r\neq s} \hat{I}_{K_1}(rs) = \sum_{r\neq s} \min(\hat{I}_{J_1}(r), \hat{I}_{J_1}(s))$

$$\sum_{r\neq s} \widehat{F}_{K_1}(rs) + \sum_{r\neq s} \widehat{F}_{K_1}(rs) = \sum_{r\neq s} \max(\widehat{F}_{J_1}(r), \widehat{F}_{J_1}(s))$$

$$2\sum_{r\neq s} \widehat{T}_{K_1}(rs) = \sum_{r\neq s} \min(\widehat{T}_{J_1}(r), \widehat{T}_{J_1}(s))$$

$$2\sum_{r\neq s} \hat{I}_{K_1}(rs) = \sum_{r\neq s} \min(\hat{I}_{J_1}(r), \hat{I}_{J_1}(s))$$

$$2\sum_{r\neq s} \widehat{F}_{K_1}(rs) = \sum_{r\neq s} \max(\widehat{F}_{J_1}(r), \widehat{F}_{J_1}(s))$$

from the equation of the proposition (4.8) holds.

Proposition 4.8 Let G_1 and G_2 be strong neutrosophic vague graph $\overline{G_1} \approx \overline{G_2}$(isomorphism)

Proof. Assume that G_1 and G_2 are isomorphic there exist a bijective map $f: J_1 \to J_2$ satisfying,

$$\widehat{T}_{J_1}(r) = \widehat{T}_{J_2}(f(r)), \hat{I}_{J_1}(r) = \hat{I}_{J_2}(f(r)), \widehat{F}_{J_1}(r) = \widehat{F}_{J_2}(f(r)), \text{forallr} \in J_1$$

and

$$\widehat{T}_{K_1}(rs) = \widehat{T}_{K_2}(f(r), f(s))$$
$$\hat{I}_{K_1}(rs) = \hat{I}_{K_2}(f(r), f(s))$$
$$\widehat{F}_{K_1}(r) = \widehat{F}_{K_2}(f(r), f(s)) \forall rs \in K_1$$

by definition (4.3) we have

$\mathrm{T}^{\mathrm{c}}_{\mathrm{K}_1}(\mathrm{rs}) = \min(\mathrm{T}_{\mathrm{J}_1}(\mathrm{r}), \mathrm{T}_{\mathrm{J}_1}(\mathrm{s})) - \mathrm{T}_{\mathrm{K}_1}(\mathrm{rs})$

$= \min(\mathrm{T}_{\mathrm{J}_2}\mathrm{f}(\mathrm{r}), \mathrm{T}_{\mathrm{J}_2}\mathrm{f}(\mathrm{s})) - \mathrm{T}_{\mathrm{K}_2}(\mathrm{f}(\mathrm{r})\mathrm{f}(\mathrm{s}))$

$= \mathrm{T}^{\mathrm{c}}_{\mathrm{K}_2}(\mathrm{f}(\mathrm{r})\mathrm{f}(\mathrm{s}))$

$\mathrm{I}^{\mathrm{c}}_{\mathrm{K}_1}(\mathrm{rs}) = \min(\mathrm{I}_{\mathrm{J}_1}(\mathrm{r}), \mathrm{I}_{\mathrm{J}_1}(\mathrm{s})) - \mathrm{I}_{\mathrm{K}_1}(\mathrm{rs})$

$= \min(\mathrm{I}_{\mathrm{J}_2}\mathrm{f}(\mathrm{r}), \mathrm{I}_{\mathrm{J}_2}\mathrm{f}(\mathrm{s})) - \mathrm{I}_{\mathrm{K}_2}(\mathrm{f}(\mathrm{r})\mathrm{f}(\mathrm{s}))$

$= \mathrm{I}^{\mathrm{c}}_{\mathrm{K}_2}(\mathrm{f}(\mathrm{r})\mathrm{f}(\mathrm{s}))$

$\mathrm{F}^{\mathrm{c}}_{\mathrm{K}_1}(\mathrm{rs}) = \max(\mathrm{F}_{\mathrm{J}_1}(\mathrm{r}), \mathrm{F}_{\mathrm{J}_1}(\mathrm{s})) - \mathrm{F}_{\mathrm{K}_1}(\mathrm{rs})$

$= \max(\mathrm{F}_{\mathrm{J}_2}\mathrm{f}(\mathrm{r}), \mathrm{F}_{\mathrm{J}_2}\mathrm{f}(\mathrm{s})) - \mathrm{F}_{\mathrm{K}_2}(\mathrm{f}(\mathrm{r})\mathrm{f}(\mathrm{s}))$

$= \mathrm{F}^{\mathrm{c}}_{\mathrm{K}_2}(\mathrm{f}(\mathrm{r})\mathrm{f}(\mathrm{s}))$

Hence $\mathrm{G}_1^{\mathrm{c}} \approx \mathrm{G}_2^{\mathrm{c}}$ for all $(\mathrm{rs}) \in \mathrm{K}_1$

Definition 4.9 A neutrosophic vague graph $\mathrm{G} = (\mathrm{J}, \mathrm{K})$ is complete if

$$\mathrm{T}_{\mathrm{K}}^{-}(\mathrm{rs}) = \{\mathrm{T}_{\mathrm{J}}^{-}(\mathrm{r}) \wedge \mathrm{T}_{\mathrm{J}}^{-}(\mathrm{s})\}$$

$$\mathrm{I}_{\mathrm{K}}^{-}(\mathrm{rs}) = \{\mathrm{I}_{\mathrm{J}}^{-}(\mathrm{r}), \mathrm{I}_{\mathrm{J}}^{-}(\mathrm{s})\}$$

$$\mathrm{F}_{\mathrm{K}}^{-}(\mathrm{rs}) = \{\mathrm{F}_{\mathrm{J}}^{-}(\mathrm{r}) \vee \mathrm{F}_{\mathrm{J}}^{-}(\mathrm{s})\},$$

similarly,

$$\mathrm{T}_{\mathrm{K}}^{+}(\mathrm{rs}) = \{\mathrm{T}_{\mathrm{J}}^{+}(\mathrm{r}) \wedge \mathrm{T}_{\mathrm{J}}^{+}(\mathrm{s})\}$$

$$\mathrm{I}_{\mathrm{K}}^{+}(\mathrm{rs}) = \{\mathrm{I}_{\mathrm{J}}^{+}(\mathrm{r}) \wedge \mathrm{I}_{\mathrm{J}}^{+}(\mathrm{s})\}$$

$$\mathrm{F}_{\mathrm{K}}^{+}(\mathrm{rs}) = \{\mathrm{F}_{\mathrm{J}}^{+}(\mathrm{r}) \vee \mathrm{F}_{\mathrm{J}}^{+}(\mathrm{s})\}\mathrm{for r, s} \in \mathrm{J}.$$

Example 4.10 Consider a neutrosophic vague graph $\mathrm{G} = (\mathrm{J}, \mathrm{K})$ such that $\mathrm{J} = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}$ and $\mathrm{K} = \{\mathrm{ab}, \mathrm{bc}, \mathrm{cd}, \mathrm{da}\}$ defined by

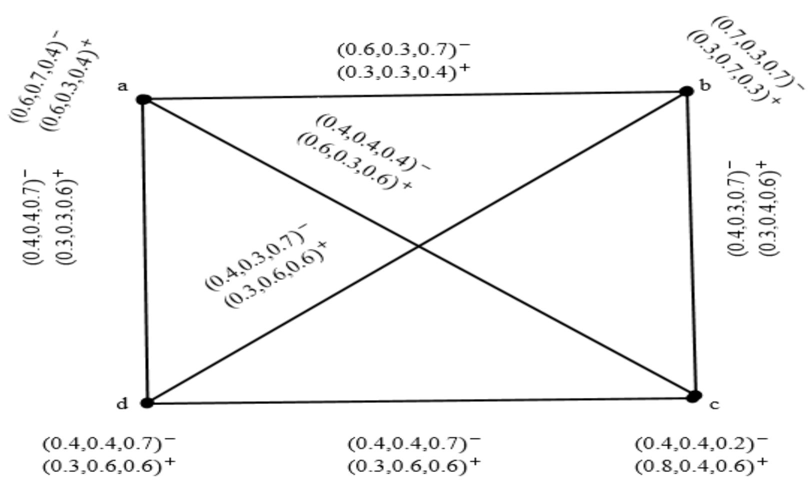


Figure 7

COMPLETE NEUTROSOPHIC VAGUE GRAPH

Definition 4.11 The complement of neutrosophic vague graph $G = (J, K)$ of $G^* = (V, E)$ is a neutrosophic vague complete graph $G = (J^c, K^c)$ on $G^* = (R, S^c)$ where

(1) $J^c(r_i) = J(r_i)$

(2) $\widehat{T}_J^c(r_i) = \widehat{T}_J(r_i), \hat{I}_J^c(r_i) = \hat{I}_J(r_i), \widehat{F}_J^c(r_i) = \widehat{F}_J(r_i)$ for all $r_i \in J$

(3)$\widehat{T}_K^c(r_i s_j) = (\widehat{T}_J(r_i) \wedge \widehat{T}_J(s_j)) - \widehat{T}_K(r_i s_j)$

$$\hat{I}_K^c(r_i s_j) = (\hat{I}_J(r_i) \wedge \hat{I}_J(s_j)) - \hat{I}_K(r_i, s_j)$$

$$\widehat{F}_K^c(r_i s_j) = (\widehat{F}_J(r_i) \vee \widehat{F}_J(s_j)) - \widehat{F}_K(r_i s_j) \quad \text{for all } (r_i s_j) \in K$$

Proposition 4.12 The complement of complete neutrosophic vague graph with no edge. or if G is complete then G^c the edge is empty.

Proof. Let $G = (J, K)$ be a complete neutrosophic vague graph so

$$\widehat{T}_K(r_i s_j) = (\widehat{T}_J(r_i) \wedge \widehat{T}_J(s_j))$$

$$\hat{I}_K(r_i s_j) = (\hat{I}_J(r_i) \wedge \widehat{T}_J(s_j))$$

$$\widehat{F}_K(r_i s_j) = (\widehat{F}_J(r_i) \vee \widehat{F}_J(s_j)) \forall (r_i, s_j) \in J$$

Hence in G^c. Now,

$$\widehat{T}_K^c(r_i s_j) = (\widehat{T}_J(r_i) \wedge \widehat{T}_J(s_j)) - \widehat{T}_K(r_i s_j)$$

$= (\widehat{T}_J(r_i) \wedge \widehat{T}_J(s_j)) - (\widehat{T}_J(r_i) \wedge \widehat{T}_J(s_j)) \ \forall\ i, j, \ldots, n$

$= 0\ \forall, i, j, \ldots, n.$

and

$\hat{I}_K^c(r_i s_j) = (\hat{I}_J(r_i) \wedge \hat{I}_J(s_j)) - \hat{I}_K(r_i s_j)$

$= (\hat{I}_J(r_i) \wedge \hat{I}_J(s_j)) - (\hat{I}_J(r_i) \wedge \hat{I}_J(s_j)) \forall\ i, j, \ldots, n$

$= 0 \forall\ i, j, \ldots, n.$

Similarly $\widehat{F}_K^c(r_i, s_j) = 0$. Thus,$(\widehat{T}_K(r_i, s_j), \hat{I}_K(r_i, s_j), \widehat{F}_K(r_i, s_j)) = (0,0,0)$

Hence, the edge set of G^c is empty if G is a complete neutrosophic vague graph.

Conclusion and futute directions:

This work dealt with the new concept of neutrosophic vague graphs. Moreover, some remarkable properties of strong neutrosophic vague graphs, complete neutrosophic vague graphs and self-complementary neutrosophic vague graphs have been investigated and the proposed concepts were described with suitable examples. Further we can extend to investigate the regular and isomorphic properties of the proposed graph. This can be applied to social network model and operations research.

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Neutrosophic soft cubic Subalgebras of G-algebras

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Abstract: In this paper, neutrosophic soft cubic G-subalgebra is studied through P-union, P-intersection, R-union and R-intersection etc. furthermore we study the notion of homomorphism on G-algebra with some results.

Keywords: G-algebra, Neutrosophic soft cubic set, Neutrosophic soft cubic G-subalgebra, Homomorphism of neutrosophic soft cubic subalgebra.

1 Introduction

Zadeh was the introducer of the fuzzy set and interval-valued fuzzy theory [2] in 1965. Many researchers afterward followed the notions of Zadeh. The cubic set was defined by Jun et al. [9, 10] They used the notion of cubic sets in group and initiated the idea of cubic subgroups. The algebraic structures like BCK/BCI-algebra was introduced by Imai et al. [1] in 1966. This algebra was a field of propositional calculus. Many algebraic structures like G-algebra, BG-algebra, etc. [19, 4] are structured as an extension of Q-algebra. Quadratic B-algebra was investigated by Park et al. [22]. Molodstov gave the concept of soft sets [14] in 1999. Cubic soft set with application and subalgebra in BCK/BCI-algebra were studied by Muhiuddin et al. [15,16]. Senapati et al. [13] generalized the concept of cubic set to B-subalgebra with cubic subalgebra and cubic closed ideals. Subalgebra, ideal are studied with the help of cubic set by Jun et al. [12]. The intuitionistic fuzzy G-subalgebra is studied by Jana et al. [18]. L-fuzzy G-subalgebra was studied by Senapati et al. [7]. As an extension of B-algebra, lots of work on BG-algebra have been done by the Senapati et al. [8]. The idea of a neutrosophic set which was the extension of intuitionistic fuzzy set theory and neutrosophic probability were introduced by Smarandache [20,21]. The notion of neutrosophic cubic set introduced truth-internal and truth-external were extended by Jun et al. [11] and related properties were also investigated by him. Rosenfeld's fuzzy subgroup with an interval-valued membership function was studied by Biswas [3]. The characteristics of the neutrosophic cubic soft set were studied by Pramanik et al. [5]. Cubic G-subalgebra with significent results were investigated by jana et al. [17]. The bipolar fuzzy structure of BG-algebra was interrogated by Senapati [6]. Neutrosophic cubic soft expert sets were studied for its applications in games by Gulistan M et al. [23]. Neutrosophic cubic graphs and find out the applications of neutrosophic cubic graphs in the industry by defining the model which are based on the present time and future predictions was studied by Gulistan M et al. [24]. Complex neutrosophic subsemigroups with the Cartesian product, complex neutrosophic (left, right, interior, ideal, characteristic function and direct product was observed by Gulistan M et al. [25]. Results showed the most preferred and the lowest preferred metrics in order to evaluate the sustainability of the supply chain strategy are studied by Abdel-Basset et al. [26]. Hybrid combination between analytical hierarchical process (AHP) as an MCDM method and neutrosophic theory to successfully detect and handle the uncertainty and inconsistency challenges proposed by Abdel-Basset et al. [27].

In this paper, the notion of neutrosophic soft cubic subalgebras (**NSCSU)** of G-algebras is introduced. And some relevant properties are studied. This study also discussed the homomorphism of neutrosophic soft cubic subalgebras and several related properties.

2 Preliminaries

Definition 2.1 [13] A nonempty set Y with a constant 0 and a binary operation $*$ is said to be G-algebra if it fulfills these axioms.

G1: $t_1 * t_1 = 0$.

G2: $t_1 * (t_1 * t_2) = t_2$, for all $t_1, t_2 \in Y$.

A G-algebra is denoted by $(Y, *, 0)$.

Definition 2.2 [3] A nonempty subset S of G-algebra Y is called a subalgebra of Y if $t_1 * t_2 \in S\ \forall\ t_1, t_2 \in S$.

Definition 2.3 [3] Mapping $\tau | Y \to X$ of G-algebras is called homomorphism if $\tau(t_1 * t_2) = \tau(t_1) * \tau(t_2)\ \forall\ t_1, t_2 \in Y$.

Note that if $\tau | Y \to X$ is a g-homomorphism, then $\tau(0) = 0$.

Definition 2.4 [11] A nonempty set A in Y of the $A = \{< t_1, \vartheta_A(t_1) > | t_1 \in Y\}$, is called fuzzy set, where $\vartheta_A(t_1)$ is called the existence value of t_1 in A and $\vartheta_A(t_1) \in [0,1]$.

For a family $A_i = \{< t_1, \vartheta_{A_i}(t_1) > | t_1 \in Y\}$ of fuzzy sets in Y, where $i \in h$ and h is index set, we define the join ($\vee$) meet ($\wedge$) operations like this:

$$\bigvee_{i \in h} A_i = (\bigvee_{i \in h} \vartheta_{A_i})(t_1) = \sup\{\vartheta_{A_i} | i \in h\},$$

and

$$\bigwedge_{i \in h} A_i = (\bigwedge_{i \in h} \vartheta_{A_i})(t_1) = \inf\{\vartheta_{A_i} | i \in h\} \text{ respectively, } \forall\ t_1 \in Y.$$

Definition 2.5 [11] A nonempty set A over Y of the form $A = \{< t_1, \tilde{\vartheta}_A(t_1) > | t_1 \in Y\}$, is called IVFS where $\tilde{\vartheta}_A | Y \to D[0,1]$, here $D[0,1]$ is the collection of all subintervals of [0,1].

The intervals $\tilde{\vartheta}_A t_1 = [\vartheta_A^-(t_1), \vartheta_A^+(t_1)]\ \forall\ t_1 \in Y$ denote the degree of existence of the element t_1 to the set A. Also $\tilde{\vartheta}_A^c = [1 - \vartheta_A^-(t_1), 1 - \vartheta_A^+(t_1)]$ represents the complement of $\tilde{\vartheta}_A$.

For a family $\{A_i | i \in k\}$ of IVFSs in Y where h is an index set, the union $G = \bigcup_{i \in h} \tilde{\vartheta}_{A_i}(t_1)$ and the intersection $F = \bigcap_{i \in h} \tilde{\vartheta}_{A_i}(t_1)$ are defined below:

$$G(t_1) = (\cup_{i \in h} \tilde{\vartheta}_{A_i})(t_1) = \text{rsup}\{\tilde{\vartheta}_{A_i}(t_1) | i \in h\}$$

and

$$F(t_1) = (\cap_{i \in h} \tilde{\vartheta}_{A_i})(t_1) = \text{rinf}\{\tilde{\vartheta}_{A_i}(t_1) | i \in k\}, \text{ respectively, } \forall\ t_1 \in Y.$$

Definition 2.6 [12] Consider two elements $K_1, K_2 \in D[0,1]$. If $K_1 = [f_1^-, f_1^+]$ and $K_2 = [f_2^-, f_2^+]$, then $\text{rmax}(K_1, K_2) = [\max(f_1^-, f_2^-), \max(f_1^+, f_2^+)]$ which is denoted by $K_1 \vee^r K_2$ and $\text{rmin}(K_1, K_2) = [\min(f_1^-, f_2^-), \min(f_1^+, f_2^+)]$ which is denoted by $K_1 \wedge^r K_2$. Thus, if $K_i = [f_i^-, f_i^+] \in K[0,1]$ for $i = 1,2,3, \ldots,$ then we define $\text{rsup}_i(K_i) = [\sup_i(f_i^-), \sup_i(f_i^+)]$, i.e., $\vee_i^r K_i = [\vee_i (f_i^-), \vee_i (f_i^+)]$. Similarly we define $\text{rinf}_i(K_i) = [\inf_i(f_i^-), \inf_i(f_i^+)]$, i.e., $\wedge_i^r K_i = [\wedge_i (f_i^-), \wedge_i (f_i^+)]$. Now $K_1 \geq K_2 \Leftarrow f_1^- \geq f_2^-$ and $f_1^+ \geq f_2^+$. Similarly the relations $K_1 \leq K_2$ and $K_1 = K_2$ are defined.

Definition 2.7 [13] A fuzzy set $A = \{< t_1, \vartheta_A(t_1) > | t_1 \in Y\}$ is called a fuzzy subalgebra of Y if $\vartheta_A(t_1 * t_2) \geq \min\{\vartheta_A(t_1), \vartheta_A(t_2)\}\ \forall\ t_1, t_2 \in Y$.

Definition 2.8 [22] A pair $\tilde{\mathcal{P}}_k = (\mathbf{A}, \Lambda)$ is called NCS where $\mathbf{A} = \{\langle t_1; A_T(t_1), A_I(t_1), A_F(t_1)\rangle\ | t_1 \in Y\}$ is an INS in Y and $\Lambda = \{\langle t_1; \lambda_T(t_1), \lambda_I(t_1), \lambda_F(t_1)\rangle |\ t_1 \in Y\}$ is a neutrosophic set in Y.

Definition 2.9 [3] Let $C = \{\langle t_1, A(t_1), \lambda(t_1)\rangle\}$ be a cubic set, where $A(t_1)$ is an IVFS in Y, $\lambda(t_1)$ is a fuzzy set in Y and Y is subalgebra. Then A is cubic subalgebra under binary operation * if it fulfills these axioms:

C1: $A(t_1 * t_2) \geq \text{rmin}\{A(t_1), A(t_2)\}$,

C2: $\lambda(t_1 * t_2) \leq \max\{\lambda(t_1), \lambda(t_2)\}\ \forall\ t_1, t_2 \in Y$.

Definition 3.0 [14] Let U be an universe set. Let $NC(U)$ represents the set of all neutrosophic cubic sets and E be the collection of parameters. Let $K \subset E$ then $\tilde{P}_K = \{\langle t_1, A_{e_i}(t_1), \lambda_{e_i}(t_1)\rangle | t_1 \in U, e_i \in K\}$, where $A_{e_i}(t_1) = \{\langle A^T_{e_i}(t_1), (A)^I_{e_i}(t_1), (A)^F_{e_i}(t_1)\rangle | t_1 \in U\}$, is an interval neutrosophic soft set, $\lambda_{e_i}(t_1) = \{\langle \lambda^T_{e_i}(t_1), (A)^I_{e_i}(t_1)(t_1), (\lambda)^F_{e_i}(t_1)\rangle | t_1 \in U\}$ is a neutrosophic soft set. $\tilde{P}_k$ is named as the neutrosophic soft cubic set over U where $\tilde{P}$ is a mapping given by $\tilde{P}|K \rightarrow NC(U)$. The sets of all neutrosophic soft cubic sets over U will be denoted by C^N_U.

3 Neutrosophic Soft Cubic Subalgebras of G-Algebra

Definition 3.1 Let $\tilde{\mathcal{P}}_k = (\mathbf{A}_{e_i}, \Lambda_{e_i})$ be a neutrosophic soft cubic set, where Y is subalgebra. Then $\tilde{\mathcal{P}}_k$ is **NSCSU** under binary operation $*$ if it holds the following conditions:

N1:

$$A^T_{e_i}(t_1 * t_2) \geq \text{rmin}\{A^T_{e_i}(t_1), A^T_{e_i}(t_2)\}$$
$$A^I_{e_i}(t_1 * t_2) \geq \text{rmin}\{A^I_{e_i}(t_1), A^I_{e_i}(t_2)\}$$
$$A^F_{e_i}(t_1 * t_2) \geq \text{rmin}\{A^F_{e_i}(t_1), A^F_{e_i}(t_2)\},$$

N2:

$$\Lambda^T_{e_i}(t_1 * t_2) \leq \max\{\Lambda^T_{e_i}(t_1), \Lambda^T_{e_i}(t_2)\}$$
$$\Lambda^I_{e_i}(t_1 * t_2) \leq \max\{\Lambda^I_{e_i}(t_1), \Lambda^I_{e_i}(t_2)\}$$
$$\Lambda^F_{e_i}(t_1 * t_2) \leq \max\{\Lambda^F_{e_i}(t_1), \Lambda^F_{e_i}(t_2)\}.$$

For simplicity we introduced new notation for neutrosophic soft cubic set as

$$\tilde{\mathcal{P}}_k = (A^{T,I,F}_{e_i}, \lambda^{T,I,F}_{e_i}) = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i}) = \{\langle t_1, A^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_1)\rangle\}$$

and for conditions N1, N2 as

N1: $A^{\varrho}_{e_i}(t_1 * t_2) \geq \text{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}$,

N2: $\lambda^{\varrho}_{e_i}(t_1 * t_2) \leq \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}$.

Example 3.2 Let $Y = \{0, c_1, c_2, c_3, c_4, c_5\}$ be a G-algebra with the following Cayley table.

$*$	0	c_1	c_2	c_3	c_4	c_5
0	0	c_5	c_4	c_3	c_2	c_1
c_1	c_1	0	c_5	c_4	c_3	c_2
c_2	c_2	c_1	0	c_5	c_4	c_3
c_3	c_3	c_2	c_1	0	c_5	c_4
c_4	c_4	c_3	c_2	c_1	0	c_5
c_5	c_5	c_4	c_3	c_2	c_1	0

A NSCS $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ of Y is defined by

$*$	0	c_1	c_2	c_3	c_4	c_5
$A^T_{e_i}$	[0.6,0.8]	[0.5,0.7]	[0.6,0.8]	[0.5,0.7]	[0.6,0.8]	[0.5,0.7]
$A^I_{e_i}$	[0.5,0.4]	[0.4,0.3]	[0.5,0.4]	[0.4,0.3]	[0.5,0.4]	[0.4,0.3]
$A^F_{e_i}$	[0.5,0.7]	[0.3,0.6]	[0.5,0.7]	[0.3,0.6]	[0.5,0.7]	[0.3,0.6],

and

$*$	0	c_1	c_2	c_3	c_4	c_5
$\lambda^T_{e_i}$	0.3	0.5	0.3	0.5	0.3	0.5
$\Lambda^I_{e_i}$	0.5	0.7	0.5	0.7	0.5	0.7
$\Lambda^F_{e_i}$	0.7	0.8	0.7	0.8	0.7	0.8.

Definition 3.1 is satisfied by the set $\tilde{\mathcal{P}}_k$. Thus $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ is a **NSCSU** of Y.

Proposition 3.3 Let $\tilde{\mathcal{P}}_k = \{\langle t_1, A^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_1)\rangle\}$ is a **NSCSU** of Y, then $\forall\ t_1 \in Y$, $A^{\varrho}_{e_i}(t_1) \geq A^{\varrho}_{e_i}(0)$ and $\lambda^{\varrho}_{e_i}(0) \leq \lambda^{\varrho}_{e_i}(t_1)$. Thus, $A^{\varrho}_{e_i}(0)$ and $\lambda^{\varrho}_{e_i}(0)$ are the upper bounds and lower bounds of $A^{\varrho}_{e_i}(t_1)$ and $\lambda^{\varrho}_{e_i}(t_1)$ respectively.

Proof. For all $t_1 \in Y$, we have $A^{\varrho}_{e_i}(0) = A^{\varrho}_{e_i}(t_1 * t_1) \geq \mathrm{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_1)\} = A^{\varrho}_{e_i}(t_1) \Rightarrow A^{\varrho}_{e_i}(0) \geq A^{\varrho}_{e_i}(t_1)$ and $\lambda^{\varrho}_{e_i}(0) = \lambda^{\varrho}_{e_i}(t_1 * t_1) \leq \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_1)\} = \lambda^{\varrho}_{e_i}(t_1) \Rightarrow \lambda^{\varrho}_{e_i}(0) \leq \lambda^{\varrho}_{e_i}(t_1)$.

Theorem 3.4 Let $\tilde{\mathcal{P}}_k = \{\langle t_1, A^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_1)\rangle\}$ be a **NSCSU** of Y. If there exists a sequence $\{(t_1)_n\}$ of Y such that $\lim_{n\to\infty} A^{\varrho}_{e_i}((t_1)_n) = [1,1]$ and $\lim_{n\to\infty} \lambda^{\varrho}_{e_i}((t_1)_n) = 0$. Then $A^{\varrho}_{e_i}(0) = [1,1]$ and $\lambda^{\varrho}_{e_i}(0) = 0$.

Proof. Using Proposition 3.3, $A^{\varrho}_{e_i}(0) \geq A^{\varrho}_{e_i}(t_1)\ \forall\ t_1 \in Y$, $\therefore\ A^{\varrho}_{e_i}(0) \geq A^{\varrho}_{e_i}((t_1)_n)$ for $n \in Z^+$. Consider, $[1,1] \geq A^{\varrho}_{e_i}(0) \geq \lim_{n\to\infty} A^{\varrho}_{e_i}((t_1)_n) = [1,1]$. Hence, $A^{\varrho}_{e_i}(0) = [1,1]$. Again, using Proposition 3.3, $\lambda^{\varrho}_{e_i}(0) \leq \lambda^{\varrho}_{e_i}(t_1)\ \ \forall\ \ t_1 \in Y$, $\therefore\ \lambda^{\varrho}_{e_i}(0) \leq \lambda^{\varrho}_{e_i}((t_1)_n)$ for $n \in Z^+$. Consider, $0 \leq \lambda^{\varrho}_{e_i}(0) \leq \lim_{n\to\infty} \lambda^{\varrho}_{e_i}((t_1)_n) = 0$. Hence, $\lambda^{\varrho}_{e_i}(0) = 0$.

Theorem 3.5 The R-intersection of any set of **NSCSU** of Y is also a **NSCSU** of Y.

Proof. Let $\tilde{\mathcal{P}}_k = \{\langle t_1, A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i}\rangle | t_1 \in Y\}$ where $i \in k$, be set of **NSCSU** of Y and $t_1, t_2 \in Y$. Then

$$\begin{aligned}
(\cap A^{\varrho}_{e_i})(t_1 * t_2) &= \mathrm{rinf} A^{\varrho}_{e_i}(t_1 * t_2)\\
&\geq \mathrm{rinf}\{\mathrm{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}\}\\
&= \mathrm{rmin}\{\mathrm{rinf} A^{\varrho}_{e_i}(t_1), \mathrm{rinf} A^{\varrho}_{e_i}(t_2)\}\\
&= \mathrm{rmin}\{(\cap\ A^{\varrho}_{e_i})(t_1), (\cap\ A^{\varrho}_{e_i})(t_2)\}\\
&\Rightarrow (\cap\ A^{\varrho}_{e_i})(t_1 * t_2) \geq \mathrm{rmin}\{(\cap\ A^{\varrho}_{e_i})(t_1), (\cap\ A^{\varrho}_{e_i})(t_2)\}
\end{aligned}$$

and

$$\begin{aligned}
(\vee \lambda^{\varrho}_{e_i})(t_1 * t_2) &= \sup \lambda^{\varrho}_{e_i}(t_1 * t_2)\\
&\leq \sup\{\max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}\}\\
&= \max\{\sup \lambda^{\varrho}_{e_i}(t_1), \sup \lambda^{\varrho}_{e_i}(t_2)\}\\
&= \max\{(\vee \lambda^{\varrho}_{e_i})(t_1), (\vee \lambda^{\varrho}_{e_i})(t_2)\}\\
&\Rightarrow (\vee \lambda^{\varrho}_{e_i})(t_1 * t_2) \leq \max\{(\vee \lambda^{\varrho}_{e_i})(t_1), (\vee \lambda^{\varrho}_{e_i})(t_2)\},
\end{aligned}$$

which show that R-intersection of $\tilde{\mathcal{P}}_k$ is a **NSCSU** of Y.

Remark 3.6 This is not compulsary that R-union, P-intersection and P-union of **NSCSU** are also the **NSCSU**.

Example 3.7 Let $Y = \{0, c_1, c_2, c_3, c_4, c_5\}$ be a G-algebra with the following Cayley table.

$*$	0	c_1	c_2	c_3	c_4	c_5
0	0	c_2	c_1	c_3	c_4	c_5
c_1	c_1	0	c_2	c_5	c_3	c_4
c_2	c_2	c_1	0	c_4	c_5	c_3
c_3	c_3	c_4	c_5	0	c_1	c_2
c_4	c_4	c_5	c_3	c_2	0	c_1
c_5	c_5	c_3	c_4	c_1	c_2	0.

Let $\mathcal{A}_{e_1} = (A^{\varrho}_{e_1}, \lambda^{\varrho}_{e_1})$ and $\mathcal{A}_{e_2} = (A^{\varrho}_{e_2}, \lambda^{\varrho}_{e_2})$ are neutrosophic soft cubic sets of Y defined by

	0	c_1	c_2	c_3	c_4	c_5
$A_{e_1}T$	[0.5,0.4]	[0.1,0.2]	[0.1,0.2]	[0.5,0.4]	[0.1,0.2]	[0.1,0.2]
$A_{e_1}I$	[0.6,0.7]	[0.2,0.3]	[0.2,0.3]	[0.6,0.7]	[0.2,0.3]	[0.2,0.3]
$A_{e_1}F$	[0.7,0.8]	[0.3,0.4]	[0.3,0.4]	[0.7,0.8]	[0.3,0.4]	[0.3,0.4]
$A_{e_2}T$	[0.6,0.7]	[0.2,0.3]	[0.2,0.3]	[0.6,0.7]	[0.2,0.3]	[0.2,0.3]
$A_{e_2}I$	[0.5,0.4]	[0.1,0.2]	[0.1,0.2]	[0.1,0.2]	[0.5,0.4]	[0.1,0.2]
$A_{e_2}F$	[0.4,0.3]	[0.2,0.4]	[0.2,0.4]	[0.2,0.4]	[0.4,0.5]	[0.2,0.4]

and

	0	c_1	c_2	c_3	c_4	c_5
$\lambda_{e_1}T$	0.2	0.8	0.8	0.3	0.8	0.8
$\lambda_{e_1}I$	0.3	0.7	0.7	0.4	0.7	0.7
$\lambda_{e_1}F$	0.5	0.6	0.6	0.5	0.6	0.6
$\lambda_{e_2}T$	0.3	0.5	0.5	0.5	0.4	0.5
$\lambda_{e_2}I$	0.4	0.7	0.7	0.7	0.5	0.7
$\lambda_{e_2}F$	0.5	0.9	0.9	0.9	0.6	0.9

Then $\mathcal{A}_{e_1}$ and $\mathcal{A}_{e_2}$ are neutrosophic soft cubic subalgebras of Y but R-union, P-union and P-intersection of $\mathcal{A}_{e_1}$ and $\mathcal{A}_{e_2}$ are not neutrosophic soft cubic subalgebras of Y. $\left(\cup A^{\varrho}_{e_i}\right)(c_3 * c_4) = ([0.2,0.5],[0.2,0.3],[0.3,0.4])_{\varrho} = \operatorname{rmin}\{\left(\cup A^{\varrho}_{e_i}\right)(c_3), \left(\cup A^{\varrho}_{e_i}\right)(c_4)\}$ *and* $\left(\wedge\lambda^{\varrho}_{e_i}\right)(c_3 * c_4) = (0.7,0.6,0.8)_{\varrho} \nleq (0.1,0.2,0,3)_{\varrho} = \max\{(\wedge\lambda^{\varrho}_{e_i})(c_3), (\wedge\lambda^{\varrho}_{e_i})(c_4)\}$.

We give the conditions that R-union, P-union and P-intersection of **NSCSU** are also **NSCSU**. Which are at Theorem 3.8, 3.9, 3.10.

Theorem 3.8 Let $\tilde{\mathcal{P}}_k = \{\langle t_1, A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i}\rangle | t_1 \in Y\}$ where $i \in k$ be set of **NSCSU** of Y, where $i \in k$. If $\inf\{\max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}\} = \max\{\inf\lambda^{\varrho}_{e_i}(t_1), \inf\lambda^{\varrho}_{e_i}(t_2)\}\ \forall\ t_1 \in Y$. Then the P-intersection of $\tilde{\mathcal{P}}_k$ is also a **NSCSU** of Y.

Proof. Suppose that $\tilde{\mathcal{P}}_k = \{\langle t_1, A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i}\rangle | t_1 \in Y\}$ where $i \in k$ be set of **NSCSU** of Y such that $\inf\{\max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}\} = \max\{\inf\lambda^{\varrho}_{e_i}(t_1), \inf\lambda^{\varrho}_{e_i}(t_2)\}\ \forall\ t_1, t_2 \in Y$. Then for $t_1, t_2 \in Y$. Then $(\cap A^{\varrho}_{e_i})(t_1 * t_2) = r\inf A^{\varrho}_{e_i}(t_1 * t_2) \geq r\inf\{r\min\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}\} = r\min\{r\inf A^{\varrho}_{e_i}(t_1), r\inf A^{\varrho}_{e_i}(t_2)\} = r\min\{(\cap A^{\varrho}_{e_i})(t_1), (\cap A^{\varrho}_{e_i})(t_2)\} \Rightarrow (\cap A^{\varrho}_{e_i})(t_1 * t_2) \geq r\min\{(\cap A^{\varrho}_{e_i})(t_1), (\cap A^{\varrho}_{e_i})(t_2)$ and $(\wedge\lambda^{\varrho}_{e_i})(t_1 * t_2) = \inf\lambda^{\varrho}_{e_i}(t_1 * t_2) \leq \inf\{\max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}\} = \max\{\inf\lambda^{\varrho}_{e_i}(t_1), \inf\lambda^{\varrho}_{e_i}(t_2)\} = \max\{(\wedge\lambda^{\varrho}_{e_i})(t_1), (\wedge\lambda^{\varrho}_{e_i})(t_2)\} \Rightarrow (\wedge\lambda^{\varrho}_{e_i})(t_1 * t_2) \leq \max\{(\wedge\lambda^{\varrho}_{e_i})(t_1), (\wedge\lambda^{\varrho}_{e_i})(t_2)\}$, which show that P-intersection of $\tilde{\mathcal{P}}_k$ is a **NSCSU** of Y.

Theorem 3.9 Let $\tilde{\mathcal{P}}_k = \{\langle t_1, A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i}\rangle | t_1 \in Y\}$ where $i \in k$ be set of **NSCSU** of Y. If $\sup\{r\min\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}\} = r\min\{\sup A^{\varrho}_{e_i}(t_1), \sup A^{\varrho}_{e_i}(t_2)\}\ \forall\ t_1, t_2 \in Y$. Then the P-union of $\tilde{\mathcal{P}}_k$ is also a **NSCSU** of Y.

Proof. Let $\tilde{\mathcal{P}}_k = \{\langle t_1, A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i}\rangle | t_1 \in Y\}$ where $i \in k$ be set of **NSCSU** of Y such that $\sup\{r\min\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}\} = r\min\{\sup A^{\varrho}_{e_i}(t_1), \sup A^{\varrho}_{e_i}(t_2)\}\ \forall\ t_1 \in Y$. Then for $t_1, t_2 \in Y, (\cup A^{\varrho}_{e_i})(t_1 * t_2) = r\sup A^{\varrho}_{e_i}(t_1 * t_2) \geq r\sup\{r\min\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}\} = r\min\{r\sup A^{\varrho}_{e_i}(t_1), r\sup A^{\varrho}_{e_i}(t_2)\} = r\min\{(\cup A^{\varrho}_{e_i})(t_1), (\cup A^{\varrho}_{e_i})(t_2)\} \Rightarrow (\cup A^{\varrho}_{e_i})(t_1 * t_2) \geq r\min\{(\cup A^{\varrho}_{e_i})(t_1), (\cup A^{\varrho}_{e_i})(t_2)\}$

an $(\vee\lambda^{\varrho}_{e_i})(t_1 * t_2) = \sup\lambda^{\varrho}_{e_i}(t_1 * t_2) \leq \sup\left\{\max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}\right\} = \max\{\sup\lambda^{\varrho}_{e_i}(t_1), \sup\lambda^{\varrho}_{e_i}(t_2)\} =$

$\max\{(\vee\lambda^{\varrho}_{e_i})(t_1), (\vee\lambda^{\varrho}_{e_i})(t_2)\} \Rightarrow (\vee\lambda^{\varrho}_{e_i})(t_1 * t_2) \leq \max\{(\vee\lambda^{\varrho}_{e_i})(t_1), (\vee\lambda^{\varrho}_{e_i})(t_2)\}$, which show that P-union of $\tilde{\mathcal{P}}_k$ is a **NSCSU** of Y.

Theorem 3.10 Let $\tilde{\mathcal{P}}_k = \{\langle t_1, A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i}\rangle | t_1 \in Y\}$ where $i \in k$ be set of **NSCSU** of Y. If $\inf\{\max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}\} = \max\{\inf\lambda^{\varrho}_{e_i}(t_1), \inf\lambda^{\varrho}_{e_i}(t_1)\}$ and $\sup\{r\min\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}\} = r\min\{\sup A^{\varrho}_{e_i}(t_1), \sup A^{\varrho}_{e_i}(t_2)\}\ \forall\ t_1, t_2 \in Y$. Then the R-union of $\tilde{\mathcal{P}}_k$ is also a **NSCSU** of Y.

Proof. Let $\tilde{\mathcal{P}}_k = \{\langle t_1, A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i}\rangle | t_1 \in Y\}$ where $i \in k$ be set of **NSCSU** of Y such that $\inf\{\max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}\} = \max\{\inf\lambda^{\varrho}_{e_i}(t_1), \inf\lambda^{\varrho}_{e_i}(t_1)\}$ and $\sup\{r\min\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}\} = r\min\{\sup\lambda^{\varrho}_{e_i}(t_1), \sup\lambda^{\varrho}_{e_i}(t_1)\}\ \forall\ t_1 \in Y$. Then for $t_1, t_2 \in Y$, $(\cup A^{\varrho}_{e_i})(t_1 * t_2) = r\sup A^{\varrho}_{e_i}(t_1 * t_2) \geq r\sup\{r\min\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}\} = r\min\{r\sup A^{\varrho}_{e_i}(t_1), r\sup A^{\varrho}_{e_i}(t_2)\} = r\min\{(\cup A^{\varrho}_{e_i})(t_1), (\cup A^{\varrho}_{e_i})(t_2)\} \Rightarrow (\cup A^{\varrho}_{e_i})(t_1 * t_2) \geq r\min\{(\cup A^{\varrho}_{e_i})(t_1), (\cup A^{\varrho}_{e_i})(t_2)\}$ and $(\wedge\lambda^{\varrho}_{e_i})(t_1 * t_2) = \inf\lambda^{\varrho}_{e_i}(t_1 * t_2) \leq \inf\{\max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}\} = \max\{\inf\lambda^{\varrho}_{e_i}(t_1), \inf\lambda^{\varrho}_{e_i}(t_2)\} = \max\{(\wedge\lambda^{\varrho}_{e_i})(t_1), (\wedge\lambda^{\varrho}_{e_i})(t_2)\} \Rightarrow (\wedge\lambda^{\varrho}_{e_i})(t_1 * t_2) \leq \max\{(\wedge\lambda^{\varrho}_{e_i})(t_1), (\wedge\lambda^{\varrho}_{e_i})(t_2)\}$, which show that R-union of $\tilde{\mathcal{P}}_k$ is a **NSCSU** of Y.

Proposition 3.11 If a neutrosophic soft cubic set $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ of Y is a subalgebra. Then $\forall\ t_1 \in Y$, $A^{\varrho}_{e_i}(0 * t_1) \geq A^{\varrho}_{e_i}(t_1)$ and $\lambda^{\varrho}_{e_i}(0 * t_1) \leq \lambda^{\varrho}_{e_i}(t_1)$.

Proof. For all $t_1 \in Y$, $A^{\varrho}_{e_i}(0 * t_1) \geq \mathrm{rmin}\{A^{\varrho}_{e_i}(0), A^{\varrho}_{e_i}(t_1)\} = \mathrm{rmin}\{A^{\varrho}_{e_i}(t_1 * t_1), A^{\varrho}_{e_i}(t_1)\} \geq \mathrm{rmin}\{\mathrm{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_1)\}, A^{\varrho}_{e_i}(t_1)\} = A^{\varrho}_{e_i}(t_1)$ and similarly $\lambda^{\varrho}_{e_i}(0 * t_1) \leq \max\{\lambda^{\varrho}_{e_i}(0), \lambda^{\varrho}_{e_i}(t_1)\} = \lambda^{\varrho}_{e_i}(t_1)$.

Lemma 3.12 If a netrosophic soft cubic set $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ of Y is a subalgebra. Then $\tilde{\mathcal{P}}_k(t_1 * t_2) = \tilde{\mathcal{P}}_k(t_1 * (0 * (0 * t_2)))$ $\forall$ $t_1, t_2 \in Y$.

Proof. Let Y be a G-algebra and $t_1, t_2 \in Y$. Then $t_2 = 0 * (0 * t_2)$ by ([9], Lemma 3.1). Hence $A^{\varrho}_{e_i}(t_1 * t_2) = A^{\varrho}_{e_i}(t_1 * (0 * (0 * t_2)))$ and $\lambda^{\varrho}_{e_i}(t_1 * t_2) = \lambda^{\varrho}_{e_i}(t_1 * (0 * (0 * t_2)))$. Therefore, $\tilde{\mathcal{P}}_k(t_1 * t_2) = \tilde{\mathcal{P}}_k(t_1 * (0 * (0 * t_2)))$

Proposition 3.13 If a NSCS $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ of Y is **NSCSU**. Then $\forall$ $t_1, t_2 \in Y$, $A^{\varrho}_{e_i}(t_1 * (0 * t_2)) \geq \mathrm{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}$ and $\lambda^{\varrho}_{e_i}(t_1 * (0 * t_2)) \leq \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}$.

Proof. Let $t_1, t_2 \in Y$. Then we have $A^{\varrho}_{e_i}(t_1 * (0 * t_2)) \geq \mathrm{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(0 * t_2)\} \geq \mathrm{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}$ and $\lambda^{\varrho}_{e_i}(t_1 * (0 * t_2)) \leq \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(0 * t_2)\} \leq \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}$ by Definition 3.1 and Proposition 3.11. Hence proof is completed.

Theorem 3.14 If a NSCS $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ of Y satisfies the following conditions. Then $\tilde{\mathcal{P}}_k$ refers to a **NSCSU** of Y.

1. $A^{\varrho}_{e_i}(0 * t_1) \geq A^{\varrho}_{e_i}(t_1)$ and $\lambda^{\varrho}_{e_i}(0 * t_1) \leq \lambda^{\varrho}_{e_i}(x)$ $\forall$ $t_1 \in Y$.
2. $A^{\varrho}_{e_i}(t_1 * (0 * t_2)) \geq \mathrm{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}$ and $\lambda^{\varrho}_{e_i}(t_1 * (0 * t_2)) \leq \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}$ $\forall$ $t_1, t_2 \in Y$.

Proof. Assume that the neutrosophic soft cubic set $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ of Y satisfies the above conditions. Then by Lemma 3.12, $A^{\varrho}_{e_i}(t_1 * t_2) = A^{\varrho}_{e_i}(t_1 * (0 * (0 * t_2))) \geq \mathrm{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(0 * t_2)\} \geq \mathrm{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}$ and $\lambda^{\varrho}_{e_i}(t_1 * t_2) = \lambda^{\varrho}_{e_i}(t_1 * (0 * (0 * t_2))) \leq \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(0 * t_2)\} \leq \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}$ $\forall$ $t_1, t_2 \in Y$. Hence $\tilde{\mathcal{P}}_k$ is **NSCSU** of Y.

Theorem 3.15 A neutrosophic soft cubic set $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ of Y is **NSCSU** of Y iff $(A^{\varrho}_{e_i})^-, (A^{\varrho}_{e_i})^+$ and $\lambda^{\varrho}_{e_i}$ are fuzzy subalgebras of Y.

Proof. Let $(A^{\varrho}_{e_i})^-, (A^{\varrho}_{e_i})^+$ and $\lambda^{\varrho}_{e_i}$ are fuzzy subalgebra of Y and $t_1, t_2 \in Y$ then $(A^{\varrho}_{e_i})^-(t_1 * t_2) \geq \min\{(A^{\varrho}_{e_i})^-(t_1), (A^{\varrho}_{e_i})^-(t_2)\}$, $(A^{\varrho}_{e_i})^+(t_1 * t_2) \geq \min\{(A^{\varrho}_{e_i})^+(t_1), (A^{\varrho}_{e_i})^+(t_2)\}$ and $\lambda^{\varrho}_{e_i}(t_1 * t_2) \leq \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}$. Now, $A^{\varrho}_{e_i}(t_1 * t_2) = [(A^{\varrho}_{e_i})^-(t_1 * t_2), (A^{\varrho}_{e_i})^+(t_1 * t_2)] \geq [\min\{(A^{\varrho}_{e_i})^-(t_1), (A^{\varrho}_{e_i})^-(t_2)\}, \min\{(A^{\varrho}_{e_i})^+(t_1), (A^{\varrho}_{e_i})^+(t_2)\}] \geq \mathrm{rmin}\{[(A^{\varrho}_{e_i})^-(t_1), (A^{\varrho}_{e_i})^+(t_1)], [(A^{\varrho}_{e_i})^-(t_2), (A^{\varrho}_{e_i})^+(t_2)]\} = \mathrm{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}$. Therefore, $\tilde{\mathcal{P}}_k$ is **NSCSU** of Y.

Conversely, assume that $\tilde{\mathcal{P}}_k$ is a **NSCSU** of Y. For any $t_1, t_2 \in Y$, $[(A^{\varrho}_{e_i})^-(t_1 * t_2), (A^{\varrho}_{e_i})^+(t_1 * t_2)] = A^{\varrho}_{e_i}(t_1 * t_2) \geq \mathrm{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\} = \mathrm{rmin}\{[(A^{\varrho}_{e_i})^-(t_1), (A^{\varrho}_{e_i})^+(t_1)], [(A^{\varrho}_{e_i})^-(t_2), (A^{\varrho}_{e_i})^+(t_2)]\} = [\min\{(A^{\varrho}_{e_i})^-(t_1), (A^{\varrho}_{e_i})^-(t_2)\}, \min\{(A^{\varrho}_{e_i})^+(t_1), (A^{\varrho}_{e_i})^+(t_2)\}]$. Thus, $(A^{\varrho}_{e_i})^-(t_1 *$

$t_2) \geq \min\{(A^{\varrho}_{e_i})^-(t_1), (A^{\varrho}_{e_i})^-(t_2)\}$, $(A^{\varrho}_{e_i})^+(t_1 * t_2) \geq \min\{(A^{\varrho}_{e_i})^+(t_1), (A^{\varrho}_{e_i})^+(t_2)\}$ and $\lambda^{\varrho}_{e_i}(t_1 * t_2) \leq \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}$. Hence $(A^{\varrho}_{e_i})^-, (A^{\varrho}_{e_i})^+$ and $\lambda^{\varrho}_{e_i}$ are fuzzy subalgebras of Y.

Theorem 3.16 Let $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ be a **NSCSU** of Y and let $n \in \mathbb{Z}^+$. Then

i) $A^{\varrho}_{e_i}(\coprod n\, t_1 * t_1) \geq A^{\varrho}_{e_i}(t_1)$ for $n \in \mathbb{O}$.

ii) $\lambda^{\varrho}_{e_i}(\coprod n\, t_1 * t_1) \leq A^{\varrho}_{e_i}(t_1)$ for $n \in \mathbb{O}$.

iii) $A^{\varrho}_{e_i}(\coprod n\, t_1 * t_1) = A^{\varrho}_{e_i}(t_1)$ for $n \in \mathbb{E}$.

iv) $\lambda^{\varrho}_{e_i}(\coprod n\, t_1 * t_1) = A^{\varrho}_{e_i}(t_1)$ for $n \in \mathbb{E}$.

Proof. Let $t_1 \in Y$ and suppose that n is odd. Then $n = 2p - 1$ for some $p \in Z^+$. We prove the theorem by induction.

Now $A^{\varrho}_{e_i}(t_1 * t_1) = A^{\varrho}_{e_i}(0) \geq A^{\varrho}_{e_i}(t_1)$ and $\lambda^{\varrho}_{e_i}(t_1 * t_1) = \lambda^{\varrho}_{e_i}(0) \leq \lambda^{\varrho}_{e_i}(t_1)$. Suppose that $A^{\varrho}_{e_i}((\coprod_{2p-1})(t_1 * t_1)) \geq A^{\varrho}_{e_i}(t_1)$ and $\lambda^{\varrho}_{e_i}((\coprod_{2p-1})(t_1 * t_1)) \leq \lambda^{\varrho}_{e_i}(t_1)$. Then by assumption, $A^{\varrho}_{e_i}((\coprod_{2(p+1)-1}(t_1 * t_1)) = A^{\varrho}_{e_i}((\coprod_{2p+1})(t_1 * t_1)) = A^{\varrho}_{e_i}(\coprod_{2p-1} t_1 * (t_1 * (t_1 * t_1))) = A^{\varrho}_{e_i}((\coprod_{2p-1})(t_1 * t_1)) \geq A^{\varrho}_{e_i}(t_1)$ and $\lambda^{\varrho}_{e_i}((\coprod_{2(p+1)-1}(t_1 * t_1)) = \lambda^{\varrho}_{e_i}((\coprod_{2p+1}(t_1 * t_1)) = \lambda^{\varrho}_{e_i}(\coprod_{2p-1} t_1 * (t_1 * (t_1 * t_1))) = \lambda^{\varrho}_{e_i}(\coprod_{2p-1} t_1 * t_1) \leq \lambda^{\varrho}_{e_i}(t_1)$, which proves (1) and (2). Similarly, cases (3) and (4) has the same proofs.

These sets denoted by $I_{A^{\varrho}_{e_i}}$ and $I_{\lambda^{\varrho}_{e_i}}$ are subalgebras of Y. Which were defined as

$$I_{A^{\varrho}_{e_i}} = \{t_1 \in Y | A^{\varrho}_{e_i}(t_1) = A^{\varrho}_{e_i}(0)\},\ I_{\lambda^{\varrho}_{e_i}} = \{t_1 \in Y | \lambda^{\varrho}_{e_i}(t_1) = \lambda^{\varrho}_{e_i}(0)\}.$$

Theorem 3.17 Let $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ be a **NSCSU** of Y. Then the sets $I_{A^{\varrho}_{e_i}}$ and $I_{\lambda^{\varrho}_{e_i}}$ are subalgebras of Y.

Proof. Let $t_1, t_2 \in I_{A^{\varrho}_{e_i}}$. Then $A^{\varrho}_{e_i}(t_1) = A^{\varrho}_{e_i}(0) = A^{\varrho}_{e_i}(t_2)$ and so, $A^{\varrho}_{e_i}(t_1 * t_2) \geq \text{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}$

$= A^{\varrho}_{e_i}(0)$. By using Proposition 3.3, we know that $A^{\varrho}_{e_i}(t_1 * t_2) = A^{\varrho}_{e_i}(0)$ or equivalently $t_1 * t_2 \in I_{A^{\varrho}_{e_i}}$.

Again suppose $t_1, t_2 \in I_{A^{\varrho}_{e_i}}$. Then $\lambda^{\varrho}_{e_i}(t_1) = \lambda^{\varrho}_{e_i}(0) = \lambda^{\varrho}_{e_i}(t_2)$ and so, $\lambda^{\varrho}_{e_i}(t_1 * t_2) \leq \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\} = \lambda^{\varrho}_{e_i}(0)$. Again by using Proposition 3.3, we know that $\lambda^{\varrho}_{e_i}(t_1 * t_2) = \lambda^{\varrho}_{e_i}(0)$ or equivalently $t_1 * t_2 \in I_{A^{\varrho}_{e_i}}$. Hence the sets $I_{A^{\varrho}_{e_i}}$ and $\lambda_{A^{\varrho}_{e_i}}$ are subalgebras of Y.

Theorem 3.18 Assume B is a nonempty subset of Y and $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ be a neutrosophic soft cubic set of Y defined by $A^{\varrho}_{e_i}(t_1) = \begin{cases} [\xi_{T,I,F_1}, \xi_{T,I,F_2}], & \text{if } t_1 \in B \\ [\beta_{T,I,F_1}, \beta_{T,I,F_2}], & \text{otherwise,} \end{cases}$ $\Lambda^{T}_{e_i}(t_1) = \begin{cases} \gamma_{\varrho}, & \text{if } t_1 \in B \\ \delta_{\varrho}, & \text{otherwise,} \end{cases}$

$\forall\ [\xi_{T,I,F_1}, \xi_{T,I,F_2}], [\beta_{T,I,F_1}, \beta_{T,I,F_2}] \in D[0,1]$ and $\gamma_{\varrho}, \delta_{\varrho} \in [0,1]$ with $[\xi_{T,I,F_1}, \xi_{T,I,F_2}] \geq [\beta_{T,I,F_1}, \beta_{T,I,F_2}]$ and $\gamma_{\varrho} \leq \delta_{\varrho}$. Then $\tilde{\mathcal{P}}_k$ is a nuetrosophic soft cubic subalgebra of $Y \Leftarrow B$ is a subalgebra of Y. Moreover, $I_{A^{\varrho}_{e_i}} = B = I_{\lambda^{\varrho}_{e_i}}$.

Proof. Let $\tilde{\mathcal{P}}_k$ be a **NSCSU** of Y. Let $t_1, t_2 \in Y$ such that $t_1, t_2 \in B$. Then $A^{\varrho}_{e_i}(t_1 * t_2) \geq \text{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\} = \text{rmin}\{[\xi_{T,I,F_1}, \xi_{T,I,F_2}], [\xi_{T,I,F_1}, \xi_{T,I,F_2}]\} = [\xi_{T,I,F_1}, \xi_{T,I,F_2}]$ and $\lambda^{\varrho}_{e_i}(t_1 * t_2) \leq \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\} = \max\{\gamma_{\varrho}, \gamma_{\varrho}\} = \gamma_{\varrho}$. Therefore $t_1 * t_2 \in B$. Hence, B is a subalgebra of Y.

Conversely, assume that B is a subalgebra of Y. Let $t_1, t_2 \in Y$. Now take two cases.

Case 1: If $t_1, t_2 \in B$, then $t_1 * t_2 \in B$, thus $A^{\varrho}_{e_i}(t_1 * t_2) = [\xi_{T,I,F_1}, \xi_{T,I,F_2}] = \text{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}$ and $\lambda^{\varrho}_{e_i}(t_1 * t_2) = \gamma_{\varrho} = \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}$.

Case 2: If $t_1 \notin B$ or $t_2 \notin B$, then $A^{\varrho}_{e_i}(t_1 * t_2) \geq [\beta_{T,I,F_1}, \beta_{T,I,F_2}] = \text{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}$ and $\lambda^{\varrho}_{e_i}(t_1 * t_2) \leq \delta_{\varrho} = \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}$. Hence $\tilde{\mathcal{P}}_k$ is a **NSCSU** of Y.

Now, $I_{A^{\varrho}_{e_i}}=\{t_1 \in Y, A^{\varrho}_{e_i}(t_1) = A^{\varrho}_{e_i}(0)\}= \{t_1 \in Y, A^{\varrho}_{e_i}(t_1) = [\xi_{T,I,F_1}, \xi_{T,I,F_2}]\} = B$ and $I_{\lambda^{\varrho}_{e_i}}=\{t_1 \in Y, \lambda^{\varrho}_{e_i}(t_1) = \lambda^{\varrho}_{e_i}(0)\}=\{t_1 \in Y, \lambda^{\varrho}_{e_i}(t_1) = \gamma_{\varrho}\}=B$.

Definition 3.19 Let $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ be a neutrosophic soft cubic set of Y. For $[w_{T_1}, w_{T_2}], [w_{I_1}, w_{I_2}], [w_{F_1}, w_{F_2}] \in D[0,1]$ and $t_{T_1}, t_{I_1}, t_{F_1} \in [0,1]$, the set $U(A^{\varrho}_{e_i}|([w_{T_1}, w_{T_2}], [w_{I_1}, w_{I_2}], [w_{F_1}, w_{F_2}])) = \{t_1 \in Y | A^T_{e_i}(t_1) \geq [w_{T_1}, w_{T_2}], A^I_{e_i}(t_1) \geq [w_{I_1}, w_{I_2}], A^F_{e_i}(t_1) \geq [w_{F_1}, w_{F_2}]\}$ is called upper $([w_{T_1}, w_{T_2}], [w_{I_1}, w_{I_2}], [w_{F_1}, w_{F_2}])$-level of $\tilde{\mathcal{P}}_k$ and $L(\lambda^{\varrho}_{e_i}|(t_{T_1}, t_{I_1}, t_{F_1}))$ $=\{t_1 \in Y | \Lambda^T_{e_i}(t_1) \leq t_{T_1}, \Lambda^I_{e_i}(t_1) \leq t_{I_1}, \Lambda^F_{e_i}(t_1) \leq t_{F_1}\}$ is called lower $(t_{T_1}, t_{I_1}, t_{F_1})$-level of $\tilde{\mathcal{P}}_k$.

For convenience, we introduced the new notions for upper level and lower level of $\tilde{\mathcal{P}}_k$ as, $U(A^{\varrho}_{e_i}|[w_{T,I,F_1}, w_{T,I,F_2}]=\{t_1 \in Y | A^{\varrho}_{e_i}(t_1) \geq [w_{T,I,F_1}, w_{T,I,F_2}]\}$ is called upper $([w_{T,I,F_1}, w_{T,I,F_2}])$-level of $\tilde{\mathcal{P}}_k$ and $L(\lambda^{\varrho}_{e_i}|t_{T,I,F_1})=\{t_1 \in Y | \lambda^{\varrho}_{e_i}(t_1) \leq t_{T,I,F_1}\}$ is called lower t_{T,I,F_1}-level of $\tilde{\mathcal{P}}_k$.

Theorem 3.20 If $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ is neutrosophic soft cubic subalgebra of Y, then the upper $[w_{T,I,F_1}, w_{T,I,F_2}]$-level and lower t_{T,I,F_1}-level of $\tilde{\mathcal{P}}_k$ are subalgebras of Y.

Proof. Let $t_1, t_2 \in U(A^{\varrho}_{e_i}|[w_{T,I,F_1}, w_{T,I,F_2}])$. Then $A^{\varrho}_{e_i}(t_1) \geq [w_{T,I,F_1}, w_{T,I,F_2}]$ and $A^{\varrho}_{e_i}(t_2) \geq [w_{T,I,F_1}, w_{T,I,F_2}]$. It follows that $A^{\varrho}_{e_i}(t_1 * t_2) \geq \text{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\} \geq [w_{T,I,F_1}, w_{T,I,F_2}] \Rightarrow t_1 * t_2 \in U(A^{\varrho}_{e_i}|[w_{T,I,F_1}, w_{T,I,F_2}])$. Hence, $U(A^{\varrho}_{e_i}|[w_{T,I,F_1}, w_{T,I,F_2}]$ is a subalgebra of Y.
Let $t_1, t_2 \in L(\lambda^{\varrho}_{e_i}|t_{T,I,F_1})$. Then $\lambda^{\varrho}_{e_i}(t_1) \leq t_{T,I,F_1}$ and $\lambda^{\varrho}_{e_i}(t_2) \leq t_{T,I,F_1}$. It follows that $\lambda^{\varrho}_{e_i}(t_1 * t_2) \leq \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\} \leq t_{T,I,F_1} \Rightarrow t_1 * t_2 \in L(\lambda^{\varrho}_{e_i}|t_{T,I,F_1})$. Hence $L(\lambda^{\varrho}_{e_i}|t_{T,I,F_1})$ is a subalgebra of Y.

Corollary 3.21 Let $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ is NSCSU of Y. Then $A([w_{T,I,F_1}, w_{T,I,F_2}], t_{T,I,F_1}) = U(A^{\varrho}_{e_i}|[w_{T,I,F_1}, w_{T,I,F_2}]) \cap L(\lambda^{\varrho}_{e_i}|t_{T,I,F_1})=\{t_1 \in Y | A^{\varrho}_{e_i}(t_1) \geq [w_{T,I,F_1}, w_{T,I,F_2}], \lambda^{\varrho}_{e_i}(t_1) \leq t_{T,I,F_1}\}$ is a subalgebra of Y.

Proof. We can prove it by using Theorem 3.20.
This example shows that the converse of Corollary 3.21 is not true
Example 3.22 Let $Y = \{0, c_1, c_2, c_3, c_4, c_5\}$ be a G-algebra in Remark 3.6 and $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ is a neutrosophic soft cubic set defined by

	0	c_1	c_2	c_3	c_4	c_5
$A^T_{e_i}$	[0.3,0.5]	[0.3,0.4]	[0.3,0.4]	[0.3,0.4]	[0.1,0.2]	[0.1,0.2]
$A^I_{e_i}$	[0.5,0.7]	[0.2,0.3]	[0.2,0.3]	[0.5,0.7]	[0.1,0.1]	[0.1,0.1]
$A^F_{e_i}$	[0.4,0.6]	[0.2,0.5]	[0.2,0.5]	[0.2,0.5]	[0.1,0.2]	[0.1,0.2],

and

	0	c_1	c_2	c_3	c_4	c_5
$\Lambda^T_{e_i}$	0.1	0.4	0.4	0.6	0.4	0.6
$\Lambda^I_{e_i}$	0.2	0.5	0.5	0.7	0.5	0.7
$\Lambda^F_{e_i}$	0.3	0.6	0.6	0.8	0.6	0.8

We take $[w_{T,I,F_1}, w_{T,I,F_2}] = ([0.41,0.48],[0.30,0.36],[0.13,0.17])$ and $t_{T,I,F_1} = (0.3,0.4,0.5)$. Then $A([w_{T,I,F_1}, w_{T,I,F_2}], t_{T,I,F_1}) = U(A^\varrho_{e_i}|[w_{T,I,F_1}, w_{T,I,F_2}]) \cap L(\lambda^\varrho_{e_i}|t_{T,I,F_1}) = \{t_1 \in Y | A^\varrho_{e_i}(t_1) \geq [w_{T,I,F_1}, w_{T,I,F_2}], \lambda^\varrho_{e_i}(t_1) \leq t_{T,I,F_1}\} = \{0, c_1, c_2, c_3\} \cap \{0, c_1, c_2, c_4\} = \{0, c_1, c_2\}$ is a subalgebra of Y, but $\tilde{\mathcal{P}}_k = (A^\varrho_{e_i}, \lambda^\varrho_{e_i})$ is not a **NSCSU**, since $A^T_{e_i}(c_1 * c_3) = [0.2,0.3] \ngeq [0.4,0.5] = \mathrm{rmin}\{A^T_{e_i}(c_1), A^T_{e_i}(c_3)\}$ and $\Lambda^T_{e_i}(c_2 * c_4) = 0.4 \nleq 0.3 = \max\{\Lambda^T_{e_i}(c_2), \Lambda^T_{e_i}(c_4)\}$.

Theorem 3.23 Let $\tilde{\mathcal{P}}_k = (A^\varrho_{e_i}, \lambda^\varrho_{e_i})$ be a neutrosophic soft cubic set of Y, such that the sets $U(A^\varrho_{e_i}|[w_{T,I,F_1}, w_{T,I,F_2}])$ and $L(\lambda^\varrho_{e_i}|t_{T,I,F_1})$ are subalgebras of Y for every $[w_{T,I,F_1}, w_{T,I,F_2}] \in D[0,1]$ and $t_{T,I,F_1} \in [0,1]$. Then $\tilde{\mathcal{P}}_k = (A^\varrho_{e_i}, \lambda^\varrho_{e_i})$ is NSCSU of Y.

Proof. Let $U(A^\varrho_{e_i}|[w_{T,I,F_1}, w_{T,I,F_2}])$ and $L(\lambda^\varrho_{e_i}|t_{T,I,F_1})$ are subalgebras of Y for every $[w_{T,I,F_1}, w_{T,I,F_2}] \in D[0,1]$ and $t_{T,I,F_1} \in [0,1]$. On the contrary, let $(t_1)_0, (t_2)_0 \in Y$ be such that $A^\varrho_{e_i}((t_1)_0 * (t_2)_0) < \mathrm{rmin}\{A^\varrho_{e_i}((t_1)_0), A^\varrho_{e_i}((t_2)_0)\}$. Let $A^\varrho_{e_i}((t_1)_0) = [\phi_1, \phi_2]$, $A^\varrho_{e_i}((t_2)_0) = [\phi_3, \phi_4]$ and $A^\varrho_{e_i}((t_1))_0 * (t_2)_0) = [w_{T,I,F_1}, w_{T,I,F_2}]$. Then $[w_{T,I,F_1}, w_{T,I,F_2}] < \mathrm{rmin}\{[\phi_1, \phi_2], [\phi_3, \phi_4]\} = [\min\{\phi_1, \phi_3\}, \min\{\phi_2, \phi_4\}]$. So, $w_{T,I,F_1} < \mathrm{rmin}\{\phi_1, \phi_3\}$ and $w_{T,I,F_2} < \min\{\phi_2, \phi_4$. Let us consider, $[\rho_1, \rho_2] = \frac{1}{2}[A^\varrho_{e_i}((t_1)_0 * (t_2)_0) + \mathrm{rmin}\{A^\varrho_{e_i}((t_1)_0), A^\varrho_{e_i}((t_2)_0)\}] = \frac{1}{2}[[w_{T,I,F_1}, w_{T,I,F_2}] + [\min\{\phi_1, \phi_3\}, \min\{\phi_2, \phi_4\}]] = [\frac{1}{2}(w_{T,I,F_1} + \min\{\phi_1, \phi_3\}), \frac{1}{2}(w_{T,I,F_2} + \min\{\phi_2, \phi_3\})]$. Therefore, $\min\{\phi_1, \phi_3\} > \rho_1 = \frac{1}{2}(w_{T,I,F_1} + \min\{\phi_1, \phi_3\}) > w_{T,I,F_1}$ and $\min\{\phi_2, \phi_4\} > \rho_2 = \frac{1}{2}(w_{T,I,F_2} + \min\{\phi_2, \phi_4\}) > w_{T,I,F_2}$. Hence, $[\min\{\phi_1, \phi_3\}, \min\{\phi_2, \phi_4\}] > [\rho_1, \rho_2] > [w_{T,I,F_1}, w_{T,I,F_2}]$ so that $(t_1)_0 * (t_2)_0 \notin U(A^\varrho_{e_i}|[w_{T,I,F_1}, w_{T,I,F_2}])$ which is a contradiction since $A^\varrho_{e_i}((t_1)_0) = [\phi_1, \phi_2] \geq [\min\{\phi_1, \phi_3\}, \min\{\phi_2, \phi_4\}] > [\rho_1, \rho_2]$ and $A^\varrho_{e_i}((t_2)_0) = [\phi_3, \phi_4] \geq [\min\{\phi_1, \phi_3\}, \min\{\phi_2, \phi_4\}] > [\rho_1, \rho_2]$. This implies $(t_1)_0 * (t_2)_0 \in U(A^\varrho_{e_i}|[w_{T,I,F_1}, w_{T,I,F_2}])$. Thus $A^\varrho_{e_i}(t_1 * t_2) \geq \mathrm{rmin}\{A^\varrho_{e_i}(t_1), A^\varrho_{e_i}(t_2)\}\ \forall\ t_1, t_2 \in Y$.

Again, let $(t_1)_0, (t_2)_0 \in Y$ be such that $\lambda^\varrho_{e_i}((t_1)_0 * (t_2)_0) > \max\{\lambda^\varrho_{e_i}((t_1)_0), \lambda^\varrho_{e_i}(0)\}$. Let $\lambda^\varrho_{e_i}((t_1)_0) = \eta_{T,I,F_1}$, $\lambda^\varrho_{e_i}((t_2)_0) = \eta_{T,I,F_2}$ and $\lambda^\varrho_{e_i}((t_1)_0 * (t_2)_0) = t_{T,I,F_1}$. Then $t_{T,I,F_1} > \max\{\zeta_{T,I,F_1}. \zeta_{T,I,F_2}\}$. Let us consider $t_{T,I,F_2} = \frac{1}{2}[\lambda^\varrho_{e_i}((t_1)_0 * \hat{v}_0) + \max\{\lambda^\varrho_{e_i}((t_1)_0), \lambda^\varrho_{e_i}(0)\}]$. We get that $t_{T,I,F_2} = \frac{1}{2}(t_{T,I,F_1} + \max\{\zeta_{T,I,F_1}, \zeta_{T,I,F_2}\})$. Therefore, $\zeta_{T,I,F_1} < t_{T,I,F_2} = \frac{1}{2}(t_{T,I,F_1} + \max\{\zeta_{T,I,F_1}, \zeta_{T,I,F_2}\}) < t_{T,I,F_1}$ and $\zeta_{T,I,F_2} < t_{T,I,F_2} = \frac{1}{2}(t_{T,I,F_1} + \max\{\zeta_{T,I,F_1}, \zeta_{T,I,F_2}\}) < t_{T,I,F_1}$. Hence, $\max\{\zeta_{T,I,F_1}, \zeta_{T,I,F_2}\} < t_{T,I,F_2} < t_{T,I,F_1} = \lambda^\varrho_{e_i}((t_1)_0, (t_2)_0)$, so that $(t_1)_0 * (t_2)_0 \notin L(\lambda^\varrho_{e_i}|t_{T,I,F_1})$ which is a contradiction since $\lambda^\varrho_{e_i}((t_1)_0) = \zeta_{T,I,F_1} \leq \max\{\zeta_{T,I,F_1}, \zeta_{T,I,F_2}\} < t_{T,I,F_2}$ and $\lambda^\varrho_{e_i}((t_2)_0) = \zeta_{T,I,F_2} \leq \max\{\zeta_{T,I,F_1}, \zeta_{T,I,F_2}\} < t_{T,I,F_2}$. This implies $(t_1)_0, (t_2)_0 \in L(\lambda^\varrho_{e_i}|t_{T,I,F_1})$. Thus $\lambda^\varrho_{e_i}(t_1 *$

$t_2) \leq \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}$ $\forall$ $t_1, t_2 \in Y$. Therefore, $U(A^{\varrho}_{e_i}|[w_{T,I,F_1}, w_{T,I,F_2}])$ and $L(\lambda^{\varrho}_{e_i}|t_{T,I,F_1})$ are subalgebras of Y. Hence, $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ is **NSCSU** of Y.

Theorem 3.24 Any subalgebra of Y can be consider as both the upper $[w_{T,I,F_1}, w_{T,I,F_2}]$- level and lower t_{T,I,F_1}-level of some **NSCSU** of Y.

Proof. Let $\tilde{\mathcal{N}}_k$ be a **NSCSU** of Y, and $\tilde{\mathcal{P}}_k$ be a neutrosophic soft cubic set on Y defined by

$$A^{\varrho}_{e_i} = \begin{cases} [\xi_{T,I,F_1}, \xi_{T,I,F_2}] & \text{if } t_1 \in \tilde{\mathcal{N}}_k \\ [0,0] & \text{otherwise}. \end{cases}, \lambda^{\varrho}_{e_i} = \begin{cases} \beta_{T,I,F_1} & \text{if } t_1 \in \tilde{\mathcal{N}}_k \\ 0 & \text{otherwise}. \end{cases}$$

$\forall$ $[\xi_{T,I,F_1}, \xi_{T,I,F_2}] \in D[0,1]$ and $\beta_{T,I,F_1} \in [0,1]$. We consider the following cases.

Case1 : If $\forall$ $t_1, t_2 \in \tilde{\mathcal{N}}_k$ then $A^{\varrho}_{e_i}(t_1) = [\xi_{T,I,F_1}, \xi_{T,I,F_2}]$, $\lambda^{\varrho}_{e_i}(t_1) = \beta_{T,I,F_1}$ and $A^{\varrho}_{e_i}(t_2) = [\xi_{T,I,F_1}, \xi_{T,I,F_2}]$, $\lambda^{\varrho}_{e_i}(t_2) = \beta_{T,I,F_1}$. Thus $A^{\varrho}_{e_i}(t_1 * t_2) = [\xi_{T,I,F_1}, \xi_{T,I,F_2}] = \text{rmin}\{[\xi_{T,I,F_1}, \xi_{T,I,F_2}], [\xi_{T,I,F_1}, \xi_{T,I,F_2}]\} = \text{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}$ and $\lambda^{\varrho}_{e_i}(t_1 * t_2) = \beta_{T,I,F_1} = \max\{\beta_{T,I,F_1}, \beta_{T,I,F_1}\} = \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}$.

Case2: If $t_1 \in \tilde{\mathcal{N}}_k$ and $t_2 \notin \tilde{\mathcal{N}}_k$, then $A^{\varrho}_{e_i}(t_1) = [\xi_{T,I,F_1}, \xi_{T,I,F_2}]$, $\lambda^{\varrho}_{e_i}(t_1) = \beta_{T,I,F_1}$ and $A^{\varrho}_{e_i}(t_2) = [0,0]$, $\lambda^{\varrho}_{e_i}(t_2) = 1$. Thus $A^{\varrho}_{e_i}(t_1 * t_2) \geq [0,0] = \text{rmin}\{[\xi_{T,I,F_1}, \xi_{T,I,F_2}], [0,0]\} = \text{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}$ and $\lambda^{\varrho}_{e_i}(t_1 * t_2) \leq 1 = \max\{\beta_{T,I,F_1}, 1\} = \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}$.

Case3 : If $t_1 \notin \tilde{\mathcal{N}}_k$ and $t_2 \in \tilde{\mathcal{N}}_k$, then $A^{\varrho}_{e_i}(t_1) = [0,0]$, $\lambda^{\varrho}_{e_i}(t_1) = 1$ and $A^{\varrho}_{e_i}(t_2) = [\xi_{T,I,F_1}, \xi_{T,I,F_2}]$, $\lambda^{\varrho}_{e_i}(t_2) = \beta_{T,I,F_1}$. Thus $A^{\varrho}_{e_i}(t_1 * t_2) \geq [0,0] = \text{rmin}\{[0,0], [\xi_{T,I,F_1}, \xi_{T,I,F_2}]\} = \text{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}$ and $\lambda^{\varrho}_{e_i}(t_1 * t_2) \leq 1 = \max\{1, \beta_{T,I,F_1}\} = \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}$.

Case4: If $t_1 \notin \tilde{\mathcal{N}}_k$ and $t_2 \notin \tilde{\mathcal{N}}_k$, then $A^{\varrho}_{e_i}(t_1) = [0,0]$, $\lambda^{\varrho}_{e_i}(t_1) = 1$ and $A^{\varrho}_{e_i}(t_2) = [0,0]$, $\lambda^{\varrho}_{e_i}(t_2) = 1$. Thus $A^{\varrho}_{e_i}(t_1 * t_2) \geq [0,0] = \text{rmin}\{[0,0], [0,0]\} = \text{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}$ and $\lambda^{\varrho}_{e_i}(t_1 * t_2) \leq 1 = \max\{1,1\} = \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2)\}$. Therefore, $\tilde{\mathcal{P}}_k$ is a **NSCSU** of Y.

Theorem 3.25 Let $\tilde{\mathcal{N}}_k$ be a subset of Y and $\tilde{\mathcal{P}}_k$ be a neutrosophic soft cubic set on Y which is given in the proof of Theorem 3.24. If $\tilde{\mathcal{P}}_k$ is realized as lower level subalgebra and upper level subalgebra of some **NSCSU** of Y, then $\tilde{\mathcal{N}}_k$ is a neutrosophic soft cubic one of Y.

Proof. Let $\tilde{\mathcal{P}}_k$ be a **NSCSU** of Y, and $t_1, t_2 \in \tilde{\mathcal{N}}_k$. Then $A^{\varrho}_{e_i}(t_1) = A^{\varrho}_{e_i}(t_2) = [\xi_{T,I,F_1}, \xi_{T,I,F_2}]$ and $\lambda^{\varrho}_{e_i}(t_1) = \lambda^{\varrho}_{e_i}(t_2) = \beta_{T,I,F_1}$. Thus $A^{\varrho}_{e_i}(t_1 * t_2) \geq \text{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\} = \text{rmin}\{[\xi_{T,I,F_1}, \xi_{T,I,F_2}], [\xi_{T,I,F_1}, \xi_{T,I,F_2}]\} = [\xi_{T,I,F_1}, \xi_{T,I,F_2}]$ and $\lambda^{\varrho}_{e_i}(t_1 * t_2) \leq \max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_2) = \max\{\beta_{T,I,F_1}, \beta_{T,I,F_1}\} = \beta_{T,I,F_1} \Rightarrow t_1 * t_2 \in \tilde{\mathcal{N}}_k$. Hence $\tilde{\mathcal{N}}_k$ is a neutrosophic soft cubic one of Y.

4 Homomorphism of Neutrosophic Soft Cubic Subalgebras

Suppose τ be a mapping from a set Y into a set Y and $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ be a neurosophic soft cubic set in Y. Then the inverse-image of $\tilde{\mathcal{P}}_k$ is defined as $\tau^{-1}(\tilde{\mathcal{P}}_k) = \{\langle t_1, \tau^{-1}(A^{\varrho}_{e_i}), \tau^{-1}(\lambda^{\varrho}_{e_i})\rangle | t_1 \in Y\}$ and $\tau^{-1}(A^{\varrho}_{e_i})(t_1) = A^{\varrho}_{e_i}(\tau(t_1))$ and $\tau^{-1}(\lambda^{\varrho}_{e_i})(t_1) = \lambda^{\varrho}_{e_i}(\tau(t_1))$. It is clear that $\tau^{-1}(\tilde{\mathcal{P}}_k)$ is a neutrosophic soft cubic set.

Theorem 4.1 Let $\tau \mid Y \to X$ is a homomorphic mapping of G-algebra. If $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ is a **NSCSU** of X. Then the pre-image $\tau^{-1}(\tilde{\mathcal{P}}_k) = \{\langle t_1, \tau^{-1}(A^{\varrho}_{e_i}), \tau^{-1}(\lambda^{\varrho}_{e_i})\rangle | t_1 \in X\}$ of $\tilde{\mathcal{P}}_k$ under τ is a **NSCSU** of Y.

Proof. Assume that $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ is a **NSCSU** of Y and $t_1, t_2 \in Y$. Then $\tau^{-1}(A^{\varrho}_{e_i})(t_1 * t_2) = A^{\varrho}_{e_i}(\tau(t_1 * t_2)) = A^{\varrho}_{e_i}(\tau(t_1) * \tau(t_2)) \geq \text{rmin}\{A^{\varrho}_{e_i}(\tau(t_1)), A^{\varrho}_{e_i}(\tau(t_2))\} = \text{rmin}\{\tau^{-1}(A^{\varrho}_{e_i})(t_1), \tau^{-1}(A^{\varrho}_{e_i})(t_2)\}$ and $\tau^{-1}(\lambda^{\varrho}_{e_i})(t_1 * t_2) = \lambda^{\varrho}_{e_i}(\tau(t_1 * t_2)) = \lambda^{\varrho}_{e_i}(\tau(t_1) * \tau(t_2)) \leq \max\{\lambda^{\varrho}_{e_i}(\tau(t_1)), \lambda^{\varrho}_{e_i}(\tau(t_2))\} = \max\{\tau^{-1}(\lambda^{\varrho}_{e_i})(t_1), \tau^{-1}(\lambda^{\varrho}_{e_i})(t_2)\}$. Hence $\tau^{-1}(\tilde{\mathcal{P}}_k) = \{\langle t_1, \tau^{-1}(A^{\varrho}_{e_i}), \tau^{-1}(\lambda^{\varrho}_{e_i})\rangle | t_1 \in Y\}$ is **NSCSU** of Y.

Theorem 4.2 Let $\tau \mid Y \to X$ is a homomorphic mapping of G-algebra and $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_j}, \lambda^{\varrho}_{e_j})$ is a **NSCSU** of X where $j \in k$. If $\inf\{\max\{\lambda^{\varrho}_{e_j}(t_2), \lambda^{\varrho}_{e_j}(t_2)\}\} = \max\{\inf \lambda^{\varrho}_{e_j}(t_2), \inf \lambda^{\varrho}_{e_j}(t_2)\}$ $\forall$ $t_2 \in Y$. Then $\tau^{-1}(\bigcap_{R\, j\in k} \tilde{\mathcal{P}}_k)$ is a **NSCSU** of Y.

Proof. Let $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_j}, \lambda^{\varrho}_{e_j})$ be a **NSCSU** of Y where $j \in k$ satisfying $\inf\left\{\max\left\{\lambda^{\varrho}_{e_j}(t_2), \lambda^{\varrho}_{e_j}(t_2)\right\}\right\} = \max\{\inf \lambda^{\varrho}_{e_j}(t_2), \inf \lambda^{\varrho}_{e_j}(t_2)\}$ $\forall$ $t_2 \in Y$. Then by Theorem 3.8, $\bigcap_{R\, j\in k} \tilde{\mathcal{P}}_k$ is a **NSCSU** of Y. Hence $\tau^{-1}(\bigcap_{R\, j\in k} \tilde{\mathcal{P}}_k)$ is also a **NSCSU** of Y.

Definition 4.3 A neutrosophic soft cubic set $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ in Y is said to have sup-property and inf-property if for any subset S of Y, $\exists$ $s_0 \in T$ such that $A^{\varrho}_{e_i}(s_0) = \underset{s_0 \in S}{\text{rsup}} A^{\varrho}_{e_i}(s_0)$ and $\lambda^{\varrho}_{e_i}(s_0) = \underset{t_0 \in T}{\inf} \lambda^{\varrho}_{e_i}(t_0)$ respectively.

Definition 4.4 Let τ be the mapping from the set Y to the set X. If $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ is neutrosphic cubic set of Y, then the image of $\tilde{\mathcal{P}}_k$ under τ denoted by $\tau(\tilde{\mathcal{P}}_k)$ and is defined as $\tau(\tilde{\mathcal{P}}_k)=\{\langle t_1, \tau_{\text{rsup}}(A^{\varrho}_{e_i}), \tau_{\inf}(\lambda^{\varrho}_{e_i})\rangle | t_1 \in Y\}$, where

$$\tau_{\text{rsup}}(A^{\varrho}_{e_i})(t_2) = \begin{cases} \underset{t_1 \in \tau^{-1}(t_2)}{A^{\varrho}_{e_i}}(t_1), & \text{if } \tau^{-1}(t_2) \neq \phi \\ [0,0], & \text{otherwise}, \end{cases}$$

and

$$\tau_{\inf}(\lambda^{\varrho}_{e_i})(t_2) = \begin{cases} \underset{t_1 \in \tau^{-1}(t_2)}{\lambda^{\varrho}_{e_i}}(t_1), & \text{if } \tau^{-1}(t_2) \neq \phi \\ 1, & \text{otherwise}. \end{cases}$$

Theorem 4.5 Assume $\tau \mid Y \to X$ is a homomorphic mapping of G − algebra and $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ is a **NSCSU** of Y, where $i \in k$. If $\inf\{\max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_1)\}\} = \max\{\inf \lambda^{\varrho}_{e_i}(t_1), \inf \lambda^{\varrho}_{e_i}(t_1)\}$ $\forall$ $t_1 \in Y$. Then $\tau(\bigcap_{P\, i\in k} \tilde{\mathcal{P}}_k)$ is a **NSCSU** of Y.

Proof. Let $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ be **NSCSU** of Y where $i \in k$ satisfying $\inf\{\max\{\lambda^{\varrho}_{e_i}(t_1), \lambda^{\varrho}_{e_i}(t_1)\}\} = \max\{\inf \lambda^{\varrho}_{e_i}(t_1), \inf \lambda^{\varrho}_{e_i}(t_1)\}$ $\forall$ $t_1 \in Y$. Then by Theorem 3.8, $\bigcap_{P\, i\in k} \tilde{\mathcal{P}}_k$ is a **NSCSU** of Y. Hence $\tau(\bigcap_{P\, i\in k} \tilde{\mathcal{P}}_{ki})$ is a **NSCSU** of Y.

Theorem 4.6 Suppose $\tau \mid Y \to X$ is a homomorphic mapping of G-algebra. Let $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ be **NSCSU** of Y where $i \in k$. If $\text{rsup}\{\text{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}\} = \text{rmin}\{\text{rsup}A^{\varrho}_{e_i}(t_1), \text{rsup}A^{\varrho}_{e_i}(t_2)\}$ $\forall$ $t_1, t_2 \in X$. Then $\tau(\bigcup_{P\, i\in k} \tilde{\mathcal{P}}_k)$ is a **NSCSU** of X.

Proof. Let $\tilde{\mathcal{P}}_k = (A^{\varrho}_{e_i}, \lambda^{\varrho}_{e_i})$ be **NSCSU** of Y where $i \in k$ satisfying $\mathrm{rsup}\{\mathrm{rmin}\{A^{\varrho}_{e_i}(t_1), A^{\varrho}_{e_i}(t_2)\}\}$=$\mathrm{rmin}\{\mathrm{rsup}A^{\varrho}_{e_i}(t_1), \mathrm{rsup}A^{\varrho}_{e_i}(t_2)\}$ $\forall$ $t_1, t_2 \in Y$. Then by Theorem 3.8, $\bigcup_{P\, i\in k} \tilde{\mathcal{P}}_k$ is a **NSCSU** of Y. Hence $\tau(\bigcup_{P\, i\in k} \tilde{\mathcal{P}}_k)$ is a **NSCSU** of X.

Corollary 4.7 For a homomorphism $\tau \mid Y \to X$ of G-algebras, these results hold:

1. If $\forall$ $i \in k$, $\tilde{\mathcal{P}}_k$ are **NSCSU** of Y, then $\tau(\bigcap_{R\, i\in k} (\tilde{\mathcal{P}}_k))$ is **NSCSU** of X

2. If $\forall$ $i \in k$, $\widetilde{\mathcal{N}}_k$ are **NSCSU** of X, then $\tau^{-1}(\bigcap_{R\, i\in k} (\widetilde{\mathcal{N}}_k))$ is **NSCSU** of Y.

Proof. Straigtforward.

Theorem 4.8 Let τ be an isomorphic mapping from a G-algebra Y to a G-algebra X. If $\tilde{\mathcal{P}}_k$ is a **NSCSU** of Y. Then $\tau^{-1}(\tau(\tilde{\mathcal{P}}_k)) = \tilde{\mathcal{P}}_k$.

Proof. For any $t_1 \in Y$, let $\tau(t_1) = t_2$. Since τ is an isomorphism, $\tau^{-1}(t_2) = \{t_1\}$. Thus $\tau(\tilde{\mathcal{P}}_k)(\tau(t_1)) = \tau(\tilde{\mathcal{P}}_k)(t_2) = \bigcup_{t_1\in\tau^{-1}(t_2)} \tilde{\mathcal{P}}_k(t_1) = \tilde{\mathcal{P}}_k(t_1)$. For any $t_2 \in Y$, since τ is an isomorphism, $\tau^{-1}(t_2) = \{t_1\}$ so that $\tau(t_1) = t_2$. Thus $\tau^{-1}(\tilde{\mathcal{P}}_k)(t_1) = \tilde{\mathcal{P}}_k(\tau(t_1)) = \tilde{\mathcal{P}}_k(t_2)$. Hence, $\tau^{-1}(\tau(\tilde{\mathcal{P}}_k)) = \tau^{-1}(\tilde{\mathcal{P}}_k) = \tilde{\mathcal{P}}_k$.

5. Conclusions

In this paper, the concept of neutrosophic soft cubic subalgebra of G-algebra was investigated through several useful results. Homomorhic properties of NSCSU were also investigated. For future work this study will provide base for t-soft cubic subalgebra, t-neutrosophic soft cubic subalgebra.

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Neutrosophic ℵ −ideals in semigroups

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Abstract: The aim of this paper is to introduce the notion of neutrosophic ℵ −ideals in semigroups and investigate their properties. Conditions for neutrosophic ℵ −structure to be a neutrosophic ℵ −ideal are provided. We also discuss the concept of characteristic neutrosophic ℵ −structure of semigroups and its related properties.

Keywords: Semigroup; neutrosophic ℵ − structure; neutrosophic ℵ − ideals, neutrosophic ℵ −product.

1. Introduction

Throughout this paper, S denotes a semigroup and for any subsets A and B of S, the multiplication of A and B is defined as $AB = \{ab | a \in A \; and \; b \in B\}$. A nonempty subset A of S is called a subsemigroup of S if $A^2 \subseteq A$. A subsemigroup A of S is called a left (resp., right) ideal of S if $AX \subseteq A$ (resp., $XA \subseteq A$). A subset A of S is called two-sided ideal or ideal of S if it is both a left and right ideal of S.

L.A. Zadeh introduced the concept of fuzzy subsets of a well-defined set in his paper [17] for modeling the vague concepts in the real world. K. T. Atanassov [1] introduced the notion of an Intuitionistic fuzzy set as a generalization of a fuzzy set. In fact from his point of view for each element of the universe there are two degrees, one a degree of membership to a vague subset and the other is a degree of non-membership to that given subset. Many researchers have been working on the theory of this subject and developed it in interesting different branches.

As a more general platform which extends the notions of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued (intuitionistic) fuzzy set, Smarandache introduced the notion of neutrosophic sets (see [15, 16]), which is useful mathematical tool for dealing with incomplete, inconsistent and indeterminate information. This concept has been extensively studied and investigated by several authors in different fields (see [2 – 8] and [10 – 14]).

For further particulars on neutrosophic set theory, we refer the readers to the site http://fs.gallup.unm.edu/FlorentinSmarandache.htm

In [9], M. Khan et al. introduced the notion of neutrosophic ℵ −subsemigroup in semigroup and investigated several properties. It motivates us to define the notion of neutrosophic ℵ −ideal in semigroup. In this paper, the notion of neutrosophic ℵ −ideals in semigroups is introduced and several properties are investigated. Conditions for neutrosophic ℵ −structure to be neutrosophic ℵ −ideal are provided. We also discuss the concept of characteristic neutrosophic ℵ −structure of semigroups and its related properties.

2. Neutrosophic ℵ - structures

This section explains some basic definitions of neutrosophic $\aleph-$ structures of a semigroup S that have been used in the sequel and introduce the notion of neutrosophic $\aleph-$ ideals in semigroups.

The collection of function from a set S to $[-\mathbf{1},\mathbf{0}]$ is denoted by $\mathfrak{I}(\boldsymbol{S},[-\mathbf{1},\mathbf{0}])$. An element of $\mathfrak{I}(\boldsymbol{S},[-\mathbf{1},\mathbf{0}])$ is called a negative-valued function from S to $[-\mathbf{1},\mathbf{0}]$ (briefly, $\aleph-$ function on S). By a $\aleph-$structure, we mean an ordered pair $(\boldsymbol{S},\boldsymbol{g})$ of S and a $\aleph-$function g on S.

For any family $\{x_i \mid i \in \Lambda\}$ of real numbers, we define:

$$\bigvee\{x_i \mid i \in \Lambda\} := \begin{cases} \max\{x_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite} \\ \sup\{x_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is infinite} \end{cases}$$

and

$$\bigwedge\{x_i \mid i \in \Lambda\} := \begin{cases} \min\{x_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite} \\ \inf\{x_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is infinite} \end{cases}$$

For any real numbers x and y, we also use $x \vee y$ and $x \wedge y$ instead of $\vee\{\boldsymbol{x},\boldsymbol{y}\}$ **and** $\wedge\{\boldsymbol{x},\boldsymbol{y}\}$ respectively.

Definition 2.1. [9] A neutrosophic $\aleph-$ structure over S defined to be the structure:

$$\boldsymbol{S_N} := \frac{S}{(T_N,\ I_N,\ F_N)} = \left\{\frac{x}{T_N(x),\ I_N(x),\ F_N(x)} \mid x \in S\right\},$$

where $\boldsymbol{T_N}$, $\boldsymbol{I_N}$ and $\boldsymbol{F_N}$ are $\aleph-$ functions on S which are called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function, respectively, on S. It is clear that for any neutrosophic $\aleph-$ structure $\boldsymbol{S_N}$ over S, we have $-\mathbf{3} \le \boldsymbol{T_N(y)} + \boldsymbol{I_N(y)} + \boldsymbol{F_N(y)} \le \mathbf{0}$ for all $y \in \boldsymbol{S}$.

Definition 2.2. [9] Let $\boldsymbol{S_N} := \frac{S}{(T_N,\ I_N,\ F_N)}$ and $\boldsymbol{S_M} := \frac{S}{(T_M,\ I_M,\ F_M)}$ be neutrosophic $\aleph-$structures over S. Then

(i) $\boldsymbol{S_N}$ is called a neutrosophic $\aleph-$ substructure of $\boldsymbol{S_M}$ over S, denote by $\boldsymbol{S_N} \subseteq \boldsymbol{S_M}$, if $\boldsymbol{T_N(s)} \ge \boldsymbol{T_M(s)}$, $\boldsymbol{I_N(s)} \le \boldsymbol{I_M(s)}$, $\boldsymbol{F_N(s)} \ge \boldsymbol{F_M(s)}$ for all $s \in \boldsymbol{S}$.

If $\boldsymbol{S_N} \subseteq \boldsymbol{S_M}$ and $\boldsymbol{S_M} \subseteq \boldsymbol{S_N}$, then we say that $\boldsymbol{S_N} = \boldsymbol{S_M}$.

(ii) The neutrosophic $\aleph-$ product of $\boldsymbol{S_N}$ and $\boldsymbol{S_M}$ is defined to be a neutrosophic $\aleph-$structure over S

$$\boldsymbol{S_N} \odot \boldsymbol{S_M} := \frac{S}{(T_{N\circ M},\ I_{N\circ M},\ F_{N\circ M})} = \left\{\frac{s}{T_{N\circ M}(s),\ I_{N\circ M}(s),\ F_{N\circ M}(s)} \mid s \in S\right\},$$

where

$$T_{N\circ M}(s) = \begin{cases} \bigwedge_{s=uv}\{T_N(u) \vee T_M(v)\} & \text{if } \exists\, u,v \in S \text{ such that } s = uv \\ 0 & \text{otherwise}, \end{cases}$$

$$I_{N\circ M}(s) = \begin{cases} \bigvee_{s=uv}\{I_N(u) \wedge I_M(v)\} & \text{if } \exists\, u,v \in S \text{ such that } s = uv \\ 0 & \text{otherwise}, \end{cases}$$

$$F_{N\circ M}(s) = \begin{cases} \bigwedge_{s=uv}\{F_N(u) \vee F_M(v)\} & \text{if } \exists\, u,v \in S \text{ such that } s = uv \\ 0 & \text{otherwise}. \end{cases}$$

For $s \in \mathbf{S}$, the element $\frac{s}{(T_{N\circ M},\ I_{N\circ M},\ F_{N\circ M})}$ is simply denoted by $(\mathbf{S_N} \odot \mathbf{S_M})(\mathbf{s}) = (\mathbf{T_{N\circ M}(s)},\ \mathbf{I_{N\circ M}(s)},\ \mathbf{F_{N\circ M}(s)})$ for the sake of convenience.

(iii) The union of $\boldsymbol{S_N}$ and $\boldsymbol{S_M}$ is defined to be a neutrosophic $\aleph-$structure over S

$$\boldsymbol{S_{N\cup M}} = (\boldsymbol{S}; \boldsymbol{T_{N\cup M}},\ \boldsymbol{I_{N\cup M}},\ \boldsymbol{F_{N\cup M}}),$$

where

$$\boldsymbol{T_{N\cup M}(a)} = \boldsymbol{T_N(a)} \wedge \boldsymbol{T_M(a)},$$

$$I_{N\cup M}(a) = I_N(a) \vee I_M(a),$$
$$F_{N\cup M}(a) = F_N(a) \wedge F_M(a) \text{ for all } a \in S.$$

(iv) The intersection of S_N and S_M is defined to be a neutrosophic $\aleph$ −structure over S

$$S_{N\cap M} = (S; T_{N\cap M},\ I_{N\cap M},\ F_{N\cap M}),$$

where

$$T_{N\cap M}(a) = T_N(a) \vee T_M(a),$$
$$I_{N\cap M}(a) = I_N(a) \wedge I_M(a),$$
$$F_{N\cap M}(a) = F_N(a) \vee F_M(a) \text{ for all } a \in S.$$

Definition 2.3. [9] A neutrosophic $\aleph-$ structure S_N over *S* is called a neutrosophic $\aleph$ −subsemigroup of *S* if it satisfies:

$$(\forall\, a, b \in S)\begin{pmatrix} T_N(ab) \leq T_N(a) \vee T_N(b) \\ I_N(ab) \geq I_N(a) \vee I_N(b) \\ F_N(ab) \leq F_N(a) \vee F_N(b) \end{pmatrix}.$$

Definition 2.4. A neutrosophic $\aleph$ −structure S_N over *S* is called a neutrosophic $\aleph$ −left (resp., right) ideal of *S* if it satisfies:

$$(\forall\, a, b \in S)\begin{pmatrix} T_N(ab) \leq T_N(a)\ (resp., T_N(ab) \leq T_N(b)) \\ I_N(ab) \geq I_N(a)\ (resp.,\ I_N(ab) \geq I_N(b)) \\ F_N(ab) \leq F_N(a)\ (resp.,\ F_N(ab) \leq F_N(b)) \end{pmatrix}.$$

If S_N is both a neutrosophic $\aleph-$ left and neutrosophic $\aleph$ −right ideal of *S*, then it called a neutrosophic $\aleph$ −ideal of *S*.

It is clear that every neutrosophic $\aleph$ −left and neutrosophic $\aleph$ −right ideal of *S* is a neutrosophic $\aleph-$ subsemigroup of *S*, but neutrosophic $\aleph-$ subsemigroup of *S* is need not to be either a neutrosophic $\aleph$ −left or a neutrosophic $\aleph$ −right ideal of *S* as can be seen by the following example.

Example 2.5. Let *S*= $\{0, 1, 2, 3, 4, 5\}$ be a semigroup with the following multiplication table:

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	3	1	1
3	0	1	1	1	2	3
4	0	1	4	5	1	1
5	0	1	1	1	4	5

Then $S_N = \left\{\frac{0}{(-0.9,-0.1,-0.8)}, \frac{1}{(-0.5,-0.2,-0.6)}, \frac{2}{(-0.1,-0.8,-0.1)}, \frac{3}{(-0.3,-0.6,-0.4)}, \frac{4}{(-0.1,-0.8,-0.1)}, \frac{5}{(-0.4,-0.3,-0.5)}\right\}$ is a neutrosophic $\aleph-$ subsemigroup of S, but not a neutrosophic $\aleph-$ left ideal of S as $T_N(3.5) \nleq T_N(5), I_N(3.5) \ngeq I_N(5)$ and $F_N(3.5) \nleq F_N(5)$.

□

Example 2.6. Let $S = \{a, b, c,\ d\}$ be a semigroup with the following multiplication table:

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Then $S_N = \left\{\frac{a}{(-0.9,\ -0.1,\ -0.8)}, \frac{b}{(-0.5,\ -0.2,\ -0.6)}, \frac{c}{(-0.3,\ -0.3,\ -0.4)}, \frac{d}{(-0.4,\ -0.2,\ -0.5)}\right\}$ is a neutrosophic $\aleph$ −ideal of *S*. □

Definition 2.7. For a subset A of S, consider the neutrosophic $\aleph$ −structure

$$\chi_A(S_N) = \frac{S}{(\chi_A(T)_N,\ \chi_A(I)_N,\ \chi_A(F)_N)}$$

where

$$\chi_A(T)_N : S \to [-1, 0],\ s \to \begin{cases} -1 \ if\ s \in A \\ 0 \ \ otherwise \end{cases}$$

$$\chi_A(I)_N : S \to [-1, 0],\ s \to \begin{cases} 0 \ if\ s \in A \\ -1 \ otherwise \end{cases}$$

$$\chi_A(F)_N : S \to [-1, 0],\ s \to \begin{cases} -1 \ if\ s \in A \\ 0 \ \ otherwise \end{cases}$$

which is called the characteristic neutrosophic $\aleph$ −structure of S.

Definition 2.8. [9] Let S_N be a neutrosophic $\aleph-$ structure over S and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \le \alpha + \beta + \gamma \le 0$. Consider the following sets:

$$T_N^{\alpha} = \{s \in S : T_N(s) \le \alpha\},$$
$$I_N^{\beta} = \{s \in S : I_N(s) \ge \beta\},$$
$$F_N^{\gamma} = \{s \in S : F_N(s) \le \gamma\}.$$

The set $S_N(\alpha, \beta, \gamma) := \{s \in S \mid T_N(s) \le \alpha,\ I_N(s) \ge \beta,\ F_N(s) \le \gamma\}$ is called a (α, β, γ)-level set of S_N. Note that $S_N(\alpha, \beta, \gamma) = T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$.

3. Neutrosophic $\aleph-$ ideals

Theorem 3.1 Let S_N be a neutrosophic $\aleph-$ structure over S and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \le \alpha + \beta + \gamma \le 0$. If S_N is a neutrosophic $\aleph-$ left (resp., right) ideal of S, then $(\alpha, \beta, \gamma)-$ level set of S_N is a neutrosophic left (resp., right) ideal of S whenever it is non-empty.

Proof: Assume that $S_N(\alpha, \beta, \gamma) \ne \emptyset$ for $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \le \alpha + \beta + \gamma \le 0$. Let S_N be a neutrosophic $\aleph-$ left ideal of S and let $x, y \in S_N(\alpha, \beta, \gamma)$. Then $T_N(xy) \le T_N(x) \le \alpha; I_N(xy) \ge I_N(x) \ge \beta$ and $F_N(xy) \le F_N(x) \le \gamma$ which imply $xy \in S_N(\alpha, \beta, \gamma)$. Therefore $S_N(\alpha, \beta, \gamma)$ is a neutrosophic $\aleph-$ left ideal of S. □

Theorem 3.2. Let S_N be a neutrosophic $\aleph-$ structure over S and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \le \alpha + \beta + \gamma \le 0$. If T_N^{α}; I_N^{β} and F_N^{γ} are left (resp., right) ideals of S, then S_N is a neutrosophic $\aleph$ −left (resp., right) ideal of S whenever it is non-empty.

Proof: If there are a, $b \in S$ such that $T_N(ab) > T_N(a)$. Then $T_N(ab) > t_{\alpha} \ge T_N(a)$ for some $t_{\alpha} \in [-1, 0)$. Thus $a \in T_N^{t_{\alpha}}(a)$, but $ab \notin T_N^{t_{\alpha}}(a)$, a contradiction. So $T_N(ab) \le T_N(a)$. Similar way we can get $T_N(ab) \le T_N(b)$.

If there are $a, b \in S$ such that $I_N(ab) < I_N(a)$. Then $I_N(ab) < t_{\beta} \le I_N(a)$ for some $t_{\beta} \in (-1,\ 0]$. Thus $a \in I_N^{t_{\beta}}(a)$, but $ab \notin I_N^{t_{\beta}}(a)$, a contradiction. So $I_N(ab) \ge I_N(a)$. Similar way we can get $I_N(ab) \ge I_N(b)$.

If there are $a, b \in S$ such that $F_N(ab) > F_N(a)$. Then $F_N(ab) > t_{\gamma} \ge F_N(a)$ for some $t_{\gamma} \in [-1, 0)$. Thus $a \in F_N^{t_{\gamma}}(a)$, but $ab \notin F_N^{t_{\gamma}}(a)$, a contradiction. So $F_N(ab) \le F_N(a)$. Similar way we can get $F_N(ab) \le F_N(b)$.

Hence S_N is a neutrosophic $\aleph$ −left ideal of S. □

Theorem 3.3. Let S be a semigroup. Then the intersection of two neutrosophic $\aleph$ −left (resp., right) ideals of S is also a neutrosophic $\aleph$ −left (resp., right) ideal of S.

Proof: Let $S_N := \frac{S}{(T_N,\ I_N,\ F_N)}$ and $S_M := \frac{S}{(T_M,\ I_M,\ F_M)}$ be neutrosophic $\aleph$ −left ideals of S. Then for any $x, y \in S$, we have

$$T_{N\cap M}(xy) = T_N(xy) \vee T_M(xy) \leq T_N(y) \vee T_M(y) = T_{N\cap M}(y),$$
$$I_{N\cap M}(xy) = I_N(xy) \wedge I_M(xy) \geq I_N(y) \wedge I_M(y) = I_{N\cap M}(y),$$
$$F_{N\cap M}(xy) = F_N(xy) \vee F_M(xy) \leq F_N(y) \vee F_M(y) = F_{N\cap M}(y).$$

Therefore $X_{N\cap M}$ is a neutrosophic $\aleph$ −left ideal of *S*.

□

Corollary 3.4. Let *S* be a semigroup. Then $\{X_{N_i} | i \in \mathbb{N}\}$ is a family of neutrosophic $\aleph$ −left (resp., right) ideals of *S*, then so is $X_{\cap N_i}$.

Theorem 3.5. For any non-empty subset *A* of *S*, the following conditions are equivalent:

(i) A is a neutrosophic $\aleph$ −left (resp., right) ideal of *S*,

(ii) The characteristic neutrosophic $\aleph$ −structure $\chi_A(X_N)$ over *S* is a neutrosophic $\aleph$ −left (resp., right) ideal of *S*.

Proof: Assume that *A* is a neutrosophic $\aleph$ −left ideal of *S*. For any $x, y \in A$.

If $y \notin A$, then $\chi_A(T)_N(xy) \leq 0 = \chi_A(T)_N(y)$; $\chi_A(I)_N(xy) \geq -1 = \chi_A(I)_N(y)$ and $\chi_A(F)_N(xy) \leq 0 = \chi_A(F)_N(y)$. Otherwise $y \in A$. Then $xy \in A$, so $\chi_A(T)_N(xy) = -1 = \chi_A(T)_N(y)$; $\chi_A(I)_N(xy) = 0 = \chi_A(I)_N(y)$ and $\chi_A(F)_N(xy) = -1 = \chi_A(F)_N(y)$. Therefore $\chi_A(S_N)$ is a neutrosophic $\aleph$ −left ideal of *S*.

Conversely, assume that $\chi_A(S_N)$ is a neutrosophic $\aleph$ −left ideal of *S*. Let $a \in A$ and $x \in S$. Then $\chi_A(T)_N(xa) \leq \chi_A(T)_N(a) = -1$, $\chi_A(I)_N(xa) \geq \chi_A(I)_N(a) = 0$ and $\chi_A(F)_N(xa) \leq \chi_A(F)_N(a) = -1$. Thus $\chi_A(T)_N(xa) = -1$, $\chi_A(I)_N(xa) = 0$ and $\chi_A(F)_N(xa) = -1$ and hence $xa \in A$. Therefore *A* is a neutrosophic $\aleph$ −left ideal of *S*. □

Theorem 3.6. Let $\chi_A(S_N)$ and $\chi_B(S_N)$ be characteristic neutrosophic $\aleph$ −structure over *S* for subsets A and B of *S*. Then

(i) $\chi_A(S_N) \cap \chi_B(S_N) = \chi_{A\cap B}(S_N)$.

(ii) $\chi_A(S_N) \odot \chi_B(S_N) = \chi_{AB}(S_N)$.

Proof: (i) Let s $\in S$.

If s $\in A \cap B$, then

$$(\chi_A(T)_N \cap \chi_B(T)_N)(s) = \chi_A(T)_N(s) \vee \chi_B(T)_N(s) = -1 = \chi_{A\cap B}(T)_N(s),$$
$$(\chi_A(I)_N \cap \chi_B(I)_N)(s) = \chi_A(I)_N(s) \wedge \chi_B(I)_N(s) = 0 = \chi_{A\cap B}(I)_N(s),$$
$$(\chi_A(F)_N \cap \chi_B(F)_N)(s) = \chi_A(F)_N(s) \vee \chi_B(F)_N(s) = -1 = \chi_{A\cap B}(F)_N(s).$$

Hence $\chi_A(S_N) \cap \chi_B(S_N) = \chi_{A\cap B}(S_N)$.

If s $\notin A \cap B$, then s $\notin A$ or s $\notin B$. Thus

$$(\chi_A(T)_N \cap \chi_B(T)_N)(s) = \chi_A(T)_N(s) \vee \chi_B(T)_N(s) = 0 = \chi_{A\cap B}(T)_N((s)),$$
$$(\chi_A(I)_N \cap \chi_B(I)_N)(s) = \chi_A(I)_N(s) \wedge \chi_B(I)_N(s) = -1 = \chi_{A\cap B}(I)_N(s),$$
$$(\chi_A(F)_N \cap \chi_B(F)_N)(s) = \chi_A(F)_N(s) \vee \chi_B(F)_N(s) = 0 = \chi_{A\cap B}(F)_N(s).$$

Hence $\chi_A(S_N) \cap \chi_B(S_N) = \chi_{A\cap B}(S_N)$.

(ii) Let $x \in S$. If $x \in AB$, then $x = ab$ for some $a \in A$ and $b \in B$.

Now

$$\begin{aligned}
(\chi_A(T)_N \circ \chi_B(T)_N)(x) &= \wedge_{x=st}\{\chi_A(T)_N(s) \vee \chi_B(T)_N(t)\} \\
&\leq \chi_A(T)_N(a) \vee (\chi_B(T)_N(b) \\
&= -1 = \chi_{AB}(T)_N(x), \\
(\chi_A(I)_N \circ \chi_B(I)_N)(x) &= \bigvee_{x=st}\{\chi_A(I)_N(s) \vee \chi_B(I)_N(t)\} \\
&\geq \chi_A(I)_N(a) \vee \chi_B(I)_N(b) \\
&= 0 = \chi_{AB}(I)_N(x), \\
(\chi_A(F)_N \circ \chi_B(F)_N)(x) &= \bigwedge_{x=st}\{\chi_A(F)_N(s) \vee \chi_B(F)_N(t)\}
\end{aligned}$$

$$\leq \chi_A(F)_N(a) \vee (\chi_B(F)_N(b)$$
$$= -1 = \chi_{AB}(F)_N(x).$$

Therefore $\chi_A(S_N) \odot \chi_B(S_N) = \chi_{AB}(S_N)$. □

Note 3.7. Let $S_N := \frac{S}{(T_N,\ I_N,\ F_N)}$ and $S_M := \frac{S}{(T_M,\ I_M,\ F_M)}$ be neutrosophic $\aleph$ −structures over S. Then for any subsets A and B of S, we have

(i) $\chi_{A\cap B}(S_N \cap S_M) = (S{:}\, \chi_{A\cap B}(T)_{N\cap M}, \chi_{A\cap B}(I)_{N\cap M}, \chi_{A\cap B}(F)_{N\cap M})$,

where

$$\chi_{A\cap B}(T)_{N\cap M}(s) = \chi_{A\cap B}(T)_N(s) \vee \chi_{A\cap B}(T)_M(s),$$
$$\chi_{A\cap B}(I)_{N\cap M}(s) = \chi_{A\cap B}(I)_N(s) \wedge \chi_{A\cap B}(I)_M(s),$$
$$\chi_{A\cap B}(F)_{N\cap M}(s) = \chi_{A\cap B}(F)_N(s) \vee \chi_{A\cap B}(F)_M(s)\ \boldsymbol{for\ s\ \varepsilon\ S}.$$

(ii) $\chi_{A\cup B}(S_N \cap S_M) = (S{:}\, \chi_{A\cup B}(T)_{N\cup M}, \chi_{A\cup B}(I)_{N\cup M}, \chi_{A\cup B}(F)_{N\cup M})$,

where

$$\chi_{A\cup B}(T)_{N\cup M}(s) = \chi_{A\cup B}(T)_N(s) \wedge \chi_{A\cup B}(T)_M(s),$$
$$\chi_{A\cup B}(I)_{N\cup M}(s) = \chi_{A\cup B}(I)_N(s) \vee \chi_{A\cup B}(I)_M(s),$$
$$\chi_{A\cup B}(F)_{N\cup M}(s) = \chi_{A\cup B}(F)_N(s) \wedge \chi_{A\cup B}(F)_M(s)\ \boldsymbol{for\ s\ \varepsilon\ S}.$$

Theorem 3.8. Let S_M be a neutrosophic $\aleph-$ structure over S. Then S_M is a neutrosophic $\aleph-$ left ideal of S if and only if $S_N \odot S_M \subseteq S_M$ for any neutrosophic $\aleph-$ structure S_N over S.

Proof: Assume that S_M is a neutrosophic $\aleph-$ left ideal of S and let $s, t, u \in S$. If $s = tu$, then

$T_M(s) = T_M(tu) \leq T_M(u) \leq T_M(t) \vee T_M(u)$ which implies $T_M(s) \leq T_{N\circ M}(s)$. Otherwise $s \neq tu$. Then $T_M(s) \leq 0 = T_{N\circ M}(s)$.

$I_M(s) = I_M(tu) \geq I_M(u) \geq I_M(t) \wedge I_M(t)$ which implies $I_M(s) \geq I_{N\circ M}(s)$. Otherwise $s \neq tu$. Then $I_M(s) \geq -1 = I_{N\circ M}(s)$.

$F_M(s) = F_M(tu) \leq F_M(u) \leq F_M(t) \vee F_M(u)$ which implies $F_M(s) \leq F_{N\circ M}(s)$. Otherwise $s \neq tu$. Then $F_M(s) \leq 0 = F_{N\circ M}(s)$.

Conversely, assume that S_M is a neutrosophic $\aleph-$ structure over S such that $S_N \odot S_M \subseteq S_M$ for any neutrosophic $\aleph-$ structure S_N over S. Let $x, y \in S$. If $a = xy$, then

$$T_M(xy) = T_M(a) \leq (\chi_X(T)_N \circ T_M)(a) = \bigwedge_{a=st}\{\chi_X(T)_N(s) \vee T_M(t)\} \leq \chi_X(T)_N(x) \vee T_M(y) = T_M(y),$$

$$I_M(xy) = I_M(a) \geq (\chi_X(I)_N \circ I_M)(a) = \bigvee_{a=st}\{\chi_X(I)_N(s) \wedge I_M(t)\} \geq \chi_X(I)_N(x) \vee I_M(y) = I_M(y),$$

$$F_M(xy) = F_M(a) \leq (\chi_X(F)_N \circ F_M)(a) = \bigwedge_{a=st}\{\chi_X(F)_N(s) \vee F_M(t)\} \leq \chi_X(F)_N(x) \vee F_M(y) = F_M(y).$$

Therefore S_M is a neutrosophic $\aleph-$ left ideal of S. □

Similarly, we have the following.

Theorem 3.9. Let S_M be a neutrosophic $\aleph-$ structure over S. Then S_M is a neutrosophic $\aleph-$ left ideal of S if and only if $S_M \odot S_N \subseteq S_M$ for any neutrosophic $\aleph-$ structure S_N over S.

Theorem 3.10. Let S_M and S_N be neutrosophic $\aleph-$ structures over S. If S_M is a neutrosophic $\aleph-$ left ideal of S, then so is the $S_M \odot S_N$.

Proof: Assume that S_M is a neutrosophic $\aleph-$ left ideal of S and let $x, y \in S$. If there exist $a, b \in S$ such that $y = ab$, then $xy = x(ab) = (xa)b$.

Now,

$$(T_N \circ T_M)(y) = \bigwedge_{y=ab}\{T_N(a) \vee T_M(b)\}$$

$$\leq \bigwedge_{xy=(xa)b} \{T_N(xa) \vee T_M(b)$$

$$= \bigwedge_{xy=cb} \{T_N(c) \vee T_M(b)\} = (T_N \circ T_M)(xy),$$

$$(I_N \circ I_M)(y) = \bigvee_{y=ab} \{I_M(b) \wedge I_M(b)\}$$

$$\geq \bigvee_{xy=(xa)b} \{I_M(xa) \wedge I_M(b)\}$$

$$= \bigvee_{xy=cb} \{I_M(c) \wedge I_M(b)\} = (I_N \circ I_M)(xy),$$

$$(F_N \circ F_M)(y) = \bigwedge_{y=ab} \{F_N(a) \vee F_M(b)\}$$

$$\leq \bigwedge_{xy=(xa)b} \{F_N(xa) \vee F_M(b)$$

$$= \bigwedge_{xy=cb} \{F_N(c) \vee F_M(b)\} = (F_N \circ F_M)(xy).$$

Therefore $S_M \odot S_N$ is a neutrosophic $\aleph -$ left ideal of *S*. □

Similarly, we have the following.

Theorem 3.11. Let S_M and S_N be neutrosophic $\aleph -$ structures over *S*. If S_M is a neutrosophic $\aleph -$ right ideal of *S*, then so is the $S_M \odot S_N$.

Conclusions

In this paper, we have introduced the notion of neutrosophic $\aleph -$ideals in semigroups and investigated their properties, and discussed characterizations of neutrosophic $\aleph -$ideals by using the notion of neutrosophic $\aleph -$ product, also provided conditions for neutrosophic $\aleph -$structure to be a neutrosophic $\aleph -$ ideal in semigroup. We have also discussed the concept of characteristic neutrosophic $\aleph -$structure of semigroups and its related properties. Using this notions and results in this paper, we will define the concept of neutrosophic $\aleph -$bi-ideals in semigroups and study their properties in future.

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Fuzzy Multi Item Inventory Model with Deterioration and Demand Dependent Production Cost Under Space Constraint: Neutrosophic Hesitant Fuzzy Programming Approach

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Abstract. In this paper, we have developed a multi-objective inventory model with constant demand rate, under the limitation on storage of space. Production cost is considered in demand dependent and the deterioration cost is considered in average inventory level dependent. Also inventory holding cost is dependent on time. Due to uncertainty, all cost parameters are taken as generalized trapezoidal fuzzy number. Our proposed model is solved by both neutrosophic hesitant fuzzy programing approach and fuzzy non-linear programming technique. Numerical example has been given to illustrate the model. Finally sensitivity analysis has been presented graphically.

Keywords: Inventory, Deterioration, Multi-item, Generalized trapezoidal fuzzy number, Neutrosophic Hesitant fuzzy programming approach.

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1. Introduction

An inventory model deal with decision that minimum the total average cost or maximum the total average profit. In that way to construct a real life mathematical inventory model on base on various assumptions and notations and approximation.

In ordinary inventory system inventory cost i.e set-up cost, holding cost, deterioration cost, etc. are taken fixed amount but in real life inventory system these cost not always fixed. So consideration of fuzzy variable is more realistic and interesting.

Inventory problem for deteriorating items have been widely studied, deterioration is defined as the spoilage, damage, dryness, vaporization etc., this result in decrease of usefulness of the commodity. Economic order quantity model was first introduced in February 1913 by Harris [1], afterwards many researchers developed EOQ model in inventory systems like as Singh, T., Mishra, P.J. and Pattanayak, H. [4], Jong Wuu Wu & Wen Chuan Lee [5] etc.

Deterioration of an item is the most important factor in the inventory systems. Ghare and Schrader [15], developed the inventory model by considering the constant demand rate and constant deterioration rate. Jong-Wuu Wu, Chinho Lin, Bertram Tan & Wen-Chuan Lee [6] developed an EOQ inventory model with time-varying demand and Weibull deterioration with shortages. Mishra, U. [13] presented a paper on an inventory model for Weibull deterioration with stock and price dependent demand. Jong Wuu Wu & Wen Chuan Lee [5] discussed an EOQ inventory model for items with Weibull deterioration, shortages and time varying demand. Singh, T., Mishra, P.J. and Pattanayak, H. [4] studied an EOQ inventory model for deteriorating items with time-dependent deterioration rate, ramp-type demand rate and shortages. Roy, T. K. & Maity, M, N. K. Mondal [14] discussedc inventory model of deteriorating items with a constraint: A geometric programming approach.

The concept of fuzzy set theory was first introduced by Zadeh, L.A. [16]. Afterward Zimmermann, H.J [17], [18] applied the fuzzy set theory concept with some useful membership functions to solve the linear programming problem with some objective functions. Then the various ordinary inventory model transformed to fuzzy versions model by various authors such as Roy, T. K. & Maity, M [2] presented on a fuzzy inventory model with constraints.

Also Smarandache, F. introduced the nutrosophic set. Smarandache, F [19] presented Neutrosophic Set, a generalization of the intuitionistic fuzzy set also discussed a geometric interpretation of the NS set a generalization of the intuitionistic fuzzy set. Multi-item and limitations of spaces is the important in the business world. Ye, J. [7] studied on multiple-attribute decision-making method under a single-valued neutrosophic hesitant fuzzy environment. Firoz Ahmad, Ahmad Yusuf Adhami, F. Smarandache [12] established single valued Neutrosophic Hesitant Fuzzy Computational Algorithm for multi objective nonlinear optimization problem. Islam, S. and Mandal, W. A. [10] considered a fuzzy inventory model with unit production cost, time depended holding cost, with-out shortages under a space constraint: a parametric geometric programming approach. B. Mondal, C. Kar, A. Garai, T. Kr. Roy [8, 25] studied on optimization of EOQ Model with Limited Storage Capacity by neutrosophic geometric programming. Mullai, M. and Surya, R. [22] developed neutrosophic EOQ model with price break. Mohana, K., Christy, V. and Smarandache, F. [24] discussed on multi-criteria decision making problem via bipolar single-valued neutrosophic. Nabeeh, N. A.; Abdel-Basset, M.; El-Ghareeb, H. A.; Aboelfetouh, A. Discussed neutrosophic multi-criteria decision making approach for iot-based enterprises. Biswas, P., Pramanik, S., Giri, B. C. [11] presented multi-attribute group decision making based on expected value of neutrosophic trapezoidal numbers. Pramanik, S., Mallick, R., Dasgupta, [9] discussed a Contributions of selected Indian researchers to multi attribute decision making in neutrosophic environment: an over view.

In this paper, we have considered the constant demand rate, under the restriction on storage area. Production cost is considered in demand dependent and the deterioration cost is considered in average inventory and also holding cost is time dependent. Due to uncertainty, all the required parameters are considered generalized trapezoidal fuzzy number. The formulated inventory problem has been solved by using FNLP and crisp and neutrosophic hesitant fuzzy programing approach. Finally numerical example has been given to illustrate the model.

2. Preliminaries

2.1 Definition of Fuzzy Set

Let X be a collection of objects called the universe of discourse. A fuzzy set is a subset of X denoted by $\tilde{A}$ and is defined by a set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$. Here $\mu_{\tilde{A}} : X \to [0,1]$ is a function which is called the membership function of the fuzzy set $\tilde{A}$ and $\mu_{\tilde{A}}(x)$ is called the grade of membership of $x \in X$ in the fuzzy set $\tilde{A}$.

2.2 Union of two fuzzy sets

The union of $\tilde{A}$ and $\tilde{B}$ is fuzzy set in X, denoted by $\tilde{A} \cup \tilde{B}$, and defined by the membership function $\mu_{\tilde{A}\cup\tilde{B}}(x) = \mu_{\tilde{A}}(x) \vee \mu_{\tilde{B}}(x) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$ for each x ∈ X.

2.3 Intersection of two fuzzy sets

The intersection of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ in X, denoted by $\tilde{A} \cap \tilde{B}$, and defined by the membership function $\mu_{\tilde{A}\cap\tilde{B}}(x) = \mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(x) = \min\{\mu_{\tilde{A}}(x),\ \mu_{\tilde{B}}(x)\}$ for each x ∈ X.

2.4 Generalized Trapezoidal Fuzzy Number (GTrFN)

A generalized trapezoidal fuzzy number (GTrFN) $\tilde{A} \equiv$(a,b,c,d; w) is a fuzzy set of the real line R whose membership function $\mu_{\tilde{A}}$ (x) : R → [0,w] is defined as

$$\mu_{\tilde{A}}(x) = \begin{cases} \mu_{LA}^{w}(x) = w\left(\frac{x-a}{b-a}\right) & \text{for } a \le x \le b \\ w & \text{for } b \le x \le c \\ \mu_{RA}^{w}(x) = w\left(\frac{d-x}{d-c}\right) & \text{for } c \le x \le d \\ 0 & \text{for otherwise} \end{cases}$$

where a < b < c < d and w ∈ (0, 1]. If w = 1, the generalized fuzzy number $\tilde{A}$ is called a trapezoidal fuzzy number (TrFN) denoted $\tilde{A} \equiv$(a, b, c, d).

2.5 Definition of Neutrosophic Set (NS)

Let X be a collection of objects called the universe of discourse. A neutrosophic set A in X is defined by

$$A = \{(x, T_A(x), I_A(x), F_A(x)) | x \in X\}$$

Here $T_A(x), I_A(x)$ *and* $F_A(x)$ are called truth, indeterminacy and falsity membership function respectively. This membership functions are defined by

$$T_A(x): X \to]0^-, 1^+[\,,\ I_A(x): X \to]0^-, 1^+[,\ F_A(x): X \to]0^-, 1^+[\ \text{ so we have}$$
$$0^- \le sup T_A(x) + sup I_A(x) + sup F_A(x) \le 3^+$$

2.6 Definition of Single valued Neutrosophic Set (SVNS)

Let X be a collection of objects called the universe of discourse. A single valued neutrosophic set A in X is defined by $A = \{(x, T_A(x), I_A(x), F_A(x)) | x \in X\}$. Here $T_A(x): X \to [0,1]$, $I_A(x): X \to [0,1]$, $F_A(x): X \to [0,1]$ and $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$ for all $x \in X$.

2.7 Hesitant Fuzzy Set (HFS)

Let X be a non-empty reference set, a hesitant fuzzy set A on X is defined in terms of a function $h_A(x)$ which is applied to X returns a finite subset of [0,1]. It's mathematical representation is

$$A = \{\langle x, h_A(x)\rangle | x \in X\}$$

Where $h_A(x)$ is a set of some different values in [0,1], representing the possible membership degree of the element $x \in X$ to A. the set $h_A(x)$ is called the hesitant fuzzy element (HFE).

Example 1: Let $X = \{x_1, x_2, x_3\}$ be a reference set, $h_A(x_1) = \{0.4, 0.7, 0.8\}$, $h_A(x_2) = \{0.7, 0.5, 0.6\}$, $h_A(x_3) = \{0.3, 0.8, 0.9, 0.7\}$ be hesitant fuzzy element of x_1, x_2, x_3 respectively to a set A. Then hesitant fuzzy set A is $A = \{\langle x_1, \{0.4,0.7,0.8\}\rangle, \langle x_2, \{0.7,0.5,0.6\}\rangle, \langle x_3, \{0.3,0.8,0.9,0.7\}\rangle\}$.

2.8 Definition of Single valued Neutrosophic Hesitant Fuzzy Set (SVNHFS)

It is based on the combination of SVNS and HFS. Concept of SVNHFS is proposed by Ye [7].

Let X be a non-empty reference set, an single valued neutrosophic hesitant fuzzy set A on X is defined as

$$A = \{\langle x, \widetilde{T_A}(x), \widetilde{I_A}(x), \widetilde{F_A}(x)\rangle | x \in X\}$$

Where $\widetilde{T_A}(x) = \{\alpha | \alpha \in \widetilde{T_A}(x)\}$, $\widetilde{I_A}(x) = \{\beta | \beta \in \widetilde{I_A}(x)\}$ and $\widetilde{F_A}(x) = \{\gamma | \gamma \in \widetilde{F_A}(x)\}$ are three sets of some different values in [0,1], denoting the possible truth membership hesitant, indeterminacy membership hesitant and falsity membership hesitant degree of $x \in X$ to the set A respectively. This are satisfied the following conditions

$$\alpha, \beta, \gamma \subseteq [0,1] \text{ and } 0 \le sup\ \alpha^+ + sup\beta^+ + sup\gamma^+ \le 3$$

Where $\alpha^+ = \cup_{\alpha\in\widetilde{T_A}(x)} max\{\alpha\}$, $\beta^+ = \cup_{\beta\in\widetilde{T_A}(x)} max\{\beta\}$ and $\alpha^+ = \cup_{\gamma\in\widetilde{T_A}(x)} max\{\gamma\}$ for $x \in X$.

2.9 Union of two SVNS sets

Let X be a collection of objects called the universe of discourse and A and B are any two subsets of X. Here $T_A(x): X \rightarrow [0,1]$, $I_A(x): X \rightarrow [0,1]$, $F_A(x): X \rightarrow [0,1]$ are called truth, indeterminacy and falsity membership function of A respectively. The union of A and B denoted by $A \cup B$ and define by

$$A \cup B = \left\{ \left(x, max\left(T_A(x),\ T_B(x) \right), max\left(I_A(x),\ I_B(x) \right), min\left(F_A(x),\ F_B(x) \right) \right) \middle| x \in X \right\}$$

2.10 Intersection of two SVNS sets

Let X be a collection of objects called the universe of discourse and A and B are any two subsets of X. Here $T_A(x): X \rightarrow [0,1]$, $I_A(x): X \rightarrow [0,1]$, $F_A(x): X \rightarrow [0,1]$ are called truth, indeterminacy and falsity membership function of A respectively. The intersection of A and B denoted by $A \cap B$ and define by

$$A \cap B = \left\{ \left(x, min\left(T_A(x),\ T_B(x) \right), min\left(I_A(x),\ I_B(x) \right), max\left(F_A(x),\ F_B(x) \right) \right) \middle| x \in X \right\}$$

3. Mathematical model formulation for ith item

3.1 Notations

c_i: Ordering cost per order for ith item.
$H_i (= h_i t)$: Holding cost per unit per unit time for ith item.
θ_i: Constant deterioration rate for the ith item.
θ_c: Deterioration cost depend average inventory level.
T_i: The length of cycle time for i^{th}item, $T_i > 0$.
D_i : Demand rate per unit time for the ith item.
$I_i(t)$: Inventory level of the ith item at time t.
Q_i: The order quantity for the duration of a cycle of length T_i for ith item.
$TAC_i(T_i, D_i)$: Total average profit per unit for the ith item.
w_i: Storage space per unit time for the ith item.
W: Total area of space.
$\widetilde{c_i}$: Fuzzy ordering cost per order for the ith item.
$\widetilde{\theta_i}$: Fuzzy deterioration rate for the ith item.
$\widetilde{w_i}$: Fuzzy storage space per unit time for the ith item.
$\widetilde{H_i} (= \widetilde{h_i} t)$: Fuzzy holding cost per unit per unit time for the ith item.
$\widetilde{TAC_i}(T_i, D_i)$: Fuzzy total average cost per unit for the ith item.
$\widehat{c_i}$: Defuzzyfication of the fuzzy ordering cost per order for the ith item.
$\widehat{\theta_i}$: Defuzzyfication of the fuzzy deterioration rate for the ith item.
$\widehat{w_i}$: Defuzzyfication of the fuzzy storage space per unit time for the ith item.
$\widehat{H_i} (= \widehat{h_i} t)$: Defuzzyfication of the fuzzy holding cost per unit per unit time for the ith item.
$\widehat{TAC_i}\ (T_i, D_i)$: Defuzzyfication of the fuzzy total average cost per unit for the ith item.

3.2 Assumptions

1. The inventory system involves multi-item.

2. The replenishment occurs instantaneously at infinite rate.

3. The lead time is negligible.

4. Shortages are not allowed.

5. The unit production cost C_p^i of i^{th} item is inversely related to the demand rate D_i. So we take the following form $C_p^i(\ D_i) = \alpha_i {D_i}^{-\beta_i}$, where $\alpha_i > 0$ and $\beta_i > 1$ are constant real number.

6. The deterioration cost is proportionality related to the average inventory level. So we take the form

$\theta_c{}^i(Q) = \gamma_i \left(\frac{Q_i}{2}\right)^{\delta_i}$ where $0 < \gamma_i$ and $0 < \delta_i \ll 1$ are constant real number.

3.3 Model formation in scrip model

The inventory level for ith item is illustrated in Figure-1. During the period $[0, T_i]$ the inventory level reduces due to demand rate and deterioration rate for ith item. In this time period, the inventory level is described by the differential equation-

$$\frac{dI_i(t)}{dt} + \theta_i I_i(t) = -D, 0 \le t \le T_i \tag{1}$$

With boundary condition, $I_i(0) = Q_i,\ I_i(T_i) = 0$.

Solving (1) we have,

$$I_i(t) = \frac{D_i}{\theta_i}\left[e^{\theta_i(T_i - t)} - 1\right] \tag{2}$$

$$Q_i = \frac{D_i}{\theta_i}\left(e^{\theta_i T_i} - 1\right) \tag{3}$$

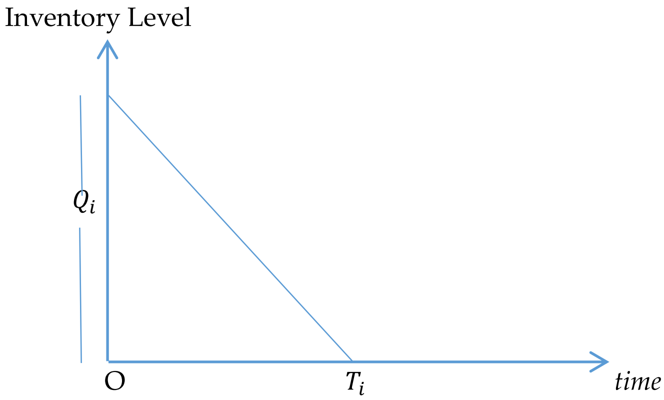

Figure-1 (Inventory level for the ith item.)

Now calculating various cost for ith item

i) Production cost $(PC_i) = \frac{Q_i C_p^i(D_i)}{T_i}$

$$= \frac{D_i{}^{(1-\beta_i)}\alpha_i}{\theta_i T_i}\left(e^{\theta_i T_i} - 1\right)$$

ii) Inventory holding cost $(HC_i) = \frac{1}{T_i}\int_0^{T_i} h_i t I_i(t) dt$

$$= \frac{D_i h_i}{\theta_i T_i}\left\{-\frac{T_i}{\theta_i} + \frac{1}{\theta_i{}^2}\left(e^{\theta_i T_i} - 1\right) - \frac{T_i{}^2}{2}\right\}$$

iv) Deterioration cost $(DC_i) = \theta_i \gamma_i \left(\frac{Q_i}{2}\right)^{\delta_i}$

$$= \theta_i \gamma_i \left(\frac{D_i}{2\theta_i}\left(e^{\theta_i T_i} - 1\right)\right)^{\delta_i}$$

v) Ordering cost $(OC_i) = \frac{c_i}{T_i}$

Total average cost per unit time for ith item

$$TAC_i(T_i, D_i) = (PC_i + HC_i + DC_i + OC_i)$$

$$=\frac{D_i^{(1-\beta_i)}\alpha_i}{\theta_i T_i}\left(e^{\theta_i T_i}-1\right)+\frac{D_i h_i}{\theta_i T_i}\left\{-\frac{T_i}{\theta_i}+\frac{1}{\theta_i^2}\left(e^{\theta_i T_i}-1\right)-\frac{T_i^2}{2}\right\}+\theta_i\gamma_i\left(\frac{D_i}{2\theta_i}\left(e^{\theta_i T_i}-1\right)\right)^{\delta_i}+\frac{c_i}{T_i} \quad (4)$$

A multi-item inventory model (MIIM) can be written as:

Min

$$TAC_i(T_i,D_i)=\frac{D_i^{(1-\beta_i)}\alpha_i}{\theta_i T_i}\left(e^{\theta_i T_i}-1\right)+\frac{D_i h_i}{\theta_i T_i}\left\{-\frac{T_i}{\theta_i}+\frac{1}{\theta_i^2}\left(e^{\theta_i T_i}-1\right)-\frac{T_i^2}{2}\right\}+\theta_i\gamma_i\left(\frac{D_i}{2\theta_i}\left(e^{\theta_i T_i}-1\right)\right)^{\delta_i}+\frac{c_i}{T_i}$$

Subject to, $\sum_{i=1}^{n} w_i \frac{D_i}{\theta_i}\left(e^{\theta_i T_i}-1\right)\leq W$, for $i=1,2,\ldots\ldots\ldots..n.$ (5)

4. Fuzzy Model

Generally the parameters for holding cost, unit production cost, and storage spaces, deterioration are not particularly known to us. Due to uncertainty, we assume all the parameters $(\alpha_i,\beta_i,\theta_i,h_i,\gamma_i,\delta_i,c_i)$ and storage space w_i as generalized trapezoidal fuzzy number (GTrFN)$(\widetilde{\alpha_i},\widetilde{\beta_i},\widetilde{\theta_i},\widetilde{h_i},\widetilde{\gamma_i},\widetilde{\delta_i},\widetilde{c_i}\widetilde{w_i})$. Let us assume,

$\widetilde{\alpha_i}=\left(\alpha_i^1,\alpha_i^2,\alpha_i^3,\alpha_i^4;\omega_{\alpha_i}\right), 0<\omega_{\alpha_i}\leq 1$; $\widetilde{\theta_i}=\left(\theta_i^1,\theta_i^2,\theta_i^3,\theta_i^4;\omega_{\theta_i}\right), 0<\omega_{\theta_i}\leq 1$;

$\widetilde{\beta_i}=\left(\beta_i^1,\beta_i^2,\beta_i^3,\beta_i^4;\omega_{\beta_i}\right), 0<\omega_{\beta_i}\leq 1$; $\widetilde{h_i}=\left(h_i^1,h_i^2,h_i^3,h_i^4;\omega_{h_i}\right), 0<\omega_{h_i}\leq 1$;

$\widetilde{\gamma_i}=\left(\gamma_i^1,\gamma_i^2,\gamma_i^3,\gamma_i^4;\omega_{\gamma_i}\right), 0<\omega_{\gamma_i}\leq 1$; $\widetilde{w_i}=\left(w_i^1,w_i^2,w_i^3,w_i^4;\omega_{w_i}\right), 0<\omega_{w_i}\leq 1$;

$\widetilde{\delta_i}=\left(\delta_i^1,\delta_i^2,\delta_i^3,\delta_i^4;\omega_{\delta_i}\right), 0<\omega_{\delta_i}\leq 1$; $\widetilde{c_i}=\left(c_i^1,c_i^2,c_i^3,c_i^4;\omega_{c_i}\right), 0<\omega_{c_i}\leq 1$; $(i=1,2,\ldots\ldots\ldots,n)$.

Then the above crisp inventory model (5) becomes the fuzzy model as

$$\text{Min } \widetilde{TAC_i}(T_i,D_i)=\frac{D_i^{(1-\widetilde{\beta_i})}\widetilde{\alpha_i}}{\widetilde{\theta_i}T_i}\left(e^{\widetilde{\theta_i}T_i}-1\right)+\frac{D_i\widetilde{h_i}}{\widetilde{\theta_i}T_i}\left\{-\frac{T_i}{\widetilde{\theta_i}}+\frac{1}{\widetilde{\theta_i}^2}\left(e^{\widetilde{\theta_i}T_i}-1\right)-\frac{T_i^2}{2}\right\}+\widetilde{\theta_i}\widetilde{\gamma_i}\left(\frac{D_i}{2\widetilde{\theta_i}}\left(e^{\widetilde{\theta_i}T_i}-1\right)\right)^{\widetilde{\delta_i}}+\frac{\widetilde{c_i}}{T_i}$$

Subject to, $\sum_{i=1}^{n}\widetilde{w_i}\frac{D_i}{\widetilde{\theta_i}}\left(e^{\widetilde{\theta_i}T_i}-1\right)\leq W$, for $i=1,2,\ldots\ldots\ldots..n.$ (6)

In defuzzification of fuzzy number technique, if we consider a GTrFN $\tilde{A}=(a,b,c,d;\omega)$, then the total λ- integer value of $\tilde{A}=(a,b,c,d;\omega)$ is

$$I_\lambda^w(\tilde{A})=\lambda\omega\frac{c+d}{2}+(1-\lambda)\omega\frac{a+b}{2}$$

Taking $=0.5$, therefore we get approximated value of a GTrFN $\tilde{A}=(a,b,c,d;\omega)$ is $\omega\left(\frac{a+b+c+d}{4}\right)$.

So using approximated value of GTrFN, we have the approximated values $(\widehat{\alpha_i},\widehat{\beta_i},\widehat{\theta_i},\widehat{h_i},\widehat{\gamma_i},\widehat{\delta_i},\widehat{c_i},\widehat{w_i})$ of the GTrFN parameters $(\widetilde{\alpha_i},\widetilde{\beta_i},\widetilde{\theta_i},\widetilde{h_i},\widetilde{\gamma_i},\widetilde{\delta_i},\widetilde{c_i},\widetilde{w_i})$. So the above model (6) reduces to

$$\text{Min } \widehat{TAC_i}(T_i,D_i)=\frac{D_i^{(1-\widehat{\beta_i})}\widehat{\alpha_i}}{\widehat{\theta_i}T_i}\left(e^{\widehat{\theta_i}T_i}-1\right)+\frac{D_i\widehat{h_i}}{\widehat{\theta_i}T_i}\left\{-\frac{T_i}{\widehat{\theta_i}}+\frac{1}{\widehat{\theta_i}^2}\left(e^{\widehat{\theta_i}T_i}-1\right)-\frac{T_i^2}{2}\right\}+\widehat{\theta_i}\widehat{\gamma_i}\left(\frac{D_i}{2\widehat{\theta_i}}\left(e^{\widehat{\theta_i}T_i}-1\right)\right)^{\widehat{\delta_i}}+\frac{\widehat{c_i}}{T_i}$$

Subject to, $\sum_{i=1}^{n}\widehat{w_i}\frac{D_i}{\widehat{\theta_i}}\left(e^{\widehat{\theta_i}T_i}-1\right)\leq W$, for $i=1,2,\ldots\ldots\ldots..n.$ (7)

5. Neutrosophic hesitant fuzzy programming technique to solve multi item inventory model (MIIM). (That is NHFNP method)

Solve the MIIM (7) as a single objective NLP using only one objective at a time and we ignoring the others. So we get the ideal solutions.

From the above results, we find out the corresponding values of every objective function at each solution obtained. With these values the pay-off matrix can be prepared as follows:

$$\begin{array}{c} \\ (T_1{}^1, D_1^1) \\ (T_2{}^2, D_2^2) \\ \\ \\ (T_n{}^n, D_n^n) \end{array} \begin{array}{c} \begin{array}{cccc} TAC_1(T_1, D_1) & TAC_2(T_2, D_2) & \ldots\ldots\ldots\ldots\ldots & TAC_n(T_n, D_n) \end{array} \\ \begin{pmatrix} TAC_1{}^*(T_1{}^1, D_1^1) & TAC_2(T_1{}^1, D_1^1) & \ldots\ldots\ldots & TAC_n(T_1{}^1, D_1^1) \\ TAC_1(T_2{}^2, D_2^2) & TAC_2{}^*(T_2{}^2, D_2^2) & \ldots\ldots\ldots & TAC_n(T_2{}^2, D_2^2) \\ \ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots \\ TAC_1(T_n{}^n, D_n^n) & TAC_2(T_n{}^n, D_n^n) & \ldots\ldots\ldots & TAC_n{}^*(T_n{}^n, D_n^n) \end{pmatrix} \end{array}$$

Let $U^k = \max\{TAC_k(T_i{}^i, D_i^i), i = 1,2, \ldots, n\}$ for $k = 1,2, \ldots, n$

and $L^k = TAC_k{}^*(T_k{}^k, D_k^k)$, $k = 1,2, \ldots, n$.

There $L^k \le TAP_k(T_i{}^i, D_i^i) \le U^k$, for $i = 1,2, \ldots, n$; $k = 1,2, \ldots, n$. (8)

Now we define the different hesitant membership function more elaborately under neutrosophic hesitant fuzzy environment as follows

The truth hesitant- membership function:

$$T_{\bar{h}}^{E_1}(TAC_k(T_k, D_k)) = \begin{cases} 1 & if\ TAC_k(T_k, D_k) < L^k \\ \sigma_1 \dfrac{(U^k)^t - (TAC_k(T_k, D_k))^t}{(U^k)^t - (L^k)^t} & if\ L^k \le TAC_k(T_k, D_k) \le U^k \\ 0 & if\ U^k < TAC_k(T_k, D_k) \end{cases}$$

$$T_{\bar{h}}^{E_2}(TAC_k(T_k, D_k)) = \begin{cases} 1 & if\ TAC_k(T_k, D_k) < L^k \\ \sigma_2 \dfrac{(U^k)^t - (TAC_k(T_k, D_k))^t}{(U^k)^t - (L^k)^t} & if\ L^k \le TAC_k(T_k, D_k) \le U^k \\ 0 & if\ U^k < TAC_k(T_k, D_k) \end{cases}$$

$$T_{\bar{h}}^{E_n}(TAC_k(T_k, D_k)) = \begin{cases} 1 & if\ TAC_k(T_k, D_k) < L^k \\ \sigma_n \dfrac{(U^k)^t - (TAC_k(T_k, D_k))^t}{(U^k)^t - (L^k)^t} & if\ L^k \le TAC_k(T_k, D_k) \le U^k \\ 0 & if\ U^k < TAC_k(T_k, D_k) \end{cases}$$

The indeterminacy hesitant- membership function:

$$I_{\bar{h}}^{E_1}(TAC_k(T_k, D_k)) = \begin{cases} 1 & if\ TAC_k(T_k, D_k) < L^k - s^k \\ \rho_1 \dfrac{(U^k)^t - (TAC_k(T_k, D_k))^t}{(s^k)^t} & if\ U^k - s^k \le TAC_k(T_k, D_k) \le U^k \\ 0 & if\ U^k < TAC_k(T_k, D_k) \end{cases}$$

$$I_{\bar{h}}^{E_2}(TAC_k(T_k, D_k)) = \begin{cases} 1 & if\ TAC_k(T_k, D_k) < L^k - s^k \\ \rho_2 \dfrac{(U^k)^t - (TAC_k(T_k, D_k))^t}{(s^k)^t} & if\ U^k - s^k \le TAC_k(T_k, D_k) \le U^k \\ 0 & if\ U^k < TAC_k(T_k, D_k) \end{cases}$$

$$I_{h^-}^{E_n}\left(TAC_k(T_k,D_k)\right)=\begin{cases}1 & if\ TAC_k(T_k,D_k)<L^k-s^k\\ \rho_n\dfrac{(U^k)^t-\left(TAC_k(T_k,D_k)\right)^t}{(s^k)^t} & if\ U^k-s^k\leq TAC_k(T_k,D_k)\leq U^k\\ 0 & if\ U^k<TAC_k(T_k,D_k)\end{cases}$$

The falsity hesitant- membership function:

$$F_{h^-}^{E_1}\left(TAC_k(T_k,D_k)\right)=\begin{cases}0 & if\ TAC_k(T_k,D_k)<L^k+v^k\\ \tau_1\dfrac{\left(TAC_k(T_k,D_k)\right)^t-(L^k)^t-(v^k)^t}{(U^k)^t-(L^k)^t-(v^k)^t} & if\ L^k+v^k\leq TAC_k(T_k,D_k)\leq U^k\\ 1 & if\ U^k<TAC_k(T_k,D_k)\end{cases}$$

$$F_{h^-}^{E_2}\left(TAC_k(T_k,D_k)\right)=\begin{cases}0 & if\ TAC_k(T_k,D_k)<L^k+v^k\\ \tau_2\dfrac{\left(TAC_k(T_k,D_k)\right)^t-(L^k)^t-(v^k)^t}{(U^k)^t-(L^k)^t-(v^k)^t} & if\ L^k+v^k\leq TAC_k(T_k,D_k)\leq U^k\\ 1 & if\ U^k<TAC_k(T_k,D_k)\end{cases}$$

$$F_{h^-}^{E_n}\left(TAC_k(T_k,D_k)\right)=\begin{cases}0 & if\ TAC_k(T_k,D_k)<L^k+v^k\\ \tau_1\dfrac{\left(TAC_k(T_k,D_k)\right)^t-(L^k)^t-(v^k)^t}{(U^k)^t-(L^k)^t-(v^k)^t} & if\ L^k+v^k\leq TAC_k(T_k,D_k)\leq U^k\\ 1 & if\ U^k<TAC_k(T_k,D_k)\end{cases}$$

Where parameter $t>0$ and $s^k, v^k \epsilon (0,1)\ \forall\ k=1,2,3,\dots\dots,n$ are indeterminacy and falsity tolerance values, which are assigned by decision making and h^- represent the minimization type hesitant objective function.

$T_{h^-}^{E_1}\left(TAC_k(T_k,D_k)\right)$, $I_{h^-}^{E_1}\left(TAC_k(T_k,D_k)\right)$, $F_{h^-}^{E_1}\left(TAC_k(T_k,D_k)\right)$ are truth, indeterminacy and falsity hesitant membership degrees assigned by 1st expert.

$T_{h^-}^{E_2}\left(TAC_k(T_k,D_k)\right)$, $I_{h^-}^{E_2}\left(TAC_k(T_k,D_k)\right)$, $F_{h^-}^{E_2}\left(TAC_k(T_k,D_k)\right)$ are truth, indeterminacy and falsity hesitant membership degrees assigned by 2nd expert.

$T_{h^-}^{E_n}\left(TAC_k(T_k,D_k)\right)$, $I_{h^-}^{E_n}\left(TAC_k(T_k,D_k)\right)$, $F_{h^-}^{E_n}\left(TAC_k(T_k,D_k)\right)$ are truth, indeterminacy and falsity hesitant membership degrees assigned by nth expert.

Using the above membership function, the multi-item nonlinear inventory problem formulated as follows

Max $\frac{\sum_1^n \sigma_i}{n}$

Max $\frac{\sum_1^n \rho_i}{n}$

Min $\frac{\sum_1^n \tau_i}{n}$

Subject to

$$T_{h^-}^{E_i}\big(TAC_k(T_k, D_k)\big) \geq \sigma_i, I_{h^-}^{E_i}\big(TAC_k(T_k, D_k)\big) \geq \rho_i, F_{h^-}^{E_i}\big(TAC_k(T_k, D_k)\big) \leq \tau_i$$

$$\sum_{i=1}^n w_i \frac{D_i}{\theta_i}\left(e^{\theta_i T_i} - 1\right) \leq W, \sigma_i + \rho_i + \tau_i \leq 3, \sigma_i \geq \rho_i, \sigma_i \geq \tau_i, \forall\, i = 1,2,3, \dots, n \qquad (9)$$

Using above linear membership function, we can written as

Max $\frac{\sigma_1+\sigma_2+\,\dots\dots+\sigma_n}{n} + \frac{\rho_1+\rho_2+\,\dots\dots\dots+\rho_n}{n} - \frac{\tau_1+\tau_2+\,\dots\dots\dots+\tau_n}{n}$

Subject to

$T_{h^-}^{E_i}\big(TAC_k(T_k, D_k)\big) \geq \sigma_i, I_{h^-}^{E_i}\big(TAC_k(T_k, D_k)\big) \geq \rho_i, F_{h^-}^{E_i}\big(TAC_k(T_k, D_k)\big) \leq \tau_i$

$\sum_{i=1}^n w_i \frac{D_i}{\theta_i}\left(e^{\theta_i T_i} - 1\right) \leq W, \sigma_i + \rho_i + \tau_i \leq 3, \sigma_i \geq \rho_i, \sigma_i \geq \tau_i, \forall\, i = 1,2,3, \dots, n$,

$0 \leq \sigma_1, \sigma_2, \dots \dots, \sigma_n \leq 1, 0 \leq \rho_1, \rho_2, \dots \dots \dots, \rho_n \leq 1, 0 \leq \tau_1, \tau_2, \dots \dots \dots, \tau_n \leq 1, T_k \geq 0, D_k \geq 0 \qquad (10)$

Above gives the solution D_i^* , T_i^* and then TAC_i^* for $i = 1,2,3, \dots, n$.

6. Algorithm to solve MIIM in Neutrosophic hesitant fuzzy programming technique

Following steps have been used to solve MIIM in neutrosophic hesitant fuzzy programming Technique.

Step-1: Solve only one objective at time and ignoring the others and using the all restrictions. These solutions are known as ideal solution.

Step-2: Form pay-off matrix using the step-1.

Step-3: Determine U^k and L^k. (U^k and L^k are the upper and lower bounds of the k-th item respectively)

Step-4: Using U^k and L^k define all hesitant membership function, i.e truth hesitant membership function $T_{h^-}^{E_i}(TAC_k)$, Indeterminacy hesitant membership function $I_{h^-}^{E_i}(TAC_k)$, Falsity hesitant membership function $F_{h^-}^{E_i}(TAC_k)$, $i = 1,2,3, \dots, n$, $k = 1,2,3, \dots, n$

Step-5: Ask for the truth hesitant, Indeterminacy hesitant and falsity hesitant membership degrees from different experts E_i, $i = 1,2,3, \dots, n$.

Step-6: Formulate multi-objective non-linear programming problem under neutrosophic hesitant fuzzy system.

Step-7: Solve multi-objective non-linear programming problem using suitable technique or optimization software package.

7. Fuzzy programming technique (Multi-Objective on max-min operators) to solve MIIM. (That is FNLP method)

Firstly derive (8) and then we use following way for solving the problem (7)

Now objective functions of the problem (7) are considered as fuzzy constraints. Therefore fuzzy linear membership function $\mu_{TAC_k}\big(TAC_k(T_k, D_k)\big)$ for the k[th] objective function $TAC_k(T_k, D_k)$ is defined as follows:

$$\mu_{TAC_k}\big(TAC_k(T_k, D_k)\big) = \begin{cases} 1 & for\ TAC_k(T_k, D_k) < L^k \\ \dfrac{U^k - TAC_k\big(T_{k,}D_k\big)}{U^k - L^k} & for\ L^k \le TAC_k(T_k, D_k) \le U^k \\ 0 & for\ TAC_k(T_k, D_k) > U^k \end{cases}$$

for $k = 1,2, \ldots, n.$

Using the above membership function, fuzzy non-linear programming problem is formulated as follows:

$Max\ \alpha$

Subject to

$\alpha\ (U^k - L^k) + TAC_k(T_k, D_k) \le U^k$

$0 \le \alpha \le 1,\ T_k \ge 0, D_k \ge 0$ for $k = 1,2, \ldots, n$ (11)

And same constraints and restrictions of the problem (7).

This problem (11) can be solved easily and we shall get the optimal solution of (7).

8. Numerical Example

Let us consider an inventory model which consist two items with following parameter values in proper units. Total storage area $W = 900m^2$.

Table-1

Input imprecise data for shape parameters

Parameters	Item I	Item II
$\widetilde{\alpha_i}$	(10000,12000,11000,13000; 0.7)	(20000,22000,25000,30000; 0.8)
$\widetilde{\theta_i}$	(0.02,0.05,0.09,0.07; 0.9)	(0.02,0.05,0.09,0.07; 0.9)
$\widetilde{\beta_i}$	(30,40,60,70; 0.8)	(50,70,80,90; 0.9)
$\widetilde{h_i}$	(0.6,0.8,0.9,0.5; 0.9)	(0.4,0.6,0.9,0.7; 0.7)
$\widetilde{\gamma_i}$	(8000,10000,12000,15000; 0.9)	(10000,13000,15000,18000; 0.9)
$\widetilde{\delta_i}$	(0.04,0.06,0.08,0.07; 0.8)	((0.04,0.06,0.08,0.07; 0.8))
$\widetilde{c_i}$	(100,150,200,210; 0.7)	(150,180,190,200; 0.9)
$\widetilde{w_i}$	(10,11,12,13; 0.9)	(20,21,24,23; 0.8)

The problem (7) reduces to the following:

$$\text{Min } TAC_1(T_1, D_1) = \frac{D_1^{-39}8050}{0.05.T_1}(e^{0.05.T_1} - 1) + \frac{D_1.0.63}{0.05.T_1}\left\{-\frac{T_1}{0.05} + \frac{1}{0.05^2}(e^{0.05.T_1} - 1) - \frac{{T_1}^2}{2}\right\} + 393.75\left(\frac{D_1}{0.1}(e^{0.05.T_1} - 1)\right)^{0.05} + \frac{115.50}{T_1}$$

$$\text{Min } TAC_2(T_2, D_2) = \frac{D_2^{-64.25}19400}{0.05.T_2}(e^{0.05.T_2} - 1) + \frac{D_2 0.46}{0.05.T_2}\left\{-\frac{T_2}{0.05} + \frac{1}{0.05^2}(e^{0.05.T_2} - 1) - \frac{{T_2}^2}{2}\right\} + 630\left(\frac{D_2}{0.1}(e^{0.05.T_2} - 1)\right)^{0.05} + \frac{162}{T_2}$$

Subject to, $210(e^{0.05.T_1} - 1)D_1 + 352(e^{0.05.T_2} - 1)D_2 \le 900$, T_1, D_1, T_2, D_2 are positive. (12)

Table –2

Optimal solutions of MIIM (12) using different methods

Methods	$D_1{}^*$	$T_1{}^*$	$TAC_1{}^*$	$D_2{}^*$	$T_2{}^*$	$TAC_2{}^*$
CRISP	1.28	4.12	446.51	1.18	4.07	703.52
FNLP	1.28	4.07	446.52	1.18	4.07	703.52
NHFNP	1.28	4.06	446.52	1.18	4.07	703.52

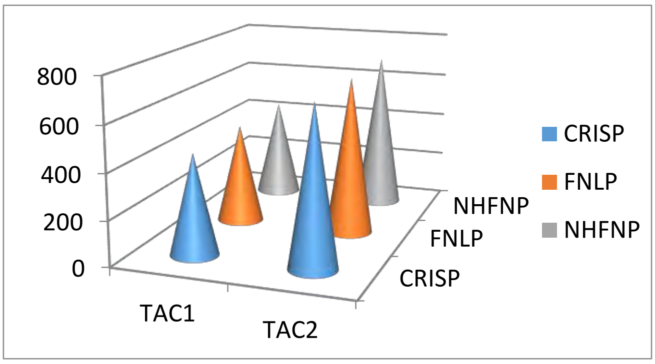

Figure 2. Minimizing cost of 1st and 2nd item using different methods

From the above Figure 2 shows that CRISP, FNLP and NHFNP method gives the almost same result of MIIM.

9. Sensitivity Analysis

The optimal solutions of the MIIM (7) by CRISP, FNLP and NHFNP techniques for different values of θ, h are given in Tables-3 and 4 respectively.

Table – 3

Optimal solutions of MIIM (7) by CRISP, FNLP and NHFNP for different values of θ

Methods	θ	$D_1{}^*$	$T_1{}^*$	$TAC_1{}^*$	$D_2{}^*$	$T_2{}^*$	$TAC_2{}^*$
CRISP	0.05	1.28	4.12	446.51	1.18	4.07	703.52
	0.10	1.26	2.43	858.92	1.17	2.23	1358.15
	0.15	1.25	1.69	1262.44	1.16	1.52	1997.44
	0.20	1.24	1.29	1659.65	1.16	1.16	2626.53
FNLP	0.05	1.28	4.07	446.52	1.18	4.07	703.52
	0.10	1.26	2.42	858.92	1.18	2.19	1358.34
	0.15	1.25	1.69	1262.44	1.16	1.52	1997.44
	0.20	1.24	1.31	1659.45	1.16	1.15	2626.50
NHFNP	0.05	1.28	4.06	446.52	1.18	4.07	703.52
	0.10	1.26	2.43	858.92	1.18	2.21	1358.34
	0.15	1.25	1.68	1262.42	1.16	1.53	1997.45
	0.20	1.24	1.29	1659.65	1.17	1.16	2626.52

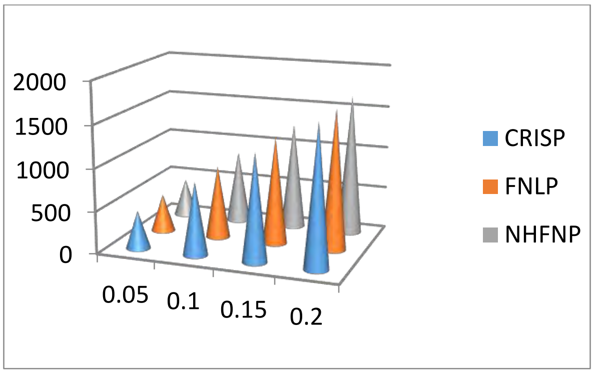

Figure 3. total average cost (TAC_1) of 1st item using different methods for different values of θ.

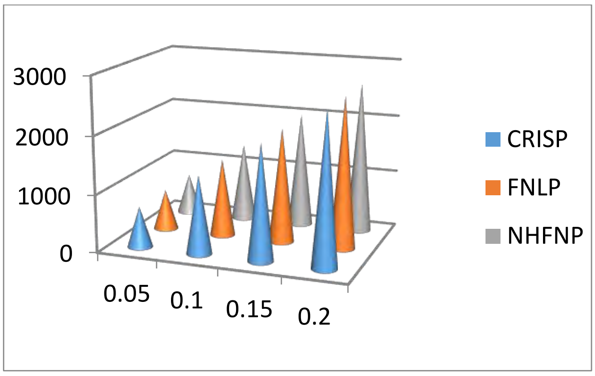

Figure 4. total average cost (TAC_2) of 2nd item using different methods for different values of θ.

From the above Figure 3 and Figure 4 shows that for all different methods when θ is increased then minimum cost of both items are increased

Table – 4

Optimal solution of MIIM (7) by CRISP, FNLP and NHFNP for different values of h_1 and h_2.

Methods	h_1	h_2	$D_1{}^*$	$T_1{}^*$	$TAC_1{}^*$	$D_2{}^*$	$T_2{}^*$	$TAC_2{}^*$
CRISP	0.63	0.46	1.28	4.12	446.51	1.18	4.07	703.52
	1.13	0.96	1.28	3.81	448.26	1.18	3.83	705.13
	1.63	1.46	1.28	3.59	449.79	1.18	3.66	706.57
	2.13	1.96	1.28	3.42	451.15	1.18	3.52	707.89
FNLP	0.63	0.46	1.28	4.07	446.52	1.18	4.07	703.52
	1.13	0.96	1.28	3.81	448.26	1.18	3.80	705.13
	1.63	1.46	1.28	3.65	449.80	1.18	3.67	706.57
	2.13	1.96	1.28	3.43	451.17	1.18	3.50	707.92
NHFNP	0.63	0.46	1.28	4.06	446.52	1.18	4.07	703.52
	1.13	0.96	1.27	3.80	448.30	1.18	3.78	705.33
	1.63	1.46	1.28	3.65	449.80	1.18	3.67	706.57
	2.13	1.96	1.28	3.42	451.19	1.18	3.49	707.91

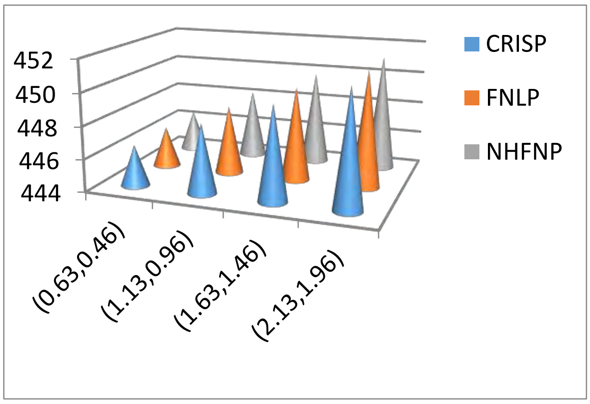

Figure 5. total average cost (TAC_1) of 1st item using different methods for different values of h_1 and h_2

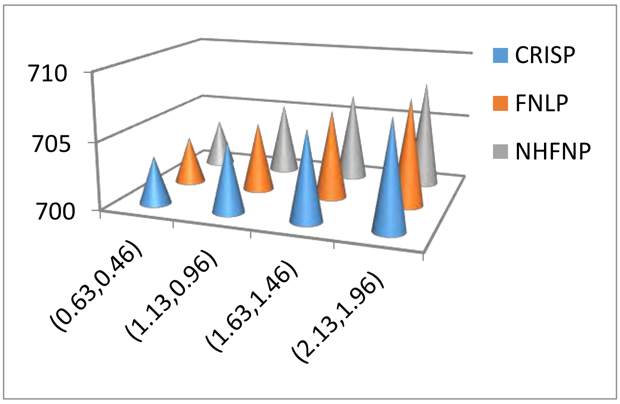

Figure-6. total average cost (TAC_2) of 2nd item using different methods for different values of h_1 and h_2

From the above Figure 5 and Figure 6 shows that all different methods, when h_1 and h_2 are continuously increasing then minimum cost of both items are continuously increasing.

10. Conclusions

Here we have considered the constant demand rate, under the restriction on storage area. Production cost is considered in demand dependent and the deterioration cost is considered in average inventory also holding cost is time dependent. The model have been formulated using multi-items. Due to uncertainty all the required parameters are taken as generalized trapezoidal fuzzy number. Multi-objective inventory model is solved by using neutrosophic hesitant fuzzy programing approach and fuzzy non-linear programming technique.

In the future study, it is hoped to further incorporate the proposed model into more realistic assumptions, such as probabilistic demand, ramp type demand, power demand, shortages, under two-level credit period strategy etc. Also inflation can be used to develope the model. Other type of fuzzy numbers like as triangular fuzzy number, Parabolic Flat Fuzzy Number (PfFN), Pentagonal Fuzzy Number etc. may be used for all cost parameters of the model to form the fuzzy model. Generalised single valued neutrosophic Number and its application can be used in this model.

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