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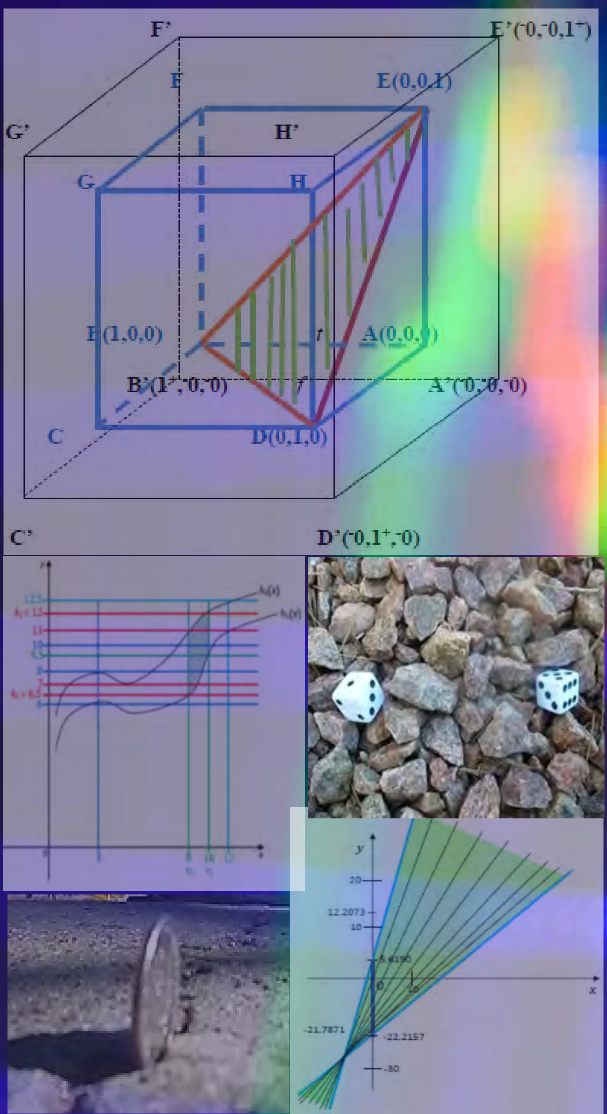

<A> <neutA> <antiA>

Florentin Smarandache . Mohamed Abdel-Basset . Said Broumi
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*"*Neutrosophic Sets and Systems*"* has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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Law of Included Infinitely-Many-Middles within the frame of Neutrosophy

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Abstract: In this paper, we extend, for the first time, the Law of Included Multiple-Middles to the Law of Infinitely-Many-Middles. And we present several practical applications. Also, we discuss Aristotle's Syllogism, Principle of Identity, and Principle of NonContradiction.

Keywords: Excluded Middle; Included Middle; Included Multiple-Middles; Included Infinitely-Many-Middles; Syllogism; Many-Valued Syllogism; Identity; NonIdentity; NonContradiction, Anti-NonContradiction.

A. Short History

We present below the evolution from the Law of Excluded Middle to the Law of Included Infinitely-Many-Multiples.

1. Law of Excluded Middle

The **Law of Excluded Middle** was enounced by the Ancient Greek philosopher Aristotle (384 - 322 BC) on his opus on logic and reasoning [1, 2] that was based on analysis and dichotomy:

> "There cannot be an indeterminate between contraries, but of one subject we must either affirm or deny anyone predicate".

Therefore, a proportion is either 100% true or 100% false, as in Boolean logic. Or, an element either belong 100% to a set, or does not belong 100% to the set (as in the classical set theory).

2. Law of Included Middle

The **Law of Included Middle** is the denial of previous, and it supports the idea that between contraries there may be a middle. It is based on trichotomy. Several philosophers and logicians developed it, such as Stephane Lupasco's logic of contradiction, using the non-standard logic, followed by Basarab Nicolescu's levels of reality, and J.-J. Wunenburger. Gonseth pleads for a low necessity in using the logic of contradiction. [5]

With the introduction of modern sets and logics, such as fuzzy set/logic (Zadeh, 1965), intuitionistic fuzzy set/logic (Atanassov, 1983), neutrosophic set/logic/probability (Smarandache, 1995), the Law of Included Middle became evident and useful in our everyday life where we deal with approximate partial membership/non-membership/truth/falsehood while in neutrosophic probability besides the chance of occurrence of an event, there has been added the middle term: indeterminate-chance of occurrence or not.

Neutrosophic set and logic explicitly presented the *middle term I (indeterminacy or neutrality)* in between the opposite terms (membership/truth), and *F* (non-membership/falsehood).

3. Law of Included Multiple-Middles

The **Law of Included Multiple-Middles** is an extension of the previous, and it was enounced by Smarandache [4] in 2014.

Neutrosophy [6] is a branch of philosophy that studies the dynamics of the opposites <A> and <antiA> together with their neutrality <neutA>, where <A> is an item (idea, proposition, theory, etc.), <antiA> is its opposite, while <neutA> is the neutrality in between them (i.e. neither <A> nor <antiA>).

Of course, we are referring to the neutrosophic triads <A>, <neutA>, <antiA> that make sense in our real world.

Neutrosophy, together with Neutrosophic Set/logic/probability, have been refined [7] in 2013, by refining splitting/multiplicating <A>, <neutA>, <antiA> as follows:

<A> as <A_1>, <A_2>, ..., <A_p>;

<neutA> as <neutA_1>, <neutA_2>, ..., <neutA_r>;

and <antiA> as <antiA_1>, <antiA_2>, ..., <antiA_s>;

where $p, r, s \geq 0$ are integers,

and at least one of p, r, s is $\geq$ in order to ensure that at least one neutrosophic component amongst <A>, <neutA>, <antiA> is refined/split/multiplicated.

This definition also permits the refinement of fuzzy set/logic (for $p \geq 2$ and $r = s = 0$), and of intuitionistic fuzzy set/logic (for $p \geq 1$, $r = 0$, $s \geq 1$ and at least one of p or s is ≥ 2).

By taking $p = 1$, $r \geq 2$, and $s = 1$, we defined the Law of Included Multiple-Middles:

Between the opposites <A> and <antiA> there are included multiple-middles: <neutA_1>, <neutA_2>, ..., <neutA_r>.

It is of course based on a multichotomical analysis.

i) Between the opposite colors White and Black there are many colors such as: yellow, rose, red, blue, etc.

ii) Pentagonal Neutrosophic logic, where each proposition is characterized by five degrees of truth, such as (T, C, V, U, F)

where the opposites are:

T=degree of truth and F=degree of falsehood,

and the three included-middles are:

C=degree of contradiction

V=degree of vagueness

U=degree of unknowingness

For example, the logical proposition:

P=Artificial intelligence will take over the world, evaluated by experts, this proposition may be 40% true (T), 20% contradictory (C), 30% vague (V), 60% unknown (U) and 50% false (F).

P (0.4, 0.2, 0.3, 0.6, 0.5).

4. Law of Included Infinitely-Many-Middles

In between the opposites <A> and <antiA> there are infinitely many middles, denoted by

<neutA_i>, i =1, 2, ..., ∞.

(i) Practical Example

Between the White and Black colors there are infinitely-many nuances of colors.

(ii) Between 100% True and 100% False, there are included infinitely many middles, which are truth-values of the form: d% True and (1-d)% False, thus a logical proposition may be, for example:

1% True and 99% False, 2% True and 98% False, etc.

(iii) Similarly, between 100% membership and 100% non-membership, there are included infinitely many middles of the form: d% membership and (1-d%) non-membership.

5. Syllogism

Aristotle studied it:

if "A→B" and "B→C" are totally true, then "A→C" is also totally true.

This is in classical logic.

6. Many-Valued Syllogism

In many-valued logics, where "A→B" and "B→C" are partially true, then "A→C" is partially true as well.

7. Principle of NonContradiction

It was enounced by Aristotle [1, 2], that <A> and <antiA> cannot be true at the same time:

> "the same attribute cannot at the same time belong and not belong to the same subject and in the same respect",

and

> "it is impossible for anyone to believe the same thing to be and not to be, as something Heraclitus says".

8. Principle of Anti NonContradiction

We name this principle as "Anti Contradiction" that occur in the many-valued logics in order to distinguish it from the Principle of NonContradiction (also called Contradiction).

The above principle, related to the Law of Excluded Middle, does not work any longer in the modern theories. Again, with the introduction of modern set theories, it is possible to have both, degree of belonging and degree of not-belonging simultaneously of an element to a set, for example John (0.6, 0.4), meaning that John belongs (works) only 60% for his company and 40% does not. And similarly with respect to the modern logics, where a logical proposition may be partially true and partially false.

In fuzzy and fuzzy extension theories (except neutrosophic theories), <A> and <antiA> may be partially (not totally) true at the same time.

In fuzzy logic, if a proposition P is 50% true, then its negation $\neg$P is also 100%-50% = 50% true.

In neutrosophic logic, if a proposition has the truth-value P is $(a, 0.5, a)$, where $0 \le a \le 1$, then its negation $\neg$P is also $(a, 1 - 0.5, a) = (a, 0.5, a)$.

In neutrosophic theories, <A> and <antiA> may be partially or totally true at the same time. For example, a paradox is proposition that is 100% true and 100% false at the same time, therefore <A> and <antiA> are totally true at the same time.

9. Principle of Identity

As enounced by Aristotle:

A=A (an item is equal to itself).

This is true if one considers the item <A> under the same parameters that characterize it, and having the same corresponding values:

$$A(P_1 = v_1, P_2 = v_2, \ldots, P_n = v_n) \quad = \quad A(P_1 = v_1, P_2 = v_2, \ldots, P_n = v_n).$$

10. Principle of NonIdentity

The Principle of Identity, by Aristotle, that A = A, works when the entity A is compared to itself with respect to the same parameters that characterize A, with each parameter measured at the same scale and on the same time.

But, if the parameters that characterize A are different, or their corresponding values are different, then one has non-equality.

For example, if A=Andrew, then Andrew(at age 5) ≠ Andrew(at age 70) physically, intellectually, and psychically.

As such, one may also define a Principle of NonIdentity, when A is different from A in at least one circumstance. And, in general, an item (person, animal, object, etc.) is not equal to itself at different times:

item (at time $t1$) $\neq$ item (at time $t2$).

B. Conclusion

We have presented the Law of Excluded Middle by Aristotle, then the Law of Included Middle, Law of Included Multiple-Middles, and we introduced for the first time the Law of Included Infinitely-Many-Middles.

Afterwards, several comments we made on Aristotle's Syllogism, Principle of NonContradiction, and Principle of Identity, that, in the many-valued logics, may have degrees of partial truth and partial falsehood even partial indeterminacy - depending on each application.

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Conflicts of Interest: The author declares no conflict of interest.

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Neutrosophic Overlap Function and Its Derived Neutrosophic Residual Implication

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Abstract: A new concept of neutrosophic overlap function is given, furthermore a neutrosophic residual implication derived from it is also introduced. Firstly, we give new concept of neutrosophic overlap function and some classical examples which are introduced on the lattice. Secondly, the concept of representable neutrosophic overlap function and its pertinent examples are given, meanwhile the general method of constructing representable neutrosophic overlap function by using intuitionistic overlap function is given. Finally, neutrosophic residual implication induced by neutrosophic overlap function and its basic properties are studied.

Keywords: neutrosophic overlap function; lattice; representable neutrosophic overlap function; neutrosophic residual implication

1. Introduction

In 1998, smarandache added an independent membership degree of uncertainty to intuitionistic fuzzy set (IFS) [1], thus putting forward the neutrosophic set (NS) initially. NS is such a robust formal frame extending those concepts of typical set, fuzzy set, IFS and interval-valued IFS from philosophical viewpoint. Because IFS and interval-valued IFS that can solely address incomplete information, but can't address uncertainty and the lack of consistent information which exists in reality. Hence, NS is introduced. Its uncertainty can be explicitly quantified, and its true affiliation, uncertain affiliation and false affiliation are expressed independent. However, its application is hard to solve the actual problems, some scholars have brought forward that notion of single-valued NS [2], as one specific case of NS. And those relevant contents of using single-valued NS to address decision-making issues are as follows [3-8].

Since the triangular norm has a broad range of applications in solving pragmatic issues, it is also important to study the wide range of forms of the triangular norm in applications. The overlap function is a generalization of triangular norm that fulfils continuity [9]. Bustince et al. gave accurate definition of overlap(grouping) function in [10,11]. Over the past period of time, overlap function and grouping function evolved rapidly in theory and practice. See the following literature [12-16] for the rich achievements in the field of theoretical research about overlap function and grouping function. In decision problems, image processing and other fields of wide application see the following literature [17-20]. In an effort to better handle inconclusive information, some scholars extend the overlap function [12, 21] into the IFS, while introducing the method at [22].

Fuzzy implication and fuzzy residual implication play an integral part in traditional fuzzy logic. Fuzzy implication [23] generalizes classical implication into fuzzy logic via the consideration of truth values varying in [0, 1] as opposed to {0, 1}. Fuzzy implication is one of the important components of fuzzy logic and acts as a very crucial part in some fields, such as image processing, fuzzy control, data mining sees the following literature [24-26], etc. Based on the wide application of fuzzy implication, it is necessary to research it from the theoretical viewpoint [19]. There are a few various models of fuzzy implication for example the R-implication induced by triangular norm [27], (S-N)-implication induced by triangular conorm and fuzzy negation [28], etc. Because overlap function is closely related to triangular norm, in view of the research of neutrosophic triangular norm on neutrosophic fuzzy residual implication, and referring to the research of neutrosophic triangular norm derived residual implication in Hu and Zhang [18], it is natural to consider the neutrosophic residual implication (NRI) induced by neutrosophic overlap function.

The second section mainly introduces the basic knowledge that needs to be used, such as overlap function, grouping function and NS etc. And in the third section, the new concept and related examples of neutrosophic overlap function are given. In addition, the notions and relevant examples of representable neutrosophic overlap function and non-representable neutrosophic overlap function are presented, respectively. Furthermore, the new concept of neutrosophic negation and De Morgan neutrosophic triple which can express the dual relationship between neutrosophic overlap function and neutrosophic grouping function is introduced. The general method of constructing representable neutrosophic overlap functions by intuitionistic overlap functions is given. The fourth section focuses on NRI induced by neutrosophic overlap function, and concludes that every NRI induced by neutrosophic overlap function must be a neutrosophic implication. The final section summarizes the research content.

2. Preliminaries

Definition 2.1 ([29]) O is referred to as an overlap function, if the binary map O: [0, 1] × [0, 1]→[0, 1] fulfils prerequisites below, $\forall s, t, v\in [0, 1]$:

(a) O fulfils exchangeability;

(b) $O(s, t) = 0$ when and only when $st= 0$;

(c) $O(s, t) = 1$ when and only when $st= 1$;

(d) $O(s, t) \leq_1 O(s, v)$ if $t \leq_1 v$;

(e) O fulfils continuity.

Example 2.1 The bivariate functions below are overlap functions, $\forall s, t\in [0, 1]$:

(a) $O_{\mathrm{mM}}(s,t) = \min(s,t)\max(s^2,t^2)$;

(b) $O_p(s,t) = s^p t^p$, for $p > 0$ and $p \neq 1$;

(c) $O_{\mathrm{DB}}(s,t) = \begin{cases} \dfrac{2st}{s+t}, & \text{if } s+t \neq 0, \\ 0 \quad , & \text{if } s+t = 0. \end{cases}$;

(d) $O_{\min}(s,t) = \min\{\sqrt{s}, \sqrt{t}\}$.

Definition 2.2 ([30]) The bivariate function G: [0, 1] × [0, 1]→[0, 1] is referred to as the grouping function, when it fulfils prerequisites below, $\forall s, t, v\in [0, 1]$:

(a) G fulfils exchangeability;

(b) $G(s, t) = 0$ when and only when $s= 0$ and $t= 0$;

(c) $G(s, t) = 1$ when and only when $s= 1$ or $t= 1$;

(d) $G(s, t) \leq_1 G(s, v)$ if $t \leq_1 v$;

(e) G fulfils continuity.

Example 2.2 The bivariate functions below are grouping functions, $\forall s, t\in [0, 1]$:

(a) $G_{\mathrm{mM}}(s,t) = 1 - \min(1-s, 1-t)\max((1-s)^2, (1-t)^2)$;

(b) $G_p(s,t) = 1 - (1-s)^p(1-t)^p$, for $p > 0$ and $p \neq 1$;

(c) $G_{\mathrm{DB}}(s,t) = \begin{cases} \dfrac{s+t-2st}{2-s-t}, & \text{if } s+t \neq 2, \\ 1, & \text{if } s+t = 2. \end{cases}$;

(d) $G_{\min}(s,t) = 1 - \min\{\sqrt{1-s}, \sqrt{1-t}\}$.

Definition 2.3 ([31]) An affiliation function $\mu_E(s)$ and a non-affiliation function $\nu_E(s)$ portray an IFS E in S. S is a set that is not empty. And the IFS E be denoted as

$$E = \{(s, \mu_E(s), \nu_E(s)) \mid s \in S\}.$$

In which $\mu_E(s)$, $\nu_E(s)\in [0, 1]$ and satisfies the term of $0\leq \mu_E(s)+ \nu_E(s)\leq 1$.

Definition 2.4 ([2]) Truth-affiliation function $T_E(s)$, uncertainty-affiliation function $U_E(s)$ and falsity -affiliation function $F_E(s)$ portray the single-valued NS E in S. S is a set that is not empty. And the single-valued NS E is defined as

$$E = \{\langle s, T_E(s), U_E(s), F_E(s)\rangle \mid s \in S\}.$$

In which $T_E(s)$, $U_E(s)$, $F_E(s)\in [0, 1]$ and satisfies the term of $0 \leq T_E(s)+U_E(s)+F_E(s) \leq 3$.

Definition 2.5 ([32]) The overlap function is a map O: $L^2\to L$ on $(L; \leq_L)$ which fulfils monotonicity, commutative and continuity, while it fulfils $O(s, 0_L)= 0_L$, $O(0_L, t)= 0_L$ and $O(1_L, 1_L)= 1_L$, $\forall s, t\in L$; the grouping function is a map G: $L^2\to L$ on $(L; \leq_L)$ which fulfils monotonicity, commutative and continuity, while it fulfils $G(s, 1_L)= 1_L$, $G(1_L, t)= 1_L$ and $G(0_L, 0_L)= 0_L$, $\forall s, t\in L$.

Definition 2.6 ([18]) Define the set D^* in the following way,

$$D^* = \{s = (s_1, s_2, s_3) \mid s_1, s_2, s_3 \in [0,1]\}.$$

If $s\in D^*$, as above, then s has three components s_1, s_2 and s_3.

$\forall s, t\in D^*$, where $s= (s_1, s_2, s_3)$, t is analogous to s. $\leq_1$ on D^* is defined as the order relation below,

$$s \leq_1 t \text{ iff } s_1 \leq t_1,\ s_2 \geq t_2,\ s_3 \geq t_3.$$

And the definition of the first type inclusion relation $\subseteq_1$ is analogous to the definition of $\leq_1$.

Proposition 2.1 ([18]) $(D^*; \leq_1)$ is a complete lattice.

Definition 2.7 ([18]) The supplement of s is written as below, $\forall s\in D^*$,

$$s^c= (s_3, 1-s_2, s_1).$$

In particular, $1_{D^*}= (1, 0, 0)$ and $0_{D^*}= (0, 1, 1)$ represent the maximum and minimum in $(D^*; \leq_1)$, respectively.

Proposition 2.2 ([18]) $s\wedge_1 t$ is defined as maximum lower bound of s, t, and expressed as inf(s, t); $s\vee_1 t$ is defined as minimum upper bound of s, t, and expressed as sup(s, t), $\forall s, t\in D^*$.

3. Neutrosophic overlap function

This section proposes new concept of neutrosophic overlap function and provides relevant examples, giving the concept and examples of representable and non-representable neutrosophic overlap function. Finally, a new method for constructing representable neutrosophic overlap function through intuitionistic fuzzy overlap function (IFO) is proposed.

Definition 3.1 A neutrosophic overlap function is a map $\boldsymbol{O}$: $D^*\times D^*\rightarrow D^*$ which fulfils prerequisites below, $\forall s, t, v\in D^*$:

(NO1) $\boldsymbol{O}$ fulfils exchangeability;
(NO2) $\boldsymbol{O}(s, t) \leq_1 \boldsymbol{O}(s, v)$ if $t \leq_1 v$;
(NO3) $\boldsymbol{O}(0_{D^*}, t) = 0_{D^*}$ or $\boldsymbol{O}(s, 0_{D^*}) = 0_{D^*}$;
(NO4) $\boldsymbol{O}(1_{D^*}, 1_{D^*}) = 1_{D^*}$;
(NO5) $\boldsymbol{O}$ fulfils continuity.

Definition 3.2 A neutrosophic grouping function is a map $\boldsymbol{G}$: $D^*\times D^*\rightarrow D^*$ which fulfils prerequisites below, $\forall s, t, v\in D^*$:

(NG1) $\boldsymbol{G}$ fulfils exchangeability;
(NG2) $\boldsymbol{G}(s, t) \leq_1 \boldsymbol{G}(s, v)$ if $t \leq_1 v$;
(NG3) $\boldsymbol{G}(1_{D^*}, t) = 1_{D^*}$ or $\boldsymbol{G}(s, 1_{D^*}) = 1_{D^*}$;
(NG4) $\boldsymbol{G}(0_{D^*}, 0_{D^*}) = 0_{D^*}$;
(NG5) $\boldsymbol{G}$ fulfils continuity.

Example 3.1 The following binary functions are neutrosophic overlap functions, $\forall s, t\in D^*$,

(1) $\boldsymbol{O}_{\text{mM}}(s,t) = (O_{\text{mM}}(s_1,t_1), G_{\text{mM}}(s_2,t_2), G_{\text{mM}}(s_3,t_3))$

$= (\min(s_1,t_1)\max(s_1^2,t_1^2), 1-\min(1-s_2,1-t_2)\max((1-s_2)^2,(1-t_2)^2), 1-\min(1-s_3,1-t_3)\max((1-s_3)^2,(1-t_3)^2))$;

(2) $\boldsymbol{O}_p(s,t) = (O_p(s_1,t_1), G_p(s_2,t_2), G_p(s_3,t_3)) = (s_1^p t_1^p, 1-(1-s_2)^p(1-t_2)^p, 1-(1-s_3)^p(1-t_3)^p)$, for $p > 0$ and $p\neq 1$;

(3) $\boldsymbol{O}_{\text{DB}}(s,t) = (O_{\text{DB}}(s_1,t_1), G_{\text{DB}}(s_2,t_2), G_{\text{DB}}(s_3,t_3)) = (\frac{2s_1t_1}{s_1+t_1}, \frac{s_2+t_2-2s_2t_2}{2-s_2-t_2}, \frac{s_3+t_3-2s_3t_3}{2-s_3-t_3})$, for $s_1 \neq t_1 \neq 0$,

$s_2 \neq t_2 \neq 1$ and $s_3 \neq t_3 \neq 1$;

(4) $\boldsymbol{O}_{\min}(s,t) = (O_{\min}(s_1,t_1), G_{\min}(s_2,t_2), G_{\min}(s_3,t_3))$

$= (\min\{\sqrt{s_1}, \sqrt{t_1}\}, 1-\min\{\sqrt{1-s_2}, \sqrt{1-t_2}\}, 1-\min\{\sqrt{1-s_3}, \sqrt{1-t_3}\})$.

Example 3.2 The following binary functions are neutrosophic grouping functions, $\forall s, t\in D^*$,

(1) $\boldsymbol{G}_{\text{mM}}(s,t) = (G_{\text{mM}}(s_1,t_1), O_{\text{mM}}(s_2,t_2), O_{\text{mM}}(s_3,t_3))$

$= (1-\min(1-s_1,1-t_1)\max((1-s_1)^2,(1-t_1)^2, \min(s_2,t_2)\max(s_2^2,t_2^2), \min(s_3,t_3)\max(s_3^2,t_3^2))$;

(2) $\boldsymbol{G}_p(s,t)=(G_p(s_1,t_1),O_p(s_2,t_2),O_p(s_3,t_3))=(1-(1-s_1)^p(1-t_1)^p, s_2{}^p t_2{}^p, s_3{}^p t_3{}^p)$, for $p>0$ and $p\neq 1$;

(3) $\boldsymbol{G}_{\mathrm{DB}}(s,t)=(G_{\mathrm{DB}}(s_1,t_1),O_{\mathrm{DB}}(s_2,t_2),O_{\mathrm{DB}}(s_3,t_3))=(\frac{s_1+t_1-2s_1t_1}{2-s_1-t_1},\frac{2s_2t_2}{s_2+t_2},\frac{2s_3t_3}{s_3+t_3})$, for $s_1\neq t_1\neq 1$, $s_2\neq t_2\neq 0$ and $s_3\neq t_3\neq 0$;

(4) $\boldsymbol{G}_{\min}(s,t)=(G_{\min}(s_1,t_1),O_{\min}(s_2,t_2),O_{\min}(s_3,t_3))$

$=(1-\min\{\sqrt{1-s_1},\sqrt{1-t_1}\},\min\{\sqrt{s_2},\sqrt{t_2}\},\min\{\sqrt{s_3},\sqrt{t_3}\})$.

Theorem 3.1 Let $\boldsymbol{O}$ is a bivariate operation on D^*, $\forall s, t\in D^*$,

$$\boldsymbol{O}(s,t)=(O(s_1,t_1),G_1(s_2,t_2),G_2(s_3,t_3)).$$

Then $\boldsymbol{O}$ is a neutrosophic overlap function, where O is the overlap function, G_1 and G_2 are grouping functions on [0, 1].

Proof. $\forall s, t, v\in D^*$ in which $s=(s_1, s_2, s_3)$, t and v are analogous to s.

(NO1) Since G_1 and G_2 are grouping functions, O is the overlap function, then $O(s_1, t_1) = O(t_1, s_1)$, $G_1(t_2, s_2) = G_1(s_2, t_2)$ and $G_2(t_3, s_3) = G_2(s_3, t_3)$, thus $\boldsymbol{O}$ fulfils exchangeability.

(NO2) Let $t \leq_1 v$, then $O(s_1, t_1) \leq O(s_1, v_1)$, $G_1(s_2, t_2) \geq G_1(s_2, v_2)$, $G_2(s_3, t_3) \geq G_2(s_3, v_3)$. Therefore, $\boldsymbol{O}(s, t) \leq_1 \boldsymbol{O}(s, v)$.

(NO3) $\boldsymbol{O}(0_{D^*}, t)= (O(0, t_1), G_1(1, t_2), G_2(1, t_3))=(0, 1, 1)= 0_{D^*}$, and $\boldsymbol{O}(s, 0_{D^*})= (O(s_1, 0), G_1(s_2, 1), G_2(s_3, 1))=(0, 1, 1)= 0_{D^*}$.

(NO4) $\boldsymbol{O}(1_{D^*}, 1_{D^*})= (O(1, 1), G_1(0, 0), G_2(0, 0))= (1, 0, 0)= 1_{D^*}$.

(NO5) Firstly, we prove left continuous. That is, prove that this equation $\boldsymbol{O}(s, \vee_{i\in I}t_i)= \vee_{i\in I}\boldsymbol{O}(s, t_i)$ holds. Because the overlap function and the grouping function are continuous, so $O(s_1, \vee_{i\in I}t_i)= \vee_{i\in I}O(s_1, t_i)$, $G_2(s_3, \vee_{i\in I}t_i)= \vee_{i\in I}G_2(s_3, t_i)$ and $G_1(s_2, \vee_{i\in I}t_i)= \vee_{i\in I}G_1(s_2, t_i)$ is valid.

So we can get

$$\begin{aligned}\boldsymbol{O}(s,\vee_{i\in I}t_i)&=(O(s_1,\vee_{i\in I}t_i),G_1(s_2,\vee_{i\in I}t_i),G_2(s_3,\vee_{i\in I}t_i))\\&=(\vee_{i\in I}O(s_1,t_i),\vee_{i\in I}G_1(s_2,t_i),\vee_{i\in I}G_2(s_3,t_i))\\&=\vee_{i\in I}\boldsymbol{O}(s,t_i)\end{aligned}$$

In this way, show that $\boldsymbol{O}$ is left continuous.

Likewise, $\boldsymbol{O}(s, \wedge_{i\in I}t_i)= \wedge_{i\in I}\boldsymbol{O}(s, t_i)$ is simple to prove, state clearly that $\boldsymbol{O}$ is right continuous. To sum up, $\boldsymbol{O}$ is shown to be a neutrosophic overlap function.

Theorem 3.2 $\boldsymbol{G}$ is a bivariate operation on D^*, $\forall s, t\in D^*$,

$$\boldsymbol{G}(s,t)=(G(s_1,t_1),O_1(s_2,t_2),O_2(s_3,t_3)).$$

Then $\boldsymbol{G}$ can be called the neutrosophic grouping function, where G is the grouping function, O_1 and O_2 are overlap functions on [0, 1].

Proof. The procedure for proving analogy **Theorem 3.1.**

Above **Theorem 3.1** supplies the measure for constructing neutrosophic overlap function using overlap function O and grouping functions G_1, G_2 which are defined on [0, 1]. But it requires a condition that $\boldsymbol{O}= (O, G_1, G_2)$ holds. According to this condition, we bring in the concept of representable neutrosophic overlap function.

Definition 3.3 A neutrosophic overlap function $\boldsymbol{O}$ is referred to as representable, when and only when, there exists O which is an overlap function on [0, 1] and G_1, G_2 which are grouping functions on [0, 1] satisfying, $\forall s, t \in D^*$,

$$\boldsymbol{O}(s,t) = (O(s_1,t_1), G_1(s_2,t_2), G_2(s_3,t_3)).$$

Example 3.3 The representable neutrosophic overlap function is shown below, $\forall s, t \in D^*$,

$$\boldsymbol{O}(s,t) = (O_{\mathrm{DB}}(s_1,t_1), G_p(s_2,t_2), G_{\mathrm{mM}}(s_3,t_3)).$$

Proof. The first step verifies that $\boldsymbol{O}$ is a neutrosophic overlap function holds. $\forall s, t, v \in D^*$ in which $s = (s_1, s_2, s_3)$, t and v are analogous to s.

(NO1) Let $G_1 = G_p$, $G_2 = G_{\mathrm{mM}}$ ($p=2$) are grouping functions, $O = O_{\mathrm{DB}}$ is an overlap function on [0, 1]. Since $O_{\mathrm{DB}}(s_1, t_1) = O_{\mathrm{DB}}(t_1, s_1)$, $G_p(s_2, t_2) = G_p(t_2, s_2)$, $G_{\mathrm{mM}}(s_3, t_3) = G_{\mathrm{mM}}(s_3, t_3)$, thus $\boldsymbol{O}$ fulfils exchangeability.

(NO2) Let $t \leq_1 v$, then $O_{\mathrm{DB}}(s_1, t_1) \leq O_{\mathrm{DB}}(s_1, v_1)$, $G_p(s_2, t_2) \geq G_p(s_2, v_2)$, $G_{\mathrm{mM}}(s_3, t_3) \geq G_{\mathrm{mM}}(s_3, v_3)$. Therefore, $\boldsymbol{O}(s, t) \leq_1 \boldsymbol{O}(s, v)$.

(NO3) $\boldsymbol{O}(0_{D^*}, t) = (O_{\mathrm{DB}}(0, t_1), G_p(1, t_2), G_{\mathrm{mM}}(1, t_3)) = (0, 1, 1) = 0_{D^*}$, and $\boldsymbol{O}(s, 0_{D^*}) = (O_{\mathrm{DB}}(s_1, 0), G_p(s_2, 1), G_{\mathrm{mM}}(s_3, 1)) = (0, 1, 1) = 0_{D^*}$.

(NO4) $\boldsymbol{O}(1_{D^*}, 1_{D^*}) = (O_{\mathrm{DB}}(1, 1), G_p(0, 0), G_{\mathrm{mM}}(0, 0)) = (1, 0, 0) = 1_{D^*}$.

(NO5) Firstly, we prove left continuous. That is, prove that this equation $\boldsymbol{O}(s, \vee_{i\in I} t_i) = \vee_{i\in I} \boldsymbol{O}(s, t_i)$ holds. Because of an overlap function O_{DB} and grouping functions G_p and G_{mM} are continuous, so $O_{\mathrm{DB}}(s_1, \vee_{i\in I} t_i) = \vee_{i\in I} O_{\mathrm{DB}}(s_1, t_i)$, $G_{\mathrm{mM}}(s_3, \vee_{i\in I} t_i) = \vee_{i\in I} G_{\mathrm{mM}}(s_3, t_i)$ and $G_p(s_2, \vee_{i\in I} t_i) = \vee_{i\in I} G_p(s_2, t_i)$ is valid.

So we can get

$$\begin{aligned} \boldsymbol{O}(s, \vee_{i\in I} t_i) &= (O_{DB}(s_1, \vee_{i\in I} t_i), G_p(s_2, \vee_{i\in I} t_i), G_{\mathrm{mM}}(s_3, \vee_{i\in I} t_i)) \\ &= (\vee_{i\in I} O_{DB}(s_1, t_i), \vee_{i\in I} G_p(s_2, t_i), \vee_{i\in I} G_{\mathrm{mM}}(s_3, t_i)) \\ &= \vee_{i\in I} \boldsymbol{O}(s, t_i) \end{aligned}$$

In this way, show that $\boldsymbol{O}$ is left continuous.

Likewise, $\boldsymbol{O}(s, \wedge_{i\in I} t_i) = \wedge_{i\in I} \boldsymbol{O}(s, t_i)$ is simple to prove, state clearly that $\boldsymbol{O}$ is right continuous. To sum up, $\boldsymbol{O}$ is shown to be the neutrosophic overlap function.

Finally, it is simple to show that fulfils $\boldsymbol{O}(s, t) = (O_{\mathrm{DB}}(s_1, t_1), G_p(s_2, t_2), G_{\mathrm{mM}}(s_3, t_3))$, so it must be the representable neutrosophic overlap function.

Definition 3.4 The neutrosophic overlap function $\boldsymbol{O}$ is known as standard representable, when and only when, there exists G which is a grouping function on [0, 1] and O which is an overlap function on [0, 1] satisfying, $\forall s, t \in D^*$,

$$\boldsymbol{O}(s,t) = (O(s_1,t_1), G(s_2,t_2), G(s_3,t_3)).$$

Example 3.4 The standard representable neutrosophic overlap function is as follows, $\forall s, t \in D^*$,

$$\boldsymbol{O}(s,t) = (O_{\mathrm{DB}}(s_1,t_1), G_p(s_2,t_2), G_p(s_3,t_3)).$$

Proof. This procedure for proving analogy **Example 3.3.**

Definition 3.5 The N-dual representable neutrosophic overlap function $\boldsymbol{O}$ by the following being defined by, $\forall s, t \in D^*$,

$$\boldsymbol{O}(s,t) = (O(s_1,t_1), G(s_2,t_2), G(s_3,t_3)).$$

O and G has dual relation as follows,

$$O(s_1,t_1) = 1 - G(1-s_1, 1-t_1).$$

Example 3.5 The N-dual representable neutrosophic overlap function is as follows, $\forall s, t \in D^*$,

$$\boldsymbol{O}(s,t) = (O_p(s_1,t_1), G_p(s_2,t_2), G_p(s_3,t_3)).$$

Proof. This procedure for proving analogy **Example 3.3**.

Definition 3.6 $\boldsymbol{G}$ is referred to as representable neutrosophic grouping function, when and only when, there obtains the grouping function G and the overlap functions O_1, O_2 on [0, 1] satisfying, $\forall s, t\in D^*$,

$$\boldsymbol{G}(s,t)=(G(s_1,t_1),O_1(s_2,t_2),O_2(s_3,t_3)).$$

Other concepts can be derived from the analogy of the neutrosophic overlap function.

In recent years, there have been many extensions of overlap functions. However, due to the limitations of existing definitions in addressing practical issues by using intuitionistic fuzzy information, scholars have proposed IFO. In the preceding paragraphs, the representable neutrosophic overlap function is proposed and further the below propositions propose a method to construct new representable neutrosophic overlap function (grouping function) with IFO (intuitionistic fuzzy grouping function).

Proposition 3.1 Where $m=(m_1, m_3)$, $n=(n_1, n_3)$, $\forall m, n\in L$. $\mathbf{O}$ is an IFO while satisfying $\mathbf{O}(m, n)=(O(m_1, n_1), G_2(m_3, n_3))$, with O being an overlap function on [0, 1], G_2 being a grouping function on [0, 1]. Suppose G_1 is a grouping function on [0, 1] satisfying

$$0\le O(s_1,t_1)+G_1(s_2,t_2)+G_2(s_3,t_3)\le 3.$$

Then $\boldsymbol{O}(s, t)=(O(s_1, t_1), G_1(s_2, t_2), G_2(s_3, t_3))$ is called the representable neutrosophic overlap function, $\forall s, t\in D^*$.

Proof. First, we can get $\mathbf{O}(m, n)=(O(m_1, n_1), G_2(m_3, n_3))$ which is an IFO, and then we add another grouping function G_1, satisfying $0\le O(s_1, t_1)+G_1(s_2, t_2)+G_2(s_3, t_3)\le 3$.

$\forall s, t, v\in D^*$ in which $s=(s_1, s_2, s_3)$, t and v are analogous to s.

(NO1) Since $O(s_1, t_1)=O(t_1, s_1)$, $G_2(s_3, t_3)=G_2(t_3, s_3)$, $G_1(s_2, t_2)=G_1(s_2, t_2)$, then $\boldsymbol{O}(s, t)=\boldsymbol{O}(t, s)$, thus it shown that $\boldsymbol{O}$ fulfils exchangeability.

(NO2) Let $t\le_1 v$, then $O(s_1, t_1)\le O(s_1, v_1)$, $G_2(s_3, t_3)\ge G_2(s_3, v_3)$, $G_1(s_2, t_2)\ge G_1(s_2, v_2)$. Therefore, $\boldsymbol{O}(s, t)\le_1 \boldsymbol{O}(s, v)$.

(NO3) $\boldsymbol{O}(0_{D^*}, t)=(O(0, t_1), G_1(1, t_2), G_2(1, t_3))=(0, 1, 1)=0_{D^*}$, and $\boldsymbol{O}(s, 0_{D^*})=(O(s_1, 0), G_1(s_2, 1), G_2(s_3, 1))=(0, 1, 1)=0_{D^*}$.

(NO4) $\boldsymbol{O}(1_{D^*}, 1_{D^*})=(O(1, 1), G_1(0, 0), G_2(0, 0))=(1, 0, 0)=1_{D^*}$.

(NO5) Firstly, we prove left continuous. That is, prove that this equation $\boldsymbol{O}(s, \vee_{i\in I}t_i)=\vee_{i\in I}\boldsymbol{O}(s, t_i)$ holds. Because the overlap function O and the grouping functions G_2, G_1 are continuous, $O(s_1, \vee_{i\in I}t_i)=\vee_{i\in I}O(s_1, t_i)$, $G_2(s_3, \vee_{i\in I}t_i)=\vee_{i\in I}G_2(s_3, t_i)$ and $G_1(s_2, \vee_{i\in I}t_i)=\vee_{i\in I}G_1(s_2, t_i)$ is holding.

So we can get

$$\begin{aligned}\boldsymbol{O}(s,\vee_{i\in I}t_i)&=(O(s_1,\vee_{i\in I}t_i),G_1(s_2,\vee_{i\in I}t_i),G_2(s_3,\vee_{i\in I}t_i))\\&=(\vee_{i\in I}O(s_1,t_i),\vee_{i\in I}G_1(s_2,t_i),\vee_{i\in I}G_2(s_3,t_i))\\&=\vee_{i\in I}\boldsymbol{O}(s,t_i)\end{aligned}$$

In this way, show that $\boldsymbol{O}$ is left continuous.

Likewise, $\boldsymbol{O}(s, \wedge_{i\in I}t_i)=\wedge_{i\in I}\boldsymbol{O}(s, t_i)$ is simple to prove, state clearly that $\boldsymbol{O}$ is right continuous. To sum up, $\boldsymbol{O}$ is shown to be the neutrosophic overlap function.

It is simple to show that satisfies $\boldsymbol{O}(s, t)=(O(s_1, t_1), G_1(s_2, t_2), G_2(s_3, t_3))$, so $\boldsymbol{O}$ is a representable neutrosophic overlap function.

Proposition 3.2 Where $m=(m_1, m_3)$, $n=(n_1, n_3)$, $\forall m, n\in L$. $\mathbf{G}$ is an intuitionistic fuzzy grouping function, while satisfying the fact that $\mathbf{G}(m, n)=(G(m_1, n_1), O_2(m_3, n_3))$, with G being a grouping function on [0, 1], O_2 being an overlap function on [0, 1]. Suppose O_1 is an overlap function on [0, 1] satisfying,

$$0\le G(s_1,t_1)+O_1(s_2,t_2)+O_2(s_3,t_3)\le 3.$$

Then $\boldsymbol{G}(s, t)=(G(s_1, t_1), O_1(s_2, t_2), O_2(s_3, t_3))$ is a representable neutrosophic grouping function, $\forall s, t\in D^*$.

Proof. This procedure for proving analogy **Proposition 3.1.**

The dual relation between triangular norm and triangular conorm in relation to fuzzy negation can be characterized by De Morgan triple, which is a proper expression for the relationship between triangular norm, triangular conorm and fuzzy negation [18]. There are also corresponding studies on NS. Based on the close connection between triangular norm and overlap function, one can naturally consider De Morgan neutrosophic triple about neutrosophic overlap function, neutrosophic grouping function and neutrosophic negation. First, neutrosophic negation as an extension of fuzzy negation can be denoted by the method below.

Definition 3.7 ([18]) A neutrosophic negaton is a map $\boldsymbol{N}: D^* \to D^*$ that fulfils prerequisites below:
(a) $\boldsymbol{N}(t) \geq_1 \boldsymbol{N}(s)$, $\forall s, t \in D^*$ such as $t \leq_1 s$;
(b) $\boldsymbol{N}(0_{D^*}) = 1_{D^*}$;
(c) $\boldsymbol{N}(1_{D^*}) = 0_{D^*}$.

$\boldsymbol{N}$ is referred to as the involutive neutrosophic negaton when and only when that fulfils $\boldsymbol{N}(\boldsymbol{N}(s)) = s$, $\forall s \in D^*$.

An involutive neutrosophic negaton $\boldsymbol{N}$g: $D^* \to D^*$ satisfies the following, where $s = (s_1, s_2, s_3)$, $\forall s \in D^*$,

$$\boldsymbol{N}\mathrm{g}(s_1, s_2, s_3) = (s_3, 1-s_2, s_1).$$

Further, we define such $\boldsymbol{N}$g as the standard neutrosophic negaton.

Definition 3.8 $\boldsymbol{O}$, $\boldsymbol{G}$ and $\boldsymbol{N}$ are a neutrosophic overlap function, a neutrosophic grouping function, and a neutrosophic negation, respectively.

For this triple $(\boldsymbol{O}, \boldsymbol{N}, \boldsymbol{G})$ if the conditions below holding true, $\forall s, t \in D^*$,

$$\boldsymbol{N}(\boldsymbol{O}(s, t)) = \boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(t));$$
$$\boldsymbol{N}(\boldsymbol{G}(s, t)) = \boldsymbol{O}(\boldsymbol{N}(s), \boldsymbol{N}(t)).$$

Then such the triple is referred to as De Morgan neutrosophic triple. In addition, $\boldsymbol{O}$ and $\boldsymbol{G}$ have a dual relationship in relation to $\boldsymbol{N}$.

Theorem 3.3 Suppose neutrosophic negaton $\boldsymbol{N}$ is involutory, that it fulfils $\boldsymbol{N}(\boldsymbol{N}(s)) = s$, $\forall s \in D^*$.
(a) Assume $\boldsymbol{G}$ is the neutrosophic grouping function, $\boldsymbol{O}$ is expressed in the following form, $\forall s, t \in D^*$,

$$\boldsymbol{O}(s, t) = \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(t))).$$

Then $\boldsymbol{O}$ is the neutrosophic overlap function. Moreover, $(\boldsymbol{O}, \boldsymbol{N}, \boldsymbol{G})$ is De Morgan neutrosophic triple.
(b) Assume $\boldsymbol{O}$ is the neutrosophic overlap function, $\boldsymbol{G}$ is expressed in the following form, $\forall s, t \in D^*$,

$$\boldsymbol{G}(s, t) = \boldsymbol{N}(\boldsymbol{O}(\boldsymbol{N}(s), \boldsymbol{N}(t))).$$

Then $\boldsymbol{G}$ is the neutrosophic grouping function. Moreover, $(\boldsymbol{O}, \boldsymbol{N}, \boldsymbol{G})$ is De Morgan neutrosophic triple.

Proof. (a) Suppose $\boldsymbol{N}$, $\boldsymbol{G}$ are the involutory neutrosophic negaton and the neutrosophic grouping function, respectively. $\forall s, t, v \in D^*$ in which $s = (s_1, s_2, s_3)$, t and v are analogous to s.

(NO1) It is pretty simple to justify that $\boldsymbol{O}(s, t) = \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(t))) = \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(t), \boldsymbol{N}(s))) = \boldsymbol{O}(t, s)$, $\boldsymbol{O}$ fulfils exchangeability.

(NO2) Let $t \leq_1 v$, $\boldsymbol{O}(s, t) = \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(t)))$, $\boldsymbol{O}(s, v) = \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(v)))$, because $\boldsymbol{N}$ is non-increasing, then $\boldsymbol{N}(t) \geq_1 \boldsymbol{N}(v)$. Moreover $\boldsymbol{G}(s, t) \leq_1 \boldsymbol{G}(s, v)$ and when $t \leq_1 v$, then $\boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(t)) \geq_1 \boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(v))$. Hence $\boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(t))) \leq_1 \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(v)))$, then $\boldsymbol{O}(s, t) \leq_1 \boldsymbol{O}(s, v)$.

(NO3) $\boldsymbol{O}(0_{D^*}, t) = \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(0_{D^*}), \boldsymbol{N}(t))) = \boldsymbol{N}(\boldsymbol{G}(1_{D^*}, \boldsymbol{N}(t))) = \boldsymbol{N}(1_{D^*}) = 0_{D^*}$, similarly $\boldsymbol{O}(s, 0_{D^*}) = \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(0_{D^*}))) = 0_{D^*}$.

(NO4) $\boldsymbol{O}(1_{D^*}, 1_{D^*}) = \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(1_{D^*}), \boldsymbol{N}(1_{D^*}))) = \boldsymbol{N}(\boldsymbol{G}(0_{D^*}, 0_{D^*})) = \boldsymbol{N}(0_{D^*}) = 1_{D^*}$.

(NO5) Firstly, we prove left continuous. That is, prove that this equation $\boldsymbol{O}(s, \vee_{i\in I} t_i) = \vee_{i\in I}\boldsymbol{O}(s, t_i)$ holds. As a result of $\boldsymbol{O}(s, \vee_{i\in I} t_i) = \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(\vee_{i\in I} t_i))) = \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(s), \wedge_{i\in I}\boldsymbol{N}(t_i))) = \boldsymbol{N}(\wedge_{i\in I}\boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(t_i))) =$

$\vee_{i\in I}\boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(t_i))) = \vee_{i\in I}\boldsymbol{O}(s, t_i)$. Then we could get $\boldsymbol{O}(s, \vee_{i\in I}t_i)= \vee_{i\in I}\boldsymbol{O}(s, t_i)$. In this way, show that $\boldsymbol{O}$ is left continuous.

Likewise, $\boldsymbol{O}(s, \wedge_{i\in I}t_i)= \wedge_{i\in I}\boldsymbol{O}(s, t_i)$ is simple to prove, state clearly that $\boldsymbol{O}$ is right continuous. To sum up, $\boldsymbol{O}(s, t)$ is shown to be the neutrosophic overlap function.

Moreover, $(\boldsymbol{O}, \boldsymbol{N}, \boldsymbol{G})$ is the De Morgan neutrosophic triple.

(b) Likewise, suppose $\boldsymbol{O}$ is the neutrosophic overlap function and that $\boldsymbol{G}$ can be shown to be the neutrosophic grouping function, $(\boldsymbol{O}, \boldsymbol{N}, \boldsymbol{G})$ would be the De Morgan neutrosophic triple.

Example 3.6 The following functions are the neutrosophic overlap(grouping) functions, which are dual in relation to $\boldsymbol{N}$g, $\forall s, t\in D^*$,

(1) $\boldsymbol{O}_{\mathrm{mM}}(s,t)=(O_{\mathrm{mM}}(s_1,t_1),G_{\mathrm{mM}}(s_2,t_2),G_{\mathrm{mM}}(s_3,t_3))$ and $\boldsymbol{G}_{\mathrm{mM}}(s,t)=(G_{\mathrm{mM}}(s_1,t_1),O_{\mathrm{mM}}(s_2,t_2),O_{\mathrm{mM}}(s_3,t_3))$;

In fact, $\boldsymbol{O}_{\mathrm{mM}}(\boldsymbol{N}(s), \boldsymbol{N}(t))= \boldsymbol{O}_{\mathrm{mM}}((s_3, 1-s_2, s_1), (t_3, 1-t_2, t_1))= (O_{\mathrm{mM}}(s_3, t_3), G_{\mathrm{mM}}(1-s_2, 1-t_2), G_{\mathrm{mM}}(s_1, t_1))$, then $\boldsymbol{N}(\boldsymbol{O}_{\mathrm{mM}}(\boldsymbol{N}(s), \boldsymbol{N}(t)))= \boldsymbol{N}(O_{\mathrm{mM}}(s_3, t_3), G_{\mathrm{mM}}(1-s_2, 1-t_2), G_{\mathrm{mM}}(s_1, t_1))= (G_{\mathrm{mM}}(s_1, t_1), 1-G_{\mathrm{mM}}(1-s_2, 1-t_2), O_{\mathrm{mM}}(s_3, t_3))= (G_{\mathrm{mM}}(s_1, t_1), O_{\mathrm{mM}}(s_2, t_2), O_{\mathrm{mM}}(s_3, t_3))= \boldsymbol{G}_{\mathrm{mM}}(s, t)$. Thus, $\boldsymbol{O}_{\mathrm{mM}}$ and $\boldsymbol{G}_{\mathrm{mM}}$ are dual with respect to $\boldsymbol{N}$g.

(2) $\boldsymbol{O}_p(s,t)=(O_p(s_1,t_1),G_p(s_2,t_2),G_p(s_3,t_3))$ and $\boldsymbol{G}_p(s,t)=(G_p(s_1,t_1),O_p(s_2,t_2),O_p(s_3,t_3))$;

(3) $\boldsymbol{O}_{\mathrm{DB}}(s,t)=(O_{\mathrm{DB}}(s_1,t_1),G_{\mathrm{DB}}(s_2,t_2),G_{\mathrm{DB}}(s_3,t_3))$ and $\boldsymbol{G}_{\mathrm{DB}}(s,t)=(G_{\mathrm{DB}}(s_1,t_1),O_{\mathrm{DB}}(s_2,t_2),O_{\mathrm{DB}}(s_3,t_3))$;

(4) $\boldsymbol{O}_{\min}(s,t)=(O_{\min}(s_1,t_1),G_{\min}(s_2,t_2),G_{\min}(s_3,t_3))$ and $\boldsymbol{G}_{\min}(s,t)=(G_{\min}(s_1,t_1),O_{\min}(s_2,t_2),O_{\min}(s_3,t_3))$;

(5) $\boldsymbol{O}(s,t)=(O_{\mathrm{DB}}(s_1,t_1),G_p(s_2,t_2),G_p(s_3,t_3))$ and $\boldsymbol{G}(s,t)=(G_p(s_1,t_1),O_p(s_2,t_2),O_{\mathrm{DB}}(s_3,t_3))$.

We give the following theorem for non-representable neutrosophic overlap function.

Theorem 3.4 Let $\boldsymbol{O}$ be a map on D^* below, $\forall s, t\in D^*$,

$$\boldsymbol{O}(s,t)=\begin{cases}0_{D^*}, & \text{if } s=0_{D^*} \text{ or } t=0_{D^*},\\ 1_{D^*}, & \text{if } s=t=1_{D^*},\\ (s_1t_1, s_3t_3, s_3t_3), & \text{otherwise.}\end{cases}$$

Then $\boldsymbol{O}$ is a non-representable neutrosophic overlap function.

Proof. The first step is to verify that $\boldsymbol{O}$ is the neutrosophic overlap function. $\forall s, t, v, u\in D^*$ in which $s= (s_1, s_2, s_3)$, t, u and v are analogous to s.

(NO1) The proof that $\boldsymbol{O}$ fulfils exchangeability is very straightforward.

(NO2) Let $t \leq_1 v$. The obvious one is $\boldsymbol{O}(s, t) \leq_1 \boldsymbol{O}(s, v)$.

(NO3) $\boldsymbol{O}(0_{D^*}, t)= \boldsymbol{O}(s, 0_{D^*})= (0, 1, 1)= 0_{D^*}$.

(NO4) $\boldsymbol{O}(1_{D^*}, 1_{D^*})= (1, 0, 0)= 1_{D^*}$.

(NO5) Firstly, we prove left continuous. That is, prove that this equation $\boldsymbol{O}(s, \vee_{i\in I}t_i)= \vee_{i\in I}\boldsymbol{O}(s, t_i)$ holds. As a result of $\boldsymbol{O}(s, \vee_{i\in I}t_i)= (s_1*\max(t_i), s_3*\max(t_i), s_3*\max(t_i))$; $\vee_{i\in I}\boldsymbol{O}(s, t_i)= (s_1*t_1, s_3*t_1, s_3*t_1)\vee (s_1*t_2, s_3*t_3, s_3*t_3)\vee (s_1*t_3, s_3*t_3, s_3*t_3)= (s_1*\max(t_i), s_3*\max(t_i), s_3*\max(t_i))$. We can get $\boldsymbol{O}(s, \vee_{i\in I}t_i)= \vee_{i\in I}\boldsymbol{O}(s, t_i)$. Therefore, it is show that $\boldsymbol{O}$ fulfils left continuity.

Likewise, $\boldsymbol{O}(s, \wedge_{i\in I}t_i)= \wedge_{i\in I}\boldsymbol{O}(s, t_i)$ is simple to prove, state clearly that $\boldsymbol{O}$ is right continuous. To sum up, $\boldsymbol{O}(s, t)$ is shown to be the neutrosophic overlap function.

And then, verify that for the representable neutrosophic overlap function $\boldsymbol{O}$ whether there has the overlap function O and grouping functions G_1, G_2 on [0, 1] fulfilling the form $\boldsymbol{O}= (O, G_1, G_2)$.

Have $s= (0.3, 0.5, 0.6)$, $u= (0.3, 0.5, 0.2)$ and $t= (0.4, 0.5, 0.8)$ respectively. From $\boldsymbol{O}(s, t)= (0.12, 0.48, 0.48)$ and $\boldsymbol{O}(u, t)= (0.12, 0.16, 0.16)$. We get $G_1(s_2, t_2)= 0.48$ and $G_1(u_2, t_2)= 0.16$, so $G_1(u_2, t_2)\neq G_1(s_2, t_2)$. Thus, G_1 is not independent from s_3, which suggests that $\boldsymbol{O}$ is non-representable.

In addition, the neutrosophic grouping function $\boldsymbol{G}$ is the dual of $\boldsymbol{O}$ in relation to the standard neutrosophic negaton $\boldsymbol{N}$g, defined as below, $\forall s, t\in D^*$,

$$G(s,t)=\begin{cases}0_{D^*}, & \text{if } s=t=0_{D^*},\\ 1_{D^*}, & \text{if } s=1_{D^*} \text{ or } t=1_{D^*},\\ (1-(1-s_1)(1-t_1),1-(1-s_3)(1-t_3),1-(1-s_3)(1-t_3)), & \text{otherwise.}\end{cases}$$

Then G is a non-representable neutrosophic grouping function.

4. NRI derived from neutrosophic overlap function

This section would bring in the concept of NRI on D^* and research fundamental properties of NRI. First, the notion of neutrosophic implication is introduced on D^*.

Definition 4.1 ([18]) The map I: $(D^*)^2\to D^*$ is known as the neutrosophic implication when it fulfils the prerequisites below, $\forall s, u, t, v \in D^*$:
(a) I is non-increasing for the first variable component (in relation to the order relation $\leq_1$), which means that when $s \leq_1 u$, there is $I(s, t) \geq_1 I(u, t)$;
(b) I is non-decreasing for the second variable component (in relation to the order relation $\leq_1$), which means that when $t \leq_1 v$, there is $I(s, t) \leq_1 I(s, v)$;
(c) $I(0_{D^*}, 0_{D^*})= 1_{D^*}$;
(d) $I(1_{D^*}, 1_{D^*})= 1_{D^*}$;
(e) $I(1_{D^*}, 0_{D^*})= 0_{D^*}$.

Definition 4.2 Suppose I: $(D^*)^2\to D^*$ is a binary map. A neutrosophic overlap function O exists which enables the following condition to hold, $\forall s, t, h\in D^*$,

$$I(s,t)=\sup\{h \mid h\in D^*, O(s,h)\leq_1 t\}.$$

Thus such I: $(D^*)^2\to D^*$ is referred to as the NRI.

When I is a NRI derived from a neutrosophic overlap function O, it is written as I_O.

Additionally, a neutrosophic overlap function O fulfils the residual principle, $\forall s, t, h\in D^*$:

$$h \leq_1 I_O(s, t) \text{ iff } O(s, h) \leq_1 t.$$

Example 4.1 The functions below are NRIs derived from neutrosophic overlap functions in **Example 3.1**, $\forall s, t\in D^*$,

(1)

$$I_{O_{mM}}(s,t)=\begin{cases}(1,0,0), & \text{if } s_1\le t_1, s_2\ge t_2 \text{ and } s_3\ge t_3,\\ (1,\max\{1-\sqrt{\frac{1-t_2}{1-s_2}},1-\frac{1-t_2}{(1-s_2)^2}\},\max\{1-\sqrt{\frac{1-t_3}{1-s_3}},1-\frac{1-t_3}{(1-s_3)^2}\}), & \text{if } s_1\le t_1, s_2< t_2 \text{ and } s_3< t_3,\\ (1,0,\max\{1-\sqrt{\frac{1-t_3}{1-s_3}},1-\frac{1-t_3}{(1-s_3)^2}\}), & \text{if } s_1\le t_1, s_2\ge t_2 \text{ and } s_3< t_3\\ (1,\max\{1-\sqrt{\frac{1-t_2}{1-s_2}},1-\frac{1-t_2}{(1-s_2)^2}\},0\}), & \text{if } s_1\le t_1, s_2< t_2 \text{ and } s_3\ge t_3,\\ (\min\{\sqrt{\frac{t_1}{s_1}},\frac{t_1}{s_1^2}\},\max\{1-\sqrt{\frac{1-t_2}{1-s_2}},1-\frac{1-t_2}{(1-s_2)^2}\},\max\{1-\sqrt{\frac{1-t_3}{1-s_3}},1-\frac{1-t_3}{(1-s_3)^2}\}), & \text{if } s_1> t_1, s_2< t_2 \text{ and } s_3< t_3,\\ (\min\{\sqrt{\frac{t_1}{s_1}},\frac{t_1}{s_1^2}\},0,0), & \text{if } s_1> t_1, s_2\ge t_2 \text{ and } s_3\ge t_3,\\ (\min\{\sqrt{\frac{t_1}{s_1}},\frac{t_1}{s_1^2}\},\max\{1-\sqrt{\frac{1-t_2}{1-s_2}},1-\frac{1-t_2}{(1-s_2)^2}\},0), & \text{if } s_1> t_1, s_2< t_2 \text{ and } s_3\ge t_3,\\ (\min\{\sqrt{\frac{t_1}{s_1}},\frac{t_1}{s_1^2}\},0,\max\{1-\sqrt{\frac{1-t_3}{1-s_3}},1-\frac{1-t_3}{(1-s_3)^2}\}), & \text{if } s_1> t_1, s_2\ge t_2 \text{ and } s_3< t_3.\end{cases}$$

(2)

$$I_{O_{p=2}}(s,t)=\begin{cases}(\frac{\sqrt{t_1}}{s_1},1-\frac{\sqrt{1-t_2}}{1-s_2},1-\frac{\sqrt{1-t_3}}{1-s_3}), & \text{if } s_1>\sqrt{t_1}, s_2<1-\sqrt{1-t_2} \text{ and } s_3<1-\sqrt{1-t_3},\\ (\frac{\sqrt{t_1}}{s_1},0,0), & \text{if } s_1>\sqrt{t_1}, s_2\ge 1-\sqrt{1-t_2} \text{ and } s_3\ge 1-\sqrt{1-t_3},\\ (\frac{\sqrt{t_1}}{s_1},1-\frac{\sqrt{1-t_2}}{1-s_2},0), & \text{if } s_1>\sqrt{t_1}, s_2<1-\sqrt{1-t_2} \text{ and } s_3\ge 1-\sqrt{1-t_3},\\ (\frac{\sqrt{t_1}}{s_1},0,1-\frac{\sqrt{1-t_3}}{1-s_3}), & \text{if } s_1>\sqrt{t_1}, s_2\ge 1-\sqrt{1-t_2} \text{ and } s_3<1-\sqrt{1-t_3},\\ (1,1-\frac{\sqrt{1-t_2}}{1-s_2},1-\frac{\sqrt{1-t_3}}{1-s_3}), & \text{if } s_1\le\sqrt{t_1}, s_2<1-\sqrt{1-t_2} \text{ and } s_3<1-\sqrt{1-t_3},\\ (1,0,0), & \text{if } s_1\le\sqrt{t_1}, s_2\ge 1-\sqrt{1-t_2} \text{ and } s_3\ge 1-\sqrt{1-t_3},\\ (1,0,1-\frac{\sqrt{1-t_3}}{1-s_3}), & \text{if } s_1\le\sqrt{t_1}, s_2\ge 1-\sqrt{1-t_2} \text{ and } s_3<1-\sqrt{1-t_3},\\ (1,1-\frac{\sqrt{1-t_2}}{1-s_2},0), & \text{if } s_1\le\sqrt{t_1}, s_2<1-\sqrt{1-t_2} \text{ and } s_3\ge 1-\sqrt{1-t_3}.\end{cases}$$

(3)

$$I_{O_{DB}}(s,t)=\begin{cases}(\min\{1,\frac{s_1t_1}{2s_1-t_1}\},\frac{2t_2-s_2t_2-s_2}{1-2s_2+t_2},\frac{2t_3-s_3t_3-s_3}{1-2s_3+t_3}), & \text{if } s_1>\frac{1}{2}t_1, s_2<\frac{1}{2}+\frac{1}{2}t_2 \text{ and } s_3<\frac{1}{2}+\frac{1}{2}t_3,\\ (\min\{1,\frac{s_1t_1}{2s_1-t_1}\},0,\frac{2t_3-s_3t_3-s_3}{1-2s_3+t_3}), & \text{if } s_1>\frac{1}{2}t_1, 1>s_2\ge\frac{1}{2}+\frac{1}{2}t_2 \text{ and } s_3<\frac{1}{2}+\frac{1}{2}t_3,\\ (\min\{1,\frac{s_1t_1}{2s_1-t_1}\},\frac{2t_2-s_2t_2-s_2}{1-2s_2+t_2},0), & \text{if } s_1>\frac{1}{2}t_1, s_2<\frac{1}{2}+\frac{1}{2}t_2 \text{ and } 1>s_3\ge\frac{1}{2}+\frac{1}{2}t_3,\\ (\min\{1,\frac{s_1t_1}{2s_1-t_1}\},0,0), & \text{if } s_1>\frac{1}{2}t_1, 1>s_2\ge\frac{1}{2}+\frac{1}{2}t_2 \text{ and } 1>s_3\ge\frac{1}{2}+\frac{1}{2}t_3,\\ (1,\frac{2t_2-s_2t_2-s_2}{1-2s_2+t_2},\frac{2t_3-s_3t_3-s_3}{1-2s_3+t_3}), & \text{if } 0<s_1\le\frac{1}{2}t_1, s_2<\frac{1}{2}+\frac{1}{2}t_2 \text{ and } s_3<\frac{1}{2}+\frac{1}{2}t_3,\\ (1,0,\frac{2t_3-s_3t_3-s_3}{1-2s_3+t_3}), & \text{if } 0<s_1\le\frac{1}{2}t_1, 1>s_2\ge\frac{1}{2}+\frac{1}{2}t_2 \text{ and } s_3<\frac{1}{2}+\frac{1}{2}t_3,\\ (1,\frac{2t_2-s_2t_2-s_2}{1-2s_2+t_2},0), & \text{if } 0<s_1\le\frac{1}{2}t_1, s_2<\frac{1}{2}+\frac{1}{2}t_2 \text{ and } 1>s_3\ge\frac{1}{2}+\frac{1}{2}t_3,\\ (1,0,0), & \text{if } 0<s_1\le\frac{1}{2}t_1, 1>s_2\ge\frac{1}{2}+\frac{1}{2}t_2 \text{ and } 1>s_3\ge\frac{1}{2}+\frac{1}{2}t_3,\\ (0,1,1) & \text{if } s_1=t_1=0, s_2=t_2=1 \text{ and } s_3=t_3=1.\end{cases}$$

(4)

$$I_{O_{\min}}(s,t)=\begin{cases}(1,0,0), & \text{if } s_1\le t_1^2, s_2\ge 1-(1-t_2)^2 \text{ and } s_3\ge 1-(1-t_3)^2,\\ (1,1-(1-t_2)^2,0), & \text{if } s_1\le t_1^2, s_2<1-(1-t_2)^2 \text{ and } s_3\ge 1-(1-t_3)^2,\\ (1,0,1-(1-t_3)^2), & \text{if } s_1\le t_1^2, s_2\ge 1-(1-t_2)^2 \text{ and } s_3<1-(1-t_3)^2,\\ (1,1-(1-t_2)^2,1-(1-t_3)^2), & \text{if } s_1\le t_1^2, s_2<1-(1-t_2)^2 \text{ and } s_3<1-(1-t_3)^2,\\ (t_1^2,0,0), & \text{if } s_1>t_1^2, s_2\ge 1-(1-t_2)^2 \text{ and } s_3\ge 1-(1-t_3)^2,\\ (t_1^2,1-(1-t_2)^2,0), & \text{if } s_1>t_1^2, s_2<1-(1-t_2)^2 \text{ and } s_3\ge 1-(1-t_3)^2,\\ (t_1^2,0,1-(1-t_3)^2), & \text{if } s_1>t_1^2, s_2\ge 1-(1-t_2)^2 \text{ and } s_3<1-(1-t_3)^2,\\ (t_1^2,1-(1-t_2)^2,1-(1-t_3)^2), & \text{if } s_1>t_1^2, s_2<1-(1-t_2)^2 \text{ and } s_3<1-(1-t_3)^2.\end{cases}$$

Theorem 4.1 Suppose $\boldsymbol{O}$ is the neutrosophic overlap function on D^*, $\forall s, t, h \in D^*$:

$$\boldsymbol{I}_O(s,t) = \sup\{h \mid h \in D^*, \boldsymbol{O}(s,h) \leq_1 t\}.$$

Thus, $\boldsymbol{I}_O$ is a neutrosophic implication.

Proof. $\forall s, u, t, h, v \in D^*$ with $s = (s_1, s_2, s_3)$, t, h, u and v are analogous to s.

(a) Let $s \leq_1 u$ and since $\boldsymbol{O}$ is non-decreasing, $\{h \mid h \in D^*, \boldsymbol{O}(s, h) \leq_1 t\} \supseteq_1 \{h \mid h \in D^*, \boldsymbol{O}(u, h) \leq_1 t\}$ then $\sup\{h \mid h \in D^*, h \leq_1 \boldsymbol{I}_O(s, t)\} \geq_1 \sup\{h \mid h \in D^*, h \leq_1 \boldsymbol{I}_O(u, t)\}$. Thus $\boldsymbol{I}_O(s, t) \geq_1 \boldsymbol{I}_O(u, t)$. In other words, the first variable of $\boldsymbol{I}_O$ regarding $\leq_1$ is non-increasing.

(b) Let $t \leq_1 v$ and since $\boldsymbol{O}$ is non-decreasing, $\{h \mid h \in D^*, \boldsymbol{O}(s, h) \leq_1 t\} \subseteq_1 \{h \mid h \in D^*, \boldsymbol{O}(s, h) \leq_1 v\}$ then $\sup\{h \mid h \in D^*, h \leq_1 \boldsymbol{I}_O(s, t)\} \leq_1 \sup\{h \mid h \in D^*, h \leq_1 \boldsymbol{I}_O(s, v)\}$. Thus $\boldsymbol{I}_O(s, t) \leq_1 \boldsymbol{I}_O(s, v)$. In other words, the second variable of $\boldsymbol{I}_O$ regarding $\leq_1$ is non-decreasing.

(c) $\boldsymbol{I}_O(0_{D^*}, 0_{D^*}) = \sup\{h \mid h \in D^*, \boldsymbol{O}(0_{D^*}, h) \leq_1 0_{D^*}\} = 1_{D^*}$;

(d) $\boldsymbol{I}_O(1_{D^*}, 1_{D^*}) = \sup\{h \mid h \in D^*, \boldsymbol{O}(1_{D^*}, h) \leq_1 1_{D^*}\} = \sup\{h \mid h \in D^*, h \leq_1 1_{D^*}\} = 1_{D^*}$;

(e) $\boldsymbol{I}_O(1_{D^*}, 0_{D^*}) = \sup\{h \mid h \in D^*, \boldsymbol{O}(1_{D^*}, h) \leq_1 0_{D^*}\} = \sup\{h \mid h \in D^*, h \leq_1 0_{D^*}\} = 0_{D^*}$.

The NRI has the following important properties.

Theorem 4.2 Assume that $\boldsymbol{I}_O$ is NRI, $\boldsymbol{O}$ is neutrosophic overlap function on D^*. $\forall s, t, h \in D^*$, the follows properties are valid,

(1) $\boldsymbol{I}_O(0_{D^*}, t) = 1_{D^*}$;
(2) $\boldsymbol{I}_O(s, 1_{D^*}) = 1_{D^*}$;
(3) $\boldsymbol{I}_O(s, s) = 1_{D^*}$;
(4) $\boldsymbol{I}_O(1_{D^*}, t) = t$;
(5) $\boldsymbol{I}_O(s, t) \geq_1 t$;
(6) $\boldsymbol{I}_O(s, t) = 1_{D^*}$ iff $s \leq_1 t$;
(7) $s \leq_1 \boldsymbol{I}_O(t, h)$ iff $t \leq_1 \boldsymbol{I}_O(s, h)$;
(8) $s \leq_1 \boldsymbol{I}_O(t, \boldsymbol{I}_O(s, t))$.

Proof. $\forall s, t, h \in D^*$ in which $s = (s_1, s_2, s_3)$, t and h are analogous to s.

(1) $\boldsymbol{I}_O(0_{D^*}, t) = 1_{D^*}$ is the same thing as $\boldsymbol{I}_O(0_{D^*}, t) = \sup\{h \mid h \in D^*, \boldsymbol{O}(0_{D^*}, h) \leq_1 t\} = 1_{D^*}$, then $h = 1_{D^*}$ for $\boldsymbol{O}(0_{D^*}, h) \leq_1 t$. Then this formula is proved.

The proofs of **(2)–(4)** is similar to that of **(1)**.

(5) $\boldsymbol{I}$ is non-increasing for the first variable component (in relation to the order relation $\leq_1$), then $\boldsymbol{I}_O(s, t) \geq_1 \boldsymbol{I}_O(1_{D^*}, t) = t$.

(6) $\boldsymbol{I}_O(s, t) = 1_{D^*}$ iff $s \leq_1 t$. Let $s \leq_1 t$, $\boldsymbol{O}(1_{D^*}, s) \leq_1 t$, then $\boldsymbol{I}_O(s, t) = 1_{D^*}$. In contrast, let $\boldsymbol{I}_O(s, t) = 1_{D^*}$,thus $\boldsymbol{O}(1_{D^*}, s) \leq_1 t$, hence $s \leq_1 t$.

(7) $s \leq_1 \boldsymbol{I}_O(t, h)$ iff $t \leq_1 \boldsymbol{I}_O(s, h)$. Since $s \leq_1 \boldsymbol{I}_O(t, h)$, $\boldsymbol{O}(t, s) \leq_1 h$. Thus, $t \leq_1 \boldsymbol{I}_O(s, h)$. Likewise, $s \leq_1 \boldsymbol{I}_O(t, h)$ can be proved from $t \leq_1 \boldsymbol{I}_O(s, h)$.

(8) $s \leq_1 \boldsymbol{I}_O(t, \boldsymbol{I}_O(s, t))$. Since $\boldsymbol{O}(t, s) \leq_1 \boldsymbol{O}(s, t)$, then $s \leq_1 \boldsymbol{I}_O(t, \boldsymbol{I}_O(s, t))$.

Example 4.2 These concrete cases about NRI deduced from neutrosophic overlap function as shown in **Example 4.1** are given. And it is readily proved that NRI deduced from neutrosophic overlap functions satisfy the properties characterised by **Theorem 4.2**.

Furthermore, for the non-representable neutrosophic overlap function, using the neutrosophic overlap function got from **Theorem 3.4** as an example, $\forall s, t \in D^*$,

$$\boldsymbol{I}_O(s,t) = \begin{cases} 1_{D^*}, & \text{if } s = 0_{D^*} \text{ or } t = 0_{D^*}; \text{or } s = t = 1_{D^*}, \\ (1, [0,1], \max\{\frac{t_3}{s_2}, \frac{t_3}{s_3}\}), & \text{if } s_1 \leq t_1, \\ (\frac{t_1}{s_1}, [0,1], \max\{\frac{t_3}{s_2}, \frac{t_3}{s_3}\}), & \text{if } s_1 > t_1. \end{cases}$$

Then it follows that $I_O(s, t)$ is a neutrosophic implication, while the properties from **Theorem 4.2** is satisfied.

5. Conclusions

As an important part of NS theory, neutrosophic logic plays a significant part in it. Neutrosophic overlap function, neutrosophic grouping function and neutrosophic implication which are crucial neutrosophic logic operators. For the first kind of inclusion relationship, the definitions of neutrosophic overlap function (neutrosophic grouping function) on $(D^*; \leq_1)$ are defined and related examples are given. At the same time, new definitions of representable and non-representable neutrosophic overlap function are proposed. In the next place, based on the close relationship between overlap function and triangular norm, a new description of neutrosophic negation is offered through analogy research, then the dual relationship between neutrosophic overlap function and neutrosophic grouping function on neutrosophic negation is described. Moreover, we show that definition of neutrosophic implication is given based on $(D^*; \leq_1)$ and the basic properties of NRI are studied. Finally, the result that NRI induced by neutrosophic overlap function must be neutrosophic implication is proved. Based on these results and some new results [33-44], we consider applying them to generalized neutrosophic overlap function and neutrosophic inference systems of the future.

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Convert between neutrosophic complex numbers forms

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Abstract: Complex numbers have been studied in previous papers, but the papers did not deal with the conversion from algebraic form of a neutrosophic complex number form to exponential or trigonometric form and vice versa, and this prompted us to search for a method that facilitates this conversion process, as this paper dealt with how to move from algebraic form of a neutrosophic complex number to exponential or trigonometric form and vice versa, also, we discussed the roots from order n of a neutrosophic complex number, whether this number is given in algebraic or trigonometric form.

Keywords: neutrosophic complex numbers, algebraic form, roots of a neutrosophic complex number, exponential form.

1. Introduction

As Smarandache proposed the Neutrosophic Logic as an alternative to the existing logics to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, and contradiction. Smarandache introduced the concept of neutrosophy as a new school of philosophy [4][8]. He presented the definition of the standard form of neutrosophic real number and conditions for the division of two neutrosophic real numbers to exist [3][5], studying the concept of the Neutrosophic probability [4][6], the Neutrosophic statistics [5][7], and professor Smarandache entered the concept of preliminary calculus of the differential and integral calculus, where he introduced for the first time the notions of neutrosophic mereo-limit, mereo-continuity, mereoderivative, and mereo-integral [1][8]. Y.Alhasan presented the definition of the concept of neutrosophic complex numbers and its properties including the conjugate of neutrosophic complex number, division of neutrosophic complex numbers, the inverted neutrosophic complex number and the absolute value of a neutrosophic complex number and Theories related to the conjugate of neutrosophic complex numbers, the product of a neutrosophic complex number by its conjugate equals the absolute value of number and he studied the general exponential form of a neutrosophic complex number [2-11]. Madeleine Al- Taha presented results on single valued neutrosophic (weak) polygroups [10]. An algebraic approach to neutrosophic euclidean geometry is presented [12].

Complex numbers play a significant role in daily life because they make it much easier to perform mathematical operations and give us a way to solve equations for which there are no real-number-group solutions. The electrical engineering field makes extensive use of complex numbers to calculate electric voltage and measure alternating current.

Paper is divided into four pieces. provides an introduction in the first portion, which includes a review of neutrosophic science. A few definitions and hypotheses of a neutrosophic complex number are covered in the second section. The third section describes the transformation of a neutrosophic complex number from its exponential or trigonometric form to its algebraic form, and vice versa. The paper's conclusion is provided in the fourth section.

2. Preliminaries

2.1. The general exponential form of a neutrosophic complex number [11]

Theorem 1

The general exponential form of the neutrosophic complex number is given by the formula:

$$z = |z|e^{i(\theta+\vartheta I)} = re^{i(\theta+\vartheta I)}$$

Whereas $r = |z|$ is the absolute value of a neutrosophic complex number.

2.2 The general Trigonometric form of a neutrosophic complex number [11]

Definition 1

The following formula:

$$z = \mathrm{r} \ \ (\cos(\theta + \vartheta I) + sin \ \ (\theta + \vartheta I) \ \ i)$$

is called the general trigonometric form of a neutrosophic complex number

Definition 2 [12]

Let f: $R(I) \ \rightarrow \ R(I)$; $f = f(X)$ and $X = x + yI \in R(I)$ the f is called a neutrosophic real function with one neutrosophic variable. a neutrosophic real function $f(X)$ written as follows:

$$f(X) = f(x + yI) = f(x) + I[f(x + y) - f(x)]$$

3. Conversion from exponential or trigonometric form of a neutrosophic complex number to algebraic form

$$z = \acute{r}e^{i(\acute{\theta}+\acute{\vartheta}I)}$$

$$= \acute{r}\left(\ cos\left(\acute{\theta} + \acute{\vartheta}I\right) + i \ \ sin(\acute{\theta} + \acute{\vartheta}I)\right)$$

$$= \acute{r}\left(\ cos\left(\acute{\theta}\right) + I\left[cos\left(\acute{\theta} + \acute{\vartheta}\right) - \ cos\left(\acute{\theta}\right)\right] + i \ \ sin\left(\acute{\theta}\right) + i \ \ I\left[sin\left(\acute{\theta} + \acute{\vartheta}\right) - \ sin\left(\acute{\theta}\right)\right]\right)$$

Whereas $\left(\acute{\theta} + \acute{\vartheta}I\right)$ is the indeterminate angle between two indeterminate parts of the coordinate axes ($x - axis \ \ and \ \ y - axis$).

Example 1

$$z = 4e^{i\left(\frac{\pi}{6}+\frac{\pi}{3}I\right)}$$

$$= 4\left(cos\left(\frac{\pi}{6}+\frac{\pi}{3}I\right) + i \ \ sin\left(\frac{\pi}{6}+\frac{\pi}{3}I\right)\right)$$

$$= 4\left(cos\frac{\pi}{6} + I\left[cos\left(\frac{\pi}{6}+\frac{\pi}{3}\right) - cos\frac{\pi}{6}\right] + i\, sin\frac{\pi}{6} + i \ \ I\left[sin\left(\frac{\pi}{6}+\frac{\pi}{3}\right) - sin\frac{\pi}{6}\right]\right)$$

$$= 4\left(\frac{\sqrt{3}}{2} + I\left[0 - \frac{\sqrt{3}}{2}\right] + i\frac{1}{2} + i \ \ I\left[1 - \frac{1}{2}\right]\right)$$

$$= 2\sqrt{3} - 2\sqrt{3} \ \ I + (2 + 2 \ \ I)i$$

Example 2

$$z = \frac{1}{6}e^{i(\pi - 2\pi I)}$$

$$= \frac{1}{6}\left(cos(\pi - 2\pi I) + i \ \ sin(\pi - 2\pi I)\right)$$

$$= \frac{1}{6}(cos\pi + I[cos(\pi - 2\pi) - cos\,\pi] + i\, sin\,\pi + i \ \ I[sin(\pi - 2\pi) - sin\pi])$$

$$= \frac{1}{6}(-1 + I[-1 + 1] + i(0) + i \ \ I[0 - 0]) = -\frac{1}{6}$$

3.1 Conversion from algebraic form of a neutrosophic complex number to exponential or trigonometric form

Let $z = \ddot{a} + \ddot{c}I + \ddot{b}i + \ddot{d}iI$, then:

$$\acute{r} = \sqrt{(\ddot{a} + \ddot{c}I)^2 + \left(\ddot{b} + \ddot{d}I\right)^2}$$

$$cos\left(\acute{\theta} + \acute{\vartheta}I\right) = \frac{\ddot{a} + \ddot{c}I}{\acute{r}}$$

$$sin\left(\acute{\theta} + \acute{\vartheta}I\right) = \frac{\ddot{b} + \ddot{d}I}{\acute{r}}$$

We can find angle whereas: $\acute{\theta} + \acute{\vartheta}I = \acute{\mathfrak{I}} + \acute{\varphi}I + 2\pi k \ \ ; k \in Z$
hence:

$$\text{z} = \acute{r}e^{i(\acute{\mathfrak{I}} + \acute{\varphi}I)}$$

Example 3

$$z = \left(\frac{-1}{\sqrt{2}} + \frac{2}{\sqrt{2}}I\right) + \left(\frac{1}{\sqrt{2}}\right)i$$

$$\acute{r} = \sqrt{(\ddot{a} + \ddot{c}I)^2 + \left(\ddot{b} + \ddot{d}I\right)^2}$$

$$= \sqrt{\left(\frac{-1}{\sqrt{2}} + \frac{2}{\sqrt{2}}I\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \sqrt{\left(\frac{1}{2} - 2\frac{1}{\sqrt{2}}\frac{2}{\sqrt{2}}I + 2I\right) + \frac{1}{2}} = 1$$

then:

$$cos(\acute{\theta}+\acute{\vartheta}I)=\frac{\ddot{a}+\ddot{c}I}{\acute{r}}=\frac{-1}{\sqrt{2}}+\frac{2}{\sqrt{2}}I$$

$$sin(\acute{\theta}+\acute{\vartheta}I)=\frac{\ddot{b}+\ddot{d}I}{\acute{r}}=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \acute{\theta}+\acute{\vartheta}I=\frac{3\pi}{4}+\frac{3\pi}{2}I$$

hence:

$$z=e^{i\left(\frac{3\pi}{4}+\frac{3\pi}{2}I\right)}$$

to check the solution: if we take some values of I, we get on the following

#	$z=e^{i\left(\frac{3\pi}{4}+\frac{3\pi}{2}I\right)}$	$cos\left(\frac{3\pi}{4}+\frac{3\pi}{2}I\right)=\frac{-1}{\sqrt{2}}+\frac{2}{\sqrt{2}}I$	$sin\left(\frac{3\pi}{4}+\frac{3\pi}{2}I\right)=\frac{1}{\sqrt{2}}$	**Result**
$I=0$	$z=e^{i\left(\frac{3\pi}{4}\right)}$	$cos\left(\frac{3\pi}{4}\right)=\frac{-1}{\sqrt{2}}$	$sin\left(\frac{3\pi}{4}\right)=\frac{1}{\sqrt{2}}$	✓
$I=1$	$z=e^{i\left(\frac{9\pi}{4}\right)}$	$cos\left(\frac{9\pi}{4}\right)=\frac{1}{\sqrt{2}}$	$sin\left(\frac{9\pi}{4}\right)=\frac{1}{\sqrt{2}}$	✓

and so on…

Example 4

Let:

$$z_1=\frac{\sqrt{3}-\sqrt{3}\ I}{2}+\frac{1+I}{2}i \qquad and \qquad z_2=\left(\frac{\sqrt{6}+\sqrt{2}}{4}+\frac{\sqrt{2}-\sqrt{6}}{4}I\right)+\left(\frac{\sqrt{6}-\sqrt{2}}{4}+\frac{3\sqrt{2}-\sqrt{6}}{4}I\right)\ i$$

Write of each z_1, z_2 by exponential form.

Solution:

$$z_1=\frac{\sqrt{3}-\sqrt{3}\ I}{2}+\frac{1+I}{2}i$$

$$\acute{r}_1=\sqrt{(\ddot{a}+\ddot{c}I)^2+(\ddot{b}+\ddot{d}I)^2}$$

$$=\sqrt{\left(\frac{\sqrt{3}-\sqrt{3}\ I}{2}\right)^2+\left(\frac{1+I}{2}\right)^2}$$

$$=\sqrt{\left(\frac{3-6I+3I}{4}\right)+\frac{1+2I+I}{4}}=1$$

then:

$$cos(\acute{\theta}+\acute{\vartheta}I)=\frac{\ddot{a}+\ddot{c}I}{\acute{r}}=\frac{\sqrt{3}-\sqrt{3}\ I}{2}=\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}I$$

$$sin(\acute{\theta} + \acute{\vartheta}I) = \frac{\ddot{b} + \ddot{d}I}{\acute{r}} = \frac{1 + I}{2} = \frac{1}{2} + \frac{1}{2}I$$

$$\Longrightarrow \qquad \acute{\theta} + \acute{\vartheta}I = \frac{\pi}{6} + \frac{\pi}{3}I$$

hence:

$$z_1 = e^{i\left(\frac{\pi}{6}+\frac{\pi}{3}I\right)}$$

to check the solution: if we take some values of I, we get on the following

#	$z = e^{i\left(\frac{\pi}{6}+\frac{\pi}{3}I\right)}$	$cos\left(\frac{\pi}{6} + \frac{\pi}{3}I\right) = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}I$	$sin\left(\frac{\pi}{6} + \frac{\pi}{3}I\right) = \frac{1}{2} + \frac{1}{2}I$	**Result**
$I = 0$	$z = e^{i\left(\frac{\pi}{6}\right)}$	$cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$	$sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$	✓
$I = 1$	$z = e^{i\left(\frac{\pi}{2}\right)}$	$cos\left(\frac{\pi}{2}\right) = 0$	$sin\left(\frac{\pi}{2}\right) = 1$	✓

and so on…

now:

$$z_2 = \left(\frac{\sqrt{6} + \sqrt{2}}{4} + \frac{\sqrt{2} - \sqrt{6}}{4}I\right) + \left(\frac{\sqrt{6} - \sqrt{2}}{4} + \frac{3\sqrt{2} - \sqrt{6}}{4}I\right) i$$

$$\acute{r}_2 = \sqrt{(\ddot{a} + \ddot{c}I)^2 + \left(\ddot{b} + \ddot{d}I\right)^2}$$

$$= \sqrt{\left(\frac{\sqrt{6} + \sqrt{2}}{4} + \frac{\sqrt{2} - \sqrt{6}}{4}I\right)^2 + \left(\frac{\sqrt{6} - \sqrt{2}}{4} + \frac{3\sqrt{2} - \sqrt{6}}{4}I\right)^2}$$

$$= \sqrt{\left(\frac{6 + 4\sqrt{3} \;\; + 2 + 2I - 4\sqrt{3} \;\; I + 6I}{16} - \frac{8I}{16}\right) + \left(\frac{6 - 4\sqrt{3} \;\; + 2 + 18I - 12\sqrt{3} \;\; I + 6I}{16} + \frac{16\sqrt{3} \;\; I - 24I}{16}\right)}$$
$$= 1$$

then:

$$cos(\acute{\theta} + \acute{\vartheta}I) = \frac{\ddot{a} + \ddot{c}I}{\acute{r}} = \frac{\sqrt{6} + \sqrt{2}}{4} + \frac{\sqrt{2} - \sqrt{6}}{4}I$$

$$sin(\acute{\theta} + \acute{\vartheta}I) = \frac{\ddot{b} + \ddot{d}I}{\acute{r}} = \frac{\sqrt{6} - \sqrt{2}}{4} + \frac{3\sqrt{2} - \sqrt{6}}{4}I$$

$$\Longrightarrow \qquad \acute{\theta} + \acute{\vartheta}I = \frac{\pi}{12} + \frac{\pi}{6}I$$

hence:

$$z_2 = e^{i\left(\frac{\pi}{12}+\frac{\pi}{6}I\right)}$$

to check the solution: if we take some values of I, we get on the following

#	$\boldsymbol{z = e^{i\left(\frac{\pi}{12}+\frac{\pi}{6}I\right)}}$	$\boldsymbol{cos\left(\frac{\pi}{12}+\frac{\pi}{6}I\right) = \frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{2}-\sqrt{6}}{4}I}$	$\boldsymbol{sin\left(\frac{\pi}{12}+\frac{\pi}{6}I\right) = \frac{\sqrt{6}-\sqrt{2}}{4} + \frac{3\sqrt{2}-\sqrt{6}}{4}I}$	**Result**
$\boldsymbol{I = 0}$	$z = e^{i\left(\frac{\pi}{12}\right)}$	$cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$	$sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$	✓
$\boldsymbol{I = 1}$	$z = e^{i\left(\frac{\pi}{4}\right)}$	$cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$	$sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$	✓

and so on…

Example 5

Let:

$$z_1 = \frac{\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2}I + \left(\frac{-1}{2} + \frac{1-\sqrt{3}}{2}I\right)i \qquad and \qquad z_2 = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}I\right) + \left(\frac{-\sqrt{2}}{2} + \frac{-2+\sqrt{2}}{2}I\right)i$$

a) Write of each $z_1, z_2, \frac{z_1}{z_2}$ by exponential and trigonometric form.
b) Write $\frac{z_1}{z_2}$ by algebraic form.
c) conclude that:

$$cos\left(\frac{\pi}{12} + \frac{\pi}{12}I\right) = \frac{\sqrt{6}+\sqrt{2}}{4} + \frac{2\sqrt{3}-\sqrt{2}-\sqrt{6}}{4}I$$

$$sin\left(\frac{\pi}{12} + \frac{\pi}{12}I\right) = \frac{\sqrt{6}-\sqrt{2}}{4} + \frac{2+\sqrt{2}-\sqrt{6}}{4}I$$

Solution:

a)

$$z_1 = \frac{\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2}I + \left(\frac{-1}{2} + \frac{1-\sqrt{3}}{2}I\right)i$$

$$\acute{r}_1 = \sqrt{(\ddot{a} + \ddot{c}I)^2 + \left(\ddot{b} + \ddot{d}I\right)^2}$$

$$= \sqrt{\left(\frac{\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2}I\right)^2 + \left(\frac{-1}{2} + \frac{1-\sqrt{3}}{2}I\right)^2} = 1$$

then:

$$cos\left(\acute{\theta} + \acute{\vartheta}I\right) = \frac{\ddot{a} + \ddot{c}I}{\acute{r}} = \frac{\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2}I$$

$$sin\left(\acute{\theta} + \acute{\vartheta}I\right) = \frac{\ddot{b} + \ddot{d}I}{\acute{r}} = \frac{-1}{2} + \frac{1-\sqrt{3}}{2}I$$

$$\Longrightarrow \qquad \acute{\theta} + \acute{\vartheta}I = -\frac{\pi}{6} - \frac{\pi}{6}I$$

hence:

$$z_1 = e^{-i\left(\frac{\pi}{6}+\frac{\pi}{6}I\right)}$$

now:

$$z_2=\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}I\right)+\left(\frac{-\sqrt{2}}{2}+\frac{-2+\sqrt{2}}{2}I\right)i$$

$$\acute{r}_2=\sqrt{(\ddot{a}+\ddot{c}I)^2+\left(\ddot{b}+\ddot{d}I\right)^2}$$

$$=\sqrt{\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}I\right)^2+\left(\frac{-\sqrt{2}}{2}+\frac{-2+\sqrt{2}}{2}I\right)^2}=1$$

then:

$$cos\left(\acute{\theta}+\acute{\vartheta}I\right)=\frac{\ddot{a}+\ddot{c}I}{\acute{r}}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}I$$

$$sin\left(\acute{\theta}+\acute{\vartheta}I\right)=\frac{\ddot{b}+\ddot{d}I}{\acute{r}}=\frac{-\sqrt{2}}{2}+\frac{-2+\sqrt{2}}{2}I$$

$$\Longrightarrow \quad \acute{\theta}+\acute{\vartheta}I=-\frac{\pi}{4}-\frac{\pi}{4}I$$

hence:

$$z_2=e^{-i\left(\frac{\pi}{4}+\frac{\pi}{4}I\right)}$$

$$\Longrightarrow \quad \frac{z_1}{z_2}=\frac{e^{-i\left(\frac{\pi}{6}+\frac{\pi}{3}I\right)}}{e^{-i\left(\frac{\pi}{4}+\frac{\pi}{4}I\right)}}=e^{-i\left(\frac{\pi}{6}+\frac{\pi}{3}I\right)+i\left(\frac{\pi}{12}+\frac{\pi}{6}I\right)}=e^{i\left(\frac{\pi}{12}+\frac{\pi}{12}I\right)}$$

b)

$$\frac{z_1}{z_2}=\frac{\frac{\sqrt{3}}{2}+\frac{1+\sqrt{3}}{2}I+\left(\frac{-1}{2}+\frac{1-\sqrt{3}}{2}I\right)i}{\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}I\right)+\left(\frac{-\sqrt{2}}{2}+\frac{-2+\sqrt{2}}{2}I\right)i}$$

$$=\frac{\left(\frac{\sqrt{3}}{2}+\frac{1+\sqrt{3}}{2}I+\left(\frac{-1}{2}+\frac{1-\sqrt{3}}{2}I\right)i\right)\left(\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}I\right)+\left(\frac{\sqrt{2}}{2}+\frac{2-\sqrt{2}}{2}I\right)i\right)}{\left(\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}I\right)+\left(\frac{-\sqrt{2}}{2}+\frac{-2+\sqrt{2}}{2}I\right)i\right)\left(\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}I\right)+\left(\frac{\sqrt{2}}{2}+\frac{2-\sqrt{2}}{2}I\right)i\right)}$$

$$\frac{z_1}{z_2}=\left(\frac{\sqrt{6}+\sqrt{2}}{4}+\frac{2\sqrt{3}-\sqrt{2}-\sqrt{6}}{4}I\right)+\left(\frac{\sqrt{6}-\sqrt{2}}{4}+\frac{\sqrt{2}+2-\sqrt{6}}{4}I\right)i \qquad (*)$$

c)

$$\frac{z_1}{z_2}=e^{i\left(\frac{\pi}{12}+\frac{\pi}{12}I\right)}=cos\left(\frac{\pi}{12}+\frac{\pi}{12}I\right)+i\ \ sin\left(\frac{\pi}{12}+\frac{\pi}{12}I\right)$$

compared with $(*)$, we find:

$$cos\left(\frac{\pi}{12}+\frac{\pi}{12}I\right)+i\ \ sin\left(\frac{\pi}{12}+\frac{\pi}{12}I\right)$$
$$=\left(\frac{\sqrt{6}+\sqrt{2}}{4}+\frac{2\sqrt{3}-\sqrt{2}-\sqrt{6}}{4}I\right)+\left(\frac{\sqrt{6}-\sqrt{2}}{4}+\frac{\sqrt{2}+2-\sqrt{6}}{4}I\right)i$$

hence:

$$cos\left(\frac{\pi}{12}+\frac{\pi}{12}I\right)=\frac{\sqrt{6}+\sqrt{2}}{4}+\frac{2\sqrt{3}-\sqrt{2}-\sqrt{6}}{4}I$$

$$sin\left(\frac{\pi}{12}+\frac{\pi}{12}I\right)=\frac{\sqrt{6}-\sqrt{2}}{4}+\frac{\sqrt{2}+2-\sqrt{6}}{4}I$$

3.2 The roots from order n of a neutrosophic complex number

The roots from order n of a complex number y are the set of complex numbers that satisfy:

$$z^n = z_0$$

let z be one of these roots whear:

$$z = \acute{r}e^{i(\acute{\theta}+\acute{\vartheta}I)}$$

then:

$$\acute{r}^n e^{in(\acute{\theta}+\acute{\vartheta}I)} = \acute{r}_0 e^{i(\acute{\theta}_0+\acute{\vartheta}_0 I)}$$

$$\acute{r}^n = \acute{r}_0 \qquad\qquad \Longrightarrow \qquad\qquad \acute{r} = \sqrt[n]{\acute{r}_0}$$

there is $k \in \mathbb{Z}$, where:

$$n\left(\acute{\theta}+\acute{\vartheta}I\right) = \acute{\theta}_0 + \acute{\vartheta}_0 I + 2\pi k$$

$$\acute{\theta}+\acute{\vartheta}I = \frac{\acute{\theta}_0}{n}+\frac{\acute{\vartheta}_0 I}{n}+\frac{2\pi k}{n}$$

so the general formula for the roots is:

$$z_k = \sqrt[n]{\acute{r}_0}\ \ e^{i\left(\frac{\acute{\theta}_0}{n}+\frac{\acute{\vartheta}_0 I}{n}+\frac{2\pi k}{n}\right)} \qquad\qquad ;k = 0\ \ ,1\ \ ,2\ \ ,\ldots\ \ ,n-1$$

Example 6

Find the cube roots of:

$$z = \frac{1}{\sqrt{2}}-\frac{2}{\sqrt{2}}I+\frac{1}{\sqrt{2}}i$$

Solution:

$$\acute{r} = \sqrt{(\ddot{a}+\ddot{c}I)^2+\left(\ddot{b}+\ddot{d}I\right)^2}$$

$$= \sqrt{\left(\frac{1}{\sqrt{2}}-\frac{2}{\sqrt{2}}I\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \sqrt{\left(\frac{1}{2}-2\frac{1}{\sqrt{2}}\frac{2}{\sqrt{2}}I+2I\right)+\frac{1}{2}} = 1$$

then:

$$cos\left(\acute{\theta}+\acute{\vartheta}I\right) = \frac{\ddot{a}+\ddot{c}I}{\acute{r}} = \frac{1}{\sqrt{2}}-\frac{2}{\sqrt{2}}I$$

$$\sin(\acute{\theta}+\acute{\vartheta}I)=\frac{\ddot{b}+\ddot{d}I}{\acute{r}}=\frac{1}{\sqrt{2}}$$

$$\Longrightarrow \quad \acute{\theta}+\acute{\vartheta}I=\frac{\pi}{4}+\frac{\pi}{2}I$$

hence:

$$z=e^{i\left(\frac{\pi}{4}+\frac{\pi}{2}I\right)}$$

$$z_k=\sqrt[n]{\acute{r}_0}\ e^{i\left(\frac{\acute{\theta}_0}{n}+\frac{\acute{\vartheta}_0 I}{n}+\frac{2\pi k}{n}\right)} \qquad ;k=0\ ,1\ ,2\ ,\dots\ ,n-1$$

$$z_k=\sqrt[3]{1}\ e^{i\left(\frac{\pi}{12}+\frac{\pi}{6}I+\frac{2\pi k}{3}\right)} \quad ;\quad k=0\ ,1\ ,2$$

$$k=0 \quad \Longrightarrow \quad z_0=\ e^{i\left(\frac{\pi}{12}+\frac{\pi}{6}I\right)}=\cos\left(\frac{\pi}{12}+\frac{\pi}{6}I\right)+i\ \sin\left(\frac{\pi}{12}+\frac{\pi}{6}I\right)$$

z_0

$$=\frac{\sqrt{6}+\sqrt{2}}{4}+\frac{\sqrt{2}-\sqrt{6}}{4}I+\left(\frac{\sqrt{6}-\sqrt{2}}{4}+\frac{3\sqrt{2}-\sqrt{6}}{4}I\right)i$$

$$k=1 \quad \Longrightarrow \quad z_1=\ e^{i\left(\frac{3\pi}{4}+\frac{\pi}{6}I\right)}=\cos\left(\frac{3\pi}{4}+\frac{\pi}{6}I\right)+i\ \sin\left(\frac{3\pi}{4}+\frac{\pi}{6}I\right)$$

$$=\cos\frac{3\pi}{4}+I\left[\cos\left(\frac{3\pi}{4}+\frac{\pi}{6}\right)-\cos\frac{3\pi}{4}\right]+i\ \left(\sin\frac{3\pi}{4}+I\left[\sin\left(\frac{3\pi}{4}+\frac{\pi}{6}\right)-\sin\frac{3\pi}{4}\right]\right)$$

$$=\cos\frac{3\pi}{4}+I\left[\cos\frac{11\pi}{12}-\cos\frac{3\pi}{4}\right]+i\ \left(\sin\frac{3\pi}{4}+I\left[\sin\frac{11\pi}{12}-\sin\frac{3\pi}{4}\right]\right)$$

$$=\frac{-\sqrt{2}}{2}+I\left[\frac{-\sqrt{6}-\sqrt{2}}{4}+\frac{\sqrt{2}}{2}\right]+\left(\frac{\sqrt{2}}{2}+I\left[\frac{\sqrt{6}-\sqrt{2}}{4}-\frac{\sqrt{2}}{2}\right]\right)i$$

$$z_1=\frac{-\sqrt{2}}{2}+\frac{-\sqrt{6}+\sqrt{2}}{4}I+\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{6}-3\sqrt{2}}{4}I\right)i$$

$$k=2 \quad \Longrightarrow \quad z_2=\ e^{i\left(\frac{17\pi}{12}+\frac{\pi}{6}I\right)}=\cos\left(\frac{17\pi}{12}+\frac{\pi}{6}I\right)+i\ \sin\left(\frac{17\pi}{12}+\frac{\pi}{6}I\right)$$

$$=\cos\frac{17\pi}{12}+I\left[\cos\left(\frac{17\pi}{12}+\frac{\pi}{6}\right)-\cos\frac{17\pi}{12}\right]+i\ \left(\sin\frac{17\pi}{12}+I\left[\sin\left(\frac{17\pi}{12}+\frac{\pi}{6}\right)-\sin\frac{17\pi}{12}\right]\right)$$

$$= cos\frac{17\pi}{12} + I\left[\cos\frac{19\pi}{12} - cos\frac{17\pi}{12}\right] + i \quad \left(sin\frac{17\pi}{12} + I\left[sin\frac{19\pi}{12} - sin\frac{17\pi}{12}\right]\right)$$

$$= \frac{-\sqrt{6}+\sqrt{2}}{4} + I\left[\frac{\sqrt{6}-\sqrt{2}}{4} - \left(\frac{-\sqrt{6}+\sqrt{2}}{4}\right)\right] + \left(\frac{-\sqrt{6}-\sqrt{2}}{4} + I\left[\frac{-\sqrt{6}-\sqrt{2}}{4} - \left(\frac{-\sqrt{6}-\sqrt{2}}{4}\right)\right]\right)i$$

$$z_2 = \frac{-\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}-\sqrt{2}}{2}I + \frac{-\sqrt{6}-\sqrt{2}}{4}i$$

4. Conclusions

The importance of this paper comes from the fact that it presented a scientific method for converting from the algebraic form of a neutrosophic complex number to the exponential or trigonometric form and vice versa. Where this method was harnessed to find the roots from order n of a neutrosophic complex number and write its by algebraic form of a neutrosophic complex number. This article is regarded as one of the key studies on neutrosophic complex numbers.

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Modelling the progression of Alzheimer's disease using Neutrosophic hidden Markov models

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Abstract
Alzheimer's disease is the primary cause of dementia. Due to the sluggish rate of progression of Alzheimer's disease, individuals have the opportunity to start receiving therapy early through routine testing. procedures since they are pricy and difficult to find. For many slowly advancing disorders, such as Alzheimer's disease (AD), the capacity to recognise changes in disease progression is essential. Machine learning methods with a high degree of modularity were used throughout the pipeline. We propose the use of Neutrosophic hidden Markov models (NHMMs) to simulate disease progression in a more thorough manner than the clinical phases of the disease. Due to the complexity and ambiguity of reality, decision-makers find it challenging to draw conclusions from precise data. Since they cannot be computed directly, the variables are encoded using a single interval Neutrosophic set. We showed that the trained HMM can imitate sickness development more accurately than the commonly acknowledged clinical phases.

Keywords: Alzheimer disease, Neutrosophic, hidden Markov model, Decision making ,Brain disorders.

1. Introduction

The retina, on the other hand, offers a simple window for obtaining possible biomarkers of Alzheimer's disease. Only the retina may be directly viewed in vivo as a part of the nervous system. One of these retinal indications that is usually linked to Alzheimer's disease is retinal vasculature. It is possible that Alzheimer's disease patients' aberrant narrowing of the retinal venous blood column diameter and decreased blood flow accounts for the subjects with mild cognitive impairment's observed reduced retinal oxygen metabolism rate. A fundus vascular network that is sparse and has reduced fractal dimensions is highly linked to dementia. It is also found that Alzheimer's patients have deteriorated retinal neurovascular coupling when compared to healthy ageing. Based on recently published studies, Yet, two prevalent issues have hindered earlier retinal imaging investigations. Recent investigations show that abnormal characteristics associated with the early stages of neurodegenerative disorders are visible in retinal fundus pictures. More than 5% of people over 65 have Alzheimer's disease (AD), a long-term neurological disease that is the most common cause of dementia. An important and early step in the pathogenic cascade of Alzheimer's disease is the build up of multimers of the misfolded beta amyloid (Aß) peptide (4kDa peptides). The crucial function of the Aß peptide in the genesis of AD has been further reinforced by genetic discoveries during the past ten years. According to statistics, between 60 and 70 percent of dementia cases are caused by Alzheimer's disease (AD). It is still a neurodegenerative phenotype that is progressing[1]. People who are affected show a considerable deterioration in cognitive function, which has a negative impact on their quality of life and general health[2] .Contrary to other neurological disorders, managing AD has also shown to be associated with a significant financial burden in Asia, North America6, and even globally. Additionally, it has been reported that by the year 2030, the estimated global total cost of dementia-related expenses will be close to USD 2.0 trillion[3]. For many slowly progressing conditions, including Alzheimer's disease, a disease progression model that can be used to swiftly assess disease progression or lack thereof can help in the development of prospective treatments. This implies the model should be able to detect more specific disease phases as opposed to illness stages that match to clinical diagnoses. Modeling and prediction of disease progression has been the focus of several research projects [4-6]. whereas in [5], the authors put out a non-linear model based on the progression of the scores on the Alzheimer's disease Assessment Scale, however, has been underlined by recent study[8]. According to one study, 31% of patients with MCI return to a cognitively normal state within two years[9]. The proper targeting of treatment drugs can be aided by an understanding of the characteristics of MCI patients who are more likely to develop AD and those who are more likely to return to normal cognition. Since it is believed that disease-modifying therapies may work better in people who have not yet developed AD and have not yet seen neuronal loss, it is especially crucial to identify risk factors for conversion to MCI. The objective in this case is to estimate the link between patient-level characteristics and the rate of conversion from MCI to AD and reversion from MCI to normal cognition. Machine learning offers a technique for automatic classification by identifying complex and nuanced patterns from highly dimensional data. In AD research, such algorithms are frequently developed to do automatic diagnosis and predict a person's future clinical status based on biomarkers. These algorithms aim to enhance medical decision-making by providing a possibly more objective diagnosis than that offered by common clinical criteria[10]. There is a wealth of information on the classification of AD and its prodromal stage, moderate cognitive impairment (MCI)[11,12]. Indicating strong performance for the classification of AD patients and control individuals, the area under the receiver operating characteristic curve (AUC) for classification approaches varies from 85 to 98 percent.

Following the trends and advancements in medical image analysis and machine learning, convolutional neural networks (CNN) in particular have seen increased application as neural network classifiers over the past few years[13]. In studies on AD, many researchers have discovered shrinking of grey matter in the brain's temporal lobe and hippocampal areas. The easiest way to define Random Forest (RF), an ensemble machine learning technique, is as "a collection of tree predictors such that each tree depends on the values of a random vector generated independently and with the same distribution for all trees in the forest"[14]. Regarding the handling of extremely non-linear biological data, tolerance to noise, tuning simplicity (relative to other ensemble learning algorithms), and the potential for efficient parallel processing, this algorithm offers a considerable edge over other approaches. In many applications, it also yields one of the finest accuracies to date [15]. Due of the numerous redundant features in high-dimensional situations, RF is a perfect contender for managing these issues. Although RF is an efficient feature selection technique in and of itself, other methods for feature set reduction both inside and outside of RF have been presented to further enhance its performance[16]. Researchers are particularly interested in finding risk factors for changing disease stages in multistate disease processes like Alzheimer's disease (AD). In this case, three kinds of cognitive function can be distinguished: normal cognition, moderate cognitive impairment (MCI), and Alzheimer's disease (AD). In AD clinical trials, MCI has been identified as a significant transitory illness. Markov process models are ideal for application in research on AD because they evaluate the rate of transition between distinct disease states while taking into account the potential reversibility of some stages, such MCI, and the competing danger of mortality. The competing risk of death has been largely disregarded when determining the rate of progression to MCI and AD, and earlier research have demonstrated how biassed this is [17]. The observational aspect of AD studies, where cognitive status is frequently tested at regular clinic visits, making interval censored data, makes Markov process models the most suitable choice. Additionally, if patients switch between different illness states in between clinic appointments, the full disease history won't be provided. The failure to see the entire course of the disease as well as interval censoring can both be easily accommodated by discrete time Markov process models. In studies of disease, it is frequently observed that the rate of transition between states rises with age. Markov models, which are characterised by time-varying transition rates, can be used to estimate disease processes[18]. Hidden Markov models (HMMs) are well adapted for the task of describing longitudinal data as a set of hidden states. The impact of state-specific factors on responses are examined using a conditional regression model. A transition model to depict the dynamic change of hidden states makes up the second part of HMMs. Due to their ability to simultaneously display the longitudinal association structure and dynamic variability of the observed process, HMMs and its variants have attracted significant interest from the medical, behavioural, and academic fields[19–21]. As a result, neutrosophic techniques prove to be superior in the medical industry. Neutral hidden Markov models are a new field of innovation for uncertainty. The uncertainty information cannot be taken into account by the hidden Markov models now in use [22]. [23] discussed on Plithogenic sets in decision making.[24] ,this paper to introduced a new measure on nutrosophic centroid .[25]in this paper explained a Review on biomedicine .[26] explained an Markov chain in forecasting for stock trend. The structure of this paper is organized as follows.

We suggest modelling illness progression in a more detailed manner than the clinical stages of the disease using Neutrosophic hidden Markov models (NHMMs). Due to the complexity and ambiguity of reality, decision-makers find it challenging to draw conclusions from precise data. Since they cannot be computed directly, the variables are encoded using a single interval Neutrosophic set. In an actual circumstance, Neutrosophic sets are used to deal with indeterminacy. We train our NHMM in an unsupervised manner, in contrast to many existing uses of Neutrosophic Hidden Markov Models, and then assess the model's efficiency in revealing underlying statistical trends in disease development by referring to NHMM states as disease stages. In this study, we concentrate on AD and demonstrate that our model can identify more detailed disease phases than the three officially recognised clinical stages of "Normal," "MCI" (Mild Cognitive Impairment), and "AD" when tested on the cross validation data.

Consider the space X, which is composed of universal components that exhibit x. The structure of the neutrosophic set is

$N = \{(T_N(x), I_N(x), F_N(x) \mid x \in X\}$, where the 3 grades of memberships are from X to of the element $x \in X$ to the set X, with the criterion:

2. $$^{-}0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3^{+}.$$

The functions $T_N(x)$, $I_N(x)$ and $F_N(x)$ are the truth, indeterminate and falsity grades lie in real standard/non-standard subsets of $]^{-}0,1^{+}[$.

3. Operations on Interval-Valued Neutrosophic Numbers [5, 33]

Let $N_1 = \langle [T_1^L, T_1^U], [I_1^L, I_1^U], [F_1^L, F_1^U] \rangle$ and $N_2 = \langle [T_2^L, T_2^U], [I_2^L, I_2^U], [F_2^L, F_2^U] \rangle$ be two interval neutrosophic numbers then

Addition:

$$N_1 \oplus N_2 = \langle [T_1^L + T_2^L - T_1^L T_2^L, T_1^U + T_2^U - T_1^U T_2^U], [I_1^L I_2^L, I_1^U I_2^U], [F_1^L F_2^L, F_1^U F_2^U] \rangle$$

Multiplication:

$$N_1 \otimes N_2 = \langle [T_1^L T_2^L, T_1^U T_2^U], [I_1^L + I_2^L - I_1^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U], [F_1^L + F_2^L - F_1^L F_2^L, F_1^U + F_2^U - F_1^U F_2^U] \rangle$$

Multiplication Neutrosophic probability:

$$(x_1, y_1, z_1).(x_2, y_2, z_2) = (x_1 x_2, Min\{y_1 y_2\}, Max\{z_1 z_2\})$$

Addition Neutrosophic probability:

$$(x_1, y_1, z_1).(x_2, y_2, z_2) = (x_1 + x_2, Min\{y_1 y_2\}, Min\{z_1 z_2\})$$

2.8 Interval Neutrosophic Markov Chain [25]

An interval neutrosophic stochastic process $\{X(n): n \in \mathrm{N}\}$ is said to be an interval neutrosophic Markov chain if it satisfies the Markov property:

$$\beta(X_{n+1} = j \mid X_{n-1} = i, X_n = k, \ldots, X_0 = m) = \beta(X_{n+1} = j \mid X_{n-1} = i)$$

where i, j, k establish the state space S of the process.

Here $\tilde{P}_{ij} = \beta(X_{n+1} = j \mid X_n = i)$ are called the interval-valued neutrosophic probabilities of moving from state i to state j in one step. Hence $\tilde{P}_{ij} = \left(\left[T_{\tilde{P}_{ij}}^L, T_{\tilde{P}_{ij}}^U\right], \left[I_{\tilde{P}_{ij}}^L, I_{\tilde{P}_{ij}}^U\right], \left[F_{\tilde{P}_{ij}}^L, F_{\tilde{P}_{ij}}^U\right]\right)$, where $T_{\tilde{P}_{ij}}^L, T_{\tilde{P}_{ij}}^U$ are the lower and upper truth membership of the transition from state i to state j, respectively, $I_{\tilde{P}_{ij}}^L, I_{\tilde{P}_{ij}}^U$ are the lower and upper indeterminate membership of the transition from state i to state j respectively and $F_{\tilde{P}_{ij}}^L, F_{\tilde{P}_{ij}}^U$ are the lower and upper falsity membership of the transition from state i to state j. The matrix $P = (\tilde{P}_{ij})$ is called the interval-valued neutrosophic transition probability matrix.

The connections between the states in a hidden markov model are controlled by a set of transitional probabilities. Since a Markov Chain's states could be observed, a Hidden Markov Model's states are statistical and have associated probability distributions known as observation probability density functions. This makes it possible to distinguish a Hidden Markov Model from a Markov Model or a Markov Chain. The multidimensional vector that encodes the observation is typically created using the HMM feature vector, which is a collection of characteristics. Continuous and discrete observation density functions are the two different types.

In this study, we take a slightly different strategy to training and testing the HMM. Since our objective is to identify and model illness stages, the HMM training is conducted in an unsupervised manner using the time signals from all participants, regardless of their clinical diagnosis at any given time.

The goal is to use the HMM's training strategy to take advantage of patterns in the biomarker feature vector that are present both temporally and across individual biomarker features in order to group subjects with similar conditions into one state and those with dissimilar conditions into different states.

A1:Memory loss and forgetfulness: Alzheimer's disease, a type of dementia, affects one's behaviour, thinking, and memory. Alzheimer's disease is characterised by memory loss, particularly of recent memories, which gets worse over time. Along with this, other cognitive symptoms are commonly noticeable, such as difficulties with language, problem-solving, and visual-spatial abilities.

A2: Challenges with language and communication: Alzheimer's disease is a type of dementia that affects thinking, behaviour, and memory. Alzheimer's disease is characterised by memory loss, particularly of recent memories, which gets worse over time. Along with this, other cognitive symptoms are commonly noticeable, such as difficulties with language, problem-solving, and visual-spatial abilities. When trying to communicate, a person with Alzheimer's may have problems finding the right words. Following a discussion may be challenging for someone with Alzheimer's disease, particularly if the conversation is complex or fast-paced. They keep doing it: A person suffering from Alzheimer's disease may repeat themselves or ask the same question. Alzheimer's disease can impact a person's ability to modulate their tone and inflection, which could cause them to speak in a different way. A person with Alzheimer's may have trouble communicating their needs and wants because they find it difficult to articulate their thoughts and ideas in a straightforward manner.

A3: problems with decision-making and problem-solving due to disorientation and confusion disorientation, and difficulties with problem-solving and decision-making are common symptoms of Alzheimer's disease. The ability to think, reason, and make decisions may be compromised as the condition worsens. Some typical issues associated with Alzheimer's disease include the following: Confusion and disorientation: These signs and symptoms might appear in someone with Alzheimer's disease, especially in unfamiliar or unexpected settings. Problem-solving challenges: Alzheimer's disease can impair a person's ability to think critically and solve problems. They could struggle to find solutions to simple or complex problems.

$$\pi = [< [0.05\ 0.1\ \ 0.05] >, < [0.3 \quad 0.2 \quad 0.1] >]$$

It demonstrates that Alzhimer is [0.3 0.2 0.1] and that the initial single value neutrosophic of Normal is [0.05 0.1 0.05]. It demonstrates that the value of obesity is greater than the single-valued neutrosophic value of overweight. The state's probability transition diagram. The state N is normal and A is Alzheimer diseases.

Single-valued Neutrosophic transition probability value is

$$\begin{matrix} N \\ A \end{matrix}\begin{bmatrix} N & A \\ \langle[0.6,0.1,0.1]\rangle & \langle[0.4,0.05,0.05]\rangle \\ \langle[0.2,0.05,0.1]\rangle & \langle[0.6,0.1\ ,0]\rangle \end{bmatrix}$$

The observer's state is represented by 1, which stands for memory loss and forgetfulness, , which stands for difficulties with language and communication, and 3, which stands for difficulties with decision-making and problem-solving. The state's emission likelihood is

$$\begin{matrix} N \\ A \end{matrix}\begin{bmatrix} 1 & 2 & 3 \\ \langle[0.2,0.1\ ,0]\rangle & \langle[0.3,0.1,0.05]\rangle & \langle[0.1,0.05\ ,0.1]\rangle \\ \langle[0.7,0.2\ ,0.1]\rangle & \langle[0.6,0.05,0]\rangle & \langle[0.8,0.05\ ,0]\rangle \end{bmatrix}$$

There are two concealed states, and there are three observations. Choose a sequence 132 with the probability value as follows: The majority of participants depend on 1, which stands for memory loss and forgetfulness, 3, which represents for issues with decision-making and problem-solving, and 2, which stands for difficulties with language and communication.

$$P(O,Q)) = \prod P\left(\frac{O}{Q}\right)P(Q)$$

4. *Single-valued Neutrosophic Hidden Markov Model*

$$P(132,NNN) = P(1/N)P(3/N)P(2/N)P(N)P(N/N)P(N/N) = [\langle 0.000216,0.05,0.5\rangle]$$

$$P(132,NNA) = P(1/N)P(3/N)P(2/A)P(N)P(N/N)P(A/N) = [\langle 0.00144,0.05,0.1\rangle]$$

$$P(132,\text{NAN}) = P(1/N)P(3/A)P(2/N)P(N)P(A/N)P(N/A) = [\langle 0.0023,0.05,0.1\rangle]$$

$$P(132,\text{ANN}) = P(1/A)P(3/N)P(2/N)P(A)P(N/A)P(N/N) = [\langle 0.00075,0.05,0.1\rangle]$$

$$P(132,\text{AAA}) = P(1/\text{A})P(3/A)P(2/A)P(A)P\left(\frac{A}{A}\right)P\left(\frac{A}{A}\right) = [\langle 0.03628,0.05,0.1\rangle]$$

$$P(132,\text{AAN}) = P(1/A)P(3/A)P(2/N)P(A)P(A/N)P(N/A) = [\langle 0.0040.05,0.1\rangle]$$

$$P(132,\text{NAN}) = P(1/A)P(3/N)P(2/A)P(A)P(N/A)P\left(\frac{A}{N}\right) = [\langle 0.000504,0.05,0.1\rangle]$$

$$P(132,NAA) = P(1/\text{N})P(3/A)P(2/A)P(N)P(A/N)P(A/A) = [\langle 0.01151,0.05,0.05\rangle]$$

The greatest likelihood of the above values is [0.036288, 0.05, 0.1] for the probability value of the sequence 132, and the maximum probability of the combination is P(132,AAA).

Find the likelihood of any interval combination in a similar manner. Viterbi algorithm verification of this probability of sequence 132.

5. *Calculation for Single-valued Neutrosophic Hidden Markov Model*

$$P(1,N) = P(1/N)P(N) = [\langle 0.1,0.1,0.05\rangle]$$
$$P(1,A) = P(1/A)P(A) = [\langle 0.21,0.2,0.1\rangle]$$
$$P\left(3,\frac{N}{N}\right) = P(3/N)P\left(\frac{N}{N}\right) = [\langle 0.06,0.05,0.1\rangle]$$
$$P\left(3,\frac{N}{C}\right) = P(3/N)P\left(\frac{N}{A}\right) = [\langle 0.02,0.05,0.1\rangle]$$
$$P\left(3,\frac{A}{N}\right) = P(3/A)P\left(\frac{A}{N}\right) = [\langle 0.48,0.05,0.05\rangle]$$
$$P\left(3,\frac{A}{A}\right) = P(3/A)P\left(\frac{A}{A}\right) = [\langle 0.48,0.05,0\rangle]$$
$$P\left(2,\frac{N}{N}\right) = P(2/N)P\left(\frac{N}{N}\right) = [\langle 0.18,0.1,0.1\rangle]$$
$$P\left(2,\frac{A}{N}\right) = P(2/A)P\left(\frac{A}{N}\right) = [\langle 0.24,0.05,0.05\rangle]$$
$$P\left(2,\frac{N}{A}\right) = P(2/\mathrm{N})P\left(\frac{N}{A}\right) = [\langle 0.06,0.05,0.1\rangle]$$
$$P\left(2,\frac{A}{A}\right) = P(2/A)P\left(\frac{A}{A}\right) = [\langle 0.36,0.05,0\rangle]$$
$$V2 = [\langle 0.0042,0.1008\rangle]$$
$$V3 = [\langle 0.0064,0.0217\rangle]$$

The Viterbi algorithm is used to validate the probability. It demonstrates that sequence 132 has a 0.21 probability. The combination has a P(132,AAA) maximum probability, and the path is A_A_A. It demonstrates that the Alzheimer disease's disease sequence path.

6. Conclusion

Uncertainty, ambiguity, and indeterminacy are common factors in real-world decision-making issues, and Neutrosophic devotes a lot of effort to fixing them. The neutrosophic Hidden Markov model (NHMM) has been used as a key mathematical mode for ambiguity, redundancy, and uncertainty. Indeterminacy is formally quantified by NHMM. Truth, ambiguity, and falsity exist independently. These characteristics are important for the biological diagnosis of the condition. In the medical industry, the decision is built using NHMM. Three components serve as the original representation of NHMM probability in the proposed framework, and three memberships carry out the transformation, which was developed to address the issue of Alzhemer. On the foundation of a Hidden Markov Model framework, we provided a model for illness progression. We trained an HMM in an unsupervised manner for Alzheimer's disease in an effort to identify more precise stages in disease development. We demonstrated that the trained HMM can more precisely simulate illness development than the conventionally recognised clinical phases.

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Neutrosophic Totally Semi-Continuous Functions

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Abstract: The principal objective of this article is to develop the perception of neutrosophic totally continuous and neutrosophic totally-semi-continuous functions in neutrosophic topological space using neutrosophic semi-open and neutrosophic semi-closed sets. Neutrosophic strongly semi-continuous and slightly semi-continuous functions have been presented and investigated. Some important properties of these functions are also given in neutrosophic topological spaces. Relations between these newly defined functions and other classes of neutrosophic functions are established.

Keywords: Neutrosophic semi-open set, Neutrosophic semi-closed set, Neutrosophic clopen, Neutrosophic semi-clopen

1. Introduction

The contribution of mathematics to present-day Technology in researching fast trends must be addressed. The theories presented differently from classical methods in studies by Zadeh[19]; a fuzzy set was investigated as a mathematical tool for handling uncertainties, with each element having a degree of membership, truth(t), an intuitionistic fuzzy introduced by Atanassov [2] utilizing falsehood (f), the degree of non-membership, Neutrosophic Set in which Smarandache [14,15,16] presented Neutrality (i), the degree of indeterminacy, as an independent concept have great importance in this contribution of mathematics in recent years.

The neutrosophic set is concerned with the origin, nature, and scope of neutrality. Research on the neutrosophic set is crucial since it provides access to a variety of scientific and technical applications. The universe is full of uncertainty, therefore the neutrosophic set can locate a suitable location for investigation. The neutrosophic set on the real line is distinguished by single-valued neutrosophic numbers (SVN) and single-valued bipolar neutrosophic numbers (SVbN). When a human decision is based on positive and negative ideas, the SVbN number is crucial in decision-making problems. All SVbN number membership functions (truth, indeterminacy, and falsity) are contained in the positive and negative sections between -1 and 0 and 0 and 1, respectively. Using these concepts, the authors of [6, 7, 8, 9, 10] have created applications in the health field, such as multi-criteria decision-making in COVID-19 vaccines and the possibility of multiattribute decision-making on water resource management challenges, in recent years. The neutrosophic crisp set was converted by Salama et al. [12, 13] into neutrosophic topological spaces. In [13], Salama et al. presented continuous functions and neutrosophic closed sets. Generalized neutrosophic closed sets were provided in [4]. [1] Presents a set of neutrosophic semi-open, pre-open, and semipre-open sets. The authors explored the concept of neutrosophic almost -contra-continuous functions in [5].. With its potential application in practical settings, neutrosophic topology has now become a fertile field for study.

The review of the literature reveals that there is still work to be done on the qualities of neutrosophic totally and totally semi-continuous functions. By observing these, we are directed to work in continuous and semi-continuous functions based on the neutrosophic set. In addition, neutrosophic strongly semi-continuous and slightly semi-continuous functions are introduced, and these continuous functions are characterized.

For the sake of presentation clarity, abbreviations are used throughout this article.

Abbreviations are listed here.

Array of Words	Shortenings
Neutrosophic Topology	NT
Neutrosophic Topological Space	NTS
Neutrosophic Set	NS
Neutrosophic Sets	NSs
Neutrosophic Open Set	NOS
Neutrosophic Closed Set	NCS
Neutrosophic Point	NP
Neutrosophic Semi-Open Set	NSOS
Neutrosophic Semi-Closed Set	NSCS
Neutrosophic Semi-Open Sets	NSOSs

2. Methodologies

Definition: 2.1[14,15]: A NS Ψ over a finite non-empty set S_1 has the form $\Psi = \{t, \ T_\Psi(t), I_\Psi \ (t), \ F_\Psi(t): t\epsilon S_1\}$, wherein $T, I, F: \ S_1 \longrightarrow]0^-, 1^+[$ are the truth, indeterminacy, besides false membership functions, in that order.

Definition: 2.2[14,15]: Let $S_1 \neq \emptyset$ and the NSs Ψ and Ω be defined as

$\Psi = \{\langle d, \mu_\Psi(\mathrm{d}), \sigma_\Psi(\mathrm{d}), \Omega_\Psi(d)\rangle : \mathrm{d} \in S_1\}$, $\Omega = \{\langle d, \mu_\Omega(\mathrm{d}), \sigma_\Omega(\mathrm{d}), \Omega_\Omega(\mathrm{d})\rangle : x \in S_1\}$. Then

I. $\Psi \subseteq \Omega$ *iff* $\mu_\Psi(d) \leq \mu_\Omega(d)$, $\sigma_\Psi(d) \leq \sigma_\Omega(d)$ *and* $\Omega_\Psi(d) \geq \Omega_\Omega(d)$ *for all* $d \in S_1$;

II. $\Psi = \Gamma$ *iff* $\Psi \subseteq \Omega$ *and* $\Omega \subseteq \Psi$;

III. $\overline{\Psi} = \{\langle x, \Gamma_\Psi(d), \sigma_\Psi(d), \mu_\Psi(d)\rangle : d \in S_1\}$; *[Complement of* Ψ*]*

IV. $\Psi \cap \Omega = \{\langle d, \mu_\Psi(d) \wedge \mu_\Gamma(d), \sigma_\Psi(d) \wedge \sigma_\Omega(d), \Gamma_\Psi(d) \vee \Omega_\Omega(d)\rangle : d \in S_1\}$;

V. $\Psi \cup \Omega = \{\langle d, \mu_\Psi(d) \vee \mu_\Omega(d), \sigma_\Psi(d) \vee \sigma_\Omega(d), \Gamma_\Psi(d) \wedge \gamma_\Omega(d)\rangle : d \in S_1\}$;

VI. $[\;]\;\Psi = \{\langle d, \mu_\Psi(d), \sigma_\Psi(d), 1 - \mu_\Psi(d)\rangle : d \in S_1\}$;

VII. $\langle\rangle\;\Psi = \{\langle d, 1 - \Gamma_\Psi(d), \sigma_\Psi(d), \Gamma_\Psi(d)\rangle : d \in S_1\}$.

The primary goal is to create the resources for NTS development, so we make the NSs $0_\aleph$ along with $1_\aleph$ in S_1 as follows:

Definition: 2.3[15,16]: $0_\aleph = \{\langle \omega, 0,0,1\rangle : \omega \in S_1\}$ along with $1_\aleph = \{\langle \omega, 1,1,0\rangle : \omega \in S_1\}$.

Definition: 2.4[12]: A NT $S_1 \neq \emptyset$ is a family ξ_1 of NSs in S_1 observing the rules listed below:

I. $0_\aleph, 1_\aleph \in \xi_1$,

II. $W_1 \cap W_2 \in T$ *being* $W_1, W_2 \in \xi_1$,

III. $\cup\, W_i \in \xi_1$ *for random family* $\{W_i | i \in \Psi\} \subseteq \xi_1$.

Here (S_1, ξ_1) or just S_1 is labeled as NTS, with each NS in ξ_1 being noted as NOS. The complement $\overline{\Omega}$ of a NOS Ω in S_1 is noted as NCS in S_1.

Definition: 2.5[12, 13]: Consider an NS Ω in NTS S_1. Accordingly,
$\aleph int(\Omega) = \cup\{E | E$ is NOS in S_1 with $E \subseteq \Omega\}$ is titled as neutrosophic interior ($\aleph int$ in short) of Ω;
$\aleph cl(\Omega) = \cap\{\mathfrak{B} | \mathfrak{B}$ is a NCS in S_1 with $\mathfrak{B} \supseteq \Omega\}$ is entitled as neutrosophic closure (in short $\aleph cl$) of Ω.

Definition: 2.6[1]: For r, t, s the real standard else non-standard members of $]0^-, 1^+[$. A NS $\mho_{r,t,s}$ is termed a neutrosophic point (NP) over S_1 defined by $\mho_{r,t,s}\,(\mathrm{q}) = \begin{cases} (r,t,s), & if \quad \mho = \mathrm{q} \\ (0,0,1) & if \quad \mho \neq \mathrm{q} \end{cases}$ for q $\in S_1$, where r, t, s are the truth, indeterminacy, and falsity membership values of $\mho$.

Definition 2.7[1]: Consider a $\aleph$S Ψ in an NTS (S_1, Γ), neutrosophic semi-open set (NSOS) whenever $\Psi \subseteq \aleph cl(\aleph int(\Psi))$. The complement of NSOS is known as NSCS.

Definition 2.8[11]: A space S_1 is mentioned as neutrosophic semi-connected if S_1 cannot be expressed as the union of two non-empty disjoint NSOSs.

Definition 2.9[3]: Conder a mapping $\eta: S_1 \rightarrow S_2$ is neutrosophic semi-continuous whenever $\eta^{-1}(\Omega)$ is NSOS of S_1 for each $\Omega \epsilon S_2$

3. Results and Discussion.

This section introduces neutrosophic totally continuous, neutrosophic totally semi-continuous, and neutrosophic strongly semi-continuous functions, and their characterizations and interrelationships are discussed.

Definition 3.1: A neutrosophic function $\eta: S_1 \rightarrow S_2$ is termed neutrosophic totally continuous (in short (Nt-c.) (resp. neutrosophic totally-semi-continuous in short (Nts-c.) whenever the inverse image of every NOS (resp. NSOS) of S_2 is a neutrosophic clopen (NOS and NCS) (resp. neutrosophic semi-clopen) subset of S_1

Remark 3.2: Every Nt-c function is also a Nts-c. But the opposite doesn't have to be true.

Example 3.3: Let $S_1 = S_2 = \{a, b\}$ we define a mapping $\eta: (S_1, \tau) \rightarrow (S_2, \sigma)$ by $\eta^{-1}(C) = C, \eta^{-1}(D) = D, \eta^{-1}(E) = E, \eta^{-1}(0_\aleph) = 0_\aleph, \eta^{-1}(1_\aleph) = 1_\aleph$ where $\tau = \{A, B, C, D, 0_\aleph, 1_\aleph\}$ and $\sigma = \{C, D, E, 0_\aleph, 1_\aleph\}$. Here $A = \{\langle 0.7,0.3,0.8\rangle\langle 0.5,0.8,0.9\rangle\}, B = \{\langle 0.8,0.7,0.7\rangle\langle 0.9,0.2,0.5\rangle\}, C = \{\langle 0.8,0.3,0.7\rangle\langle 0.9,0.2,0.5\rangle\}$, $D = \{\langle 0.7,0.3,0.8\rangle\langle 0.5,0.2,0.9\rangle\}$, $E = \{\langle 0.7,0.7,0.8\rangle\langle 0.5,0.8,0.9\rangle\}$. Here $C, D, and\ E$ are neutrosophic semi-clopen in S_2, whereas E is not neutrosophic semi-clopen in S_1. Hence each Nts-c need not be Nt-c.

Definition 3.4: A neutrosophic function $\eta: S_1 \rightarrow S_2$ is known as the neutrosophic strongly semi-continuous (in brief, Nss-c) iff inverse image of each neutrosophic subset of S_2 is a neutrosophic semi-clopen subset of S_1

Remark 3.5: Neutrosophic strongly semi-continuity → Neutrosophic totally-semi-continuity → Neutrosophic semi-continuity

Example 3.6: The example that follows is not Nss-c, but rather Nts-c functoin.

Let $S_1 = S_2 = \{a, b\}$ define a map $\eta: (S_1, \tau) \rightarrow (S_2, \sigma)$ by $\eta^{-1}(C) = C$, $\eta^{-1}(D) = D$, $\eta^{-1}(0_\aleph) = 0_\aleph$, $\eta^{-1}(1_\aleph) = 1_\aleph$ where $\tau = \{A, B, C, D, 0_\aleph, 1_\aleph\}$ and $\sigma = \{C, D, 0_\aleph, 1_\aleph\}$, where $A = \{\langle 0.4,0.6,0.8\rangle\langle 0.3,0.5,0.7\rangle\}$, $B = \{\langle 0.8,0.4,0.4\rangle\langle 0.7,0.5,0.3\rangle\}, C = \{\langle 0.4,0.6,0.8\rangle\langle 0.3,0.5,0.8\rangle\}$,

$D = \{\langle 0.7,0.3,0.8\rangle\langle 0.5,0.8,0.9\rangle\}$ and $E = \{\langle 0.6,0.5,0.4\rangle\langle 0.7,0.5,0.4\rangle\}$. Here $C, D, and\ E$ are neutrosophic clopen in S_1, but E is not neutrosophic semi-clopen in S_2. Hence every Nss-c function need not be Nss-c .

Proposition 3.7: Each and every Nss-c function into NT_1-space is a Nts-c function.

Prof: In an NT_1-space, singletons are neutrosophic closed sets. Henceforth $\eta^{-1}(D_1)$ is neutrosophic semi-clopen in S_1 for every subset D_1 of S_2.

Remark 3.8: It is very evident that the classes of Nts-c functions along with Nss-c functions coincide when a range is NT_1-space.

Definition 3.9: A neutrosophic function $\eta: S_1 \rightarrow S_2$ is termed as neutrosophic slightly semi-continuous(in short Nsls-c) if for each $p \in S_1$ and every neutrosophic clopen subset Q of S_2 containing $\eta(Q)$, there exists an NSOS, R of S_1 such that $\eta(R) \subseteq Q$.

Proposition 3.10: Every Nsls-c function that enters a discrete space is considered to be a Nss-c function.

Proof: Let $\eta: S_1 \rightarrow S_2$ be a Nsls-c function from a space S_1 into a discrete space S_2. Consider D_1 be any neutrosophic subset of S_2. Then D_1 is a neutrosophic-clopen subset of S_2. Hence $\eta^{-1}(D_1)$ is the neutrosophic semi-clopen of S_1. Thus, η is Nss-c.

Proposition 3.11: If η is a Nts-c function from a neutrosophic semi-connected space S_1 onto any space S_2, then S_2 is an indiscrete space.

Proof: If possible, suppose that S_2 is not indiscrete. Let D_1 be a proper non-empty neutrosophic open subset of S_2. Then $\eta^{-1}(D_1)$ is a proper non-empty neutrosophic semi-clopen subset of S_1, which is a discrepancy to the fact that S_1 is neutrosophic semi-connected.

Proposition 3.12: If $\eta: S_1 \rightarrow S_2$ be a Nsls-c and $\mu: S_2 \rightarrow S_3$ is Nt-c, then $(\mu \ o \ \eta)$ is Nts-c function.

Proof: Let D_1 be a NOS of S_3. Then $\mu^{-1}(D_1)$ is an NSCS of S_2. Since η Nsls-c, therefore $\eta^{-1}(\mu^{-1}(D_1)) = (\eta o \mu)^{-1}(D_1)$ is a neutrosophic semi-clopen subset of S_1. Hence $(\mu \ o \ \eta)$ is Nts-c function.

Definition 3.13: A NTS S_1 is said to be neutrosophic semi-T_2 if, for any pair of distinct points q_1, q_2 of S_1, there exist disjoint NSOSs Q_1 and Q_2 such that $q_1 \in Q_1$ and $q_2 \in Q_2$.

Remark 3.14: A NTS S_1 is neutrosophic semi-T_2 if and only if for any pair of different points q_1, q_2 of S_1 such that $q_1 \in Q_1$, $q_2 \in Q_2$, and $Nscl(Q_1) \cap Nscl(Q_2) = \emptyset$.

Proposition 3.15: Let $\eta: S_1 \rightarrow S_2$ be a Nts-c injection. If S_2 is NT_0, then S_1 is neutrosophic semi-T_2.

Proof: Consider q_1 and q_2 to be any pair of different points of S_1. Then $\eta(q_1) \neq \eta(q_2)$. As S_2 is NT_0, there exists a NOS, Q containing, say, $\eta(q_1)$ but not $\eta(q_2)$. Then $q_1 \in \eta^{-1}(Q)$ and $q_2 \notin \eta^{-1}(Q)$. Since η is Nts-c, $\eta^{-1}(Q)$ is a neutrosophic semi-clopen subset of S_1. Also, $q_1 \in \eta^{-1}(Q)$ and $q_2 \in S_1 - \eta^{-1}(Q)$. By Remark 3.14, it follows that S_1 is neutrosophic semi-T_2.

4. Comparison Analysis

To highlight the primary contributions of the current study, comparisons with pertinent earlier studies are shown and discussed in this section. Currently, a NS is being created to express incomplete, inconsistent, partial, and ambiguous facts. A NS is expressed to deal with uncertainty using the membership function of Truth, Indeterminacy, and Falsehood. In many practical difficulties, the neutrosophic set leads to more logical conclusions. The neutrosophic set reveals data inconsistencies and can address practical issues. Real or complex continuous functions fail to deal with data containing uncertainty, but Neutrosophic totally semi-continuous functions deal with this. Continuous and neutrosophic continuous functions cannot be compared as both sets are different in these concepts. One is a crisp set, and another is a NS. Even Fuzzy totally continuous functions cannot handle uncertainty. The idea of neutrosophic totally a semi-continuity function is crucial to both functional analysis as well as fixed-point theory. It has numerous applications in information sciences, artificial intelligence, economic theory, decision theory, etc. The key difference is that if h is real, it can only approach zero from the left and right direction in a real line. If h is complex, it can approach zero from infinite directions and any spiral path, etc., in a complex plane. If h is neutrosophic, it can approach zero not only from an endless number of directions in a neutrosophic plane.

5. Conclusion

The Neutrosophic method makes use of all three membership functions (truth, indeterminacy, and falsity). Prepared to face adversity. The perception of the classic set, fuzzy set, and intuitionistic fuzzy set are all generalized by the neutrosophic set. Neutrosophic totally continuous and semi-continuous functions are defined and characterized using neutrosophic semi-open and semi-closed sets. Its properties are analyzed, and some implications are given. Counterexamples also show that the converse statements of these implications are only sometimes true. In addition, neutrosophic strongly semi-continuous along with neutrosophic slightly semi-continuous functions are presented. In the future, we would like to use the neutrosophic semi and nearly continuous mapping to investigate various characteristics in the neutrosophic semi and almost topological group. These ideas will open up new avenues for future research and development of neutrosophic soft topological spaces.

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Neutrosophic Soft Generalized b-Closed Sets in Neutrosophic Soft Topological Spaces

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Abstract: The notion of neutrosophic soft generalized sets is introduced as a general mathematical tool that incorporates useful properties of both neutrosophic generalized sets and soft sets. In this study, we acquaint the notion of neutrosophic soft generalized b-closed (open) (gb, in short) sets in neutrosophic soft topological spaces. In addition, the relations of this set with other neutrosophic soft closed and open sets and various properties are examined. Furthermore, the properties of the gb-closure and gb-interior operator in neutrosophic soft sets and gb-neighborhood in neutrosophic soft topological spaces are investigated.

Keywords: Neutrosophic soft sets; Neutrosophic soft topology; Neutrosophic soft generalized sets; Neutrosophic soft generalized b-closed (open) sets; Neutrosophic soft generalized b-closure (interior); Neutrosophic soft point; Neutrosophic soft generalized b-neighborhood

1. Introduction

Model uncertainties in the solution of problems in many different fields such as science, social science, engineering, and medicine, to cope with the structural difficulties of the classical sets, to evaluate the problems in terms of uncertain situations, to model set structures that bring a new approach different from the classical set structure have been defined. One of these set structures is the fuzzy set theory defined by Zadeh [1]. Later, the fuzzy set is generalized as the concept of the intuitionistic fuzzy set by Atanassov [2] in 1986. The fuzzy set is defined by the membership function. However, since it is not practical to create a membership function for each state, Molodotsov [3] defined soft set theory in 1999. This theory is more functional than compared to other structures in practice for decision-making and solving problems involving uncertainties. In the fuzzy set and intuitionistic fuzzy set theories, the values of an element such as being a member and not being a member are emphasized, while the uncertain values are not emphasized. To meet this need, Smarandache

[4] defined the neutrosophic set theory for solving problems involving imprecise, ambiguous, and inconsistent data. Later, many researchers have done successful studies on different combinations of theories such as soft sets, intuitionistic sets, fuzzy sets, and neutrosophic sets [5-12]. One of these combinations is the neutrosophic soft set theory, first described by Maji [13] and later edited by Deli and Broumi [14]. On the other hand, Salama and Alblowi [15] used the concept of a generalized set [16] in neutrosophic sets and described the generalized neutrosophic set and neutrosophic topological space. Broumi [17] defined the generalized neutrosophic soft set by combining the notion of the generalized neutrosophic set proposed by Salama and Alblowi in [15], and the notion of the soft set proposed by Molodtsov in [3].

Applications of the topology of neutrosophic depend on the neutrosophic internal and closing properties of the neutrosophic open and closed sets. For this reason, topologists have identified new set structures in neutrosophic soft sets [18,19] and generalized sets [20-23] using the properties of neutrosophic open and closed sets. In recent years, studies on the properties of these set structures and the properties of their different combinations, which are defined in neutrosophic soft and various topographical spaces, have diversified, and become important research [24-32] topic. In addition, some research [33-40] has been done on how to use neutrosophic and other topological spaces and different sets in fields such as image processing, medical diagnosis, decision-making, information systems, data analysis, industry, graphic structures, applied mathematics, and computer coding.

The b-sets identified by Dimitrije [41] in 1996 are one of these defined structures and have a stronger relationship with other set structures. Akdag and Ozkan [42] defined and studied the concept of the soft b-closed set in 2014. In addition, Ebenanjar [18] et al. studied the notion of the neutrosophic soft b-open set. Das and Pramanik [43] studied generalized neutrosophic b-open sets. In recent years, many studies on b-sets in neutrosophic spaces have been carried out by many researchers. Soft b-separation axioms were studied by Khattak et al. [44], generalized b-closed sets in fuzzy neutrosophic bitopological spaces were studied by Mohammed and Raheem [30], neutrosophic b-generalized sets and their continuity were studied by Maheswari and Chandrasekar [31], and neutrosophic soft $*$b open sets studied by Mehmood [45]. Later, studies using b-sets diversified [46-51].

For modern topology, which is heavily dependent on set theory ideas, in this work, we present a set, so-called "neutrosophic soft generalized b-closed (open)", and its basic properties. We then present the basics of the properties of the gb-closure and gb-interior operator in neutrosophic soft sets and gb-neighborhood in neutrosophic soft topological spaces. This set can be applied in different neutrosophic topological spaces and different set types in the future, and be considered as the starting point for the expansion of concepts such as continuity, compactness, connectedness, and separation axioms through these sets.

2. Preliminaries

In this section, some descriptions and properties that will be used in the article will be given.

Definition 2.1 [3]. Let's assume that $\mathcal{U}$ and E are the initial universe and the set of parameters, respectively. Let the symbol $P(\mathcal{U})$ denote the power set of $\mathcal{U}$. In this case, the pair (F, A) defined over $\mathcal{U}$, where A is a subset of E parameters, is called a soft set. Where $F: A \to P(\mathcal{U})$ is a mapping.

Definition 2.2 [4]. A neutrosophic set (NS, in short) A on the universe of discourse $\mathcal{U}$ is defined as $A = \{< u, T_A(u), I_A(u), F_A(u) >: u \in \mathcal{U}\}$, where T_A, I_A, F_A: $\mathcal{U} \to]^{-}0,1[^{+}$ and

$$^{-}0 \leq T_A(u) + I_A(u) + F_A(u) \leq 3^{+}.$$

Where $T_A(u)$, $I_A(u)$, and $F_A(u)$ which represent the degree of membership function (or Truth), the degree of indeterminacy, and the degree of non-membership (or Falsehood) respectively of each element $u \in \mathcal{U}$ to the set A.

We take the neutrosophic set in the subset of [0,1] since it is not feasible to use a neutrosophic set with values from $]^{-}0,1[^{+}$ in real-life applications such as scientific and engineering calculations.

Definition 2.3 [14]. Let's assume that $\mathcal{U}$ and E are the initial universe and the set of parameters, respectively. Let the symbol $P(\mathcal{U})$ denote the set of all neutrosophic sets of $\mathcal{U}$. Then $(\tilde{F}, E)$ is called a neutrosophic soft set (NSS, in short) over $\mathcal{U}$, where $\tilde{F}: E \to P(\mathcal{U})$ is a mapping.

It can be defined as a parametrized family of some elements of the set $P(\mathcal{U})$ and written as a set of ordered pairs,

$$(\tilde{F}, E) = \left\{\left(e < u, T_{\tilde{F}(e)}(u), I_{\tilde{F}(e)}(u), F_{\tilde{F}(e)}(u) >: u \in \mathcal{U}\right): e \in E\right\}$$

where $T_{\tilde{F}(e)}(u), I_{\tilde{F}(e)}(u), F_{\tilde{F}(e)}(u) \in [0,1]$, respectively called the truth-membership, indeterminacy-membership, and the falsity-membership function of $(\tilde{F}, E)$. The inequality

$$0 \leq T_{\tilde{F}(e)}(u) + I_{\tilde{F}(e)}(u) + F_{\tilde{F}(e)}(u) \leq 3$$

is satisfied.

Throughout this paper, the symbol $NSS(\mathcal{U}_E)$ will indicate the class of all neutrosophic soft sets on $\mathcal{U}$, and $\tilde{F}_E$ will be replaced instead of $(\tilde{F}, E)$.

Definition 2.4 [52]. Let $\tilde{F}_E$, $\tilde{G}_E \in NSS(\mathcal{U}_E)$. Then

(i) $\tilde{F}_E$ is said to be a *null* set if $T_{\tilde{F}(e)}(u) = 0$, $I_{\tilde{F}(e)}(u) = 0$, $F_{\tilde{F}(e)}(u) = 1$; for all $e \in E$, for all $u \in \mathcal{U}$.

It is denoted by $\emptyset_E$. Obviously $(\emptyset_E)^c = 1_E$.

(ii) $\tilde{F}_E$ is said to be an *absolute* set if $T_{\tilde{F}(e)}(u) = 1$, $I_{\tilde{F}(e)}(u) = 1$, $F_{\tilde{F}(e)}(u) = 0$; for all $e \in E$, for all $u \in \mathcal{U}$.

It is denoted by 1_E. Obviously $(1_E)^c = \emptyset_E$.

(iii) the neutrosophic soft union of $\tilde{F}_E$ and $\tilde{G}_E$, denoted $\tilde{F}_E \cup \tilde{G}_E = \tilde{H}_E$, is defined as $\tilde{H}_E = \{(e < u, T_{\tilde{H}(e)}(u), I_{\tilde{H}(e)}(u), F_{\tilde{H}(e)}(u) >: u \in \mathcal{U}): e \in E\}$ where,

$$T_{\tilde{H}(e)}(u) = max\{T_{\tilde{F}(e)}(u), T_{\tilde{G}(e)}(u)\},$$
$$I_{\tilde{H}(e)}(u) = max\{I_{\tilde{F}(e)}(u), I_{\tilde{G}(e)}(u)\},$$
$$F_{\tilde{H}(e)}(u) = min\{F_{\tilde{F}(e)}(u), F_{\tilde{G}(e)}(u)\}.$$

(iv) the neutrosophic soft intersection of $\tilde{F}_E$ and $\tilde{G}_E$, denoted $\tilde{F}_E \cap \tilde{G}_E = \tilde{H}_E$, is defined as $\tilde{H}_E = \{(e < u, T_{\tilde{H}(e)}(u), I_{\tilde{H}(e)}(n), F_{\tilde{H}(e)}(u) >: u \in \mathcal{U}): e \in E\}$ where,

$$T_{\tilde{H}(e)}(u) = min\{T_{\tilde{F}(e)}(u), T_{\tilde{G}(e)}(u)\},$$
$$I_{\tilde{H}(e)}(u) = min\{I_{\tilde{F}(e)}(u), I_{\tilde{G}(e)}(u)\},$$
$$F_{\tilde{H}(e)}(u) = max\{F_{\tilde{F}(e)}(u), F_{\tilde{G}(e)}(u)\}.$$

(v) $\tilde{F}_E$ is a subset of $\tilde{G}_E$, denoted by $\tilde{F}_E \subseteq \tilde{G}_E$. If for all $e \in E$, for all $u \in \mathcal{U}$; $T_{\tilde{F}(e)}(u) \leq T_{\tilde{G}(e)}(u)$, $I_{\tilde{F}(e)}(u) \leq I_{\tilde{G}(e)}(u)$, $F_{\tilde{F}(e)}(u) \geq F_{\tilde{G}(e)}(u)$.

(vi) the neutrosophic soft complement of $\tilde{F}_E$, denoted $(\tilde{F}_E)^c$, is defined as

$$(\tilde{F}_E)^c = \{(e < u, F_{\tilde{F}(e)}(u), 1 - I_{\tilde{F}(e)}(u), T_{\tilde{F}(e)}(u) >: u \in \mathcal{U}): e \in E\}$$

Obvious that $((\tilde{F}_E)^c)^c = \tilde{F}_E$.

(vii) the neutrosophic soft difference of $\tilde{F}_E$ and $\tilde{G}_E$, denoted $\tilde{F}_E \setminus \tilde{G}_E = \tilde{H}_E$, is defined as $\tilde{H}_E = \{(e < u, T_{\tilde{H}(e)}(u), I_{\tilde{H}(e)}(u), F_{\tilde{H}(e)}(u) >: u \in \mathcal{U}): e \in E\}$ where,

$$T_{\tilde{H}(e)}(u) = min\{T_{\tilde{F}(e)}(u), T_{\tilde{G}(e)}(u)\},$$
$$I_{\tilde{H}(e)}(u) = min\{I_{\tilde{F}(e)}(u), 1 - I_{\tilde{G}(e)}(u)\},$$
$$F_{\tilde{H}(e)}(u) = max\{F_{\tilde{F}(e)}(u), F_{\tilde{G}(e)}(u)\}.$$

Definition 2.5 [52, 53]. Let $\tilde{\tau}_{NSS} \subset NSS(\mathcal{U}_E)$. Then $\tilde{\tau}_{NSS}$ is said to be a neutrosophic soft topology (NST, in short) on $\mathcal{U}$ if

(i) $\emptyset_E$ and 1_E belong to $\tilde{\tau}_{NSS}$,

(ii) $\cup_{i \in I} (\tilde{F}_E)_i \in \tilde{\tau}_{NSS}$ for each $(\tilde{F}_E)_i \in \tilde{\tau}_{NSS}$,

(iii) $\tilde{F}_E \cap \tilde{G}_E \in \tilde{\tau}_{NSS}$ for any $\tilde{F}_E, \tilde{G}_E \in \tilde{\tau}_{NSS}$.

In this case, the triplet $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is called a neutrosophic soft topological space (NSTS, in short) over $\mathcal{U}$. The members of $\tilde{\tau}_{NSS}$ are said to be a neutrosophic soft open set (NSOS, in short) and their complements are said to be a neutrosophic soft closed set (NSCS, in short).

Definition 2.6 [53]. Let $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be an NSTS over $\mathcal{U}$ and $\tilde{F}_E \in NSS(\mathcal{U}_E)$. Then,

(i) the interior of $\tilde{F}_E$, denoted $int(\tilde{F}_E)$, is described as

$$int(\tilde{F}_E) = \cup \{\tilde{G}_E : \tilde{G}_E \text{ is an NSOS in } \mathcal{U} \text{ and } \tilde{G}_E \subseteq \tilde{F}_E\}.$$

(ii) the closure of $\tilde{F}_E$, denoted $cl(\tilde{F}_E)$, is defined as

$$cl(\tilde{F}_E) = \cap \{\tilde{K}_E : \tilde{K}_E \text{ is an NSCS in } \mathcal{U} \text{ and } \tilde{K}_E \supseteq \tilde{F}_E\}.$$

Definition 2.7 [54]. The NSS $u^e_{(\alpha,\beta,\gamma)}$ is said to be a neutrosophic soft point, for every $u \in \mathcal{U}$, $0 < \alpha, \beta, \gamma \leq 1$, $e \in E$, and is described as

$$u^e_{(\alpha,\beta,\gamma)}(e')(y) = \begin{cases} (\alpha,\beta,\gamma) & if \;\; e' = e \;\; and \;\; y = u \\ (0,0,1) & if \;\; e' \neq e \;\; or \;\; y \neq u. \end{cases}$$

Definition 2.8 [54]. Let $\tilde{F}_E$ be an NSS over $\mathcal{U}$. We say that $u^e_{(\alpha,\beta,\gamma)} \in \tilde{F}_E$ read as belonging to the NSS $\tilde{F}_E$ whenever

$$\alpha \leq T_{\tilde{F}(e)}(u), \beta \leq I_{\tilde{F}(e)}(u) \text{ and } \gamma \geq F_{\tilde{F}(e)}(u).$$

Definition 2.9 [54]. Let $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be an NSTS over $\mathcal{U}$ and $\tilde{F}_E \in NSS(\mathcal{U}_E)$. If there exists an NSOS $\tilde{G}_E$ such that $u^e_{(\alpha,\beta,\gamma)} \in \tilde{G}_E \subset \tilde{F}_E$, then $\tilde{F}_E$ is called a neutrosophic soft neighborhood of the neutrosophic soft point $u^e_{(\alpha,\beta,\gamma)} \in \tilde{F}_E$.

3. Neutrosophic Soft b-Closed Sets

In this section, we introduce the elementary descriptions and outcomes of the netrosophic soft closed and netrosophic soft-b-closed set theories that will be required in the future chapter.

Definition 3.1 [18, 19]. Let $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be an NSTS over $\mathcal{U}$ and $\tilde{F}_E \in NSS(\mathcal{U}_E)$. Then

(i) $\tilde{F}_E$ is called a neutrosophic soft regular closed (open) set (NS-rCS (NS-rOS), in short) if $\tilde{F}_E = cl\left(int(\tilde{F}_E)\right)$ ($\tilde{F}_E = int\left(cl(\tilde{F}_E)\right)$).

(ii) $\tilde{F}_E$ is called a neutrosophic soft pre-closed (open) set (NS-pCS (NS-pOS), in short) if $cl\left(int(\tilde{F}_E)\right) \subseteq \tilde{F}_E$ ($\tilde{F}_E \subseteq int\left(cl(\tilde{F}_E)\right)$).

(iii) $\tilde{F}_E$ is called a neutrosophic soft semi-closed (open) set (NS-sCS (NS-sOS), in short) if $int\left(cl(\tilde{F}_E)\right) \subseteq \tilde{F}_E$ ($\tilde{F}_E \subseteq cl\left(int(\tilde{F}_E)\right)$).

(iv) $\tilde{F}_E$ is called a neutrosophic soft α-closed (open) set (NS-αCS (NS-αOS), in short) if $cl\left(int\left(cl(\tilde{F}_E)\right)\right) \subseteq \tilde{F}_E$ ($\tilde{F}_E \subseteq int\left(cl\left(int(\tilde{F}_E)\right)\right)$).

Definition 3.2. Let $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be an NSTS over $\mathcal{U}$ and $\tilde{F}_E \in NSS(\mathcal{U}_E)$. Then,

(i) the regular closure of $\tilde{F}_E$, denoted $cl_r(\tilde{F}_E)$, is defined as

$$cl_r(\tilde{F}_E) = \cap \{\tilde{K}_E : \tilde{K}_E \text{ is an NS-rCS in } \mathcal{U} \text{ and } \tilde{K}_E \supseteq \tilde{F}_E\}.$$

(ii) the regular interior of $\tilde{F}_E$, denoted $int_r(\tilde{F}_E)$, is defined as

$$int_r(\tilde{F}_E) = \cup \{\tilde{G}_E : \tilde{G}_E \text{ is an NS-rOS in } \mathcal{U} \text{ and } \tilde{G}_E \subseteq \tilde{F}_E\}.$$

(iii) the pre-closure of $\tilde{F}_E$, denoted $cl_p(\tilde{F}_E)$, is defined as

$$cl_p(\tilde{F}_E) = \cap \{\tilde{K}_E : \tilde{K}_E \text{ is an NS-pCS in } \mathcal{U} \text{ and } \tilde{K}_E \supseteq \tilde{F}_E\}.$$

(iv) the pre-interior of $\tilde{F}_E$, denoted $int_p(\tilde{F}_E)$, is defined as

$$int_p(\tilde{F}_E) = \cup \{\tilde{G}_E : \tilde{G}_E \text{ is an NS-pOS in } \mathcal{U} \text{ and } \tilde{G}_E \subseteq \tilde{F}_E\}.$$

(v) the semi-closure of $\tilde{F}_E$, denoted $cl_s(\tilde{F}_E)$, is defined as

$$cl_s(\tilde{F}_E) = \cap \{\tilde{K}_E: \tilde{K}_E \text{ is an NS-sCS in } \mathcal{U} \text{ and } \tilde{K}_E \supseteq \tilde{F}_E\}.$$

(vi) the semi-interior of $\tilde{F}_E$, denoted $int_s(\tilde{F}_E)$, is defined as

$$int_s(\tilde{F}_E) = \cup \{\tilde{G}_E: \tilde{G}_E \text{ is an NS-sOS in } \mathcal{U} \text{ and } \tilde{G}_E \subseteq \tilde{F}_E\}.$$

(vii) the α-interior of $\tilde{F}_E$, denoted $int_\alpha(\tilde{F}_E)$, is defined as

$$int_\alpha(\tilde{F}_E) = \cup \{\tilde{G}_E: \tilde{G}_E \text{ is an NS-}\alpha\text{OS in } \mathcal{U} \text{ and } \tilde{G}_E \subseteq \tilde{F}_E\}.$$

(viii) the α-closure of $\tilde{F}_E$, denoted $cl_\alpha(\tilde{F}_E)$, is defined as

$$cl_\alpha(\tilde{F}_E) = \cap \{\tilde{K}_E: \tilde{K}_E \text{ is an NS-}\alpha\text{CS in } \mathcal{U} \text{ and } \tilde{K}_E \supseteq \tilde{F}_E\}.$$

Theorem 3.1. [18] Let $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be an NSTS over $\mathcal{U}$ and $\tilde{F}_E \in NSS(\mathcal{U}_E)$. Then,

(i) $cl_p(\tilde{F}_E) = \tilde{F}_E \cup cl\left(int(\tilde{F}_E)\right)$, $int_p(\tilde{F}_E) = \tilde{F}_E \cap int\left(cl(\tilde{F}_E)\right)$,

(ii) $cl_s(\tilde{F}_E) = \tilde{F}_E \cup int\left(cl(\tilde{F}_E)\right)$, $int_s(\tilde{F}_E) = \tilde{F}_E \cap cl\left(int(\tilde{F}_E)\right)$,

(iii) $cl_\alpha(\tilde{F}_E) = \tilde{F}_E \cup cl\left(int\left(cl(\tilde{F}_E)\right)\right)$, $int_\alpha(\tilde{F}_E) = \tilde{F}_E \cap int\left(cl\left(int(\tilde{F}_E)\right)\right)$.

Definition 3.3 [18]. Let $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be a NSTS over $\mathcal{U}$ and $\tilde{F}_E \in NSS(\mathcal{U}_E)$. Then

(i) $\tilde{F}_E$ is called a neutrosophic soft b-closed set (NS-bCS, in short) if $int\left(cl(\tilde{F}_E)\right) \cap cl\left(int(\tilde{F}_E)\right) \subseteq \tilde{F}_E$.

(ii) $\tilde{F}_E$ is called a neutrosophic soft b-open set (NS-bOS, in short) if $\tilde{F}_E \subseteq int\left(cl(\tilde{F}_E)\right) \cup cl\left(int(\tilde{F}_E)\right)$.

Example 3.1. Let $\mathcal{U} = \{u_1, u_2\}$, $\mathrm{E} = \{\mathrm{e}_1, \mathrm{e}_2\}$ and $\tau_{NSS} = \{\emptyset_E, 1_E, (\tilde{F}_E)_1\}$ where $(\tilde{F}_E)_1$ is an NSS over $\mathcal{U}$, defined as

$$(\tilde{F}_E)_1 = \begin{Bmatrix} e_1 = \{< u_1, 0.6,0.4,0.7 >, < u_2, 0.2,0.5,0.5 >\} \\ e_2 = \{< u_1, 0.6,0.5,0.8 >, < u_2, 0.4,0.3,0.5 >\} \end{Bmatrix}.$$

Then τ_{NSS} defines an NST on $\mathcal{U}$, and so $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is an NSTS over $\mathcal{U}$. An NSS $\tilde{G}_E$ in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is defined as

$$\tilde{G}_E = \begin{Bmatrix} e_1 = \{< u_1, 0.7,0.8,0.9 >, < u_2, 0.8,0.2,0.1 >\} \\ e_2 = \{< u_1, 0.3,0.3,0.3 >, < u_2, 0.7,0.1,0.1 >\} \end{Bmatrix}.$$

Then, $\tilde{G}_E$ is an NS-bCS in $\mathcal{U}$ since $int\left(cl(\tilde{G}_E)\right) \cap cl\left(int(\tilde{G}_E)\right) = int(1_E) \cap cl(\emptyset_E) = \emptyset_E \subseteq \tilde{G}_E$. Also, an NSS $\tilde{K}_E$ in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is defined as

$$\tilde{K}_E = \begin{Bmatrix} e_1 = \{< u_1, 0.3,0.2,0.7 >, < u_2, 0.1,0.4,0.5 >\} \\ e_2 = \{< u_1, 0.2,0.3,0.9 >, < u_2, 0.2,0.2,0.8 >\} \end{Bmatrix}.$$

Then, $\tilde{K}_E$ is an NS-bOS in $\mathcal{U}$ because $\tilde{K}_E \subseteq int\left(cl(\tilde{K}_E)\right) \cup cl\left(int(\tilde{K}_E)\right) = int\left((\tilde{F}_E)_1{}^c\right) \cup cl(\emptyset_E) = (\tilde{F}_E)_1$.

Definition 3.4 [18]. Let $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be an NSTS over $\mathcal{U}$ and $\tilde{F}_E \in NSS(\mathcal{U}_E)$. Then,

(i) the b-interior of $\tilde{F}_E$, denoted $int_b(\tilde{F}_E)$, is defined as

$$int_b(\tilde{F}_E) = \cup \{\tilde{G}_E : \tilde{G}_E \text{ is an NS-bOS in } \mathcal{U} \text{ and } \tilde{G}_E \subseteq \tilde{F}_E\}.$$

(ii) the b-closure of $\tilde{F}_E$, denoted $cl_b(\tilde{F}_E)$, is defined as

$$cl_b(\tilde{F}_E) = \cap \{\tilde{K}_E : \tilde{K}_E \text{ is an NS-bCS in } \mathcal{U} \text{ and } \tilde{K}_E \supseteq \tilde{F}_E\}.$$

Example 3.2. Let $\mathcal{U} = \{u_1, u_2\}$, $\mathrm{E} = \{\mathrm{e}_1, \mathrm{e}_2\}$ and $\tau_{\mathrm{NSS}} = \{\emptyset_{\mathrm{E}}, 1_{\mathrm{E}}, (\tilde{\mathrm{F}}_{\mathrm{E}})_1, (\tilde{\mathrm{F}}_{\mathrm{E}})_2\}$ where $(\tilde{\mathrm{F}}_{\mathrm{E}})_1$ and $(\tilde{\mathrm{F}}_{\mathrm{E}})_2$ are NSSs over $\mathcal{U}$, defined as

$$(\tilde{\mathrm{F}}_{\mathrm{E}})_1 = \begin{Bmatrix} \mathrm{e}_1 = \{< u_1, 0.7,0.7,0.5 >, < u_2, 0.7,0.5,0.5 >\} \\ \mathrm{e}_2 = \{< u_1, 0.5,0.6,0.4 >, < u_2, 0.6,0.4,0.6 >\} \end{Bmatrix},$$

$$(\tilde{\mathrm{F}}_{\mathrm{E}})_2 = \begin{Bmatrix} \mathrm{e}_1 = \{< u_1, 0.4,0.2,0.8 >, < u_2, 0.3,0.5,0.8 >\} \\ \mathrm{e}_2 = \{< u_1, 0.4,0.3,0.6 >, < u_2, 0.6,0.3,0.6 >\} \end{Bmatrix}.$$

Then τ_{NSS} defines an NST on $\mathcal{U}$, and thus $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is an NSTS over $\mathcal{U}$. An NSS $\tilde{\mathrm{F}}_{\mathrm{E}}$ in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is defined as

$$\tilde{\mathrm{F}}_{\mathrm{E}} = \begin{Bmatrix} \mathrm{e}_1 = \{< u_1, 0.2,0.1,0.8 >, < u_2, 0.2,0.3,0.7 >\} \\ \mathrm{e}_2 = \{< u_1, 0.1,0.2,0.9 >, < u_2, 0.2,0.4,0.6 >\} \end{Bmatrix}.$$

By Definition 3.3 (i), $\emptyset_{\mathrm{E}}, 1_{\mathrm{E}}, (\tilde{\mathrm{F}}_{\mathrm{E}})_1, (\tilde{\mathrm{F}}_{\mathrm{E}})_2$,and $\tilde{\mathrm{F}}_{\mathrm{E}}$ are NS-bCSs in $\mathcal{U}$ and by Definition 3.4 (ii) $(\tilde{\mathrm{F}}_{\mathrm{E}})_1 \supseteq \tilde{F}_E$, $1_{\mathrm{E}} \supseteq \tilde{F}_E$, and $\tilde{\mathrm{F}}_{\mathrm{E}} \supseteq \tilde{F}_E$. Hence, $cl_b(\tilde{F}_E) = (\tilde{\mathrm{F}}_{\mathrm{E}})_1 \cap 1_{\mathrm{E}} \cap \tilde{\mathrm{F}}_{\mathrm{E}} = \tilde{\mathrm{F}}_{\mathrm{E}}$.

Example 3.3. Let $\mathcal{U} = \{u_1, u_2, u_3\}$, $\mathrm{E} = \{\mathrm{e}_1, \mathrm{e}_2\}$ and $\tau_{\mathrm{NSS}} = \{\emptyset_{\mathrm{E}}, 1_{\mathrm{E}}, (\tilde{\mathrm{F}}_{\mathrm{E}})_1\}$ where $(\tilde{\mathrm{F}}_{\mathrm{E}})_1$ is NSS over $\mathcal{U}$, defined as

$$(\tilde{\mathrm{F}}_{\mathrm{E}})_1 = \begin{Bmatrix} \mathrm{e}_1 = \{< u_1, 0.5,0.6,0.3 >, < u_2, 0.6,0.5,0.2 >, < u_3, 0.5,0.3,0.4 >\} \\ \mathrm{e}_2 = \{< u_1, 0.7,0.8,0.2 >, < u_2, 0.7,0.4,0.3 >, < u_3, 0.2,0.4,0.1 >\} \end{Bmatrix}.$$

Then τ_{NSS} defines an NST on $\mathcal{U}$, and hence $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is an NSTS over $\mathcal{U}$. An NSS $\tilde{\mathrm{F}}_{\mathrm{E}}$ in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is defined as

$$\tilde{\mathrm{F}}_{\mathrm{E}} = \begin{Bmatrix} \mathrm{e}_1 = \{< u_1, 0.3,0.4,0.6 >, < u_2, 0.1,0.5,0.7 >, < u_3, 0.4,0.6,0.5 >\} \\ \mathrm{e}_2 = \{< u_1, 0.3,0.1,0.8 >, < u_2, 0.2,0.5,0.7 >, < u_3, 0.1,0.2,0.3 >\} \end{Bmatrix}.$$

By Definition 3.3 (ii), $\emptyset_{\mathrm{E}}$, 1_{E}, $(\tilde{\mathrm{F}}_{\mathrm{E}})_1$, and $\tilde{\mathrm{F}}_{\mathrm{E}}$ are NS-bOSs in $\mathcal{U}$ and by Definition 3.4 (i) $\emptyset_{\mathrm{E}} \subseteq \tilde{F}_E$, $\tilde{\mathrm{F}}_{\mathrm{E}} \subseteq \tilde{F}_E$. Thus, $int_b(\tilde{F}_E) = \emptyset_{\mathrm{E}} \cup \tilde{\mathrm{F}}_{\mathrm{E}} = \tilde{\mathrm{F}}_{\mathrm{E}}$.

Theorem 3.2 [18]. Let $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be an NSTS over $\mathcal{U}$ and $\tilde{F}_E \in NSS(\mathcal{U}_E)$. Then

(i) $\tilde{F}_E$ is a neutrosophic soft b-closed set iff $\tilde{F}_E = cl_b(\tilde{F}_E)$,

(ii) $\tilde{F}_E$ is a neutrosophic soft b-open set iff $\tilde{F}_E = int_b(\tilde{F}_E)$,

(iii) $cl_b(\emptyset_E) = \emptyset_E, cl_b(1_E) = 1_E$,

(iv) $int_b(\emptyset_E) = \emptyset_E, int_b(1_E) = 1_E$,

(v) $\tilde{F}_E$ is a neutrosophic soft b-closed set iff $cl_b\left(cl_b(\tilde{F}_E)\right) = cl_b(\tilde{F}_E)$,

(vi) $\left(int_b(\tilde{F}_E)\right)^c = cl_b(\tilde{F}_E)^c$,

(vii) $\left(cl_b(\tilde{F}_E)\right)^c = int_b(\tilde{F}_E)^c$.

Example 3.4. Let us take into account the topology, NS-bCS, and NS-bOS that are given in Example 3.1. By Definition 3.3 (i), $\emptyset_E$, 1_E, $(\tilde{F}_E)_1$, and $\tilde{G}_E$ are NS-bCSs in $\mathcal{U}$ and by Definition 3.4 (ii) $1_E \supseteq \tilde{G}_E$ and $\tilde{G}_E \supseteq \tilde{G}_E$. Then, $cl_b(\tilde{G}_E) = 1_E \cap \tilde{G}_E = \tilde{G}_E$. Hence, $cl_b(\tilde{G}_E) = \tilde{G}_E$.

By Definition 3.3 (ii), $\emptyset_E$, 1_E, $(\tilde{F}_E)_1$, and $\widetilde{N}_E$ are NS-bOSs in $\mathcal{U}$ and by Definition 3.4 (i), $\emptyset_E \subseteq \widetilde{N}_E$, $\widetilde{N}_E \subseteq \widetilde{N}_E$. Then, $int_b(\widetilde{N}_E) = \emptyset_E \cup \widetilde{N}_E = \widetilde{N}_E$. Thus, $int_b(\widetilde{N}_E) = \widetilde{N}_E$.
By Definition 3.4 (ii), $1_E \supseteq 1_E$, $cl_b(1_E) = 1_E$. Also, By Definition 3.3 (i), $\emptyset_E, 1_E$, and $(\tilde{F}_E)_1$ are NS-bCSs in $\mathcal{U}$ and by Definition 3.4 (ii), $\emptyset_E \supseteq \emptyset_E$, $1_E \supseteq \emptyset_E$, and $(\tilde{F}_E)_1 \supseteq \emptyset_E$. Then, $cl_b(\emptyset_E) = 1_E \cap \emptyset_E \cap (\tilde{F}_E)_1 = \emptyset_E$. Similarly, by taking the complement, Theorem 3.2 (iv) also provides. By Theorem 3.2 (i), $cl_b(\tilde{G}_E) = \tilde{G}_E$. So, $cl_b\left(cl_b(\tilde{G}_E)\right) = cl_b(\tilde{G}_E)$ is obtained. By Theorem 3.2 (ii), $int_b(\widetilde{N}_E) = \widetilde{N}_E$. Therefore, $\left(int_b(\widetilde{N}_E)\right)^c = (\widetilde{N}_E)^c$ is obtained. Also, By Definition 3.3 (i), $\emptyset_E$, 1_E, $(\tilde{F}_E)_1$, $(\widetilde{N}_E)^c$ are NS-bCSs in $\mathcal{U}$ and by Definition 3.4 (ii) $1_E \supseteq (\widetilde{N}_E)^c$ and $(\widetilde{N}_E)^c \supseteq (\widetilde{N}_E)^c$. Thus, $cl_b((\widetilde{N}_E)^c) = 1_E \cap (\widetilde{N}_E)^c = (\widetilde{N}_E)^c$. Hence, $cl_b((\widetilde{N}_E)^c) = (\widetilde{N}_E)^c$. Therefore, $\left(int_b(\widetilde{N}_E)\right)^c = cl_b((\widetilde{N}_E)^c)$ is obtained. Similarly, by taking the complement, Theorem 3.2 (vii) also provides.

Theorem 3.3 [18]. Let $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be an NSTS over $\mathcal{U}$ and $\tilde{G}_E, \widetilde{K}_E \in NSS(\mathcal{U}_E)$. Then,

(i) $int_b(\tilde{G}_E) \subseteq int_b(\widetilde{K}_E)$ if $\tilde{G}_E \subseteq \widetilde{K}_E$,
(ii) $cl_b(\tilde{G}_E) \subseteq cl_b(\widetilde{K}_E)$ if $\tilde{G}_E \subseteq \widetilde{K}_E$,
(iii) $cl_b(\tilde{G}_E \cup \widetilde{K}_E) \supseteq cl_b(\tilde{G}_E) \cup cl_b(\widetilde{K}_E)$.
(iv) $cl_b(\tilde{G}_E \cap \widetilde{K}_E) \subseteq cl_b(\tilde{G}_E) \cap cl_b(\widetilde{K}_E)$.
(v) $int_b(\tilde{G}_E \cup \widetilde{K}_E) \supseteq int_b(\tilde{G}_E) \cup int_b(\widetilde{K}_E)$.
(vi) $int_b(\tilde{G}_E \cap \widetilde{K}_E) \subseteq int_b(\tilde{G}_E) \cap int_b(\widetilde{K}_E)$.

Theorem 3.4 [18]. Let $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be an NSTS over $\mathcal{U}$ and $\tilde{F}_E \in NSS(\mathcal{U}_E)$. Then,

(i) $cl_b(\tilde{F}_E) = \tilde{F}_E \cup [int(cl(\tilde{F}_E)) \cap cl(int(\tilde{F}_E))]$,
(ii) $int_b(\tilde{F}_E) = \tilde{F}_E \cap [int(cl(\tilde{F}_E)) \cup cl(int(\tilde{F}_E))]$.

4. Neutrosophic Soft Generalized b-Closed Sets

In this section, we present and examine the description of the neutrosophic soft generalized b-closed set in neutrosophic soft topological spaces and its related properties. In addition, we give generalized definitions of the neutrosophic soft regular, pre, semi, and α sets and their relations with the neutrosophic soft generalized b-closed set.

Definition 4.1. Let $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be an NSTS over $\mathcal{U}$ and $\tilde{F}_E \in NSS(\mathcal{U}_E)$. Then

(i) $\tilde{F}_E$ is called a neutrosophic soft generalized closed set (NS-gCS, in short) if $cl(\tilde{F}_E) \subseteq \tilde{G}_E$ whenever $\tilde{F}_E \subseteq \tilde{G}_E$ and $\tilde{G}_E$ is an NSOS in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$.
(ii) $\tilde{F}_E$ is called a neutrosophic soft generalized b-closed set (NS-gbCS, in short) if $cl_b(\tilde{F}_E) \subseteq \tilde{G}_E$ whenever $\tilde{F}_E \subseteq \tilde{G}_E$ and $\tilde{G}_E$ is an NSOS in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$.

Theorem 4.1. Let $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be an NSTS over $\mathcal{U}$ and $\tilde{F}_E \in NSS(\mathcal{U}_E)$. Every NS-gCS is an NS-gbCS.

Proof: Let $\tilde{F}_E \subseteq \tilde{G}_E$ and $\tilde{G}_E$ be an NSOS in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$. Then, since $\tilde{F}_E$ is an NS-gCS, $cl(\tilde{F}_E) \subseteq \tilde{G}_E$. Therefore, $cl_b(\tilde{F}_E) \subseteq cl(\tilde{F}_E)$ and $cl(\tilde{F}_E) \subseteq \tilde{G}_E$. Thus, $\tilde{F}_E$ is an NS-gbCS in $\mathcal{U}$.

Remark 4.1. Example 4.1 shows that every NS-gCS is an NS-gbCS but the converse is not always true. Moreover, nor can we say that every non-NS-gbCS must be an NS-gCS.

Example 4.1. Let $\mathcal{U} = \{u_1, u_2\}$, $\mathrm{E} = \{\mathrm{e}_1, \mathrm{e}_2\}$ and $\tau_{\mathrm{NSS}} = \{\emptyset_{\mathrm{E}}, 1_{\mathrm{E}}, (\widetilde{\mathrm{N}}_{\mathrm{E}})_1, (\widetilde{\mathrm{N}}_{\mathrm{E}})_2\}$ where $(\widetilde{\mathrm{N}}_{\mathrm{E}})_1$ and $(\widetilde{\mathrm{N}}_{\mathrm{E}})_2$ are NSSs over $\mathcal{U}$, defined as

$$(\widetilde{\mathrm{N}}_{\mathrm{E}})_1 = \begin{Bmatrix} \mathrm{e}_1 = \{< u_1, 0.8,0.8,0.3 >, < u_2, 0.6,0.5,0.4 >\} \\ \mathrm{e}_2 = \{< u_1, 0.7,0.9,0.2 >, < u_2, 0.7,0.5,0.6 >\} \end{Bmatrix},$$

$$(\widetilde{\mathrm{N}}_{\mathrm{E}})_2 = \begin{Bmatrix} \mathrm{e}_1 = \{< u_1, 0.7,0.7,0.4 >, < u_2, 0.6,0.4,0.5 >\} \\ \mathrm{e}_2 = \{< u_1, 0.5,0.6,0.3 >, < u_2, 0.7,0.3,0.8 >\} \end{Bmatrix}.$$

Then τ_{NSS} defines an NST on $\mathcal{U}$, and therefore $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is an NSTS over $\mathcal{U}$. An NSS $\tilde{\mathrm{F}}_{\mathrm{E}}$ in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is defined as

$$\tilde{F}_E = \begin{Bmatrix} \mathrm{e}_1 = \{< u_1, 0.1,0.1,0.9 >, < u_2, 0.3,0.2,0.7 >\} \\ \mathrm{e}_2 = \{< u_1, 0.1,0.1,0.8 >, < u_2, 0.2,0.4,0.6 >\} \end{Bmatrix}.$$

Then, for the NSOS $(\widetilde{\mathrm{N}}_{\mathrm{E}})_1$, we have $\tilde{F}_E \subseteq (\widetilde{\mathrm{N}}_{\mathrm{E}})_1$. By Theorem 3.4. (i) $cl_b(\tilde{F}_E) = \tilde{F}_E \cup \left[int\left(cl(\tilde{F}_E)\right) \cap cl\left(int(\tilde{F}_E)\right)\right] = \tilde{F}_E \cup [int(1_{\mathrm{E}}) \cap cl(\emptyset_{\mathrm{E}})] = \tilde{F}_E \subseteq (\widetilde{\mathrm{N}}_{\mathrm{E}})_1$. $cl(\tilde{F}_E) = 1_{\mathrm{E}} \nsubseteq (\widetilde{\mathrm{N}}_{\mathrm{E}})_1$ is obtained according to Definition 4.1 (i). So, $\tilde{F}_E$ is an NS-gbCS in $\mathcal{U}$ but not NS-gCS. Intercalarily, an NSS $\widetilde{\mathrm{K}}_{\mathrm{E}}$ in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is defined as

$$\widetilde{\mathrm{K}}_{\mathrm{E}} = \begin{Bmatrix} \mathrm{e}_1 = \{< u_1, 0.7,0.8,0.4 >, < u_2, 0.6,0.5,0.5 >\} \\ \mathrm{e}_2 = \{< u_1, ,0.5,0.7,0.2 >, < u_2, 0.7,0.4,0.7 >\} \end{Bmatrix}.$$

Now, we have $\widetilde{K}_E \subseteq (\widetilde{N}_{\mathrm{E}})_1$. Because $cl_b(\widetilde{\mathrm{K}}_{\mathrm{E}}) = \widetilde{K}_E \cup \left[int\left(cl(\widetilde{K}_E)\right) \cap cl\left(int(\widetilde{K}_E)\right)\right] = \widetilde{K}_E \cup \left[int(1_{\mathrm{E}}) \cap cl\left((\widetilde{\mathrm{N}}_{\mathrm{E}})_2\right)\right] = 1_{\mathrm{E}}$, we have $cl_b(\widetilde{\mathrm{K}}_{\mathrm{E}}) \nsubseteq (\widetilde{\mathrm{N}}_{\mathrm{E}})_1$. Hence, $\widetilde{K}_E$ is not an NS-gbCS in $\mathcal{U}$. Then since $cl(\widetilde{K}_E) = 1_{\mathrm{E}}$, we have $cl(\widetilde{K}_E) \nsubseteq (\widetilde{\mathrm{N}}_{\mathrm{E}})_1$. Thus, $\widetilde{K}_E$ is not an NS-gCS in $\mathcal{U}$.

Definition 4.2. Let $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be an NSTS over $\mathcal{U}$ and $\tilde{F}_E \in NSS(\mathcal{U}_E)$. Let $\tilde{G}_E$ be an NSOS in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$. Then

(i) $\tilde{F}_E$ is called a neutrosophic soft generalized regular closed set (NS-grCS, in short) if $cl_r(\tilde{F}_E) \subseteq \tilde{G}_E$ whenever $\tilde{F}_E \subseteq \tilde{G}_E$.

(ii) $\tilde{F}_E$ is called a neutrosophic soft generalized pre-closed set (NS-gpCS, in short) if $cl_p(\tilde{F}_E) \subseteq \tilde{G}_E$ whenever $\tilde{F}_E \subseteq \tilde{G}_E$.

(iii) $\tilde{F}_E$ is called a neutrosophic soft generalized semi-closed set (NS-gsCS, in short) if $cl_s(\tilde{F}_E) \subseteq \tilde{G}_E$ whenever $\tilde{F}_E \subseteq \tilde{G}_E$.

(iv) $\tilde{F}_E$ is called a neutrosophic soft α-generalized closed set (NS-αgCS, in short) if $cl_\alpha(\tilde{F}_E) \subseteq \tilde{G}_E$ whenever $\tilde{F}_E \subseteq \tilde{G}_E$.

Theorem 4.2. In a NSTS $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$

(i) Every NS-CS is an NS-gbCS.

(ii) Every NS-rCS is an NS-gbCS.

(iii) Every NS-αCS is an NS-gbCS.
(iv) Every NS-αgCS is an NS-gbCS.
(v) Every NS-pCS is an NS-gbCS.
(vi) Every NS-gpCS is an NS-gbCS.
(vii) Every NS-bCS is an NS-gbCS.
(viii) Every NS-sCS is an NS-gbCS.
(ix) Every NS-gsCS is an NS-gbCS.

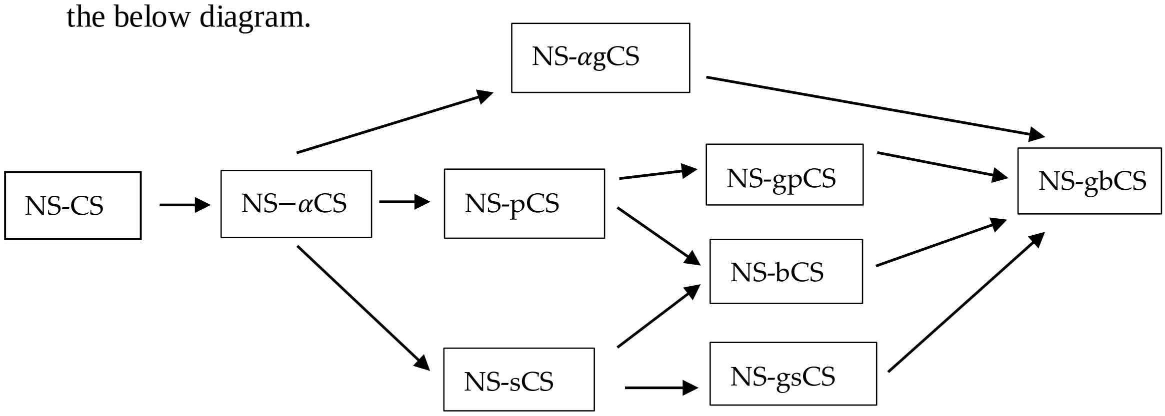
Proof: Let $\tilde{F}_E \subseteq \tilde{G}_E$ and $\tilde{G}_E$ be an NSOS in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$. Then,
(i) since $\tilde{F}_E$ is an NS-CS and $cl_b(\tilde{F}_E) \subseteq cl(\tilde{F}_E)$, $cl_b(\tilde{F}_E) \subseteq cl(\tilde{F}_E) = \tilde{F}_E \subseteq \tilde{G}_E$. Thus, $\tilde{F}_E$ is an NS-gbCS in $\mathcal{U}$.

(ii) since $\tilde{F}_E$ is an NS-rCS, $cl\left(int(\tilde{F}_E)\right) = \tilde{F}_E$ which implies $cl\left(int(\tilde{F}_E)\right) = cl(\tilde{F}_E)$. Therefore, $cl(\tilde{F}_E) = \tilde{F}_E$. Hence, $\tilde{F}_E$ is an NS-CS in $\mathcal{U}$. By Theorem 4.2 (i), $\tilde{F}_E$ is an NS-gbCS in $\mathcal{U}$.
(iii) since $\tilde{F}_E$ is an NS-αCS, $cl_\alpha(\tilde{F}_E) = \tilde{F}_E$. So, $cl_b(\tilde{F}_E) \subseteq cl_\alpha(\tilde{F}_E) = \tilde{F}_E \subseteq \tilde{G}_E$. Thus, $\tilde{F}_E$ is an NS-gbCS in $\mathcal{U}$.
(iv) since $\tilde{F}_E$ is an NS-αgCS, $cl_\alpha(\tilde{F}_E) \subseteq \tilde{G}_E$. Therefore, $cl_b(\tilde{F}_E) \subseteq cl_\alpha(\tilde{F}_E)$, $cl_b(\tilde{F}_E) \subseteq \tilde{G}_E$. Thus, $\tilde{F}_E$ is an NS-gbCS in $\mathcal{U}$.
(v) since $\tilde{F}_E$ is an NS-pCS, by Definition 3.1 (ii) $cl(int(\tilde{F}_E)) \subseteq \tilde{F}_E$ which implies $int\left(cl(\tilde{F}_E)\right) \cap cl\left(int(\tilde{F}_E)\right) \subseteq cl(\tilde{F}_E) \cap cl\left(int(\tilde{F}_E)\right) \subseteq \tilde{F}_E$. Therefore, $cl_b(\tilde{F}_E) \subseteq \tilde{G}_E$. Hence, $\tilde{F}_E$ is an NS-gbCS in $\mathcal{U}$.

(vi) since $\tilde{F}_E$ is an NS-gpCS, $cl_p(\tilde{F}_E) \subseteq \tilde{G}_E$. Thus, $cl_b(\tilde{F}_E) \subseteq cl_p(\tilde{F}_E)$, $cl_b(\tilde{F}_E) \subseteq \tilde{G}_E$. So, $\tilde{F}_E$ is an NS-gbCS in $\mathcal{U}$.
(vii)since $\tilde{F}_E$ is an NS-bCS, by Definition 3.3 (i) $int(cl(\tilde{F}_E)) \cap cl(int(\tilde{F}_E)) \subseteq \tilde{F}_E$. Therefore, $cl_b(\tilde{F}_E) = \tilde{F}_E \cup \left(int(cl(\tilde{F}_E)) \cap cl(int(\tilde{F}_E))\right) \subseteq \tilde{F}_E$. So, $cl_b(\tilde{F}_E) \subseteq \tilde{G}_E$. Thus, $\tilde{F}_E$ is an NS-gbCS in $\mathcal{U}$.
(viii) since $\tilde{F}_E$ is an NS-sCS, Definition 3.1 (iii) $int(cl(\tilde{F}_E)) \subseteq \tilde{F}_E$ which implies $int(cl(\tilde{F}_E)) \cap cl(int(\tilde{F}_E)) \subseteq \tilde{F}_E$. Therefore, $\tilde{F}_E$ is an NS-bCS in $\mathcal{U}$. By Theorem 4.2 (vii), $\tilde{F}_E$ is an NS-gbCS in $\mathcal{U}$.
(ix) since $\tilde{F}_E$ is an NS-gsCS, $cl_s(\tilde{F}_E) \subseteq \tilde{G}_E$. Hence, $cl_b(\tilde{F}_E) \subseteq cl_s(\tilde{F}_E)$, $cl_b(\tilde{F}_E) \subseteq \tilde{G}_E$. Thus, $\tilde{F}_E$ is an NS-gbCS in $\mathcal{U}$.

Remark 4.2. Let $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be a NSTS over $\mathcal{U}$ and $\tilde{F}_E \in NSS(\mathcal{U}_E)$. Then, every neutrosophic soft regular, closed, α, pre, semi, g, αg, gs, gp, and the b-closed set is NS-gbCS.

❖ We can also see the relationships between NS-gbCS and NS-CS sets with the help of the below diagram.



Remark 4.3. Example 4.2 and 4.3 show that the inverse of the applications in the diagram above is not always true.

Example 4.2. Let $\mathcal{U} = \{u_1, u_2, u_3\}$, $\mathrm{E} = \{e_1, e_2\}$ and $\tau_{NSS} = \{\emptyset_E, 1_E, (\widetilde{N}_E)_1, (\widetilde{N}_E)_2\}$ where $(\widetilde{N}_E)_1$ and $(\widetilde{N}_E)_2$ are NSSs over $\mathcal{U}$, defined as

$$(\widetilde{N}_E)_1 = \begin{cases} e_1 = \{< u_1, 0.5,0.4,0.4 >, < u_2, 0.3,0.6,0.4 >, < u_3, 0.2,0.4,0.4 >\} \\ e_2 = \{< u_1, 0.7,0.6,0.2 >, < u_2, 0.2,0.3,0.6 >, < u_3, 0.2,0.5,0.2 >\} \end{cases},$$

$$(\widetilde{N}_E)_2 = \begin{cases} e_1 = \{< u_1, 0.2,0.1,0.8 >, < u_2, 0.3,0.3,0.5 >, < u_3, 0.2,0.3,0.5 >\} \\ e_2 = \{< u_1, 0.2,0.2,0.9 >, < u_2, 0.1,0.2,0.7 >, < u_3, 0.1,0.5,0.4 >\} \end{cases}.$$

Then τ_{NSS} defines an NST on $\mathcal{U}$, and therefore $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is an NSTS over $\mathcal{U}$. An NSS $\widetilde{K}_E$ in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is defined as

$$\widetilde{K}_E = \begin{cases} e_1 = \{< u_1, 0.9,0.8,0.2 >, < u_2, 0.4,0.6,0.3 >, < u_3, 0.7,0.8,0.1 >\} \\ e_2 = \{< u_1, 0.7,0.6,0.1 >, < u_2, 0.5,0.7,0.2 >, < u_3, 0.3,0.6,0.2 >\} \end{cases}.$$

Then, $\widetilde{K}_E$ is an NS-gbCS in $\mathcal{U}$ but not an NS-bCS in $\mathcal{U}$ since $int(cl(\widetilde{K}_E)) \cap cl(int(\widetilde{K}_E)) = ((\widetilde{N}_E)_2)^c \nsubseteq \widetilde{K}_E$. Also, by Definition 3.1 $\widetilde{K}_E$ is not an NS-sCS, NS-pCS, NS-αCS, respectively.

Example 4.3. Let $\mathcal{U} = \{u_1, u_2\}$, $\mathrm{E} = \{e_1, e_2\}$ and $\tau_{NSS} = \{\emptyset_E, 1_E, (\widetilde{N}_E)_1, (\widetilde{N}_E)_2, (\widetilde{N}_E)_3, (\widetilde{N}_E)_4, (\widetilde{N}_E)_5\}$ where $(\widetilde{N}_E)_1, (\widetilde{N}_E)_2, (\widetilde{N}_E)_3, (\widetilde{N}_E)_4$, and $(\widetilde{N}_E)_5$ are NSSs over $\mathcal{U}$, defined as

$$(\widetilde{N}_E)_1 = \begin{cases} e_1 = \{< u_1, 0.3,0.2,0.6 >, < u_2, 0.2,0.1,0.4 >\} \\ e_2 = \{< u_1, 0.1,0.2,0.7 >, < u_2, 0.4,0.2,0.6 >\} \end{cases},$$

$$(\widetilde{N}_E)_2 = \begin{cases} e_1 = \{< u_1, 0.4,0.2,0.5 >, < u_2, 0.6,0.4,0.3 >\} \\ e_2 = \{< u_1, 0.1,0.3,0.6 >, < u_2, 0.4,0.2,0.5 >\} \end{cases},$$

$$(\widetilde{N}_E)_3 = \begin{cases} e_1 = \{< u_1, 0.5,0.6,0.2 >, < u_2, 0.7,0.5,0.3 >\} \\ e_2 = \{< u_1, 0.4,0.5,0.3 >, < u_2, 0.5,0.5,0.5 >\} \end{cases},$$

$$(\widetilde{N}_E)_4 = \begin{cases} e_1 = \{< u_1, 0.5,0.8,0.2 >, < u_2, 0.7,0.6,0.3 >\} \\ e_2 = \{< u_1, 0.4,0.7,0.3 >, < u_2, 0.6,0.7,0.2 >\} \end{cases},$$

$$(\widetilde{N}_E)_5 = \begin{cases} e_1 = \{< u_1, 0.5,0.3,0.5 >, < u_2, 0.6,0.4,0.3 >\} \\ e_2 = \{< u_1, 0.2,0.4,0.6 >, < u_2, 0.5,0.3,0.5 >\} \end{cases}.$$

Then τ_{NSS} defines an NST on $\mathcal{U}$, and therefore $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is an NSTS over $\mathcal{U}$. Then, for the NSOS $(\widetilde{N}_E)_1$, we have $(\widetilde{N}_E)_1 \subseteq (\widetilde{N}_E)_1$. By Theorem 3.4. (i) $cl_b((\widetilde{N}_E)_1) = (\widetilde{N}_E)_1 \cup$

$\left[int\left(cl\left((\widetilde{N}_E)_1\right)\right) \cap cl\left(int\left((\widetilde{N}_E)_1\right)\right)\right] = (\widetilde{N}_E)_1 \cup \left[int\left({(\widetilde{N}_E)_1}^c\right) \cap cl\left((\widetilde{N}_E)_1\right)\right] = (\widetilde{N}_E)_1 \subseteq$ $(\widetilde{N}_E)_1$. Thus, by Definition 4.1 (ii) $(\widetilde{N}_E)_1$ is an NS-gbCS in $\mathcal{U}$. But since $cl\left(int\left(cl(\widetilde{N}_E)_1\right)\right) = {(\widetilde{N}_E)_1}^c \nsubseteq (\widetilde{N}_E)_1$, hence $(\widetilde{N}_E)_1$ is not an NS-αCS in $\mathcal{U}$. Intercalarily, by Definition 3.1. and Definition 4.2, $(\widetilde{N}_E)_1$ is not an NS-pCS, NS-rCS, NS-gpCS, NS-αgCS, respectively.

Remark 4.4. Example 4.4 shows that the union and intersection of any two NS-gbCSs need not be NS-gbCS.

Example 4.4. Let $\mathcal{U} = \{u_1, u_2\}$, $\mathrm{E} = \{\mathrm{e}_1, \mathrm{e}_2\}$ and $\tau_{\mathrm{NSS}} = \{\emptyset_{\mathrm{E}}, 1_{\mathrm{E}}, (\tilde{\mathrm{F}}_{\mathrm{E}})_1\}$ where $(\tilde{\mathrm{F}}_{\mathrm{E}})_1$ is NSS over $\mathcal{U}$, defined as

$$(\tilde{\mathrm{F}}_{\mathrm{E}})_1 = \begin{Bmatrix} \mathrm{e}_1 = \{< u_1, 0.8,0.8,0.1 >, < u_2, 0.6,0.5,0.4 >\} \\ \mathrm{e}_2 = \{< u_1, 0.7,0.9,0.2 >, < u_2, 0.4,0.5,0.4 >\} \end{Bmatrix}.$$

Then τ_{NSS} defines an NST on $\mathcal{U}$, and so $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is an NSTS over $\mathcal{U}$. Two NSSs $\widetilde{\mathrm{G}}_{\mathrm{E}}$ and $\widetilde{\mathrm{K}}_{\mathrm{E}}$ in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ are defined as

$$\widetilde{\mathrm{G}}_{\mathrm{E}} = \begin{Bmatrix} \mathrm{e}_1 = \{< u_1, 0.1,0.2,0.9 >, < u_2, 0.2,0.3,0.8 >\} \\ \mathrm{e}_2 = \{< u_1, 0.1,0.3,0.7 >, < u_2, 0.4,0.3,0.5 >\} \end{Bmatrix},$$

$$\widetilde{K}_{\mathrm{E}} = \begin{Bmatrix} \mathrm{e}_1 = \{< u_1, 0.8,0.8,0.1 >, < u_2, 0.6,0.5,0.4 >\} \\ \mathrm{e}_2 = \{< u_1, 0.7,0.9,0.2 >, < u_2, 0.3,0.5,0.4 >\} \end{Bmatrix}.$$

By Definition 4.1 (ii), NSSs $\widetilde{\mathrm{G}}_{\mathrm{E}}$ and $\widetilde{\mathrm{K}}_{\mathrm{E}}$ are NS-gbCSs in $\mathcal{U}$, but since $\widetilde{\mathrm{G}}_{\mathrm{E}} \cup \widetilde{K}_{\mathrm{E}} \subseteq (\tilde{\mathrm{F}}_{\mathrm{E}})_1$ and $cl_b\left(\widetilde{\mathrm{G}}_{\mathrm{E}} \cup \widetilde{K}_{\mathrm{E}}\right) = 1_{\mathrm{E}} \nsubseteq (\tilde{\mathrm{F}}_{\mathrm{E}})_1$, and thus $\widetilde{\mathrm{G}}_{\mathrm{E}} \cup \widetilde{K}_{\mathrm{E}}$ is not an NS-gbCS in $\mathcal{U}$.

Now, two NSSs $\widetilde{\mathrm{M}}_{\mathrm{E}}$ and $\widetilde{\mathrm{N}}_{\mathrm{E}}$ in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ are defined as

$$\widetilde{\mathrm{M}}_{\mathrm{E}} = \begin{Bmatrix} \mathrm{e}_1 = \{< u_1, 0.9,0.9,0.1 >, < u_2, 0.6,0.7,0.3 >\} \\ \mathrm{e}_2 = \{< u_1, 0.7,1.0,0.1 >, < u_2, 0.5,0.5,0.4 >\} \end{Bmatrix},$$

$$\widetilde{N}_{\mathrm{E}} = \begin{Bmatrix} \mathrm{e}_1 = \{< u_1, 0.8,0.8,0.1 >, < u_2, 0.7,0.5,0.4 >\} \\ \mathrm{e}_2 = \{< u_1, 0.8,0.9,0.2 >, < u_2, 0.4,0.6,0.4 >\} \end{Bmatrix}.$$

By Definition 4.1 (ii), NSSs $\widetilde{M}_{\mathrm{E}}$ and $\widetilde{N}_{\mathrm{E}}$ are NS-gbCSs in $\mathcal{U}$, but since $\widetilde{M}_{\mathrm{E}} \cap \widetilde{N}_{\mathrm{E}} \subseteq (\tilde{\mathrm{F}}_{\mathrm{E}})_1$ and $cl_b\left(\widetilde{M}_{\mathrm{E}} \cap \widetilde{N}_{\mathrm{E}}\right) = 1_{\mathrm{E}} \nsubseteq (\tilde{\mathrm{F}}_{\mathrm{E}})_1$, and hence $\widetilde{M}_{\mathrm{E}} \cap \widetilde{N}_{\mathrm{E}}$ is not an NS-gbCS in $\mathcal{U}$.

Theorem 4.3. Let $\tilde{F}_E$ be an NS-gbCS in an NSTS $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$. If $\tilde{F}_E \subseteq \tilde{G}_E \subseteq cl_b(\tilde{F}_E)$, then $\tilde{G}_E$ is also an NS-gbCS in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$.

Proof. Let $\widetilde{H}_E$ be an NSOS in $\mathcal{U}$ such that $\tilde{G}_E \subseteq \widetilde{H}_E$, then $\tilde{F}_E \subseteq \widetilde{H}_E$. Since $\tilde{F}_E$ is an NS-gbCS set in $\mathcal{U}$, it follows $cl_b(\tilde{F}_E) \subseteq \widetilde{H}_E$. Now, $\tilde{G}_E \subseteq cl_b(\tilde{F}_E)$ implies $cl_b(\tilde{G}_E) \subseteq cl_b(cl_b(\tilde{F}_E)) = cl_b(\tilde{F}_E)$. Thus, $cl_b(\tilde{G}_E) \subseteq \widetilde{H}_E$. Hence, $\tilde{G}_E$ is an NS-gbCS in $\mathcal{U}$.

Definition 4.3. An NSS $\tilde{F}_E$ of an NSTS $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is called a neutrosophic soft gb-open set (NS-gbOS, in short) if $(\tilde{F}_E)^c$ is an NS-gbCS in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$.

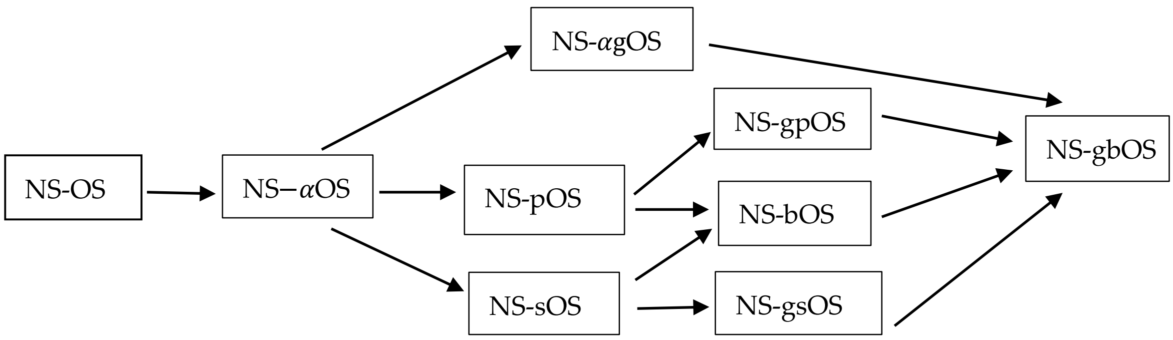
Theorem 4.4. An NSS $\widetilde{N}_E$ of an NSTS $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is an NS-gbOS if and only if $\widetilde{M}_E \subseteq int_b(\widetilde{N}_E)$ whenever $\widetilde{M}_E \subseteq \widetilde{N}_E$ and $\widetilde{M}_E$ is a neutrosophic soft closed set in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$.

Proof. Suppose $\widetilde{N}_E$ is an NS-gbOS in $\mathcal{U}$. Then $(\widetilde{N}_E)^c$ is an NS-gbCS in $\mathcal{U}$. Let $\widetilde{M}_E$ be a neutrosophic soft b-closed set in $\mathcal{U}$ such that $\widetilde{M}_E \subseteq \widetilde{N}_E$. Then $(\widetilde{N}_E)^c \subseteq (\widetilde{M}_E)^c$, $(\widetilde{M}_E)^c$ is a neutrosophic soft b-open set in $\mathcal{U}$. Since $(\widetilde{N}_E)^c$ is an NS-gbCS, $cl_b(\widetilde{N}_E)^c \subseteq (\widetilde{M}_E)^c$, which implies $(int_b(\widetilde{N}_E))^c \subseteq (\widetilde{M}_E)^c$. Thus, $\widetilde{M}_E \subseteq int_b(\widetilde{N}_E)$.

Conversely, assume that $\widetilde{M}_E \subseteq int_b(\widetilde{N}_E)$, whenever $\widetilde{M}_E \subseteq \widetilde{N}_E$ and $\widetilde{M}_E$ be a neutrosophic soft b-closed set in $\mathcal{U}$. Then $(int_b(\widetilde{N}_E))^c \subseteq (\widetilde{M}_E)^c \subseteq \widetilde{H}_E$, where $\widetilde{H}_E$ is a neutrosophic soft b-open set in $\mathcal{U}$. Hence, $cl_b(\widetilde{N}_E)^c \subseteq \widetilde{H}_E$, which implies $(\widetilde{N}_E)^c$ is an NS-gbCS. Therefore, $\widetilde{N}_E$ is an NS-gbOS.

Remark 4.5. An NSS $\tilde{F}_E$ is called NS-gCS, NS-grCS NS-gpCS, NS-gsCS, NS-αgCS if the complement of $(\tilde{F}_E)^c$ is a neutrosophic soft generalized open set, neutrosophic soft generalized regular open set, neutrosophic soft generalized pre-open set, neutrosophic soft generalized semi-open set and neutrosophic soft α-generalized open set (NS-gOS, NS-grOS, NS-gpOS, NS-gsOS, NS-αgOS, in short resp.), respectively.

❖ In the diagram, we have shown the relationship between NS-gbOS and NS-OSs.



Remark 4.6. Example 4.5 and 4.6 show that the inverse of the applications in the diagram above is not always true.

Example 4.5. Let $\mathcal{U} = \{u_1, u_2\}$, $\mathrm{E} = \{\mathrm{e}_1, \mathrm{e}_2\}$ and $\tau_{\mathrm{NSS}} = \{\emptyset_{\mathrm{E}}, 1_{\mathrm{E}}, (\tilde{\mathrm{F}}_{\mathrm{E}})_1\}$ where $(\tilde{\mathrm{F}}_{\mathrm{E}})_1$ is NSS over $\mathcal{U}$, defined as

$$(\tilde{F}_E)_1 = \begin{Bmatrix} e_1 = \{< u_1, 0.4,0.5,0.6 >, < u_2, 0.5,0.4,0.7 >\} \\ e_2 = \{< u_1, 0.1,0.2,0.6 >, < u_2, 0.5,0.5,0.6 >\} \end{Bmatrix}.$$

Then τ_{NSS} defines an NST on $\mathcal{U}$, and hence $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is an NSTS over $\mathcal{U}$. An NSS $\widetilde{\mathrm{K}}_{\mathrm{E}}$ in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is defined as

$$\widetilde{K}_{\mathrm{E}} = \begin{Bmatrix} e_1 = \{< u_1, 0.3,0.4,0.4 >, < u_2, 0.7,0.4,0.5 >\} \\ e_2 = \{< u_1, 0.3,0.8,0.1 >, < u_2, 0.4,0.4,0.6 > \} \end{Bmatrix}.$$

By Theorem 3.4 (ii) and Theorem 4.4, $\tilde{G}_E = \emptyset_{\mathrm{E}} \subseteq int_b(\widetilde{K}_{\mathrm{E}})$ and $\tilde{G}_E = \emptyset_{\mathrm{E}} \subseteq \widetilde{K}_{\mathrm{E}}$, so $\widetilde{K}_{\mathrm{E}}$ is an NS-gbOS in $\mathcal{U}$, but not NS-bOS in $\mathcal{U}$ since $\widetilde{K}_{\mathrm{E}} \nsubseteq int\left(cl(\widetilde{K}_{\mathrm{E}})\right) \cup cl\left(int(\widetilde{K}_{\mathrm{E}})\right) = (\tilde{F}_E)_1$. Although NSS $\widetilde{K}_{\mathrm{E}}$ is an NS-gbOS in $\mathcal{U}$, it is not NS-αOS, NS-OS, respectively.

Example 4.6. Let $\mathcal{U} = \{u_1, u_2\}$, $\mathrm{E} = \{\mathrm{e}_1, \mathrm{e}_2\}$ and $\tau_{\mathrm{NSS}} = \{\emptyset_{\mathrm{E}}, 1_{\mathrm{E}}, (\tilde{\mathrm{F}}_{\mathrm{E}})_1\}$ where $(\tilde{\mathrm{F}}_{\mathrm{E}})_1$ is NSS over $\mathcal{U}$, defined as

$$(\tilde{F}_E)_1 = \begin{Bmatrix} e_1 = \{< u_1, 0.5,0.5,0.6 >, < u_2, 0.4,0.4,0.7 >\} \\ e_2 = \{< u_1, 0.1,0.3,0.4 >, < u_2, 0.6,0.5,0.6 >\} \end{Bmatrix}.$$

Then τ_{NSS} defines an NST on $\mathcal{U}$, and hence $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is an NSTS over $\mathcal{U}$. An NSS $\widetilde{K}_E$ in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is defined as

$$\widetilde{K}_E = \begin{Bmatrix} e_1 = \{< u_1, 0.6,0.6,0.3 >, < u_2, 0.8,0.7,0.1 >\} \\ e_2 = \{< u_1, 0.6,0.9,0.1 >, < u_2, 0.7,0.5,0.4 >\} \end{Bmatrix}.$$

By Theorem 4.4 and Definition 4.2, $(\tilde{F}_E)_1{}^c \subseteq \widetilde{K}_E$ and $(\tilde{F}_E)_1{}^c \subseteq int_b(\widetilde{K}_E) = \widetilde{K}_E \cap \left[int\left(cl(\widetilde{K}_E)\right) \cup cl\left(int(\widetilde{K}_E)\right)\right] = \widetilde{K}_E \cap [int(1_E) \cup cl((\tilde{F}_E)_1)] = \widetilde{K}_E$, thus $\widetilde{K}_E$ is an NS-gbOS in $\mathcal{U}$, but not NS-αgOS in $\mathcal{U}$ since $(\tilde{F}_E)_1{}^c \subseteq \widetilde{K}_E$ and $(\tilde{F}_E)_1{}^c \nsubseteq int_\alpha(\widetilde{K}_E) = \widetilde{K}_E \cap \left[int\left(cl\left(int(\widetilde{K}_E)\right)\right)\right] = (\tilde{F}_E)_1$. Further, although NSS $\widetilde{K}_E$ is an NS-gbOS in $\mathcal{U}$, it is not NS-sOS in $\mathcal{U}$.

Theorem 4.5. Let $\widetilde{M}_E$ be an NS-gbOS in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$. If $int_b(\widetilde{M}_E) \subseteq \widetilde{N}_E \subseteq \widetilde{M}_E$, then $\widetilde{N}_E$ is an NS-gbOS in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$.

Proof. Let $\widetilde{M}_E$ be NS-gbOS and $\widetilde{N}_E$ be any NSS in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ such that $int_b(\widetilde{M}_E) \subseteq \widetilde{N}_E \subseteq \widetilde{M}_E$. Thus, $(\widetilde{M}_E)^c$ is an NS-gbCS, and $(\widetilde{M}_E)^c \subseteq (\widetilde{N}_E)^c \subseteq cl_b(\widetilde{M}_E)^c$. So, $(\widetilde{M}_E)^c$ is an NS-gbCS. Hence, $\widetilde{N}_E$ is an NS-gbOS of $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$.

Remark 4.7. Example 4.7 shows that the union and intersection of any two NS-gbOSs need not be NS-gbOSs.

Example 4.7. Let $\mathcal{U} = \{u_1, u_2\}$, $E = \{e_1, e_2\}$ and $\tau_{NSS} = \{\emptyset_E, 1_E, (\tilde{F}_E)_1\}$ where $(\tilde{F}_E)_1$ is NSS over $\mathcal{U}$, defined as follows

$$(\tilde{F}_E)_1 = \begin{Bmatrix} e_1 = \{< u_1, 0.7,0.8,0.2 >, < u_2, 0.6,0.7,0.5 >\} \\ e_2 = \{< u_1, 0.8,0.6,0.2 >, < u_2, 0.4,0.5,0.4 >\} \end{Bmatrix}.$$

Then τ_{NSS} defines an NST on $\mathcal{U}$, and hence $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is an NSTS over $\mathcal{U}$. Two NSS $\widetilde{G}_E$ and $\widetilde{K}_E$ in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ are defined as

$$\widetilde{G}_E = \begin{Bmatrix} e_1 = \{< u_1, 0.1,0.2,0.8 >, < u_2, 0.4,0.3,0.7 >\} \\ e_2 = \{< u_1, 0.1,0.3,0.8 >, < u_2, 0.4,0.5,0.4 >\} \end{Bmatrix},$$

$$\widetilde{K}_E = \begin{Bmatrix} e_1 = \{< u_1, 0.2,0.2,0.7 >, < u_2, 0.5,0.1,0.6 >\} \\ e_2 = \{< u_1, 0.2,0.4,1.0 >, < u_2, 0.3,0.4,0.7 >\} \end{Bmatrix}.$$

It can be easily seen that both $\widetilde{G}_E$ and $\widetilde{K}_E$ are NS-gbOSs in $\mathcal{U}$, but $\widetilde{G}_E \cup \widetilde{K}_E$ is not NS-gbOS in $\mathcal{U}$ because $(\tilde{F}_E)_1{}^c \nsubseteq int_b(\widetilde{G}_E \cup \widetilde{K}_E) = \emptyset_E$.

Now, two NSSs $\widetilde{M}_E$ and $\widetilde{N}_E$ in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ are defined as

$$\widetilde{M}_E = \begin{Bmatrix} e_1 = \{< u_1, 0.8,0.8,0.1 >, < u_2, 0.7,0.7,0.3 >\} \\ e_2 = \{< u_1, 0.9,0.7,0.5 >, < u_2, 0.4,0.6,0.4 >\} \end{Bmatrix},$$

$$\widetilde{N}_E = \begin{Bmatrix} e_1 = \{< u_1, 0.2,0.2,0.7 >, < u_2, 0.5,0.3,0.6 >\} \\ e_2 = \{< u_1, 0.2,0.4,0.8 >, < u_2, 0.5,0.5,0.3 >\} \end{Bmatrix}.$$

It can be easily seen that both $\widetilde{M}_E$ and $\widetilde{N}_E$ are NS-gbOSs in $\mathcal{U}$, but $\widetilde{M}_E \cap \widetilde{N}_E$ is not NS-gbOS in $\mathcal{U}$ since $(\tilde{F}_E)_1{}^c \nsubseteq int_b(\widetilde{M}_E \cap \widetilde{N}_E) = \emptyset_E$.

Definition 4.4. Let $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be an NSTS over $\mathcal{U}$ and $\tilde{F}_E \in NSS(\mathcal{U}_E)$. Then,

(i) the neutrosophic soft generalized b- interior of $\tilde{F}_E$, denoted $int_{gb}(\tilde{F}_E)$, is defined as $int_{gb}(\tilde{F}_E) = \cup \{\tilde{G}_E : \tilde{G}_E$ is an NS-gbCS in $\mathcal{U}$ and $\tilde{G}_E \subseteq \tilde{F}_E\}$.

(ii) the neutrosophic soft generalized b-closure of $\tilde{F}_E$, denoted $cl_{gb}(\tilde{F}_E)$, is defined as $cl_{gb}(\tilde{F}_E) = \cap \{\tilde{K}_E : \tilde{K}_E$ is an NS-gbCS in $\mathcal{U}$ and $\tilde{K}_E \supseteq \tilde{F}_E\}$.

Theorem 4.6. Let $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be an NSTS over $\mathcal{U}$ and $\tilde{N}_E \in NSS(\mathcal{U}_E)$. Then,

(i) $\tilde{N}_E$ is an NS-gbCS iff $\tilde{N}_E = cl_{gb}(\tilde{N}_E)$,

(ii) $\tilde{N}_E$ is an NS-gbOS iff $\tilde{N}_E = int_{gb}(\tilde{N}_E)$,

(iii) $int_{gb}(\emptyset_E) = \emptyset_E$, $int_{gb}(1_E) = 1_E$,

(iv) $cl_{gb}(\emptyset_E) = \emptyset_E$, $cl_{gb}(1_E) = 1_E$,

(v) $\left(int_{gb}(\tilde{N}_E)\right)^c = cl_{gb}\left((\tilde{N}_E)^c\right)$,

(vi) $\left(cl_{gb}(\tilde{N}_E)\right)^c = int_{gb}\left((\tilde{N}_E)^c\right)$,

(vii) $\left(int_{gb}(\tilde{N}_E)^c\right)^c = cl_{gb}(N_E)$,

(viii) $\left(cl_{gb}(\tilde{N}_E)^c\right)^c = int_{gb}(\tilde{N}_E)$.

Proof.

(i) Suppose $\tilde{N}_E = cl_{gb}(\tilde{N}_E) = \cap \{\tilde{K}_E : \tilde{K}_E$ is an NS-gbCS in $\mathcal{U}$ and $\tilde{K}_E \supseteq \tilde{N}_E\}$ then $\tilde{N}_E \in \cap \{\tilde{K}_E : \tilde{K}_E$ is an NS-gbCS in $\mathcal{U}$ and $\tilde{K}_E \supseteq \tilde{N}_E\}$ which implies $\tilde{N}_E$ is an NS-gbCS.
Conversely, suppose $\tilde{N}_E$ is an NS-gbCS in $\mathcal{U}$. We take $\tilde{N}_E \subseteq \tilde{N}_E$ and $\tilde{N}_E$ is an NS-gbCS. $\tilde{N}_E \in \cap \{\tilde{K}_E : \tilde{K}_E$ is an NS-gbCS in $\mathcal{U}$ and $\tilde{K}_E \supseteq \tilde{N}_E\}$. $\tilde{N}_E \subseteq \tilde{K}_E$ implies $N_E \subseteq \cap \{\tilde{K}_E : \tilde{K}_E$ is an NS-gbCS in $\mathcal{U}$ and $\tilde{K}_E \supseteq \tilde{N}_E\} = cl_{gb}(\tilde{N}_E)$. This proves (i).
(ii) Proved by taking complement in (i).
(iii) Since the sets 1_E and $\emptyset_E$ are NS-gbOSs, the largest NS-gbOS neutrosophic subset of $\mathcal{U}$ is the set $int_{gb}(1_E)$, and the largest NS-gbOS neutrosophic subset of $\emptyset_E$ is the set $int_{gb}(\emptyset_E)$. Thus, $int_{gb}(\emptyset_E) = \emptyset_E$ and $int_{gb}(1_E) = 1_E$.

(iv) As $cl_{gb}(\emptyset_E)$ is the smallest NS-gbCS on $\mathcal{U}$ containing $\tilde{\emptyset}_E$ and $cl_{gb}(1_E)$ is the smallest NS-gbCS on $\mathcal{U}$ containing 1_E, we have $cl_{gb}(\emptyset_E) = \emptyset_E$, $cl_{gb}(1_E) = 1_E$.

(v) Since $int_{gb}(\tilde{N}_E) = \cup \{\tilde{M}_E : \tilde{M}_E$ is an NS-gbOS in $\mathcal{U}$ and $\tilde{M}_E \subseteq \tilde{N}_E\}$ then $\left(int_{gb}(\tilde{N}_E)\right)^c = \cap \{(\tilde{M}_E)^c : (\tilde{M}_E)^c$ is an NS-gbCS in $\mathcal{U}$ and $(\tilde{M}_E)^c \supseteq (\tilde{N}_E)^c\}$. Replacing $(\tilde{M}_E)^c$ by $\tilde{K}_E$, we get $\left(int_{gb}(\tilde{N}_E)\right)^c = \cap \{\tilde{K}_E : \tilde{K}_E$ is an NS-gbCS in $\mathcal{U}$ and $\tilde{K}_E \supseteq (\tilde{N}_E)^c\}$.

Therefore, $\left(int_{gb}(\tilde{N}_E)\right)^c = cl_{gb}\left((\tilde{N}_E)^c\right)$. This proves (v).

(vi) Proof is similar to the above part (v).

(vii) Since $int_{gb}(\widetilde{N}_E) = \cup\{\widetilde{M}_E : \widetilde{M}_E$ is an NS-gbOS in $\mathcal{U}$ and $\widetilde{M}_E \subseteq \widetilde{N}_E\}$ then $\left(int_{gb}(\widetilde{N}_E)\right)^c = \cap\{(\widetilde{M}_E)^c : (\widetilde{M}_E)^c$ is an NS-gbCS in $\mathcal{U}$ and $(\widetilde{N}_E)^c \subseteq (\widetilde{M}_E)^c\} = cl_{gb}((\widetilde{N}_E)^c)$. Then replacing $\widetilde{N}_E$ by $(\widetilde{N}_E)^c$, we get $\left(int_{gb}(\widetilde{N}_E)^c\right)^c = \cap\{(\widetilde{M}_E)^c : (\widetilde{M}_E)^c$ is an NS-gbCS in $\mathcal{U}$ and $\left((\widetilde{N}_E)^c\right)^c \subseteq (\widetilde{M}_E)^c\} = cl_{gb}(((\widetilde{N}_E)^c)^c) = cl_{gb}(\widetilde{N}_E)$.

(viii) Proof is similar to the above part (vii).

Example 4.8. Let $\mathcal{U} = \{u_1, u_2\}$, $\mathrm{E} = \{e_1, e_2\}$ and $\tau_{NSS} = \{\emptyset_E, 1_E, (\tilde{F}_E)_1, (\tilde{F}_E)_2\}$ where $(\tilde{F}_E)_1$ and $(\tilde{F}_E)_2$ are NSSs over $\mathcal{U}$, defined as

$$(\tilde{F}_E)_1 = \begin{cases} e_1 = \{< u_1, 0.4,0.6,0.6 >, < u_2, 0.5,0.6,0.4 >\} \\ e_2 = \{< u_1, 0.2,0.5,0.3 >, < u_2, 0.3,0.6,0.5 >\} \end{cases},$$

$$(\tilde{F}_E)_2 = \begin{cases} e_1 = \{< u_1, 0.4,0.5,0.7 >, < u_2, 0.3,0.4,0.5 >\} \\ e_2 = \{< u_1, 0.1,0.4,0.5 >, < u_2, 0.2,0.4,0.6 >\} \end{cases}.$$

Then τ_{NSS} defines an NST on $\mathcal{U}$, and hence $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is an NSTS over $\mathcal{U}$. An NSS $\widetilde{N}_E$ in $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is defined as

$$\widetilde{N}_E = \begin{cases} e_1 = \{< u_1, 0.5,0.5,0.7 >, < u_2, 0.6,0.6,0.2 >\} \\ e_2 = \{< u_1, 0.6,0.4,0.2 >, < u_2, 0.4,0.5,0.4 >\} \end{cases}.$$

By Definition 4.1 (ii), $\emptyset_E, 1_E, (\tilde{F}_E)_1, (\tilde{F}_E)_2$, and $\widetilde{N}_E$ are NS-gbCSs in $\mathcal{U}$ and $1_E \supseteq \widetilde{N}_E$, $\widetilde{N}_E \supseteq \widetilde{N}_E$. So, $cl_{gb}(\widetilde{N}_E) = 1_E \cap \widetilde{N}_E = \widetilde{N}_E$. Hence, $cl_{gb}(\widetilde{N}_E) = \widetilde{N}_E$. Similarly, by taking the complement, Theorem 4.6 (ii) also provides.

By Theorem 4.4, $\emptyset_E$, 1_E, $(\tilde{F}_E)_1$, $(\tilde{F}_E)_2$, and $(\widetilde{N}_E)^c$ are NS-gbOSs in $\mathcal{U}$ and $\emptyset_E, (\widetilde{N}_E)^c \subseteq (\widetilde{N}_E)^c$. So, $int_{gb}((\widetilde{N}_E)^c) = \emptyset_E \cup (\widetilde{N}_E)^c = (\widetilde{N}_E)^c$. Also, by Theorem 4.6 (i), $\left(cl_{gb}(\widetilde{N}_E)\right)^c = (\widetilde{N}_E)^c$. Hence, $\left(cl_{gb}(\widetilde{N}_E)\right)^c = int_{gb}((\widetilde{N}_E)^c)$ is obtained. Similarly, using the complement, Theorem 4.6 (v) is obtained. In addition, the conditions of Theorem 4.6 (vii) and Theorem 4.6 (viii) are satisfied by replacing $\widetilde{N}_E$ with $(\widetilde{N}_E)^c$.

Definition 4.5. Let $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be an NSTS over $\mathcal{U}$. An NSS $\tilde{F}_E$ over $\mathcal{U}$ is said to be a neutrosophic soft gb-neighborhood of the neutrosophic soft point $e_{\tilde{F}} \in \tilde{F}_E$ if there exits an NS-gbOS $\tilde{G}_E$ such that $e_{\tilde{F}} \in \tilde{G}_E \subseteq \tilde{F}_E$.

All neutrosophic soft gb-neighborhoods of neutrosophic soft point $e_{\tilde{F}}$ are called its neutrosophic soft gb-neighborhood system and are denoted by $gb - N_\tau(e_{\tilde{F}})$.

Definition 4.6. Let $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be an NSTS over $\mathcal{U}$. An NSS $\tilde{F}_E$ over $\mathcal{U}$ is called neutrosophic soft gb-neighborhood of an NSS $\tilde{G}_E$ if there exists an NS-gbOS $\widetilde{H}_E$ such that $\tilde{G}_E \subseteq \widetilde{H}_E \subseteq \tilde{F}_E$.

Example 4.9. Let $\mathcal{U} = \{u_1, u_2\}$, $\mathrm{E} = \{e_1, e_2, e_3\}$ and $\tau_{NSS} = \{\emptyset_E, 1_E, (\widetilde{N}_E)_1, (\widetilde{N}_E)_2, (\widetilde{N}_E)_3\}$ where $(\widetilde{N}_E)_1$, $(\widetilde{N}_E)_2$, and $(\widetilde{N}_E)_3$ are NSSs over $\mathcal{U}$, defined as

$$(\widetilde{N}_E)_1 = \begin{Bmatrix} e_1 = \{< u_1, 0.8,0.8,0.1 >, < u_2, 0.6,0.5,0.4 >\} \\ e_2 = \{< u_1, 0.7,0.9,0.2 >, < u_2, 0.4,0.5,0.4 >\} \end{Bmatrix},$$

$$(\widetilde{N}_E)_2 = \begin{Bmatrix} e_1 = \{< u_1, 0.1,0.1,0.9 >, < u_2, 0.4,0.3,0.7 >\} \\ e_2 = \{< u_1, 0.1,0.3,0.7 >, < u_2, 0.3,0.5,0.6 >\} \end{Bmatrix},$$

$$(\widetilde{N}_E)_3 = \begin{Bmatrix} e_1 = \{< u_1, 0.1,0.1,0.9 >, < u_2, 0.3,0.2,0.8 >\} \\ e_2 = \{< u_1, 0.1,0.1,0.8 >, < u_2, 0.2,0.4,0.6 >\} \end{Bmatrix}.$$

Then τ_{NSS} defines an NST on $\mathcal{U}$, and hence $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is an NSTS over $\mathcal{U}$. Since $(\widetilde{N}_E)_3 \subseteq (\widetilde{N}_E)_2 \subseteq (\widetilde{N}_E)_1$, thus $(\widetilde{N}_E)_1$ is a neutrosophic soft gb-neighborhood of $(\widetilde{N}_E)_3$.

Theorem 4.7. If $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be an NSTS over $\mathcal{U}$. $\widetilde{N}_E$ is an NS-gbOS. Then $\widetilde{N}_E$ is a neutrosophic soft gb-neighborhood of each of its neutrosophic soft points.

Proof. Let $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be an NSTS over $\mathcal{U}$ and $\widetilde{N}_E$ be an NS-gbOS over $\mathcal{U}$. Suppose that $\widetilde{N}_E$ is an NS-gbOS. Then for every neutrosophic soft gb-point $e_{\tilde{F}} \in \widetilde{N}_E$, we have $e_{\tilde{F}} \in \widetilde{N}_E \subseteq \widetilde{N}_E$, and so $\widetilde{N}_E$ is a neighborhood of $e_{\tilde{F}}$. Thus, N_E is a neighborhood of each of its neutrosophic soft points.

Next, suppose that $\widetilde{N}_E$ is a neutrosophic soft gb-neighborhood of its neutrosophic soft points. If $\widetilde{N}_E = \widetilde{\emptyset}_E$ then $\widetilde{N}_E$ is a neutrosophic soft open as $\widetilde{\emptyset}_E \in \tilde{\tau}_{NSS}$. But if $\widetilde{N}_E \neq \widetilde{\emptyset}$ then for each $e_{\tilde{F}} \in \widetilde{N}_E$ there exists a neutrosophic soft point $\widetilde{M}_E$ such that $e_{\tilde{F}} \in (\widetilde{M}_E)_{e_{\tilde{F}}} \subseteq \widetilde{N}_E$. $\widetilde{N}_E = \cup$ $(\widetilde{M}_E)_{e_{\tilde{F}}}$ and so $\widetilde{N}_E$ is an NS-gbOS in $\mathcal{U}$, being of the union of NS-gbOSs in $\mathcal{U}$. Hence, it's proved.

Theorem 4.8. Every neutrosophic soft neighborhood of a neutrosophic soft point $e_{\tilde{F}}$ of an NSTS $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ over $\mathcal{U}$ is a neutrosophic soft gb-neighborhood of $e_{\tilde{F}}$.

Proof. Let $\tilde{F}_E$ be a neutrosophic soft gb-neighborhood of $e_{\tilde{F}} \in \tilde{F}_E$. Then there exists an NSOS $\tilde{G}_E$ such that $e_{\tilde{F}} \in \tilde{G}_E \subseteq \tilde{F}_E$. Since we know that every NSOS set is a neutrosophic soft gb-open set, $\tilde{G}_E$ is an NS-gbOS and hence $\tilde{F}_E$ is a neutrosophic soft gb-neighborhood of $e_{\tilde{F}}$. Thus, it's proved.

Remark 4.8. Example 4.10 shows that the converse of Theorem 4.8 is not always true.

Example 4.10. Let $\mathcal{U} = \{u_1, u_2\}$, $E = \{e_1, e_2\}$ and $\tau_{NSS} = \{\emptyset_E, 1_E, (\widetilde{N}_E)_1, (\widetilde{N}_E)_2, (\widetilde{N}_E)_3, (\widetilde{N}_E)_4, (\widetilde{N}_E)_5\}$ where $(\widetilde{N}_E)_1$, $(\widetilde{N}_E)_2$, $(\widetilde{N}_E)_3$, $(N_E)_4$, and $(\widetilde{N}_E)_5$ are NSSs over $\mathcal{U}$, defined as

$$(\widetilde{N}_E)_1 = \begin{Bmatrix} e_1 = \{< u_1, 0.1,0.2,0.7 >, < u_2, 0.1,0.2,0.5 >\} \\ e_2 = \{< u_1, 0.3,0.3,0.6 >, < u_2, 0.4,0.5,0.6 >\} \end{Bmatrix},$$

$$(\widetilde{N}_E)_2 = \begin{Bmatrix} e_1 = \{< u_1, 0.7,0.4,0.5 >, < u_2, 0.6,0.5,0.3 >\} \\ e_2 = \{< u_1, 0.5,0.5,0.3 >, < u_2, 0.5,0.5,0.5 >\} \end{Bmatrix},$$

$$(\widetilde{N}_E)_3 = \begin{Bmatrix} e_1 = \{< u_1, 0.8,0.5,0.3 >, < u_2, 0.7,0.6,0.3 >\} \\ e_2 = \{< u_1, 0.6,0.6,0.3 >, < u_2, 0.5,0.5,0.5 >\} \end{Bmatrix},$$

$$(\widetilde{N}_E)_4 = \begin{Bmatrix} e_1 = \{< u_1, 0.8, 0.9,0.1 >, < u_2, 0.8,0.7,0.3 >\} \\ e_2 = \{< u_1, 0.6,0.6,0.3 >, < u_2, 0.6,0.7,0.2 >\} \end{Bmatrix},$$

$$(\widetilde{N}_E)_5 = \begin{Bmatrix} e_1 = \{< u_1, 0.6,0.4,0.5 >, < u_2, 0.6,0.2,0.3 >\} \\ e_2 = \{< u_1, 0.4,0.5,0.6 >, < u_2, 0.5,0.5,0.5 >\} \end{Bmatrix}.$$

Then τ_{NSS} defines an NST on $\mathcal{U}$, and hence $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ is an NSTS over $\mathcal{U}$. Here $e_{2_F} = \{< u_1, 0.7, 0.5, 0.4 >, < u_2, 0.4, 0.5, 0.4 >\}$ is a neutrosophic soft point over $\mathcal{U}$. Obviouslyerse $e_{2_F} \in \tilde{G}_E$, where

$$\tilde{G}_E = \begin{Bmatrix} e_1 = \{< u_1, 0.7, 0.8, 0.2 >, < u_2, 0.3, 0.3, 0.4 >\} \\ e_2 = \{< u_1, 0.6, 0.7, 0.3 >, < u_2, 0.5, 0.5, 0.5 >\} \end{Bmatrix}$$

is a neutrosophic soft gb-neighborhood of e_{2_F}, since $e_{2_F} \in \tilde{K}_E \subseteq \tilde{G}_E$, where

$$\tilde{K}_E = \begin{Bmatrix} e_1 = \{< u_1, 0.7, 0.6, 0.3 >, < u_2, 0.2, 0.3, 0.2 >\} \\ e_2 = \{< u_1, 0.6, 0.7, 0.3 >, < u_2, 0.6, 0.5, 0.4 >\} \end{Bmatrix}$$

is an NS-gbOS. But $\tilde{G}_E$ is not a neutrosophic soft neighborhood of e_{2_F}.

Theorem 4.9. Let $(\mathcal{U}, \tilde{\tau}_{NSS}, E)$ be an NSTS over $\mathcal{U}$ and $e_{\tilde{F}}$ be a neutrosophic soft point. Then, $gb - N_\tau(e_{\tilde{F}})$ has the following properties.

(i) If $\tilde{G}_E \in gb - N_\tau(e_{\tilde{F}})$, then $e_{\tilde{F}} \in \tilde{G}_E$,

(ii) If $\tilde{G}_E \in gb - N_\tau(e_{\tilde{F}})$, $\tilde{G}_E \subseteq \tilde{H}_E$, then $\tilde{H}_E \in gb - N_\tau(e_{\tilde{F}})$,

(iii) If $\tilde{G}_E, \tilde{H}_E \in gb - N_\tau(e_{\tilde{F}})$, then $\tilde{G}_E \cup \tilde{H}_E \in gb - N_\tau(e_{\tilde{F}})$,

(iv) If $\tilde{G}_E \in gb - N_\tau(e_{\tilde{F}})$ then there exists $\tilde{M}_E \in gb - N_\tau(e_{\tilde{F}})$, such that $\tilde{M}_E \subseteq \tilde{G}_E$ and $\tilde{M}_E \in gb - N_\tau(e_{\tilde{G}})$ for every $e_{\tilde{G}} \in \tilde{M}_E$.

Proof. (i) Let $\tilde{G}_E \in gb - N_\tau(e_{\tilde{F}})$ then $\tilde{G}_E$is a neutrosophic soft gb-neighborhood of $e_{\tilde{F}}$ which implies $e_{\tilde{F}} \in \tilde{G}_E$.

(ii) Assume $\tilde{G}_E \in gb - N_\tau(e_{\tilde{F}})$ then $\tilde{G}_E$ is a neutrosophic soft gb-neighborhood of $e_{\tilde{F}}$ which implies there exists an NS-gbOS $\tilde{K}_E$, such that $e_{\tilde{F}} \in \tilde{K}_E \subseteq \tilde{G}_E \subseteq \tilde{H}_E$. Hence, $\tilde{H}_E$ is a neutrosophic soft gb-neighborhood of $e_{\tilde{F}}$ and so, $\tilde{H}_E \in gb - N_\tau(e_{\tilde{F}})$.

(iii) If $\tilde{G}_E, \tilde{H}_E \in gb - N_\tau(e_{\tilde{F}})$, then there exists an NS-gbOSs $\tilde{F}_E$ and $\tilde{K}_E$ such that $e_{\tilde{F}} \in \tilde{F}_E \subseteq \tilde{G}_E$ and $e_{\tilde{F}} \in \tilde{K}_E \subseteq \tilde{H}_E$. Obviously, $\tilde{F}_E$ and $\tilde{K}_E$ are contained in $\tilde{H}_E \cup \tilde{G}_E$ which implies $\tilde{G}_E \cup \tilde{H}_E \in gb - N_\tau(e_{\tilde{F}})$.

(iv) If $\tilde{G}_E \in gb - N_\tau(e_{\tilde{F}})$, then there exists an NS-gbOS $\tilde{M}_E$ such that $e_{\tilde{F}} \in \tilde{M}_E \subseteq \tilde{G}_E$. Since $\tilde{M}_E$ is an NS-gbOS and $e_{\tilde{F}} \in \tilde{M}_E \subseteq \tilde{M}_E$, then $\tilde{M}_E \in N_\tau(e_{\tilde{F}})$. Thus, $\tilde{M}_E \in gb - N_\tau(e_{\tilde{F}})$ and $\tilde{M}_E \subseteq \tilde{G}_E$.Again since $\tilde{M}_E$ is an NS-gbOS, so $\tilde{M}_E$ is a neighborhood of each of its neutrosophic soft points. Therefore, $\tilde{M}_E \in N_\tau(e_{\tilde{F}})$ for all $e_{\tilde{F}} \in \tilde{M}_E$. Hence, its proved (iv).

5. Conclusion

In this study, we have introduced the concepts of neutrosophic soft generalized b-closed (open) sets and neutrosophic soft generalized b-interior, neutrosophic soft generalized b-closure, neutrosophic soft generalized b-neighborhood, and explored some of their properties. We have also proved some theorems about neutrosophic soft generalized b-closed sets in neutrosophic soft topological space and analyzed them with appropriate examples. This study is based on theoretical operations of neutrosophic soft generalized b-sets. These sets may be the starting point for new theoretical and applied studies. Therefore, we believe that many new studies can be done in neutrosophic soft generalized b-sets and neutrosophic topological spaces. In addition, this study can be extended to analyze neutrosophic properties such as continuity, compactness, connectedness, and separation axioms by using neutrosophic soft generalized b-sets and other neutrosophic soft generalized sets.

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Secure Dominance in Neutrosophic Graphs

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Abstract: Secure dominance is a significant proportion of dominance which plays vital role in communication networks. In this article, we present and analyze an idea of the secured neutrosophic dominance and totally neutrosophic dominance number of neutrosophic graphs primarily based on strong arcs and the properties of both notions are studied. The terms 2-neutrosophic dominance number, 2- secured neutrosophic dominance variety, 2- totally neutrosophic dominance, and 2- secured neutrosophic dominance number also are defined. Some of their theoretical properties are investigated.

Keywords: Dominance set, total dominance neutrosophic number, Secure dominance neutrosophic number, neutrosophic total secure dominance number.

1. Introduction

In 1965, L.A. Zadeh [28] gave a definition of the word "fuzzy" that deals with uncertainty and fuzzy relatives. Rosenfeld [22] then observed Zadeh's fuzzy functions on fuzzy batchs and developed the idea of fuzzy networks with membership value in [0,1]. K. T. Atanassov [2] expanded the concept of fuzzy networks to intuitionistic fuzzy networks and presented intuitionistic fuzzy relationships with an additional level of indeterminancy. Bipolar fuzzy network is defined by Akram [1] and subjected to a number of procedures. As an extension of the fuzzy network and the intuitionistic fuzzy network, Florentin Smarandache et al. [24, 26, 27] introduced neutrosophic network and single valued neutrosophic network or graphs (SVNG) as a new version of graph notion. Said Broumi et al [5, 6] created the concept of SVNG and studied its additives.

Dominance is the concept that identifies the key nodes in networks that control the entire communication in the networks. Dominance is broadly studied and implemented in graph idea and its extensions. Secure connectivity is crucial in communication networks because it protects against node failure, which could affect the network's stability. Secure dominance is a subset of dominance in which a node's failure is guarded by its adjacent node, securing the network's entire communication. Ore [21] and Berge [4] pioneered the analysis of dominance batchs in graphs. Merouane and Chellali [19] proposed the secured dominance batch and the two- dominance batch. A.Somasundaram and S.Somasundaram [25] produced an idea of dominance in fuzzy networks and

acquired numerous bounds for the dominance number. M.G. Karunambigai et al [13] introduced secure dominance in fuzzy networks. We focused on introducing secured neutrosophic dominance and totally secured neutrosophic dominance in neutrosophic networks, prompted via the idea of dominance number and its applicability [3, 9, 10, 11, 13, 17, 18,23]. Secured dominance is extremely important in many fields, including e-commerce, banking, information transfer, telecommunications, and medical diagnosis.
This paper is structured as follows. Section 2 contains preliminary information, and Section 3 defines a secured neutrosophic dominance number, a totally secure neutrosophic dominance number, and a 2-secured neutrosophic dominance number of a fuzzy network, and their bounds has been formulated. The section 4 concludes the paper.

2. Preliminaries

"**Definition 2.1 [12]** A pair G=(A,B) is known as a single valued neutrosophic graph (SVN-graph) with the underlying set V.

1. The functions $T_A: V \to [0,1]$, $I_A: V \to [0,1]$, and $F_A: V \to [0,1]$, denote the degree of truth-membership, degree of indeterminacy-membership and falsity-membership of the element $v_i \in V$, respectively, and $0 \le T_A(v_i) + I_A(v_i) + F_A(v_i) \le 3$ for all $v_i \in V$.
2. The functions $T_B: E \subseteq V \times V \to [0,1]$, $I_B: E \subseteq V \times V \to [0,1]$, and $F_B: E \subseteq V \times V \to [0,1]$ are defined by $T_B(v_i, v_j) \le T_A(v_i) \wedge T_A(v_j)$, $I_B(v_i, v_j) \ge I_A(v_i) \vee I_A(v_j)$ and $F_B(v_i, v_j) \ge F_A(v_i) \vee F(v_j)$ denotes the degree of truth-membership, indeterminacy-membership and falsity-membership of the edge $(v_i, v_j) \in E$ respectively, where $0 \le T_B(v_i, v_j) + I_B(v_i, v_j) + F_B(v_i, v_j) \le 3$ for all $(v_i, v_j) \in E$ (i, j $= 1, 2, \ldots, n$). We call A the single valued neutrosophic vertex set of V, B the single valued neutrosophic edge set of E, respectively."

"**Definition 2.2 [6]** A partial SVN-subgraph of SVN-graph $G = (A, B)$ is a SVN-graph $H = (V', E')$ such that $V' \subseteq V$, where $T'_A(v_i) \le T_A(v_i)$, $I'_A(v_i) \ge I_A(v_i)$, and $F'_A(v_i) \ge F_A(v_i)$ for all $v_i \in V$ and $E' \subseteq E$, where $T'_B(v_i, v_j) \le T_B(v_i, v_j)$, $I'_B(v_i, v_j) \ge I_B(v_i, v_j)$, $F'_B(v_i, v_j) \ge F_B(v_i, v_j)$ for all $(v_i, v_j) \in E$."

"**Definition 2.3 [6]** Let $G = (A, B)$ be a SVNG. G is said to be a strong SVNG if
$T_B(u, v) = T_A(u) \wedge T_A(v)$,
$I_B(u, v) = I_A(u) \vee I_A(v)$ and
$F_B(u, v) = F_A(u) \vee F_A(v)$ for every $(u, v) \in E$.

"**Definition 2.4 [8]** Let $G = (A, B)$ be a SVNG. G is said to be a complete SVNG if
$T_B(u, v) = T_A(u) \wedge T_A(v)$,
$I_B(u, v) = I_A(u) \vee I_A(v)$ and
$F_B(u, v) = F_A(u) \vee F_A(v)$ for every $u, v \in V$."

"**Definition 2.5 [8]** Let $G = (A, B)$ be a SVNG. $\bar{G} = (\bar{A}, \bar{B})$ is the complement of an SVNG if
$\bar{A} = A$ *and* $\bar{B}$ is computed as below.
$\overline{T_B(u, v)} = T_A(u) \wedge T_A(v) - T_B(u, v)$,

$\overline{I_B(u,v)} = I_A(u) \vee I_A(v) - I_B(u,v)$
and $\overline{F_B(u,v)} = F_A(u) \vee F_A(v) - F_B(u,v)$ for every $(u,v) \in E$.
Here, $\overline{T_B(u,v)}$, $\overline{I_B(u,v)}$ and $\overline{F_B(u,v)}$ denote the true, intermediate, and false membership degree for edge (u,v) of $\bar{G}$."

"Definition 2.6 [16] Let $G = (A,B)$ be a SVNG on V, then the neutrosophic vertex cardinality of G is defined by"

$$|V| = \sum_{(u,v)\in V} \frac{1 + T_A(u,v) + I_A(u,v) - F_A(u,v)}{2}$$

"Definition 2.7 [16] Let $G = (A,B)$ be a SVNG on V, then the neutrosophic edge cardinality of G is defined by"

$$|E| = \sum_{(u,v)\in E} \frac{1+T_B(u,v)+I_B(u,v)-F_B(u,v)}{2}$$ "

"Definition 2.8 [20] An arc (u,v) of a SVNG G is called strong arc if
$T_B(u,v) = T_A(u) \wedge T_A(v)$,
$I_B(u,v) = I_A(u) \vee I_A(v)$ and
$F_B(u,v) = F_A(u) \vee F_A(v)$"

"Definition 2.9[16] Let $G = (A,B)$ be a SVNG on V. Let $(u,v) \in V$, we say that u dominates v in G, if there exist a strong arc between them."

"Definition 2.10 [16] Given $S \subset V$ is called a dominating set in G if for every vertex $v \in V - S$ there exists a vertex $u \in S$ such that u dominates v. for all $e \in A, u, v \in V$."

"Definition 2.11[16] A dominance set S of an Neutrosophic soft graph is said to be minimal dominance set if no proper subset of S is a dominance set. for all $e \in A, u, v \in V$."

"Definition 2.13 [14] Let $G = (V,E)$ be a fuzzy graph. Let $u, v \in V$ and we say that u dominates v in G if $\mu(u,v) = \sigma(u) \vee \sigma(v)$. .A subset S of V is called dominance set in G if for every $v \in V - S$, there exists $u \in S$ such that u dominates v. The minimum fuzzy cardinality of a dominating set in G is called the dominance number of G and is denoted by $\gamma(G)$."

"Definition 2.14 [15] Let $G = (V,E)$ be a fuzzy graph. A dominance set S of V is a secure dominance set if for each vertex $u \in V - S$ is adjacent to a vertex $v \in S$ such that $(S - \{v\}) \cup \{u\}$ is dominating set. The secure dominance number of G is minimum fuzzy cardinality taken over all secure dominance sets of G and is denoted by $\gamma_s(G)$. "

"Definition 2.15 [13] A subset S of V is a 2- dominance set in G if every vertex of $V-S$ has atleast two neighbour in S.The 2- dominance number of G is minimum fuzzy cardinality taken over all 2- dominance sets of G and is denoted by $\gamma_2(G)$."

"Theorem 2.16 [13] Every arc in a complete fuzzy graph is a strong arc."

Table 1: Some basic notations

Notation	Meaning
$G^* = (V, E)$	Fuzzy Network or Graph
$G = (A, B)$	Neutrosophic Network or Graph
V	Vertex batch
E	Edge batch
$T_A(v)$, $I_A(v), F_A(v)$	True, indeterminacy, and falsity membership value of the node v of G.
$T_B(u,v), I_B(u,v), F_B(u,v)$	True, indeterminacy, and falsity membership value of the link (u,v) of G.
$\gamma_{nd}(G)$	Neutrosophic dominance number
$\gamma_{snd}(G)$	Secured neutrosophic dominance number
$\gamma_{tnd}(G)$	Totally neutrosophic dominance number
$\gamma_{tsnd}(G)$	Totally secure neutrosophic dominance number
$\gamma_{2nd}(G)$	2- neutrosophic dominance number
$\gamma_{2snd}(G)$	2- secured neutrosophic dominance number
$\gamma_{2tnd}(G)$	2-totally neutrosophic dominance number
$\gamma_{2tsnd}(G)$	2-totally secured neutrosophic dominance number

2. Secured Dominance in Neutrosophic Graphs

Definition 3.1 [16]

Let $G = (A, B)$ of $G^* = (V, E)$ be a unique value neutrosophic network. consider a subset S of V such that $u \epsilon S$ dominating v for every $v \epsilon V - S$, then that subset is known to be a neutrosophic dominance batch in G. The neutrosophic dominance number of G is given by $\gamma_{nd}(G)$. It is the minimal cardinality across all the dominance sets of G.

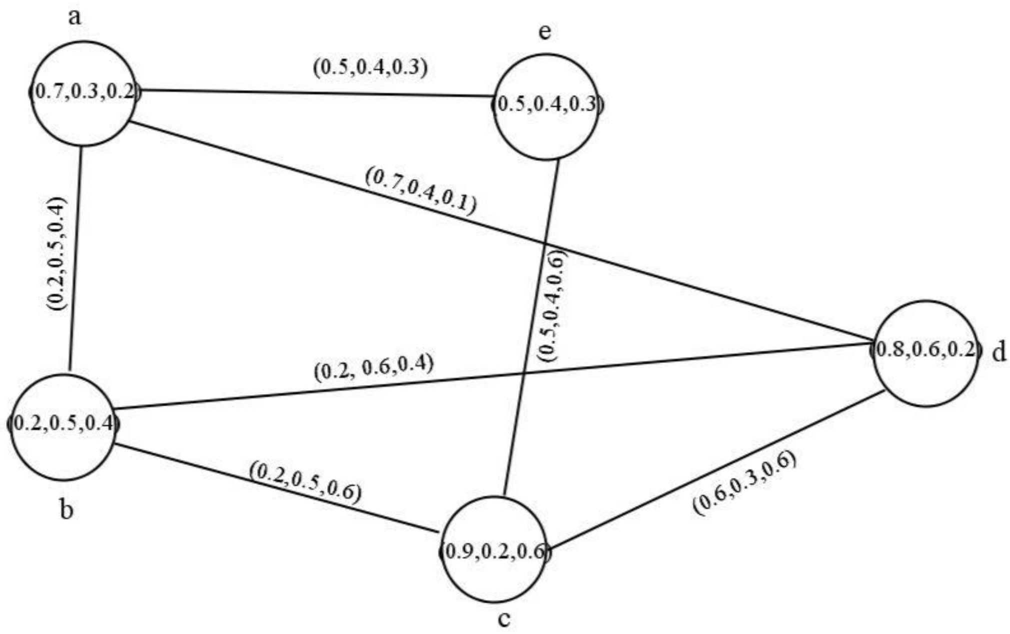

Fig.1 Dominance in a SVNG

Here {a,b},{b,c},{b,e} ,{a,b,c},{a,b,e} are few dominance batchs of G and $\gamma_{nd}(G) = 1.4$.

Definition 3.2

Let $G = (A, B)$ of G^* be a SVNG. Consider a node $u \in V - S$ is contiguous to any node $v \in S$ such that $(S - \{v\}) \cup \{u\}$ is a batch which is again a dominance batch, then the neutrosophic batch S of vertex V is said to be a secured neutrosophic dominance batch. Secured neutrosophic dominance number of G is the number with the lowest vertex cardinality in all secured neutrosophic dominance batchs of G, and it is represented by the symbol $\gamma_{snd}(G)$.

From figure 1, {b,e},{a,b,c},{a,b,e} are secure neutrosophic dominance batchs of G and $\gamma_{snd}(G) = 1.45$.

Definition 3.3

Let $G = (A, B)$ of G^* be a connected SVNG.

Consider a subgraph $< S >$ induced by a batch S of V is also connected, then the neutrosophic dominance set S is known to be a totally neutrosophic dominance batch. Totally neutrosophic dominance number of G is the number with the lowest vertex cardinality among all total neutrosophic dominance batchs of G, and it is represented by the symbol $\gamma_{tnd}(G)$.

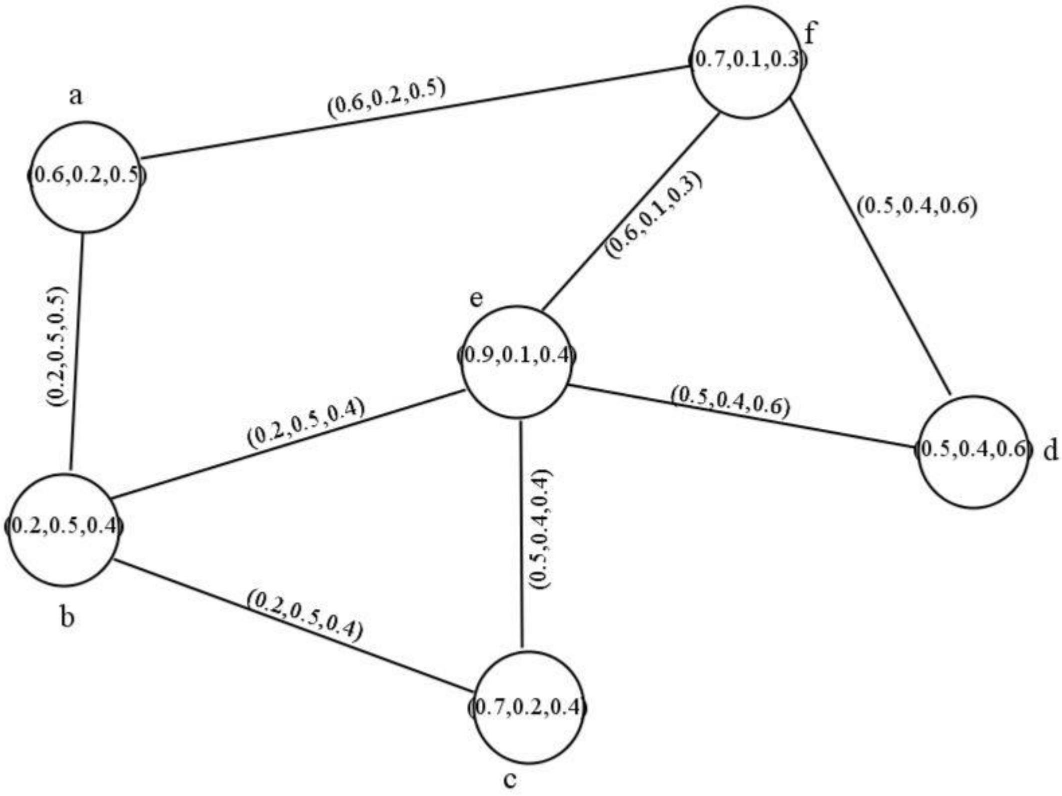

Fig.2 Total neutrosophic dominance in a SVNG

Here {a,b,e},{b,e,d},{a,b,f},{a,b,e,d},{a,b,f,d},{a,d,e,f} are total neutrosophic dominance batchs and $\gamma_{tnd}(G) = 2.05$.

Definition 3.4

Consider the connected SVNG $G = (A, B)$ of $G^* = (V, E)$. If the subgraph $< S >$ induced by the neutrosophic set S of V is also connected, then the neutrosophic batch S of V is known to be a totally secured neutrosophic dominance batch. The term totally secure neutrosophic dominance number of G, which is abbreviated as $\gamma_{tsnd}(G)$, refers to the totally secure neutrosophic dominance batch of G with the least vertex cardinality.

From Figure 2, {a,b,c,e,f},{a,b,c,e,d},{b,c,e,d,f},{a,b,c,d,f} are total secure neutrosophic dominance batchs of G and $\gamma_{tsnd}(G) = 3.45$.

Definition 3.5

Let SVNG be $G = (A, B)$ of $G^* = (V, E)$. If every node of $V - S$ has at least two neighbours in S, then the subset S of V is a 2-neutrophic dominance batch in G. The 2- neutrosophic dominance number of G, which is shown by the symbol $\gamma_{2nd}(G)$ is the set of 2- neutrosophic dominance batchs with the least vertex cardinality.

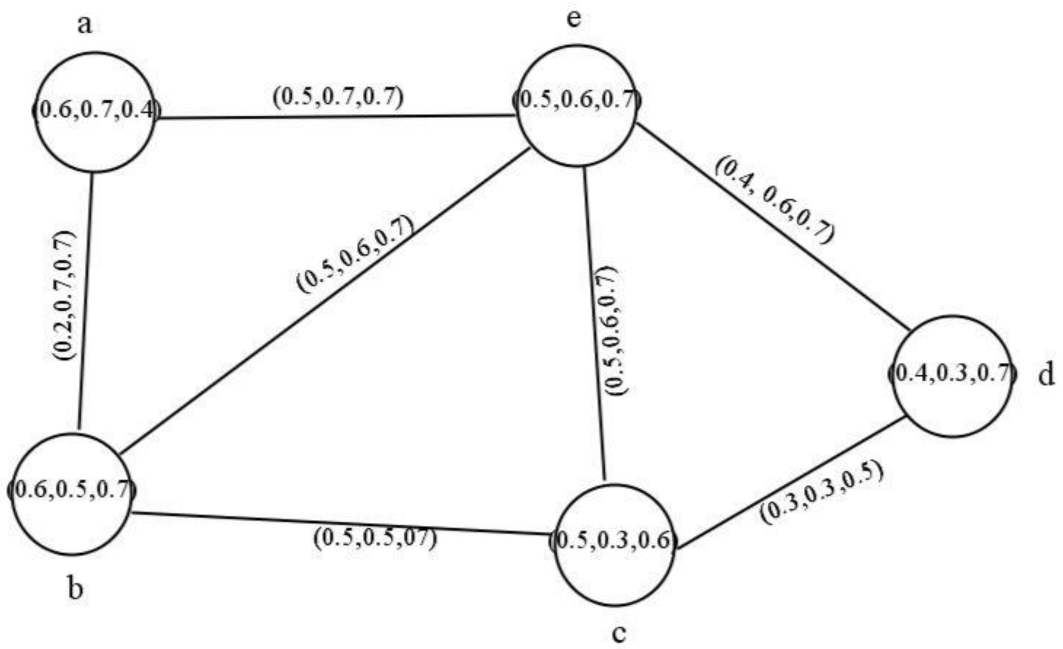

Fig.3 2- neutrosophic dominance in a SVNG

Here {a,b,d},{a,e,d},{a,c,d},{a,b,c,d} are 2- neutrosophic dominance batchs of G and $\gamma_{2nd}(G) = 2.05$.

Definition 3.6

Let SVNG be $G = (A, B)$ of $G^* = (V, E)$. Consider a node $u \in V - S$ is neighbouring to other node $v \in S$ such that ($S - \{v\}) \cup \{u\}$ is a batch which is again a 2-dominance batch, then the 2-neutrosophic batch S of V is known to be a 2-secured neutrosophic dominance batch. The 2-secured neutrosophic dominance number of G, which is represented by the symbol $\gamma_{2snd}(G)$, is the batch of 2-secure neutrosophic dominance batchs of G with the least vertex cardinality.

From Figure 3, {a,b,c,d} are 2- secure neutrosophic dominance batchs of G and $\gamma_{2snd}(G) = 3.45$.

Definition 3.7

Consider the connected SVNG $G = (A, B)$ of $G^* = (V, E)$.

Consider a subgraph $< S >$ induced by a subset S is connected, then the subset S is a 2-total neutrosophic dominance batch. The 2-totally neutrosophic dominance number of G, which is shown by the symbol $\gamma_{2tnd}(G)$, is the batch of 2-totally neutrosophic dominance batchs of G with the least vertex cardinality.

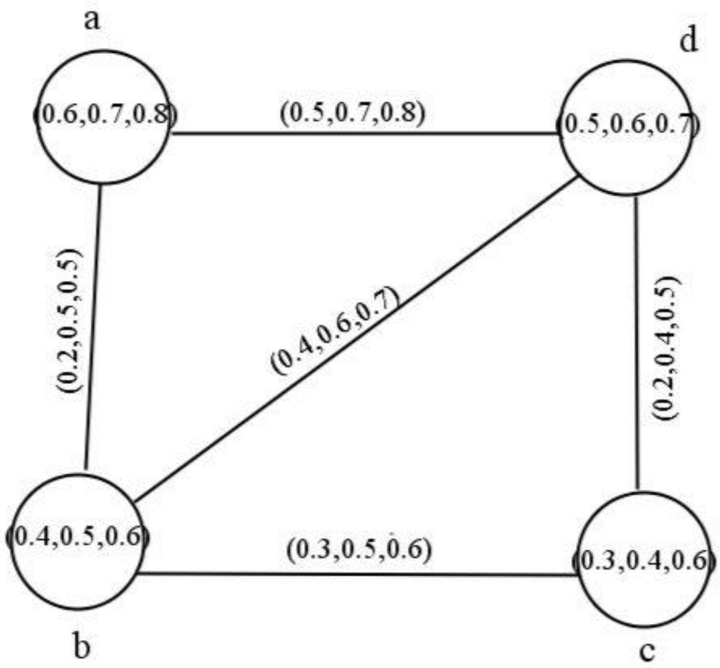

Fig.4 2-Total dominance in a SVNG

Here {a,b,c} and {a,b} are 2- totally neutrosophic dominance batchs and $\gamma_{2tnd}(G) = 1.2$.

Definition 3.8

Consider the connected SVNG $G = (A, B)$ of $G^* = (V, E)$. If the subgraph $< S >$ produced by a neutrosophic set S of V is also connected, then the batch S is a 2-secured dominance and is referred to as a 2-totally secured neutrosophic dominance batch. The 2-totally secured neutrosophic dominance number of G is represented by the symbol $\gamma_{2tsnd}(G)$ and is the set of 2-totally secured neutrosophic dominance batchs of G with the least vertex cardinality.

From Figure 4, {a,b,c} are 2-secured total neutrosophic dominance batchs of G and $\gamma_{2tsnd}(G) = 1.95$.

Theorem 3.9 In a complete SVNG G, each neutrosophic dominance batch is a secure neutrosophic dominance batch.

Proof:

If $G = (A, B)$ is a complete neutrosophic network, then for every $u, v \in V$ we obtain $T_B(u, v) = T_A(u) \wedge T_A(v)$, $I_B(u, v) = I_A(u) \vee I_A(v)$ and $F_B(u, v) = F_A(u) \vee F_A(v)$.

Here, SVNG $G = (A, B)$ has only strong arcs.

Any neutrosophic dominance batch defined in $G = (A, B)$ is given by S. Currently, any vertex $v \in V - S$ is next to every vertex of S, and $(S - \{v\}) \cup \{u\}$ is a neutrosophic dominance batch for all $u \in S$. Hence, S is a secured neutrosophic dominance batch of G.

Theorem 3.10

For a neutrosophic netwok G = (A, B), any 2-secured neutrosophic dominance batch of G is a secure neutrosophic dominance batch of G.

Proof:

If G = (A, B) is a neutrosophic netwok, then let S be a 2-secured neutrosophic dominance batch of that network. Every node $u \in V - S$ is then next to a node $v \in S$ resulting in $(S - \{v\}) \cup \{u\}$ being a 2-neutosophic dominance batch. Since G is a 2-secured neutrosophic dominance batch, G is a 2- neutrosophic dominance batch, and every 2- neutrosophic dominance batch is a dominance batch. As a result, $(S - \{v\}) \cup \{u\}$ is a neutrosophic dominance batch since every node $u \in V - S$ is close to a node $v \in S$. As a result, S is a secured neutrosophic dominance batch of G.

Theorem 3.11

In a SVNG G, the complement of a neutrosophic dominance batch of G is a neutrosophic dominance batch.

Proof:

Let G represent a neutrosophic network.

Any subset S of V is considered a neutrosophic dominance batch in G, according to the notion of dominance, if for every $v \in V - S$, there is an $u \in S$ such that u dominating v. If $\bar{S}$ is the complement of S's neutrosophic dominance batch, then $v \in \bar{S}$ and $u \in V - \bar{S}$ such that v dominating u. i.e., for every $v \in \bar{S}$ and $u \in V - \bar{S}$ such that v dominating u in $\bar{G}$. Therefore, $\bar{S}$ is a neutrosophic dominance batch of G.

Theorem 3.12

In a SVNG G, the complement of any 2- neutrosophic dominance batch is a 2- neutrosophic dominance batch of G.

Proof:

Let G represent a neutrosophic network.

According to Theorem 3.11, if $\bar{S}$ is the complement of S's neutrosophic dominance batch, then $\bar{S}$ is also one of G's neutrosophic dominance batchs. Every 2-neutrosophic dominance batch of G is a neutrosophic dominance batch of G. Therefore, any 2-neutosophic dominance batch in a SVNG has a complement that is a neutrosophic dominance batch of G.

Theorem 3.13

For any SVNG G, $\gamma_{nd}(G) \leq \gamma_{snd}(G) \leq \gamma_{2snd}(G) \leq \gamma_{2nd}(G)$.

Proof:

Consider S is a least neutrosophic dominance batch of neutrosophic network G and $\gamma_{nd}(G) = k$. If S is also a least secure neutrosophic dominance batch of neutrosophic graph G then $\gamma_{snd}(G) = k$. $i.e., \gamma_{nd}(G) \leq \gamma_{snd}(G)$. Suppose S is not a least secured neutrosophic dominance batch and if $S'$is a least secured neutrosophic dominance batch then $\gamma_{snd}(G) > k$.

Thus, $\gamma_{nd}(G) \leq \gamma_{snd}(G)$. (1)

Let D be a least 2- neutrosophic dominance batch of neutrosophic graph G and $\gamma_{2nd}(G) = l$. If D is also a least secure 2- neutrosophic dominance batch of neutrosophic graph G then $\gamma_{2snd}(G) =$

$l.$ $i.e., \gamma_{2snd}(G) \leq \gamma_{2nd}(G)$. Suppose D is not a least 2-secured neutrosophic dominance batch and if D'**is** a least secured 2- neutrosophic dominance batch then $\gamma_{2snd}(G) > l$.
Thus, $\gamma_{2snd}(G) \leq \gamma_{2nd}(G)$. (2)
Let Q be a least secured neutrosophic dominance batch of neutrosophic graph G and $\gamma_{snd}(G) = n$. Every 2-secured neutrosophic dominance batch is a secured neutrosophic dominance batch of G. If Q is also a least secured 2- dominance batch of neutrosophic graph G then $\gamma_{2snd}(G) = n.$ $i.e., \gamma_{snd}(G) \leq \gamma_{2snd}(G)$. Suppose Q is not a least 2-secured neutrosophic dominance batch and if Q'**is** a least secured 2- neutrosophic dominance batch then $\gamma_{2snd}(G) > n$. Thus, $\gamma_{snd}(G) \leq \gamma_{2snd}(G)$. (3)

From (1), (2), (3) we get, $\gamma_{nd}(G) \leq \gamma_{snd}(G) \leq \gamma_{2snd}(G) \leq \gamma_{2nd}(G)$.

Theorem 3.14
For any SVNG G, $\gamma_{tnd}(G) \leq \gamma_{tsnd}(G) \leq \gamma_{2tsnd}(G) \leq \gamma_{2tnd}(G)$.

Proof:
Consider S is a least totally neutrosophic dominance batch of neutrosophic network G and $\gamma_{tnd}(G) = k$. If S is also a least totally secured neutrosophic dominance batch of neutrosophic graph G then $\gamma_{tsnd}(G) = k.$ $i.e., \gamma_{tnd}(G) \leq \gamma_{tsnd}(G)$. Suppose S fails to be a least secured totally neutrosophic dominance batch and consider S' is a least totally secured neutrosophic dominance batch then $\gamma_{tsnd}(G) > k$.
Thus, $\gamma_{tnd}(G) \leq \gamma_{tsnd}(G)$. (1)
Consider D is a least 2-totally neutrosophic dominance batch of neutrosophic graph G and $\gamma_{2tnd}(G) = l$. If D is also a least 2- totally secured neutrosophic dominance batch of neutrosophic graph G then $\gamma_{2tsnd}(G) = l.$ $i.e., \gamma_{2tsnd}(G) \leq \gamma_{2tnd}(G)$. Suppose D is not a least 2-totally secured neutrosophic dominance batch and if D'**is** a least 2- totally secured neutrosophic dominance batch then $\gamma_{2tsnd}(G) > l$.
Thus, $\gamma_{2tsnd}(G) \leq \gamma_{2tnd}(G)$. (2)
Let Q be a least totally secured neutrosophic dominance batch of neutrosophic graph G and $\gamma_{tsnd}(G) = n$. Every 2- totally secured neutrosophic dominance batch is a totally secure neutrosophic dominance batch of G. If Q is also a least 2- totally secure neutrosophic dominance batch of neutrosophic graph G then $\gamma_{2tsnd}(G) = n.$ $i.e., \gamma_{tsnd}(G) \leq \gamma_{2tsnd}(G)$. Suppose Q is not a least 2-totally secured neutrosophic dominance batch and if $Q'$is a least 2-total secured neutrosophic dominance batch then $\gamma_{2tsnd}(G) > n$.
Thus, $\gamma_{tsnd}(G) \leq \gamma_{2tsnd}(G)$. (3)

From (1), (2), (3) we get, $\gamma_{tnd}(G) \leq \gamma_{tsnd}(G) \leq \gamma_{2tsnd}(G) \leq \gamma_{2tnd}(G)$.

4. Conclusion

Neurosophic network theory is being extensively used in a wide range of scientific and technological domains, including cognitive field, genetic methodss, optimisation techniques, clustering, medical treatments, and decision trees. A neutrosophic network is initiated by Florentin Smarandache from neutrosophic groups. When compared to other generic and fuzzy analogs, neurosophic analogs give the system greater accuracy, adaptable, and capable. However, the independence of indeterminacy-membership occasionally allows real data to be unbounded. The idea of secure neutrosophic network dominance is developed in this work, and we intend to keep developing the application that would secure social network connectivity in the neutrosophic environment.

"Compliance with Ethical Standards

Conflict of Interest

The authors declare that they do not have any financial or associative interest indicating a conflict of interest in about submitted work."

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Unification of some generalized aggregation operators in neutrosophic cubic environment and its applications in multi expert decision-making analysis

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Abstract: This manuscript aims to provide a platform that merges different aggregation operators into a concise and computationally trackable single aggregation operator. This generalization can produce numerous aggregation operators while the complex framework can be streamlined and analyzed more efficiently. Generalized aggregation operators have become increasingly important in decision-making (DM) theory due to the growing complexity and diversity of DM problems.

These unified aggregation operators are furnished upon numerical examples from MEMADM and MADM problems as applications. Finally, a comparative analysis with some existing methods is conducted to examine the properties and behavior of the proposed aggregation operator.

Keywords: Neutrosophic Cubic Set (NCS); Decision Making (DM); Multi Attribute Decision making (MADM); Multi Expert Multi Attribute Decision making (MEMADM).

1. Introduction

Complex phenomena such as uncertainty occur in daily life problems. Due to the vagueness that is inevitably involved in many areas of life problems, the common notions of sets still need to deal with the situation. Many successful attempts have been made to handle uncertainty in the system description. Zadeh initiated his idea of a Fuzzy set (FS) [22]. In recent years, FS attracted researchers, and it has been applied in the many fields like medical sciences, social sciences and engineering. Over time, it is further extended onto the interval-valued fuzzy set (IVFS)[23], intuitionistic fuzzy set(IFS)[2], interval-valued intuitionistic fuzzy set(IVIFS)[3], and cubic set(CS)[7]. Smrandache proposed a neutrosophic set (NS) [15], which is a generalization of IFS[16]. NS is characterized by three independent constituents, the truth, indeterminacy, and falsehood. NS was classified into the single-valued neutrosophic set (SVNS)[20]. Later the neutrosophic interval set (INS) [19] was proposed. Jun characterized the NS and INS to form a neutrosophic cubic set (NCS)[8]. NCS is a generalization of NS, and INS [9], enables us to express more information. This makes NCS a persuasive tool to deal with uncertain and imprecise data more efficiently.

The role of aggregation operators in DM is essential. The DM is a challenging job in an inconsistent and vague environment. NCS can minimize the uncertain situation. NCS provides a platform to handle a complex frame of the environment to its structure of both interval and crisp value simultaneously. This complexion of NCS attracted many researchers to apply it in different fields of DM. Khan *et al.*, [9] proposed Einstein geometric operators in NCS. Zhan *et al.*, [24] used NCS on MCDM. Banerjee *et al.* [4] worked on the grey rational analysis (GRA) method in NCS. Lu and Ye [12] proposed a cosine measure of NCS. Pramanik *et al.* [13] defined similarity measures on NCS. Shi and Ji [14] suggested Dombi aggregation operators on NCS.

Khan et al. generalized some aggregation operators on NCS in their work as generalized aggregation operators[11]. In addition, khan et al. generalized Shapley and Choquet integral aggregation operators in NCS[10]. These two generalizations are a precious platform for deducing aggregation operators.

The aggregation operators have a vast scope in data analysis, machine learning, artificial intelligence, and decision-making theory. The aggregation operators enable one to arrive at a single decision, considering various degrees of uncertainty and imprecision in complex work frames. The development of a new generalized aggregation operator with improved performance and flexibility in the active research area can further expand their scope of application.

1.1 Motivation

The Choquet integral can handle overall interaction, and shapley takes the interaction among the criteria and can measure the weights [17]. Motivated by this ([10,11]) work, the author(s) attempted to further generalization of these generalizations so that a single platform is provided to choose an appropriate aggregation operator. This will allow one to not only measure the weight but also choose an aggregation operator of choice according to the situation.

1.2 Contribution

The aggregation operators play a vital role in DM theory to allow a MADM to make a single decision. The choice of an aggregation operator significantly impacts the final decision. Overall aggregation operators are crucial to handling imprecise and complex DM problems. To address such challenges, generalized aggregation operators are needed so that these challenges can be dealt with under one platform. Recently, the author(s) have generalized some aggregation operators like neutrosophic cubic generalized aggregation operators, induced generalized Shapley Choquet integral operator.

The methodologies to measure the unified aggregation in NCS.

- The generalization-induced generalized neutrosophic cubic unified aggregation operator is provided.
- Shapley measures are used to determine the interactive and overall weights.
- These aggregation operators are applied and compared with existing methods to test their validity.

Organization The remaining paper is patterned as follows. The preliminaries are given in section 2. Section 3 deals with the IGNCUSCI aggregation operator proposed, and other generalizations are deduced. In section 4, a numerical example is furnished upon proposed aggregation operators as an application. The data is both tabulated and graphically interpreted. Finally, a comparative analysis is conducted to validate the results obtained by the current study.

Table1.1: Table of abbreviation and notations

Abbreviation/notation	**Description**
DM	Decision Making

MADM	Multi Attribute Decision Making
MEMADM	Multi Expert Multi Attribute Decision making
FS	Fuzzy set
IVFS	Interval value fuzzy set
IFS	Intuitionistic fuzzy set
IVIFS	Interval valued Intuitionistic fuzzy set
CS	Cubic set
NS	Neutrosophic set
INS	Interval Neutrosophic set
NCS	Neutrosophic cubic set
NCWA	Neutrosophic cubic weighted arithmetic
NCWG	Neutrosophic cubic weighted geometric
IGNCSCIA	Induced generalized neutrosophic cubic Shapley choquet integral arithmetic
IGNCSCIG	Induced generalized neutrosophic cubic Shapley choquet integral geometric
GNCU	Generalized neutrosophic cubic unified
IGNCUSCI	induced generalized neutrosophic cubic unified Shapley choquet integral
Scr	Score
Acu	accuracy

2. Preliminaries

In this section some preliminaries are considered.

Definition 2.1[22] A set $\Gamma = \{(\ell, \mu(\ell)) \mid \ell \in \Xi\}$, where the mapping $\mu : \Xi \rightarrow [0,1]$ called a FS and μ is called a membership degree.

Definition 2.2[2] A pair $\{(\ell, (\mu(\ell), \nu(\ell))) \mid \ell \in \Xi\}$ of mappings where $\mu : \Xi \rightarrow [0,1]$ and $\nu : \Xi \rightarrow [0,1]$ is called an IFS. Where μ and ν are a membership and non-membership function, simply denoted by (μ, ν).

Definition 2.3[15] A set $\wp = \{(\ell, (\wp_T(\ell), \wp_I(\ell), \wp_F(\ell))) \mid \ell \in \Xi\}$ is called an NS. Where

$\wp_T : \Xi \to]0^-, 1^+[$, $\wp_I : \Xi \to]0^-, 1^+[$, $\wp_F : \Xi \to]0^-, 1^+[$ such that

$0^- \le \wp_T(\ell) + \wp_I(\ell) + \wp_F(\ell) \le 3^+$ and $\wp_T, \wp_I, \wp_F$ are called truth, indeterminacy, and falsity.

Definition 2.4[8] A set $\wp = \{(\ell, (\tilde{\wp}_T(\ell), \tilde{\wp}_I(\ell), \tilde{\wp}_F(\ell)), (\wp_T(\ell), \wp_I(\ell), \wp_F(\ell))) \mid \ell \in \Xi\}$ is NCS. Where $(\tilde{\wp}_T, \tilde{\wp}_I, \tilde{\wp}_F)$ is an INS and $(\wp_T, \wp_I, \wp_F)$ is a NS such that $\tilde{0} \le \tilde{\wp}_T(\ell) + \tilde{\wp}_I(\ell) + \tilde{\wp}_F(\ell) \le \tilde{3}$ and $0 \le \wp_T(\ell) + \wp_I(\ell) + \wp_F(\ell) \le 3$. Denoted by $\wp = (\tilde{\wp}_T, \tilde{\wp}_I, \tilde{\wp}_F, \wp_T, \wp_I, \wp_F) = ([\wp_T^-, \wp_T^+], [\wp_I^-, \wp_I^+], [\wp_F^-, \wp_F^+], \wp_T, \wp_I, \wp_F)$.

Definition 2.5[9] The sum, product, scalar multiplication and $\sigma - th$ power on two NCS $\wp = ([\wp_T^-, \wp_T^+], [\wp_I^-, \wp_I^+], [\wp_F^-, \wp_F^+], \wp_T, \wp_I, \wp_F)$ and $N = ([N_T^-, N_T^+], [N_I^-, N_I^+], [N_F^-, N_F^+], N_T, N_I, N_F)$ is defined by

$$\wp \oplus N = \begin{pmatrix} [\wp_T^- + N_T^- - \wp_T^- N_T^-, \wp_T^+ + N_T^+ - \wp_T^+ N_T^+], [\wp_I^- + N_I^- - \wp_I^- N_I^-, \wp_I^+ + N_I^+ - \wp_I^+ N_I^+], \\ [\wp_F^- N_F^-, \wp_F^+ N_F^+], \wp_T N_T, \wp_I N_I, \wp_F + N_F - \wp_F N_F \end{pmatrix}$$

$$\wp \otimes N = \begin{pmatrix} [\wp_T^- N_T^-, \wp_T^+ N_T^+], [\wp_I^- N_I^-, \wp_I^+ N_I^+], [\wp_F^- + N_F^- - \wp_F^- N_F^-, \wp_F^+ + N_F^+ - \wp_F^+ N_F^+], \\ \wp_T + N_T - \wp_T N_T, \wp_I + N_I - \wp_I N_I, \wp_F N_F \end{pmatrix}$$

$$\sigma\wp = \begin{pmatrix} [1-(1-\wp_T^-)^\sigma, 1-(1-\wp_T^+)^\sigma], [1-(1-\wp_I^-)^\sigma, 1-(1-\wp_I^+)^\sigma], \\ [(\wp_F^-)^\sigma, (\wp_F^+)^\sigma], (\wp_T)^\sigma, (\wp_I)^\sigma, 1-(1-\wp_F)^\sigma \end{pmatrix}$$

$$\wp^\sigma = \begin{pmatrix} [(\wp_T^-)^\sigma, (\wp_T^+)^\sigma], [(\wp_I^-)^\sigma, (\wp_I^+)^\sigma], [1-(1-\wp_F^-)^\sigma, 1-(1-\wp_F^+)^\sigma], \\ 1-(1-\wp_T)^\sigma, 1-(1-\wp_I)^\sigma, (\wp_F)^\sigma \end{pmatrix}$$

where σ is scalar.

Definition 2.8[11] The GNCU aggregation operator is generalization of some aggregation operators defined by, $GNCU(\wp_1, \wp_2, \ldots, \wp_n) = \left(\left(\sum_{j=1}^{m} C_j \left(\sum_{i=1}^{n} w_i^j \wp_i^{\lambda_j} \right)^{1/\lambda_j} \right)^q \right)^{1/q}$

C_j is relevance of sub-aggregation operator with $0 \le C_j \le 1$ and $\sum C_j = 1$, w_p^q is the pth weight of qth weight vector with $0 \le w_p^q \le 1$ and $\sum_{i=1}^{m} w_i^j = 1$, $\lambda_j \in R$ is a parameter and A_i is NCS. The parameter q ranges, $q \in (0, \infty)$. Usually, λ_j remain unchanged however in complex types of aggregations one can assigned different values to λ_j.

Definition 2.9 [5,17] The Shapley index is defined as

$$\varpi_S^{Sh}(\theta, N) = \sum_{T \subseteq N \setminus S} \frac{(\wp - u - v)!t!}{(\wp - u + 1)!t!}\left(\theta(K \cup L) - \theta(L)\right), \qquad \forall K \subseteq N$$

Where u, $\wp$ and v respectively denotes the cardinalities of N K and L. θ denotes fuzzy function of fuzzy measure λ on N.

Definition 2.10[5,17] Meng proposed generalized Shapley index by $\lambda-$ fuzzy measure θ_λ on N by

$$\varpi_S^{Sh}(\theta_\lambda, N) = \sum_{T \subseteq N \setminus S} \frac{(\wp - u - v)!v!}{(\wp - u + 1)!v!}\left(\theta_\lambda(K \cup L) - \theta_\lambda(L)\right), \forall K \subseteq N$$

where $\theta_\lambda -$ fuzzy measure expressed as

$$\theta_\lambda(A) = \begin{cases} \frac{1}{\lambda}\left(\prod_{k \in \mathcal{T}}\left((\lambda\theta_\lambda(\text{ʌ}) + 1) - 1\right)\right) & if\, \lambda \neq 0 \\ \sum_{k \in \mathcal{T}} \lambda\theta_\lambda(k) & if\, \lambda = 0 \end{cases}$$

where $\prod_{c \in N}(1 + \theta_\lambda\{c_k\}) = 1 + \theta_\lambda$ measure of λ. Furthermore, if $S = \{k\}$, then

$$\varpi_k^{Sh}(\theta_\lambda, N) = \sum_{T \subseteq N \setminus S} \frac{t!(\wp - t - 1)!}{t!(\wp)!}\left(\theta_\lambda(\text{ʌ})\left(\prod_{j \in T}(1 + \lambda\theta_\lambda(\text{ʌ}))\right)\right), \ \forall K \subseteq N$$

Arithmetic λ −Shapley Choquet integral operator as

$$C_{\varpi_S^{Sh}(\theta_\lambda, N)}\left(k\left(m_{\rho(\text{ʌ})}\right)\right) = \sum_{\text{ъ}=1}^{\ddot{n}} k\left(m_{\rho(\text{ʌ})}\right) \Theta\left(\hat{\Upsilon}_{(\hat{\mathcal{T}})_{\rho(\text{ʌ})}}\right)$$
$$where\left(\hat{\Upsilon}_{(\hat{\mathcal{T}})_{\rho(\text{ʌ})}}\right) = \left(\varpi^{Sh}_{(\hat{\mathcal{T}})_{\rho(\text{ʌ})}}(\theta_\lambda, N) - \varpi^{Sh}_{(\hat{\mathcal{T}})_{\rho(\text{ʌ}+1)}}(\theta_\lambda, N)\right)$$

for $\rho(\text{ʌ})$ as $(\rho(1), ..., \rho(\wp))$ present the permutation of $(1, ..., \wp)$ such that $k(m_{(1)}) \leq ... \leq k(m_{(\wp)})$ and $\hat{\mathcal{T}} = \{m_1, ..., m_\wp\}$ with $\hat{\mathcal{T}}_{\rho(\wp+1)} = \varphi$.

Geometric λ −Shapley Choquet integral operator as

$$C_{\varpi_S^{Sh}(\eta_\lambda, N)}\left(k\left(m_{\rho(\text{ʌ})}\right)\right) = \prod_{\text{ʌ}=1}^{\wp}\left(k\left(m_{\rho(\text{ʌ})}\right)\right)^{\Theta\left(\hat{\Upsilon}_{(\hat{\mathcal{T}})_{\rho(\text{ʌ})}}\right)}$$
$$where\left(\hat{\Upsilon}_{(\hat{\mathcal{T}})_{\rho(\text{ʌ})}}\right) = \left(\varpi^{Sh}_{(\hat{\mathcal{T}})_{\rho(\text{ʌ})}}(\theta_\lambda, N) - \varpi^{Sh}_{(\hat{\mathcal{T}})_{\rho(\text{ʌ}+1)}}(\theta_\lambda, N)\right)$$

for $\rho(\text{ʌ})$ as $(\rho(1), ..., \rho(\wp))$ present the permutations of $(1, ..., \wp)$ such that $k(m_{(1)}) \leq ... \leq k(m_{(\wp)})$ and $\hat{\mathcal{T}} = \{m_1, ..., m_\wp\}$ with $\hat{\mathcal{T}}_{\rho(\wp+1)} = \varphi$.

Definition 2.11[10] Let $\Xi_\Lambda = \left(\left[\left(\Xi_\Lambda{}^L\right)_T, \left(\Xi_\Lambda{}^U\right)_T\right], \left[\left(\Xi_\Lambda{}^L\right)_I, \left(\Xi_\Lambda{}^U\right)_I\right], \left[\left(\Xi_\Lambda{}^L\right)_F, \left(\Xi_\Lambda{}^U\right)_F\right], (\Xi_\Lambda)_T, (\Xi_\Lambda)_I, (\Xi_\Lambda)_F\right)$, where $(\Lambda = 1,...,\hbar)$ be the collection of NCS, and θ be a fuzzy measure on $\hat{T} = \{\Xi_1,...,\Xi_\hbar\}$ such that $\theta(\Xi_\Lambda) = \hat{u}_\Lambda$, then the IGNCSCIA operator is defined as

$$IGNCSCIA_{\theta_\lambda}\left(\langle \hat{u}_1, \Xi_1\rangle, ..., \langle \hat{u}_\hbar, \Xi_\hbar\rangle\right) = \left(\bigoplus_{\Lambda=1}^{\hbar} k\left(\Xi_\Lambda\right)^q \Theta\left(\hat{\Upsilon}_{\rho(\hat{T})_{(\Lambda)}}\right)^{1/q}\right)$$

, where $q \in (0,\infty)$ and $\rho(\Lambda)$ as $(\rho(1),...,\rho(\Lambda),...,\rho(\hbar))$ being the permutation of $(1,..,\Lambda,..,\hbar)$ such that, $k(\Xi_{(1)}) \le ... \le k(\Xi_{(\hbar)})$ and $\hat{T} = \{\Xi_1,...,\Xi_\hbar\}$ with $\hat{T}_{\rho(\hbar+1)} = \varphi$.

Definition 2.12 [10]Let $\Xi_\Lambda = \left(\left[\left(\Xi_\Lambda{}^L\right)_T, \left(\Xi_\Lambda{}^U\right)_T\right], \left[\left(\Xi_\Lambda{}^L\right)_I, \left(\Xi_\Lambda{}^U\right)_I\right], \left[\left(\Xi_\Lambda{}^L\right)_F, \left(\Xi_\Lambda{}^U\right)_F\right], (\Xi_\Lambda)_T, (\Xi_\Lambda)_I, (\Xi_\Lambda)_F\right)$, where $(\Lambda = 1,...,\hbar)$ be a collection of NC values, and θ be a fuzzy measure on $\hat{T} = \{\Xi_1,...,\Xi_\hbar\}$ such that $\theta(\Xi_\Lambda) = \hat{u}_\Lambda$, then the IGNCSCIG, operator is defined as operator is defined as

$$IGNCSCIG_{\theta_\lambda}\left(\langle \hat{u}_1, \Xi_1\rangle, ..., \langle \hat{u}_\hbar, \Xi_\hbar\rangle\right) = \frac{1}{q}\left(\bigotimes_{\Lambda=1}^{\hbar}\left(q\left(k\left(\Xi_{\rho(\Lambda)}\right)\right)\right)^{\Theta\left(\hat{\Upsilon}_{\rho(\hat{T})_{\rho(\Lambda)}}\right)}\right)$$

, where $q \in (0,\infty)$ and $\rho(\Lambda)$ as $(\rho(1),...,\rho(\Lambda),...,\rho(\hbar))$ extant the permutation of $(1,..,\Lambda,..,\hbar)$ such that, $k(\Xi_{(1)}) \le ... \le k(\Xi_{(\hbar)})$ and $\hat{T} = \{\Xi_1,...,\Xi_\hbar\}$ with $\hat{T}_{\rho(\hbar+1)} = \varphi$.

Definition 2.13 [9] For an NCS $\wp = \left(\left[\wp_T^-, \wp_T^+\right], \left[\wp_I^-, \wp_I^+\right], \left[\wp_F^-, \wp_F^+\right], \wp_T, \wp_I, \wp_F\right)$, the score is defined as

$$Scr(\wp) = \left[\wp_T^- - \wp_F^- + \wp_T^+ - \wp_F^+ + \wp_T - \wp_F\right]$$

Definition 2.14 [9] For an NCS $\wp = \left(\left[\wp_T^-, \wp_T^+\right], \left[\wp_I^-, \wp_I^+\right], \left[\wp_F^-, \wp_F^+\right], \wp_T, \wp_I, \wp_F\right)$, the accuracy is defined as

$$Acu(\wp) = \frac{1}{9}\left\{\wp_T^- + \wp_I^- + \wp_F^- + \wp_T^+ + \wp_I^+ + \wp_F^+ + \wp_T + \wp_I + \wp_F\right\}.$$

Definition 2.15 [9] Let $\wp_1$, $\wp_2$ be two NCS. Then

1). $Scr(\wp_1) > Scr(\wp_2) \Rightarrow \wp_1 > \wp_2$

2). If $Scr(\wp_1) = Scr(\wp_2)$

i). $Acu(\wp_1) > Acu(\wp_2) \Rightarrow \wp_1 > \wp_2$

ii). $Acu(\wp_1) = Acu(\wp_2) \Rightarrow \wp_1 = \wp_2$

3 Induced Generalized Neutrosophic Cubic Unified Shapley Choquet Integral Aggregation Operator

A complex frame of work in decision-making is a scenario that typically requires consideration of multiple and conflicting objectives, uncertainty, and ambiguity. The goal is to find a solution that balances these factors and satisfies the decision-maker's preferences as much as possible. To handle such complexity, the generalization provides a better platform. Hence, aggregation operators are a crucial part of DM theory. Many aggregation operators like arithmetic, geometric, hybrid, Shapley Choquet integral operators, and many more are defined so for. The central theme of this section is to unify the generalized aggregation operator [11] and induced generalized Shapley-Choquet (arithmetic and geometric) aggregation operator [10] to a single aggregation operator that balances inconsistent and uncertain information in the data and satisfies the decision maker's preferences as much as possible. This generalization is named an induced generalized neutrosophic cubic unified Shapley Choquet integral (IGNCUSCI) aggregation operator.
The IGNCUSCI operator is defined as;

$$IGNCUSCI_{\theta_\lambda}\left(\langle \hat{u}_1, \Xi_1\rangle, \dots, \langle \hat{u}_n, \Xi_n\rangle\right) = \left(\left(\sum_{j=1}^{m} C_j \left(\sum_{i=1}^{n}\left(\Theta\left(\hat{\Upsilon}_{\rho(\hat{T})_{(i)}}\right)\right)^{j} \left(k(\Xi_i)\right)^{\lambda_j}\right)^{1/\lambda_j}\right)^{q}\right)^{1/q} \quad (3.1)$$

C_j is relevance of sub-aggregation operator with $0 \le C_j \le 1$ and $\sum_{j=1}^{m} C_j = 1$, w_i^j is the ith weight of jth weighing vector with $w_i^j \in [0,1]$ and $\sum_{i=1}^{m} w_i^{\ j} = 1$, $\lambda_j \in R$ is a parameter and A_i is NCS. The parameter q ranges, $q \in (0,\infty)$. The parameter q ranges, $q \in (0,\infty)$. Usually, λ_j remains same but different values will be assigned in complex types of aggregations.

3.1 The Generalized Neutrosophic Cubic Unified Aggregation Operator

For $\hat{u}_i = \Xi_i = \wp_i$ for all i, shapley measures and choquet integral are considered, the equation (3) reduces into GNCU aggregation operator. The GNCU aggregation operator is generalization of some aggregation operators defined by,

$$GNCU(\wp_1, \wp_2, \dots, \wp_n) = \left(\left(\sum_{j=1}^{m} C_j \left(\sum_{i=1}^{n} w_i^j \wp_i^{\lambda_j}\right)^{1/\lambda_j}\right)^{q}\right)^{1/q}$$

(3.1..1)

C_j is relevance of sub-aggregation operator with $0 \le C_j \le 1$ and $\sum C_j = 1$, w_p^q is the pth weight of qth vector with $0 \le w_p^q \le 1$ and $\sum_p w_p^q = 1$, $\lambda_j \in R$ is a parameter and A_i is NCS. The parameter q ranges, $q \in (0,\infty)$. Usually, λ_j remain unchanged however in complex types of aggregations different values can be assigned.
Considering the types of problem under discussion different families of aggregation operators are analyzed by values are assign to λ_j and q. Some aggregation operators are deduced by assigning the values.

- If $\lambda = q = 1$ and $C_1 = 1, C_2 = C_3 = \cdots = C_n = 0$, the GNCU aggregation operator reduces to neutrosophic cubic weighted arithmetic NCWA operator.

$$NCWA(\wp_1, \wp_2, \dots, \wp_n) = \sum_{i=1}^{n} w_i \wp_i$$

- If $\lambda = q = 1$ and $C_2 = 1, C_1 = C_3 = \cdots = C_n = 0$, the GNCU aggregation operator reduces to neutrosophic cubic weighted arithmetic NCOWA operator.

  $NCWA(\wp_1, \wp_2, ..., \wp_n) = \sum_{i=1}^{n} w_{\sigma(i)}\wp_i$, where $\sigma(i)$ represents ordering position.

- If $\lambda \to 1, q = 1$ and $C_1 = 1, C_2 = C_3 = \cdots = C_n = 0$, the GNCU aggregation operator reduces to neutrosophic cubic weighted arithmetic NCWA operator.

$$NCWG(\wp_1, \wp_2, ..., \wp_n) = \prod_{i=1}^{n} (\wp_i)^{w_i}.$$

- If $\lambda \to 1, q = 1$ and $C_2 = 1, C_1 = C_3 = \cdots = C_n = 0$, the GNCU aggregation operator reduces to neutrosophic cubic weighted arithmetic NCWA operator.

$$NCWG(\wp_1, \wp_2, ..., \wp_n) = \prod_{i=1}^{n} (\wp_i)^{w_{\sigma(i)}},$$ where $\sigma(i)$ represents ordering position.

- If $\lambda = 2, q = 1$ and $C_1 = 1, C_2 = C_3 = \cdots = C_n = 0$, the GNCU aggregation operator reduces into NCQA.

$$NCQA(\wp_1, \wp_2, ..., \wp_n) = \left(\sum_{i=1}^{n} w_i^j \wp_i^2\right)^{1/2}$$

Assigning different values to λ_j and δ, some other family of aggregation operators can be reduced. These values depend on the type of problem under discussion.

The averaging aggregation operators have a most practical operator among their competitors, but in some situations, other operators like geometric, quadratic, cubic operators are in a much better position to evaluate the values.

- If $\lambda = \delta = 1$ and for all *j*, the aggregation operator is deduced to NCUA.

$$NCUA(\wp_1, \wp_2, ..., \wp_n) = \sum_{j=1}^{m} C_j \sum_{i=1}^{n} w_i^j \wp_i$$

- If $\lambda \to 1, \delta = 1$ and for all *j*, the aggregation operator is deduced to NCUG.

$$NCUG(\wp_1, \wp_2, ..., \wp_n) = \sum_{j=1}^{m} C_j \prod_{i=1}^{n} \wp_i^{w_j}$$

- If $\lambda = 2, \delta = 1$ and for all *j*, the aggregation operator is deduced to NCUQA.

$$NCUQA(\wp_1, \wp_2, ..., \wp_n) = \sum_{j=1}^{m} C_j \left(\sum_{i=1}^{n} w_i^j \wp_i^2\right)^{1/2}$$

- If $\lambda \to 0, \delta \to 0$ and for all *j*, the aggregation operator is deduced to NCGUG.

$$NCGUG(\wp_1, \wp_2, ..., \wp_n) = \prod_{j=1}^{m} C_j \prod_{i=1}^{n} \wp_i^{w_j}$$

- If $\lambda = 2, \delta = 2$ and for all *j*, the aggregation operator is deduced to NCQUQA.

$$NCQUQA(\wp_1, \wp_2, ..., \wp_n) = \left(\sum_{j=1}^{m} C_j^2 \left(\sum_{i=1}^{n} w_i^j \wp_i^2\right)^{1/2}\right)^{1/2}$$

Note that the complex and simple aggregation operators can be studied by assigning different values not only to λ and δ but to the weight as well.

3.2 Induced generalized neutrosophic cubic unified choquet integral aggregation operator.

In this section some complex aggregation operators are obtained from IGNCUSCI. Observe that different scenario may be constructed by assigning different values to λ and δ. The families of aggregation operators can be deduced by assigning different values to λ and q in IGNCUSCI and $C_1 = 1, C_j = 0$ for $j > 1$ .the following family of aggregations operators can be obtained.

3.2.1 Induced generalized neutrosophic cubic Shapley choquet arithmetic aggregation operators.

In this subsection the family of IGNCSCA operator is reduced from IGNCUSCI.

- If $\lambda = 1$, the IGNCUSCI (3.1.1)is reduced to IGNCSCIA operator.

$$IGNCSCIA_{\theta_\lambda}\left(\langle \hat{u}_1, \Xi_1\rangle, ..., \langle \hat{u}_n, \Xi_n\rangle\right) = \left(\overset{\wp}{\underset{i=1}{\oplus}} k\left(\Xi_i\right)^q \ominus \left(\hat{\Upsilon}_{\rho(\mp)_{(i)}}\right)\right)^{1/q} \tag{3.2.1}$$

- If $q = 1$, IGNCSCIA(3.2.1) reduced into INCSCIA.

$$INCSCIA_{\theta_\lambda}\left(\langle \hat{u}_1, \Xi_1\rangle, ..., \langle \hat{u}_n, \Xi_n\rangle\right) = \left(\overset{\wp}{\underset{i=1}{\oplus}} k\left(\Xi_i\right) \ominus \left(\hat{\Upsilon}_{\rho(\mp)_{(i)}}\right)\right)$$

- If $\hat{u}_i = \Xi_i$ for each i, IGNCSCIA(3.2.1) reduced into GNCSCIA.

$$IGNCSCIA_{\theta_\lambda}\left(A_1, ..., A_n\right) = \left(\overset{\wp}{\underset{i=1}{\oplus}} k\left(\Xi_i\right)^q \ominus \left(\hat{\Upsilon}_{\rho(\mp)_{(i)}}\right)\right)^{1/q}$$

- If $\hat{u}_i = \Xi_i$ and $q = 1$ for each i, IGNCSCIA (3.2.1) reduced into NCSCIA.

$$INCSCIA_{\theta_\lambda}\left(A_1, ..., A_n\right) = \left(\overset{\wp}{\underset{i=1}{\oplus}} k\left(\Xi_i\right) \ominus \left(\hat{\Upsilon}_{\rho(\mp)_{(i)}}\right)\right)$$

3.2.1 Induced generalized neutrosophic cubic Shapley choquet geometric aggregation operators.

In this subsection the family of IGNCSCG operator is reduced from IGNCUSCI.

- If $\lambda \to 1$, the IGNCUSCI reduced into IGNCSCIG.

$$IGNCSCIG_{\theta_\lambda}\left(\langle \hat{u}_1, \Xi_1\rangle, ..., \langle \hat{u}_\wp, \Xi_\wp\rangle\right) = \frac{1}{q}\left(\overset{\wp}{\underset{i=1}{\otimes}}\left(q\left(k\left(\Xi_{\rho(i)}\right)\right)\right)^{\ominus\left(\hat{\Upsilon}_{\rho(\mp)_{\rho(i)}}\right)}\right) \tag{3.2.2}$$

- $q = 1$, IGNCSCIG (3.1.2) reduce into INCSCIG.

$$IGNCSCIG_{\theta_\lambda}\left(\langle \hat{u}_1, \Xi_1\rangle, ..., \langle \hat{u}_\wp, \Xi_\wp\rangle\right) = \left(\overset{\wp}{\underset{i=1}{\otimes}}\left(k\left(\Xi_{\rho(i)}\right)\right)^{\ominus\left(\hat{\Upsilon}_{\rho(\mp)_{\rho(i)}}\right)}\right)$$

- $\hat{u}_i = \Xi_i$ for each i, IGNCSCIG (3.1.2) reduce into GNCSCIG.

$$IGNCSCIG_{\theta_\lambda}\left(A_1, ..., A_n\right) = \frac{1}{q}\left(\overset{\wp}{\underset{i=1}{\otimes}}\left(q\left(k\left(\Xi_{\rho(i)}\right)\right)\right)^{\ominus\left(\hat{\Upsilon}_{\rho(\mp)_{\rho(i)}}\right)}\right)$$

- $\hat{u}_i = \Xi_i$ and $q = 1$ for each i, IGNCSCIG (3.1.2) reduce into NCSCIG.

$$IGNCSCIG_{\theta_\lambda}\left(A_1, ..., A_n\right) = \left(\overset{\wp}{\underset{i=1}{\otimes}}\left(k\left(\Xi_{\rho(i)}\right)\right)^{\ominus\left(\hat{\Upsilon}_{\rho(\mp)_{\rho(i)}}\right)}\right)$$

4 Application 1

In this section a numerical example is furnished upon proposed IGNCUSCI aggregation operator as an application. A company is interested in expanding its foreign investment. There is a list of five possible alternatives (countries) $N = \{₪_1, ₪_2, ₪_3, ₪_4, ₪_5\}$, and to choose the best country(alternative) from given list. The attribute set is listed as $\Lambda = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ and attributes are resources, policies, economy, and infrastructure. Khan et al., applied different techniques. The proposed operators are applied to the data. The comparative analysis is performed at the end for validation of results.

First the data is transformed into NC form the NC values are.

$$\left(\sigma_1 \begin{bmatrix} b_{11} = ([0.70, 0.80], [0.50, 0.70], [0.10, 0.20], 0.90, 0.70, 0.20) \\ b_{12} = ([0.60, 0.80], [0.40, 0.60], [0.40, 0.60], 0.20, 0.30, 0.70) \\ b_{13} = ([0.40, 0.50], [0.50, 0.60], [0.40, 0.60], 0.30, 0.30, 0.30) \\ b_{14} = ([0.60, 0.80], [0.50, 0.60], [0.40, 0.50], 0.50, 0.40, 0.50) \\ b_{15} = ([0.60, 0.70], [0.40, 0.50], [0.40, 0.50], 0.50, 0.40, 0.50) \end{bmatrix}\right)$$

$$\left(\sigma_2 \begin{bmatrix} b_{21} = ([0.60, 0.80], [0.40, 0.50], [0.30, 0.30], 0.90, 0.80, 0.20) \\ b_{22} = ([0.50, 0.70], [0.40, 0.60], [0.10, 0.30], 0.60, 0.40, 0.20) \\ b_{23} = ([0.60, 0.70], [0.40, 0.60], [0.30, 0.40], 0.80, 0.70, 0.50) \\ b_{24} = ([0.5, 0.60], [0.30, 0.40], [0.40, 0.50], 0.40, 0.40, 0.30) \\ b_{25} = ([0.80, 0.90], [0.30, 0.40], [0.10, 0.20], 0.70, 0.40, 0.30) \end{bmatrix}\right)$$

$$\left(\sigma_3 \begin{bmatrix} b_{31} = ([0.80, 0.80], [0.40, 0.60], [0.10, 0.20], 0.80, 0.90, 0.10) \\ b_{32} = ([0.60, 0.60], [0.20, 0.30], [0.40, 0.50], 0.50, 0.10, 0.30) \\ b_{33} = ([0.70, 0.80], [0.60, 0.70], [0.10, 0.20], 0.50, 0.40, 0.40) \\ b_{34} = ([0.60, 0.80], [0.50, 0.60], [0.10, 0.20], 0.50, 0.70, 0.30) \\ b_{35} = ([0.70, 0.80], [0.50, 0.60], [0.10, 0.20], 0.50, 0.30, 0.40) \end{bmatrix}\right)$$

$$\left(\sigma_4 \begin{bmatrix} b_{41} = ([0.70, 0.70], [0.30, 0.40], [0.20, 0.20], 0.80, 0.60, 0.20) \\ b_{42} = ([0.60, 0.80], [0.40, 0.40], [0.20, 0.40], 0.70, 0.30, 0.50) \\ b_{43} = ([0.50, 0.60], [0.50, 0.60], [0.20, 0.30], 0.60, 0.60, 0.40) \\ b_{44} = ([0.80, 0.90], [0.30, 0.40], [0.10, 0.20], 0.70, 0.50, 0.40) \\ b_{45} = ([0.50, 0.70], [0.50, 0.50], [0.20, 0.30], 0.60, 0.50, 0.30) \end{bmatrix}\right)$$

The Shapley measure is used to evaluate the interactive dependence between attribute, so their weight are measured by Shapley measure presented in the table.

Table 4.1: *The Criteria along with Fuzzy Shapley Measures*

Λ	$\mu_\lambda(\Lambda)$	Λ	$\mu_\lambda(\Lambda)$

ϕ	0	$\{\sigma_1\}$	0.664
$\{\sigma_2\}$	0.077	$\{\sigma_3\}$	0.593
$\{\sigma_4\}$	0.260	$\{\sigma_1, \sigma\}$	0.144
$\{\sigma_1, \sigma_3\}$	0.659	$\{\sigma_1, \sigma_4\}$	0.327
$\{\sigma_2, \sigma_3\}$	0.671	$\{\sigma_2, \sigma_4\}$	0.3384
$\{\sigma_3, \sigma_4\}$	0.858	$\{\sigma_1, \sigma_2, \sigma_3\}$	0.739
$\{\sigma_1, \sigma, \sigma_4\}$	0.405	$\{\sigma, \sigma_3, \sigma\}$	0.922
$\{\sigma_2, \sigma_3, \sigma_4\}$	0.993	Λ	1

These interactive weights are used to measure the weight for the attributes (2.10).

$$\varpi_1^{Sh} = 0.099, \varpi_2^{Sh} = 0.010, \varpi_3^{Sh} = 0.599, \varpi_4^{Sh} = 0.292 .$$

In order to apply IGNCUSCI to this numerical problem first $\lambda = 1$, this reduce IGNCUSCI(3.1) into IGNCSCIA(3.2.1) operator. Since INCSCIA has different values for q so different values are assigned to q and the aggregated values is obtained. Here the values obtained for $q = 1$ are written and other values obtained from different values of q just tabulated in table so that unnecessary length is being avoided.

Then the aggregated values of IGNCUSCI calculated for $\lambda = 1$, $q = 1$.

$$= \begin{bmatrix} \tilde{u}_1 = ([0.735, 0.862], [0.412, 0.557], [0.158, 0.204], 0.172, 0.267, 0.191) \\ \tilde{u}_2 = ([0.594, 0.746], [0.383, 0.468], [0.137, 0.329], 0.531, 0.207, 0.375) \\ \tilde{u}_3 = ([0.648, 0.750], [0.528, 0.670], [0.165, 0.270], 0.588, 0.516, 0.480) \\ \tilde{u}_4 = ([0.662, 0.801], [0.406, 0.511], [0.156, 0.267], 0.444, 0.449, 0.358) \\ \tilde{u}_5 = ([0.761, 0.872], [0.451, 0.528], [0.093, 0.188], 0.573, 0.348, 0.369) \end{bmatrix}$$

The alternatives are ranked in IGNCUSCI $(\lambda = 1)$ for different values of q and rankings are tabulated in Table 4.2.

Table 4.2: The Ranking for different values of r

Tabulated view of IGNCUSCI $(\lambda = 1)$ for different values of q	
q	Ranks
$q = 0.1$	$₪_1 > ₪_3 > ₪_5 > ₪_4 > ₪_2$
$q = 0.5$	$₪_1 > ₪_3 > ₪_5 > ₪_4 > ₪_2$
$q = 1$	$₪_1 > ₪_3 > ₪_5 > ₪_4 > ₪_2$
$q = 2$	$₪_1 > ₪_3 > ₪_5 > ₪_4 > ₪_2$
$q = 3$	$₪_1 > ₪_3 > ₪_5 > ₪_4 > ₪_2$
$q = 5$	$₪_1 > ₪_3 > ₪_5 > ₪_4 > ₪_2$

The graphical presentation is given below for IGNCUSCI $(\lambda = 1)$ and different values of q operators.

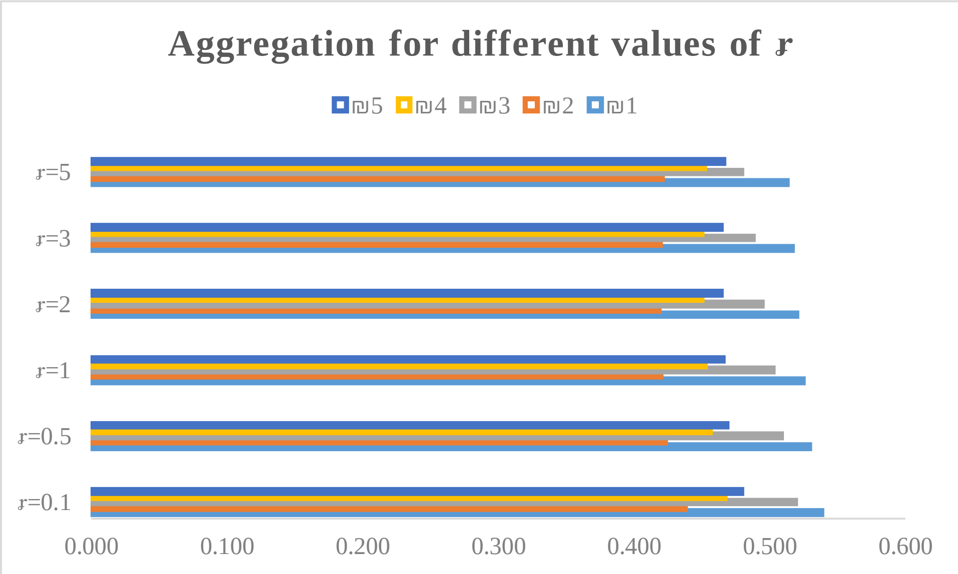

Figure 4.1 The Graphical Presentation of IGNCSCIA

To apply IGNCUSCI to this numerical problem first $\lambda \to 1$ this reduce IGNCUSCI operator into IGNCSCIG operator.

Then the aggregated values of IGNCUSCI $(\lambda \to 1)$ calculated. ($ɼ = 1$)

$$=\begin{bmatrix} \tilde{u}_{31} = \left([0.644, 0.803], [0.339, 0.468], [0.251, 0.271], 0.888, 0.843, 0.869\right) \\ \tilde{u}_{32} = \left([0.518, 0.663], [0.286, 0.372], [0.268, 0.421], 0.634, 0.343, 0.757\right) \\ \tilde{u}_{33} = \left([0.560, 0.664], [0.436, 0.597], [0.261, 0.370], 0.731, 0.646, 0.598\right) \\ \tilde{u}_{34} = \left([0.540, 0.664], [0.311, 0.414], [0.317, 0.415], 0.560, 0.573, 0.714\right) \\ \tilde{u}_{35} = \left([0.645, 0.786], [0.350, 0.436], [0.158, 0.260], 0.665, 0.432, 0.708\right) \end{bmatrix}$$

The alternatives are ranked in IGNCUSCI $(\lambda \to 1)$ for different values of ɼ and rankings are tabulated in Table 4.3.

Table 4.3: The Ranking Based on values of ɼ

Tabulated view of IGNCUSCI $(\lambda \to 1)$ for different values of ɼ	
ɼ	Ranks
$ɼ = 0.1$	$₪_1 > ₪_3 > ₪_5 > ₪_4 > ₪_2$
$ɼ = 0.5$	$₪_1 > ₪_3 > ₪_5 > ₪_4 > ₪_2$
$ɼ = 1$	$₪_1 > ₪_3 > ₪_5 > ₪_4 > ₪_2$
$ɼ = 2$	$₪_1 > ₪_3 > ₪_4 > ₪_5 > ₪_2$
$ɼ = 3$	$₪_3 > ₪_1 > ₪_5 > ₪_4 > ₪_2$
$ɼ = 5$	$₪_3 > ₪_1 > ₪_5 > ₪_4 > ₪_2$

₪The graphical presentation is given below for IGNCUSCI $(\lambda \to 1)$ and different values of ɼ.

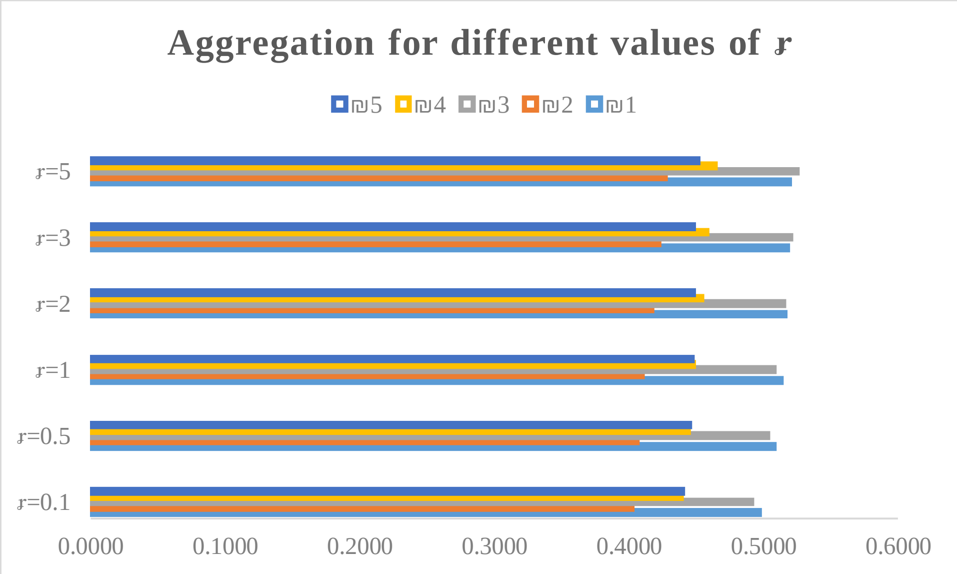

Figure 4.2: The Graphical Presentation of IGNCUSCI $(\lambda \rightarrow 1)$

Sensitivity Analysis

The table and figure show that by IGNCSCIG, the aggregated values have different rankings for an alternative. Which makes it a sensitive choice to use it for different values of q. Assigning different values to q and comparing them with other MADM methods, it is observed that the values near 0 in the aggregation operators give results that better match existing methods. This can be overcome by applying a distance-based method like the NCCODAS (see [10]). Otherwise, IGNCSCIA has a ranking that agrees with most existing methods.

Comparative Analysis

To determine the validity of results obtained by current study different existing techniques are applied on above problem. It is worth mentioning that the weight used in these methods were considered from fuzzy Shapley measures. The rankings are tabulated in a table.

Table 4.4: Comparative Analysis with Existence Methods

Method	**Ranking**
NC Einstein geometric aggregation operator in MCDM	$₪_1 > ₪_3 > ₪_2 > ₪_5 > ₪_4$
NC averaging aggregation operators with application MADM	$₪_1 > ₪_3 > ₪_5 > ₪_4 > ₪_2$
GRA MADM in NCS [14]	$₪_1 > ₪_5 > ₪_3 > ₪_2 > ₪_4$
Cosine measure of NCS for MCDM [15]	$₪_1 > ₪_2 > ₪_5 > ₪_3 > ₪_4$
NCCODAS [10]	$₪_1 > ₪_3 > ₪_5 > ₪_4 > ₪_2$
Proposed IGNCSCIA	$₪_1 > ₪_3 > ₪_5 > ₪_4 > ₪_2$
Proposed IGNCSCIG $(q \rightarrow 0)$	$₪_1 > ₪_3 > ₪_5 > ₪_4 > ₪_2$
Proposed IGNCSCIG $(q > 2)$	$₪_3 > ₪_1 > ₪_5 > ₪_4 > ₪_2$

From the table it can be observed that the ranking of proposed aggregation operators matches with MADM methods in table with best alternative. Whereas in case of worse alternative at agree with NCCODAS and NC averaging. The second-best alternative is also agreed to NC averaging, NC geometric and NCCODAD method. It is also observed that IGNCSCIG agree to IGNCSCIA for value $(q \leq 2)$ and for $(q > 2)$ the IGNCSCIG ranks different than other method in table. By applying IGNCSCIG one must be aware of sensitive due to different values of q.

Application 2

Cellular companies play a important role in country's stock market. The capital marcket is affected by the performance of these celluar compnies. A company is interested to invest his capital levy in listed companies $\{A_1, A_2, A_3, A_4\}$. Two types of experts are aquired. Attorney D_a look legal matters and market maker D_m expertise in market. Data are collected on the basis of stock market analysis and growth in different areas. The three alternatives are (Ω_1) trends of stock market, (Ω_2) policy directions and (Ω_3) annual performance. The two experts proposed their DM matrices consist of NCS.

The equation 3.1.1 for $\lambda \to 1, q = 1$ is applied throughout this application.

Step 1. We construct the decision maker matrices.

DM matrix for the D_a is

$$\begin{pmatrix} & \Omega_1 & \Omega_2 & \Omega_3 \\ A_1 & \begin{pmatrix} [0.20,0.60],[0.40,0.60], \\ [0.50,0.80],0.70,0.40,0.30 \end{pmatrix} & \begin{pmatrix} [0.10,0.40],[0.50,0.80], \\ [0.40,0.80],0.60,0.70,0.50 \end{pmatrix} & \begin{pmatrix} [0.40,0.60],[0.20,0.70], \\ [0.50,0.90],0.40,0.50,0.30 \end{pmatrix} \\ A_2 & \begin{pmatrix} [0.30,0.50],[0.60,0.90], \\ [0.30,0.60],0.30,0.60,0.70 \end{pmatrix} & \begin{pmatrix} [0.50,0.90],[0.10,0.30], \\ [0.40,0.80],0.80,0.30,0.60 \end{pmatrix} & \begin{pmatrix} [0.20,0.70],[0.10,0.60], \\ [0.40,0.70],0.50,0.40,0.70 \end{pmatrix} \\ A_3 & \begin{pmatrix} [0.60,0.90],[0.20,0.70], \\ [0.40,0.90],0.50,0.50,0.60 \end{pmatrix} & \begin{pmatrix} [0.20,0.60],[0.30,0.70], \\ [0.30,0.80],0.40,0.60,0.50 \end{pmatrix} & \begin{pmatrix} [0.50,0.90],[0.70,0.90], \\ [0.10,0.50],0.50,0.60,0.40 \end{pmatrix} \\ A_4 & \begin{pmatrix} [0.40,0.80],[0.50,0.90], \\ [0.30,0.80],0.50,0.80,0.50 \end{pmatrix} & \begin{pmatrix} [0.20,0.70],[0.40,0.90], \\ [0.50,0.70],0.60,0.40,0.50 \end{pmatrix} & \begin{pmatrix} [0.30,0.50],[0.50,0.90], \\ [0.30,0.70],0.30,0.30,0.80 \end{pmatrix} \end{pmatrix}$$

DM matrix for D_m is

$$\begin{pmatrix} & \Omega_1 & \Omega_2 & \Omega_3 \\ A_1 & \begin{pmatrix} [0.30,0.60],[0.20,0.60], \\ [0.20,0.60],0.80,0.70,0.20 \end{pmatrix} & \begin{pmatrix} [0.30,0.80],[0.40,0.80], \\ [0.30,0.80],0.60,0.70,0.40 \end{pmatrix} & \begin{pmatrix} [0.20,0.70],[0.20,0.60], \\ [0.30,0.80],0.50,0.30,0.50 \end{pmatrix} \\ A_2 & \begin{pmatrix} [0.20,0.50],[0.60,0.90], \\ [0.30,0.70],0.40,0.80,0.70 \end{pmatrix} & \begin{pmatrix} [0.40,0.90],[0.10,0.40], \\ [0.50,0.80],0.60,0.50,0.70 \end{pmatrix} & \begin{pmatrix} [0.40,0.90],[0.10,0.40], \\ [0.50,0.80],0.60,0.50,0.70 \end{pmatrix} \\ A_3 & \begin{pmatrix} [0.50,0.90],[0.20,0.60], \\ [0.30,0.80],0.70,0.70,0.80 \end{pmatrix} & \begin{pmatrix} [0.20,0.50],[0.20,0.70], \\ [0.50,0.80],0.60,0.70,0.20 \end{pmatrix} & \begin{pmatrix} [0.30,0.50],[0.30,0.90], \\ [0.20,0.50],0.60,0.50,0.40 \end{pmatrix} \\ A_4 & \begin{pmatrix} [0.30,0.50],[0.30,0.90], \\ [0.20,0.50],0.60,0.50,0.40 \end{pmatrix} & \begin{pmatrix} [0.40,0.70],[0.20,0.80], \\ [0.30,0.70],0.60,0.70,0.70 \end{pmatrix} & \begin{pmatrix} [0.20,0.60],[0.50,0.90], \\ [0.20,0.80],0.40,0.40,0.80 \end{pmatrix} \end{pmatrix}$$

Step2. Let $W = (0.4, 0.6)^T$, then the single matrix corresponding to weight W is.

$$\begin{pmatrix} & \Omega_1 & \Omega_2 & \Omega_3 \\ \mathsf{A}_1 & \begin{pmatrix}[0.255,0.600],\\ [0.288,0.673],\\ [0.337,0.699],\\ 0.765,0.604,0.235\end{pmatrix} & \begin{pmatrix}[0.193,0.606],\\ [0.443,0.800],\\ [0.349,0.868],\\ 0.600,0.700,0.477\end{pmatrix} & \begin{pmatrix}[0.269,0.658],\\ [0.120,0.638],\\ [0.388,0.848],\\ 0.462,0.388,0.222\end{pmatrix} \\ \mathsf{A}_2 & \begin{pmatrix}[0.235,0.558],\\ [0.600,0.900],\\ [0.300,0.663],\\ 0.362,0.736,0.700\end{pmatrix} & \begin{pmatrix}[0.235,0.558],\\ [0.600,0.900],\\ [0.300,0.663],\\ 0.362,0.736,0.700\end{pmatrix} & \begin{pmatrix}[0.525,0.867],\\ [0.152,0.600],\\ [0.462,0.845],\\ 0.337,0.462,0.330\end{pmatrix} \\ \mathsf{A}_3 & \begin{pmatrix}[0.538,0.900],\\ [0.356,0.838],\\ [0.342,0.848],\\ 0.632,0.632,0.713\end{pmatrix} & \begin{pmatrix}[0.200,0.538],\\ [0.235,0.700],\\ [0.428,0.800],\\ 0.530,0.663,0.288\end{pmatrix} & \begin{pmatrix}[0.368,0.632],\\ [0.421,0.900],\\ [0.161,0.500],\\ 0.563,0.543,0.400\end{pmatrix} \\ \mathsf{A}_4 & \begin{pmatrix}[0.510,0.800],\\ [0.346,0.632],\\ [0.242,0.745],\\ 0.500,0.613,0.381\end{pmatrix} & \begin{pmatrix}[0.303,0.700],\\ [0.264,0.838],\\ [0.388,0.700],\\ 0.600,0.604,0.612\end{pmatrix} & \begin{pmatrix}[0.235,0.558],\\ [0.500,0.900],\\ [0.242,0.765],\\ 0.362,0.362,0.800\end{pmatrix} \end{pmatrix}$$

Step3. Let the weights of attributes are $W = \{0.350, 0.300, 0.350\}$, the aggregated value is.

$$\begin{pmatrix} \mathsf{A}_1 & \begin{pmatrix}[0.238,0.620],\\ [0.288,0.792],\\ [0.357,0.815],\\ 0.632,0.576,0.285\end{pmatrix} \\ \mathsf{A}_2 & \begin{pmatrix}[0.443,0.766],\\ [0.216,0.592],\\ [0.538,0.780],\\ 0.483,0.573,0.528\end{pmatrix} \\ \mathsf{A}_3 & \begin{pmatrix}[0.350,0.662],\\ [0.334,0.814],\\ [0.313,0.750],\\ 0.579,0.613,0.444\end{pmatrix} \\ \mathsf{A}_4 & \begin{pmatrix}[0.333,0.677],\\ [0.363,0.779],\\ [0.289,0.740],\\ 0.491,0.536,0.569\end{pmatrix} \end{pmatrix}$$

Step4. The alternatives are ranked by score function.

$$S(\mathsf{A}_1) = 0.032, S(\mathsf{A}_2) = 0.055, S(\mathsf{A}_3) = 0.084 \text{ and } S(\mathsf{A}_4) = -0.097,$$

$\mathsf{A}_3 > \mathsf{A}_2 > \mathsf{A}_1 > \mathsf{A}_4$. The desirable alternative is A_3.

Conclusion

The generalization of aggregation operators plays a critical role in DM and make DM theory more manageable. In this work a unified generalized aggregation operator IGNCUSCI is proposed to counter such situation. The IGNCUSCI can generate a bunch of aggregation operators by assigning value to constraints. Application 1 is evaluated by (3.2.1, 3.2.2) and tested with existing method so that their validity and applicability be tested. It is found that in the case of (3.2.2) a sensitivity arises which needs to be kept in mind.

Application 2 is evaluated using MEMADM applied on (3.1.1). Both applications provide plate form to apply aggregation operators on MADM and MEMADM.

Remark

This idea can be extend to some other aggregation operators like Bonferroni Shapley choquet integral aggregation operators.

Compliance with ethical standards

Conflicts of Interest: The authors declare no conflict of interest.

Ethical approval: This article does not contain any sudies with humain participants or animals performed by any of the authors.

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An Inventory Control Model in the Framework of COVID-19 Disruptions Considering Overage Items with Neutrosophic Fuzzy Uncertainty

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Abstract: With the goal of profit maximization and overage control, a mathematical model in a single valued triangular neutrosophic fuzzy environment has been designed to fulfil the demands of the industrial sectors during pandemic situations. This research presents that overage is a crucial factor because even in well-arranged businesses, some proportion of the items might be in overage during a prescribed time interval. Well-planned inventory is important to the successful operation of healthiest businesses. The costs of occurrence of overage factors are converted to fuzzy model, and the overall profit is calculated using the signed distance technique and compared using a numerical example. To determine how the overage will influence the overall system, a sensitivity analysis is undertaken. The ideal amount that provides the maximum value of the projected profit per unit of time is found in both crisp and fuzzy models. The suggested maximizing model will undoubtedly aid decision-makers in dealing with overage circumstances induced by pandemic social distance. The new method of this research is to model the merger of profit maximization and overage concept in the framework of COVID-19 to benefit the inventory management and supply chain sectors.

Keywords: EOQ model; Neutrosophic fuzzy; Profit maximization; COVID-19; Signed distance method

1. Introduction

COVID-19 (Coronavirus disease 2019) is a disease which has become a global pandemic and is rapidly spreading [1]. The disease had spread to almost all countries, territories and areas by 8 October 2020. Despite the installation of new production technology, production sectors require human resources to run and oversee the manufacturing process, as well as supervise and exercise inventory management to minimise shortages. Guo et al. [2] describes the status update on COVID-19 origin and transmission. When lockdown is imposed, however, the manufacturing process is slowed and inventory levels are left uncontrolled. Every manufacturing company has many layers of inventory, such as raw items [3]. Economic decline occurs because delivery of these types of items is extremely unlikely. These new comers collect the bare minimum of goods. The issue now is that if the situation is resolved, the inventory replenishment plan would be disrupted. The stock on hand accumulates, when the same system of inventory up gradation is continued, or the inventory police planner, can alter the system after calculating the required inventory level to be purchased while taking into consideration the current extra stock on hand. There is a stumbling block in managerial decision-making process; it is the issue of inventory quality. The manufacturing atmosphere is shattered by a mix of inventories with varying degrees of quality. Following the regular inventory renewal cycle is the best way to avoid such quality impediment situations. However, the next pressing concern is what to do with the surplus [4]. Inventory management strategies of a Mexican industrial goods firm during the COVID 19 epidemic to retain liquidity and improve customer service levels are discussed in [5]. In reference [6], the inventory management at functioning in Polish collective buying organisations during the COVID-19 epidemic were studied.

Relph [7] used the phrase "overage" to characterize it. He highlighted the elements that contribute to overage inventory in his study work on the subject, and said that overage inventory has some beneficial effects on effective supervision. Ritha and Jayanthi [8] expanded the concept to a fuzzy model since it was so compatible. These models focused on single-item inventory, but when multi-item inventory is considered, the single-item overage inventory model becomes an overage inventory management model for multi-item inventory. This study offers an inventory approach to address inventory management errors induced by COVID-19 social distance. Harris [9] was the first to establish an economic order quantity model, and Taft [10] was the first to develop a production inventory model. These two models serve as the foundation for inventory models. Models of inventory with shortfall costs, Backlogs, price breaks, trade discounts, deteriorating products, and environmental sustainability were all formulated to tackle the issues individually or concurrently. There are several shortage inventory models, and they have been coupled with other inventory management issues. Many studies have been published in the previous two decades in which demand is price dependent. Fuzzy EOQ model was developed by many researcher, for instance, [11], [12].

Pakhira et al. [13] during a shortage, applied the effect of memory to an inventory problem describing the demand function varying with price. Under fuzzy choice factors, Garai [14] presented an inventory problem involving time-varying holding cost. Indrajitsingha et al. [15] discussed a two-warehouse problem in partial- backlogging and fuzzy Environment. Many articles have been written about price-dependent demand in [16], [17] and [18]. Indrajitsingha et al. [19] has developed an EOQ model for perishable products during the COVID-19 pandemic. Dey et al. [20] developed a model assuming the discrete setup cost, varying safety factor, and the demand as a function of price. In the midst of a pandemic, a method for managing disruptions in supply chain was developed in

[21]. Afterwards, Deshmukh and Haleem [22] designed a mathematical model for manufacturing in the after-COVID-19 period.

A customer's daily need may vary from day to day in reality. Because of a lack of historical data or an abundance of information, evaluating a demand distribution is useful. Demand and different inventory model elements have lately been viewed as separate forms of fuzzy numbers by certain scholars. The neutrosophic set is concerned with its relationships with other aspects of the conceptual spectrum. A new strategy for resolving completely neutrosophic linear programming problem has been developed Abdel-Basset [23]. The neutrosophic set is a versatile and widely used formal framework for analysing data sets that contain uncertainty. Mondal et al. [24] presented optimal policy for stock problem with restricted storage capacity using Neutrosophic sets. Mullai and Surya [25] formulated a problem with price break using neutrosophic sets, Mullai et al. [26] and Mullai et. al. [27] used a single valued neutrosophic set to construct multiple inventory models. Surya and Mullai [28] originated on the premise of quick return for faulty items in their neutrosophic lot-size model. Pal and Chakraborty *[29-30] developed a triangular neutrosophic-based EOQ model for non-instantaneously decaying items under scarcity conditions. Authors in [32-35] developed many optimization model in neutosophic environment.* In a neutrosophic paradigm, the work could readily manage any company's inventory system. When uncertain and unexpected circumstances arise in the inventory system, this approach offers more accurate results than prior methodologies. To demonstrate the model's outputs, a sensitivity analysis for crisp and neutrosophic sets is provided, and the findings are briefly reviewed. However, overage models are not well-defined, thus this study sheds light on the state of overage inventory of several goods, with a particular focus on the pandemic epidemic. Overage management, trash disposal, and product dispersion are all part of this profit maximization inventory model.

This paper answers key questions as follows:

(1) Why does this paper apply COVID-19 inventory crisis combating model in neutrosophic environment instead of any other models?

> COVID-19's spread has sparked one of the most devastating pandemics in contemporary human past times. Humanity is still now in the early stages of learning about this terrible sickness, and dealing properly with such a big public disaster is a critical issue. Although there are differences in human natural semantics, the lack of knowledge makes it worse. The Neutrosophic set approaches are a useful tool for representing the ambiguity of genuine human semantic expression, as well as the analysis of partial and uncertain data. For example, a person providing their opinion on a product's quality, with 0.7 indicating "possibility that the quality is good," 0.2 indicating "possibility that he or she is unsure about quality," and 0.5 indicating "possibility that the quality is not good." Then this set is represented by a single neutrosophic number (0.7, 0.2, 0.5), where. $0 \leq 0.7 + 0.2 + 0.5 \leq 3$.
>
> So the benefit of the SVNS is that any indeterminacy updating in your day-to-day life if a

few people work on decisions that aren't certain, that set corresponds to the neutrosophic set. Other fuzzy models do not take into account this uncertain fact. Because it parallels human thinking and decision-making, it gives a remarkably efficient solution to difficult issues in all areas of life.

(2) What is the practical implication of this model?

The suggested maximizing model will undoubtedly aid decision-makers in dealing with overage circumstances induced by pandemic social distance.

The present paper is arranged below: Basic definitions involving Neutrosophic sets are provided in section 2. In section 3, we present some assumptions & notations. In section 4, the proposed model as well as methodology is demonstrated. Numerical experimentation is executed in section 5. In sections 6 and 7, graphical representation and sensitivity analysis have been given, respectively. In Section 8, some result and discussion are done based on the numerical study and in section 9 concluding remarks of the whole model and future scope have been discussed.

2. Preliminaries

In this section, the required preliminaries are explained here, and they are particularly relevant for the suggested model.

Definition 2.1. [31] **Neutrosophic set**

A neutrosophic set A in the universal set U defined by a truth-membership function T_A, an indeterminacy-membership function I_A, and a falsity-membership function F_A $T_A(x), I_A(x)$ and $F_A(x)$ are real members of [0,1]. Then,

$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in U, T_A(x), I_A(x), F_A(x) \in]0^-, 1^+[\}$. No restriction is applied on the sum of $T_A(x), I_A(x)$ $F_A(x)$ and, so $0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.

Definition 2.2. *[29]* **Triangular single valued neutrosophic numbers of Type 1.**

A triangular single valued Neutrosophic number of Type 1 is referred as $A_{Neu} = (a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3)$ whose truth membership, indeterminacy, and falsity membership is defined as follows:

$$T_{A_{Neu}}(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1} & when\, a_1 \le x \le a_2 \\ 1 & when \quad x = a_2 \\ \dfrac{a_3-x}{a_3-a_1} & when\, a_2 \le x \le a_3 \\ 0 & otherwise \end{cases}$$

$$I_{A_{Neu}}(x) = \begin{cases} \dfrac{b_2-x}{b_2-b_1} & when\, b_1 \le x \le b_2 \\ 0 & when \quad x = a_2 \\ \dfrac{x-a_2}{a_3-a_2} & when\, a_2 \le x \le a_3 \\ 1 & otherwise \end{cases}$$

$$F_{A_{Neu}}(x) = \begin{cases} \dfrac{c_2-x}{c_2-c_1} & when\, c_1 \le x \le c_2 \\ 0 & when \quad x = c_2 \\ \dfrac{x-c_2}{c_3-c_2} & when\, c_2 \le x \le c_3 \\ 1 & otherwise \end{cases}$$

Here, $0 \le T_{A_{Neu}}(x) + I_{A_{Neu}}(x) + F_{A_{Neu}}(x) \le 3, x \in A_{Neu}$

Definition 2.3. [29] Alpha cut

The parametric form of the triangular single-valued neutrosophic number A_{Neu} is presented as:

$$\left(A_{Neu}\right)_{\alpha,\beta,\gamma} = \left[A_L(\alpha), A_R(\alpha); A_L(\beta), A_R(\beta); A_L(\gamma), A_R(\gamma)\right].$$

Here, $A_L(\alpha) = a_1 + (a_2 - a_1)\alpha$, $A_R(\alpha) = a_3 - (a_3 - a_2)\alpha$ $A_L(\beta) = b_2 - (b_2 - b_1)\beta$ $A_R(\beta) = b_2 + (b_3 - b_2)\beta$ $A_L(\gamma) = c_2 - (c_2 - c_1)\gamma$ $A_R(\gamma) = c_2 + (c_3 - c_2)\gamma$ is the alpha cut of the A_{Neu}. Also, $\alpha \in [0,1], \beta \in [0,1]\, and\, \gamma \in [0,1]$.

2. Assumptions and Notations

2.1 Assumption

(a) The greatest stock level is exceeded on account of any of the following factors: insufficient transportation services, climatic barriers, natural disasters and lockdown during COVID-19 pandemics.

(b) It is assumed that demand changes as and when the selling price of the product changes. Mathematically, $D(S) = a - bS$. Here, a is a positive constant and varies with selling price (S), and $0 < b < \frac{a}{S}$. Further, a and b are assumed as triangular single valued neutrosophic fuzzy numbers.

(c) The expense of obtaining a new customer is referred to as the acquisition cost.

(d) The products are of the partly degraded kind.

(e) There is no end to the planned horizon.

3.2 Notations

Q : Order Quantity

P : The proportion of Q occurrences that cause overage in coming cycle.

$f(P)$: Density functions of probability for P.

K : Placing an order has a set price.

H : The greatest stock, per unit of time.

H_{SI} : Keeping a unique inventory of cost per unit of time.

S : Items in maximum supply are sold at a unit selling price.

V : Selling price per unit of overage products, v < s.

R : Re-novating an overage item's cost per unit.

W_{dc} : The expense of removing the goods from the landfill.

P_d : The percentage of overage times which is discarded.

$f(P_d)$: The density function.

P_{pc} : Cost of product dissemination.

A : Cost of acquiring a new consumer.

N : Total number of new clients gained.

T : Duration of the cycle.

4. Model Development

4.1. Crisp Model

The proposed model is a progression of the client acquisition and overage management paradigms. In order to restore the overage to the maximum stock level, this model takes into account the inventory levels of different items. Overage occurs when inventory surpluses and deficits exceed the maximum stock level. Consider the current situation: before to the spread of COVID 19, the manufacturing sectors were operating with their inventory levels. When a firm closes at time t, it has three inventory levels, item 1, item 2, and item 3. If the scenario is restarted at time t_1, the company's normal operations begin, which include ordering, manufacturing, and marketing. Assume these three things are the raw ingredients or partially finished goods that will be used to create the final product. The time period $t_1 - t$ influences the quality of three things. If $t_1 - t \le t^*$, then an overage situation can be avoided by using a various inventory ordering procedure. (t^*, the given time for quality verification of the products, assuming that the physical properties of the inputs are uniform) If $t_1 - t \ge t^*$, the inventory replenishment pattern remains unchanged, and overage occurs.

The models are created based on the above premise. The components in this suggested model are somewhat decaying in nature, which is one of the underlying assumptions. We have the following equations:

$$U_{TR}(Q) = S\ (1-P)\ D\ +\ \mathrm{V}\ \mathrm{P}D \tag{1}$$

$$U_{TC}(Q) = \frac{KD}{Q} + \frac{NAD}{Q} + RPD + \frac{HQ}{2} + \frac{QP^2}{2}\left(H_{SI} - H\right) + \frac{W_{dc}PP_dD}{Q} + \frac{DP_{pc}}{Q} \tag{2}$$

The total profit per unit time as:

$$U_{TP}(Q) = \left(S\ (1-P)\ D\ +\ \mathrm{V}\ \mathrm{P}D\right) - \begin{pmatrix} \dfrac{KD}{Q} + \dfrac{NAD}{Q} + RPD + \dfrac{HQ}{2} \\ + \dfrac{QP^2}{2}\left(H_{SI} - H\right) + \dfrac{W_{dc}PP_dD}{Q} + \dfrac{DP_{pc}}{Q} \end{pmatrix} \tag{3}$$

Since P_d is a random variate, therefore, the expectation of (1), $U_{ETP}(Q)$ is presented as:

$$U_{ETP}(Q) = \left(S\ (1-E(P))\ D\ +\ \mathrm{V}\ \mathrm{E(P)}D\right) - \begin{pmatrix} \dfrac{KD}{Q} + \dfrac{NAD}{Q} + RE(P)D + \dfrac{HQ}{2} \\ + \dfrac{QE(P^2)}{2}\left(H_{SI} - H\right) + \dfrac{W_{dc}E(P)E(P_d)D}{Q} + \dfrac{DP_{pc}}{Q} \end{pmatrix} \tag{4}$$

The Optimal order quantity

$$Q = \sqrt{\frac{2D\left(K + NA + W_{dc}E(P)E(P_d) + P_{pc}\right)}{H + E(P^2)\left(H_{SI} - H\right)}} \tag{5}$$

4.2 Fuzzy Model

It is uncommon t to specify each and every parameter exactly due to the ambiguity in the environment. To account for more realistic situations, it is possible to assume that a few of the parameters, such as a, b, D, H, H_S *and* R alters under certain bounds. Let a, b, D, H, H_{SI}, *and* R be fuzzy neutrosophic numbers. Then, we obtain

$$a = \left\langle \left(a_1, a_2, a_3\right), \left(a_1^{'}, a_2^{'}, a_3^{'}\right), \left(a_1^{''}, a_2^{''}, a_3^{''}\right) \right\rangle$$

$$\tilde{b} = \left\langle \left(b_1, b_2, b_3\right), \left(b_1^{'}, b_2^{'}, a_3^{'}\right), \left(b_1^{''}, b_2^{''}, b_3^{''}\right) \right\rangle$$

$$D = \left\langle \left(D_1, D_2, D_3\right), \left(D_1^{'}, D_2^{'}, D_3^{'}\right), \left(D_1^{''}, D_2^{''}, D_3^{''}\right) \right\rangle$$

$$H = \left\langle \left(H_1, H_2, H_3\right), \left(H_1^{'}, H_2^{'}, H_3^{'}\right), \left(H_1^{''}, H_2^{''}, H_3^{''}\right) \right\rangle$$

$$H_{SI} = \left\langle \left(H_{SI1}, H_{SI2}, H_{SI3}\right), \left(H_{SI1}^{'}, H_{SI2}^{'}, H_{SI3}^{'}\right), \left(H_{SI1}^{''}, H_{SI2}^{''}, H_{SI3}^{''}\right) \right\rangle$$

$$R = \left\langle \left(R_1, R_2, R_3\right), \left(R_1^{'}, R_2^{''}, R_3^{'}\right), \left(R_1^{''}, R_2^{''}, R_3^{''}\right) \right\rangle$$

We defuzzify using the signed distance method.

The signed distance of $B = \left\langle \left(B_1, B_2, B_3\right), \left(B_1^{'}, B_2^{'}, B_3^{'}\right)\left(B_1^{''}, B_2^{''}, B_3^{''}\right) \right\rangle$ is presented as

$$d\left(B, 0\right) = \frac{B_1 + 2B_2 + B_3 + B_1^{'} + 2B_2^{'} + B_3^{'} + B_1^{''} + 2B_2^{''} + B_3^{''}}{12}$$

The neutrosophic total cost is given by

$$U_{TP}\left(Q\right) = \left(S\ \left(1-P\right)\ D\ +\ \mathrm{V\ P}D\right) - \left(\begin{array}{l} \frac{KD}{Q} + \frac{NAD}{Q} + RPD + \frac{HQ}{2} \\ + \frac{QP^2}{2}\left(H_{SI} - H\right) + \frac{W_{dc}PP_d D}{Q} + \frac{DP_{pc}}{Q} \end{array}\right) \qquad (6)$$

$$= \left\{ \begin{Bmatrix} \left(S\ (1-P)\ D_1 + V\ PD_1\right) - \begin{pmatrix} \frac{KD_1}{Q} + \frac{NAD_1}{Q} + R_1 PD_1 + \frac{H_1 Q}{2} \\ + \frac{QP^2}{2}\left(H_{SI1} - H_1\right) + \frac{W_{dc} PP_d D_1}{Q} + \frac{D_1 P_{pc}}{Q} \end{pmatrix}, \\ \left(S\ (1-P)\ D_2 + V\ PD_2\right) - \begin{pmatrix} \frac{KD_2}{Q} + \frac{NAD_2}{Q} + R_2 PD_2 + \frac{H_2 Q}{2} \\ + \frac{QP^2}{2}\left(H_{SI2} - H_2\right) + \frac{W_{dc} PP_d D_2}{Q} + \frac{D_2 P_{pc}}{Q} \end{pmatrix}, \\ \left(S\ (1-P)\ D_3 + V\ PD_3\right) - \begin{pmatrix} \frac{KD_3}{Q} + \frac{NAD_3}{Q} + R_3 PD_3 + \frac{H_3 Q}{2} \\ + \frac{QP^2}{2}\left(H_{SI3} - H_3\right) + \frac{W_{dc} PP_d D_3}{Q} + \frac{D_3 P_{pc}}{Q} \end{pmatrix} \end{Bmatrix}, \begin{Bmatrix} \left(S\ (1-P)\ D'_1 + V\ PD'_1\right) - \begin{pmatrix} \frac{KD'_1}{Q} + \frac{NAD'_1}{Q} + R'_1 PD'_1 + \frac{H'_1 Q}{2} \\ + \frac{QP^2}{2}\left(H'_{SI1} - H'_1\right) + \frac{W_{dc} PP_d D'_1}{Q} + \frac{D'_1 P_{pc}}{Q} \end{pmatrix}, \\ \left(S\ (1-P)\ D'_2 + V\ PD'_2\right) - \begin{pmatrix} \frac{KD'_2}{Q} + \frac{NAD'_2}{Q} + R'_2 PD'_2 + \frac{H'_2 Q}{2} \\ + \frac{QP^2}{2}\left(H'_{SI2} - H'_2\right) + \frac{W_{dc} PP_d D'_2}{Q} + \frac{D'_2 P_{pc}}{Q} \end{pmatrix}, \\ \left(S\ (1-P)\ D'_3 + V\ PD'_3\right) - \begin{pmatrix} \frac{KD'_3}{Q} + \frac{NAD'_3}{Q} + R'_3 PD'_3 + \frac{H'_3 Q}{2} \\ + \frac{QP^2}{2}\left(H'_{SI3} - H'_3\right) + \frac{W_{dc} PP_d D'_3}{Q} + \frac{D'_3 P_{pc}}{Q} \end{pmatrix} \end{Bmatrix}, \right.$$

$$\left. \begin{Bmatrix} \left(S\ (1-P)\ D''_1 + V\ PD''_1\right) - \begin{pmatrix} \frac{KD''_1}{Q} + \frac{NAD''_1}{Q} + R''_1 PD''_1 + \frac{H''_1 Q}{2} \\ + \frac{QP^2}{2}\left(H''_{SI1} - H''_1\right) + \frac{W_{dc} PP_d D''_1}{Q} + \frac{D''_1 P_{pc}}{Q} \end{pmatrix}, \\ \left(S\ (1-P)\ D''_2 + V\ PD''_2\right) - \begin{pmatrix} \frac{KD''_2}{Q} + \frac{NAD''_2}{Q} + R''_2 PD''_2 + \frac{H''_2 Q}{2} \\ + \frac{QP^2}{2}\left(H''_{SI2} - H''_2\right) + \frac{W_{dc} PP_d D''_2}{Q} + \frac{D''_2 P_{pc}}{Q} \end{pmatrix}, \\ \left(S\ (1-P)\ D''_3 + V\ PD''_3\right) - \begin{pmatrix} \frac{KD''_3}{Q} + \frac{NAD''_3}{Q} + R''_3 PD''_3 + \frac{H''_3 Q}{2} \\ + \frac{QP^2}{2}\left(H''_{SI3} - H''_3\right) + \frac{W_{dc} PP_d D''_3}{Q} + \frac{D''_3 P_{pc}}{Q} \end{pmatrix} \end{Bmatrix} \right\}$$

The defuzzified neutrosophic total cost using above signed distance method is given by

$$
d\left(U_{TP}(Q),0\right)=\frac{1}{12}\left[\begin{array}{l}
\left(S\,(1-P)\,D_1+V\,PD_1\right)-\left(\begin{array}{l}\frac{KD_1}{Q}+\frac{NAD_1}{Q}+R_1PD_1+\frac{H_1Q}{2}\\+\frac{QP^2}{2}\left(H_{SI1}-H_1\right)+\frac{W_{dc}PP_dD_1}{Q}+\frac{D_1P_{pc}}{Q}\end{array}\right)\\
+2\left\{\left(S\,(1-P)\,D_2+V\,PD_2\right)-\left(\begin{array}{l}\frac{KD_2}{Q}+\frac{NAD_2}{Q}+R_2PD_2+\frac{H_2Q}{2}\\+\frac{QP^2}{2}\left(H_{SI2}-H_2\right)+\frac{W_{dc}PP_dD_2}{Q}+\frac{D_2P_{pc}}{Q}\end{array}\right)\right\}\\
+\left(S\,(1-P)\,D_3+V\,PD_3\right)-\left(\begin{array}{l}\frac{KD_3}{Q}+\frac{NAD_3}{Q}+R_3PD_3+\frac{H_3Q}{2}\\+\frac{QP^2}{2}\left(H_{SI3}-H_3\right)+\frac{W_{dc}PP_dD_3}{Q}+\frac{D_3P_{pc}}{Q}\end{array}\right)\\
+\left(S\,(1-P)\,D_1^{'}+V\,PD_1^{'}\right)-\left(\begin{array}{l}\frac{KD_1^{'}}{Q}+\frac{NAD_1^{'}}{Q}+R_1^{'}PD_1^{'}+\frac{H_1^{'}Q}{2}\\+\frac{QP^2}{2}\left(H_{SI1}^{'}-H_1^{'}\right)+\frac{W_{dc}PP_dD_1^{'}}{Q}+\frac{D_1^{'}P_{pc}}{Q}\end{array}\right)\\
+2\left\{\left(S\,(1-P)\,D_2^{'}+V\,PD_2^{'}\right)-\left(\begin{array}{l}\frac{KD_2^{'}}{Q}+\frac{NAD_2^{'}}{Q}+R_2^{'}PD_2^{'}+\frac{H_2^{'}Q}{2}\\+\frac{QP^2}{2}\left(H_{SI2}^{'}-H_2^{'}\right)+\frac{W_{dc}PP_dD_2^{'}}{Q}+\frac{D_2^{'}P_{pc}}{Q}\end{array}\right)\right\}\\
+\left(S\,(1-P)\,D_3^{''}+V\,PD_3^{''}\right)-\left(\begin{array}{l}\frac{KD_3^{''}}{Q}+\frac{NAD_3^{''}}{Q}+R_3^{''}PD_3^{''}+\frac{H_3^{''}Q}{2}\\+\frac{QP^2}{2}\left(H_{SI3}^{''}-H_3^{''}\right)+\frac{W_{dc}PP_dD_3^{''}}{Q}+\frac{D_3^{''}P_{pc}}{Q}\end{array}\right)\\
+\left(S\,(1-P)\,D_1^{''}+V\,PD_1^{''}\right)-\left(\begin{array}{l}\frac{KD_1^{''}}{Q}+\frac{NAD_1^{''}}{Q}+R_1^{''}PD_1^{''}+\frac{H_1^{''}Q}{2}\\+\frac{QP^2}{2}\left(H_{SI1}^{''}-H_1^{''}\right)+\frac{W_{dc}PP_dD_1^{''}}{Q}+\frac{D_1^{''}P_{pc}}{Q}\end{array}\right)\\
+2\left\{\left(S\,(1-P)\,D_2^{''}+V\,PD_2^{''}\right)-\left(\begin{array}{l}\frac{KD_2^{''}}{Q}+\frac{NAD_2^{''}}{Q}+R_2^{''}PD_2^{''}+\frac{H_2^{''}Q}{2}\\+\frac{QP^2}{2}\left(H_{SI2}^{''}-H_2^{''}\right)+\frac{W_{dc}PP_dD_2^{''}}{Q}+\frac{D_2^{''}P_{pc}}{Q}\end{array}\right)\right\}\\
+\left(S\,(1-P)\,D_3^{''}+V\,PD_3^{''}\right)-\left(\begin{array}{l}\frac{KD_3^{''}}{Q}+\frac{NAD_3^{''}}{Q}+R_3^{''}PD_3^{''}+\frac{H_3^{''}Q}{2}\\+\frac{QP^2}{2}\left(H_{SI3}^{''}-H_3^{''}\right)+\frac{W_{dc}PP_dD_3^{''}}{Q}+\frac{D_3^{''}P_{pc}}{Q}\end{array}\right)
\end{array}\right] \tag{7}
$$

To find the optimal solution by taking the derivative of $d\left(U_{TP}\left(Q\right),0\right)$ relative to Q and then equating it to zero, we get

$$Q=\sqrt{\frac{2\left(D_1+2D_2+D_3+D_1^{'}+2D_2^{'}+D_3^{'}+D_1^{''}+2D_2^{''}+D_3^{''}\right)\left(K+NA+W_{dc}PP_d+P_{pc}\right)}{\left(H_1+2H_2+H_3+H_1^{'}+2H_2^{'}+H_3^{'}+H_1^{''}+2H_2^{''}+H_3^{''}\right)+P^2\begin{pmatrix}\left(H_{SI1}-H_1\right)+2\left(H_{SI2}-H_2\right)+\left(H_{SI3}-H_3\right)\\+\left(H_{SI1}^{'}-H_1^{'}\right)+2\left(H_{SI2}^{'}-H_2^{'}\right)+\left(H_{SI3}^{'}-H_3^{'}\right)\\+\left(H_{SI1}^{''}-H_1^{''}\right)+2\left(H_{SI2}^{''}-H_2^{''}\right)+\left(H_{SI3}^{''}-H_3^{''}\right)\end{pmatrix}}} \quad (8)$$

5. Numerical Example

Case study: To demonstrate the situation, suppose the demand of a product fall due to insufficient transportation services, climatic barriers, natural disasters and lockdown during COVID-19 pandemics etc. In this situation, business model encounters the fall in the forecast requirement for the item 1, item 2 and item3. And the inventories goes up maximum into overage. A numerical example using the following inputs, as in Table 1, is used to validate the proposed model.

Table 1. Input data

Parameter	Item 1	Item 2	Item 3
a	50,000 unit / year	50,000 unit / year	50,000 unit / year
b	50	50	50
K	200 / cycle	220 / cycle	230 / cycle
H	5 / unit / year	4 / unit / year	6 / unit / year
H_{SI}	6 / unit / year	7 / unit / yea	8 / unit / yea
S	50 / unit	60 / unit	55 / unit
V	35 / unit	40 / unit	35 / unit
R	5 / item	6 / item	5 / item
A	30 / customers	35 / customers	40 / customers
N	2	3	4
W_{dc}	3 / unit	4 / unit	5 / unit
P_{pc}	300 / cycle	320 / cycle	330 / cycle

It is assumed that $P\, and\, P_d$ follows uniform distribution with the below density function:

$$f(P)=\begin{cases}4, & 0\le P\le 0.25\\ 0, & Otherwise\end{cases} \quad \text{and} \quad f(P_d)=\begin{cases}20, & 0\le P_d\le 0.05\\ 0, & Otherwise\end{cases}$$

$$E(P)=0.125, E(P^2)=0.021 and\ E(P_d)=0.1.$$

The following Table 2 demonstrates the optimal results.

Table 2. Optimal results for both models

Types of Models	Items	Optimal order Quantity	Expected Maximum profit
Crisp	Item 1	3255	2239905.732
	Item 2	3863	2651554.179
	Item 3	3355	2430817.326
Fuzzy	Item 1	3262	1913343.196
	Item 2	3893	2263803.700
	Item 3	3367	2105973.452

6. Graphical representation

Figures 1 and 2 demonstrate the cost function behavior for both models. The concavity of Figure demonstrates that it maximizes overall profit. Figure 3 depicts a comparison of both crisp and fuzzy models. Figures 4 and 5 show how different cost functions, such as acquisition and fixed costs, vary.

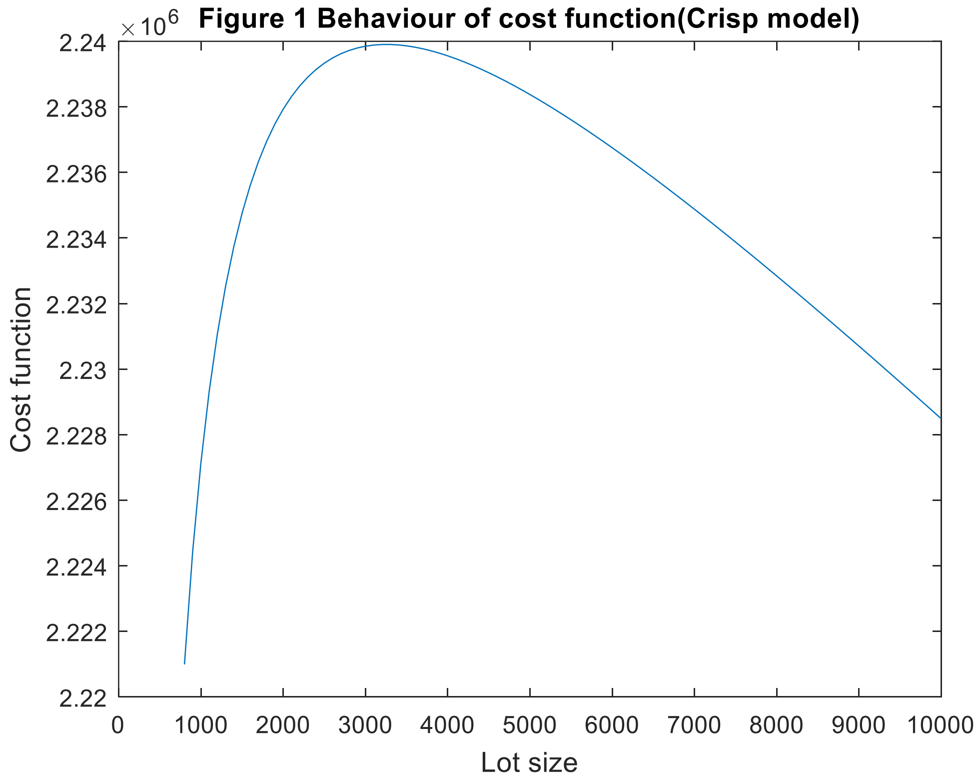


Figure 2 Behaviour of cost function(Fuzzy model)

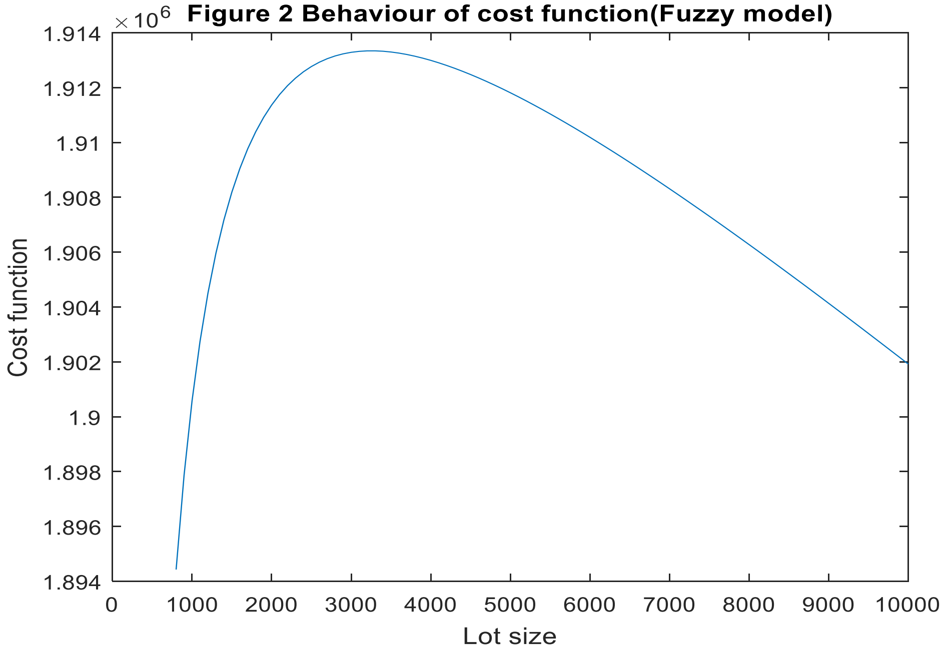


Figure 3 Behaviour of cost function in both the environment

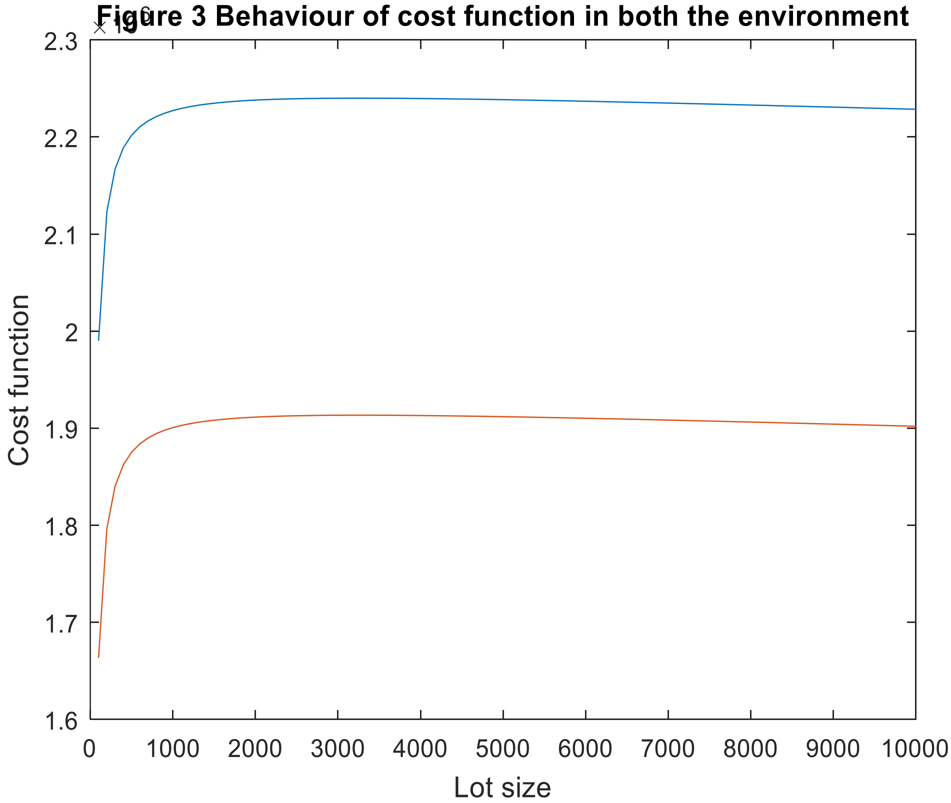


Figure 4 variation of fixed cost

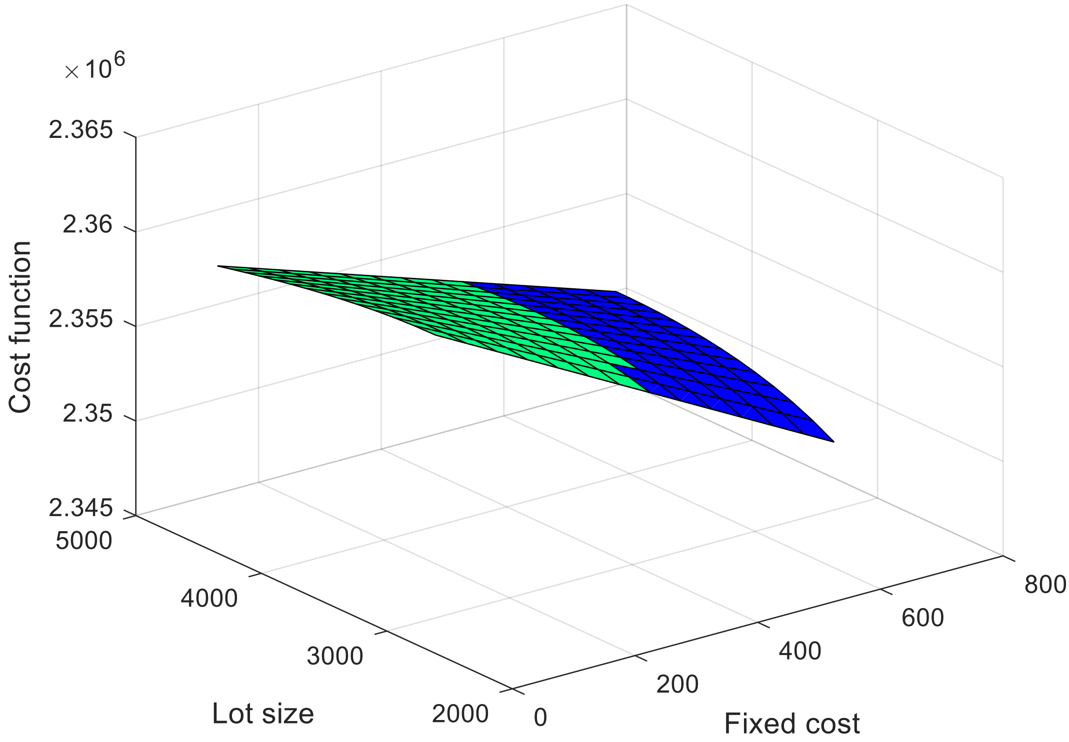
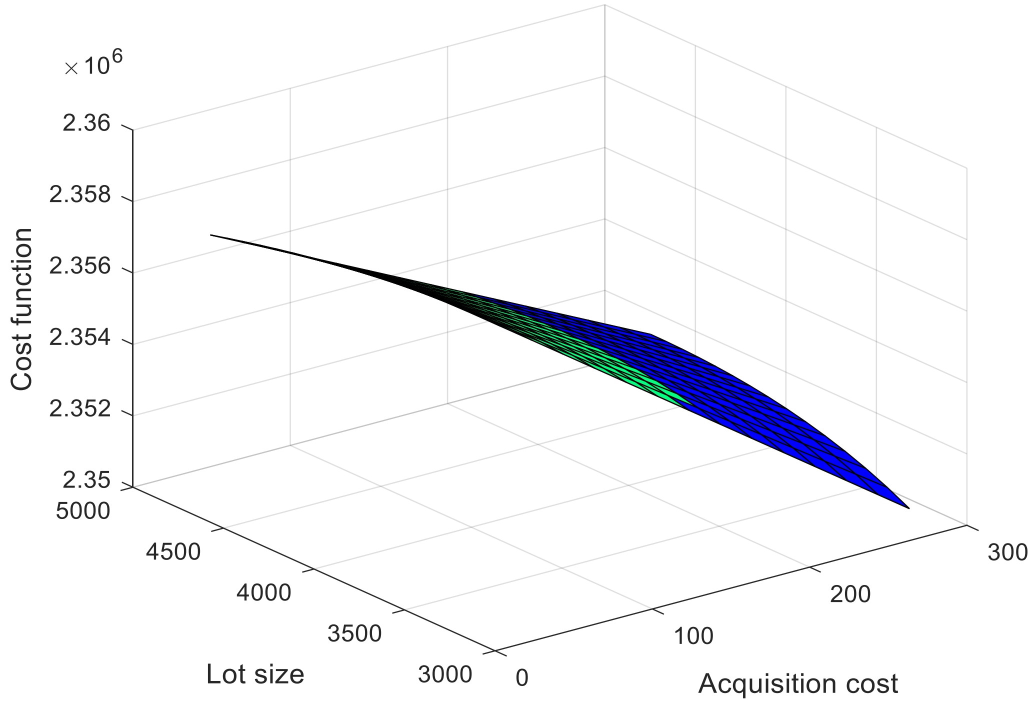


Figure 5 Variation of Acquisition cost



7. Sensitivity Analysis

We were able to observe some changes in the final cost by gently changing the parameters in this section. By changing each of the parameters by -15 %, -10 %, -5 %, 0 %, +5 %, +10 %, and +15 % this study does a sensitivity analysis. We may do a sensitivity analysis for the given situation. We changed one of the parameters' percent while leaving the rest alone. We've also looked into the whole cost's influence. The findings are summarized in Table 3.

Table 3: Percentage change in parameter

Variation of Parameters	-15	-10	-5	0	5	10	15
a	1885001	2003289	2121591	2239905	2358231	2476567	2594912
b	2257653	2251737	2245821	2239905	2233989	2228073	2222157
K	2240349	2240200	2240052	2239905	2239760	2239616	2239473
H	2241148	2240722	2240309	2239905	2239512	2239127	2238751
H_{SI}	2239936	2239926	2239915	2239905	2239895	2239885	2239874
S	1943474	2042831	2141642	2239905	2337622	2434792	2531415
V	2208733	2219124	2229515	2239905	2250296	2260686	2271077
R	2244358	2242874	2241390	2239905	2238421	2236936	2235452
A	2240037	2239993	2239949	2239905	2239862	2239818	2239774
W_{dc}	2239905	2239905	2239905	2239905	2239905	2239905	2239905
P_{pc}	2240576	2240349	2240126	2239905	2239688	2239473	2239261

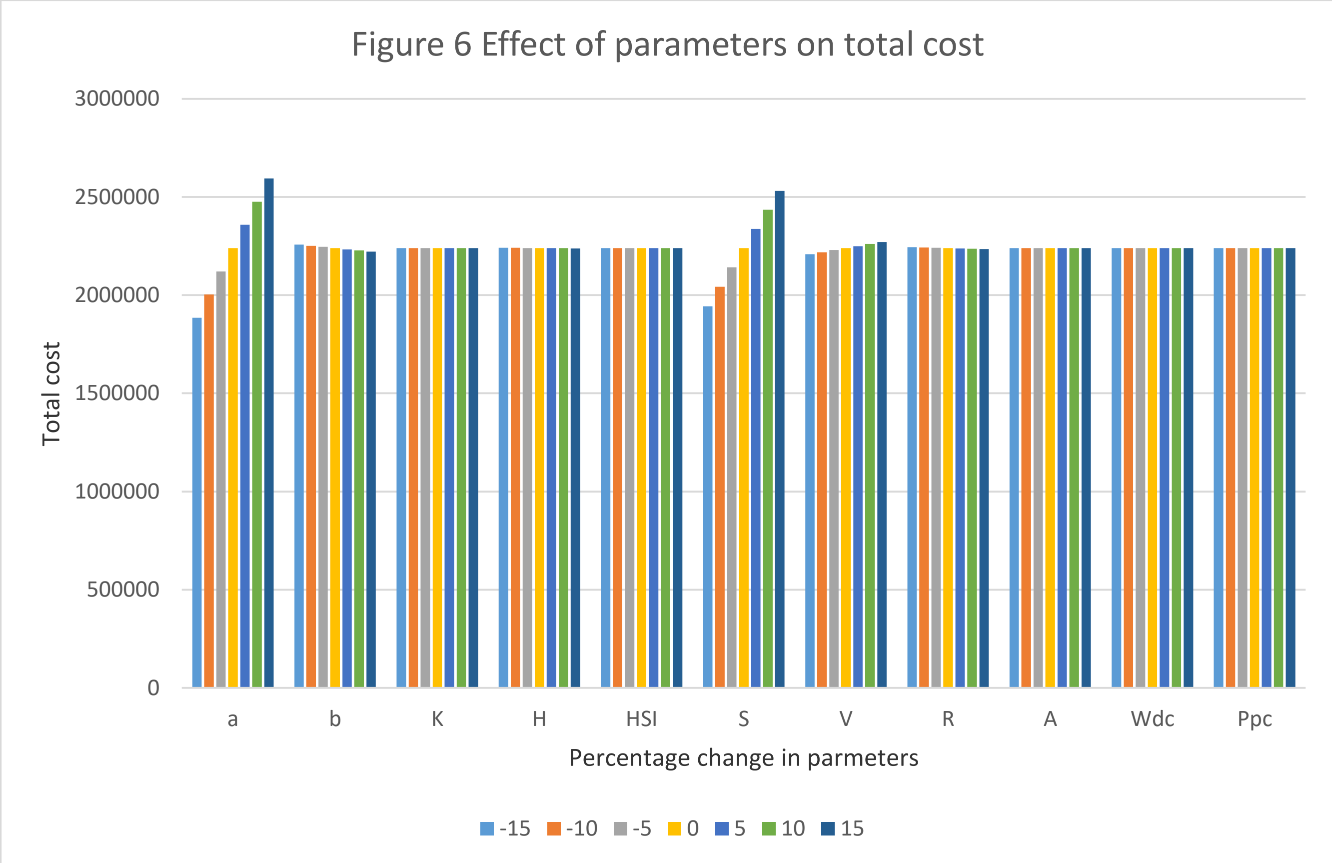
The below mentioned observations are found with the support of Table 3.

(i) An increase in the total cost is caused by a percentage change (PC) in demand parameters ***a***, selling price in maximum stock, and selling price in overage.

(ii) Percentage change (PC) in demand parameter ***b***, K, H, H_{SI}, R, A and P_{pc} lead a decrease in the total cost.

(iii) Percentage change (PC) in Waste disposal cost of the items not effect of the total cost.

Furthermore, we have illustrated below in Figure 6.



8. Results and Discussion

The difference between collective earnings and costs each cycle is the projected aggregate profit per unit of time. Table 2 shows the optimum order quantity for items 1, 2, and 3 to maximize the corresponding projected aggregate profit per unit of time. The optimum order quantity Q for each item is determined, as well as the predicted maximum profit per unit of time. In the above table 2 we also observed that the order quantity of crisp and fuzzy model partially equal but the profit decreases in fuzzy model. The total expenses include the costs of upgrading the overage goods, as well as disposal and product propagation charges. The entire expenses and income related with overage management are represented in this model. The building of inventory is a crucial challenge for the manufacturing industries. The proposed model is not specific to any type of manufacturing company since it includes all of the basic cost characteristics associated with handling excess inventory. The inclusion of environmental costs represents the model's societal relevance in encouraging environmental sustainability; moreover, including environmental costs will make the model compatible with environmental accounting procedures.

9. Conclusions

COVID-19 is a novel experience around the globe. The great nations are running out of endurance in their fight against this terrible health disaster, and the only weapon they have left is an absurd medicine called "Social Distancing." The introduction of quarantine has wreaked havoc on the manufacturing sector. The abrupt curfew interruption disrupted product manufacturing, and the pandemic effect on output resulted in overproduction. In this research, a more generic overage model is formulated to address problems of overage during pandemic outbreaks. This model is for restoring overstock of several goods to maximum stock levels. The suggested maximizing model will undoubtedly aid decision-makers in dealing with overage circumstances induced by pandemic social distance. The goal of this work was to look at price dependent linear demand inventory models in a fuzzy setting. Because the demand and quality of the overage times is not always consistent, it possesses an impact on the overall system. And, in the middle of current variations, our fuzzy model aids in estimating the best projected total profit each cycle. The model is also compared in two different environments, crisp and neutrosophic fuzzy, in the study. This hazy model stresses that customers are more valuable than money and also provides overage control measures. In addition, in the middle of current volatility, our fuzzy model aids in estimating the ideal predicted total profit each cycle. In this study, we did a sensitivity analysis in a crisp setting to show our case. The proposed model may be used to help companies establish production plans in response to pandemics and other unanticipated disasters that interrupt the manufacturing process.

The future scope of this research comprises the investigation of various planning models which allows inclusive planning of overage with different environments such as sustainable and fuzzy. The proposed model reveals a number of ways to extend the model in other dimensions such as demand inversely related to the selling price of the product [37], parabolic holding cost [38], etc. In addition, the developed model can be in multi criteria decision and supply chain problems.

Conflicts of Interest: The authors declare no conflict of interest.
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On Dominating Energy in Bipolar Single-Valued Neutrosophic Graph

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Abstract: One of the most important concepts in graph theory for dealing with unpredictable phenomena is the concept of domination and it has gained attention from many scholars. Recently, dominating energy of graph plays a vital role in the field of graph energies. If the fuzzy graph (FG) fails to give outstanding results, the neutrosophic set (NS) and neutrosophic graphs (NG) can handle the uncertainty correlated with indeterminate and inconsistent information in any real-world scenario. Recent studies related to domination energy in fuzzy environment only deal with the single membership function. It is more flexible and applicable to use bipolar neutrosophic models because they include both positive and negative influencers. Therefore, this paper is based on some developments of neutrosophic graph theory to deal with situations where imprecision is characterized by positive and negative types of membership functions. A novel concept of the dominating energy graph is proposed based on recently introduced concept of bipolar single-valued neutrosophic graphs (BSVNG). Moreover, this study analyses the concepts of dominating energy graph in BSVNG environment. More precisely, the adjacency matrix of a dominating BSVNG as well as the spectrum of the adjacency matrix and their related theory is developed with the help of illustrative examples. Further, the domination energy of BSVNG is computed. Aside from it, various operations relating to this dominating have been depicted. The complement, union, and join of dominating energy in BSVNG have been investigated by using appropriate examples and some properties of the dominating energy in BSVNG are established.

Keywords: Dominating energy; neutrosophic graph; bipolar single-valued neutrosophic graph

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1. Introduction

The energy of a graph is defined as the sum of the absolute values of its eigenvalues. This concept was proposed by Gutman, motivated by chemical applications. In chemistry, the energy of a given molecular graph is interesting because of its relation to the total π-electron energy of the molecule represented by that graph. A graph with all isolated vertices K_n^c has zero energy, while the complete graph K_n with n vertices have energy $2(n-1)$.

The domination concept in graphs may be utilised to a myriad of issues, including facility location, social network analysis, matching theory, coding theory, communication networks, security systems, clutters, and block cutters. For example, to address challenges in facility location problems in which the number of facilities such as police stations, fire stations, hospitals, and supermarkets is fixed, and one wants to shorten the route that persons must travel to reach the closest facility. Provided that the maximum distance to a facility is defined and efforts are made to lower the number of facilities required to accommodate everyone, a similar situation develops. Furthermore, the domination concept comes up in situations such as monitoring communication or electrical networks, finding sets of representatives, and surveying land.

When considering real life conditions, things are not often precise, and they cannot be described by crisp or deterministic models. Thus, the capacity of making precise statements is quite challenging. In order to handle these vague and imprecise events, Zadeh [1] introduced fuzzy sets together with degrees of membership of elements to these sets . Since that time, ordinary fuzzy sets have been extended to intuitionistic, hesitant, orthopair fuzzy sets, and neutrosophic sets.

Despite all these extensions, fuzzy sets could not handle all types of uncertainties such as indeterminate and inconsistent information. In order to tackle this inadequacy, Smarandache [2] introduced neutrosophic logic and neutrosophic sets . A neutrosophic set is composed of three subsets which are degree of truthiness (T), degree of indeterminacy (I), and degree of falsity (F). These subsets are between]−0, 1+[non-standard unit interval. Thus, a membership function of a neutrosophic number is represented by truth sub-set; non-membership function is represented by falsity sub-set; and hesitancy is represented by indeterminacy sub-set. These features constitute the superiority of neutrosophic sets over the other extensions of fuzzy sets. We utilize neutrosophic sets since their main advantage is the capability in distinguishing relativity and absoluteness of decision makers' preferences.

Bipolarity refers to the tendency of the human mind to analyze and take responsibility based on positive and negative outcomes. The positive analysis is all about reasonable, permitted, appropriate, or considered acceptable, while impossible, rejected or forbidden represents negative analyses. Furthermore, positive thoughts correspond to the preferences as they interpret which objects are preferable to others without rejecting those that do not meet the preferences. Still, negative thoughts correspond to the constraints as they interpret which values or objects must be declined. Based on these consequences, Deli et al. [3] proposed bipolar fuzzy sets and neutrosophic sets to bipolar neutrosophic sets in which positive membership degree, negative membership degree, and operations were studied. Bipolar fuzzy sets have a great value in dealing with uncertainty in real-life problems and useful in dealing with the positive and the negative membership values.

2. Literature review

In 1978, I. Gutman proposed the idea of "graph energy," which is defined as the sum of the absolute values of the eigenvalues of the graph's adjacency matrix. By linking the edge of a graph to the electron energy of a type of molecule, the energy of a graph is employed in quantum theory and many other applications in the context of energy. Later, Gutman and Zhou [4] defined the Laplacian energy of a graph as the sum of the absolute values of the differences of average vertex degree of G to the Laplacian eigenvalues of G. Details on the properties of graph energy and Laplacian energy can be found in [5]–[11].

Meanwhile, after the expansion of fuzzy sets by Zadeh [12], the concept of fuzzy graph was initially proposed by Kauffman [13] and Rosenfeld [14] to deal with the fuzziness in graphs. Fuzzy graphs are effective for representing the structures of object relationship, where the presence of a real object and the link between two objects is ambiguous or uncertain. Some application related to fuzzy graph can be found in [15]-[19]. Apart from that, Anjali and Mathew [20] first proposed the energy of a graph within the fuzzy set environment. In [21], Praba et al. extend the concept of energy in fuzzy graph to the energy of intuitionistic fuzzy graph. Later on, Naz et al. [22] proposed the novel concept of energy graph considering bipolar fuzzy environment and examined some of their properties. Akram and Naz [23] studied on energy of Pythagorean fuzzy graphs and fuzzy digraphs. Akram et al. [24] proposed the concept of energy in bipolar fuzzy graph and demonstrate multi-criteria decision-making approaches in commercial partnerships and smooth communication challenges based on the energy of bipolar fuzzy graphs. Recently, Patra et al. [25] proposed novel techniques of graph energy in interval-valued fuzzy graphs and computed eigenvalues using max–min operators.

The concept of domination is one of the most significant problems in graph theory. In 1998, the novel concepts of domination in fuzzy graphs was first introduced by Somasundaram and Somasundaram [26]. After that, Somasundaram [27] studied the domination in products of fuzzy graphs and discussed various operations on fuzzy graphs such as join, union, composition, Cartesian product and domination parameters. Ghobadi et al. [28] presented an idea of inverse dominating set in fuzzy graph whereas Natarajan and Ayyaswamy [29] initiated the concept of strong (weak) domination in fuzzy graphs. Afterwards, the concepts of cardinality, dominating set, independent set, total dominating number and independent dominating number of bipolar fuzzy graphs was investigated by Karunambigai et al. [30]. Umamageswari et al. [31] introduced the concept of multiple dominating set in bipolar fuzzy graph where the k-dominating set and its domination number in bipolar fuzzy graph were defined.

Later on, Muthuraj et al. [32] defined the non-split total strong (weak) domination in bipolar fuzzy graph and its various parameters. Muneera et al. [33] studied the domination in regular and irregular bipolar fuzzy graphs whereas Akram et al. [34] discussed the different concepts of dominating, total dominating, equitable dominating, total equitable dominating, and independent and equitable independent sets in bipolar fuzzy graphs. Equitable domination in bipolar fuzzy graph, equitable total domination in bipolar fuzzy graph and its various classifications was proposed by Muthuraj and Kanimozhi [35]. Recently, Gong et al. [36] presented the concept of domination in the fuzzy graph to the bipolar frameworks and determined the related expanded concepts of a variety of bipolar fuzzy graphs.

The study of domination has gained attention and it has now been extended to the neutrosophic environment. Hussain et al. [37] introduced the domination number of neutrosophic soft graphs and elaborate them with suitable examples by using strength of path and strength of connectedness. Later on, Banitalebi and Borzooei [38] extend the study of dominating set in the concepts of neutrosophic special dominating set, and define the neutrosophic special domination numbers, neutrosophic special cobondage set and neutrosophic special cobondage numbers in neutrosophic graphs. Ramya et al. [39] presented the concept of complementary domination in single valued neutrosophic graphs (SVNG) and studied the bounds and characteristic of an inverse domination number (IDN) in various SVNG.

Apart from that, the concept of minimum dominating energy was extended to fuzzy graph by Kartheek and Basha [40]. This study defined the properties including various upper and lower bounds for this energy on fuzzy graphs. Praba et al. [41] analyzed the spreading rate of virus on energy of dominating intuitionistic fuzzy graph whereas Kalimulla et al. [42] studied the concept energy of an IFG to dominating energy in operations on IFG. This research employed the various operations such as complement, union, join, Cartesian product and composition to obtain the value of dominating energy. Vijayaragavan et al. [43] obtained the value of dominating Laplacian energy in two products such as Cartesian product and tensor product hence studied the relation between the dominating Laplacian energy in the products in two IFG. Moreover, Sarwar et al. [44] put forward some new concepts of dominating and double dominating energy of m-Polar fuzzy graphs and demonstrate a decision model based on m-polar fuzzy preference relations to solve multi-criteria decision-making problems. Additionally, Akram et al. [45] proposed novel concepts of energy of dominating bipolar fuzzy graph and the energy of double dominating bipolar fuzzy graph.

Since the emergence of the neutrosophic set, many scholars are intended to integrate the study of energy graph, dominating set as well as neutrosophic set. Recently, study from Mullai and Broumi [46] proposed the dominating energy in single-valued neutrosophic graph (SVNG). This study employed the dominating energy in numerous operations such as complement, union and join of neutrosophic graph and provide some theorems related to dominating energy in neutrosophic. Table 1 illustrates the contributions related to domination and energy in fuzzy and neutrosophic graph. Additionally, Figure 1 shows the development of energy graph, domination concept as well as fuzzy and neutrosophic graphs.

Motivated by [45] and [46], we extend this idea and introduce the novel concepts of dominating energy of bipolar single valued neutrosophic graph (BSVNG). Using the notion of eigenvalues of bipolar relations, we study interesting properties and bounds for dominating energy. Before looking deeply into these ideas, the dominating energy of a BSVNG and the dominating energy of different operations on a BSVNG are defined with examples, and certain theorems in dominating energy of BSVNG are developed, as well as other conclusions.

The following is a breakdown of the study's structure: The initial part introduces the historical backgrounds of domination and energy graphs, as well as the concepts of the fuzzy and neutrosophic set. In Part 2, some review of the literature regarding the domination and energy graph in the fuzzy and neutrosophic environment. Part 3 offers a brief introduction to graphs and the neutrosophic set, which will be employed shortly. In Part 4, we establish the dominating energy concept in BSVNG and delve into its aspects. Part 5 demonstrates the dominating energy in various operations in BSVNG and presented new theorems. Finally, Part 6 concludes with a summary of the research's findings as well as its restrictions and a suggestion for further research.

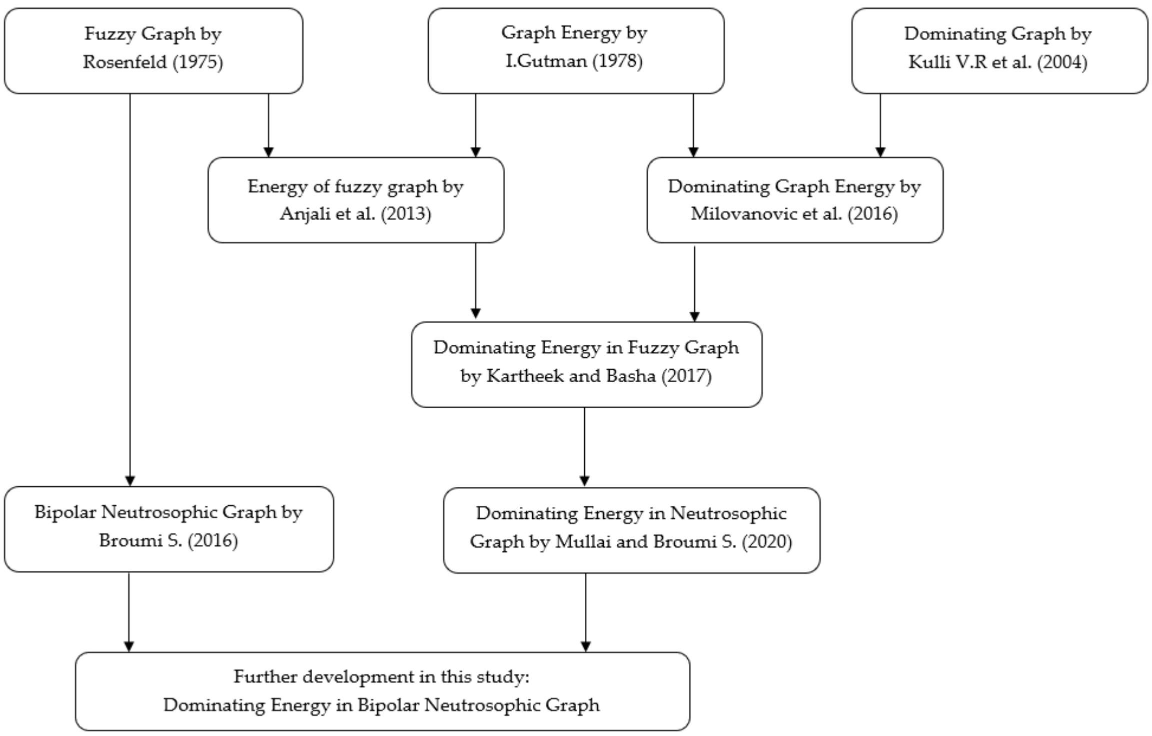

Figure 1: Development of energy, domination, and neutrosophic graphs

Table 1: Contributions related to domination and energy in fuzzy and neutrosophic graphs.

Studied by	Findings	Gaps
Somasundaram & Somasundaram (1998) [26]	Proposed domination concept in FG	Concentrate only uncertainty and ambiguity problem
Somasundaram (2005) [27]	Studied various operation of domination in FG	Concentrate only a few operations
Ghobadi et al., (2008) [28]	Proposed inverse dominating in FG	Did not consider the bipolarity concepts
Natarajan & Ayyaswamy (2010) [29]	Proposed strong (weak) domination in FG	Concentrate only uncertainty and ambiguity problem
Karunambigai et al., (2013) [30]	Studied dominating set in BFG	Did not consider the energy concepts
Umamageswari et al., (2017) [31]	Proposed multiple dominating set of BFG	Did not consider the energy concepts
Muthuraj & Kanimozhi (2019) [32]	Proposed split total strong domination of BFG	Did not investigate the various operation
Muneera et al., (2020) [33]	Studied domination concept in regular and irregular of BFG	Cannot capture the indeterminacy and inconsistent information
Akram et al., (2021) [34]	Proposed various concept of domination in BFG	Did not consider the energy concepts
Muthuraj & Kanimozhi (2020) [35]	Proposed equitable domination in BFG	Did not investigate the various operation
Gong et al., (2021) [36]	Studied domination FG to BFG	Did not consider the energy concepts
Hussain et al., (2019) [37]	Proposed domination number in neutrosophic soft graphs	Did not consider the bipolarity concepts
Banitalebi & Borzooei (2021) [38]	Proposed neutrosophic special domination number	Did not consider the bipolarity concepts
Ramya et al., (2021) [39]	Proposed complementary domination in SVNG	Did not consider the energy concepts
Kartheek & Basha (2017) [40]	Proposed minimum dominating energy in FG	Did not consider the indeterminacy and inconsistent information
Praba et al., (2017) [41]	Proposed dominating energy in IFG	Did not consider the bipolarity concepts
Kalimulla et al., (2018) [42]	Proposed dominating energy of IFG in various operation	Concentrate only a few operations
Vijayaragavan et al., (2019) [43]	Proposed dominating Laplacian energy in product of two IFG	Did not consider the indeterminacy and inconsistent information
Sarwar et al., (2020) [44]	Proposed dominating and double dominating energy of m-polar FG	Did not consider the bipolarity concepts
Akram et al., (2019) [45]	Proposed energy of dominating BFG and doble dominating BFG	Did not consider the indeterminacy and inconsistent information
Mullai & Broumi (2020) [46]	Proposed the dominating energy in SVNG	Did not consider the bipolarity concepts

3. Preliminaries

This section recalls some fundamental definitions with respect to bipolar fuzzy graph (BFG), bipolar single-valued neutrosophic graph (BSVNG) and domination that are necessary for this study.

Definition 3.1 [47]
A bipolar fuzzy graph with a finite set X as the underlying set is a pair of $G=(V,E)$, where $V=(\mu_V^+,\mu_V^-)$ is a bipolar fuzzy set on X and $E=(\mu_E^+,\mu_E^-)$ is a bipolar fuzzy relation on X such that $\mu_E^+(v_m v_n)\le \mu_V^+(v_m)\wedge \mu_V^+(v_n)$ and $\mu_E^-(v_m v_n)\ge \mu_V^+(v_m)\vee \mu_V^+(v_n)$ where $v_m v_n \in E$. We call $V=(\mu_V^+,\mu_V^-)$ the bipolar fuzzy vertex set of X and $E=(\mu_E^+,\mu_E^-)$ the bipolar fuzzy edge set of X.

Definition 3.2 [3]
A bipolar neutrosophic set A on a non-empty set X is an object of the form

$$A=\left\{\left\langle x,T_A^+(x),I_A^+(x),F_A^+(x),T_A^-(x),I_A^-(x),F_A^-(x)\right\rangle : x\in X\right\},$$

where $T_A^+,I_A^+,F_A^+ : X\to[0,1]$ and $T_A^-,I_A^-,F_A^- : X\to[-1,0]$. The positive values $T_A^+(x),I_A^+(x),F_A^+(x)$ denote the truth, indeterminacy and false-memberships degrees of an element $x\in X$, respectively, whereas, $T_A^-(x),I_A^-(x),F_A^-(x)$ denote the implicit counter property of the truth, indeterminacy and false-memberships degrees of the element $x\in X$, respectively, corresponding to the bipolar neutrosophic set A.

Definition 3.3 [48]
Let $A=(T_A^+,I_A^+,F_A^+,T_A^-,I_A^-,F_A^-)$ and $B=(T_B^+,I_B^+,F_B^+,T_B^-,I_B^-,F_B^-)$ be bipolar neutrosophic graph on a set X. If $B=(T_B^+,I_B^+,F_B^+,T_B^-,I_B^-,F_B^-)$ is a bipolar neutrosophic relation on $A=(T_A^+,I_A^+,F_A^+,T_A^-,I_A^-,F_A^-)$, then

$$T_B^+(xy)\le \min\left(T_A^+(x),T_A^+(y)\right), I_B^+(xy)\ge \max\left(I_A^+(x),I_A^+(y)\right), F_B^+(xy)\ge \max\left(F_A^+(x),F_A^+(y)\right),$$
$$T_B^-(xy)\ge \max\left(T_A^-(x),T_A^-(y)\right), I_B^-(xy)\le \min\left(I_A^-(x),I_A^-(y)\right), F_B^-(xy)\le \min\left(F_A^-(x),F_A^-(y)\right),$$
$$\forall x,y\in X.$$

Definition 3.4 [49]
Let $G=(V,E)$ be a bipolar single-valued neutrosophic graph (BSVNG) and $x,y\in V$ in G, then we say that x dominates y if

$$T_E^+(xy)=T_V^+(x)\wedge T_V^+(y), I_E^+(xy)=I_V^+(x)\vee I_V^+(y), F_E^+(xy)=F_V^+(x)\vee F_V^+(y),$$
$$T_E^-(xy)=T_V^-(x)\vee T_V^-(y), I_E^-(xy)=I_V^-(x)\wedge I_V^-(y), F_E^-(xy)=F_V^-(x)\wedge F_V^-(y).$$

4 Dominating energy in bipolar single-valued neutrosophic graphs

In this section, we consider a BSVNG $G=(V,E,\mu^+,\gamma^+,\sigma^+,\mu^-,\gamma^-,\sigma^-)$, then we define $(\mu_1^+,\gamma_1^+,\sigma_1^+): V\to[0,1], (\mu_1^-,\gamma_1^-,\sigma_1^-): V\to[-1,0]$ and prove that $(\mu_1^+,\gamma_1^+,\sigma_1^+,\mu_1^-,\gamma_1^-,\sigma_1^-)$ is a bipolar single-valued neutrosophic set (BSVNS).

Definition 4.1

Let $G=\left(V,E,\mu^{+},\gamma^{+},\sigma^{+},\mu^{-},\gamma^{-},\sigma^{-},\mu_1^{+},\gamma_1^{+},\sigma_1^{+},\mu_1^{-},\gamma_1^{-},\sigma_1^{-}\right)$ be a dominating BSVNG where $\mu_1^{+}:V\to[0,1]$, denotes a positive degree of membership, $\gamma_1^{+}:V\to[0,1]$, denotes a positive degree of indeterminacy, $\sigma_1^{+}:V\to[0,1]$, denotes a positive degree of non-membership, $\mu_1^{-}:V\to[-1,0]$, denotes a negative degree of membership, $\gamma_1^{-}:V\to[-1,0]$, denotes a negative degree of indeterminacy and $\sigma_1^{-}:V\to[-1,0]$, denotes a negative degree of non-membership defined such as

$$\mu_1^{+}(v_m)=\min_{v_n}\left[\mu^{+}(v_m,v_n)\right],\gamma_1^{+}(v_m)=\max_{v_n}\left[\gamma^{+}(v_m,v_n)\right],\sigma_1^{+}(v_m)=\max_{v_n}\left[\sigma^{+}(v_m,v_n)\right],$$
$$\mu_1^{-}(v_m)=\max_{v_n}\left[\mu^{-}(v_m,v_n)\right],\gamma_1^{-}(v_m)=\min_{v_n}\left[\gamma^{-}(v_m,v_n)\right],\sigma_1^{-}(v_m)=\min_{v_n}\left[\sigma^{+}(v_m,v_n)\right].$$

Definition 4.2

Let G be a dominating BSVNG. Let $x,y\in V$, we state that x dominates y in G if there exists strong arc from x to y. A subset $D^N\subseteq V$ is called dominating set in G if for each $y\in V-D^N$, there exists x in D^N such that x dominates y .

Definition 4.3

A dominating set D^N of BSVNG is said to be minimal dominating set if for any $x\in D^N, D^N\setminus\{x\}$ is not a dominating set.

Definition 4.4

The minimum cardinality among all minimal dominating sets is called a domination number of G and denoted by $\gamma^N(G)$.

Definition 4.5

Let $G=\left(V,E,\mu^{+},\gamma^{+},\sigma^{+},\mu^{-},\gamma^{-},\sigma^{-},\mu_1^{+},\gamma_1^{+},\sigma_1^{+},\mu_1^{-},\gamma_1^{-},\sigma_1^{-}\right)$ be a dominating BSVNG. The adjacency matrix of a dominating BSVNG G is defined as, $A_{D^N}(G)=\left[d_{jk}\right]$ where

$$d_{jk}=\begin{cases}\begin{pmatrix}\mu_{jk}^{+},\gamma_{jk}^{+},\sigma_{jk}^{+}\\ \mu_{jk}^{-},\gamma_{jk}^{-},\sigma_{jk}^{-}\end{pmatrix} & if\ (v_j,v_k)\in E\\ \begin{pmatrix}1,1,1,\\ -1,-1,-1\end{pmatrix} & if\ j=k\ and\ v_j\in D^N\\ \begin{pmatrix}0,0,0,\\ 0,0,0\end{pmatrix} & otherwise\end{cases}$$

This dominating bipolar single-valued neutrosophic adjacency matrix, $A_{D^N}(G)$ can be written as $A_{D^N}(G)=\left(\mu_{D^N}^{+}(G),\gamma_{D^N}^{+}(G),\sigma_{D^N}^{+}(G),\mu_{D^N}^{-}(G),\gamma_{D^N}^{-}(G),\sigma_{D^N}^{-}(G)\right)$ where

$$\mu_{D^N}^{+}(G)=\begin{cases}\mu_{jk}^{+} & if\ (v_j,v_k)\in E\\ 1 & if\ i=j\ and\ v_j\in D^N\\ 0 & otherwise\end{cases},\qquad \gamma_{D^N}^{+}(G)=\begin{cases}\gamma_{jk}^{+} & if\ (v_j,v_k)\in E\\ 1 & if\ i=j\ and\ v_j\in D^N\\ 0 & otherwise\end{cases},$$

$$\sigma^{+}_{D^N}(G)=\begin{cases}\sigma^{+}_{jk} & if\ (v_j,v_k)\in E\\ 1 & if\ i=j\ and\ v_j\in D^N\\ 0 & otherwise\end{cases},\qquad \mu^{-}_{D^N}(G)=\begin{cases}\mu^{-}_{jk} & if\ (v_j,v_k)\in E\\ -1 & if\ i=j\ and\ v_j\in D^N\\ 0 & otherwise\end{cases},$$

$$\gamma^{-}_{D^N}(G)=\begin{cases}\gamma^{-}_{jk} & if\ (v_j,v_k)\in E\\ -1 & if\ i=j\ and\ v_j\in D^N\\ 0 & otherwise\end{cases},\qquad \sigma^{-}_{D^N}(G)=\begin{cases}\sigma^{-}_{jk} & if\ (v_j,v_k)\in E\\ -1 & if\ i=j\ and\ v_j\in D^N\\ 0 & otherwise\end{cases}.$$

Definition 4.6

The spectrum of adjacency matrix of a dominating BSVNG G is defined as $\left(S^{\mu^+}_{D^N},S^{\gamma^+}_{D^N},S^{\sigma^+}_{D^N},S^{\mu^-}_{D^N},S^{\gamma^-}_{D^N},S^{\sigma^-}_{D^N}\right)$, where $S^{\mu^+}_{D^N},S^{\gamma^+}_{D^N},S^{\sigma^+}_{D^N},S^{\mu^-}_{D^N},S^{\gamma^-}_{D^N},S^{\sigma^-}_{D^N}$ are the sets of eigenvalues of $\mu^{+}_{D^N}(G),\gamma^{+}_{D^N}(G),\sigma^{+}_{D^N}(G),\mu^{-}_{D^N}(G)$, $\gamma^{-}_{D^N}(G)$, and $\sigma^{-}_{D^N}(G)$, respectively.

Definition 4.7

The energy of a dominating BSVNG $G=\left(V,E,\mu^+,\gamma^+,\sigma^+,\mu^-,\gamma^-,\sigma^-,\mu_1^+,\gamma_1^+,\sigma_1^+,\mu_1^-,\gamma_1^-,\sigma_1^-\right)$ is defined as

$$E_{D^N}(G)=\left(E\left(\mu^{+}_{D^N}(G)\right),E\left(\gamma^{+}_{D^N}(G)\right),E\left(\sigma^{+}_{D^N}(G)\right),E\left(\mu^{-}_{D^N}(G)\right),E\left(\gamma^{-}_{D^N}(G)\right),E\left(\sigma^{-}_{D^N}(G)\right)\right)$$
$$=\left(\sum_{p=1}^{n}|\zeta_p|,\sum_{p=1}^{n}|\tau_p|,\sum_{p=1}^{n}|\upsilon_p|,\sum_{p=1}^{n}|\vartheta_p|,\sum_{p=1}^{n}|\xi_p|,\sum_{p=1}^{n}|\varepsilon_p|\right)$$

where $S^{\mu^+}_{D^N}=\{\zeta_p\}_{p=1}^{n}, S^{\gamma^+}_{D^N}=\{\tau_p\}_{p=1}^{n}, S^{\sigma^+}_{D^N}=\{\upsilon_p\}_{p=1}^{n}, S^{\mu^-}_{D^N}=\{\vartheta_p\}_{p=1}^{n}, S^{\gamma^-}_{D^N}=\{\xi_p\}_{p=1}^{n}$ and $S^{\sigma^-}_{D^N}=\{\varepsilon_p\}_{p=1}^{n}$.

Example 4.8

Consider the BSVNG $G=(V,E)$ where $V=\{v_1,v_2,v_3,v_4,v_5\}$, $E=\{v_1v_2,v_1v_5,v_2v_3,v_2v_5,v_3v_4,v_4v_5\}$ as shown in Figure 2. Then, the above dominating BSVNG can be written as

$$G=\left(V,E,\mu^+,\gamma^+,\sigma^+,\mu^-,\gamma^-,\sigma^-,\mu_1^+,\gamma_1^+,\sigma_1^+,\mu_1^-,\gamma_1^-,\sigma_1^-\right)$$

where $\mu_1^+,\gamma_1^+,\sigma_1^+,\mu_1^-,\gamma_1^-,\sigma_1^-$ are given by $\mu_1^+:V\to[0,1]$, $\gamma_1^+:V\to[0,1]$, $\sigma_1^+:V\to[0,1]$, $\mu_1^-:V\to[-1,0]$, $\gamma_1^-:V\to[-1,0]$, and $\sigma_1^-:V\to[-1,0]$ such that

$$\mu_1^+(v_1)=\min\left[\mu^+(v_1,v_2),\mu^+(v_1,v_5)\right]=\min[0.1,0.5]=0.1$$
$$\mu_1^+(v_2)=\min\left[\mu^+(v_2,v_1),\mu^+(v_2,v_3),\mu^+(v_2,v_5)\right]=\min[0.1,0.2,0.4]=0.1$$
$$\mu_1^+(v_3)=\min\left[\mu^+(v_3,v_2),\mu^+(v_3,v_4)\right]=\min[0.2,0.3]=0.2$$
$$\mu_1^+(v_4)=\min\left[\mu^+(v_4,v_3),\mu^+(v_4,v_5)\right]=\min[0.3,0.2]=0.2$$
$$\mu_1^+(v_5)=\min\left[\mu^+(v_5,v_1),\mu^+(v_5,v_2),\mu^+(v_5,v_4)\right]=\min[0.5,0.4,0.2]=0.2$$
$$\gamma_1^+(v_1)=\max\left[\gamma^+(v_1,v_2),\gamma^+(v_1,v_5)\right]=\max[0.3,0.2]=0.3$$
$$\gamma_1^+(v_2)=\max\left[\gamma^+(v_2,v_1),\gamma^+(v_2,v_3),\gamma^+(v_2,v_5)\right]=\max[0.3,0.6,0.5]=0.6$$
$$\gamma_1^+(v_3)=\max\left[\gamma^+(v_3,v_2),\gamma^+(v_3,v_4)\right]=\max[0.4,0.1]=0.4$$
$$\gamma_1^+(v_4)=\max\left[\gamma^+(v_4,v_3),\gamma^+(v_4,v_5)\right]=\max[0.1,0.2]=0.2$$
$$\gamma_1^+(v_5)=\max\left[\gamma^+(v_5,v_1),\gamma^+(v_5,v_2),\gamma^+(v_5,v_4)\right]=\max[0.2,0.5,0.3]=0.5$$

$$\sigma_1^+(v_1) = \max\left[\sigma^+(v_1,v_2),\sigma^+(v_1,v_5)\right] = \max\left[0.2,0.5\right] = 0.5$$
$$\sigma_1^+(v_2) = \max\left[\sigma^+(v_2,v_1),\sigma^+(v_2,v_3),\sigma^+(v_2,v_5)\right] = \max\left[0.5,0.2,0.1\right] = 0.5$$
$$\sigma_1^+(v_3) = \max\left[\sigma^+(v_3,v_2),\sigma^+(v_3,v_4)\right] = \max\left[0.2,0.5\right] = 0.5$$
$$\sigma_1^+(v_4) = \max\left[\sigma^+(v_4,v_3),\sigma^+(v_4,v_5)\right] = \max\left[0.5,0.3\right] = 0.5$$
$$\sigma_1^+(v_5) = \max\left[\sigma^+(v_5,v_1),\sigma^+(v_5,v_2),\sigma^+(v_5,v_4)\right] = \max\left[0.3,0.1,0.3\right] = 0.3$$
$$\mu_1^-(v_1) = \max\left[\mu^-(v_1,v_2),\mu^-(v_1,v_5)\right] = \max\left[-0.2,-0.3\right] = -0.2$$
$$\mu_1^-(v_2) = \max\left[\mu^-(v_2,v_1),\mu^-(v_2,v_3),\mu^-(v_2,v_5)\right] = \max\left[-0.2,-0.3,-0.2\right] = -0.2$$
$$\mu_1^-(v_3) = \max\left[\mu^-(v_3,v_2),\mu^-(v_3,v_4)\right] = \max\left[-0.3,-0.4\right] = -0.3$$
$$\mu_1^-(v_4) = \max\left[\mu^-(v_4,v_3),\mu^-(v_4,v_5)\right] = \max\left[-0.4,-0.2\right] = -0.2$$
$$\mu_1^-(v_5) = \max\left[\mu^-(v_5,v_1),\mu^-(v_5,v_2),\mu^-(v_5,v_4)\right] = \max\left[-0.3,-0.2,-0.2\right] = -0.2$$
$$\gamma_1^-(v_1) = \min\left[\gamma^-(v_1,v_2),\gamma^-(v_1,v_5)\right] = \min\left[-0.5,-0.3\right] = -0.5$$
$$\gamma_1^-(v_2) = \min\left[\gamma^-(v_2,v_1),\gamma^-(v_2,v_3),\gamma^-(v_2,v_5)\right] = \min\left[-0.5,-0.4,-0.4\right] = -0.5$$
$$\gamma_1^-(v_3) = \min\left[\gamma^-(v_3,v_2),\gamma^-(v_3,v_4)\right] = \min\left[-0.4,-0.1\right] = -0.4$$
$$\gamma_1^-(v_4) = \min\left[\gamma^-(v_4,v_3),\gamma^-(v_4,v_5)\right] = \min\left[-0.1,-0.2\right] = -0.2$$
$$\gamma_1^-(v_5) = \min\left[\gamma^-(v_5,v_1),\gamma^-(v_5,v_2),\gamma^-(v_5,v_4)\right] = \min\left[-0.3,-0.4,-0.2\right] = -0.4$$
$$\sigma_1^-(v_1) = \min\left[\sigma^-(v_1,v_2),\sigma^-(v_1,v_5)\right] = \min\left[-0.6,-0.2\right] = -0.6$$
$$\sigma_1^-(v_2) = \min\left[\sigma^-(v_2,v_1),\sigma^-(v_2,v_3),\sigma^-(v_2,v_5)\right] = \min\left[-0.6,-0.2,-0.6\right] = -0.6$$
$$\sigma_1^-(v_3) = \min\left[\sigma^-(v_3,v_2),\sigma^-(v_3,v_4)\right] = \min\left[-0.2,-0.7\right] = -0.7$$
$$\sigma_1^-(v_4) = \min\left[\sigma^-(v_4,v_3),\sigma^-(v_4,v_5)\right] = \min\left[-0.7,-0.1\right] = -0.7$$
$$\sigma_1^-(v_5) = \min\left[\sigma^-(v_5,v_1),\sigma^-(v_5,v_2),\sigma^-(v_5,v_4)\right] = \min\left[-0.2,-0.6,-0.1\right] = -0.6$$

Here, v_1 dominates v_2 because

$$\mu^+(v_1v_2) \le \mu_1^+(v_1) \wedge \mu_1^+(v_2) \qquad \mu^-(v_1v_2) \le \mu_1^-(v_1) \wedge \mu_1^-(v_2)$$
$$0.1 \le 0.1 \wedge 0.1 \qquad -0.2 \le -0.2 \wedge -0.2$$
$$\gamma^+(v_1v_2) \le \gamma_1^+(v_1) \wedge \gamma_1^+(v_2) \qquad \gamma^-(v_1v_2) \le \gamma_1^-(v_1) \wedge \gamma_1^-(v_2)$$
$$0.3 \le 0.3 \wedge 0.6 \qquad -0.5 \le -0.5 \wedge -0.5$$
$$\sigma^+(v_1v_2) \le \sigma_1^+(v_1) \wedge \sigma_1^+(v_2) \qquad \sigma^-(v_1v_2) \le \sigma_1^-(v_1) \wedge \sigma_1^-(v_2)$$
$$0.5 \le 0.5 \wedge 0.5 \qquad -0.6 \le -0.6 \wedge -0.6$$

Also, v_4 dominates v_5 because

$$\mu^+(v_4v_5) \le \mu_1^+(v_4) \wedge \mu_1^+(v_5) \qquad \mu^-(v_4v_5) \le \mu_1^-(v_4) \wedge \mu_1^-(v_5)$$
$$0.2 \le 0.2 \wedge 0.2 \qquad -0.4 \le -0.4 \wedge -0.2$$
$$\gamma^+(v_4v_5) \le \gamma_1^+(v_4) \wedge \gamma_1^+(v_5) \qquad \gamma^-(v_4v_5) \le \gamma_1^-(v_4) \wedge \gamma_1^-(v_5)$$
$$0.2 \le 0.2 \wedge 0.5 \qquad -0.4 \le -0.2 \wedge -0.4$$
$$\sigma^+(v_4v_5) \le \sigma_1^+(v_4) \wedge \sigma_1^+(v_5) \qquad \sigma^-(v_4v_5) \le \sigma_1^-(v_4) \wedge \sigma_1^-(v_5)$$
$$0.3 \le 0.5 \wedge 0.3 \qquad -0.7 \le -0.7 \wedge -0.6$$

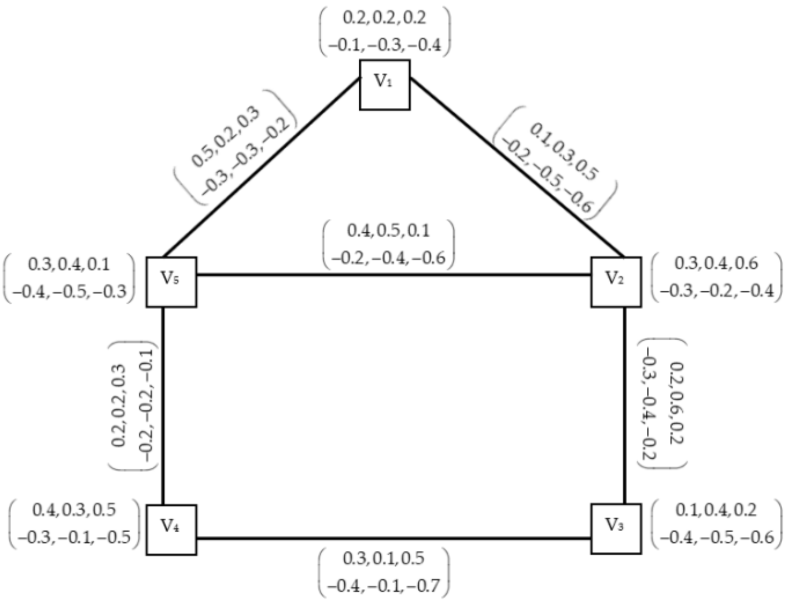

Figure 2. Dominating BSVNG

Thus, $D^N = \{v_1, v_4\}$ is a dominating set because every vertex in $V - D^N = \{v_2, v_3, v_5\}$, is dominated by at least one vertex in D^N and $|D^N| = 2$ is sum of dominating elements. The adjacency matrix of dominating BSVNG is given below

$$A_{D^N}(G) = \begin{bmatrix} \begin{pmatrix} 1,1,1 \\ -1,-1,-1 \end{pmatrix} & \begin{pmatrix} 0.1,0.3,0.5 \\ -0.2,-0.5,-0.6 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} & \begin{pmatrix} 0.5,0.2,0.3 \\ -0.3,-0.3,-0.2 \end{pmatrix} \\ \begin{pmatrix} 0.1,0.3,0.5 \\ -0.2,-0.5,-0.6 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} & \begin{pmatrix} 0.2,0.6,0.2 \\ -0.3,-0.4,-0.2 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} & \begin{pmatrix} 0.4,0.5,0.1 \\ -0.2,-0.4,-0.6 \end{pmatrix} \\ \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} & \begin{pmatrix} 0.2,0.6,0.2 \\ -0.3,-0.4,-0.2 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} & \begin{pmatrix} 0.3,0.1,0.5 \\ -0.4,-0.1,-0.7 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} \\ \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} & \begin{pmatrix} 0.3,0.1,0.5 \\ -0.4,-0.1,-0.7 \end{pmatrix} & \begin{pmatrix} 1,1,1 \\ -1,-1,-1 \end{pmatrix} & \begin{pmatrix} 0.2,0.2,0.3 \\ -0.2,-0.2,-0.1 \end{pmatrix} \\ \begin{pmatrix} 0.5,0.2,0.3 \\ -0.3,-0.3,-0.2 \end{pmatrix} & \begin{pmatrix} 0.4,0.5,0.1 \\ -0.2,-0.4,-0.6 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} & \begin{pmatrix} 0.2,0.2,0.3 \\ -0.2,-0.2,-0.1 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} \end{bmatrix}$$

This can be written in six different matrices as:

$$A\left(\mu^+_{D^N}(G)\right) = \begin{bmatrix} 1 & 0.1 & 0 & 0 & 0.5 \\ 0.1 & 0 & 0.2 & 0 & 0.4 \\ 0 & 0.2 & 0 & 0.3 & 0 \\ 0 & 0 & 0.3 & 1 & 0.2 \\ 0.5 & 0.4 & 0 & 0.2 & 0 \end{bmatrix} \qquad A\left(\mu^-_{D^N}(G)\right) = \begin{bmatrix} -1 & -0.2 & 0 & 0 & -0.3 \\ -0.2 & 0 & -0.3 & 0 & -0.2 \\ 0 & -0.3 & 0 & -0.4 & 0 \\ 0 & 0 & -0.4 & -1 & -0.2 \\ -0.3 & -0.2 & 0 & -0.2 & 0 \end{bmatrix}$$

$$A\left(\gamma^{+}_{D^N}(G)\right)=\begin{bmatrix}1 & 0.3 & 0 & 0 & 0.2\\ 0.3 & 0 & 0.6 & 0 & 0.5\\ 0 & 0.6 & 0 & 0.1 & 0\\ 0 & 0 & 0.1 & 1 & 0.2\\ 0.2 & 0.5 & 0 & 0.2 & 0\end{bmatrix} \qquad A\left(\gamma^{-}_{D^N}(G)\right)=\begin{bmatrix}-1 & -0.5 & 0 & 0 & -0.3\\ -0.5 & 0 & -0.4 & 0 & -0.4\\ 0 & -0.4 & 0 & -0.1 & 0\\ 0 & 0 & -0.1 & -1 & -0.2\\ -0.3 & -0.4 & 0 & -0.2 & 0\end{bmatrix}$$

$$A\left(\sigma^{+}_{D^N}(G)\right)=\begin{bmatrix}1 & 0.5 & 0 & 0 & 0.3\\ 0.5 & 0 & 0.2 & 0 & 0.1\\ 0 & 0.2 & 0 & 0.5 & 0\\ 0 & 0 & 0.5 & 1 & 0.3\\ 0.3 & 0.1 & 0 & 0.3 & 0\end{bmatrix} \qquad A\left(\sigma^{-}_{D^N}(G)\right)=\begin{bmatrix}-1 & -0.6 & 0 & 0 & -0.2\\ -0.6 & 0 & -0.2 & 0 & -0.6\\ 0 & -0.2 & 0 & -0.7 & 0\\ 0 & 0 & -0.7 & -1 & -0.1\\ -0.2 & -0.6 & 0 & -0.1 & 0\end{bmatrix}$$

Since,

$$Spec\left(A\left(\mu^{+}_{D^N}(G)\right)\right)=\{-0.530,-0.080,0.239,1.069,1.301\}$$
$$Spec\left(A\left(\gamma^{+}_{D^N}(G)\right)\right)=\{-0.801,-0.035,0.557,1.019,1.261\}$$
$$Spec\left(A\left(\sigma^{+}_{D^N}(G)\right)\right)=\{-0.398,-0.192,0.042,1.181,1.367\}$$
$$Spec\left(A\left(\mu^{-}_{D^N}(G)\right)\right)=\{-1.233,-1.081,-0.190,0.068,0.436\}$$
$$Spec\left(A\left(\gamma^{-}_{D^N}(G)\right)\right)=\{-1.385,-1.025,-0.272,0.073,0.609\}$$
$$Spec\left(A\left(\sigma^{-}_{D^N}(G)\right)\right)=\{-1.477,-1.322,-0.225,0.295,0.730\}$$

Therefore,

$$Spec\left(A_{D^N}(G)\right)=\left\{\begin{array}{c}(-0.530,-0.801,-0.398,-1.233,-1.385,-1.477),\\ (-0.080,-0.035,-0.192,-1.081,-1.025,-1.322),\\ (0.239,0.557,0.042,-0.190,-0.272,-0.225),\\ (1.069,1.019,1.181,0.068,0.073,0.295),\\ (1.301,1.261,1.367,0.436,0.609,0.730)\end{array}\right\}$$

The energy of BSVNG is

$$\begin{aligned}E_{D^N}(G)&=\left(E\left(\mu^{+}_{D^N}(G)\right),E\left(\gamma^{+}_{D^N}(G)\right),E\left(\sigma^{+}_{D^N}(G)\right),E\left(\mu^{-}_{D^N}(G)\right),E\left(\gamma^{-}_{D^N}(G)\right),E\left(\sigma^{-}_{D^N}(G)\right)\right)\\ &=\left(\sum_{p=1}^{n}|\zeta_p|,\sum_{p=1}^{n}|\tau_p|,\sum_{p=1}^{n}|\upsilon_p|,\sum_{p=1}^{n}|\vartheta_p|,\sum_{p=1}^{n}|\xi_p|,\sum_{p=1}^{n}|\varepsilon_p|\right)\\ &=(3.219,3.673,3.180,3.008,3.364,4.049)\end{aligned}$$

5. Dominating energy in various operation of BSVNG

5.1 Dominating energy in complement of bipolar single-valued neutrosophic graphs

Definition 5.1

The complement of BSVNG $G=(V,E)$ is a BSVNG $\bar{G}=(\bar{V},\bar{E})$ where

$$\bar{\mu}_{1i}^{+}=\mu_{1i}^{+},\bar{\gamma}_{1i}^{+}=\gamma_{1i}^{+},\bar{\sigma}_{1i}^{+}=\sigma_{1i}^{+},$$

$$\bar{\mu}_{1i}^{-}=\mu_{1i}^{-},\bar{\gamma}_{1i}^{-}=\gamma_{1i}^{-},\bar{\sigma}_{1i}^{-}=\sigma_{1i}^{-},\forall i=1,2,\dots,n$$

$$\bar{\mu}_{2ij}^{+}=\mu_{1i}^{+}\mu_{1j}^{+}-\mu_{2ij}^{+},\bar{\gamma}_{2ij}^{+}=\gamma_{1i}^{+}\gamma_{1j}^{+}-\gamma_{2ij}^{+},\bar{\sigma}_{2ij}^{+}=\sigma_{1i}^{+}\sigma_{1j}^{+}-\sigma_{2ij}^{+},$$

$$\bar{\mu}_{2ij}^{-}=\mu_{1i}^{-}\mu_{1j}^{-}-\mu_{2ij}^{-},\bar{\gamma}_{2ij}^{-}=\gamma_{1i}^{-}\gamma_{1j}^{-}-\gamma_{2ij}^{-},\bar{\sigma}_{2ij}^{-}=\sigma_{1i}^{-}\sigma_{1j}^{-}-\sigma_{2ij}^{-},\forall i,j=1,2,\dots,n.$$

Example 5.2

First, we find the dominating energy of BSVNG $G=(V,E)$ as shown in Figure 3. Consider a dominating BSVNG $G=(V,E,\mu^{+},\gamma^{+},\sigma^{+},\mu^{-},\gamma^{-},\sigma^{-},\mu_1^{+},\gamma_1^{+},\sigma_1^{+},\mu_1^{-},\gamma_1^{-},\sigma_1^{-})$ where $V=\{v_1,v_2,v_3,v_4\}$, and $\mu_1^{+}:V\to[0,1],\gamma_1^{+}:V\to[0,1],\sigma_1^{+}:V\to[0,1],\mu_1^{-}:V\to[-1,0],\gamma_1^{-}:V\to[-1,0],\sigma_1^{-}:V\to[-1,0]$ such that

$$\mu_1^{+}(v_1)=\min[\mu^{+}(v_1v_2),\mu^{+}(v_1v_4)]=\min[0.1,0.1]=0.1 \qquad \mu_1^{-}(v_1)=\max[\mu^{-}(v_1v_2),\mu^{-}(v_1v_4)]=\max[-0.2,-0.3]=-0.2$$

$$\mu_1^{+}(v_2)=\min[\mu^{+}(v_2v_1),\mu^{+}(v_2v_3)]=\min[0.1,0.3]=0.1 \qquad \mu_1^{-}(v_2)=\max[\mu^{-}(v_2v_1),\mu^{-}(v_2v_3)]=\max[-0.2,-0.2]=-0.2$$

$$\mu_1^{+}(v_3)=\min[\mu^{+}(v_3v_2),\mu^{+}(v_3v_4)]=\min[0.3,0.1]=0.1 \qquad \mu_1^{-}(v_3)=\max[\mu^{-}(v_3v_2),\mu^{-}(v_3v_4)]=\max[-0.2,-0.2]=-0.2$$

$$\mu_1^{+}(v_4)=\min[\mu^{+}(v_4v_1),\mu^{+}(v_4v_3)]=\min[0.1,0.1]=0.1 \qquad \mu_1^{-}(v_4)=\max[\mu^{-}(v_4v_1),\mu^{-}(v_4v_3)]=\max[-0.3,-0.2]=-0.2$$

$$\gamma_1^{+}(v_1)=\max[\gamma^{+}(v_1v_2),\gamma^{+}(v_1v_4)]=\max[0.5,0.5]=0.5 \qquad \gamma_1^{-}(v_1)=\min[\gamma^{-}(v_1v_2),\gamma^{-}(v_1v_4)]=\min[-0.6,-0.6]=-0.6$$

$$\gamma_1^{+}(v_2)=\max[\gamma^{+}(v_2v_1),\gamma^{+}(v_2v_3)]=\max[0.5,0.4]=0.5 \qquad \gamma_1^{-}(v_2)=\min[\gamma^{-}(v_2v_1),\gamma^{-}(v_2v_3)]=\min[-0.6,-0.6]=-0.6$$

$$\gamma_1^{+}(v_3)=\max[\gamma^{+}(v_3v_2),\gamma^{+}(v_3v_4)]=\max[0.4,0.4]=0.4 \qquad \gamma_1^{-}(v_3)=\min[\gamma^{-}(v_3v_2),\gamma^{-}(v_3v_4)]=\min[-0.6,-0.6]=-0.6$$

$$\gamma_1^{+}(v_4)=\max[\gamma^{+}(v_4v_1),\gamma^{+}(v_4v_3)]=\max[0.5,0.4]=0.5 \qquad \gamma_1^{-}(v_4)=\min[\gamma^{-}(v_4v_1),\gamma^{-}(v_4v_3)]=\min[-0.6,-0.6]=-0.6$$

$$\sigma_1^{+}(v_1)=\max[\sigma^{+}(v_1v_2),\sigma^{+}(v_1v_4)]=\max[0.6,0.6]=0.6 \qquad \sigma_1^{-}(v_1)=\min[\sigma^{-}(v_1v_2),\sigma^{-}(v_1v_4)]=\min[-0.7,-0.7]=-0.7$$

$$\sigma_1^{+}(v_2)=\max[\sigma^{+}(v_2v_1),\sigma^{+}(v_2v_3)]=\max[0.6,0.7]=0.7 \qquad \sigma_1^{-}(v_2)=\min[\sigma^{-}(v_2v_1),\sigma^{-}(v_2v_3)]=\min[-0.7,-0.7]=-0.7$$

$$\sigma_1^{+}(v_3)=\max[\sigma^{+}(v_3v_2),\sigma^{+}(v_3v_4)]=\max[0.7,0.6]=0.7 \qquad \sigma_1^{-}(v_3)=\min[\sigma^{-}(v_3v_2),\sigma^{-}(v_3v_4)]=\min[-0.7,-0.5]=-0.7$$

$$\sigma_1^{+}(v_4)=\max[\sigma^{+}(v_4v_1),\sigma^{+}(v_4v_3)]=\max[0.6,0.6]=0.6 \qquad \sigma_1^{-}(v_4)=\min[\sigma^{-}(v_4v_1),\sigma^{-}(v_4v_3)]=\min[-0.7,-0.5]=-0.7$$

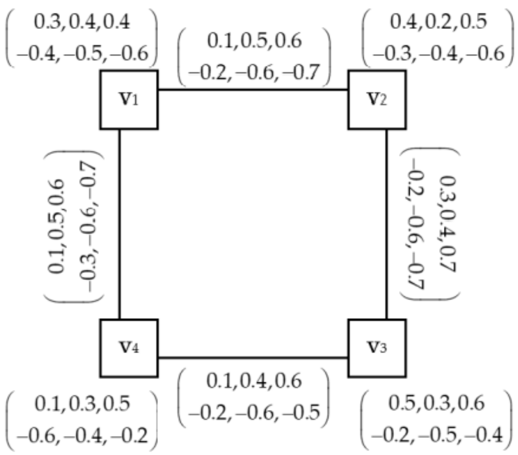

Figure 3: $G=(V,E)$

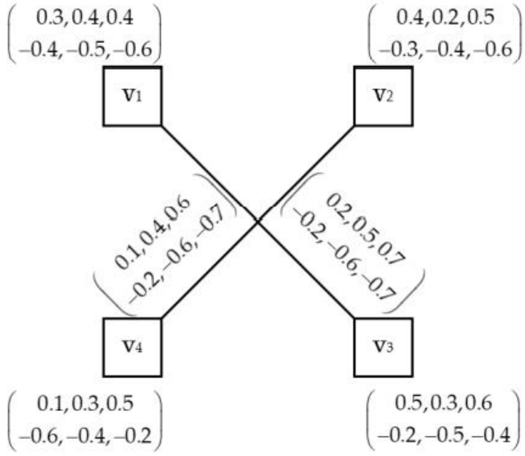

Figure 4: $\bar{G}=(\bar{V},\bar{E})$

Here, v_2 dominates v_1 because

$$\mu^+(v_2v_1)\le \mu_1^+(v_2)\wedge \mu_1^+(v_1) \qquad \mu^-(v_2v_1)\le \mu_1^-(v_2)\wedge \mu_1^-(v_1)$$
$$0.1\le 0.1\wedge 0.1 \qquad -0.3\le -0.2\wedge -0.2$$
$$\gamma^+(v_2v_1)\le \gamma_1^+(v_2)\wedge \gamma_1^+(v_1) \qquad \gamma^-(v_2v_1)\le \gamma_1^-(v_2)\wedge \gamma_1^-(v_1)$$
$$0.5\le 0.5\wedge 0.5 \qquad -0.6\le -0.6\wedge -0.6$$
$$\sigma^+(v_2v_1)\le \sigma_1^+(v_2)\wedge \sigma_1^+(v_1) \qquad \sigma^-(v_2v_1)\le \sigma_1^-(v_2)\wedge \sigma_1^-(v_1)$$
$$0.6\le 0.7\wedge 0.6 \qquad -0.7\le -0.7\wedge -0.7$$

Also, v_1 dominates v_4 because

$$\mu^+(v_1v_4)\le \mu_1^+(v_1)\wedge \mu_1^+(v_4) \qquad \mu^-(v_1v_4)\le \mu_1^-(v_1)\wedge \mu_1^-(v_4)$$
$$0.1\le 0.1\wedge 0.1 \qquad -0.3\le -0.2\wedge -0.2$$
$$\gamma^+(v_1v_4)\le \gamma_1^+(v_1)\wedge \gamma_1^+(v_4) \qquad \gamma^-(v_1v_4)\le \gamma_1^-(v_1)\wedge \gamma_1^-(v_4)$$
$$0.5\le 0.5\wedge 0.5 \qquad -0.6\le -0.6\wedge -0.6$$
$$\sigma^+(v_1v_4)\le \sigma_1^+(v_1)\wedge \sigma_1^+(v_4) \qquad \sigma^-(v_1v_4)\le \sigma_1^-(v_1)\wedge \sigma_1^-(v_4)$$
$$0.6\le 0.6\wedge 0.6 \qquad -0.7\le -0.7\wedge -0.7$$

Thus, $D^N=\{v_1,v_2\}$ is a dominating set because every vertex in $V-D^N=\{v_3,v_4\}$, is dominated by at least one vertex in D^N and $|D^N|=2$ is sum of dominating elements. The adjacency matrix of dominating BSVNG is given below;

$$A_{D^N}(G)=\begin{bmatrix} \begin{pmatrix}1,1,1\\-1,-1,-1\end{pmatrix} & \begin{pmatrix}0.1,0.5,0.6\\-0.2,-0.6,-0.7\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.1,0.5,0.6\\-0.3,-0.6,-0.7\end{pmatrix} \\ \begin{pmatrix}0.1,0.5,0.6\\-0.2,-0.6,-0.7\end{pmatrix} & \begin{pmatrix}1,1,1\\-1,-1,-1\end{pmatrix} & \begin{pmatrix}0.3,0.4,0.7\\-0.2,-0.6,-0.7\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} \\ \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.3,0.4,0.7\\-0.2,-0.6,-0.7\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.1,0.4,0.6\\-0.2,-0.6,-0.5\end{pmatrix} \\ \begin{pmatrix}0.1,0.5,0.6\\-0.3,-0.6,-0.7\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.1,0.4,0.6\\-0.2,-0.6,-0.5\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} \end{bmatrix}$$

This can be written in six different matrices as:

$$A\left(\mu_{D^N}^{+}(G)\right)=\begin{bmatrix}1&0.1&0&0.1\\0.1&1&0.3&0\\0&0.3&0&0.1\\0.1&0&0.1&0\end{bmatrix}\qquad A\left(\mu_{D^N}^{-}(G)\right)=\begin{bmatrix}-1&-0.2&0&-0.3\\-0.2&-1&-0.2&0\\0&-0.2&0&-0.2\\-0.3&0&-0.2&0\end{bmatrix}$$

$$A\left(\gamma_{D^N}^{+}(G)\right)=\begin{bmatrix}1&0.5&0&0.5\\0.5&1&0.4&0\\0&0.4&0&0.4\\0.5&0&0.4&0\end{bmatrix}\qquad A\left(\gamma_{D^N}^{-}(G)\right)=\begin{bmatrix}-1&-0.6&0&-0.6\\-0.6&-1&-0.6&0\\0&-0.6&0&-0.6\\-0.6&0&-0.6&0\end{bmatrix}$$

$$A\left(\sigma_{D^N}^{+}(G)\right)=\begin{bmatrix}1&0.6&0&0.6\\0.6&1&0.7&0\\0&0.7&0&0.6\\0.6&0&0.6&0\end{bmatrix}\qquad A\left(\sigma_{D^N}^{-}(G)\right)=\begin{bmatrix}-1&-0.7&0&-0.7\\-0.7&-1&-0.7&0\\0&-0.7&0&-0.5\\-0.7&0&-0.5&0\end{bmatrix}$$

Since,

$$Spec\left(A\left(\mu_{D^N}^{+}(G)\right)\right)=\{0.942,0.058,1.152,-0.152\},$$
$$Spec\left(A\left(\gamma_{D^N}^{+}(G)\right)\right)=\{-0.588,0.238,0.688,1.662\},$$
$$Spec\left(A\left(\sigma_{D^N}^{+}(G)\right)\right)=\{0.721,0.279,1.922,-0.922\},$$
$$Spec\left(A\left(\mu_{D^N}^{-}(G)\right)\right)=\{-0.139,-0.861,0.262,-1.262\},$$
$$Spec\left(A\left(\gamma_{D^N}^{-}(G)\right)\right)=\{-0.319,-0.681,0.881,-1.881\},$$
$$Spec\left(A\left(\sigma_{D^N}^{-}(G)\right)\right)=\{-0.706,0.906,-2.022,-0.178\}.$$

Therefore, the dominating energy of BSVNG is

$$\begin{aligned}E_{D^N}(G)&=\left(E\left(\mu_{D^N}^{+}(G)\right),E\left(\gamma_{D^N}^{+}(G)\right),E\left(\sigma_{D^N}^{+}(G)\right),E\left(\mu_{D^N}^{-}(G)\right),E\left(\gamma_{D^N}^{-}(G)\right),E\left(\sigma_{D^N}^{-}(G)\right)\right)\\&=\left(\sum_{p=1}^{n}\left|\zeta_p\right|,\sum_{p=1}^{n}\left|\tau_p\right|,\sum_{p=1}^{n}\left|\upsilon_p\right|,\sum_{p=1}^{n}\left|\vartheta_p\right|,\sum_{p=1}^{n}\left|\xi_p\right|,\sum_{p=1}^{n}\left|\varepsilon_p\right|\right)\\&=(2.304,3.176,3.844,2.524,3.762,3.812).\end{aligned}$$

Now, we find the dominating energy of complement of BSVNG $\bar{G}=(\bar{V},\bar{E})$ as shown in Figure 4;

Consider a dominating BSVNG $\bar{G}=\left(V,E,\bar{\mu}^{+},\bar{\gamma}^{+},\bar{\sigma}^{+},\bar{\mu}^{-},\bar{\gamma}^{-},\bar{\sigma}^{-},\bar{\mu}_1^{+},\bar{\gamma}_1^{+},\bar{\sigma}_1^{+},\bar{\mu}_1^{-},\bar{\gamma}_1^{-},\bar{\sigma}_1^{-}\right)$ where $V=\{v_1,v_2,v_3,v_4\}$ and given, $\bar{\mu}_1^{+}:V\to[0,1]$, $\bar{\gamma}_1^{+}:V\to[0,1]$, $\bar{\sigma}_1^{+}:V\to[0,1]$, $\bar{\mu}_1^{-}:V\to[-1,0]$, $\bar{\gamma}_1^{-}:V\to[-1,0]$, $\bar{\sigma}_1^{-}:V\to[-1,0]$ where

$$\bar{\mu}_1^{+}(v_1)=\min\left[\bar{\mu}^{+}(v_1v_3)\right]=\min[0.2]=0.2\qquad \bar{\mu}_1^{-}(v_1)=\max\left[\bar{\mu}^{-}(v_1v_3)\right]=\max[-0.2]=-0.2$$
$$\bar{\mu}_1^{+}(v_2)=\min\left[\bar{\mu}^{+}(v_2v_4)\right]=\min[0.1]=0.1\qquad \bar{\mu}_1^{-}(v_2)=\max\left[\bar{\mu}^{-}(v_2v_4)\right]=\max[-0.2]=-0.2$$
$$\bar{\mu}_1^{+}(v_3)=\min\left[\bar{\mu}^{+}(v_3v_1)\right]=\min[0.2]=0.2\qquad \bar{\mu}_1^{-}(v_3)=\max\left[\bar{\mu}^{-}(v_3v_1)\right]=\max[-0.2]=-0.2$$
$$\bar{\mu}_1^{+}(v_4)=\min\left[\bar{\mu}^{+}(v_4v_2)\right]=\min[0.1]=0.1\qquad \bar{\mu}_1^{-}(v_4)=\max\left[\bar{\mu}^{-}(v_4v_2)\right]=\max[-0.2]=-0.2$$

$$\bar{\gamma}_1^+(v_1) = \max\left[\bar{\gamma}^+(v_1v_3)\right] = \max[0.5] = 0.5 \qquad \bar{\gamma}_1^-(v_1) = \min\left[\bar{\gamma}^-(v_1v_3)\right] = \min[-0.6] = -0.6$$
$$\bar{\gamma}_1^+(v_2) = \max\left[\bar{\gamma}^+(v_2v_4)\right] = \max[0.4] = 0.4 \qquad \bar{\gamma}_1^-(v_2) = \min\left[\bar{\gamma}^-(v_2v_4)\right] = \min[-0.6] = -0.6$$
$$\bar{\gamma}_1^+(v_3) = \max\left[\bar{\gamma}^+(v_3v_1)\right] = \max[0.5] = 0.5 \qquad \bar{\gamma}_1^-(v_3) = \min\left[\bar{\gamma}^-(v_3v_1)\right] = \min[-0.6] = -0.6$$
$$\bar{\gamma}_1^+(v_4) = \max\left[\bar{\gamma}^+(v_4v_2)\right] = \max[0.4] = 0.4 \qquad \bar{\gamma}_1^-(v_4) = \min\left[\bar{\gamma}^-(v_4v_2)\right] = \min[-0.6] = -0.6$$
$$\bar{\sigma}_1^+(v_1) = \max\left[\bar{\sigma}^+(v_1v_3)\right] = \max[0.7] = 0.7 \qquad \bar{\sigma}_1^-(v_1) = \min\left[\bar{\sigma}^-(v_1v_3)\right] = \min[-0.7] = -0.7$$
$$\bar{\sigma}_1^+(v_2) = \max\left[\bar{\sigma}^+(v_2v_4)\right] = \max[0.6] = 0.6 \qquad \bar{\sigma}_1^-(v_2) = \min\left[\bar{\sigma}^-(v_2v_4)\right] = \min[-0.7] = -0.7$$
$$\bar{\sigma}_1^+(v_3) = \max\left[\bar{\sigma}^+(v_3v_1)\right] = \max[0.7] = 0.7 \qquad \bar{\sigma}_1^-(v_3) = \min\left[\bar{\sigma}^-(v_3v_1)\right] = \min[-0.7] = -0.7$$
$$\bar{\sigma}_1^+(v_4) = \max\left[\bar{\sigma}^+(v_4v_2)\right] = \max[0.6] = 0.6 \qquad \bar{\sigma}_1^-(v_4) = \min\left[\bar{\sigma}^-(v_4v_2)\right] = \min[-0.7] = -0.7$$

Here, v_1 dominates v_3 because

$$\bar{\mu}^+(v_1v_3) \le \bar{\mu}_1^+(v_1) \wedge \bar{\mu}_1^+(v_3) \qquad \bar{\mu}^-(v_1v_3) \le \bar{\mu}_1^-(v_1) \wedge \bar{\mu}_1^-(v_3)$$
$$0.2 \le 0.2 \wedge 0.2 \qquad -0.2 \le -0.2 \wedge -0.2$$
$$\bar{\gamma}^+(v_1v_3) \le \bar{\gamma}_1^+(v_1) \wedge \bar{\gamma}_1^+(v_3) \qquad \bar{\gamma}^-(v_1v_3) \le \bar{\gamma}_1^-(v_1) \wedge \bar{\gamma}_1^-(v_3)$$
$$0.5 \le 0.5 \wedge 0.5 \qquad -0.6 \le -0.6 \wedge -0.6$$
$$\bar{\sigma}^+(v_1v_3) \le \bar{\sigma}_1^+(v_1) \wedge \bar{\sigma}_1^+(v_3) \qquad \bar{\sigma}^-(v_1v_3) \le \bar{\sigma}_1^-(v_1) \wedge \bar{\sigma}_1^-(v_3)$$
$$0.7 \le 0.7 \wedge 0.7 \qquad -0.7 \le -0.7 \wedge -0.7$$

Also, v_2 dominates v_4 because

$$\bar{\mu}^+(v_2v_4) \le \bar{\mu}_1^+(v_2) \wedge \bar{\mu}_1^+(v_4) \qquad \bar{\mu}^-(v_1v_4) \le \bar{\mu}_1^-(v_1) \wedge \bar{\mu}_1^-(v_4)$$
$$0.1 \le 0.1 \wedge 0.1 \qquad -0.2 \le -0.2 \wedge -0.2$$
$$\bar{\gamma}^+(v_2v_4) \le \bar{\gamma}_1^+(v_2) \wedge \bar{\gamma}_1^+(v_4) \qquad \bar{\gamma}^-(v_1v_4) \le \bar{\gamma}_1^-(v_1) \wedge \bar{\gamma}_1^-(v_4)$$
$$0.4 \le 0.4 \wedge 0.4 \qquad -0.6 \le -0.6 \wedge -0.6$$
$$\bar{\sigma}^+(v_2v_4) \le \bar{\sigma}_1^+(v_2) \wedge \bar{\sigma}_1^+(v_4) \qquad \bar{\sigma}^-(v_1v_4) \le \bar{\sigma}_1^-(v_1) \wedge \bar{\sigma}_1^-(v_4)$$
$$0.6 \le 0.6 \wedge 0.6 \qquad -0.7 \le -0.7 \wedge -0.7$$

Thus, $D^N = \{v_1, v_2\}$ is a dominating set because every vertex in $V - D^N = \{v_3, v_4\}$, is dominated by at least one vertex in D^N and $|D^N| = 2$ is sum of dominating elements. The adjacency matrix of dominating of complement BSVNG is given below;

$$A_{D^N}(\bar{G}) = \begin{bmatrix} \begin{pmatrix} 1,1,1 \\ -1,-1,-1 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} & \begin{pmatrix} 0.2,0.5,0.7 \\ -0.2,-0.6,-0.7 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} \\ \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} & \begin{pmatrix} 1,1,1 \\ -1,-1,-1 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} & \begin{pmatrix} 0.1,0.4,0.6 \\ -0.2,-0.6,-0.7 \end{pmatrix} \\ \begin{pmatrix} 0.2,0.5,0.7 \\ -0.2,-0.6,-0.7 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} \\ \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} & \begin{pmatrix} 0.1,0.4,0.6 \\ -0.2,-0.6,-0.7 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} \end{bmatrix}$$

This can be written in six different matrices as:

$$A\left(\bar{\mu}_{D^N}^{+}\left(\bar{G}\right)\right)=\begin{bmatrix}1&0&0.2&0\\0&1&0&0.1\\0.2&0&0&0\\0&0.1&0&0\end{bmatrix}\qquad A\left(\bar{\mu}_{D^N}^{-}\left(\bar{G}\right)\right)=\begin{bmatrix}-1&0&-0.2&0\\0&-1&0&-0.2\\-0.2&0&0&0\\0&-0.2&0&0\end{bmatrix}$$

$$A\left(\bar{\gamma}_{D^N}^{+}\left(\bar{G}\right)\right)=\begin{bmatrix}1&0&0.5&0\\0&1&0&0.4\\0.5&0&0&0\\0&0.4&0&0\end{bmatrix}\qquad A\left(\bar{\gamma}_{D^N}^{-}\left(\bar{G}\right)\right)=\begin{bmatrix}-1&0&-0.6&0\\0&-1&0&-0.6\\-0.6&0&0&0\\0&-0.6&0&0\end{bmatrix}$$

$$A\left(\bar{\sigma}_{D^N}^{+}\left(\bar{G}\right)\right)=\begin{bmatrix}1&0&0.7&0\\0&1&0&0.6\\0.7&0&0&0\\0&0.6&0&0\end{bmatrix}\qquad A\left(\bar{\sigma}_{D^N}^{-}\left(\bar{G}\right)\right)=\begin{bmatrix}-1&0&-0.7&0\\0&-1&0&-0.7\\-0.7&0&0&0\\0&-0.7&0&0\end{bmatrix}$$

Since,

$$\begin{aligned}
Spec\left(A\left(\bar{\mu}_{D^N}^{+}\left(\bar{G}\right)\right)\right)&=\left\{-0.0099,-0.0385,1.0099,1.0385\right\},\\
Spec\left(A\left(\bar{\gamma}_{D^N}^{+}\left(\bar{G}\right)\right)\right)&=\left\{-0.1403,-0.2071,1.1403,1.2071\right\},\\
Spec\left(A\left(\bar{\sigma}_{D^N}^{+}\left(\bar{G}\right)\right)\right)&=\left\{-0.2810,-0.3602,1.2810,1.3602\right\},\\
Spec\left(A\left(\bar{\mu}_{D^N}^{-}\left(\bar{G}\right)\right)\right)&=\left\{0.0385,0.0385,-1.0385,-1.0385\right\},\\
Spec\left(A\left(\bar{\gamma}_{D^N}^{-}\left(\bar{G}\right)\right)\right)&=\left\{0.2810,0.2810,-1.2810,-1.2810\right\},\\
Spec\left(A\left(\bar{\sigma}_{D^N}^{-}\left(\bar{G}\right)\right)\right)&=\left\{0.3602,0.3602,-1.3602,-1.3602\right\}.
\end{aligned}$$

Therefore, the dominating energy of complement of BSVNG is;

$$\begin{aligned}
E_{D^N}\left(\bar{G}\right)&=\left(E\left(\bar{\mu}_{D^N}^{+}\left(\bar{G}\right)\right),E\left(\bar{\gamma}_{D^N}^{+}\left(\bar{G}\right)\right),E\left(\bar{\sigma}_{D^N}^{+}\left(\bar{G}\right)\right),E\left(\bar{\mu}_{D^N}^{-}\left(\bar{G}\right)\right),E\left(\bar{\gamma}_{D^N}^{-}\left(\bar{G}\right)\right),E\left(\bar{\sigma}_{D^N}^{-}\left(\bar{G}\right)\right)\right)\\
&=\left(\sum_{p=1}^{n}\left|\zeta_p\right|,\sum_{p=1}^{n}\left|\tau_p\right|,\sum_{p=1}^{n}\left|\upsilon_p\right|,\sum_{p=1}^{n}\left|\vartheta_p\right|,\sum_{p=1}^{n}\left|\xi_p\right|,\sum_{p=1}^{n}\left|\varepsilon_p\right|\right)\\
&=\left(2.0968,2.6948,3.2824,2.1540,3.1240,3.4408\right).
\end{aligned}$$

5.2 Dominating energy in union of bipolar single-valued neutrosophic graphs

Definition 5.3

Let $G_1=\left(V_1,E_1\right)$ and $G_2=\left(V_2,E_2\right)$ be two BSVNG with $V_1\cap V_2=\varnothing$ and $G=G_1\cup G_2=\left(V_1\cup V_2,E_1\cup E_2\right)$ be the union of G_1 and G_2. Then, the union of BSVNG G_1 and G_2 is defined by

$$\left(\mu_1^{+}\cup\mu_1^{'+}\right)(v)=\begin{cases}\mu_1^{+}(v)&\text{if } v\in V_1-V_2\\ \mu_1^{'+}(v)&\text{if } v\in V_2-V_1\end{cases},\qquad \left(\mu_1^{-}\cup\mu_1^{'-}\right)(v)=\begin{cases}\mu_1^{-}(v)&\text{if } v\in V_1-V_2\\ \mu_1^{'-}(v)&\text{if } v\in V_2-V_1\end{cases},$$

$$\left(\gamma_1^{+}\cup\gamma_1^{'+}\right)(v)=\begin{cases}\gamma_1^{+}(v)&\text{if } v\in V_1-V_2\\ \gamma_1^{'+}(v)&\text{if } v\in V_2-V_1\end{cases},\qquad \left(\gamma_1^{-}\cup\gamma_1^{'-}\right)(v)=\begin{cases}\gamma_1^{-}(v)&\text{if } v\in V_1-V_2\\ \gamma_1^{'-}(v)&\text{if } v\in V_2-V_1\end{cases},$$

$$\left(\sigma_1^{+}\cup\sigma_1^{'+}\right)(v)=\begin{cases}\sigma_1^{+}(v)&\text{if } v\in V_1-V_2\\ \sigma_1^{'+}(v)&\text{if } v\in V_2-V_1\end{cases},\qquad \left(\sigma_1^{-}\cup\sigma_1^{'-}\right)(v)=\begin{cases}\sigma_1^{-}(v)&\text{if } v\in V_1-V_2\\ \sigma_1^{'-}(v)&\text{if } v\in V_2-V_1\end{cases},$$

where $\left(\mu_1^+,\gamma_1^+,\sigma_1^+\right)$ and $\left(\mu_1^{'+},\gamma_1^{'+},\sigma_1^{'+}\right)$ denote the vertex of truth-membership, indeterminacy-membership and falsity-membership of G_1 and G_2, respectively. Moreover, the membership values of edges are given as follows;

$$\left(\mu_2^+\cup\mu_2^{'+}\right)\left(v_iv_j\right)=\begin{cases}\mu_2^+\left(e_{ij}\right) \text{ if } e_{ij}\in E_1-E_2\\ \mu_2^{'+}\left(e_{ij}\right) \text{ if } e_{ij}\in E_2-E_1\end{cases}, \qquad \left(\mu_2^-\cup\mu_2^{'-}\right)\left(v_iv_j\right)=\begin{cases}\mu_2^-\left(v_iv_j\right) \text{ if } e_{ij}\in E_1-E_2\\ \mu_2^{'-}\left(v_iv_j\right) \text{ if } e_{ij}\in E_2-E_1\end{cases},$$

$$\left(\gamma_2^+\cup\gamma_2^{'+}\right)\left(v_iv_j\right)=\begin{cases}\gamma_2^+\left(e_{ij}\right) \text{ if } e_{ij}\in E_1-E_2\\ \gamma_2^{'+}\left(e_{ij}\right) \text{ if } e_{ij}\in E_2-E_1\end{cases}, \qquad \left(\gamma_2^-\cup\gamma_2^{'-}\right)\left(v_iv_j\right)=\begin{cases}\gamma_2^-\left(e_{ij}\right) \text{ if } e_{ij}\in E_1-E_2\\ \gamma_2^{'-}\left(e_{ij}\right) \text{ if } e_{ij}\in E_2-E_1\end{cases},$$

$$\left(\sigma_2^+\cup\sigma_2^{'+}\right)\left(v_iv_j\right)=\begin{cases}\sigma_2^+\left(e_{ij}\right) \text{ if } e_{ij}\in E_1-E_2\\ \sigma_2^{'+}\left(e_{ij}\right) \text{ if } e_{ij}\in E_2-E_1\end{cases}, \qquad \left(\sigma_2^-\cup\sigma_2^{'-}\right)\left(v_iv_j\right)=\begin{cases}\sigma_2^-\left(e_{ij}\right) \text{ if } e_{ij}\in E_1-E_2\\ \sigma_2^{'-}\left(e_{ij}\right) \text{ if } e_{ij}\in E_2-E_1\end{cases},$$

where $\left(\mu_2^+,\gamma_2^+,\sigma_2^+\right)$ and $\left(\mu_2^{'+},\gamma_2^{'+},\sigma_2^{'+}\right)$ denote the edge of truth-membership, indeterminacy-membership and falsity-membership of G_1 and G_2, respectively.

Example 5.4

First, we find the dominating energy of BSVNG $G_1=\left(V_1,E_1\right)$ as shown in Figure 5. Consider a dominating BSVNG $G_1=\left(V_1,E_1,\mu^+,\gamma^+,\sigma^+,\mu^-,\gamma^-,\sigma^-,\mu_1^+,\gamma_1^+,\sigma_1^+,\mu_1^-,\gamma_1^-,\sigma_1^-\right)$ where $V_1=\left\{v_1,v_2,v_3,v_4\right\}$, $\mu_1^+:V_1\to\left[0,1\right],\gamma_1^+:V_1\to\left[0,1\right],\sigma_1^+:V_1\to\left[0,1\right],\mu_1^-:V_1\to\left[-1,0\right],\gamma_1^-:V_1\to\left[-1,0\right],\sigma_1^-:V_1\to\left[-1,0\right]$ such that

$$\begin{aligned}\mu_1^+\left(v_1\right)&=\min\left[\mu^+\left(v_1v_2\right),\mu^+\left(v_1v_4\right)\right]\\&=\min\left[0.2,0.2\right]=0.2\\ \mu_1^+\left(v_2\right)&=\min\left[\mu^+\left(v_2v_1\right),\mu^+\left(v_2v_3\right)\right]\\&=\min\left[0.2,0.4\right]=0.2\\ \mu_1^+\left(v_3\right)&=\min\left[\mu^+\left(v_3v_2\right),\mu^+\left(v_3v_4\right)\right]\\&=\min\left[0.4,0.2\right]=0.2\\ \mu_1^+\left(v_4\right)&=\min\left[\mu^+\left(v_4v_1\right),\mu^+\left(v_4v_3\right)\right]\\&=\min\left[0.2,0.2\right]=0.2\end{aligned}$$

$$\begin{aligned}\mu_1^-\left(v_1\right)&=\max\left[\mu^-\left(v_1v_2\right),\mu^-\left(v_1v_4\right)\right]\\&=\max\left[-0.2,-0.3\right]=-0.2\\ \mu_1^-\left(v_2\right)&=\max\left[\mu^-\left(v_2v_1\right),\mu^-\left(v_2v_3\right)\right]\\&=\max\left[-0.2,-0.2\right]=-0.2\\ \mu_1^-\left(v_3\right)&=\max\left[\mu^-\left(v_3v_2\right),\mu^-\left(v_3v_4\right)\right]\\&=\max\left[-0.2,-0.3\right]=-0.2\\ \mu_1^-\left(v_4\right)&=\max\left[\mu^-\left(v_4v_1\right),\mu^-\left(v_4v_3\right)\right]\\&=\max\left[-0.3,-0.3\right]=-0.3\end{aligned}$$

$$\begin{aligned}\gamma_1^+\left(v_1\right)&=\max\left[\gamma^+\left(v_1v_2\right),\gamma^+\left(v_1v_4\right)\right]\\&=\max\left[0.7,0.6\right]=0.7\\ \gamma_1^+\left(v_2\right)&=\max\left[\gamma^+\left(v_2v_1\right),\gamma^+\left(v_2v_3\right)\right]\\&=\max\left[0.7,0.5\right]=0.7\\ \gamma_1^+\left(v_3\right)&=\max\left[\gamma^+\left(v_3v_2\right),\gamma^+\left(v_3v_4\right)\right]\\&=\max\left[0.5,0.6\right]=0.6\\ \gamma_1^+\left(v_4\right)&=\max\left[\gamma^+\left(v_4v_1\right),\gamma^+\left(v_4v_3\right)\right]\\&=\max\left[0.6,0.6\right]=0.6\end{aligned}$$

$$\begin{aligned}\gamma_1^-\left(v_1\right)&=\min\left[\gamma^-\left(v_1v_2\right),\gamma^-\left(v_1v_4\right)\right]\\&=\min\left[-0.6,-0.6\right]=-0.6\\ \gamma_1^-\left(v_2\right)&=\min\left[\gamma^-\left(v_2v_1\right),\gamma^-\left(v_2v_3\right)\right]\\&=\min\left[-0.6,-0.5\right]=-0.6\\ \gamma_1^-\left(v_3\right)&=\min\left[\gamma^-\left(v_3v_2\right),\gamma^-\left(v_3v_4\right)\right]\\&=\min\left[-0.5,-0.6\right]=-0.6\\ \gamma_1^-\left(v_4\right)&=\min\left[\gamma^-\left(v_4v_1\right),\gamma^-\left(v_4v_3\right)\right]\\&=\min\left[-0.6,-0.6\right]=-0.6\end{aligned}$$

$$\sigma_1^+(v_1) = \max\left[\sigma^+(v_1v_2), \sigma^+(v_1v_4)\right] = \max[0.8, 0.4] = 0.8$$
$$\sigma_1^-(v_1) = \min\left[\sigma^-(v_1v_2), \sigma^-(v_1v_4)\right] = \min[-0.7, -0.7] = -0.7$$
$$\sigma_1^+(v_2) = \max\left[\sigma^+(v_2v_1), \sigma^+(v_2v_3)\right] = \max[0.8, 0.8] = 0.8$$
$$\sigma_1^-(v_2) = \min\left[\sigma^-(v_2v_1), \sigma^-(v_2v_3)\right] = \min[-0.7, -0.7] = -0.7$$
$$\sigma_1^+(v_3) = \max\left[\sigma^+(v_3v_2), \sigma^+(v_3v_4)\right] = \max[0.8, 0.7] = 0.8$$
$$\sigma_1^-(v_3) = \min\left[\sigma^-(v_3v_2), \sigma^-(v_3v_4)\right] = \min[-0.7, -0.8] = -0.8$$
$$\sigma_1^+(v_4) = \max\left[\sigma^+(v_4v_1), \sigma^+(v_4v_3)\right] = \max[0.4, 0.7] = 0.7$$
$$\sigma_1^-(v_4) = \min\left[\sigma^-(v_4v_1), \sigma^-(v_4v_3)\right] = \min[-0.7, -0.8] = -0.8$$

Here, v_1 dominates v_2 because

$$\mu^+(v_1v_2) \le \mu_1^+(v_1) \wedge \mu_1^+(v_2) \qquad 0.2 \le 0.2 \wedge 0.2$$
$$\mu^-(v_1v_2) \le \mu_1^-(v_1) \wedge \mu_1^-(v_2) \qquad -0.2 \le -0.2 \wedge -0.2$$
$$\gamma^+(v_1v_2) \le \gamma_1^+(v_1) \wedge \gamma_1^+(v_2) \qquad 0.7 \le 0.7 \wedge 0.7$$
$$\gamma^-(v_1v_2) \le \gamma_1^-(v_1) \wedge \gamma_1^-(v_2) \qquad -0.6 \le -0.6 \wedge -0.6$$
$$\sigma^+(v_1v_2) \le \sigma_1^+(v_1) \wedge \sigma_1^+(v_2) \qquad 0.8 \le 0.8 \wedge 0.8$$
$$\sigma^-(v_1v_2) \le \sigma_1^-(v_1) \wedge \sigma_1^-(v_2) \qquad -0.7 \le -0.7 \wedge -0.7$$

Also, v_3 dominates v_4 because

$$\mu^+(v_3v_4) \le \mu_1^+(v_3) \wedge \mu_1^+(v_4) \qquad 0.2 \le 0.2 \wedge 0.2$$
$$\mu^-(v_1v_4) \le \mu_1^-(v_1) \wedge \mu_1^-(v_4) \qquad -0.2 \le -0.2 \wedge -0.2$$
$$\gamma^+(v_3v_4) \le \gamma_1^+(v_3) \wedge \gamma_1^+(v_4) \qquad 0.6 \le 0.6 \wedge 0.6$$
$$\gamma^-(v_1v_4) \le \gamma_1^-(v_1) \wedge \gamma_1^-(v_4) \qquad -0.6 \le -0.6 \wedge -0.6$$
$$\sigma^+(v_3v_4) \le \sigma_1^+(v_3) \wedge \sigma_1^+(v_4) \qquad 0.7 \le 0.8 \wedge 0.7$$
$$\sigma^-(v_1v_4) \le \sigma_1^-(v_1) \wedge \sigma_1^-(v_4) \qquad -0.7 \le -0.7 \wedge -0.7$$

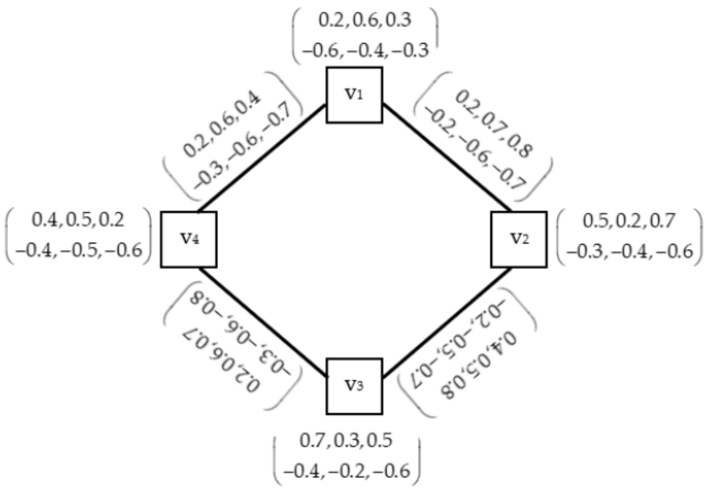

Figure 5. $G_1 = (V_1, E_1)$

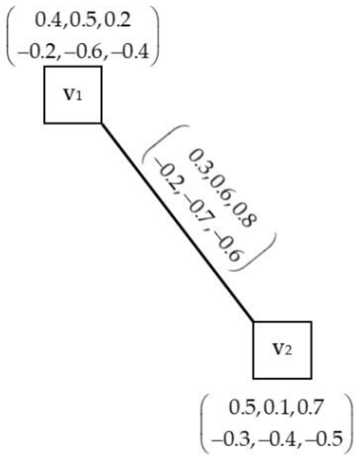

Figure 6. $G_2 = (V_2, E_2)$

Thus, $D^N = \{v_1, v_3\}$ is a dominating set because every vertex in $V - D^N = \{v_2, v_4\}$, is dominated by at least one vertex in D^N and $|D^N| = 2$ is sum of dominating elements. The adjacency matrix of dominating BSVNG $G_1 = (V_1, E_1)$ is given below;

$$A_{D^N}(G_1) = \begin{bmatrix} \begin{pmatrix} 1,1,1 \\ -1,-1,-1 \end{pmatrix} & \begin{pmatrix} 0.2,0.7,0.8 \\ -0.2,-0.6,-0.7 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} & \begin{pmatrix} 0.2,0.6,0.4 \\ -0.3,-0.6,-0.7 \end{pmatrix} \\ \begin{pmatrix} 0.2,0.7,0.8 \\ -0.2,-0.6,-0.7 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} & \begin{pmatrix} 0.4,0.5,0.8 \\ -0.2,-0.5,-0.7 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} \\ \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} & \begin{pmatrix} 0.4,0.5,0.8 \\ -0.2,-0.5,-0.7 \end{pmatrix} & \begin{pmatrix} 1,1,1 \\ -1,-1,-1 \end{pmatrix} & \begin{pmatrix} 0.2,0.6,0.7 \\ -0.3,-0.6,-0.8 \end{pmatrix} \\ \begin{pmatrix} 0.2,0.6,0.4 \\ -0.3,-0.6,-0.7 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} & \begin{pmatrix} 0.2,0.6,0.7 \\ -0.3,-0.6,-0.8 \end{pmatrix} & \begin{pmatrix} 0,0,0 \\ 0,0,0 \end{pmatrix} \end{bmatrix}$$

This can be written in six different matrices as:

$$A\left(\mu^+_{D^N}(G_1)\right) = \begin{bmatrix} 1 & 0.2 & 0 & 0.2 \\ 0.2 & 0 & 0.4 & 0 \\ 0 & 0.4 & 1 & 0.2 \\ 0.2 & 0 & 0.2 & 0 \end{bmatrix} \qquad A\left(\mu^-_{D^N}(G_1)\right) = \begin{bmatrix} -1 & -0.2 & 0 & -0.3 \\ -0.2 & 0 & -0.2 & 0 \\ 0 & -0.2 & -1 & -0.3 \\ -0.3 & 0 & -0.3 & 0 \end{bmatrix}$$

$$A\left(\gamma^+_{D^N}(G_1)\right) = \begin{bmatrix} 1 & 0.7 & 0 & 0.6 \\ 0.7 & 0 & 0.5 & 0 \\ 0 & 0.5 & 1 & 0.6 \\ 0.6 & 0 & 0.6 & 0 \end{bmatrix} \qquad A\left(\gamma^-_{D^N}(G_1)\right) = \begin{bmatrix} -1 & -0.6 & 0 & -0.6 \\ -0.6 & 0 & -0.5 & 0 \\ 0 & -0.5 & -1 & -0.6 \\ -0.6 & 0 & -0.6 & 0 \end{bmatrix}$$

$$A\left(\sigma^+_{D^N}(G_1)\right) = \begin{bmatrix} 1 & 0.8 & 0 & 0.4 \\ 0.8 & 0 & 0.8 & 0 \\ 0 & 0.8 & 1 & 0.7 \\ 0.4 & 0 & 0.7 & 0 \end{bmatrix} \qquad A\left(\sigma^-_{D^N}(G_1)\right) = \begin{bmatrix} -1 & -0.7 & 0 & -0.7 \\ -0.7 & 0 & -0.7 & 0 \\ 0 & -0.7 & -1 & -0.8 \\ -0.7 & -0.7 & -0.8 & 0 \end{bmatrix}$$

Since,

$$Spec\left(A\left(\mu^+_{D^N}(G_1)\right)\right) = \{1.2240, 1.0058, -0.2240, -0.0058\},$$
$$Spec\left(A\left(\gamma^+_{D^N}(G_1)\right)\right) = \{1.8039, 1.0098, -0.8039, -0.0098\},$$
$$Spec\left(A\left(\sigma^+_{D^N}(G_1)\right)\right) = \{1.9869, 1.0377, -0.9869, -0.0377\},$$
$$Spec\left(A\left(\mu^-_{D^N}(G_1)\right)\right) = \{-1.2141, -1.0000, 0.2141, 0.0000\},$$
$$Spec\left(A\left(\gamma^-_{D^N}(G_1)\right)\right) = \{-1.7559, -1.0027, 0.7559, 0.0027\},$$
$$Spec\left(A\left(\sigma^-_{D^N}(G_1)\right)\right) = \{-2.3055, 1.0355, -1.0023, 0.0023\}.$$

Therefore, the dominating energy of BSVNG $G_1 = (V_1, E_1)$ is;

$$\begin{aligned} E_{D^N}(G_1) &= \left(E\left(\mu^+_{D^N}(G_1)\right), E\left(\gamma^+_{D^N}(G_1)\right), E\left(\sigma^+_{D^N}(G_1)\right), E\left(\mu^-_{D^N}(G_1)\right), E\left(\gamma^-_{D^N}(G_1)\right), E\left(\sigma^-_{D^N}(G_1)\right)\right) \\ &= \left(\sum_{p=1}^{n}|\zeta_p|, \sum_{p=1}^{n}|\tau_p|, \sum_{p=1}^{n}|\upsilon_p|, \sum_{p=1}^{n}|\vartheta_p|, \sum_{p=1}^{n}|\xi_p|, \sum_{p=1}^{n}|\varepsilon_p|\right) \\ &= (2.4596, 3.6274, 4.0492, 2.4282, 3.5172, 4.3456). \end{aligned}$$

Also, we find the dominating energy of BSVNG $G_2=(V_2,E_2)$ as shown in Figure 6. Consider a dominating BSVNG $G_2=(V_2,E_2,\mu^+,\gamma^+,\sigma^+,\mu^-,\gamma^-,\sigma^-,\mu_1^+,\gamma_1^+,\sigma_1^+,\mu_1^-,\gamma_1^-,\sigma_1^-)$ where $V_2=\{v_1,v_2\}$, $\mu_1^+:V_2\to[0,1],\gamma_1^+:V_2\to[0,1],\sigma_1^+:V_2\to[0,1],\mu_1^-:V_2\to[-1,0],\gamma_1^-:V_2\to[-1,0],\sigma_1^-:V_2\to[-1,0]$ such that

$$\mu_1^+(v_1)=\min[\mu^+(v_1v_2)]=\min[0.3]=0.3 \qquad \mu_1^-(v_1)=\max[\mu^-(v_1v_2)]=\max[-0.2]=-0.2$$
$$\mu_1^+(v_2)=\min[\mu^+(v_2v_1)]=\min[0.3]=0.3 \qquad \mu_1^-(v_2)=\max[\mu^-(v_2v_1)]=\max[-0.2]=-0.2$$
$$\gamma_1^+(v_1)=\max[\gamma^+(v_1v_2)]=\max[0.6]=0.6 \qquad \gamma_1^-(v_1)=\min[\gamma^-(v_1v_2)]=\min[-0.7]=-0.7$$
$$\gamma_1^+(v_2)=\max[\gamma^+(v_2v_1)]=\max[0.6]=0.6 \qquad \gamma_1^-(v_2)=\min[\gamma^-(v_2v_1)]=\min[-0.7]=-0.7$$
$$\sigma_1^+(v_1)=\max[\sigma^+(v_1v_2)]=\max[0.8]=0.8 \qquad \sigma_1^-(v_1)=\min[\sigma^-(v_1v_2)]=\min[-0.6]=-0.6$$
$$\sigma_1^+(v_2)=\max[\sigma^+(v_2v_1)]=\max[0.8]=0.8 \qquad \sigma_1^-(v_2)=\min[\sigma^-(v_2v_1)]=\min[-0.6]=-0.6$$

Here, v_1 dominates v_2 because

$$\mu^+(v_1v_2)\le\mu_1^+(v_1)\wedge\mu_1^+(v_2) \qquad \mu^-(v_1v_2)\le\mu_1^-(v_1)\wedge\mu_1^-(v_2)$$
$$0.3\le0.3\wedge0.3 \qquad -0.2\le-0.2\wedge-0.2$$
$$\gamma^+(v_1v_2)\le\gamma_1^+(v_1)\wedge\gamma_1^+(v_2) \qquad \gamma^-(v_1v_2)\le\gamma_1^-(v_1)\wedge\gamma_1^-(v_2)$$
$$0.6\le0.6\wedge0.6 \qquad -0.7\le-0.7\wedge-0.7$$
$$\sigma^+(v_1v_2)\le\sigma_1^+(v_1)\wedge\sigma_1^+(v_2) \qquad \sigma^-(v_1v_2)\le\sigma_1^-(v_1)\wedge\sigma_1^-(v_2)$$
$$0.8\le0.8\wedge0.8 \qquad -0.6\le-0.6\wedge-0.6$$

$V_2=\{v_1,v_2\};D^N=\{v_1\};V_2-D^N=\{v_2\};|D^N|=1$ is sum of dominating element. Then, we have the adjacency matrix of dominating BSVNG $G_2=(V_2,E_2)$ is given below;

$$A_{D^N}(G_2)=\begin{bmatrix}\begin{pmatrix}1,1,1\\-1,-1,-1\end{pmatrix} & \begin{pmatrix}0.3,0.6,0.8\\-0.2,-0.7,-0.6\end{pmatrix}\\ \begin{pmatrix}0.3,0.6,0.8\\-0.2,-0.7,-0.6\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}\end{bmatrix}$$

where

$$A(\mu_{D^N}^+(G_2))=\begin{bmatrix}1&0.3\\0.3&0\end{bmatrix},A(\gamma_{D^N}^+(G_2))=\begin{bmatrix}1&0.6\\0.6&0\end{bmatrix},A(\sigma_{D^N}^+(G_2))=\begin{bmatrix}1&0.8\\0.8&0\end{bmatrix},$$
$$A(\mu_{D^N}^-(G_2))=\begin{bmatrix}-1&-0.2\\-0.2&0\end{bmatrix},A(\gamma_{D^N}^-(G_2))=\begin{bmatrix}-1&-0.7\\-0.7&0\end{bmatrix},A(\sigma_{D^N}^-(G_2))=\begin{bmatrix}-1&-0.6\\-0.6&0\end{bmatrix}.$$

Since,

$$Spec(A(\mu_{D^N}^+(G_2)))=\{1.0831,-0.0831\},Spec(A(\gamma_{D^N}^+(G_2)))=\{1.2810,-0.2810\},$$
$$Spec(A(\sigma_{D^N}^+(G_2)))=\{1.4434,-0.4434\},Spec(A(\mu_{D^N}^-(G_2)))=\{-1.0385,0.0385\},$$
$$Spec(A(\gamma_{D^N}^-(G_2)))=\{-1.3602,0.3602\},Spec(A(\sigma_{D^N}^-(G_2)))=\{-1.2810,0.2810\}.$$

Therefore, the dominating energy of BSVNG $G_2=(V_2,E_2)$ is;

$$E_{D^N}(G_2)=\left(E(\mu_{D^N}^+(G_2)),E(\gamma_{D^N}^+(G_2)),E(\sigma_{D^N}^+(G_2)),E(\mu_{D^N}^-(G_2)),E(\gamma_{D^N}^-(G_2)),E(\sigma_{D^N}^-(G_2))\right)$$
$$=\left(\sum_{p=1}^{n}|\zeta_p|,\sum_{p=1}^{n}|\tau_p|,\sum_{p=1}^{n}|\upsilon_p|,\sum_{p=1}^{n}|\vartheta_p|,\sum_{p=1}^{n}|\xi_p|,\sum_{p=1}^{n}|\varepsilon_p|\right)$$
$$=(1.1662,1.5620,1.8868,1.077,1.7204,1.5620).$$

Now, we find the dominating energy of union of two BSVNG $G_1 \cup G_2$ as shown in Figure 7.

$$\mu_1^+(v_1) = \min\left[\mu^+(v_1v_2), \mu^+(v_1v_4)\right] = \min[0.2, 0.2] = 0.2 \qquad \mu_1^-(v_1) = \max\left[\mu^-(v_1v_2), \mu^-(v_1v_4)\right] = \max[-0.2, -0.3] = -0.2$$

$$\mu_1^+(v_2) = \min\left[\mu^+(v_2v_1), \mu^+(v_2v_3)\right] = \min[0.2, 0.4] = 0.2 \qquad \mu_1^-(v_2) = \max\left[\mu^-(v_2v_1), \mu^-(v_2v_3)\right] = \max[-0.2, -0.2] = -0.2$$

$$\mu_1^+(v_3) = \min\left[\mu^+(v_3v_2), \mu^+(v_3v_4)\right] = \min[0.4, 0.2] = 0.2 \qquad \mu_1^-(v_3) = \max\left[\mu^-(v_3v_2), \mu^-(v_3v_4)\right] = \max[-0.2, -0.3] = -0.2$$

$$\mu_1^+(v_4) = \min\left[\mu^+(v_4v_1), \mu^+(v_4v_3)\right] = \min[0.2, 0.2] = 0.2 \qquad \mu_1^-(v_4) = \max\left[\mu^-(v_4v_1), \mu^-(v_4v_3)\right] = \max[-0.3, -0.3] = -0.3$$

$$\mu_1^+(u_1) = \min\left[\mu^+(u_1u_2)\right] = \min[0.3] = 0.3 \qquad \mu_1^-(u_1) = \max\left[\mu^-(u_1u_2)\right] = \max[-0.2] = -0.2$$

$$\mu_1^+(u_2) = \min\left[\mu^+(u_2u_1)\right] = \min[0.3] = 0.3 \qquad \mu_1^-(u_2) = \max\left[\mu^-(u_2u_1)\right] = \max[-0.2] = -0.2$$

$$\gamma_1^+(v_1) = \max\left[\gamma^+(v_1v_2), \gamma^+(v_1v_4)\right] = \max[0.7, 0.6] = 0.7 \qquad \gamma_1^-(v_1) = \min\left[\gamma^-(v_1v_2), \gamma^-(v_1v_4)\right] = \min[-0.6, -0.6] = -0.6$$

$$\gamma_1^+(v_2) = \max\left[\gamma^+(v_2v_1), \gamma^+(v_2v_3)\right] = \max[0.7, 0.5] = 0.7 \qquad \gamma_1^-(v_2) = \min\left[\gamma^-(v_2v_1), \gamma^-(v_2v_3)\right] = \min[-0.6, -0.5] = -0.6$$

$$\gamma_1^+(v_3) = \max\left[\gamma^+(v_3v_2), \gamma^+(v_3v_4)\right] = \max[0.5, 0.6] = 0.6 \qquad \gamma_1^-(v_3) = \min\left[\gamma^-(v_3v_2), \gamma^-(v_3v_4)\right] = \min[-0.5, -0.6] = -0.6$$

$$\gamma_1^+(v_4) = \max\left[\gamma^+(v_4v_1), \gamma^+(v_4v_3)\right] = \max[0.6, 0.6] = 0.6 \qquad \gamma_1^-(v_4) = \min\left[\gamma^-(v_4v_1), \gamma^-(v_4v_3)\right] = \min[-0.6, -0.6] = -0.6$$

$$\gamma_1^+(u_1) = \max\left[\gamma^+(u_1u_2)\right] = \max[0.6] = 0.6 \qquad \gamma_1^-(u_1) = \min\left[\gamma^-(u_1u_2)\right] = \min[-0.7] = -0.7$$

$$\gamma_1^+(u_2) = \max\left[\gamma^+(u_2u_1)\right] = \max[0.6] = 0.6 \qquad \gamma_1^-(u_2) = \min\left[\gamma^-(u_2u_1)\right] = \min[-0.7] = -0.7$$

$$\sigma_1^+(v_1) = \max\left[\sigma^+(v_1v_2), \sigma^+(v_1v_4)\right] = \max[0.8, 0.4] = 0.8 \qquad \sigma_1^-(v_1) = \min\left[\sigma^-(v_1v_2), \sigma^-(v_1v_4)\right] = \min[-0.7, -0.7] = -0.7$$

$$\sigma_1^+(v_2) = \max\left[\sigma^+(v_2v_1), \sigma^+(v_2v_3)\right] = \max[0.8, 0.8] = 0.8 \qquad \sigma_1^-(v_2) = \min\left[\sigma^-(v_2v_1), \sigma^-(v_2v_3)\right] = \min[-0.7, -0.7] = -0.7$$

$$\sigma_1^+(v_3) = \max\left[\sigma^+(v_3v_2), \sigma^+(v_3v_4)\right] = \max[0.8, 0.7] = 0.8 \qquad \sigma_1^-(v_3) = \min\left[\sigma^-(v_3v_2), \sigma^-(v_3v_4)\right] = \min[-0.7, -0.8] = -0.8$$

$$\sigma_1^+(v_4) = \max\left[\sigma^+(v_4v_1), \sigma^+(v_4v_3)\right] = \max[0.4, 0.7] = 0.7 \qquad \sigma_1^-(v_4) = \min\left[\sigma^-(v_4v_1), \sigma^-(v_4v_3)\right] = \min[-0.7, -0.8] = -0.8$$

$$\sigma_1^+(u_1) = \max\left[\sigma^+(u_1u_2)\right] = \max[0.8] = 0.8 \qquad \sigma_1^-(u_1) = \min\left[\sigma^-(u_1u_2)\right] = \min[-0.6] = -0.6$$

$$\sigma_1^+(u_2) = \max\left[\sigma^+(u_2u_1)\right] = \max[0.8] = 0.8 \qquad \sigma_1^-(u_2) = \min\left[\sigma^-(u_2u_1)\right] = \min[-0.6] = -0.6$$

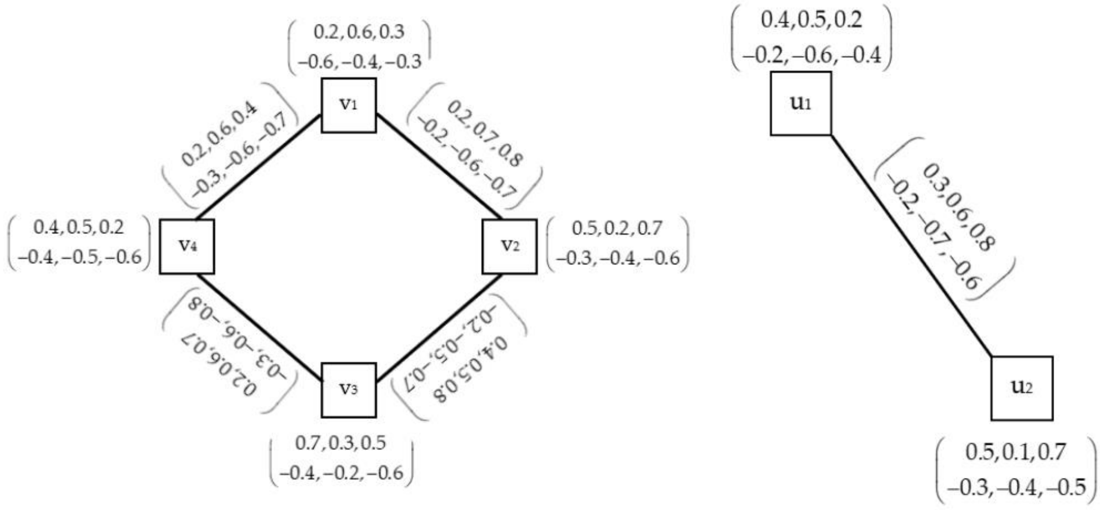

Figure 7. $G_1 \cup G_2$

Here, v_1 dominates v_2 because

$$\mu^+(v_1v_2) \le \mu_1^+(v_1) \wedge \mu_1^+(v_2) \qquad \mu^-(v_1v_2) \le \mu_1^-(v_1) \wedge \mu_1^-(v_2)$$
$$0.2 \le 0.2 \wedge 0.2 \qquad -0.2 \le -0.2 \wedge -0.2$$
$$\gamma^+(v_1v_2) \le \gamma_1^+(v_1) \wedge \gamma_1^+(v_2) \qquad \gamma^-(v_1v_2) \le \gamma_1^-(v_1) \wedge \gamma_1^-(v_2)$$
$$0.7 \le 0.7 \wedge 0.7 \qquad -0.6 \le -0.6 \wedge -0.6$$
$$\sigma^+(v_1v_2) \le \sigma_1^+(v_1) \wedge \sigma_1^+(v_2) \qquad \sigma^-(v_1v_2) \le \sigma_1^-(v_1) \wedge \sigma_1^-(v_2)$$
$$0.8 \le 0.8 \wedge 0.8 \qquad -0.7 \le -0.7 \wedge -0.7$$

Also, v_3 dominates v_4 because

$$\mu^+(v_3v_4) \le \mu_1^+(v_3) \wedge \mu_1^+(v_4) \qquad \mu^-(v_1v_4) \le \mu_1^-(v_1) \wedge \mu_1^-(v_4)$$
$$0.2 \le 0.2 \wedge 0.2 \qquad -0.2 \le -0.2 \wedge -0.2$$
$$\gamma^+(v_3v_4) \le \gamma_1^+(v_3) \wedge \gamma_1^+(v_4) \qquad \gamma^-(v_1v_4) \le \gamma_1^-(v_1) \wedge \gamma_1^-(v_4)$$
$$0.6 \le 0.6 \wedge 0.6 \qquad -0.6 \le -0.6 \wedge -0.6$$
$$\sigma^+(v_3v_4) \le \sigma_1^+(v_3) \wedge \sigma_1^+(v_4) \qquad \sigma^-(v_1v_4) \le \sigma_1^-(v_1) \wedge \sigma_1^-(v_4)$$
$$0.7 \le 0.8 \wedge 0.7 \qquad -0.7 \le -0.7 \wedge -0.7$$

Also, u_1 dominates u_2 because

$$\mu^+(u_1u_2) \le \mu_1^+(u_1) \wedge \mu_1^+(u_2) \qquad \mu^-(u_1u_2) \le \mu_1^-(u_1) \wedge \mu_1^-(u_2)$$
$$0.3 \le 0.3 \wedge 0.3 \qquad -0.2 \le -0.2 \wedge -0.2$$
$$\gamma^+(u_1u_2) \le \gamma_1^+(u_1) \wedge \gamma_1^+(u_2) \qquad \gamma^-(u_1u_2) \le \gamma_1^-(u_1) \wedge \gamma_1^-(u_2)$$
$$0.6 \le 0.6 \wedge 0.6 \qquad -0.7 \le -0.7 \wedge -0.7$$
$$\sigma^+(u_1u_2) \le \sigma_1^+(u_1) \wedge \sigma_1^+(u_2) \qquad \sigma^-(u_1u_2) \le \sigma_1^-(u_1) \wedge \sigma_1^-(u_2)$$
$$0.8 \le 0.8 \wedge 0.8 \qquad -0.6 \le -0.6 \wedge -0.6$$

$V = \{v_1, v_2, v_3, v_4, u_1, u_2\}; D^N = \{v_1, v_3, u_1\}; V - D^N = \{v_2, v_4, u_2\}; |D^N| = 3$ is sum of dominating element. Then, we have the adjacency matrix of dominating BSVNG $G_1 \cup G_2$ is given below;

$$A_{D^N}\left(G_1 \cup G_2\right)=$$

$$\begin{bmatrix}
\begin{pmatrix}1,1,1\\-1,-1,-1\end{pmatrix} & \begin{pmatrix}0.2,0.7,0.8\\-0.2,-0.6,-0.7\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.2,0.6,0.4\\-0.3,-0.6,-0.7\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}\\
\begin{pmatrix}0.2,0.7,0.8\\-0.2,-0.6,-0.7\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.4,0.5,0.8\\-0.2,-0.5,-0.7\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}\\
\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.4,0.5,0.8\\-0.2,-0.5,-0.7\end{pmatrix} & \begin{pmatrix}1,1,1\\-1,-1,-1\end{pmatrix} & \begin{pmatrix}0.2,0.6,0.7\\-0.3,-0.6,-0.8\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}\\
\begin{pmatrix}0.2,0.6,0.4\\-0.3,-0.6,-0.7\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.2,0.6,0.7\\-0.3,-0.6,-0.8\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}\\
\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}1,1,1\\-1,-1,-1\end{pmatrix} & \begin{pmatrix}0.3,0.6,0.8\\-0.2,-0.7,-0.6\end{pmatrix}\\
\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.3,0.6,0.8\\-0.2,-0.7,-0.6\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}
\end{bmatrix}$$

where

$$A\left(\mu^+_{D^N}\left(G_1 \cup G_2\right)\right)=\begin{bmatrix}1&0.2&0&0.2&0&0\\0.2&0&0.4&0&0&0\\0&0.4&1&0.2&0&0\\0.2&0&0.2&0&0&0\\0&0&0&0&1&0.3\\0&0&0&0&0.3&0\end{bmatrix}\quad A\left(\mu^-_{D^N}\left(G_1 \cup G_2\right)\right)=\begin{bmatrix}-1&-0.2&0&-0.3&0&0\\-0.2&0&-0.2&0&0&0\\0&-0.2&-1&-0.3&0&0\\-0.3&0&-0.3&0&0&0\\0&0&0&0&-1&-0.2\\0&0&0&0&-0.2&0\end{bmatrix}$$

$$A\left(\gamma^+_{D^N}\left(G_1 \cup G_2\right)\right)=\begin{bmatrix}1&0.7&0&0.6&0&0\\0.7&0&0.5&0&0&0\\0&0.5&1&0.6&0&0\\0.6&0&0.6&0&0&0\\0&0&0&0&1&0.6\\0&0&0&0&0.6&0\end{bmatrix}\quad A\left(\gamma^-_{D^N}\left(G_1 \cup G_2\right)\right)=\begin{bmatrix}-1&-0.6&0&-0.6&0&0\\-0.6&0&-0.5&0&0&0\\0&-0.5&-1&-0.6&0&0\\-0.6&0&-0.6&0&0&0\\0&0&0&0&-1&-0.7\\0&0&0&0&-0.7&0\end{bmatrix}$$

$$A\left(\sigma^+_{D^N}\left(G_1 \cup G_2\right)\right)=\begin{bmatrix}1&0.8&0&0.4&0&0\\0.8&0&0.8&0&0&0\\0&0.8&1&0.7&0&0\\0.4&0&0.7&0&0&0\\0&0&0&0&1&0.8\\0&0&0&0&0.8&0\end{bmatrix}\quad A\left(\sigma^-_{D^N}\left(G_1 \cup G_2\right)\right)=\begin{bmatrix}-1&-0.7&0&-0.7&0&0\\-0.7&0&-0.7&0&0&0\\0&-0.7&-1&-0.8&0&0\\-0.7&0&-0.8&0&0&0\\0&0&0&0&-1&-0.6\\0&0&0&0&-0.6&0\end{bmatrix}$$

Since,

$$Spec\left(A\left(\mu^+_{D^N}\left(G_1 \cup G_2\right)\right)\right)=\{1.2239, 1.0831, 1.0058, -0.2239, -0.0831, -0.0058\},$$

$$Spec\left(A\left(\gamma^+_{D^N}\left(G_1 \cup G_2\right)\right)\right)=\{1.8039, 1.2810, 1.0098, -0.8039, -0.2810, -0.0098\},$$

$$Spec\left(A\left(\sigma^+_{D^N}\left(G_1 \cup G_2\right)\right)\right)=\{1.9662, 1.4434, 1.0295, -0.9662, -0.4434, -0.0295\},$$

$$Spec\left(A\left(\mu^-_{D^N}\left(G_1 \cup G_2\right)\right)\right)=\{-1.2141, -1.0385, -1.0000, 0.2141, 0.0385, 0.0000\},$$

$$Spec\left(A\left(\gamma^-_{D^N}\left(G_1 \cup G_2\right)\right)\right)=\{-1.7559, -1.3602, -1.0027, 0.7559, 0.3602, 0.0027\},$$

$$Spec\left(A\left(\sigma^-_{D^N}\left(G_1 \cup G_2\right)\right)\right)=\{-2.0355, -1.2810, 1.0355, -1.0023, 0.2810, 0.0023\}.$$

Therefore, the dominating energy of union BSVNG $G_1 \cup G_2$ is;

$$E_{D^N}(G_1 \cup G_2) = \begin{pmatrix} E\left(\mu^+_{D^N}(G_1 \cup G_2)\right), E\left(\gamma^+_{D^N}(G_1 \cup G_2)\right), E\left(\sigma^+_{D^N}(G_1 \cup G_2)\right), E\left(\mu^-_{D^N}(G_1 \cup G_2)\right), \\ E\left(\gamma^-_{D^N}(G_1 \cup G_2)\right), E\left(\sigma^-_{D^N}(G_1 \cup G_2)\right) \end{pmatrix}$$
$$= \left(\sum_{p=1}^{n}|\zeta_p|, \sum_{p=1}^{n}|\tau_p|, \sum_{p=1}^{n}|\upsilon_p|, \sum_{p=1}^{n}|\vartheta_p|, \sum_{p=1}^{n}|\xi_p|, \sum_{p=1}^{n}|\varepsilon_p|\right)$$
$$= (3.6256, 5.1894, 5.8782, 3.5052, 5.2376, 5.6376).$$

5.3 Dominating energy in join of bipolar single-valued neutrosophic graphs

Definition 5.5

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two BSVNG. The join of two BSVNG $\hat{G} = (\hat{V}, \hat{E}) = G_1 + G_2$ such that $\hat{V} = V_1 + V_2 = V_1 \cup V_2$, defined by

$$(\mu_1^+ + \mu_1^{'+})(v) = (\mu_1^+ \cup \mu_1^{'+})(v) \text{ if } v \in V_1 \cup V_2; \qquad (\mu_1^- + \mu_1^{'-})(v) = (\mu_1^- \cup \mu_1^{'-})(v) \text{ if } v \in V_1 \cup V_2;$$
$$(\gamma_1^+ + \gamma_1^{'+})(v) = (\gamma_1^+ \cup \gamma_1^{'+})(v) \text{ if } v \in V_1 \cup V_2; \qquad (\gamma_1^- + \gamma_1^{'-})(v) = (\gamma_1^- \cup \gamma_1^{'-})(v) \text{ if } v \in V_1 \cup V_2;$$
$$(\sigma_1^+ + \sigma_1^{'+})(v) = (\sigma_1^+ \cup \sigma_1^{'+})(v) \text{ if } v \in V_1 \cup V_2; \qquad (\sigma_1^- + \sigma_1^{'-})(v) = (\sigma_1^- \cup \sigma_1^{'-})(v) \text{ if } v \in V_1 \cup V_2.$$

and $\hat{E} = E_1 + E_2 = E_1 \cup E_2$ be the set of all edges joining the vertices of G_1 and G_2.

Example 5.6

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two BSVNG as shown in Figure 8 and 9, respectively. Now, we find the dominating energy of join BSVNG $\hat{G}(\hat{V}, \hat{E}) = G_1 + G_2$, as shown in Figure 10. Consider a dominating BSVNG $\hat{G} = (\hat{V}, \hat{E}, \mu^+, \gamma^+, \sigma^+, \mu^-, \gamma^-, \sigma^-, \mu_1^+, \gamma_1^+, \sigma_1^+, \mu_1^-, \gamma_1^-, \sigma_1^-)$ where $\hat{V} = (v_1, v_2, v_3, v_4, u_1, u_2)$ and $\mu_1^+ : \hat{V} \to [0,1], \gamma_1^+ : \hat{V} \to [0,1], \sigma_1^+ : \hat{V} \to [0,1], \mu_1^- : \hat{V} \to [-1,0], \gamma_1^- : \hat{V} \to [-1,0]$, $\sigma_1^- : \hat{V} \to [-1,0]$ such that

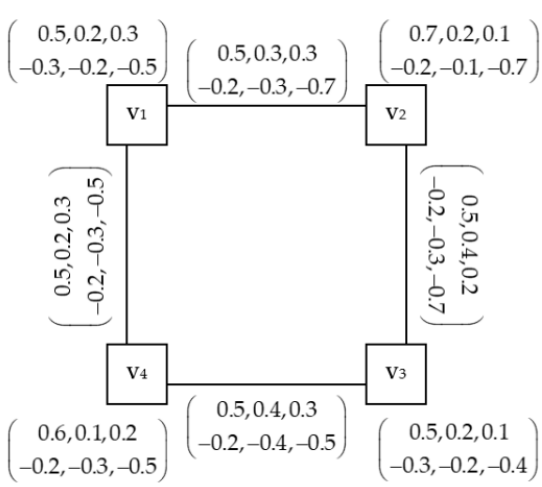

Figure 8. $G_1 = (V_1, E_1)$

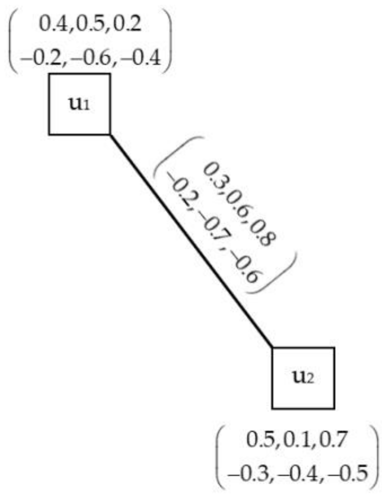

Figure 9. $G_2 = (V_2, E_2)$

$$\mu_1^+(v_1)=\min\left[\mu^+(v_1v_2),\mu^+(v_1v_4),\mu^+(v_1u_1),\mu^+(v_1u_2)\right]=\min(0.5,0.5,0.4,0.5)=0.4$$
$$\mu_1^+(v_2)=\min\left[\mu^+(v_2v_1),\mu^+(v_2v_3),\mu^+(v_2u_1),\mu^+(v_2u_2)\right]=\min(0.5,0.5,0.4,0.5)=0.4$$
$$\mu_1^+(v_3)=\min\left[\mu^+(v_3v_2),\mu^+(v_3v_4),\mu^+(v_3u_1),\mu^+(v_3u_2)\right]=\min(0.5,0.5,0.4,0.5)=0.4$$
$$\mu_1^+(v_4)=\min\left[\mu^+(v_4v_1),\mu^+(v_4v_3),\mu^+(v_4u_1),\mu^+(v_4u_2)\right]=\min(0.4,0.5,0.4,0.5)=0.4$$
$$\mu_1^+(u_1)=\min\left[\mu^+(u_1v_1),\mu^+(u_1v_2),\mu^+(u_1v_3),\mu^+(u_1v_4),\mu^+(u_1u_2)\right]=\min(0.4,0.4,0.4,0.4,0.5)=0.4$$
$$\mu_1^+(u_2)=\min\left[\mu^+(u_2v_1),\mu^+(u_2v_2),\mu^+(u_2v_3),\mu^+(u_2v_4),\mu^+(u_2u_1)\right]=\min(0.5,0.5,0.5,0.5,0.5)=0.5$$

$$\gamma_1^+(v_1)=\max\left[\gamma^+(v_1v_2),\gamma^+(v_1v_4),\gamma^+(v_1u_1),\gamma^+(v_1u_2)\right]=\max(0.3,0.2,0.5,0.2)=0.5$$
$$\gamma_1^+(v_2)=\max\left[\gamma^+(v_2v_1),\gamma^+(v_2v_3),\gamma^+(v_2u_1),\gamma^+(v_2u_2)\right]=\max(0.3,0.4,0.5,0.2)=0.5$$
$$\gamma_1^+(v_3)=\max\left[\gamma^+(v_3v_2),\gamma^+(v_3v_4),\gamma^+(v_3u_1),\gamma^+(v_3u_2)\right]=\max(0.4,0.4,0.5,0.2)=0.5$$
$$\gamma_1^+(v_4)=\max\left[\gamma^+(v_4v_1),\gamma^+(v_4v_3),\gamma^+(v_4u_1),\gamma^+(v_4u_2)\right]=\max(0.2,0.4,0.5,0.1)=0.5$$
$$\gamma_1^+(u_1)=\max\left[\gamma^+(u_1v_1),\gamma^+(u_1v_2),\gamma^+(u_1v_3),\gamma^+(u_1v_4),\gamma^+(u_1u_2)\right]=\max(0.5,0.5,0.5,0.5,0.2)=0.5$$
$$\gamma_1^+(u_2)=\max\left[\gamma^+(u_2v_1),\gamma^+(u_2v_2),\gamma^+(u_2v_3),\gamma^+(u_2v_4),\gamma^+(u_2u_1)\right]=\max(0.2,0.2,0.2,0.1,0.2)=0.2$$

$$\sigma_1^+(v_1)=\max\left[\sigma^+(v_1v_2),\sigma^+(v_1v_4),\sigma^+(v_1u_1),\sigma^+(v_1u_2)\right]=\max(0.3,0.3,0.3,0.7)=0.7$$
$$\sigma_1^+(v_2)=\max\left[\sigma^+(v_2v_1),\sigma^+(v_2v_3),\sigma^+(v_2u_1),\sigma^+(v_2u_2)\right]=\max(0.3,0.2,0.2,0.7)=0.7$$
$$\sigma_1^+(v_3)=\max\left[\sigma^+(v_3v_2),\sigma^+(v_3v_4),\sigma^+(v_3u_1),\sigma^+(v_3u_2)\right]=\max(0.2,0.3,0.2,0.7)=0.7$$
$$\sigma_1^+(v_4)=\max\left[\sigma^+(v_4v_1),\sigma^+(v_4v_3),\sigma^+(v_4u_1),\sigma^+(v_4u_2)\right]=\max(0.3,0.3,0.2,0.7)=0.7$$
$$\sigma_1^+(u_1)=\max\left[\sigma^+(u_1v_1),\sigma^+(u_1v_2),\sigma^+(u_1v_3),\sigma^+(u_1v_4),\sigma^+(u_1u_2)\right]=\max(0.3,0.2,0.2,0.2,0.3)=0.3$$
$$\sigma_1^+(u_2)=\max\left[\sigma^+(u_2v_1),\sigma^+(u_2v_2),\sigma^+(u_2v_3),\sigma^+(u_2v_4),\sigma^+(u_2u_1)\right]=\max(0.7,0.7,0.7,0.7,0.3)=0.7$$

$$\mu_1^-(v_1)=\max\left[\mu^-(v_1v_2),\mu^-(v_1v_4),\mu^-(v_1u_1),\mu^-(v_1u_2)\right]=\max(-0.2,-0.2,-0.2,-0.3)=-0.2$$
$$\mu_1^-(v_2)=\max\left[\mu^-(v_2v_1),\mu^-(v_2v_3),\mu^-(v_2u_1),\mu^-(v_2u_2)\right]=\max(-0.2,-0.2,-0.2,-0.2)=-0.2$$
$$\mu_1^-(v_3)=\max\left[\mu^-(v_3v_2),\mu^-(v_3v_4),\mu^-(v_3u_1),\mu^-(v_3u_2)\right]=\max(-0.2,-0.2,-0.2,-0.3)=-0.2$$
$$\mu_1^-(v_4)=\max\left[\mu^-(v_4v_1),\mu^-(v_4v_3),\mu^-(v_4u_1),\mu^-(v_4u_2)\right]=\max(-0.2,-0.2,-0.2,-0.2)=-0.2$$
$$\mu_1^-(u_1)=\max\left[\mu^-(u_1v_1),\mu^-(u_1v_2),\mu^-(u_1v_3),\mu^-(u_1v_4),\mu^-(u_1u_2)\right]=\max(-0.2,-0.2,-0.2,-0.2,-0.2)=-0.2$$
$$\mu_1^-(u_2)=\max\left[\mu^-(u_2v_1),\mu^-(u_2v_2),\mu^-(u_2v_3),\mu^-(u_2v_4),\mu^-(u_2u_1)\right]=\max(-0.3,-0.2,-0.3,-0.2,-0.2)=-0.2$$

$$\gamma_1^-(v_1)=\min\left[\gamma^-(v_1v_2),\gamma^-(v_1v_4),\gamma^-(v_1u_1),\gamma^-(v_1u_2)\right]=\min(-0.3,-0.3,-0.6,-0.4)=-0.6$$
$$\gamma_1^-(v_2)=\min\left[\gamma^-(v_2v_1),\gamma^-(v_2v_3),\gamma^-(v_2u_1),\gamma^-(v_2u_2)\right]=\min(-0.3,-0.3,-0.6,-0.4)=-0.6$$
$$\gamma_1^-(v_3)=\min\left[\gamma^-(v_3v_2),\gamma^-(v_3v_4),\gamma^-(v_3u_1),\gamma^-(v_3u_2)\right]=\min(-0.3,-0.4,-0.6,-0.4)=-0.6$$
$$\gamma_1^-(v_4)=\min\left[\gamma^-(v_4v_1),\gamma^-(v_4v_3),\gamma^-(v_4u_1),\gamma^-(v_4u_2)\right]=\min(-0.3,-0.4,-0.6,-0.4)=-0.6$$
$$\gamma_1^-(u_1)=\min\left[\gamma^-(u_1v_1),\gamma^-(u_1v_2),\gamma^-(u_1v_3),\gamma^-(u_1v_4),\gamma^-(u_1u_2)\right]=\min(-0.6,-0.6,-0.6,-0.6,-0.7)=-0.7$$
$$\gamma_1^-(u_2)=\min\left[\gamma^-(u_2v_1),\gamma^-(u_2v_2),\gamma^-(u_2v_3),\gamma^-(u_2v_4),\gamma^-(u_2u_1)\right]=\min(-0.4,-0.4,-0.4,-0.4,-0.7)=-0.7$$

$\sigma_1^-(v_1)=\min\left[\sigma^-(v_1v_2),\sigma^-(v_1v_4),\sigma^-(v_1u_1),\sigma^-(v_1u_2)\right]=\min(-0.7,-0.5,-0.5,-0.5)=-0.7$
$\sigma_1^-(v_2)=\min\left[\sigma^-(v_2v_1),\sigma^-(v_2v_3),\sigma^-(v_2u_1),\sigma^-(v_2u_2)\right]=\min(-0.7,-0.7,-0.7,-0.7)=-0.7$
$\sigma_1^-(v_3)=\min\left[\sigma^-(v_3v_2),\sigma^-(v_3v_4),\sigma^-(v_3u_1),\sigma^-(v_3u_2)\right]=\min(-0.7,-0.5,-0.4,-0.5)=-0.7$
$\sigma_1^-(v_4)=\min\left[\sigma^-(v_4v_1),\sigma^-(v_4v_3),\sigma^-(v_4u_1),\sigma^-(v_4u_2)\right]=\min(-0.5,-0.5,-0.5,-0.5)=-0.5$
$\sigma_1^-(u_1)=\min\left[\sigma^-(u_1v_1),\sigma^-(u_1v_2),\sigma^-(u_1v_3),\sigma^-(u_1v_4),\sigma^-(u_1u_2)\right]=\min(-0.5,-0.7,-0.4,-0.5,-0.6)=-0.7$
$\sigma_1^-(u_2)=\min\left[\sigma^-(u_2v_1),\sigma^-(u_2v_2),\sigma^-(u_2v_3),\sigma^-(u_2v_4),\sigma^-(u_2u_1)\right]=\min(-0.5,-0.7,-0.5,-0.5,-0.6)=-0.7$

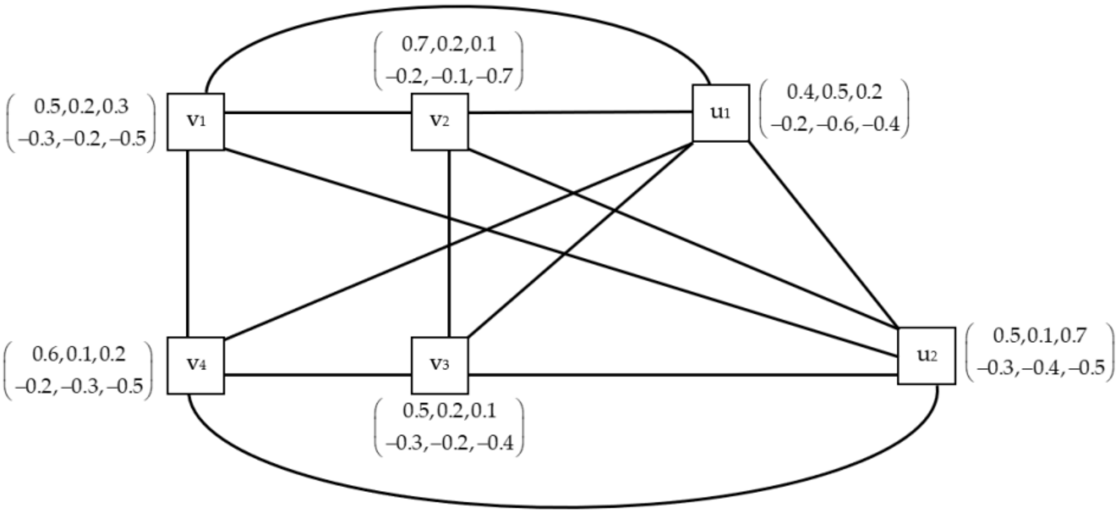

Figure 10. G_1+G_2

$V=\{v_1,v_2,v_3,v_4,u_1,u_2\};D^N=\{v_1,v_2,v_3,v_4,u_1\};V-D^N=\{u_2\};\left|D^N\right|=5$ is sum of dominating element. Then, we have the adjacency matrix of dominating BSVNG G_1+G_2 is given below;

$$A_{D^N}(G_1+G_2)=$$
$$\begin{bmatrix}
\begin{pmatrix}1,1,1\\-1,-1,-1\end{pmatrix} & \begin{pmatrix}0.5,0.3,0.3\\-0.2,-0.3,-0.7\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.5,0.2,0.3\\-0.2,-0.3,-0.5\end{pmatrix} & \begin{pmatrix}0.4,0.5,0.3\\-0.2,-0.6,-0.5\end{pmatrix} & \begin{pmatrix}0.5,0.2,0.7\\-0.3,-0.4,-0.5\end{pmatrix}\\
\begin{pmatrix}0.5,0.3,0.3\\-0.2,-0.3,-0.7\end{pmatrix} & \begin{pmatrix}1,1,1\\-1,-1,-1\end{pmatrix} & \begin{pmatrix}0.5,0.4,0.2\\-0.2,-0.3,-0.7\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.4,0.5,0.2\\-0.2,-0.6,-0.7\end{pmatrix} & \begin{pmatrix}0.5,0.2,0.7\\-0.2,-0.4,-0.7\end{pmatrix}\\
\begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.5,0.4,0.2\\-0.2,-0.3,-0.7\end{pmatrix} & \begin{pmatrix}1,1,1\\-1,-1,-1\end{pmatrix} & \begin{pmatrix}0.5,0.4,0.3\\-0.2,-0.4,-0.5\end{pmatrix} & \begin{pmatrix}0.4,0.5,0.2\\-0.2,-0.6,-0.4\end{pmatrix} & \begin{pmatrix}0.5,0.2,0.7\\-0.3,-0.4,-0.5\end{pmatrix}\\
\begin{pmatrix}0.5,0.2,0.3\\-0.2,-0.3,-0.5\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix} & \begin{pmatrix}0.5,0.4,0.3\\-0.2,-0.4,-0.5\end{pmatrix} & \begin{pmatrix}1,1,1\\-1,-1,-1\end{pmatrix} & \begin{pmatrix}0.4,0.5,0.2\\-0.2,-0.6,-0.5\end{pmatrix} & \begin{pmatrix}0.5,0.1,0.7\\-0.2,-0.4,-0.5\end{pmatrix}\\
\begin{pmatrix}0.4,0.5,0.3\\-0.2,-0.6,-0.5\end{pmatrix} & \begin{pmatrix}0.4,0.5,0.2\\-0.2,-0.6,-0.7\end{pmatrix} & \begin{pmatrix}0.4,0.5,0.2\\-0.2,-0.6,-0.4\end{pmatrix} & \begin{pmatrix}0.4,0.5,0.2\\-0.2,-0.6,-0.5\end{pmatrix} & \begin{pmatrix}1,1,1\\-1,-1,-1\end{pmatrix} & \begin{pmatrix}0.5,0.2,0.3\\-0.2,-0.3,-0.5\end{pmatrix}\\
\begin{pmatrix}0.5,0.2,0.7\\-0.3,-0.4,-0.5\end{pmatrix} & \begin{pmatrix}0.5,0.2,0.7\\-0.2,-0.4,-0.7\end{pmatrix} & \begin{pmatrix}0.5,0.2,0.7\\-0.3,-0.4,-0.5\end{pmatrix} & \begin{pmatrix}0.5,0.1,0.7\\-0.2,-0.4,-0.5\end{pmatrix} & \begin{pmatrix}0.5,0.2,0.3\\-0.2,-0.3,-0.5\end{pmatrix} & \begin{pmatrix}0,0,0\\0,0,0\end{pmatrix}
\end{bmatrix}$$

where

$$A\left(\mu^{+}_{D^N}\left(G_1+G_2\right)\right)=\begin{bmatrix}1&0.5&0&0.5&0.4&0.5\\0.5&1&0.5&0&0.4&0.5\\0&0.5&1&0.5&0.4&0.5\\0.5&0&0.5&1&0.4&0.5\\0.4&0.4&0.4&0.4&1&0.5\\0.5&0.5&0.5&0.5&0.5&0\end{bmatrix}$$

$$A\left(\mu^{-}_{D^N}\left(G_1+G_2\right)\right)=\begin{bmatrix}-1&-0.2&0&-0.2&-0.2&-0.3\\-0.2&-1&-0.2&0&-0.2&-0.2\\0&-0.2&-1&-0.2&-0.2&-0.3\\-0.2&0&-0.4&-1&-0.2&-0.2\\-0.2&-0.2&-0.2&-0.2&-1&-0.2\\-0.3&-0.2&-0.3&-0.2&-0.2&0\end{bmatrix}$$

$$A\left(\gamma^{+}_{D^N}\left(G_1+G_2\right)\right)=\begin{bmatrix}1&0.3&0&0.2&0.5&0.2\\0.3&1&0.4&0&0.5&0.2\\0&0.4&1&0.4&0.5&0.2\\0.2&0&0.4&1&0.5&0.1\\0.5&0.5&0.5&0.5&1&0.2\\0.2&0.2&0.2&0.1&0.2&0\end{bmatrix}$$

$$A\left(\gamma^{-}_{D^N}\left(G_1+G_2\right)\right)=\begin{bmatrix}-1&-0.3&0&-0.3&-0.6&-0.4\\-0.3&-1&-0.3&0&-0.6&-0.4\\0&-0.3&-1&-0.4&-0.6&-0.4\\-0.3&0&-0.4&-1&-0.6&-0.4\\-0.6&-0.6&-0.6&-0.6&-1&-0.3\\-0.4&-0.4&-0.4&-0.4&-0.3&0\end{bmatrix}$$

$$A\left(\sigma^{+}_{D^N}\left(G_1+G_2\right)\right)=\begin{bmatrix}1&0.3&0&0.3&0.3&0.7\\0.3&1&0.2&0&0.2&0.7\\0&0.2&1&0.3&0.2&0.7\\0.3&0&0.3&1&0.2&0.7\\0.3&0.2&0.2&0.2&1&0.3\\0.7&0.7&0.7&0.7&0.3&0\end{bmatrix}$$

$$A\left(\sigma^{-}_{D^N}\left(G_1+G_2\right)\right)=\begin{bmatrix}-1&-0.7&0&-0.5&-0.5&-0.5\\-0.7&-1&-0.7&0&-0.7&-0.7\\0&-0.7&-1&-0.5&-0.4&-0.5\\-0.5&0&-0.5&-1&-0.5&-0.5\\-0.5&-0.7&-0.4&-0.5&-1&-0.5\\-0.5&-0.7&-0.5&-0.5&-0.5&0\end{bmatrix}$$

Since,

$$\begin{aligned}
&Spec\left(A\left(\mu^{+}_{D^N}\left(G_1+G_2\right)\right)\right)=\left\{0,1,-0.436,0.558,2.877\right\},\\
&Spec\left(A\left(\gamma^{+}_{D^N}\left(G_1+G_2\right)\right)\right)=\left\{-0.080,0.270,0.339,0.955,1.068,2.448\right\},\\
&Spec\left(A\left(\sigma^{+}_{D^N}\left(G_1+G_2\right)\right)\right)=\left\{-0.826,0.439,0.806,0.955,1.062,2.564\right\},\\
&Spec\left(A\left(\mu^{-}_{D^N}\left(G_1+G_2\right)\right)\right)=\left\{-1,-1.838,-0.770,-0.565,0.173\right\},\\
&Spec\left(A\left(\gamma^{-}_{D^N}\left(G_1+G_2\right)\right)\right)=\left\{-0.954,-0.346,-2.809,-1.050,-0.207,0.366\right\},\\
&Spec\left(A\left(\sigma^{-}_{D^N}\left(G_1+G_2\right)\right)\right)=\left\{-1,-3.280,-1.006,-0.421,0.213,0.495\right\}.
\end{aligned}$$

Therefore, the dominating energy of union BSVNG G_1+G_2 is;

$$\begin{aligned}
E_{D^N}\left(G_1+G_2\right)&=\begin{pmatrix}E\left(\mu^{+}_{D^N}\left(G_1+G_2\right)\right),E\left(\gamma^{+}_{D^N}\left(G_1+G_2\right)\right),E\left(\sigma^{+}_{D^N}\left(G_1+G_2\right)\right),E\left(\mu^{-}_{D^N}\left(G_1+G_2\right)\right),\\E\left(\gamma^{-}_{D^N}\left(G_1+G_2\right)\right),E\left(\sigma^{-}_{D^N}\left(G_1+G_2\right)\right)\end{pmatrix}\\
&=\left(\sum_{p=1}^{n}\left|\zeta_p\right|,\sum_{p=1}^{n}\left|\tau_p\right|,\sum_{p=1}^{n}\left|\upsilon_p\right|,\sum_{p=1}^{n}\left|\vartheta_p\right|,\sum_{p=1}^{n}\left|\xi_p\right|,\sum_{p=1}^{n}\left|\varepsilon_p\right|\right)\\
&=\left(4.871,5.160,6.652,4.346,5.732,6.415\right).
\end{aligned}$$

Theorem 5.7 Let $G=\left(V,E,\mu^{+},\gamma^{+},\sigma^{+},\mu^{-},\gamma^{-},\sigma^{-},\mu_1^{+},\gamma_1^{+},\sigma_1^{+},\mu_1^{-},\gamma_1^{-},\sigma_1^{-}\right)$ be a dominating BSVNG with *n* vertices. Let $D^N=\{z_1,z_2,\ldots,z_k\}$ be a dominating set. If

$$\zeta_1,\zeta_2,\ldots\zeta_n,\tau_1,\tau_2,\ldots\tau_n,\upsilon_1,\upsilon_2,\ldots\upsilon_n,\vartheta_1,\vartheta_2,\ldots,\vartheta_n,\xi_1,\xi_2,\ldots,\xi_n,\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n$$

are the eigen values of adjacency matrix $A\left(\mu^{+}_{D^N}(G)\right),A\left(\gamma^{+}_{D^N}(G)\right),A\left(\sigma^{+}_{D^N}(G)\right),A\left(\mu^{-}_{D^N}(G)\right),\ A\left(\gamma^{-}_{D^N}(G)\right),A\left(\sigma^{-}_{D^N}(G)\right)$ respectively, then

1. $\sum_{p=1}^{n}\zeta_p=\left|D^N\right|,\sum_{p=1}^{n}\tau_p=\left|D^N\right|,\sum_{p=1}^{n}\upsilon_p=\left|D^N\right|,\sum_{p=1}^{n}\vartheta_p=-\left|D^N\right|,\sum_{p=1}^{n}\xi_p=-\left|D^N\right|,\sum_{p=1}^{n}\varepsilon_p=-\left|D^N\right|$

2. $$\sum_{p=1}^{n}\left(\zeta_p\right)^2=\sum_{p=1}^{n}\left(\mu^{+}_{pp}\right)^2+2\sum_{1\le p<q\le n}\mu^{+}_{pq}\mu^{+}_{qn},\sum_{p=1}^{n}\left(\tau_p\right)^2=\sum_{p=1}^{n}\left(\gamma^{+}_{pp}\right)^2+2\sum_{1\le p<q\le n}\gamma^{+}_{pq}\gamma^{+}_{qn},$$
$$\sum_{p=1}^{n}\left(\upsilon_p\right)^2=\sum_{p=1}^{n}\left(\sigma^{+}_{pp}\right)^2+2\sum_{1\le p<q\le n}\sigma^{+}_{pq}\sigma^{+}_{qn},\sum_{p=1}^{n}\left(\vartheta_p\right)^2=\sum_{p=1}^{n}\left(\mu^{-}_{pp}\right)^2+2\sum_{1\le p<q\le n}\mu^{-}_{pq}\mu^{-}_{qn},$$
$$\sum_{p=1}^{n}\left(\xi_p\right)^2=\sum_{p=1}^{n}\left(\gamma^{-}_{pp}\right)^2+2\sum_{1\le p<q\le n}\gamma^{-}_{pq}\gamma^{-}_{qn},\sum_{p=1}^{n}\left(\varepsilon_p\right)^2=\sum_{p=1}^{n}\left(\sigma^{-}_{pp}\right)^2+2\sum_{1\le p<q\le n}\sigma^{-}_{pq}\sigma^{-}_{qn}.$$

Proof.

1. By the trace property of matrices, where the sum of eigen values is equal to its trace, we have
$$\sum_{p=1}^{n}\zeta_p=\sum\mu^{+}_{pp}=\left|D^N\right|.$$

Analogously, we can show that
$$\sum_{p=1}^{n}\tau_p=\left|D^N\right|,\sum_{p=1}^{n}\upsilon_p=\left|D^N\right|,\sum_{p=1}^{n}\vartheta_p=-\left|D^N\right|,\sum_{p=1}^{n}\xi_p=-\left|D^N\right|,\sum_{p=1}^{n}\varepsilon_p=-\left|D^N\right|.$$

2. Equivalently, the sum of square of eigenvalues of $\left(\mu^{+}_{D^N}(G)\right)$ is equal to the trace of $\left(\mu^{+}_{D^N}(G)\right)^2$
$$\begin{aligned}tr\left(A\left(\mu^{+}_{D^N}(G)\right)^2\right)&=\sum_{p=1}^{n}\left(\zeta_p\right)^2\\&=\mu^{+}_{11}\mu^{+}_{11}+\mu^{+}_{12}\mu^{+}_{21}+\ldots+\mu^{+}_{1n}\mu^{+}_{n1}+\mu^{+}_{21}\mu^{+}_{12}+\mu^{+}_{22}\mu^{+}_{22}+\ldots+\mu^{+}_{2n}\mu^{+}_{n2}\\&\quad+\ldots+\mu^{+}_{n1}\mu^{+}_{1n}+\mu^{+}_{n2}\mu^{+}_{2n}+\ldots+\mu^{+}_{nn}\mu^{+}_{nn}\\&=\sum_{p=1}^{n}\left(\mu^{+}_{pp}\right)^2+2\sum_{1\le p<q\le n}\mu^{+}_{pq}\mu^{+}_{qn}\end{aligned}$$

Analogously, we can show that
$$\sum_{p=1}^{n}\left(\tau_p\right)^2=\sum_{p=1}^{n}\left(\gamma^{+}_{pp}\right)^2+2\sum_{1\le p<q\le n}\gamma^{+}_{pq}\gamma^{+}_{qn},\sum_{p=1}^{n}\left(\upsilon_p\right)^2=\sum_{p=1}^{n}\left(\sigma^{+}_{pp}\right)^2+2\sum_{1\le p<q\le n}\sigma^{+}_{pq}\sigma^{+}_{qn},$$
$$\sum_{p=1}^{n}\left(\vartheta_p\right)^2=\sum_{p=1}^{n}\left(\mu^{-}_{pp}\right)^2+2\sum_{1\le p<q\le n}\mu^{-}_{pq}\mu^{-}_{qn},\sum_{p=1}^{n}\left(\xi_p\right)^2=\sum_{p=1}^{n}\left(\gamma^{-}_{pp}\right)^2+2\sum_{1\le p<q\le n}\gamma^{-}_{pq}\gamma^{-}_{qn},$$
$$\sum_{p=1}^{n}\left(\varepsilon_p\right)^2=\sum_{p=1}^{n}\left(\sigma^{-}_{pp}\right)^2+2\sum_{1\le p<q\le n}\sigma^{-}_{pq}\sigma^{-}_{qn}.$$

This completes the proof. ■

We now give upper and lower bounds of dominating energy of a BSVNG in terms of the number of vertices and the sum of squares of positive truth-membership, positive indeterminacy-membership, positive falsity-membership, negative truth-membership values, negative indeterminacy-membership values, and negative falsity-membership values of the edges.

Theorem 5.8 Let $G=\left(V,E,\mu^{+},\gamma^{+},\sigma^{+},\mu^{-},\gamma^{-},\sigma^{-},\mu_1^{+},\gamma_1^{+},\sigma_1^{+},\mu_1^{-},\gamma_1^{-},\sigma_1^{-}\right)$ be a dominating BSVNG with n vertices. If $D^N=\{z_1,z_2,\ldots,z_k\}$ is the dominating set, then

1. $\sqrt{\sum_{p=1}^{n}\left(\mu_{pp}^{+}\right)^2+2\sum_{1\le p<q\le n}\mu_{pq}^{+}\mu_{qn}^{+}+n(n-1)|A|^{\frac{2}{n}}}\le E\left(\mu_{D^N}^{+}(G)\right)\le\sqrt{n\left(\sum_{p=1}^{n}\left(\mu_{pp}^{+}\right)^2+2\sum_{1\le p<q\le n}\mu_{pq}^{+}\mu_{qn}^{+}\right)}$

where $|A|$ is the determinant of $\mu_{D^N}^{+}(G)$.

2. $\sqrt{\sum_{p=1}^{n}\left(\gamma_{pp}^{+}\right)^2+2\sum_{1\le p<q\le n}\gamma_{pq}^{+}\gamma_{qn}^{+}+n(n-1)|B|^{\frac{2}{n}}}\le E\left(\gamma_{D^N}^{+}(G)\right)\le\sqrt{n\left(\sum_{p=1}^{n}\left(\gamma_{pp}^{+}\right)^2+2\sum_{1\le p<q\le n}\gamma_{pq}^{+}\gamma_{qn}^{+}\right)}$

where $|B|$ is the determinant of $\gamma_{D^N}^{+}(G)$.

3. $\sqrt{\sum_{p=1}^{n}\left(\sigma_{pp}^{+}\right)^2+2\sum_{1\le p<q\le n}\sigma_{pq}^{+}\sigma_{qn}^{+}+n(n-1)|C|^{\frac{2}{n}}}\le E\left(\sigma_{D^N}^{+}(G)\right)\le\sqrt{n\left(\sum_{p=1}^{n}\left(\sigma_{pp}^{+}\right)^2+2\sum_{1\le p<q\le n}\sigma_{pq}^{+}\sigma_{qn}^{+}\right)}$

where $|C|$ is the determinant of $\sigma_{D^N}^{+}(G)$.

4. $\sqrt{\sum_{p=1}^{n}\left(\mu_{pp}^{-}\right)^2+2\sum_{1\le p<q\le n}\mu_{pq}^{-}\mu_{qn}^{-}+n(n-1)|D|^{\frac{2}{n}}}\le E\left(\mu_{D^N}^{-}(G)\right)\le\sqrt{n\left(\sum_{p=1}^{n}\left(\mu_{pp}^{-}\right)^2+2\sum_{1\le p<q\le n}\mu_{pq}^{-}\mu_{qn}^{-}\right)}$

where $|D|$ is the determinant of $\mu_{D^N}^{-}(G)$.

5. $\sqrt{\sum_{p=1}^{n}\left(\gamma_{pp}^{-}\right)^2+2\sum_{1\le p<q\le n}\gamma_{pq}^{-}\gamma_{qn}^{-}+n(n-1)|F|^{\frac{2}{n}}}\le E\left(\gamma_{D^N}^{-}(G)\right)\le\sqrt{n\left(\sum_{p=1}^{n}\left(\gamma_{pp}^{-}\right)^2+2\sum_{1\le p<q\le n}\gamma_{pq}^{-}\gamma_{qn}^{-}\right)}$

where $|F|$ is the determinant of $\gamma_{D^N}^{-}(G)$.

6. $\sqrt{\sum_{p=1}^{n}\left(\sigma_{pp}^{-}\right)^2+2\sum_{1\le p<q\le n}\sigma_{pq}^{-}\sigma_{qn}^{-}+n(n-1)|H|^{\frac{2}{n}}}\le E\left(\sigma_{D^N}^{-}(G)\right)\le\sqrt{n\left(\sum_{p=1}^{n}\left(\sigma_{pp}^{-}\right)^2+2\sum_{1\le p<q\le n}\sigma_{pq}^{-}\sigma_{qn}^{-}\right)}$

where $|H|$ is the determinant of $\sigma_{D^N}^{-}(G)$.

Proof.

By Cauchy Schwarz inequality, $\left(\sum_{p=1}^{n}a_p b_p\right)^2\le\left(\sum_{p=1}^{n}a_p{}^2\right)\left(\sum_{p=1}^{n}b_p{}^2\right)$. Therefore,

Upper bound

If $a_p=1$ and $b_p=|\zeta_p|$, then $\left(\sum_{p=1}^{n}|\zeta_p|\right)^2\le\left(\sum_{p=1}^{n}1\right)\left(\sum_{p=1}^{n}\zeta_p{}^2\right)$. Thus,

$$\left(E\left(\mu_{D^N}^{+}(G)\right)\right)^2\le n\left(\sum_{p=1}^{n}\left(\mu_{pp}^{+}\right)^2+2\sum_{1\le p<q\le n}\mu_{pq}^{+}\mu_{qn}^{+}\right)$$

$$\left(E\left(\mu_{D^N}^{+}(G)\right)\right)\le\sqrt{n\left(\sum_{p=1}^{n}\left(\mu_{pp}^{+}\right)^2+2\sum_{1\le p<q\le n}\mu_{pq}^{+}\mu_{qn}^{+}\right)}.$$

Lower bound

$$\left(E\left(\mu^{+}_{D^N}(G)\right)\right)^2 = \left(\sum_{p=1}^{n}\left|\zeta_p\right|\right)^2$$
$$= \left(\sum_{p=1}^{n}\left|\mu^{+}_{pp}\right|^2 + 2\sum_{1\le p<q\le n}\left|\mu^{+}_{pq}\right|\left|\mu^{+}_{qn}\right|\right)$$
$$= \left(\sum_{p=1}^{n}\left(\mu^{+}_{pp}\right)^2 + 2\sum_{1\le p<q\le n}\mu^{+}_{pq}\mu^{+}_{qn}\right) + \frac{2n(n-1)}{2}\underset{1\le p<q\le n}{AM}\left\{\left|\zeta_p\right|\left|\zeta_q\right|\right\}.$$

Since $\underset{1\le p<q\le n}{AM}\left\{\left|\zeta_p\right|\left|\zeta_q\right|\right\} \ge \underset{1\le p<q\le n}{GM}\left\{\left|\zeta_p\right|\left|\zeta_q\right|\right\}$, hence

$$\left(E\left(\mu^{+}_{D^N}(G)\right)\right) \ge \sqrt{\sum_{p=1}^{n}\left(\mu^{+}_{pp}\right)^2 + 2\sum_{1\le p<q\le n}\mu^{+}_{pq}\mu^{+}_{qn} + n(n-1)\underset{1\le p<q\le n}{GM}\left\{\left|\zeta_p\right|\left|\zeta_q\right|\right\}}$$

where

$$\underset{1\le p<q\le n}{GM}\left\{\left|\zeta_p\right|\left|\zeta_q\right|\right\} = \left(\prod_{1\le p<q\le n}\left|\zeta_p\right|\left|\zeta_q\right|\right)^{\frac{2}{n(n-1)}}$$
$$= \left(\prod_{p=1}\left|\zeta_p\right|^{n-1}\right)^{\frac{2}{n(n-1)}}$$
$$= \left(\prod_{p=1}\left|\zeta_p\right|\right)^{\frac{2}{n}}$$
$$= \left|A\right|^{\frac{2}{n}}.$$

Therefore,

$$\left(E\left(\mu^{+}_{D^N}(G)\right)\right) \ge \sqrt{\sum_{p=1}^{n}\left(\mu^{+}_{pp}\right)^2 + 2\sum_{1\le p<q\le n}\mu^{+}_{pq}\mu^{+}_{qn} + n(n-1)\left|A\right|^{\frac{2}{n}}}.$$

Combining these bounds, we have

1. $\sqrt{\sum_{p=1}^{n}\left(\mu^{+}_{pp}\right)^2 + 2\sum_{1\le p<q\le n}\mu^{+}_{pq}\mu^{+}_{qn} + n(n-1)\left|A\right|^{\frac{2}{n}}} \le E\left(\mu^{+}_{D^N}(G)\right) \le \sqrt{n\left(\sum_{p=1}^{n}\left(\mu^{+}_{pp}\right)^2 + 2\sum_{1\le p<q\le n}\mu^{+}_{pq}\mu^{+}_{qn}\right)}.$

Analogously, we can show that

2. $\sqrt{\sum_{p=1}^{n}\left(\gamma^{+}_{pp}\right)^2 + 2\sum_{1\le p<q\le n}\gamma^{+}_{pq}\gamma^{+}_{qn} + n(n-1)\left|B\right|^{\frac{2}{n}}} \le E\left(\gamma^{+}_{D^N}(G)\right) \le \sqrt{n\left(\sum_{p=1}^{n}\left(\gamma^{+}_{pp}\right)^2 + 2\sum_{1\le p<q\le n}\gamma^{+}_{pq}\gamma^{+}_{qn}\right)},$

3. $\sqrt{\sum_{p=1}^{n}\left(\sigma^{+}_{pp}\right)^2 + 2\sum_{1\le p<q\le n}\sigma^{+}_{pq}\sigma^{+}_{qn} + n(n-1)\left|C\right|^{\frac{2}{n}}} \le E\left(\sigma^{+}_{D^N}(G)\right) \le \sqrt{n\left(\sum_{p=1}^{n}\left(\sigma^{+}_{pp}\right)^2 + 2\sum_{1\le p<q\le n}\sigma^{+}_{pq}\sigma^{+}_{qn}\right)},$

4. $\sqrt{\sum_{p=1}^{n}\left(\mu^{-}_{pp}\right)^2 + 2\sum_{1\le p<q\le n}\mu^{-}_{pq}\mu^{-}_{qn} + n(n-1)\left|D\right|^{\frac{2}{n}}} \le E\left(\mu^{-}_{D^N}(G)\right) \le \sqrt{n\left(\sum_{p=1}^{n}\left(\mu^{-}_{pp}\right)^2 + 2\sum_{1\le p<q\le n}\mu^{-}_{pq}\mu^{-}_{qn}\right)},$

5. $\sqrt{\sum_{p=1}^{n}\left(\gamma^{-}_{pp}\right)^2 + 2\sum_{1\le p<q\le n}\gamma^{-}_{pq}\gamma^{-}_{qn} + n(n-1)\left|F\right|^{\frac{2}{n}}} \le E\left(\gamma^{-}_{D^N}(G)\right) \le \sqrt{n\left(\sum_{p=1}^{n}\left(\gamma^{-}_{pp}\right)^2 + 2\sum_{1\le p<q\le n}\gamma^{-}_{pq}\gamma^{-}_{qn}\right)},$

6. $\sqrt{\sum_{p=1}^{n}\left(\sigma^{-}_{pp}\right)^2 + 2\sum_{1\le p<q\le n}\sigma^{-}_{pq}\sigma^{-}_{qn} + n(n-1)\left|H\right|^{\frac{2}{n}}} \le E\left(\sigma^{-}_{D^N}(G)\right) \le \sqrt{n\left(\sum_{p=1}^{n}\left(\sigma^{-}_{pp}\right)^2 + 2\sum_{1\le p<q\le n}\sigma^{-}_{pq}\sigma^{-}_{qn}\right)}.$

This completes the proof. ∎

Theorem 5.9 Let $G=\left(V,E,\mu^{+},\gamma^{+},\sigma^{+},\mu^{-},\gamma^{-},\sigma^{-}\right)$ be a BSVNG and

$$A(G)=\left(\mu^{+}(G),\gamma^{+}(G),\sigma^{+}(G),\mu^{-}(G),\gamma^{-}(G),\sigma^{-}(G)\right)$$

be the adjacency matrix of G. Let $G_1=\left(V,E,\mu^{+},\gamma^{+},\sigma^{+},\mu^{-},\gamma^{-},\sigma^{-},\mu_1^{+},\gamma_1^{+},\sigma_1^{+},\mu_1^{-},\gamma_1^{-},\sigma_1^{-}\right)$ be a dominating BSVNG of G and $A_{D^N}(G)=\left(\mu_{D^N}^{+}(G),\gamma_{D^N}^{+}(G),\sigma_{D^N}^{+}(G),\mu_{D^N}^{-}(G),\gamma_{D^N}^{-}(G),\sigma_{D^N}^{-}(G)\right)$ be the adjacency matrix of a dominating BSVNG G_1. Then

1. $E\left(\mu_{D^N}^{+}(G)\right)^2\le n\left(\sum_{p=1}^{n}\left(\mu_{pp}^{+}\right)^2+E\left(\mu^{+}(G)\right)^2\right)$,
2. $E\left(\gamma_{D^N}^{+}(G)\right)^2\le n\left(\sum_{p=1}^{n}\left(\gamma_{pp}^{+}\right)^2+E\left(\gamma^{+}(G)\right)^2\right)$,
3. $E\left(\sigma_{D^N}^{+}(G)\right)^2\le n\left(\sum_{p=1}^{n}\left(\sigma_{pp}^{+}\right)^2+E\left(\sigma^{+}(G)\right)^2\right)$,
4. $E\left(\mu_{D^N}^{-}(G)\right)^2\le n\left(\sum_{p=1}^{n}\left(\mu_{pp}^{-}\right)^2+E\left(\mu^{-}(G)\right)^2\right)$,
5. $E\left(\gamma_{D^N}^{-}(G)\right)^2\le n\left(\sum_{p=1}^{n}\left(\gamma_{pp}^{-}\right)^2+E\left(\gamma^{-}(G)\right)^2\right)$,
6. $E\left(\sigma_{D^N}^{-}(G)\right)^2\le n\left(\sum_{p=1}^{n}\left(\sigma_{pp}^{-}\right)^2+E\left(\sigma^{-}(G)\right)^2\right)$.

Proof.

$$E\left(\mu^{+}(G)\right)^2\ge 2\sum_{1\le p<q\le n}\mu_{pq}^{+}\mu_{qn}^{+}+n(n-1)|A|^{\frac{2}{n}}\ge 2\sum_{1\le p<q\le n}\mu_{pq}^{+}\mu_{qn}^{+}$$

$$\text{(i.e.) } 2\sum_{1\le p<q\le n}\mu_{pq}^{+}\mu_{qn}^{+}\le E\left(\mu^{+}(G)\right)^2 \tag{1}$$

Now,

$$E\left(\mu_{D^N}^{+}(G)\right)^2\le n\sum_{p=1}^{n}\left(\mu_{pp}^{+}\right)^2+2n\sum_{1\le p<q\le n}\mu_{pq}^{+}\mu_{qn}^{+}$$

$$\begin{aligned}E\left(\mu_{D^N}^{+}(G)\right)^2&\le n\left(\sum_{p=1}^{n}\left(\mu_{pp}^{+}\right)^2+2\sum_{1\le p<q\le n}\mu_{pq}^{+}\mu_{qn}^{+}\right)\\&\le n\left(\sum_{p=1}^{n}\left(\mu_{pp}^{+}\right)^2+E\left(\mu^{+}(G)\right)^2\right)\qquad(\text{by Eq (1)})\end{aligned}$$

Analogously, we can show that

2. $E\left(\gamma_{D^N}^{+}(G)\right)^2\le n\left(\sum_{p=1}^{n}\left(\gamma_{pp}^{+}\right)^2+E\left(\gamma^{+}(G)\right)^2\right)$,
3. $E\left(\sigma_{D^N}^{+}(G)\right)^2\le n\left(\sum_{p=1}^{n}\left(\sigma_{pp}^{+}\right)^2+E\left(\sigma^{+}(G)\right)^2\right)$,
4. $E\left(\mu_{D^N}^{-}(G)\right)^2\le n\left(\sum_{p=1}^{n}\left(\mu_{pp}^{-}\right)^2+E\left(\mu^{-}(G)\right)^2\right)$,
5. $E\left(\gamma_{D^N}^{-}(G)\right)^2\le n\left(\sum_{p=1}^{n}\left(\gamma_{pp}^{-}\right)^2+E\left(\gamma^{-}(G)\right)^2\right)$,
6. $E\left(\sigma_{D^N}^{-}(G)\right)^2\le n\left(\sum_{p=1}^{n}\left(\sigma_{pp}^{-}\right)^2+E\left(\sigma^{-}(G)\right)^2\right)$.

This completes the proof. ■

6. Conclusion

As part of this study, we explored a few graph-theoretic concepts and integrated a hypothesis of dominating energy with the idea of BSVNG. Specifically, in this study, we developed a new concept of adjacency matrix, as well as the spectrum of the adjacency matrix of the dominating in BSVNG. Hence, we computed the energy of dominating BSVNG. Apart from that, various operations regarding this domination have been illustrated. With appropriate instances, the complement, union and join of dominating energy in BSVNG have been examined. Finally, certain theorems regarding the dominating energy in BSVNG are established. In view of Akram et al. [49], the terms and notions discussed in this study can be extended in the framework of double domination energy in BSVNG.

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Neutrosophical Rejection and Acceptance Method for the Generation of Random Variables

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Abstract: Simulation has become a modern-day tool that helps us study many systems that its results could not have been studied or predicted through the work of these systems over time. The simulation process depends on generating a series of random numbers that are subject to a uniform probability distribution on the field [0,1], and then converting these random numbers to random variables that follow the probability distribution in which the system to be simulated, there are several methods that can be used to carry out the conversion.

In previous research, the inverse transformation method has been studied according to the Neutrosophic logic and we have reached Neutrosophic random variables, by using them we get a more accurate simulation of any system we want to simulate. It should be noted that the inverse transformation can be used if the cumulative distribution function has an inverse function, but if the system we want to simulate works according to a probability distribution, and the inverse function of the cumulative distribution function cannot be found, then we use other methods.

In this research, we are examining a study that enables us to generate Neutrosophical random variables that follow probability distributions based on Neutrosophical random numbers that follow for the uniform distribution, using the rejection and acceptance method, which depends on the largest value taken by the probability density function of the distribution in which the system to be simulated operates on its definition.

Keywords: Neutrosophic, Neutrosophical random variables, Neutrosophic logic

Introduction:

In the light of the great development in today's world, the complexity of systems and the significant material and non-material losses that can result from the operation of any system without prior study, there has to be a scientific method that enables us to know the results that we can get when operating systems and helps us to minimize these losses. The simulation process was the modern instrument through which we can predict what results we can get through the functioning of these systems over time. Since the simulation process depends on generating random numbers that follow regular distribution on the field [0, 1] and then converting these numbers into random variables that follow the probable distribution of the system to be simulated In classical logic, many methods were presented by which we were able to obtain the random variables needed for the simulation process, To keep pace with scientific development, the most important of these methods had to be reformulated using the revolutionary logic of the Neutrosophic logic established by the American philosopher and athlete Florentine Smarandache in 1995. So in previous research we prepared a study to generate random Neutrosophic numbers on the field [a, b) Based on the Neutrosophic researchers' findings in the definition of regular distribution and definition of integration according to Neutrosophic logic, and in other research, we converted these Neutrosophic random numbers into Neutrosophic random variables tracking the exponential distribution using the opposite conversion method [1-14].

In this research we will generate Neutrosophic random variables based on previous studies and use the method of rejection and acceptance to convert Neutrosophic random numbers into random variables that follow the probable distribution of the system to be emulated.

Discussion:

During the simulation process, we encounter many systems that operate on probability distributions. The reverse function of the cumulative distribution function cannot be found. Therefore, we cannot use the opposite conversion method that we have formulated according to the Neutrosophic logic. So in this research we will present a study to convert the Neutrosophic random numbers that follow the Neutrosophic regular probability distribution into Neutrosophic random variables.

Method of rejection and acceptance according to classical logic: [15]

We take the distribution of a probability $f(x)$ defined on the field $[a,b]$ with the following relationship:

$$f(x)=\begin{cases} f(x) & ; a \le x \le b \\ 0 & otherwise \end{cases}$$

We assume that the greatest value $f(x)$ takes on your area of definition is M the same as Mode, then the following inequality is achieved:

$$0 \leq f(x) \leq M \qquad ; \quad a \leq x \leq b$$

Thus:

$$0 \leq f(x) \leq M \Rightarrow 0 \leq \frac{f(x)}{M} \leq 1 \quad ; a \leq x \leq b$$

This means that the composition $\frac{f(x)}{M}$ is valid for comparison with random numbers that follow a uniform distribution on the field [0, 1], because it achieves the following relationship:

$$P\left(R < \frac{f(x)}{M}\right) = \frac{f(x)}{M}$$

We benefit from the above by applying the following algorithm:

1- We generate two random numbers R_1 , R_2 that follow the uniform distribution on the field [0, 1].

We use the mean squared method to generate the two random numbers defined as follows:

To generate random numbers, we apply the following relationship:

$$R_{i+1} = Mid\left[R_i^2\right] \quad ; i = 0,1,2,3,---- \quad (1)$$

Where *Mid* stands for the middle four places of R_i^2 and any fractional $R_\circ$ random number consisting of four places (called the seed) and not containing zero in any of its four places is chosen, to generate two random numbers R_1 , R_2 from the field $\left[0\ ,1\right]$.

2- We take one of the two numbers, let it be R_1 and from it a variable suitable for the uniform distribution on the field [a, b], by performing the following conversion:

$$x_1 = (b-a)R_1 + a$$

3- we check whether if R_2 achieves the inequality:

$$R_2 \leq \frac{f(x)}{M} \qquad (*)$$

4- if the inequality (*) is achieved We accept that x_1 is subject to the operated distribution in the system which is being simulated.

If the inequality $(*)$ is not achieved, we reject the two numbers R_1 , R_2 and return to the first step, to generate two random numbers again.

Thus, we continue to work and compare as much as we want or as required by the simulation process.

Practical example:

We want to simulate a system that operates according to the probability distribution whose probability density function is given by the following relationship:

$$f(x)=\begin{cases}\dfrac{1}{16}x & ;\ 2\le x\le 6\\ 0 & ;\ 0therwise\end{cases}$$

We calculate M which is the largest value of $f(x)$ and is equal to:

$$M=\frac{1}{16}\times 6=\frac{3}{8}$$

To perform the simulation, it is necessary to generate random variables that follow this distribution. We will use the rejection and acceptance method to achieve the desired:

1- We use the mean squared method given by the following relationship:

$$R_{i+1}=Mid\left[R_i^2\right]\quad ;i=0,1,2,3,----\quad (1)$$

Where Mid stands for the middle four places of R_i^2 and any fractional $R_\circ$ random number consisting of four places (called the seed) and not containing zero in any of its four places is chosen, to generate two random numbers R_1 , R_2 from the field $[0\ ,1]$ we take the seed $R_\circ=0,3176$ and from it we get

$$R_2=0.7551\,,\,R_1=0.0869$$

2- We take one of the two numbers, let it be $R_1=0,0869$ and form it a variable suitable for uniform distribution over the field $[a\ ,b]=[2\ ,6]$ by performing the following transformation:

$$x_1 = (b-a)R_1 + a \quad \Rightarrow$$
$$x_1 = (6-2)\times 0.0869 + 2 = 2.3476$$

We calculate $f(x_1)$:

$$f(x_1) = \frac{1}{16}\times 2.3476 = 0.1467$$

3- We test inequality $(*)$:

$$R_2 \le \frac{f(x_1)}{M} \quad \Rightarrow$$
$$0.7551 \le \frac{0.1467}{0.375} = 0.3912$$

4- We note that the inequality is not achieved and therefore the random variable x_1 does not follow the probability distribution $f(x)$ and therefore we reject the two random numbers R_1 , R_2 and return to (1).

We start again:

1- We generate two random numbers by taking the seed $R_\circ = 0{,}1234$ we get:

$$R_2 = 0.3215 \, , \, R_1 = 0.5227$$

2- We take one of the numbers, let it be $R_1 = 0{,}5227$ and form from it a variable suitable for the uniform distribution on the field $[a \, , b] = [2 \, , 6]$ by doing the following conversion:

$$x_1 = (b-a)R_1 + a \quad \Rightarrow$$
$$x_1 = (6-2)\times 0.5227 + 2 = 4.0908$$

We calculate $f(x_1)$:

$$f(x_1) = \frac{1}{16}\times 4.0908 = 0.2557$$

3- We check the inequality (*):

$$R_2 \le \frac{f(x_1)}{M} \quad \Rightarrow$$
$$0.3215 \le \frac{0.2557}{0.375} = 0.6819$$

4- We see that the inequality is achieved, therefor the random variable x_1 follows the probability distribution $f(x)$, we continue to work and compare as much as we want or as required by the simulation process.

In this research, we formulated the rejection and acceptance method according to the Neutrosophic logic, so the previous algorithm became as follows

1- We generate two random numbers R_1 , R_2 that follow the uniform distribution on the field [0,1].

2- We convert the two numbers R_1 , R_2 into two Neutrosophic random numbers on the field $[0+\delta , 1+\delta]$

According to the following formula:

$$NP(R < R_i) = NF(R_i) = R_i - \delta \quad ; \quad \delta \in [0 , m]$$

Where $\delta \in [0 , m]$ is the indefinite, and $0 < m < 1$ let's denote them by NR_2 , NR_1

We find the cumulative distribution function for the uniform distribution, which is given by the following relationship:

When the indeterminacy is in the upper and lower bounds of the field that means:

$[a_N , b_N] = [a+\varepsilon , b+\varepsilon]$ where $\varepsilon \in [0 , n]$ and $a < n < b$ then:

$$NF(x) = \frac{x-a}{b_N - a_N} - i \quad ; \quad i \in \left[0 , \frac{n}{b_N - a_N}\right]$$

In (1) and (2) we used a study that is mentioned in reference (16).

Then we use the inverse transformation method to find the random variables that follow the uniform distribution on the field $[a_N , b_N] = [a+\varepsilon , b+\varepsilon]$

We substitute the following relationship:

$$R = F(x) \Rightarrow x = F^{-1}(R)$$

We find:

$$NF(x) = NR \Rightarrow NR = \frac{x-a}{b_N - a_N} - i \Rightarrow$$
$$\frac{x-a}{b_N - a_N} = NR + i \Rightarrow (NR+i)(b_N - a_N) = x - a \Rightarrow$$
$$Nx = (NR+i)(b_N - a_N) + a$$

$$i \in \left[0,\ \frac{n}{b_N - a_N}\right]$$

Accordingly, to generate random variables that follow the Neutrosophic uniform distribution in the case of indeterminacy related to the lower and upper bounds

We use the following relationship:

$$Nx_j = (NR_j + i)(b_N - a_N) + a \quad ; \quad j = 0,1,2,3,4$$

Where $i \in \left[0,\ \frac{n}{b_N - a_N}\right]$ and ε is the indefinite where ε ∈ [0, n] and $a < n < b$

We note that the variables have become neutrosophical values, and therefore the probability density function $f(x)$ becomes a neutrosophical function that we denote by the symbol $f_N(x)$ and it is defined on the field $[a_N, b_N] = [a+\varepsilon\ , b+\varepsilon]$.

We suppose that the largest value that this function takes in its domain is M_N, then the following inequality is achieved:

$$0_N \le f_N(x) \le M_N \ ; \quad a+\varepsilon \le x \le b+\varepsilon$$

Then the following inequality is achieved:

This means that the composition $\frac{f_N(x)}{M_N}$ is valid for comparison with the Neutrosophical random numbers that follow the Neutrosophic uniform distribution on the field $[0+\delta\ , 1+\delta]$ because it achieves the following relationship:

$$NP\left(NR < \frac{f_N(x)}{M_N}\right) = \frac{f_N(x)}{M_N}$$

3- We test whether NR_2 achieves inequality:

$$NR_2 \leq \frac{f_N(Nx_1)}{M_N} \qquad (**)$$

If the inequality (**) is true, then we accept that Nx_1 is subject to the distribution $f_N(x)$ in which the system to be simulated operates.

4- If NR_2 does not achieve the inequality $(**)$, then we reject the two numbers R_1, R_2 and return to the first step of generating the two random numbers again.

Thus, we continue to work and compare as much as we want or as required by the simulation process.

Note 1: In step (2) we took the indeterminacy on the two terms of the field and used the appropriate relations for that in the process of converting random numbers into Neutrosophical random numbers. Reference [16].

Note 2: When generating the Neutrosophic random variable that follows the uniform distribution on the field, we also took the indeterminacy on the two terms of the field and then applied the inverse transformation method to find the random variable. It should be noted that we follow the same method if the indeterminacy is related to one of the field terms.

Practical example:

We want to simulate a system that operates according to the probability distribution whose probability density function is given by the following relationship:

$$f(x) = \begin{cases} \frac{1}{16}x & ;\ 2 \leq x \leq 6 \\ 0 & ;\ 0therwise \end{cases}$$

1- We use the mean squared method given by the following relationship:

$$R_{i+1} = Mid\left[R_i^2\right] \quad ; i = 0,1,2,3,---- \quad (1)$$

We take the seed $R_{\circ} = 0,3176$ and from it we form $R_2 = 0.7551\,,\, R_1 = 0.0869$

2- We transform the two numbers R_1 , R_2 into two Neutrosophic random numbers on the field $[0+\delta\,,1+\delta]$ using the following relationship:

$$NR_i = R_i - \delta$$

And in this example we take $\delta = [0\,,\,0.03]$

$$NR_i = R_i - \delta \quad \& \;\; i = 1 \;\Rightarrow$$
$$NR_1 = R_1 - \delta = 0.0869 - [0\,,\,0.03] \Rightarrow$$
$$NR_1 = [0.0569\,,\;\;0.0869]$$

$$NR_i = R_i - \delta \quad \& \quad i = 2 \;\Rightarrow$$
$$NR_2 = R_2 - \delta = 0.7551 - [0\,,\,0.03] \Rightarrow$$
$$NR_2 = [0.7251\,,\;\;0.7551]$$

We take one of the two numbers, let it be $NR_1 = [0.0569\,,\,0.0869]$ and from it we form a variable suitable for the Neutrosophic uniform distribution over the field

.Here we have taken $\varepsilon = [0\,,\,3]$ $[a_N\;,\,b_N] = [2+[0\,,\,3],\,6+[0\,,\,3]\,] = [\,[2\,,\,5],[6\,,\,9]\,]$

By performing the following conversion:

$$Nx_j = (NR_j + i)(b_N - a_N) + a \quad ; \quad j = 1,2,3,---$$

$$i \in \left[0,\; \frac{n}{b_N - a_N}\right]$$

First we calculate i :

$$i = \left[0,\; \frac{3}{b_N - a_N}\right] = \left[0\,,\, \frac{3}{[6\,,\,9]-[2\,,\,5]}\right] = \left[0\,,\frac{3}{4}\right] = [0\,,\,0.75]$$

Make up:

$$Nx_1 = (NR_1 + i)(b_N - a_N) + a \Rightarrow$$
$$Nx_1 = \left([0.0569\,,\,0.0869] + [0\,,\,0.75] \right)(4) + 2 \Rightarrow$$
$$Nx_1 = [2.2276\,,\,5.3476]$$

We calculate $f_N(Nx_1)$:

$$f_N(x) = f_N(Nx_1)$$

$$f_N(Nx_1) = \frac{1}{16}[2.2276\,,\,5.3476] = [0.1392\,,\,0.3342]$$

We calculate M_N which is the largest value of $f_N(x)$ and is equal to:

$$M_N = \frac{1}{16} \times [6\,,\,9] = [0.375\,,\,0.5625]$$

3- We test the inequality $(**)$:

$$NR_2 \le \frac{f_N(Nx_1)}{M_N} \quad \Rightarrow [0.7251\,,\,0.7551] \le \frac{[0.1392\,,\,0.3342]}{[0.375\,,\,0.5625]} = [0.3712\ ,\ 0.5941] \Rightarrow$$
$$[0.7251\,,\,0.7551] \le [0.3712\,,\ 0.5941]$$

4- We note that the inequality is not satisfied and therefore we reject the two random numbers

R_2, R_1 and return to step (1)

We start again

1- We use the mean squared method given by the following relationship:

$$R_{i+1} = Mid[R_i^2] \quad ; i = 0,1,2,3,---- \quad (1)$$

We take the seed $R_\circ = 0{,}1234$ and from it we form $R_2 = 0.3215\,,\, R_1 = 0.5227$

2- We transform the two numbers $R_1\,,\,R_2$ into two Neutrosophic random numbers on the field $[0+\delta\,,\,1+\delta]$ using the following relationship:

$$NR_i = R_i - \delta$$

We take it in this example $\delta = [0\,,\,0.03]$

$$NR_i = R_i - \delta \quad \& \quad i = 1 \Rightarrow$$
$$NR_1 = R_1 - \delta = 0.5227 - [0\,,\,0.03] \Rightarrow$$
$$NR_1 = [0.5227\,,\ 0.4927]$$

$$NR_i = R_i - \delta \quad \& \quad i = 2 \Rightarrow$$
$$NR_2 = R_2 - \delta = 0.3215 - [0\,,\,0.03] \Rightarrow$$
$$NR_2 = [0.3215\,,\ 0.2915]$$

We take one of the two numbers, let it be $NR_1 = [0.5227, 0.4927]$, and from it form a variable suitable for the Neutrosophic uniform distribution over the field

$[a_N\ , b_N] = [2 + [0\,,3], 6 + [0\,,3]\ ] = [\ [2\,,5], [6\,,9]\ ]$. Here we have taken $\varepsilon = [0\,,3]$

By performing the following conversion**:**

$$Nx_j = (NR_j + i)(b_N - a_N) + a \quad ; \quad j = 1,2,3,---$$

$$i \in \left[0,\ \frac{n}{b_N - a_N}\right]$$

First, we calculate i we find:

$$i = \left[0,\ \frac{3}{b_N - a_N}\right] = \left[0\,,\ \frac{3}{[6\,,9] - [2\,,5]}\right] = \left[0\,,\ \frac{3}{4}\right] = [0\,,\,0.75]$$

Make up:

$$Nx_1 = (NR_1 + i)(b_N - a_N) + a \Rightarrow$$
$$Nx_1 = (\ [0.5227\,,\,0.4927] + [0\,,\,0.75]\ )(4) + 2 \Rightarrow$$
$$Nx_1 = [4.0908\,,\,5.4927]$$

We calculate $f_N(Nx_1)$:

$$f_N(x) = f_N(Nx_1)$$
$$f_N(Nx_1) = \frac{1}{16}[4.0908\,,\,6.9708] = [0.2557\,,\,0.3106]$$

3- We calculate M_N which is the largest value of $f_N(x)$ and is equal to:

$$M_N = \frac{1}{16} \times [6\,,9] = [0.375\,,\,0.5625]$$

4- We test the inequality $(**)$:

$$NR_2 \le \frac{f_N(Nx_1)}{M_N} \Rightarrow [0.3215\,,\,0.2915] \le \frac{[0.2557\,,\,0.3106]}{[0.375\,,\,0.5625]} = [0.5522\,,\,0.6819] \Rightarrow$$
$$[\,0.2915\,,\,0.3215\,] \le [0.5522\,,\,0.6819]$$

We note that the inequality is achieved, and therefore we accept Nx_1 a Neutrosophical variable that follows the Neutrosophical probability distribution defined as follows:

$$f_N(x) = \begin{cases} \frac{1}{16}x & ;\ 2+\varepsilon \le x \le 6+\varepsilon \\ 0_N & ;\ 0therwise \end{cases}$$

Where $\varepsilon = [0\,,\,3]$.

We continue in the same way until we obtain the required number of Neutrosophic random variables needed for the simulation process.

Conclusion and Results:

From the previous study we note that we do not need the reverse function of the cumulative distribution function of the probable distribution of the system imposed when using the Neutrosophic rejection and acceptance method and therefore it is a suitable method to generate the Neutrosophic random variables needed for the simulation process when obtaining the inverse function of the cumulative distribution function is difficult or not possible and using Neutrosophic variables we get more accurate and appropriate simulation results for changes that can occur during the work of the system to be simulated.

We look forward in the near future to preparing studies in which we use the method of refusal of acceptance to generate random variables following popular and widely used potential distributions in applied fields such as beta distribution and other distributions.

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Identification of RHT (Reaching Height Transceiver) for effective communication using Neutrosophic Fuzzy Soft Sets in Communication Engineering Problems

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Abstract: The decision making theory is playing a vital role in various engineering problems recently. A contemporary strategy of object identification from a vague collection of multi observer data has been processed here. The strategy we used here involves neutrosophic fuzzy soft set in a parametric sense for managing to identify the best signal transceiver for the distribution. This may use to pick the better signal transceiver for effective communication and to avoid the loss in signal transmission. Reaching Height Transceiver proposed here for better result in communications techniques.

Keywords: Soft set, Neutrosophic fuzzy Soft set, Signal Transceiver, Reaching Height Transceiver

1. Introduction

Many intricate problems in engineering, medical sciences and many other fields involve uncertain data. All the issues cannot be worked out by using general mathematics. Here we need some applied mathematical techniques based on uncertainty to identify the optimum solution for these problems. Recently many theories have risen for dealing with such a problems based on uncertainty and vagueness. Molotov [1999] started off the new concept of soft set theory. This is used to discuss about uncertainty in different view. Smarandache [2005] initiated the concept of neutrosophic set (NS). Florentin Smarandache [2018] defined the extension of soft set to hypersoft set and discussed some of its properties. Maji PK and Biswas R [2001] introduced the concepts of fuzzy soft sets. Roy AR and Maji PK [2007] applied fuzzy soft sets in decision making problems.

M. Zulqarnain et al [2017, 2018, 2020 and 2021] discussed many decision making problems in various fields like medical, engineering etc., to find better solutions. In their continuous research work, They introduced TOPSIS method for decision making problems in numerous fields. From their discussion technique for order preference by similarity to ideal solution is used for Multi – criteria decision making problems and it provides expected results for the problem respectively.

Here we took a structure of Neutrosophic soft fuzzy set and its related properties. Also we tried to apply Neutrosophic Fuzzy soft set in identifying suitable transceiver for an effective communication among the multiple transceivers which involved multi parameters. To identify the best object using the property of Neutrosophic Fuzzy Soft Set (NFSS) is our proposed technique in this paper.

2. A Problem in Effective Communication – Solution by Neutrosophic soft fuzzy sets

Most of our real problems are vague, and we cannot identify the solution by using classical approach of mathematics. Especially some engineering problems with uncertainty conditions can be solved using fuzzy applied techniques. Here we tried to find out the solution for a communication problem using NFSS.

2.1 PRELIMINARIES - SOFT SET THEORY

In this section, we present the basic definitions and results of soft set theory in this section. Also we applied important properties of soft set theory which would be very useful for further development of this paper.

1. Definition

Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denote the power set of U and $G \subset E$.

A pair (F,G) is called a soft set over U, where F is a function given by $F : G \rightarrow P(U)$.

On the other hand, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in G$, $F(\varepsilon)$ may be considered as the set of ε -approximate elements of the soft set (F,G).

2. Definition

Let (F_1,G_1) and (F_2,G_2) be any two soft sets over a common universe U, if

(i) $G_1 \subset G_2$, and

(ii) $\forall\ \varepsilon \in G_1$, $F_1(\varepsilon)$ and $F_2(\varepsilon)$ are identical approximations.

then we say that (F_1,G_1) is a soft subset of (F_2,G_2). We write $(F_1,G_1) \subseteq (F_2,G_2)$. (F_2,G_2) is said to be a soft super set of (F_1,G_1).

3. Definition

If (F_1,G_1) and (F_2,G_2) be two soft sets then "(F_1,G_1) AND (F_2,G_2)" is defined and denoted by $(F_1,G_1) \wedge (F_2,G_2) = (H, G_1 \times G_2)$, where $H(\alpha, \beta) = F_1(\alpha) \cap F_2(\beta)$, $\forall(\alpha, \beta) \in G_1 \times G_2$.

4. Definition

If (F_1,G_1) and (F_2,G_2) be two soft sets then "(F_1,G_1) OR (F_2,G_2)" is defined and denoted by $(F_1,G_1) \vee (F_2,G_2) = (H, G_1 \times G_2)$, where $H(\alpha, \beta) = F_1(\alpha) \cup F_2(\beta)$, $\forall(\alpha, \beta) \in G_1 \times G_2$.

2.2 Preliminaries – Neutrosophic fuzzy sets

5. Definition

A Neutrosophic set A on the universe set X is defined and denoted as $A=\{<x,T(x),I(x),F(x)>:x\in X\}$ where T, I, F: $X\rightarrow[0,1]$ and $0\leq T(x)+I(x)+F(x)\leq 3$

3. Neutrosophic fuzzy soft sets in Communication Engineering Problem

Basic definitions of Neutrosophic fuzzy sets and some of its related properties are discussed in this segment.

Let $U = \{x_1, x_2, \ldots, x_n\}$ be a set of n objects, which may be signalized by a set of factors $\{A_1, A_2, \ldots, A_i\}$. The parameter space C may be written as $C \supseteq \{A_1 \cup A_2 \cup \cdots \cup A_i\}$. Let each parameter set A_i represent the i^{th} class of factors and the elements of Ai represents a specific property set called as fuzzy sets. Hence, we now define a fuzzy soft set (Fi,Ai) which specifies a set of items having the parameter set Ai.

6. Definition

Let $\mathbf{P}(U)$ denotes the set of all fuzzy sets of U. Let $Ai \subset E$. A pair (F_i,A_i) is called a fuzzy-soft-set over U, where Fi is a mapping given by $F_i : A_i \rightarrow \mathbf{P}(U)$.

7. Definition

Let ₦(U) denotes the set of all neutrosophic fuzzy sets of U. Let $Ai \subset E$. A pair (₦$_i$, G_i) is called a fuzzy-soft-set over U, where ₦$_i$ is a mapping given by ₦$_i$: $Gi \rightarrow$ ₦ (U).

In view of the above we may now define a NFSS (₦$_i$,G_i) which identifies a group of items having the parameter set Gi .

8. Definition

For two NFSSs (₦$_1$,G_1) and (₦$_2$,G_2) over a common universe U, if

(i) $G_1 \subset G_2$, and

(ii) $\forall\varepsilon \in G_1$, ₦$_1(\varepsilon)$ is a fuzzy subset of ₦$_2(\varepsilon)$.

Then (₦$_1$, G_1) is a fuzzy-soft-subset of (₦$_2$, G_2). We write (₦$_1$, G_1) $\subseteq$ (₦$_2$, G_2).

(₦$_2$, G_2) is said to be a fuzzy soft super set of (₦$_1$, G_1).

Problem:

The common problems that occur in communication system could find its solution in pure Mathematics using the very powerful concept based on the power sets. The problem generated in signal communicating system is rectified with the help of many modern techniques. In communication system, the failure of any of the transceiver in sending (or receiving) the signal to (or from) any one of the secondary receivers can be rectified by the powerful concept of Neutrosophic Fuzzy Soft Set applications. There are many parameters involved in finding good transceivers. For example

1. Output power.
2. Receiving sensitivity
3. Bias current

4. Extinction ratio
5. Saturated optical power
6. Working temperature and so on.

From these we have chosen some parameters, like SNR value of the transceiver, Transmitting capacity etc. So, for each parameter we have considered T, I, F values. T represents truth performance value of a particular parameter of the transceiver. T equal to output value divided by input value. Similarly, we have chosen F, the false value of the parameter (may be considered as performance of failure value of the same parameter), and I is considered as indeterminacy value.

In this research paper, we have discussed the method to identify the best signal transceiver among a group of transceivers with the help of Neutrosophic Fuzzy Soft Sets for better communication. Through this method we are able to locate the Reaching Height Transceiver to receive the signals in a better quality comparing to the other transceivers in the encircled area. RHT (Reaching Height Transceiver) is a better transceiver among the group of transceivers, which will be good in receiving signals from Main Transceiver and transmitting quality signals to the other nearing transceivers. For discussion we can consider many parameters, especially SNR (Signal to Noise Ratio) for communication process.

In general, SNR is the proportion between signal and noise powers. This proportion provides a significant and convenient indication of the grade to which the signal has been contaminated with additive noise.

The $(SNR)_C$ gives " *the ratio of the average power of the modulated signal s(t) to the average power of noise in the message bandwidth, both measured as received input filtered noise n(t)*" and could be calculated using,

$$(SNR)_c = \frac{C^2 A_C^2 P}{2WN_o} = \frac{Avearage\ power\ of\ the\ channel}{Average\ power\ of\ the\ noise\ n(t)},$$ where

C^2 - System dependent scaling factor (a constant);

A_C^2 - Carrier wave constant;

P - Average power of the original message signal m(t); &

WN_0 - The average noise power in the message bandwidth W in the receiver.

The $(SNR)_O$ gives "*the ratio of the average power of the demodulated message signal to the average power of noise, both measured at the receiver output*" and could be calculated using

$$(\text{SNR})_o = \frac{C^2 A_C^2 P/4}{WN_0/2} = \frac{C^2 A_C^2 P'}{2WN_0} = \frac{Average\ power\ of\ the\ component}{Average\ power\ of\ the\ noise\ n(t)},$$ where

C^2 - System dependent scaling factor (a constant);

A_C^2 - Carrier wave constant;

P' - Average power of the output message signal $m_o(t)$; &

WN_0 - The average noise power in the message bandwidth W in the transmitter.

Then we can obtain the figure of merit value through

$$\beta = \frac{SNR_O}{SNR_C} = \frac{\text{The output signal} - \text{to} - \text{noise ratio for a receiver using coherent detection}}{\text{The channel signal} - \text{to} - \text{noise ratio of a coherent receiver}}$$

Example:

Let X be the set of transceivers for communicating the signals and C is the set of factors. Each factor is a neutrosophic term or a sentence. Consider C={c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}} (Example, Receiving capacity, distance from Main transceiver, cost, Delivering Capacity, SNR…..etc). This problem, we define a NFSS means to point out c_1, c_2, c_3…… and so on. Suppose there are twelve transceivers in the universe set X given as, U_1 = {TR_1, TR_2, TR_3, TR_4, TR_5, TR_6} and U_2 = {TR_7, TR_8, TR_9, TR_{10}, TR_{11}, TR_{12}}. Let A = {c_1, c_2, c_3, c_4} where c_1 refers the factor 'Receiving Capacity', c_2 refers for the factor 'Delivering Capacity', c_3 stands for the factor 'Signal to Noise Ratio (β)' and the factor c_4 stands for 'Cost'.

Suppose that NFSS defined on U_1 and the parameter e_1 is given in Table.1 as f(U_1,c_1) follows.

	T	*I*	*F*
TR_1	.7	.5	.4
TR_2	.6	.4	.5
TR_3	.85	.6	.2
TR_4	.7	.6	.4
TR_5	.75	.9	.5
TR_6	.5	.7	.7

Table:1- NFSS (f(U_1,c_1))

NFSS defined on U_1 and the parameter e_2 is given in Table.2 as f(U_1,c_2)

	T	*I*	*F*
TR_1	.8	.4	.3
TR_2	.7	.5	.2
TR_3	.7	.6	.3
TR_4	.9	.2	.5
TR_5	1.0	.6	.5
TR_6	.8	.3	.4

Table:2- NFSS (f(U_1,c_2))

NFSS defined on U_1 and the parameter e_3 is given in Table.3 as f(U_1,c_3)

	T	*I*	*F*
TR_1	.4	.2	.3
TR_2	.6	.6	.1
TR_3	.8	.3	.4
TR_4	.8	.5	.5
TR_5	.3	.4	.7

TR_6	.5	.3	.5

Table:3- NFSS ($f(U_1,c_3)$)

NFSS defined on U_1 and the parameter e_4 is given in Table.3 as $f(U_1,c_4)$

	T	*I*	*F*
TR_1	.8	.3	.3
TR_2	.5	.5	.4
TR_3	.4	.5	.6
TR_4	.7	.2	.5
TR_5	.1	.4	.7
TR_6	.8	.4	.2

Table:4- NFSS ($f(U_1,c_4)$)

Note: The fuzzy values (T) used here are randomly collected from lab and the remaining values are selected according to the condition. We used moderate to high values, which may provide the better solution for the system. The study also based upon these values. In this proposed topic we used random values, but the original output of neutrosophic fuzzy values will be used in our future research work.

Figure 1 is the graphical representation of Table 1 and so on.

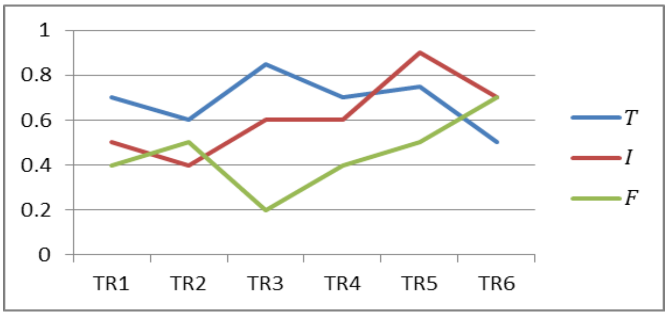

Figure.1 – Graphical representation of $f(U_1,c_1)$

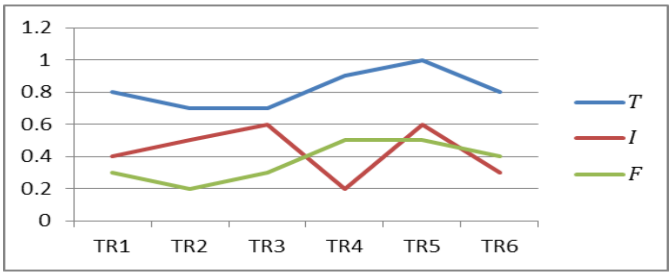

Figure.2 – Graphical representation of $f(U_1,c_2)$

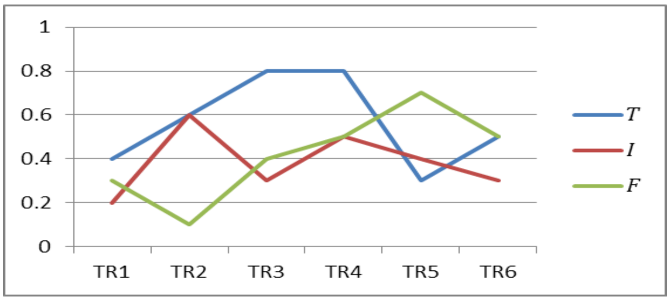

Figure.3 – Graphical representation of f(U_1,c_3)

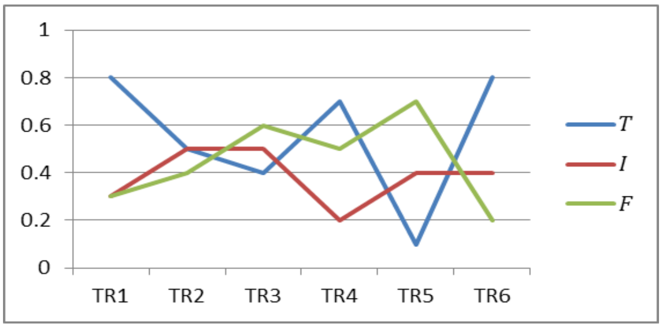

Figure.4 – Graphical representation of f(U_1,c_4)

Similarly we can define the NFSS for other group of transceivers U_2, U_3… and so on.

Also, the attributes TR $(j = 1,2,3,4,5,6)$ have the weight vector is w = $\left(\frac{1}{2},\frac{1}{4},\frac{3}{16},\frac{1}{16}\right)^T$

Then the Neutrosophic Fuzzy Weighted Average values of the Transceivers under the considered criteria are

$A_w(TR_{i1},TR_{i2},TR_{i3},TR_{i4},TR_{i5},TR_{i6})$ *where* i =1, 2, 3, 4 are

$$\tilde{a_i} = \left[1-\prod_{j=1}^{n}(1-T_j)^{w_j},\prod_{j=1}^{n}(I_j)^{w_j},\prod_{j=1}^{n}(F_j)^{w_j}\right]$$

And the Score Function $S(\tilde{a_i})$ is defined as

$$S(\tilde{a_i}) = \frac{(T_i+1-I_i+1-F_i+1)}{6}$$

For example, we can calculate $\tilde{a_i}$ values are obtained using Table 1 to Table 4 as follows.

$$\tilde{a_1} = \left[1-\left[(1-0.7)^{\frac{1}{2}}(1-0.8)^{\frac{1}{4}}(1-0.4)^{\frac{3}{16}}(1-0.8)^{\frac{1}{16}}\right];\left[(0.5)^{\frac{1}{2}}(0.4)^{\frac{1}{4}}(0.2)^{\frac{3}{16}}(0.3)^{\frac{1}{16}}\right];\left[(0.4)^{\frac{1}{2}}(0.3)^{\frac{1}{4}}(0.3)^{\frac{3}{16}}(0.3)^{\frac{1}{16}}\right]\right]$$

$$\tilde{a_2} = \left[1-\left[(1-0.6)^{\frac{1}{2}}(1-0.7)^{\frac{1}{4}}(1-0.6)^{\frac{3}{16}}(1-0.2)^{\frac{1}{16}}\right];\left[(0.4)^{\frac{1}{2}}(0.5)^{\frac{1}{4}}(0.6)^{\frac{3}{16}}(0.5)^{\frac{1}{16}}\right];\left[(0.5)^{\frac{1}{2}}(0.2)^{\frac{1}{4}}(0.1)^{\frac{3}{16}}(0.4)^{\frac{1}{16}}\right]\right]$$

$$\tilde{a_3} = \left[1-\left[(1-0.85)^{\frac{1}{2}}(1-0.7)^{\frac{1}{4}}(1-0.8)^{\frac{3}{16}}(1-0.4)^{\frac{1}{16}}\right];\left[(0.6)^{\frac{1}{2}}(0.6)^{\frac{1}{4}}(0.3)^{\frac{3}{16}}(0.5)^{\frac{1}{16}}\right];\left[(0.2)^{\frac{1}{2}}(0.3)^{\frac{1}{4}}(0.4)^{\frac{3}{16}}(0.6)^{\frac{1}{16}}\right]\right]$$

$$\widetilde{a_4} = \left[1 - \left[(1-0.7)^{\frac{1}{2}}(1-0.9)^{\frac{1}{4}}(1-0.8)^{\frac{3}{16}}(1-0.7)^{\frac{1}{16}}\right]; \left[(0.6)^{\frac{1}{2}}(0.2)^{\frac{1}{4}}(0.5)^{\frac{3}{16}}(0.2)^{\frac{1}{16}}\right]; \left[(0.4)^{\frac{1}{2}}(0.5)^{\frac{1}{4}}(0.5)^{\frac{3}{16}}(0.5)^{\frac{1}{16}}\right]\right]$$

$$\widetilde{a_5} = \left[1 \;\; -0 \qquad ; \left[(0.9)^{\frac{1}{2}}(0.6)^{\frac{1}{4}}(0.4)^{\frac{3}{16}}(0.4)^{\frac{1}{16}}\right]; \left[(0.5)^{\frac{1}{2}}(0.5)^{\frac{1}{4}}(0.7)^{\frac{3}{16}}(0.7)^{\frac{1}{16}}\right]\right]$$

$$\widetilde{a_6} = \left[1 - \left[(1-0.5)^{\frac{1}{2}}(1-0.8)^{\frac{1}{4}}(1-0.5)^{\frac{3}{16}}(1-0.8)^{\frac{1}{16}}\right]; \left[(0.7)^{\frac{1}{2}}(0.3)^{\frac{1}{4}}(0.3)^{\frac{3}{16}}(0.4)^{\frac{1}{16}}\right]; \left[(0.7)^{\frac{1}{2}}(0.4)^{\frac{1}{4}}(0.5)^{\frac{3}{16}}(0.2)^{\frac{1}{16}}\right]\right]$$

If we proceed, then we can receive the following $\widetilde{a_i}$ values which are tabulated below

	T	*I*	*F*
$\widetilde{a_1}$	*.6990*	*.3857*	*.3464*
$\widetilde{a_2}$	*.6112*	*.4627*	*.2899*
$\widetilde{a_3}$	*.7946*	*.5209*	*.2699*
$\widetilde{a_4}$	*.7887*	*.4113*	*.4472*
$\widetilde{a_5}$	*1.0000*	*.6640*	*.5438*
$\widetilde{a_6}$	*.5905*	*.4666*	*.5283*

Table: 5-NFSS Weighted average values $\widetilde{a_i}$

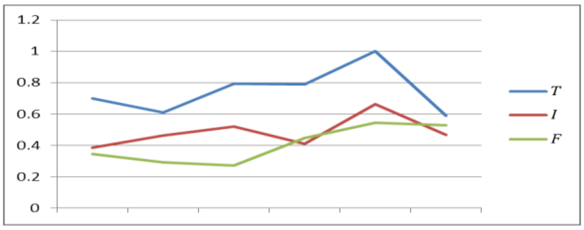

Figure.5 – Graphical representation of Table.5.

$$S(\widetilde{a_1}) = \frac{(0.6990 + 1 - 0.3857 + 1 - 0.3464 + 1)}{6} = 0.4945$$

$$S(\widetilde{a_2}) = \frac{(0.6112 + 1 - 0.4627 + 1 - 0.2899 + 1)}{6} = 0.4764$$

$$S(\widetilde{a_3}) = \frac{(0.7946 + 1 - 0.5209 + 1 - 0.2699 + 1)}{6} = 0.5006$$

$$S(\widetilde{a_4}) = \frac{(0.7887 + 1 - 0.4113 + 1 - 0.4472 + 1)}{6} = 0.4884$$

$$S(\widetilde{a_5}) = \frac{(1.0000 + 1 - 0.6640 + 1 - 0.5438 + 1)}{6} = 0.4653$$

$$S(\widetilde{a_6}) = \frac{(0.5905 + 1 - 0.4666 + 1 - 0.5283 + 1)}{6} = 0.4326$$

Hence Score function values are tabulated here

$S(\tilde{a_1})$	*.4945*
$S(\tilde{a_2})$	*.4764*
$S(\tilde{a_3})$	*.5006*
$S(\tilde{a_4})$	*.4884*
$S(\tilde{a_5})$	*.4653*
$S(\tilde{a_6})$	*.4326*

Table: 6-Score function values $S(\tilde{a_i})$

Comparing $S(\tilde{a_i})$ values, we get the following result

$$S(\tilde{a_3}) > S(\tilde{a_1}) > S(\tilde{a_4}) > S(\tilde{a_2}) > S(\tilde{a_5}) > S(\tilde{a_6})$$

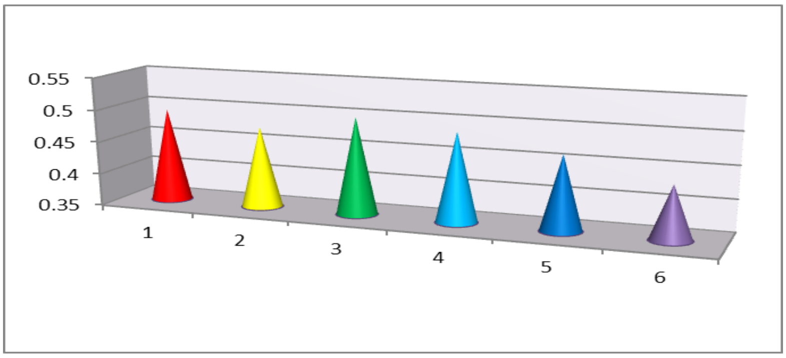

Figure.6 – Graphical representation of Table.6

So, the system can automatically choose TR_3 in the group U_1, and the main transceiver can start to deliver the data to TR_3 to distribute the signals to other transceivers in that group. It will be useful for better communication without interruption and to minimize the interference.

Conclusion:

Usually SNR values are considered to choose the best transceiver for uninterrupted communication. But in the discussed method SNR value is also considered as a parameter of the element. Hence, through the discussed method the better transceiver can be identified using Neutrosophic fuzzy soft sets. This RHT transceiver can be used for fine-tuned communication process and it may reduce loss of signals. It is very useful to increase the efficiency of the particular communication system.

Merits:

1. Proposed method used to identify the better transceiver for effective communication without loss of efficiency.
2. Neutrosophic fuzzy sets used here to identify the better transceiver using multiple parameters including SNR.
3. Score function values are used here to analyze the efficiency of the transceiver. This is better, compare with the analytic studies based upon fuzzy values on communication engineering.

Algorithm:

1. Input neutrosophic fuzzy soft sets: $\langle\langle U_1, A_1, ₦_1 \rangle, \langle U_2, A_2, ₦_2 \rangle\rangle \ldots\ldots$
2. Input the Weightage set "w" by the observer
3. Calculate the neutrosophic fuzzy weighted average $A_w(U_1), A_w(U_2), \ldots$
4. Calculate the neutrosophic fuzzy score functions $S(\widetilde{a_1})$, $S(\widetilde{a_2})$, $S(\widetilde{a_3})$ ….for a group $\langle U_1, A_1, ₦_1 \rangle$ respectively
5. Compare $S(\widetilde{a_i})$ with $S(\widetilde{a_j})$ ∀ i, j ∈k
6. Consider maximum $\widetilde{a_i}$ for all i, Then TR_i be the suitable Reaching Height transceiver for the group U_i.

Future Study:

1. Sudan Jha etal., (2019) discussed a new method to reduce the loss in signal transmission using neutrosophic philosophy in their paper entitled as "Neutrosophic approach for enhancing quality of signals". We would like to develop the new techniques to reduce the loss in signal transmission using NFSS and proposed calculative method. The output of this research will propose significant approach to minimize the loss of efficiency in signal transmitting methods.
2. M. Zulqarnain et al [2017, 2018, 2020 and 2021] discussed many decision making problems in various fields and they used TOPSIS technique to find the better results for Multi – criteria decision making problems. We would like to use TOPSIS technique for finding the suitable solution in our proposed topic.

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Authorship contributions:

M.S. Muthuraman and K.H.Manikandan proposed the idea to apply Neutrosophic fuzzy soft set in the communication engineering problems with transceiver models. M.Sridharan, G.Sabarinathan and R.Muthuraj helped to structure this paper. K.H.Manikandan worked out all the calculations and prepared the graphical representations which are useful to conclude this research article.

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Neutrosophic BWM-TOPSIS Strategy under SVNS Environment

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Abstract: Decision making under an uncertain environment is a critical task. In this article, we develop a Multi Attribute Decision Making (MADM) model using BWM and Neutrosophic-TOPSIS under Single Valued Neutrosophic Set (SVNS) environment. In developed model, BWM is utilized to find the weights of the attributes those are selected by a group of experts, and the Neutrosophic-TOPSIS is utilized to rank the alternatives.

Keywords: Decision Making; BWM; Neutrosophic-TOPSIS; Uncertainty; SVNS.

1. Introduction:

In Multi- Attribute Decision Making (MADM) decision-maker determines the best choice form a set of possible alternatives subject to multiple conflicting criteria. A strategic approach requires to be performed to deal with MADM involving uncertainty. In the MADM algorithm, a decision matrix is formed by the decision-maker to find a ranking of the alternatives.

Fuzzy set theory, proposed by Zadeh [1], has been very useful in dealing with MADM problems involving uncertainty. Decision making is a successful field of study in the fields of Medical Science, Operations Research, Data Mining, Management Science, etc. In the present era there are various popular methods like AHP [2], TODIM [3, 4], VIKOR [5, 6, 7], TOPSIS [8, 9, 10], MULTIMOORA [11], GRA [12], Cross entropy measure [13], DEMATEL [14], Subsethood measure [15], aggregation operators [16], etc. to solve the MADM under uncertain environment. Among those techniques, TOPSIS received a lot of attention in the past decade and many mathematicians studied the method for solving many MADM problems in various situations. In 2011, Pramanik and Mukhopadhyay [17] presented the Multi Attribute Group Decision Making (MAGDM approach to select the teachers based on the Grey Relational Analysis (GRA) under Intuitionistic Fuzzy Set (IFS) environment.

Chen [18] introduced the TOPSIS method in the fuzzy environment and considered the rating value of the alternative and attribute weight in terms of a triangular fuzzy number. In 2009, Boran et al. [19] extended the TOPSIS method for MAGDM under IFS environment to solve the supplier selection problem. In 2010, Ye [20] extended the TOPSIS method with an interval-valued IFS environment. In 2015, Rezaei [21] introduced the Best-Worst Method (BWM). In comparison with the current MADM methods, BWM needs more consistent comparisons, fewer comparison data with more good results. In 2020, Mohammad Javad et al. [22] presented a model of green supplier selection for the steel industry using BWM and fuzzy TOPSIS.

Till now fuzzy MADM and intuitionistic fuzzy MADM problems are studied by many researchers. Presently multiple researchers use uncertainty in the model formulation of different MADM problems. Uncertainty acts as a vital role in MADM difficulties. So neutrosophic sets should be applied in the complex environment involving uncertainty, indeterminacy and inconsistency the MADM method. Since fuzzy and intuitionistic fuzzy MADM difficulties are extensively investigated, but indeterminacy should be included in the MADM difficulties. Smarandache [23] grounded the neutrosophic set to represent the mathematical model of uncertainty, imprecision, and inconsistency. In 2010, Wang et al. [24] presented the notion of single valued neutrosophic set (SVNS). Later on, Biswas et al. [25] studied the entropy based GRA approach for MADM under SVNS environment.

In an MADM algorithm, weights of the attributes play an essential role in ranking the alternatives. In the proposed MADM algorithm, we apply BWM to find the weights of the criteria and the Neutrosophic-TOPSIS method to rank the alternatives.

The remaining paper has been split into several sections. Section 2 is on the preliminaries and the definitions. Section 3 is on neutrosophic BWM-TOPSIS by using hybrid score-accuracy values under SVNS environment. Section 4 deals with the validation of our proposed model. In this section, we consider an example to verify our proposed MADM model. Section 5 presents concluding remarks of the work and future scope research.

2. Preliminaries and Definitions:

The notion of Neutrosophic Set was grounded by Smarandache [23] in 1998. Afterwards, Wang et al. [24] introduced the concept of Single Valued Neutrosophic Set (SVNS) to deal with the events having indeterminate, incomplete information.

An SVNS W over a fixed set Ω is defined as follows:

$$W = \{(q, T_W(q), I_W(q), F_W(q)) : q \in \Omega\},$$

where T_W, I_W, F_W are functions from Ω to [0, 1] and so $0 \leq T_W(q) + I_W(q) + F_W(q) \leq 3$. For any SVNS W over a fixed set Ω, the triplet $(T_W(q), I_W(q), F_W(q))$ is called a Single Valued Neutrosophic Number (SVNN).

Assume that $W = \{(q, T_W(q), I_W(q), F_W(q)) : q \in \Omega\}$ and $M = \{(q, T_M(q), I_M(q), F_M(q)) : q \in \Omega\}$ be two SVNSs [24] over Ω. Then,

(*i*) $W^C = \{(q, 1\text{-}T_A(q), 1\text{-}I_A(q), 1\text{-}F_W(q)): q \in \Omega\}$ is called the complement of W;

(*ii*) $W \subseteq M$ if and only if $T_W(q) \leq T_M(x)$, $I_W(q) \geq I_M(q)$, and $F_W(q) \geq F_M(q)$, for all $q \in \Omega$;

(*iii*) $W = M$ if and only if $W \subseteq M$ and $M \supseteq W$;

(*iv*) $W \cup M = \{(q, T_W(q) \vee T_M(q), I_W(q) \wedge I_M(q), F_W(q) \wedge F_M(q)) : q \in \Omega\}$;

(*v*) $W \cap M = \{(q, T_W(q) \wedge T_M(q), I_W(q) \vee I_M(q), F_W(q) \vee F_M(q)) : q \in \Omega\}$.

Score Function:

In 2018, Mondal and Basu [26] proposed a new score function to solve MADM problems under the SVNS environment as follows:

The score function is defined by the following steps:

Step 1: Suppose that O is the origin and N= *(t_n, i_n, f_n), an SVNN, represents a point in three-dimensional* space. Take a translation of that point N to M = (t_m, i_m, f_m), where $t_m = t_n + r$, $i_m = i_n + r$, $f_m = f_n + r$, where r > 0, a real number such f_m never becomes 1 and unique throughout a particular problem. Consider another point M′ = (t_m,- i_m, -f_m), which is the image of (t_m, i_m, f_m), with respect to the x axis as a mirror.

Step 2: Find the score function $S_1(M)$ =Cos (a), where a is the angle between OM and OM′, O is the origin.

Step 3: If the score values $S_1(N_1)$ and $S_1(N_2)$ are same, for two different SVNNs $N_1\,(t_{n_1}, i_{n_1}, f_{n_1})$ and $N_2 = (t_{n_2}, i_{n_2}, f_{n_2})$, determine $N_1^{**} = (t_{n_1}, -i_{n_1}, -\sqrt{f_{n_1}})$ and $N_2^{**} = (t_{n_2}, -i_{n_2}, -\sqrt{f_{n_2}})$ respectively for the corresponding translated points $N_1^{*} = (t_{n_1^*}, i_{n_1^*}, f_{n_1^*})$ an $N_2^{*} = (t_{n_1^*}, i_{n_1^*}, f_{n_1^*})$,, where $t_{n_1^*} = t_{n_1} + r, f_{n_1^*} = f_{n_1} + r, i_{n_1^*} = i_{n_1} + r$ and $t_{n_2^*} = t_{n_2} + r, f_{n_2^*} = f_{n_2} + r, i_{n_2^*} = i_{n_2} + r$.

Step 4: Find Cos (b) and Cos (c), where b is the angle between ON_1^* and ON_1^{**} and c is the angle between ON_2^* and ON_2^{**}, O is the origin.

Step 5: The score function $S_2(N_1)$ = Cos(b) and $S_2(N_2)$ = Cos(c).

Example 2.1. Suppose that K_1=(0.4, 0.3, 0.2) be an SVNN. Then, score value of K_1 is $s(K_1)$ = 0.090496 , for r= 0.01.

3. Method

In this section, we describe the BWM and Neutrosophic-TOPSIS strategy. In our MADM algorithm BWM is mainly used to find the weights of the selected attributes and TOPSIS is used for ranking the set of alternatives.

3.1. BWM

In an MADM algorithm, attributes selection by the expert, and calculate the weights of those attributes is the most important and critical task. The Best-Worst Method [21] is the best suitable method to determine the values of the weights for the selected attributes.

The BWM method is stated as follows:

i. Selection of a family of m decision-makers.
ii. Selection of a family of n attributes.
iii. Selection of the best attribute and the worst attribute.
iv. Give preference of the best attribute over all the other attributes based on a scale of 1 to 9. The best-to-others vector shows the preference of the best attribute over all other attributes that can be written as: A_b=(a_{b1}, a_{b2}, ……., a_{bn}), where a_{bi} =the preference of the best attribute b over the attribute i and a_{bb}=1.
v. Assign preference of all the other attributes over the worst attribute based on a scale of 1 to 9. The others-to-worst vector shows the preference of all other attributes over the worst attribute that can be written as: A_w=(a_{1w}, a_{2w}, ……., a_{nw})^T, where a_{iw} =the preference of the attribute i over the worst attribute w and a_{ww}=1.
vi. Determine the optimal weights of all the attributes (w_1, w_2, ……, w_n).

The objective is to find the optimal weights so that the maximum absolute differences for all i are minimized of the {|w_b-$a_{bi}w_i$|, |w_i-$a_{iw}w_w$|}.

The following model is resulted considering the weights non-negativity and summation of weights constraints.

min max {|w_b-$a_{bi}w_i$|, |w_i-$a_{iw}w_w$|}

Such that

$\sum_i w_i = 1$ (1)

$w_i \geq 0$, for all i.

Now, we can be transferred the model (1) to the following linear model:

min C_R

Such that

$|w_b - a_{bi}w_i| \leq C_R$, for all i

$|w_i - a_{iw}w_w| \leq C_R$, for all i (2)

$\sum_i w_i = 1$

$w_i \geq 0$, for all i.

After solving eq. (2), we get the weights (w_1, w_2, ……, w_n) and value of C_R. The value of C_R closer to zero indicates desired consistency.

3.2. TOPSIS

Till now many MADM strategies were developed. Among them TOPSIS is one of the most popular MADM strategy to rank the set of alternatives. Also, the rank of the set of alternatives is the most necessary part of an MADM problem. In this section we describe the TOPSIS method for ranking the alternatives.

First we need to consider a set of alternatives A={W_1, W_2, W_3, …., W_m} with m≥1 and a set of attributes C={A_1, A_2, A_3, ……,A_n} with $n \geq 2$ and choose the weights w_1, w_2, …..,w_n for each attributes A_i, (i=1, 2, 3,.., n} respectively.

Decision-makers provides rating of the alternatives W_j, (j=1, 2, 3,…, m) based on the attributes A_i, (i=1, 2, 3,.., n}, which is represented in term of an SVNN. Assume that rating of j-th attribute with respect to i-th alternative is presented as follows:

$W_j^* = \left(A_i, T_{W_j}(A_i), I_{W_j}(A_i), F_{W_j}(A_i)\right)$, $j = 1, 2, \ldots\ldots, m$, where $0 \le T_{W_j}(A_i) + I_{W_j}(A_i) + F_{W_j}(A_i) \le 3$.

Here (T_{ji}, I_{ji}, F_{ji}) is denoted as an SVNN. W_j^* , (i = 1, 2, 3,..., n, and j = 1, 2, 3, …, m, where i = no of attributes and j= no of alternatives. Based on rating Therefore, we get the decision matrix:
$D^*=(W_{ji}^*)_{m\times n}$
Now, TOPSIS method is summarized as follows:

i. The score-matrix $D=\left(W_{ji}\right)_{m\times n}$ (j=1,2,…,m; i=1,2,….,n) is obtained from the decision matrix $D^*=(W_{ji}^*)_{m\times n}$ by using the following described in preliminary section:
i.e., $W_{ji} = s_1(W_{ji}^*)$.

ii. Determination of normalized decision matrix $N=(N_{ji})_{m\times n}$
where $N_{ji} = \frac{w_{ji}}{\sqrt{\sum_{j=1}^{m} w_{ji}^2}}$, j=1, 2, 3, …, m; i=1, 2, 3, …, n.

iii. Determine the weighted normalized decision matrix $V=(V_{ji})_{m\times n}$
where $V_{ji}= w_i{*}N_{ji}$, j=1, 2, 3, …, m; i=1, 2, 3, …, n.

iv. Determine the Neutrosophic Positive Ideal Solution (NPIS) and Neutrosophic Negative Ideal Solution (NNIS).
NPIS $I^+ = \{v_1^+, v_2^+, v_3^+, \ldots . v_n^+\}$, where $v_i^+ = max\, V_{ji}$, i=1, 2, 3, …, n;
NNIS $I^- = \{v_1^-, v_2^-, v_3^-, \ldots . v_n^-\}$, where $v_i^- = min\, V_{ji}$, i=1, 2, 3, …, n.

v. Determine the distance of each alternative from NPIS and NNIS using the following formula,

$$S_j^+ = \sqrt{\sum_{i=1}^{n}(V_{ji} - V_i^+)^2},\ j = 1, 2, \ldots , m$$

$$S_j^- = \sqrt{\sum_{i=1}^{n}(V_{ji} - V_i^-)^2},\ j = 1, 2, \ldots , m$$

vi. Calculate the performance score of each alternative by using the formula

$$P_j = \frac{S_j^-}{S_j^- + S_j^+}$$

vii. Put the performance score in ascending order and rank the alternatives.

4. Validation of the Proposed BMW-TOPSIS Strategy

In this section we present a numerical example to validate the developed MADM model.

Example 4.1. "Selection of Suitable Flat for a Customer".

Suppose a customer wants to buy a Flat from a set of available alternatives/Flats. The quality of the flat is very important for the customer and it is dependent on the price that is why it is a variable quantity. Which floor is ok on the basis of customer investment and the geographical position of land, ownership of land, cost of land, communication of builders, etc. To buy a flat, customers have to concentrate in some criteria and to decide the priority on the criteria. After the initial screening, customers select four possible alternatives / Flats namely W_1, W_2, W_3, W_4 for further evaluation. A decision maker selects seven attributes namely Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7 that help the customer to select the best one.

❖ **Fire safety (Q_1):**

When clients purchase an apartment on a higher floor, fire safety is the most critical consideration. To do so, the consumer must be aware that the fitness for occupancy of any apartment can be determined by whether it has received an Occupancy Certificate (OC) from the local authorities. Customers can use OC to determine whether or not a structure was built in accordance with the permitted designs. When purchasing an apartment, buyers should seek the copy of the OC. When the fire department issues clearances, the builder can seek the occupancy certificate. As a result, when clients buy a flat with OC, they know it passes the fire department's safety requirements.

❖ **Floor deviations (Q_2):**

When a customer is looking at buying a flat, floor deviation is a vital factor to consider. On occasion, unauthorized deviations from construction plans occur on the building's top floor. If clients are purchasing a flat on the top level, make sure there are no deviations from the norm. It is a good idea to inspect the floor and make sure that the property has all of the essential approvals. Another thing to keep in mind is that the lowest floors are the most pest-prone, with rats, snakes, and other animals freely entering the lower units. Higher severe road noise can sometimes be heard from all sides and only subsides beyond 12/13-th floor. So, based on the floor deviations, the customer must select which one is the best for him.

❖ **Vantage point (Q_3):**

Consider a higher floor if the view from your clients' apartment is vital to you, as they often provide the best available vantage points. The view is a major consideration for apartments nearby sea or in a scenic area. Floor rise charges will apply in an under-construction flat, making living on upper floors slightly more expensive. The benefits of living on a higher floor are numerous. When compared to those on the ground and lower floors, you receive better views of your neighbourhood, better light and ventilation, and are less affected by street-level noises. Mosquitoes and rodents are usually not a problem on higher floors (mainly rats).

❖ **Mobile network and power consumption (Q_4):**

Whole world is going to be digitalized, so that online communication is going to very first. In numerous metro towns, the construction of high-rise apartments with as much as 40 floors has become commonplace. In Mumbai, for example, high-rise buildings can reach 40 floors, while skyscrapers in Delhi-NCR average 25 floors. If you choose a higher floor, ensure sure the flat has appropriate network coverage. Lower floors are often cooler and use less energy than higher floors. This is a significant consideration in cities with long, hot summers.

❖ **Connection, service-related factors (Q_5):**

This is yet another significant factor for clients to consider. Before purchasing a property, be certain that the floor is equipped with CCTV Camera. Also, when you buy an apartment on any floor, verify the corridor area, which is the sole open place outside your flat. As a direct consequence, you should double-check that you have adequate space outside of your flat, as corridor sizes differ between apartment complexes. Aside from that, keep in mind that several service providers do not even offer on upper floors, for example, broadband.

- **Choose the right builder (Q_6):**

Before buying, choosing the right builder is very important because in some cases builder is unable to provide the customer satisfaction. They create many problems and making many false statements. So, the feedback and certificate of the builder is too important when it comes to deciding on the right project and floor to buy a flat on, make sure you to choose trusted builder with a solid track record. It is your right and responsibility as a house buyer to verify with the Real Estate Regulatory Authority to see if the builder has registered the project (RERA).

- **Local infrastructure around the society and well connectivity (Q_7):**

In determining a domestic property's current and future worth, the infrastructure in and around it is critical. Roads linking to the community and in-roads within the population, for example, should be well-built and preserved. Ascertain that the project is convenient from significant areas of the city and that governmental and non - governmental transit are well connected. Calculate routes from the city's prominent landmarks. You can do some investigation on the area and see what public transportation choices are available, such as rail, buses, and cabs. Also keep an eye out for future infrastructure plans in the area, such as a projected metro line or freeway, as well as adjacent entertainment alternatives.

The comparison of criteria by the rating from 1 to 9, are given in the following table.

Table-1: (Comparison of Criteria)

Criteria	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	
	7	9	4	3	6	2	1	Best Criteria: Q_7
Q_1								2
Q_2								1
Q_3								5
Q_4								6
Q_5								4
Q_6								8
Q_7								9
Worst Criteria: Q_2								

By using eq. (1) and eq. (2), we obtain the weights of the attributes. The weights are given in the following table.

Table-2: (Calculation of weights using BWM)

Criteria	**Weights** ($=w_i$)
Fire safety (Q_1)	0.05284016

Floor deviations (**Q_2**)	0.04109790
Vantage point (**Q_3**)	0.09247028
Mobile network and power consumption (**Q_4**)	0.1232937
Connection, service-related factors (**Q_5**)	0.06164685
Choose the right builder (**Q_6**)	0.2587700
Local infrastructure around the society and well connectivity (**Q_7**)	0.3698811
Consistency Rate (C_R)	0.6301189

Suppose the decision maker provides his/her evaluation information for the alternatives with respect to the attributes by using SVNNs, then the decision matrix is constructed.

Decision Matrix (*D*):

D	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7
W_1	(0.7,0.3,0.2)	(0.7,0.1,0.2)	(0.9,0.3,0.1)	(0.8,0.1,0.3)	(0.8,0.2,0.1)	(0.9,0.2,0.2)	(0.7,0.2,0.2)
W_2	(0.8,0.2,0.4)	(0.6,0.0,0.2)	(0.8,0.1,0.1)	(0.9,0.2,0.2)	(0.9,0.1,0.2)	(0.8,0.2,0.1)	(0.8,0.1,0.2)
W_3	(0.8,0.3,0.1)	(0.8,0.2,0.2)	(1.0,0.2,0.1)	(0.7,0.2,0.1)	(0.8,0.3,0.2)	(0.7,0.2,0.1)	(1.0,0.3,0.2)
W_4	(0.6,0.2,0.1)	(0.9,0.2,0.3)	(0.8,0.1,0.2)	(0.9,0.4,0.2)	(0.7,0.2,0.2)	(0.7,0.1,0.2)	(0.8,0.1,0.3)

Score Matrix (*D):**

D^*	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7
W_1	0.564799	0.799393	0.768877	0.716865	0.842201	0.807487	0.702178
W_2	0.511229	0.787653	0.928855	0.807487	0.872894	0.842201	0.842201
W_3	0.716865	0.762999	0.895568	0.799393	0.647871	0.799393	0.758338
W_4	0.737567	0.710420	0.842201	0.592041	0.702178	0.799393	0.716865

[In the above table, we find all the score values by taking r= 0.01]

Normalized Decision Matrix (*N*):

N	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7
W_1	0.441269	0.361863	0.446502	0.488277	0.545455	0.497031	0.463878
W_2	0.399416	0.356548	0.539404	0.550002	0.565334	0.518398	0.556381
W_3	0.560075	0.345388	0.520074	0.544489	0.419597	0.492049	0.500978
W_4	0.576250	0.321587	0.489083	0.403256	0.454769	0.492049	0.473580

Weighted Normalized Decision Matrix (*V*):

V	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7
W_1	0.023317	0.014872	0.041288	0.060201	0.033626	0.183842	0.171580
W_2	0.021105	0.014653	0.049879	0.067812	0.034851	0.191746	0.205795
W_3	0.029594	0.014195	0.048091	0.067132	0.025867	0.182000	0.185302
W_4	0.030449	0.013216	0.045226	0.049719	0.028035	0.182000	0.175168

Neutrosophic Positive Ideal Solution (I^+) and Neutrosophic Negative Ideal Solution (I^-):

	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7
W_1	0.023317	0.014872	0.041288	0.060201	0.033626	0.183842	0.171580
W_2	0.021105	0.014653	0.049879	0.067812	0.034851	0.191746	0.205795
W_3	0.029594	0.014195	0.048091	0.067132	0.025867	0.182000	0.185302
W_4	0.030449	0.013216	0.045226	0.049719	0.028035	0.182000	0.175168
I^+	0.030449	0.014872	0.049879	0.067812	0.034851	0.191746	0.205795
I^-	0.021105	0.013216	0.041288	0.049719	0.025867	0.182000	0.171580

Distance of Each Alternative from NPIS and NNIS:

	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	S_j^+	S_j^-
W_1	0.023317	0.014872	0.041288	0.060201	0.033626	0.183842	0.171580	0.037646223	0.013457438
W_2	0.021105	0.014653	0.049879	0.067812	0.034851	0.191746	0.205795	0.041828099	0.296781428
W_3	0.029594	0.014195	0.048091	0.067132	0.025867	0.182000	0.185302	0.27572937	0.27572937
W_4	0.030449	0.013216	0.045226	0.049719	0.028035	0.182000	0.175168	0.264977289	0.264977289

Performance Score:

	S_j^+	S_j^-	$P_j = \frac{S_j^-}{S_j^- + S_j^+}$
W_1	0.037646223	0.013457438	0.263336092
W_2	0.041828099	0.296781428	0.876470991
W_3	0.27572937	0.27572937	0.5
W_4	0.264977289	0.264977289	0.5

The ascending order of performance score associated with each alternative is $P_1 < P_4 = P_3 < P_2$. Hence, W_2 is the most suitable flat for the customer.

5. Conclusion

In this paper, we used some suitable attributes for decision making to a better choice of a flat among the available flats. Also, we used two important methods BWM and TOPSIS to select the appropriate flat among the available flats under the SVNS environment, where the BWM is mainly used for determining the weights of the attributes, and TOPSIS is used for ranking the possible alternatives/flats.

The data used in this paper has not taken from any source. We have just considered these numbers for the verification of our algorithm. However, this algorithm can apply for any real source data.

The developed BWM-TOPSIS can be utilized in different neutrosophic environments such as refined neutrosophic set [27], rough neutrosophic set [28], interval neutrosophic set [29], neutrosophic soft set [30], neutrosophic soft expert set [31], bipolar neutrosophic set [32], pentapartitioned neutrosophic set [33, 34], etc.

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Neutrosophic Set Hybrid MCDM Methodology for Choosing Best Surfactant-Free Microemulsion Oils within Performance and Emission Criteria Over a Wide Range of Engine Loads

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Abstract: Microemulsion fuels, also known as surfactant-free fuels, are fuels made from a combination of two immiscible liquids, hydrocarbon fuel, and water, with a trace quantity of a co-solvent. Surfactants are often used in conventional microemulsion fuels, however surfactant-free microemulsion fuels instead depend on the thermodynamic features of the combination to generate a stable emulsion. In this paper, the multi-criteria decision-making (MCDM) model for choosing microemulsion fuel with surfactant-free. Various helpful and harmful criteria were evaluated for ten fuels at varying motor loads, according to performance and emission characteristics. This paper integrated the neutrosophic set with the TOPSIS method. The neutrosophic set is used to deal with uncertain data. The TOPSIS method is used to rank the different fuels.

Keywords: Neutrosophic Set, TOPSIS Method, Microemulsions, Renewable Energy, Diesel.

1. Introduction

More fossil fuels are being used to meet the ever-increasing energy needs of a growing global population and rising quality of life. Acid rain, global warming, carbon dioxide, and other health dangers are only some of the ecological problems that may be traced back to our reliance on fossil fuels. The Paris Agreement of 2015 stipulates that countries will work together to keep global warming below 2°C. Interest in cleaner and more sustainable substitutes for power has increased in response to more stringent environmental regulations and worries about power and financial security. Most of the world's energy needs are met by fossil fuels, despite the extensive study of renewable alternatives. As its economy has grown rapidly, Asia has become one of the world's largest consumers of energy. Fossil fuel emissions are increasing the artificial aging of the atmosphere, according to studies. Renewable power has an opportunity to reduce ecological pollution, but only if it is affordable, readily accessible, and consistently used. Biodiesel and hydrogen, two examples of alternative fuels, are either costlier than traditional fuels like petrol and diesel or need a separate fuel system[1], [2].

Microemulsions are a special kind of colloidal system that has been the subject of substantial research because of its potential use in several industries, such as the pharmaceutical, cosmetic, oil recovery, and food processing sectors. Microemulsions have excellent solubilization capability for hydrophilic and hydrophobic substances, are thermodynamically stable, and are transparent. Due to these characteristics, microemulsion formulations are very desirable for several uses[3], [4].

To keep their stability, microemulsions often include a third party, such as a surfactant or co-surfactant, in addition to two immiscible liquids incompatible. The interfacial tension is decreased, and the emulsion is stabilized by the layer of surfactant molecules that is positioned at the interface between the two liquids. The co-surfactant is included to further decrease the interfacial tension and increase the solubilization capability. The resultant microemulsion has a large surface area and a tiny droplet size range (usually 10 to 100 nanometers)[4], [5].

Microemulsions' special qualities make them useful in many contexts. Due to their excellent solubility and bioavailability, microemulsions are employed in the pharmaceutical sector for medication delivery. Microemulsions have dual purposes in cosmetics, both hydrating the skin and transporting other substances. Microemulsions are used in the food business to increase the stability and longevity of various goods. Microemulsions are employed in improved oil recovery to lower interfacial tension and increase oil recovery from reservoirs[6], [7].

Several obstacles must be overcome before microemulsions may realize their full potential. The complexity of microemulsion formulation and processing contributes to its prohibitively high production cost. The surfactants employed to stabilize microemulsions may also be toxic, limiting their usefulness in certain situations. Furthermore, temperature, pH, and the presence of pollutants all have a significant role in deciding a microemulsion's stability, which might restrict its usage in certain situations.

Because of variations in efficiency and pollution output, deciding the best fuel choice often needs the use of a multi-criteria decision-making (MCDM) methodology. The TOPSIS approach was used to determine the best crop for making biodiesel[8], [9]. The choice of the best crop for making biodiesel in Egypt is based on various characteristics and performance emissions.

The term "neuropathy" was coined by Smarandache. Neutrality studies examine how various ideational spectra interact with one another and how they came to be. Classic sets, fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, paraconsistent sets, dialetheist sets, paradoxist sets, and tautological sets are all generalized by the strong and broad formal framework known as the neutrosophic set[10], [11].

The neutrosophic set is a philosophical generalization of the sets. From a scientific or technical perspective, it is necessary to specify the neutrosophic set and set-theoretic operators. Alternatively, implementing it in the actual world would be challenging. That's why Wang et al. suggested a single-valued neutrosophic set (SVNS) and laid out its set-theoretic operators and other features[12], [13].

Here is how the rest of the paper is laid out. In Section 2, we supply information about performance and emission characteristics. Section 3 supplies the alternatives to diesel. Section 4 introduces the neutrosophic TOPSIS method. Section 5 introduces the results and discussion. Concluding Section 6 presented the conclusions.

2. Performance and Emission Characteristics

Recent interest in alternative fuels has been spurred by the scarcity of supplies and the unavoidable emission rates from fossil fuels. Diesel is one widely used fossil fuel because of the great thermal effectiveness it provides to the engine. The atomization and vaporization of vegetable oils prevent them from being employed in motors as diesel substitutes. Other alternative oxygenating fuels that aid reduce CO and smoke emissions include alcohol. Nevertheless, they often split and diminish the fuel's warming potential[2], [3].

To avoid splitting of phases and ease emulsification, alcohol in diesel requires the use of surfactants. Nevertheless, emulsions are only robust kinetically, and their huge droplet dimensions lead to phase splitting if they are not disturbed. Vegetable oils and alcohols each have their own set of drawbacks, but there have been several suggested solutions, like mixing, transesterification, and micro emulsification, for working around them.

Because they need no chemical reactions during manufacture, microemulsions are among the most practical options. They also contribute to biodiesel's enhanced pour point and ignition latency. Microemulsions, in contrast to emulsions, have droplet dimensions of fewer than 200 nm and are thermally inert. Microemulsions of Colza oil, diesel, and water were shown to remain stable for more than nine months. Because the vapor depth and liquid duration of the gasoline in the vehicle are identical to those of petro-diesel, microemulsions do not need any adjustments to the motor. Even at a high insertion pressure of 1500 bar, droplet dimension distribution was found to be unaffected. Droplet dimensions in microemulsions dropped as the mixing rate rose to 1000 rpm, but thereafter rose, maybe because the continuous stage was more evenly distributed throughout the dispersed phase[14], [15].

When combined with alcohol, microemulsions have also been employed to alter the consistency of vegetable oils. For instance, increasing the proportion of ethanol in microemulsions causes the viscosity of soybean oil, coconut oil, and algal oil microemulsions to decrease. Blends of biodiesel and diesel have also made use of alcohols to enhance their low-temperature fluidity and stiffness. In addition, they aid in reducing PM's dimensions, amount, and bulk, typically by 50%, 60%, and 30%, correspondingly. Reduced NO_x and soot production is another benefit of using lower alcohols. The smoke emissions from diesel, palm-biodiesel, and alcohol mixes were all much lower than those from pure diesel. Ethanol's greater latent heat compared well to other lesser alcohols in reducing NOx emissions. However, the greater wait for ignition and lower evaporation speed of ethanol resulted in higher HC emissions.

Microemulsions of alcohol and diesel aid lower exhaust gas particulate matter, nitrogen oxide, carbon monoxide, and hydrocarbon emissions. Employing ethanol as a sustainable oxygenated ingredient, Mehta et al. developed diesel microemulsions that were stabilized by Span-80. Similar characteristics to diesel were seen in the microemulsions. Nevertheless, the calorific value, cetane index, and flash point all dropped when ethanol was added. The microemulsions enhanced thermal efficiency and had energy consumption comparable to diesel. The amount of ethanol in these microemulsions has been linked to cooler exhaust gas, which likely contributes to decreased NOx emissions[16], [17].

3. Alternatives to diesel

There are five alternatives are options of diesel-like:

Renewable biofuels are produced from various plant and animal byproducts. They are compatible with diesel engines and may result in lower emissions of greenhouse gases than conventional diesel fuel.

Electric automobiles: Unlike conventional vehicles, which rely on fuels like diesel, electric vehicles may run only on electricity. As battery technology advances, they are gaining in popularity.

Fuel cells that run on hydrogen and oxygen create energy with just water as a byproduct of the process. They may replace dirty diesel with something eco-friendlier.

Natural gas: Natural gas is a fossil fuel that, with certain adaptations to engines, may be used in place of diesel. There are fewer emissions compared to diesel.

Some kinds of engines may be converted to run-on propane since it burns cleaner than diesel. It's readily accessible, and in some places, it's even cheaper than diesel[18], [19].

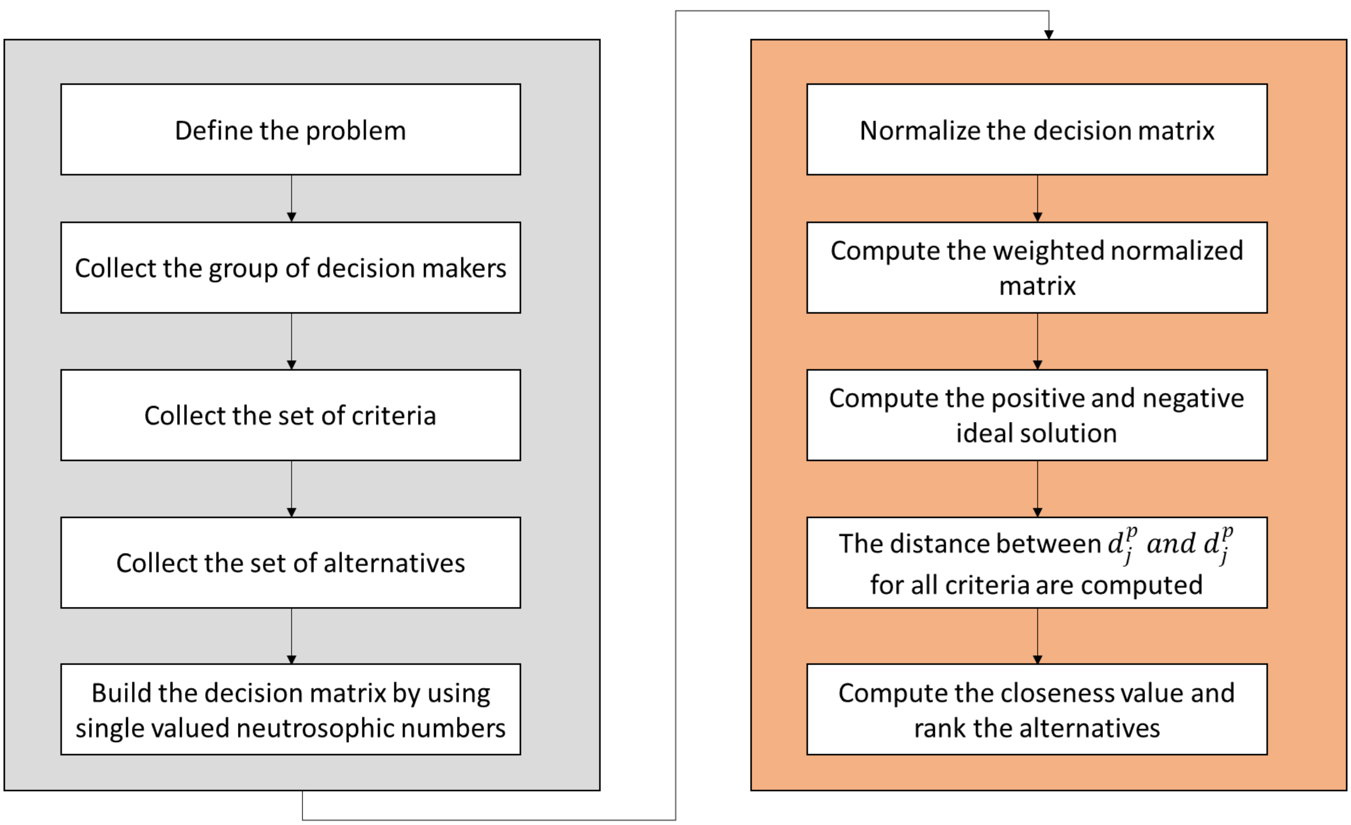

Figure 1. The framework for choice SFME fuels.

4. Neutrosophic TOPSIS Method

The broadest formal framework that generalizes the sets from a philosophical vantage point is the neutrosophic set, which is a part of neutrosophy (the study of the genesis, nature, and extent of neutralities and their interactions with diverse ideational spectra). This paper used the single-valued neutrosophic set with the TOPSIS method to rank the alternatives[20]–[22]. The steps of the proposed method are shown in Figure 1. The following steps discuss the neutrosophic TOPSIS method:

1) Build the decision matrix between criteria and alternatives

2) Normalize the decision matrix

The below equation is used to normalize the decision matrix.

$$z_{ij} = \frac{a_{ij} - a_j^-}{a_j^+ - a_j^-} \quad (1)$$

Where z_{ij} refers to the normalization value, a_{ij} refers to the value in the decision matrix, a_j^- refers to the minimum value in the decision matrix and a_j^+ refers to the maximum value in the decision matrix.

3) Compute the weighted normalized matrix

$$O_{ij} = w_j * z_{ij} \quad (2)$$

Where w_j refers to the weights of the criteria. The weights of the criteria are computed by the average method.

4) Compute the positive and negative ideal solution

The positive and negative ideal solutions are computed by using a weighted normalized decision matrix.

The positive and negative ideal solutions are computed for the positive and negative criteria.

$$d_j^p = \max_i(O_{ij}) \quad (3)$$

$$d_j^n = \min_i(O_{ij}) \quad (4)$$

5) The distance between d_j^p *and* d_j^p for all criteria to be computed

$$T_i^p = \sqrt{\sum_{j=1}^{n}\left(O_{ij} - d_j^p\right)^2} \quad (5)$$

$$T_i^n = \sqrt{\sum_{j=1}^{n}\left(O_{ij} - d_j^n\right)^2} \quad (6)$$

6) Compute the closeness value

By using the distance values, the closeness value is computed.

$$S_i = \frac{T_i^n}{T_i^n + T_i^p} \quad (7)$$

7) Rank the alternatives

5. Results and Discussion

Despite the promising future of microemulsions, there are still certain obstacles to overcome in their research and use. Microemulsions may be challenging to make and regulate due to their complicated composition. The surfactants employed to stabilize microemulsions may also be toxic, limiting their usefulness in certain situations. Another potential barrier is the costly nature of manufacturing microemulsions.

This section introduces the results of the single-valued neutrosophic set with the TOPSIS method to select the best SFME fuels. This paper used eight criteria and 10 SFME fuels.

Table 1. The matrix of normalization decision.

	$DISC_1$	$DISC_2$	$DISC_3$	$DISC_4$	$DISC_5$	$DISC_6$	$DISC_7$	$DISC_8$
$DISA_1$	0.452768	0.186445	0.253596	0.273422	0.090987	0.154616	0.37225	0.140193
$DISA_2$	0.536614	0.36025	0.113781	0.157829	0.272962	0.138913	0.365188	0.076192
$DISA_3$	0.14134	0.291518	0.113781	0.175539	0.389101	0.142537	0.238078	0.566866
$DISA_4$	0.220994	0.442412	0.465247	0.189691	0.174578	0.339431	0.258255	0.158479
$DISA_5$	0.14134	0.181705	0.314343	0.416432	0.189372	0.581623	0.241105	0.546142
$DISA_6$	0.326999	0.291518	0.060747	0.192655	0.267784	0.574979	0.530632	0.224918
$DISA_7$	0.137747	0.35709	0.362699	0.19636	0.192331	0.235548	0.37225	0.154821
$DISA_8$	0.313224	0.428192	0.412695	0.416432	0.189372	0.157032	0.232026	0.074973
$DISA_9$	0.382697	0.291518	0.270952	0.093364	0.467513	0.154616	0.238078	0.478484
$DISA_{10}$	0.218598	0.202246	0.464282	0.634281	0.579952	0.219845	0.126101	0.14385

The criteria are collected based on diesel criteria and emission criteria. The criteria collected from earlier studies are:

Diesel fuel's high energy density implies it can supply more power per liter than most other fuels can.

Diesel fuel has a high combustion efficiency, meaning it burns cleanly and with fewer pollutants compared to other fuels.

Diesel fuel is readily accessible in many regions, making it a practical choice for a variety of uses.

Diesel fuel is more affordable than other fuel options, making it a desirable choice for many consumers.

Because diesel engines were developed to run on diesel fuel, diesel fuel must be compatible with diesel engines.

Cars, lorries, and other vehicles are restricted in the quantity of pollution they may release into the atmosphere by emissions rules.

Limits on pollutant emissions from fossil fuel-burning power plants are set up by emissions regulations.

The regulations for emissions from industrial operations set up maximum allowable concentrations of pollutants released by such activities.

We start with single-value neutrosophic numbers. Then build the decision matrix between criteria and alternatives. Then compute the weights of the criteria using the average method. Then normalize the decision matrix by using an equation. (1) as shown in Table 1. Then multiply the weights of the criteria by the normalization matrix by equation. (2), as shown in Table 2. Then compute the positive and negative ideal solutions by using Equations. (3–4). Then compute the distance between positive and negative ideal solutions by using Equations. (5–6). Then compute the closeness value by using the equation. (7) as shown in Figure 2. Alternative one is the best and alternative three is the worst.

Table 2. Weighted normalized decision matrix.

	$DISC_1$	$DISC_2$	$DISC_3$	$DISC_4$	$DISC_5$	$DISC_6$	$DISC_7$	$DISC_8$
$DISA_1$	0.119474	0.015358	0.046559	0.035216	0.003906	0.013816	0.047944	0.011255
$DISA_2$	0.141599	0.029675	0.02089	0.020328	0.011719	0.012412	0.047035	0.006117
$DISA_3$	0.037296	0.024013	0.02089	0.022609	0.016705	0.012736	0.030664	0.045508
$DISA_4$	0.058315	0.036443	0.085417	0.024431	0.007495	0.03033	0.033262	0.012723
$DISA_5$	0.037296	0.014968	0.057712	0.053635	0.00813	0.051971	0.031053	0.043844
$DISA_6$	0.086287	0.024013	0.011153	0.024813	0.011496	0.051377	0.068343	0.018056
$DISA_7$	0.036348	0.029415	0.06659	0.02529	0.008257	0.021047	0.047944	0.012429
$DISA_8$	0.082652	0.035272	0.075769	0.053635	0.00813	0.014031	0.029884	0.006019
$DISA_9$	0.100984	0.024013	0.049745	0.012025	0.020071	0.013816	0.030664	0.038412
$DISA_{10}$	0.057682	0.01666	0.08524	0.081693	0.024898	0.019644	0.016241	0.011548

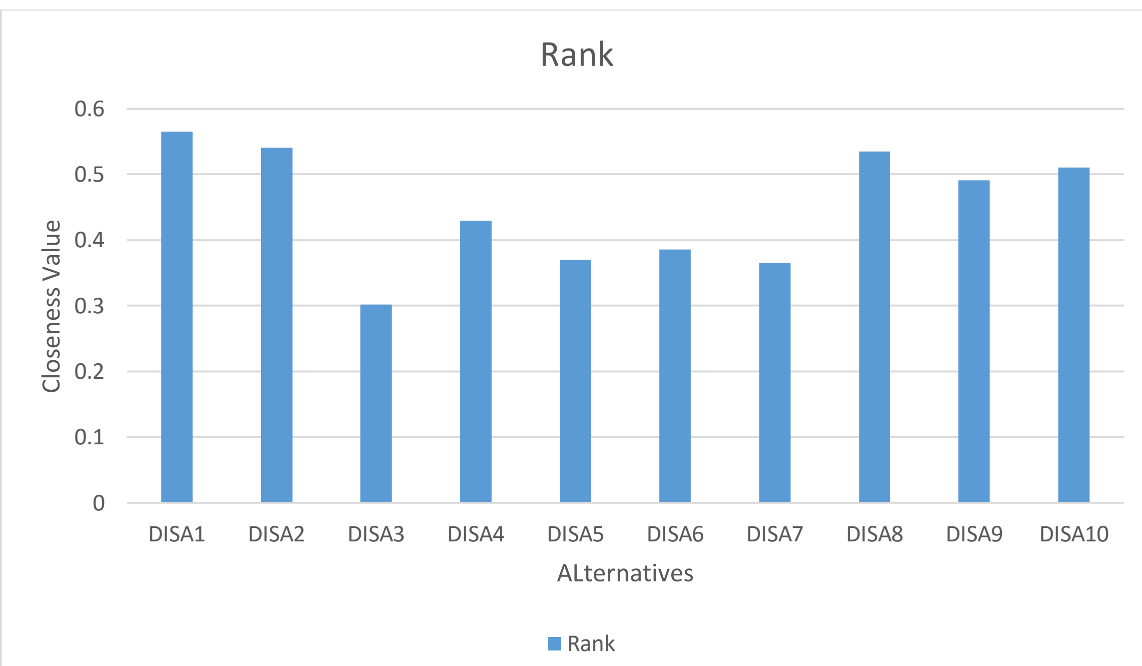

Figure 2. The closeness value by the TOPSIS method.

Surfactant-free microemulsion fuels have certain drawbacks, such as:

Surfactant-free microemulsion fuels are not as stable as their conventional counterparts and may degrade over time or under certain circumstances.

Some engine types and fuel systems may not be compatible with surfactant-free microemulsion fuels.

Some advantages of microemulsion fuels that do not need surfactants are:

Increased efficiency due to more thorough combustion because of the emulsion's tiny droplet size and enhanced atomization of the fuel.

Lower emissions of undesirable pollutants including nitrogen oxides (NOx) and particulate matter are a direct result of the higher combustion efficiency of surfactant-free microemulsion fuels.

Surfactant-free microemulsion fuels are safer than regular hydrocarbon fuels because they produce fewer explosive vapors.

6. Conclusions

Microemulsions are an interesting and potentially useful class of colloidal mixtures. The creation and usage of microemulsions present several problems, such as their complicated composition and toxicity, despite the numerous advantages they provide. This paper used the neutrosophic set with the TOPSIS method to rank the fuels of SFME. The single valued neutrosophic set is used to deal with uncertain data. Then the TOPIS method is used to compute the weights of criteria and rank the alternatives. This paper used eight criteria and ten alternatives.

The suggested MCDM framework's findings support alternative one as a long-term replacement for diesel over a wide range of engine loads and alternative three is the worst.

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Open-Pit Mine Slope Stability Clustering Analysis and Assessment Models Based on an Inverse Hyperbolic Sine Similarity Measure of SVNSs

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Abstract: Slope instability is a common and typical problem of geological hazards, often accompanied by significant losses. So, it is necessary to provide some simple and effective methods to avoid the potential geological hazards of slope instability. It is obvious that the clustering and assessment of slope stability are very crucial. However, the existing clustering and assessment methods in the scenario of single-valued neutrosophic sets (SVNSs) imply some difficulties in engineering applications, such as a lot of collective sampling work, the complex training process, and the selection issue of different types of membership functions. Regarding these problems, this paper proposes an inverse hyperbolic sine similarity measure (IHSSM) of SVNSs and its netting clustering and assessment models for slope stability clustering analysis and evaluation based on the fuzzification process of the true, false, and uncertain Gaussian membership functions for slope sample data. Finally, the proposed clustering and assessment models are applied to the clustering analysis and assessment of 20 slope samples as the case study, and then comparing the results of clustering analysis and stability evaluation of the proposed models with those of the existing relative methods by the 20 slope samples, we verify the validity, consistency, and rationality of the proposed netting clustering and evaluation models.

Keywords: single-valued neutrosophic set; netting clustering method; Gaussian membership function; similarity measure; slope stability clustering analysis; slope stability evaluation

1. Introduction

Slope instability is a common phenomenon in geological hazards. Then, the occurrence of such hazards is often accompanied by significant losses. Therefore, it is crucial to establish effective methods to eliminate the potential risk of slope instability. Slope stability evaluation methods can be divided into two main categories: deterministic and uncertain methods. The traditional limit equilibrium method or analytical calculations using numerical methods are still commonly used in modern engineering [1]. The Sweden slice method [2] is the theoretical basis of the limit equilibrium method. After the refinement and improvement of the method by Janbu [3], Bishop [4], and Morgenstern & Price [5], the calculation results of the limit equilibrium method are more reasonable and accurate. The limit equilibrium method usually assumes that the mechanical properties of the slope rock mass are rigid bodies and the potential slip surface is flat or curved. However, due to the complex geological conditions of most slopes, the use of this method to assess slopes requires technicians or researchers with extensive experience in slope engineering. The finite element method [6], the strength discounting method [7], the discrete element method [8] and so on are all relatively common methods in numerical analysis.

Since the affecting factors of slope stability contain both the internal features of slopes and the external conditions of slopes, there is the uncertain, inconsistent, and incomplete information of the affecting factors. However, they cannot be well described by traditional analogical approaches. Therefore, to avoid this deficiency of the traditional methods, some researchers proposed uncertainty methods for the analysis and assessment of slope stability. For example, a FAHP (fuzzy analytic hierarchy process) efficiency coefficient approach was used for the stability classification of rock slope [9], the artificial neural network (ANN) and fuzzy clustering methods were applied to the estimation of rainfall-induced landslides [10]. Then, clustering methods using the adaptive neuro-fuzzy inference system (ANFIS) and k-means and fuzzy c-means were applied to the clustering analysis of slope stability [11, 12]. Recently, the similarity measures of interval-valued fuzzy credibility sets were presented and applied to the slope stability assessment [13]. Since a neutrosophic number (NN) $y = v + qI$ for $I \circledcirc\in$ [inf I, sup I] [14, 15, 16], which is composed of the certain part v and the uncertain part qI with uncertainty I, can better describe the uncertainty of a real thing, Li et al. [17] introduced a probabilistic method based on NNs for the assessment of rock slope stability. Subsequently, Zhou et al. [18] and Li et al. [19] proposed some similarity measures of NNs to assess the slope stability of open-pit mines. However, these clustering analysis/assessment methods only contain fuzzy/uncertain information, but do not take into account the true, false, and uncertain information about the factors that affect slope stability.

In view of an extension of the fuzzy set (FS) [20] and (interval-valued) intuitionistic FS [21, 22], a neutrosophic set (NS) concept was proposed by Smarandache [14] and described by the true, false, and uncertain membership functions. To better apply NSs in practical engineering, Wang et al. [23, 24] proposed single-valued NSs (SVNSs) and interval NSs (IVNSs) as the subclasses of NSs. Recently, Qin et al. [25] first applied SVNS and ANFIS to open-pit mine slope stability evaluation and proposed a SVNS-ANFIS evaluation approach for assessing slope stability. Moreover, Qin et al. [26] introduced a SVNS-GPR (gaussian process regression) approach for the aassessment of open-pit mine slope stability in terms of the potential relationships between the affecting factors and the slope stability. Ding and Ye [27] presented the clustering analysis and evaluation models of slope stability based on the hyperbolic sine similarity measure (HSSM) of SVNSs and its netting clustering and assessment approaches, then applied them to the stability clustering analysis and assessment of slope sample data.

Based on the previous studies, the SVNN-ANFIS and SVNS-GPR methods [25, 26] need a lot of slope sample data to train them, and their modeling algorithms imply complexity, which presents the difficult problems of extensive sampling work and a complex training process in engineering applications. Then in existing slope stability clustering analysis and evaluation models [27], it is difficult to select many different types of true, false, and uncertain membership functions in the fuzzified process of ample slope data, which will greatly increase difficulty in practical engineering applications. To solve these problems, the objective of this paper is to propose an inverse hyperbolic sine similarity measure (IHSSM) of SVNSs and its clustering analysis and evaluation models of slope stability by a unique type of Gaussian membership functions for fuzzifying slope sample data into SVNSs, including the true, false, and uncertain membership values.

In our study, we first propose the IHSSM of SVNSs and its netting clustering and evaluation models of slope stability based on the true, false, and uncertain Gaussian membership functions for fuzzifying slope sample data into SVNSs. Through 20 slope samples collected in the Zhejiang Province, China, the proposed models are used for the clustering analysis and evaluation of their stability. By comparing with existing relevant approaches, we verify the accuracy and rationality of the proposed models in the clustering and evaluation applications of the 20 slope samples.

The rest of this paper consists of the following sections. In Section 2, some preliminaries of SVNSs are introduced. Section 3 proposes the IHSSM of SVNSs and its netting clustering model for slope stability clustering analysis. In Section 4, an assessment model based on the IHSSM of SVNSs and the clustered results is proposed for the stability assessment of slopes. Section 5 applies the proposed clustering and assessment models to the stability clustering analysis and assessment of the 20 slope samples as the case study to verify the consistency and accuracy of the clustered and evaluated results by comparing the 20 slope samples with the existing approaches. Section 6 concludes the paper and provides further research directions in the future.

2. Some preliminaries of SVNSs

Wang et al. [24] introduced SVNS as a subclass of NS.

Definition 1 [24]. Let Ψ be a universal set. The SVNS Φ in Ψ can be represented as $\Phi = \{<\psi, X_\Phi(\psi), Y_\Phi(\psi), Z_\Phi(\psi)> \mid \psi \in \Psi\}$, where $X_\Phi(\psi), Y_\Phi(\psi), Z_\Phi(\psi) \in [0, 1]$ for $\psi \in \Psi$ are the true, uncertain, and false membership functions. Then, each element $\varphi = <\psi, X_\Phi(\psi), Y_\Phi(\psi), Z_\Phi(\psi)>$ in the SVNS Φ is represented as the single-valued neutrosophic number (SVNN) $\varphi = <X, Y, Z>$.

Definition 2 [24]. If there are two SVNSs $\varphi_1 = <X_1, Y_1, Z_1>$ and $\varphi_2 = <X_2, Y_2, Z_2>$, then they include the following relations:

(1) Mutual inclusion: $\varphi_1 \subseteq \varphi_2$ if and only if $X_1 \le X_2, Y_1 \ge Y_2, Z_1 \ge Z_2$;

(2) Mutual equality: $\varphi_1 = \varphi_2$ if and only if $\varphi_1 \subseteq \varphi_2$ and $\varphi_2 \subseteq \varphi_1$;

(3) The complement of φ_1: $\varphi_1^c = \langle Z_1, 1 - Y_1, X_1 \rangle$;

(4) Union: $\varphi_1 \bigcup \varphi_2 = \langle X_1 \vee X_2, Y_1 \wedge Y_2, Z_1 \wedge Z_2 \rangle$;

(5) Intersection: $\varphi_1 \bigcap \varphi_2 = \langle X_1 \wedge X_2, Y_1 \vee Y_2, Z_1 \vee Z_2 \rangle$;

Definition 3 [28]. Assume that $\Phi_1 = \{\varphi_{11}, \varphi_{12}, \ldots, \varphi_{1n}\}$ and $\Phi_2 = \{\varphi_{21}, \varphi_{22}, \ldots, \varphi_{2n}\}$ are two SVNSs, where $\varphi_{1i} = <X_{1i}, Y_{1i}, Z_{1i}>$ and $\varphi_{2i} = <X_{2i}, Y_{2i}, Z_{2i}>$ ($i = 1, 2, \ldots, n$) are two SVNNs. Then, $\rho_i \in [0, 1]$ with $\sum_{i=1}^{n} \rho_i = 1$ is the weight of φ_{1i} and φ_{2i}. The weighted generalized distance between Φ_1 and Φ_2 is defined as follows:

$$V_\delta(\Phi_1, \Phi_2) = \left\{ \frac{1}{3} \sum_{i=1}^{n} \rho_i \left[\left| X_{1i} - X_{2i} \right|^\delta + \left| Y_{1i} - Y_{2i} \right|^\delta + \left| Z_{1i} - Z_{2i} \right|^\delta \right] \right\}^{1/\delta} \text{ for } \delta \ge 1. \quad (1)$$

The distance of Eq. (1) satisfies the following features:

(1) $0 \le V_\delta(\Phi_1, \Phi_2) \le 1$;

(2) $V_\delta(\Phi_1, \Phi_2) = 0$ if and only if $\Phi_1 = \Phi_2$;

(3) $V_\delta(\Phi_1, \Phi_2) = V_\delta(\Phi_2, \Phi_1)$;

(4) If $\Phi_1 \subseteq \Phi_2 \subseteq \Phi_3$ for any SVNS Φ_3, then $V_\delta(\Phi_1, \Phi_3) \ge V_\delta(\Phi_1, \Phi_2)$ and $V_\delta(\Phi_1, \Phi_3) \ge V_\delta(\Phi_2, \Phi_3)$.

Since the similarity measure and the distance are complementary, the similarity measure using the weighted generalized distance of SVNSs is presented as follows [28]:

$$W_\delta(\Phi_1, \Phi_2) = 1 - V_\delta(\Phi_1, \Phi_2) = 1 - \left\{ \frac{1}{3} \sum_{i=1}^{n} \rho_i \left[\left| X_{1i} - X_{2i} \right|^\delta + \left| Y_{1i} - Y_{2i} \right|^\delta + \left| Z_{1i} - Z_{2i} \right|^\delta \right] \right\}^{1/\delta}. \quad (2)$$

Thus, Eq. (2) also satisfies the following features [28]:

(1) $0 \le W_\delta(\Phi_1, \Phi_2) \le 1$;

(2) $W_\delta(\Phi_1, \Phi_2) = 1$ if and only if $\Phi_1 = \Phi_2$;

(3) $W_\delta(\Phi_1, \Phi_2) = W_\delta(\Phi_2, \Phi_1)$;

(4) If $\Phi_1 \subseteq \Phi_2 \subseteq \Phi_3$ for any SVNS Φ_3, then $W_\delta(\Phi_1,\Phi_2) \ge W_\delta(\Phi_1,\Phi_3)$ and $W_\delta(\Phi_2,\Phi_3) \ge W_\delta(\Phi_1,\Phi_3)$.

Recently, Ding and Ye [27] further presented the HSSM between the SVNSs Φ_1 and Φ_2:

$$N_\delta(\Phi_1,\Phi_2) = 1 - \sinh\left\{(\ln(1+\sqrt{2})V_\delta(\Phi_1,\Phi_2)\right\}$$
$$= 1 - \sinh\left\{\ln(1+\sqrt{2})\left(\frac{1}{3}\sum_{i=1}^{n}\rho_i\left[\left|X_{1i}-X_{2i}\right|^\delta + \left|Y_{1i}-Y_{2i}\right|^\delta + \left|Z_{1i}-Z_{2i}\right|^\delta\right]\right)^{1/\delta}\right\}. \quad (3)$$

Similarly, Eq. (3) also has the following features:

(1) $0 \le N_\delta(\Phi_1,\Phi_2) \le 1$;

(2) $N_\delta(\Phi_1,\Phi_2) = 1$ if and only if $\Phi_1 = \Phi_2$;

(3) $N_\delta(\Phi_1,\Phi_2) = N_\delta(\Phi_2,\Phi_1)$;

(4) If $\Phi_1 \subseteq \Phi_2 \subseteq \Phi_3$ for any SVNS Φ_3, then $N_\delta(\Phi_1,\Phi_2) \ge N_\delta(\Phi_1,\Phi_3)$ and $N_\delta(\Phi_2,\Phi_3) \ge N_\delta(\Phi_1,\Phi_3)$.

3. IHSSM between SVNSs and Its Netting Clustering Model of Slope Stability

Based on the weighted generalized distances of SVNSs, this section proposes the IHSSM of SVNSs and its netting clustering model of slope stability.

First, IHSSM for SVNSs is defined below.

Definition 4. Suppose that $\Phi_1 = \{\varphi_{11}, \varphi_{12}, \ldots, \varphi_{1n}\}$ and $\Phi_2 = \{\varphi_{21}, \varphi_{22}, \ldots, \varphi_{2n}\}$ are two SVNSs, where $\varphi_{1i} = \langle X_{1i}, Y_{1i}, Z_{1i}\rangle$ and $\varphi_{2i} = \langle X_{2i}, Y_{2i}, Z_{2i}\rangle$ ($i = 1, 2, \ldots, n$) are two SVNNs. Then, $\rho_i \in [0, 1]$ with $\sum_{i=1}^{n}\rho_i = 1$ is the weight of φ_{1i} and φ_{2i}. Thus, the weighted IHSSM between Φ_1 and Φ_2 is defined as follows:

$$M_\delta(\Phi_1,\Phi_2) = 1 - \frac{1}{\ln(1+\sqrt{2})}\sinh^{-1}\left\{V_\delta(\Phi_1,\Phi_2)\right\}$$
$$= 1 - \frac{1}{\ln(1+\sqrt{2})}\sinh^{-1}\left\{\frac{1}{3}\sum_{i=1}^{n}\rho_i\left[\left|X_{1i}-X_{2i}\right|^\delta + \left|Y_{1i}-Y_{2i}\right|^\delta + \left|Z_{1i}-Z_{2i}\right|^\delta\right]\right\}^{1/\delta}. \quad (4)$$

Then, the features of Eq. (4) are indicated as follows:

(1) $0 \le M_\delta(\Phi_1,\Phi_2) \le 1$;

(2) $M_\delta(\Phi_1,\Phi_2) = 1$ if and only if $\Phi_1 = \Phi_2$;

(3) $M_\delta(\Phi_1,\Phi_2) = M_\delta(\Phi_2,\Phi_1)$;

(4) If $\Phi_1 \subseteq \Phi_2 \subseteq \Phi_3$ for any SVNS Φ_3, then $M_\delta(\Phi_1,\Phi_2) \ge M_\delta(\Phi_1,\Phi_3)$ and $M_\delta(\Phi_2,\Phi_3) \ge M_\delta(\Phi_1,\Phi_3)$.

Proof: Since the features (1)–(3) are clearly valid, we only verify the feature (4).

According to the features of the distance $V_\delta(\Phi_1, \Phi_2)$, if $\Phi_1 \subseteq \Phi_2 \subseteq \Phi_3$, then $V_\delta(\Phi_1, \Phi_3) \ge V_\delta(\Phi_1, \Phi_2)$ and $V_\delta(\Phi_2, \Phi_3) \ge V_\delta(\Phi_1, \Phi_3)$ exist. Since $\sinh^{-1}(\alpha)$ for $\alpha \in [0, 1]$ is monotonically increasing, the inequalities $M_\delta(\Phi_1, \Phi_2) \ge M_\delta(\Phi_1, \Phi_3)$ and $M_\delta(\Phi_2, \Phi_3) \ge M_\delta(\Phi_1, \Phi_3)$ exist based on the complementary relationship between the distance and the similarity measure [28]. Hence, we verify that the feature (4) is correct.

In terms of the proposed weighted IHSSM of SVNSs, a netting clustering model is presented below to cluster slope sample data.

In the clustering analysis of slope sample data, $\Omega = \{\Omega_1, \Omega_2, \ldots, \Omega_m\}$ is a sample set of m slopes and $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is a set of n indices/factors that impact slope stability. Then, each affecting factor λ_i needs to consider its weight ρ_i subject to $\rho_i \in [0, 1]$ and $\sum_{i=1}^{n}\rho_i = 1$.

In order to better express the uncertain and inconsistent information of the slop sample data, we use Gaussian membership functions (e.g. Figure 1) to fuzzify the true, uncertain, and false information of the sample data. In view of the true, false, and uncertain Gaussian membership functions, the affecting factors of slope stability are fuzzified into the SVNSs $\Phi_j = \{\varphi_{j1}, \varphi_{j2}, \ldots, \varphi_{jn}\}$, where $\varphi_{ji} = <X_{ji}, Y_{ji}, Z_{ji}>$ are SVNNs for $X_{ji}, Y_{ji}, Z_{ji} \in [0, 1]$ ($j = 1, 2, \ldots, m$; $i = 1, 2, \ldots, n$).

Then, the netting clustering model based on the IHSSM of SVNSs is used for the clustering analysis of the slope sample data and presented by the following clustering procedures.

Step 1: Obtain the IHSSM matrix $B = (\beta_{js})_{m\times m}$ ($j, s = 1, 2, \ldots, m$) by Eq. (4) (usually taking $\delta = 1$), where $\beta_{js} = M_\delta(\Phi_j, \Phi_s)$ with $\beta_{js} = \beta_{sj}$ and $\beta_{jj} = 1$.

Step 2: Replace all the diagonal elements in the IHSSM matrix B by the slope samples Ω_j.

Step 3: In terms of different confidence levels of ζ, the corresponding ζ-cutting matrices $B^\zeta=(\beta_{js}^\zeta)_{m\times m}$ are gained by the equation:

$$\beta_{js}^{\zeta} = \begin{cases} 0, \beta_{js} < \zeta \\ 1, \beta_{js} \ge \zeta \end{cases} (j, s = 1, 2, \ldots, m). \tag{5}$$

In the adjusted process for ζ, all '0' are funded in the ζ-cutting matrixes and deleted, then all '1' are replaced by '*' except for the diagonal elements. Then, '*' is connected to the corresponding diagonal elements by horizontal and vertical lines. The slope samples connected by '*' are formed as a classification according to the corresponding confidence level ζ. Next, the confidence level of ζ is adjusted from large to small until the expected clustering result for the slope sample data is achieved.

4. Slope Stability Assessment Using the IHSSM of SVNSs

Because the above clustered results cannot indicate their corresponding stability risk levels of slope samples, it is necessary to evaluate the stability risk levels of the slope samples so that we can decide which risk levels the slope samples belong to. In this section, we give an assessment model of slope stability based on the IHSSM of SVNSs.

In terms of the existing knowledge and the above clustered results of slope stability, we can classify the risk states of slope stability into the corresponding risk levels (risk patterns), which can be represented as the SVNSs $\Theta_k = \{\theta_{k1}, \theta_{k2}, \ldots, \theta_{kn}\}$ including the SVNNs $\theta_{ki} = <X_{ki}, Y_{ki}, Z_{ki}>$ for $X_{ki}, Y_{ki}, Z_{ki} \in [0, 1]$ ($k = 1, 2, \ldots, r$; $i = 1, 2, \ldots, n$).

Assume that $\Omega = \{\Omega_1, \Omega_2, \ldots, \Omega_m\}$ is a sample set of m slopes and a set of n affecting factors $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ affects slope stability. Based on the true, false and uncertain Gaussian membership functions, the slope sample data can be fuzzified into the SVNSs $\Phi_j = \{\varphi_{j1}, \varphi_{j2}, \ldots, \varphi_{jn}\}$ ($j = 1, 2, \ldots, m$), which contains the SVNNs $\varphi_{ji} = <X_{ji}, Y_{ji}, Z_{ji}>$ for $X_{ji}, Y_{ji}, Z_{ji} \in [0, 1]$ ($i = 1, 2, \ldots, n$).

Considering the weight of each affecting factor in the slope stability assessment, we assign the weight of each affecting factor by $\rho_i \in [0,1]$ with $\sum_{i=1}^{n}\rho_i = 1$. Thus, the weighted IHSSM between each SVNS Φ_j ($j = 1, 2, \ldots, n$) for each slope sample Ω_j and each slope stability risk level Θ_k ($k = 1, 2, \ldots, r$) is presented by the following equation:

$$\begin{aligned} M_\delta(\Phi_j, \Theta_k) &= 1 - \frac{1}{\ln(1+\sqrt{2})}\sinh^{-1}\left\{V_\delta(\Phi_j, \Theta_k)\right\} \\ &= 1 - \frac{1}{\ln(1+\sqrt{2})}\sinh^{-1}\left\{\frac{1}{3}\sum_{i=1}^{n}\rho_i\left[\left|X_{ji}-X_{ki}\right|^\delta + \left|Y_{ji}-Y_{ki}\right|^\delta + \left|Z_{ji}-Z_{ki}\right|^\delta\right]\right\}^{1/\delta} \end{aligned}. \tag{6}$$

Applying Eq. (6), we can obtain the IHSSM results, and then use $M_\delta(\Phi_j, \Theta_{k^*}) = \max_{1\le k\le r}\left(M_\delta(\Phi_j, \Theta_k)\right)$ to judge that Ω_j should belong to Θ_{k^*}.

5. Netting Clustering and Assessment applications of Practical Cases

5.1. Netting Clustering Analysis of Practical Cases

The mountainous terrain of Zhejiang Province in China, with its subtropical monsoon climate, accounts for 74.6%, making slope instability a very common geological hazard. Therefore, the clustering analysis and assessment of slope stability show their importance and necessity. In terms of rock and topographic features in Zhejiang Province, we take into account the lithological association (λ_1), slope structure (λ_2), weathering degree of rock (λ_3), slope height (λ_4), slope angle (λ_5), and vegetation coverage (λ_6) as the main factors affecting slope stability. According to the important degrees of these affecting factors, their weight vector is assigned as $\boldsymbol{\rho}$ = (0.25, 0.21, 0.19, 0.13, 0.11, 0.11) by experts/decision makers. Then, we collected 20 slope samples as actual cases in Zhejiang Province. According to the results of engineering investigation and measurement, we also provide the actual data of 20 slope samples Ω_j (j = 1, 2, …, 20) for the clustering analysis. We can score from 0 to 10 depending on the current situation of slope stability. Thus, λ_1, λ_2, λ_3, and λ_6 are assigned by the score values, and then λ_4, and λ_5 are given by the actual measured values, which are shown in Table 1.

Table 1. Actual data of 20 slope samples

Ω_j	λ_1	λ_2	λ_3	λ_4 (m)	λ_5 (°)	λ_6
Ω_1	2.0	5.0	4.0	15.0	84.0	3.0
Ω_2	4.0	3.0	4.0	8.0	84.0	3.0
Ω_3	8.0	7.0	6.0	15.0	55.0	4.0
Ω_4	3.0	4.0	5.0	9.0	76.0	4.0
Ω_5	2.0	4.0	4.0	8.0	84.0	5.0
Ω_6	10.0	8.0	10.0	36.0	63.0	5.0
Ω_7	7.0	7.0	6.0	23.0	76.0	5.0
Ω_8	8.0	8.0	7.2	32.0	63.0	3.0
Ω_9	5.0	9.0	4.0	15.0	63.0	5.0
Ω_{10}	3.0	3.0	6.0	11.0	76.0	4.0
Ω_{11}	3.0	6.0	5.0	21.0	71.0	5.0
Ω_{12}	7.0	10.0	7.0	26.8	76.0	5.0
Ω_{13}	9.0	10.0	10.0	28.0	76.0	5.0
Ω_{14}	5.0	10.0	3.0	19.0	73.0	3.0
Ω_{15}	4.0	2.0	3.0	18.0	74.0	3.0
Ω_{16}	2.0	6.0	3.0	24.0	45.0	4.0
Ω_{17}	7.0	9.0	6.0	60.0	70.0	4.0
Ω_{18}	7.0	8.0	8.0	32.0	72.0	4.0
Ω_{19}	4.0	4.0	4.0	19.0	39.0	4.0
Ω_{20}	3.0	4.0	3.0	17.0	83.0	5.0

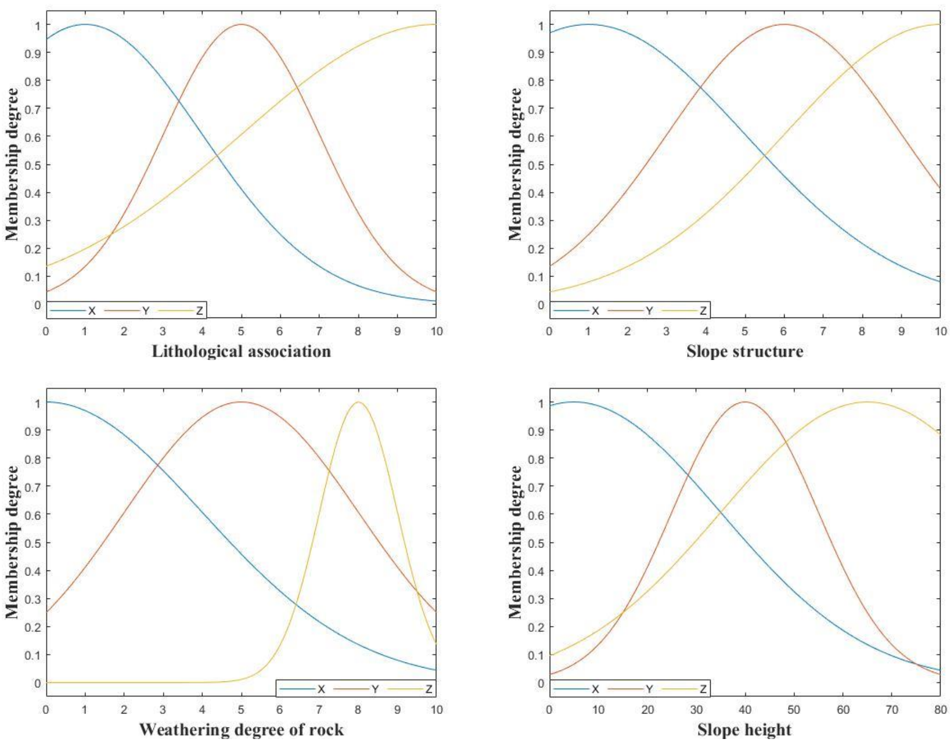
Table 2. Gaussian membership functions of the 6 affecting factors

Affecting factor	Gaussian membership function		
	X	Y	Z
(λ_1)	gaussmf[3 1]	gaussmf[2 5]	gaussmf[5 10]
(λ_2)	gaussmf[4 1]	gaussmf[3 6]	gaussmf[4 10]
(λ_3)	gaussmf[4 0]	gaussmf[3 5]	gaussmf[1 8]
(λ_4)	gaussmf[30 5]	gaussmf[15 40]	gaussmf[30 65]
(λ_5)	gaussmf[15 30]	gaussmf[25 50]	gaussmf[30 45]
(λ_6)	gaussmf[4 0]	gaussmf[3 5.5]	gaussmf[2 10]

In terms of the data of the 6 affecting factors, we use Gaussian membership functions in Table 2 to fuzzify them into the form of SVNNs. In Table 2, the true, uncertain, and false Gaussian membership functions of the six affecting factors are provided to fuzzify the slope sample data, and then the Gaussian membership degree curves of truth (X), uncertainty (Y), and falsity (Z) are shown in Figure 1. Thus, the 6 affecting factors of the 20 slope samples Ω_j (j = 1, 2, …, 20) are fuzzified into the corresponding SVNSs Φ_j, which are shown in Table 3.

Table 3. SVNSs of 20 slope samples

Φ_j	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
Φ_1	(0.946, 0.325, 0.278)	(0.607, 0.946, 0.458)	(0.607, 0.946, 0.000)	(0.946, 0.249, 0.249)	(0.002, 0.397, 0.430)	(0.755, 0.707, 0.002)
Φ_2	(0.607, 0.882, 0.487)	(0.882, 0.607, 0.216)	(0.607, 0.946, 0.000)	(0.995, 0.103, 0.164)	(0.002, 0.397, 0.430)	(0.755, 0.707, 0.002)
Φ_3	(0.066, 0.325, 0.923)	(0.325, 0.946, 0.755)	(0.325, 0.946, 0.135)	(0.946, 0.249, 0.249)	(0.249, 0.980, 0.946)	(0.607, 0.882, 0.011)
Φ_4	(0.801, 0.607, 0.375)	(0.755, 0.801, 0.325)	(0.458, 1.000, 0.011)	(0.991, 0.118, 0.175)	(0.009, 0.582, 0.586)	(0.607, 0.882, 0.011)
Φ_5	(0.946, 0.325, 0.278)	(0.755, 0.801, 0.325)	(0.607, 0.946, 0.000)	(0.995, 0.103, 0.164)	(0.002, 0.397, 0.430)	(0.458, 0.986, 0.044)
Φ_6	(0.011, 0.044, 1.000)	(0.216, 0.801, 0.882)	(0.044, 0.249, 0.135)	(0.586, 0.965, 0.627)	(0.089, 0.874, 0.835)	(0.458, 0.986, 0.044)
Φ_7	(0.135, 0.607, 0.835)	(0.325, 0.946, 0.755)	(0.325, 0.946, 0.135)	(0.835, 0.526, 0.375)	(0.009, 0.582, 0.586)	(0.458, 0.986, 0.044)
Φ_8	(0.066, 0.325, 0.923)	(0.216, 0.801, 0.882)	(0.216, 0.801, 0.607)	(0.667, 0.867, 0.546)	(0.089, 0.874, 0.835)	(0.755, 0.707, 0.002)
Φ_9	(0.411, 1.000, 0.607)	(0.135, 0.607, 0.969)	(0.607, 0.946, 0.000)	(0.946, 0.249, 0.249)	(0.089, 0.874, 0.835)	(0.458, 0.986, 0.044)
Φ_{10}	(0.801, 0.607, 0.375)	(0.882, 0.607, 0.216)	(0.325, 0.946, 0.135)	(0.980, 0.154, 0.198)	(0.009, 0.582, 0.586)	(0.607, 0.882, 0.011)
Φ_{11}	(0.801, 0.607, 0.375)	(0.458, 1.000, 0.607)	(0.458, 1.000, 0.011)	(0.867, 0.448, 0.341)	(0.024, 0.703, 0.687)	(0.458, 0.986, 0.044)
Φ_{12}	(0.135, 0.607, 0.835)	(0.080, 0.411, 1.000)	(0.216, 0.801, 0.607)	(0.768, 0.679, 0.445)	(0.009, 0.582, 0.586)	(0.458, 0.986, 0.044)
Φ_{13}	(0.029, 0.135, 0.980)	(0.080, 0.411, 1.000)	(0.044, 0.249, 0.135)	(0.745, 0.726, 0.467)	(0.009, 0.582, 0.586)	(0.458, 0.986, 0.044)
Φ_{14}	(0.411, 1.000, 0.607)	(0.080, 0.411, 1.000)	(0.755, 0.801, 0.000)	(0.897, 0.375, 0.309)	(0.016, 0.655, 0.647)	(0.755, 0.707, 0.002)
Φ_{15}	(0.607, 0.882, 0.487)	(0.969, 0.411, 0.135)	(0.755, 0.801, 0.000)	(0.910, 0.341, 0.293)	(0.014, 0.631, 0.627)	(0.755, 0.707, 0.002)
Φ_{16}	(0.946, 0.325, 0.278)	(0.458, 1.000, 0.607)	(0.755, 0.801, 0.000)	(0.818, 0.566, 0.393)	(0.607, 0.980, 1.000)	(0.607, 0.882, 0.011)
Φ_{17}	(0.135, 0.607, 0.835)	(0.135, 0.607, 0.969)	(0.325, 0.946, 0.135)	(0.186, 0.411, 0.986)	(0.029, 0.726, 0.707)	(0.607, 0.882, 0.011)
Φ_{18}	(0.135, 0.607, 0.835)	(0.216, 0.801, 0.882)	(0.135, 0.607, 1.000)	(0.667, 0.867, 0.546)	(0.020, 0.679, 0.667)	(0.607, 0.882, 0.011)
Φ_{19}	(0.607, 0.882, 0.487)	(0.755, 0.801, 0.325)	(0.607, 0.946, 0.000)	(0.897, 0.375, 0.309)	(0.835, 0.908, 0.980)	(0.607, 0.882, 0.011)
Φ_{20}	(0.801, 0.607, 0.375)	(0.755, 0.801, 0.325)	(0.755, 0.801, 0.000)	(0.923, 0.309, 0.278)	(0.002, 0.418, 0.448)	(0.458, 0.986, 0.044)



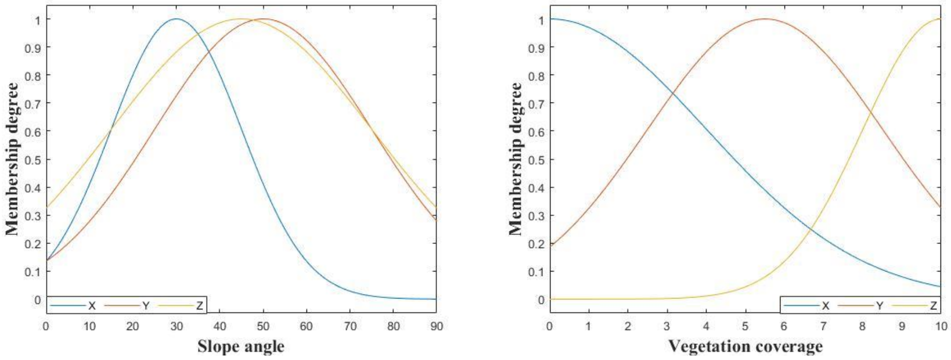


Figure 1. The Gaussian membership degree curves of the 6 affecting factors

Thus, we can calculate the IHSSM values between SVNSs of the slope samples by Eq. (4) for δ = 1, and then establish the IHSSM matrix B:

$B=$

1.0000	0.8144	0.7130	0.8608	0.9266	0.5175	0.7059	0.6064	0.6874	0.8159	0.8395	0.5655	0.5306	0.6784	0.7525	0.8356	0.5959	0.5543	0.7591	0.8629
0.8144	1.0000	0.6336	0.8659	0.8361	0.4634	0.6775	0.5506	0.7618	0.8819	0.7536	0.5884	0.5042	0.7527	0.9095	0.6540	0.6190	0.5484	0.8483	0.8435
0.7130	0.6336	1.0000	0.6995	0.6689	0.7632	0.8801	0.8300	0.7588	0.6914	0.7435	0.7345	0.7268	0.6841	0.6088	0.7345	0.7894	0.7380	0.7079	0.6548
0.8608	0.8659	0.6995	1.0000	0.9073	0.5259	0.7441	0.5889	0.7256	0.9400	0.8862	0.6242	0.5390	0.6898	0.8080	0.7542	0.6551	0.6127	0.8420	0.9211
0.9266	0.8361	0.6689	0.9073	1.0000	0.5186	0.6847	0.5593	0.6885	0.8554	0.8177	0.5666	0.5316	0.6303	0.7470	0.7905	0.5747	0.5334	0.7808	0.9085
0.5175	0.4634	0.7632	0.5259	0.5186	1.0000	0.7491	0.8373	0.6403	0.5182	0.6025	0.7274	0.8837	0.5867	0.4738	0.5818	0.7326	0.7839	0.5318	0.5342
0.7059	0.6775	0.8801	0.7441	0.6847	0.7491	1.0000	0.7923	0.7765	0.7360	0.8283	0.8527	0.7630	0.7396	0.6611	0.6958	0.8501	0.8172	0.7002	0.7325
0.6064	0.5506	0.8300	0.5889	0.5593	0.8373	0.7923	1.0000	0.6825	0.5811	0.6443	0.8371	0.7716	0.6774	0.5613	0.6459	0.7834	0.8788	0.5949	0.5751
0.6874	0.7618	0.7588	0.7256	0.6885	0.6403	0.7765	0.6825	1.0000	0.7116	0.7705	0.7475	0.6595	0.8994	0.7363	0.6784	0.7791	0.6723	0.7827	0.7257
0.8159	0.8819	0.6914	0.9400	0.8554	0.5182	0.7360	0.5811	0.7116	1.0000	0.8335	0.6455	0.5598	0.6758	0.8307	0.7101	0.6766	0.6049	0.7973	0.8759
0.8395	0.7536	0.7435	0.8862	0.8177	0.6025	0.8283	0.6443	0.7705	0.8335	1.0000	0.6840	0.5974	0.7259	0.7445	0.8430	0.7079	0.6647	0.7960	0.8667
0.5655	0.5884	0.7345	0.6242	0.5666	0.7274	0.8527	0.8371	0.7475	0.6455	0.6840	1.0000	0.8417	0.7748	0.6212	0.5821	0.8340	0.8622	0.5816	0.6327
0.5306	0.5042	0.7268	0.5390	0.5316	0.8837	0.7630	0.7716	0.6595	0.5598	0.5974	0.8417	1.0000	0.6863	0.5361	0.5470	0.7489	0.7537	0.4976	0.5473
0.6784	0.7527	0.6841	0.6898	0.6303	0.5867	0.7396	0.6774	0.8994	0.6758	0.7259	0.7748	0.6863	1.0000	0.8155	0.6672	0.7538	0.6751	0.7300	0.7183
0.7525	0.9095	0.6088	0.8080	0.7470	0.4738	0.6611	0.5613	0.7363	0.8307	0.7445	0.6212	0.5361	0.8155	1.0000	0.6856	0.6102	0.5591	0.8390	0.8371
0.8356	0.6540	0.7345	0.7542	0.7905	0.5818	0.6958	0.6459	0.6784	0.7101	0.8430	0.5821	0.5470	0.6672	0.6856	1.0000	0.6009	0.5855	0.7831	0.7831
0.5959	0.6190	0.7894	0.6551	0.5747	0.7326	0.8501	0.7834	0.7791	0.6766	0.7079	0.8340	0.7489	0.7538	0.6102	0.6009	1.0000	0.8005	0.6345	0.6212
0.5543	0.5484	0.7380	0.6127	0.5334	0.7839	0.8172	0.8788	0.6723	0.6049	0.6647	0.8622	0.7537	0.6751	0.5591	0.5855	0.8005	1.0000	0.5849	0.5988
0.7591	0.8483	0.7079	0.8420	0.7808	0.5318	0.7002	0.5949	0.7827	0.7973	0.7960	0.5816	0.4976	0.7300	0.8390	0.7831	0.6345	0.5849	1.0000	0.8294
0.8629	0.8435	0.6548	0.9211	0.9085	0.5342	0.7325	0.5751	0.7257	0.8759	0.8667	0.6327	0.5473	0.7183	0.8371	0.7831	0.6212	0.5988	0.8294	1.0000

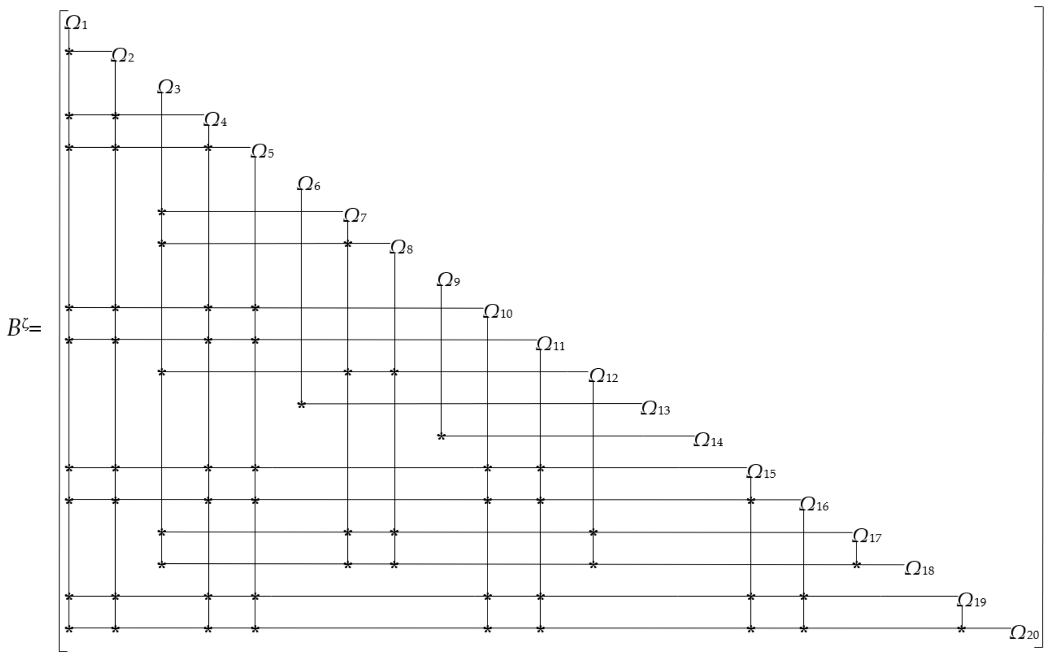


Figure 2. Clustering results based on the proposed clustering model

Using Eq. (5), we can perform the netting clustering analysis and indicate the clustering results in Figure 2, where when the confidence level takes 0.8430 ≤ ζ ≤ 1, the 20 slope samples are divided into four risk levels. In Figure 2, it is seen that {Ω_1, Ω_2, Ω_4, Ω_5, Ω_{10}, Ω_{11}, Ω_{15}, Ω_{16}, Ω_{19}, Ω_{20}}, {Ω_3, Ω_7, Ω_8, Ω_{12}, Ω_{17}, Ω_{18}}, {Ω_6, Ω_{13}}, and {Ω_9, Ω_{14}} belong to four different risk classifications, respectively.

5.2. Slope Stability Assessment of Practical Cases

According to the Chinese standard for the engineering classification of rock mass (GB/T 50218 – 2014) and the historical experience of slope stability, we provide six affecting factors and four risk levels of slope stability. Furthermore, according to the actual situation of slope engineering, we can assess the four affecting factors λ_1, λ_2, λ_3, and λ_6 in the four risk levels from 0 to 10 slope stability scores: the stable score is 0–3; the basic stable score is 3–6; the relatively unstable score is 6–8; the unstable score is 8–10. Then, the two affecting factors λ_4 and λ_5 are the actual measured values. Subsequently, all the obtained results are shown in Table 4.

Table 4. Affecting factors and risk levels of slope stability

Affecting factor	Stable state	Basic stable state	Relatively unstable state	Unstable state
Lithological association	Hard rock (0–3)	Sub-hard rock (3–6)	Sub-soft rock (6–8)	Soft rock (8–10)
Slope structure	Homogeneous structure (0–3)	Blocky structure (3–6)	Stratified structure (6–8)	Loose structure (8–10)
Weathering degree of rock	Weak weathering (0–3)	Moderate weathering (3–6)	Intense weathering (6–8)	Complete weathering (8–10)
Slope height (m)	0–20	20–40	40–60	60–80
Slope angle (°)	0–10	10–30	30–60	60–90
Vegetation coverage	Very high (0–3)	High (3–6)	Low (6–8)	Very low (8–10)

From the above classified results of slope stability, we can classify the risk states of slope stability into four categories/levels: the unstable state (Θ_1), the relatively stable state (Θ_2), the basically stable state (Θ_3), and the stable state (Θ_4), where the unstable state implies that the slope is damaged or prone to damage, the basically stable and relatively unstable states reflect that the slope is between safety and damaged, then the stable state means that the slope is safe.

Based on the knowledge and experience of slope stability, the 6 affecting factors of the 4 risk levels are fuzzified into SVNNs in Table 5, then their SVNSs are represented as follows:

Θ_1 = {s_{11}, s_{12}, s_{13}, s_{14}, s_{15}, s_{16}} = {<0.02, 0.90, 0.99>, <0.15, 0.61, 0.94>, <0.04, 0.25, 0.14>, <0.67, 0.85, 0.55>, <0.05, 0.73, 0.71>, <0.46, 0.99, 0.04>};

Θ_2 = {s_{21}, s_{22}, s_{23}, s_{24}, s_{25}, s_{26}} = {<0.11, 0.51, 0.86>, <0.22, 0.73, 0.86>, <0.26, 0.84, 0.44>, <0.68, 0.60, 0.52>, <0.07, 0.74, 0.72>, <0.58, 0.89, 0.02>};

Θ_3 = {s_{31}, s_{32}, s_{33}, s_{34}, s_{35}, s_{36}} = {<0.41, 1.00, 0.61>, <0.11, 0.51, 0.98>, <0.39, 0.97, 0.07>, <0.92, 0.31, 0.28>, <0.05, 0.76, 0.74>, <0.68, 0.79, 0.01>};

Θ_4 = {s_{41}, s_{42}, s_{43}, s_{44}, s_{45}, s_{46}} = {<0.79, 0.61, 0.38>, <0.73, 0.78, 0.35>, <0.59, 0.91, 0.02>, <0.93, 0.28, 0.26>, <0.15, 0.60, 0.62>, <0.61, 0.86, 0.02>}.

Table 1. SVNNs of the 4 risk levels for slope stability

θ_{ki}	Θ_1			Θ_2			Θ_3			Θ_4		
	X_{k1}	Y_{k1}	Z_{k1}	X_{k2}	Y_{k2}	Z_{k2}	X_{k3}	Y_{k3}	Z_{k3}	X_{k4}	Y_{k4}	Z_{k4}
θ_{k1}	0.02	0.90	0.99	0.11	0.51	0.86	0.41	1.00	0.61	0.79	0.61	0.38
θ_{k2}	0.15	0.61	0.94	0.22	0.73	0.86	0.11	0.51	0.98	0.73	0.78	0.35
θ_{k3}	0.04	0.25	0.14	0.26	0.84	0.44	0.39	0.97	0.07	0.59	0.91	0.02
θ_{k4}	0.67	0.85	0.55	0.68	0.60	0.52	0.92	0.31	0.28	0.93	0.28	0.26
θ_{k5}	0.05	0.73	0.71	0.07	0.74	0.72	0.05	0.76	0.74	0.15	0.60	0.62
θ_{k6}	0.46	0.99	0.04	0.58	0.89	0.02	0.68	0.79	0.01	0.61	0.86	0.02

Table 2. Assessed results of the risk levels for 20 slope samples

Ω_j	$M_\delta(\Phi_j, \Theta_1)$	$M_\delta(\Phi_j, \Theta_2)$	$M_\delta(\Phi_j, \Theta_3)$	$M_\delta(\Phi_j, \Theta_4)$	Risk level
Ω_1	0.494570	0.623006	0.677886	**0.873638**	Θ_4
Ω_2	0.564537	0.610318	0.752149	**0.853195**	Θ_4
Ω_3	0.713772	**0.832699**	0.756454	0.711081	Θ_2
Ω_4	0.551858	0.665090	0.728133	**0.951406**	Θ_4
Ω_5	0.495114	0.605004	0.642938	**0.886262**	Θ_4
Ω_6	**0.865189**	0.797835	0.626400	0.542880	Θ_1
Ω_7	0.776073	**0.875305**	0.782408	0.746375	Θ_2
Ω_8	0.785857	**0.904950**	0.704720	0.607567	Θ_2
Ω_9	0.743878	0.749130	**0.931893**	0.754172	Θ_3
Ω_{10}	0.572259	0.667849	0.728380	**0.910704**	Θ_4
Ω_{11}	0.628578	0.725045	0.761082	**0.882138**	Θ_4
Ω_{12}	0.803960	**0.884157**	0.773955	0.634420	Θ_2
Ω_{13}	**0.872155**	0.782644	0.685452	0.548984	Θ_1
Ω_{14}	0.714061	0.729656	**0.922054**	0.724081	Θ_3
Ω_{15}	0.575320	0.615914	0.752868	**0.834918**	Θ_4
Ω_{16}	0.534628	0.662498	0.646964	**0.791051**	Θ_4
Ω_{17}	0.793529	**0.865970**	0.808385	0.665402	Θ_2
Ω_{18}	0.810222	**0.888952**	0.700942	0.624027	Θ_2
Ω_{19}	0.582546	0.649881	0.750508	**0.869809**	Θ_4
Ω_{20}	0.559702	0.648989	0.690287	**0.935418**	Θ_4

Calculating the IHSSM values between Θ_k (k=1, 2, 3, 4) and Φ_j (j=1, 2, …, 20) by Eq. (6) for $\delta = 1$, we can gain all the IHSSM values, as shown in Table 6. Therefore, we can use the maximum measure value between Φ_j and Θ_k to decide the risk levels of these slope samples. The assessed results in Table 6 show that the 20 slope samples are clustered into the following four types of risk levels:

(1) {Ω_1, Ω_2, Ω_4, Ω_5, Ω_{10}, Ω_{11}, Ω_{15}, Ω_{16}, Ω_{19}, Ω_{20}} belongs to the risk level Θ_4;

(2) {Ω_3, Ω_7, Ω_8, Ω_{12}, Ω_{17}, Ω_{18}} belongs to the risk level Θ_2;

(3) {Ω_6, Ω_{13}} belongs to the risk level Θ_1;

(4) {Ω_9, Ω_{14}} belongs to the risk level Θ_3.

Clearly, the assessed results and the netting clustering results of the 20 slope samples are identical. Therefore, the slope stability evaluation model based on the netting clustering analysis verifies its rationality and accuracy.

5.3. Relative Comparison

In our comparative analysis, we apply the weighted similarity measures of Eqs. (2) and (3) [27, 28] to the risk level assessment of the 20 slope samples to verify the validity and accuracy of our new slope stability evaluation model.

Regarding the 20 slope samples, we calculate the weighted generalized distance-based similarity measure values between Θ_k (k = 1, 2, 3, 4) and Φ_j (j = 1, 2, …, 20) by Eq. (2) for δ = 1 [28], and then the assessed results are shown in Table 7. It can be clearly seen that there is consistency between the assessed results of Eq. (2) and the assessed results of the proposed IHSSM of SVNSs for the 20 slope samples Ω_j.

Table 7. Assessed results using Eq. (2)

Ω_j	$W_\delta(\Phi_j, \Theta_1)$	$W_\delta(\Phi_j, \Theta_2)$	$W_\delta(\Phi_j, \Theta_3)$	$W_\delta(\Phi_j, \Theta_4)$	Risk level
Ω_1	0.539646	0.661580	0.712268	**0.888397**	Θ_4
Ω_2	0.606702	0.649753	0.779809	**0.870249**	Θ_4
Ω_3	0.745041	**0.852010**	0.783693	0.742593	Θ_2
Ω_4	0.594669	0.700514	0.758084	**0.957157**	Θ_4
Ω_5	0.540175	0.644786	0.680074	**0.899586**	Θ_4
Ω_6	**0.880902**	0.820873	0.664736	0.586118	Θ_1
Ω_7	0.801353	**0.889876**	0.807043	0.774595	Θ_2
Ω_8	0.810137	**0.916128**	0.736800	0.647182	Θ_2
Ω_9	0.772339	0.777083	**0.939936**	0.781635	Θ_3
Ω_{10}	0.614006	0.703051	0.758308	**0.921215**	Θ_4
Ω_{11}	0.666760	0.755283	0.787864	**0.895933**	Θ_4
Ω_{12}	0.826355	**0.897722**	0.799449	0.672183	Θ_2
Ω_{13}	**0.887082**	0.807254	0.719201	0.591934	Θ_1
Ω_{14}	0.745305	0.759465	**0.931247**	0.754408	Θ_3
Ω_{15}	0.616897	0.654974	0.780458	**0.853987**	Θ_4
Ω_{16}	0.578235	0.698128	0.683798	**0.814795**	Θ_4
Ω_{17}	0.817016	**0.881594**	0.830311	0.700801	Θ_2
Ω_{18}	0.831953	**0.901969**	0.733355	0.662529	Θ_2
Ω_{19}	0.623709	0.686493	0.778328	**0.885001**	Θ_4
Ω_{20}	0.602119	0.685669	0.723624	**0.943048**	Θ_4

Table 8. Assessed results using Eq. (3)

Ω_j	$N_\delta(\Phi_j, \Theta_1)$	$N_\delta(\Phi_j, \Theta_2)$	$N_\delta(\Phi_j, \Theta_3)$	$N_\delta(\Phi_j, \Theta_4)$	Risk level
Ω_1	0.581600	0.697253	0.742503	**0.901390**	Θ_4
Ω_2	0.646720	0.688072	0.805233	**0.886962**	Θ_4
Ω_3	0.774547	**0.871636**	0.807972	0.769010	Θ_2
Ω_4	0.635925	0.735100	0.786431	**0.962682**	Θ_4
Ω_5	0.583890	0.683484	0.714810	**0.912949**	Θ_4
Ω_6	**0.896410**	0.838217	0.701436	0.627532	Θ_1
Ω_7	0.822104	**0.902188**	0.830551	0.800261	Θ_2
Ω_8	0.824908	**0.926346**	0.767125	0.684412	Θ_2
Ω_9	0.796648	0.802231	**0.945901**	0.806259	Θ_3
Ω_{10}	0.655512	0.738880	0.786225	**0.929453**	Θ_4
Ω_{11}	0.699053	0.781021	0.812508	**0.908627**	Θ_4
Ω_{12}	0.841484	**0.909690**	0.821160	0.705050	Θ_2
Ω_{13}	**0.901922**	0.824129	0.747833	0.629507	Θ_1
Ω_{14}	0.770840	0.784476	**0.936499**	0.781144	Θ_3
Ω_{15}	0.653407	0.689608	0.801997	**0.869884**	Θ_4
Ω_{16}	0.614510	0.726679	0.716873	**0.837449**	Θ_4
Ω_{17}	0.840246	**0.899429**	0.858115	0.740423	Θ_2
Ω_{18}	0.843693	**0.909883**	0.759968	0.694463	Θ_2
Ω_{19}	0.659384	0.718694	0.802222	**0.899089**	Θ_4
Ω_{20}	0.639969	0.718193	0.750376	**0.948368**	Θ_4

Regarding the 20 slope samples, we also calculate the weighted HSSM values between Θ_k (k = 1, 2, 3, 4) and Φ_j (j = 1, 2, …, 20) by Eq. (3) for δ = 1 [27], and then all the assessed results are shown in Table 8. It is obvious that the assessed risk results using Eq. (3) in Table 8 are the same risk results as using the proposed IHSSM, which proves the validity, rationality, and accuracy of our new assessment model in the scenario of SVNSs.

6. Conclusions

This article first presented the IHSSM of SVNSs, and then proposed its netting clustering and slope stability evaluation models to realize the risk level clustering analysis and assessment of slope stability in the scenario of SVNSs. Next, the proposed netting clustering and slope stability evaluation models are applied in the case study of the 20 slope samples. The assessment and comparison results verified the validity, rationality, and accuracy of the proposed new models in the scenario of SVNSs. However, the proposed new models can avoid the defects of the existing ANN, ANFIS, and SVNN-ANFIS evaluation methods [10, 11, 25] because they need a lot of training samples and the complex modeling process. Therefore, the new models proposed in this paper are not only simpler and more convenient in the evaluation process, but also more suitable for practical applications, which is the main advantage.

In future research, we will further investigate the slope stability clustering and evaluation problems with big sample data and verify the reasonableness and validity of the netting clustering results and the evaluation results. In addition, we shall also research on clustering and assessment methods with multi-layer affecting factors in neutrosophic scenarios and verify their accuracy and validity under the environment of big sample data.

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Softmax function based neutrosophic aggregation operators and application in multi-attribute decision making problem

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Abstract. The softmax function is a well-known generalization of the logistic function. It has been extensively applied in various probabilistic classification methods such as softmax regression, linear discriminant analysis, naive Bayes classifiers, and artificial neural networks. Inspired by the advantages of the softmax function, we have developed the softmax function-based single-valued neutrosophic aggregation operators. Then we have established some essential properties of aggregation operators based on the softmax function with the neutrosophic set. Additionally, we have defined a multi-attribute decision-making method based on the proposed aggregation operators. Using the proposed MCDM method, we have developed a novel algorithm. This algorithm helps to examine FD-risk assessment problems. Also, the proposed algorithm process is a reasonable strategy for the decision-making problem. It is easy to recognize when choosing a neutrosophic set of information for a practical decision problem. We used this proposed MADM method to exercise a realistic MADM problem with neutrosophic information. Finally, we have considered one numerical illustration to show the validity and reliability of the proposed methods.

Keywords: Softmax function; Single valued neutrosophic set, Aggregation operator, Multi attribute decision making strategy

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1. **Introduction**

An intuitionistic fuzzy set (Atanassov 1986) is an effectual generalization of the fuzzy set (Zadeh 1965). But single-valued neutrosophic (SVN) set (Wang 2010) is a successful generalization of the fuzzy set (Zadeh 1965). SVN set each element is expressed by a triplet of membership degrees which are membership, indeterminacy, and falsity degrees. The Sum of the membership degrees value lies between 0 and 3. Day by day, SVNS has received more intent from the researchers due to its structure formation. In the decision of some modern science, real problems depend on multi-attribute decision making (MADM). Because MADM provide the best choice option to select the alternatives with respect to the attributes. Expressing the attribute's value in the decision-making problem is a significantly important factor. Sometimes in the decision-making problem, so many uncertainties and complexity occur. In this case, the SVN set has a significant role in expressing this information. The possiblistic mean-variance of the SVN set was developed by Garai et al. (2020a). Recently many researchers proposed various strategies for their work considered by the SVN set, such as Jun (2013), Garai & Garg (2022a), Sod (2018), Wei (2018), Ren (2017), Biswas (2016), Pramanik (2017), Garai & Garg (2022b) and so on.

In an uncertain environment, some decision-making (DM) problems handle by the aggregation operator. Using the different aggregation operators, many researchers recently have on the DM problem under the SVN environment. For instance, Garg and Nancy (2018) developed some new hybrid aggregation operators using arithmetic and geometric aggregation operators. They also solved one MADM problem in the SVN environment. Ji et al. (2016) proposed the SVNS-Frank normalized Bonferroni mean (SVNFNPBM) operator to aggregate all values. This SVNFNPBM operator applied to choose the third-party logistics example. Some arithmetic operations of SVN numbers use frank norm operators as defined by Nancy and Garg (2016). Also, it applied to MADM problems. Sodenkamp et al. (2018) present an aggregation strategy for multi-attribute group decision making (MAGDM) problems under an SVN environment. Liu et al. (2014) defined some aggregation operators by combining Hamacher operations and generalized aggregation operators in the SVN environment. Recently, Chen and Ye (2017) considered two operators: Dombi weighted geometric average and Dombi weighted arithmetic average operators under an SVN environment.

Liu et al. (2019) developed a new single-valued neutrosophic Schweizer-sklar prioritized weighted averaging (SVNSSPRWA) operator. After that, he studied some basic properties of the proposed aggregation operators. It also gave the two decision-making models for showing the effectiveness of these novel operators. Further, Lui et al. (2020) developed the novel weighted single-valued neutrosophic power dual muirhead mean (WSVNPDMM) operator and single-valued neutrosophic power dual muirhead mean (SVNPDMM) operator. Further, they proposed a new technique for the MAGDM problem based on these aggregation operators. Tan and Zang (2020) defined a new distance measure, similarity measure, and neutrosophic entropy for straight SVN sets. Rong et al. (2020) defined several new operational laws of SVN number depending on Archimedean copula and co-copula (ACC) and discussed their related properties. They proposed some novel power aggregation operators (AOs) to merge SVN information, i.e., SVN copula power geometric (SVNCPG), weighted SVNCPA (WSVNCPA) operator, etc. Also, he has proposed MADM problems with SVN information using these operators.

Nowadays, many researchers are developing some operators in the SVN environment. Based on the dombi t-norm and t-conorm, Chen and Ye (2017) developed the SVNDWGAA operator to deal with the aggregation of SVN numbers in the MADM process. Li et al. (2016) improved a generalized weighted geometric heronian mean (IGWGHM) operator. Also, Li et al. (2016) proposed the improved weighted heronian mean (NNIGWHM) operator and improved generalized weighted geometric heronian mean (NNIGWGHM) operator for neutrosophic numbers. And these operators applied to MADM problems. Garai et al. (2020b) proposed the new ranking of SVN-number and used it for the MADM problem. Recently, Wei and Wei (2018) presented some SVN-dombi prioritized average (SVNDPA) operators and SVN-Dombi prioritized geometric (SVNDPG) operators. They utilized these operators to solve MADM problems in SVNS environment.

The paper is structured as follows: In Section 2, some basic concepts and definitions related to NS, and SVNS are discussed, and also presented score function and accuracy function of SVNNs. In Section 3, SVNWA, SVNWG, GSSVNWA, and GSSVNWG are defined and introduced as some basic properties and examples. Section 4 presented a MADM strategy based on the proposed aggregation operators. In section 5, we solved a numerical model to check the validity and applicability of the proposed method. Finally, this study's conclusions and future research direction are presented in Section 6.

1.1. *Motivation*

So, the above discussion says that many aggregation operators are extended with the different single-valued neutrosophic information. Then some researchers are successfully applied to many MADM problems and multi-attribute group decision making (MAGDM) problems under SVNS environments. But in this weighted aggregation, operators have certain restrictions because most of the aggregations are not applicable without SVNNs. Hence some operators cannot be relevant in some real-life problems. The softmax function handles these types of restrictions.

Previously, many researchers worked on MADM under a fuzzy environment using different usual aggregation operators. Torres et al. (2014) applied the softmax function in decision-making problems under an uncertain fuzzy set environment. He proposes a series of aggregation operators based on the softmax function. Later, Yu (2016) extended the softmax function-based aggregation operator in an intuitionistic fuzzy set environment. He developed the series of aggregation operators and applied these to a MADM problem. When we ranked the different alternatives to real MADM problems under the SVN environment, some had difficulties raised that time. We cannot organize the alternatives to the MADM problem using standard ranking methods like a fuzzy number. Now, how can we rank the alternatives with single-valued neutrosophic information? Also, how can we apply the softmax function-based aggregation operator in MADM? What is the usefulness of the softmax function-based aggregation operator in the MADM problem? When we studied some articles related to this research, a few questions arose in our minds. Therefore from that place, we try to establish a best ranking method with the help of a softmax function-based aggregation operator.

This paper has developed the softmax function-based single-valued neutrosophic aggregation operators. This aggregation operator is an extension of IF aggregation operators. We have proposed a softmax SVN weighted average (SVNWA) operator; Softmax SVN weighted geometric (SVNWG). In addition, we also developed some aggregation operators: generalized softmax single-valued neutrosophic weighted average (GSSVNWA) operator and generalized softmax SVN weighted geometric (GSSVNIFWG) operator. Some fundamental properties of softmax-based aggregation operators are developed here. We have introduced a novel MADM method using the proposed softmax-based aggregation operators. Finally, To check the importance of the proposed MADM method numerically.

1.2. *Novelty*

This paper extends the softmax function-based intuitionistic fuzzy (IF) aggregation operators to softmax function-based SVN aggregation operators. Additionally, some softmax function-based aggregation operators are developed here, which are the softmax SVN weighted average (SVNWA) operator, Softmax SVN weighted geometric (SVNWG) operator, and generalized softmax single-valued neutrosophic weighted average (GSSVNWA) operator and generalized softmax SVN weighted geometric (GSSVNIFWG) operator. Then, we proposed the essential properties of the proposed softmax-based aggregation operators. Further, this decision-making technique is applied to real MADM problems.

The main contributions of the paper is that:

- We extend the SIFWA operator to SSVNWA operator.
- We extend the SIFWG operator to SSVNWG operator.
- We extend the GSIFWA operator to GSSVNWA operator.
- We extend the GSIFWG operator to GSSVNWG operator.
- We develop a MADM strategy based on the proposed operators.
- To check the validity of MADM strategy we solved one real MADM problem.

2. Basic Preliminaries

Let X be a universe set. A nutrosophic (Smarandache 1998) sates $\tilde{E}$ over X is defined by $\tilde{E} = \{\langle x, (T_{\tilde{E}}(x), I_{\tilde{E}}(x),\ F_{\tilde{E}}(x))\rangle : x \in X\}$, where $T_{\tilde{E}}(x), I_{\tilde{E}}(x)$ and $F_{\tilde{E}}(x)$ are called truth membership function, indeterminacy-membership function and falsity membership functions respectively. They are defined as

$$T_{\tilde{E}} : X \to]^{-}0, 1^{+}[,\ I_{\tilde{E}} : X \to]^{-}0, 1^{+}[,\ F_{\tilde{E}} : X \to]^{-}0, 1^{+}[$$

such that $0^{-} \leq T_{\tilde{E}}(x) + I_{\tilde{E}}(x) + F_{\tilde{E}}(x) \leq 3^{+}$ Let X be a universe set (Wang 2010). A SVN-set $\tilde{E}$ over X is a neutroophic set, but the truth-membership function, indeterminacy-membership function and falsity-membership function are respectively defined by

$$T_{\tilde{E}} : X \to [0,1],\ I_{\tilde{E}} : X \to [0,1],\ F_{\tilde{E}} : X \to [0,1]$$

such that $0 \leq T_{\tilde{E}}(x) + I_{\tilde{E}}(x) + F_{\tilde{E}}(x) \leq 3$. For convenience, a SVNN can be expressed to be $\tilde{E} = (T_{\tilde{E}}, I_{\tilde{E}}, F_{\tilde{E}}), T_{\tilde{E}} \in [0,1], I_{\tilde{E}} \in [0,1], F_{\tilde{E}} \in [0,1]$ and $0 \leq T_{\tilde{E}} + I_{\tilde{E}} + F_{\tilde{E}} \leq 3$. Let $\tilde{C} = \{\langle x, (T_{\tilde{C}}(x), I_{\tilde{C}}(x),\ F_{\tilde{C}}(x))\rangle : x \in X\}$ and $\tilde{E} = \{\langle x, (T_{\tilde{E}}(x), I_{\tilde{E}}(x),\ F_{\tilde{E}}(x))\rangle : x \in X\}$ be two SVN-sets in X, then operations between them defined (Wang 2010) as follows:

(i) $\tilde{C} \subseteq \tilde{E}$ iff $T_{\tilde{C}}(x) \leq T_{\tilde{E}}(x), I_{\tilde{C}}(x) \geq I_{\tilde{E}}(x), T_{\tilde{C}}(x) \geq T_{\tilde{E}}(x)$ for all $x \in X$.
(ii) $\tilde{C} = \tilde{E}$ iff $\tilde{C} \subseteq \tilde{E}$ and $\tilde{E} \subseteq \tilde{C}$ for all $x \in X$.
(iii) $\tilde{E}^{c} = \{\langle x, (F_{\tilde{E}}(x), 1 - I_{\tilde{E}}(x),\ T_{\tilde{E}}(x))\rangle : x \in X\}$ for all $x \in X$.
(iv) $\tilde{C} \cup \tilde{E} = \{\langle x, max(T_{\tilde{C}}(x), T_{\tilde{E}}(x)), min(I_{\tilde{C}}(x), I_{\tilde{E}}(x)), min(F_{\tilde{C}}(x), F_{\tilde{E}}(x))\rangle : x \in X\}$ for all $x \in X$.
(v) $\tilde{C} \cap \tilde{E} = \{\langle x, min(T_{\tilde{C}}(x), T_{\tilde{E}}(x)), max(I_{\tilde{C}}(x), I_{\tilde{E}}(x)), max(F_{\tilde{C}}(x), F_{\tilde{E}}(x))\rangle : x \in X\}$ for all $x \in X$.

Let $\tilde{E}, \tilde{E}_1, \tilde{E}_2$ be three SVN-sets in X. Then, the arithmetic (Wang 10) operations are defined as follows:

(i) $\tilde{E}_1 + \tilde{E}_2 = \{\langle x, T_{\tilde{E}_1}(x) + T_{\tilde{E}_2}(x) - T_{\tilde{E}_1}(x).T_{\tilde{E}_2}(x), I_{\tilde{E}_1}(x).I_{\tilde{E}_2}(x), F_{\tilde{E}_1}(x).F_{\tilde{E}_2}(x)\rangle : x \in X\}$ for all $x \in X$.
(ii) $\tilde{E}_1.\tilde{E}_2 = \{\langle x, T_{\tilde{E}_1}(x).T_{\tilde{E}_2}(x), I_{\tilde{E}_1}(x) + I_{\tilde{E}_2}(x) - I_{\tilde{E}_1}(x).I_{\tilde{E}_2}(x), F_{\tilde{E}_1}(x) + F_{\tilde{E}_2}(x) - F_{\tilde{E}_1}(x).F_{\tilde{E}_2}(x)\rangle : x \in X\}$ for all $x \in X$.
(iii) $\lambda.\tilde{E} = \{\langle x, (1 - (1 - T_{\tilde{E}}(x))^{\lambda}, (I_{\tilde{E}}(x))^{\lambda}, (F_{\tilde{E}}(x))^{\lambda}\rangle : x \in X\}$ for all $x \in X$.
(iv) $\tilde{E}^{\lambda} = \{\langle x, (T_{\tilde{E}}(x))^{\lambda}, 1 - (1 - I_{\tilde{E}}(x))^{\lambda}, 1 - (1 - F_{\tilde{E}}(x))^{\lambda}\rangle : x \in X\}$ for all $x \in X$, Where $\lambda > 0$ is a parameter.

For any SVN set, the ranking method is very significant and many research results have been received (Zhang et al. 2014, Wang et al. 2010). Zhang et al. 2014 given a method based on score function and accuracy function. For any SVN-set $A = (T_A, I_A, F_A)$, the accuracy and

score function defined as:
The score function of $\tilde{E}$ is

$$S(\tilde{E}) = \frac{2 + T_{\tilde{E}} - I_{\tilde{E}} - F_{\tilde{E}}}{3}, S(\tilde{E}) \in [0, 1] \tag{1}$$

and the accuracy function of $\tilde{E}$ is

$$H(\tilde{E}) = T_{\tilde{E}} - F_{\tilde{E}}, H(\tilde{E}) \in [-1, 1] \tag{2}$$

Zhang et al. 2014 gave an order relation between two SVN numbers, which is defined as follows:
Let $\tilde{C} = (T_{\tilde{C}}, I_{\tilde{C}}, F_{\tilde{C}})$ and $\tilde{E} = (T_{\tilde{E}}, I_{\tilde{E}}, F_{\tilde{E}})$ be two SVNNs.
Now, if $S(A) > S(B)$, then $\tilde{C} \succ \tilde{E}$. Again if $S(\tilde{C}) = S(\tilde{E})$, then

(i) *If $H(\tilde{C}) = H(\tilde{C})$, then $\tilde{C} \approx \tilde{E}$.*
(ii) *If $H(\tilde{C}) > H(\tilde{E})$, then $\tilde{C} \succ \tilde{E}$.*

3. **Softmax function based aggregation operators**

This section has discussed the softmax function and its essential properties. Here we established some rigorous methods related to the softmax function-based aggregation operator.

3.1. *Softmax function*

A softmax function is a generalization form of the logistic process in the area of mathematics. It has been progressively applied to many research fields, for instance, machine learning (Jacobs 1991, Torres 2003) and decision making (Torres 14, Yu 16). The mathematical form of the softmax function is represented as follows:

$$\phi_k(j, \vartheta_1, \vartheta_2, ..., \vartheta_n) = \phi_k^j = \frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{n} \exp(\vartheta_j/k)}, k > 0. \tag{3}$$

For the SVN-sets $\alpha_j (j = 1, 2, 3, ..., n)$, S_j is the score value of SVN-number α_j. Every ϑ_j is formulated by given the equation

$$\vartheta_j = \begin{cases} \prod_{i=1}^{j-1} S_i, & \text{j} = 2, 3, ..., n \\ 1 & \text{j} = 1 \end{cases} \tag{4}$$

where k is the modulation parameter. *Some properties of softmax function (Yu 2016) are defined as follows:*

(i) $0 \leq \phi_k^j \leq 1$.
(ii) $\sum_{j=1}^{n} \phi_k^j = 1$.

Softmax function Torres 2014 has the non linear characteristic, monotonous and boundedness properties.

3.2. *SVN-sets aggregation operators based on softmax function*

In this section, we have extended the softmax function based on IF aggregation operators, such as softmax IF weighted average operator (SIFWA), softmax IF weighted geometric (SIFWG) operator, generalized softmax IF weighted average (GSIFWA) operator, and generalized softmax IF weighted geometric (GSIFWG) operator to softmax SVN weighted average (SVNWA) operator; softmax SVN weighted geometric (SVNWG) operator, generalized softmax SVN weighted average (GSSVNWA) operator, and generalized softmax SVN weighted geometric (GSSVNIFWG) operator, respectively. *Let $\alpha_j (j = 1, 2, ..., n)$ be a collection of SVNNs. Then softmax single valued neutrosophic weighted average (SSVNWA) operator is a function from $\alpha^n \to \alpha$ such that*

$$SSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) = \oplus_{j=1}^{n}(\phi_k^j \alpha_j) = (\phi_k^1 \alpha_1) \oplus (\phi_k^2 \alpha_2) \oplus ... \oplus (\phi_k^n \alpha_n)$$

where, $\phi_k^j = \frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{n}\exp(\vartheta_j/k)}, \vartheta_j = \begin{cases} \prod_{i=1}^{j-1} S_i, & \text{j} = 2, 3, ..., n \\ 1 & \text{j} = 1 \end{cases}$, S_i is the score function of the SVN-number α_i.

Theorem 3.1. Let $\alpha_j (j = 1, 2, ..., n)$ be a collection of SVN-numbers, then aggregated value of SVN-numbers using the SSVNWA operation is also a SVN-number. The SSVNWA operator can be generated as:

$$SSVNWA\left(\alpha_1, \alpha_2, ..., \alpha_n\right) = \left(1 - \prod_{j=1}^{n}\left(1 - T_j\right)^{\frac{\exp\left(\vartheta_j/k\right)}{\sum_{j=1}^{n}\exp\left(\vartheta_j/k\right)}}, \prod_{j=1}^{n}\left(I_j\right)^{\frac{\exp\left(\vartheta_j/k\right)}{\sum_{j=1}^{n} exp\left(\vartheta_j/k\right)}}, \prod_{j=1}^{n}\left(F_j\right)^{\frac{\exp\left(\vartheta_j/k\right)}{\sum_{j=1}^{n}\exp\left(\vartheta_j/k\right)}}\right) \tag{5}$$

Proof. : *We proof the above theorem 1 by using mathematical induction. For $n = 1$, we have:*

$$SSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) = \left(1 - \prod_{j=1}^{1}(1 - T_j), \prod_{j=1}^{1}(I_j), \prod_{j=1}^{1}(F_j)\right) = (T_1, I_1, F_1).$$

Since for $n = 1$, $\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{n}\exp(\vartheta_j/k)} = 1$. Thus Eq.(5) holds for n = 1. Assume that the Eq. (5) holds for n = m,

$$SSVNWA(\alpha_1, \alpha_2, ..., \alpha_m) = \left(1 - \prod_{j=1}^{m}(1 - T_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}}, \prod_{j=1}^{m}(I_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}}, \prod_{j=1}^{m}(F_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}}\right)$$

Now we prove that the Eq. (5) holds for $n = m+1$.

$$\begin{aligned} SSVNWA(\alpha_1, \alpha_2, ..., \alpha_m, \alpha_{m+1}) &= (\phi_k^1\alpha_1) \oplus (\phi_k^2\alpha_2) ... \oplus (\phi_k^m\alpha_m) \oplus (\phi_k^{m+1}\alpha_{m+1}) \\ &= \Bigg(\Bigg(1 - \prod_{j=1}^{m}(1-T_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}}\Bigg) + \Bigg(1 - (1-T_{m+1})^{\frac{\exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}}\Bigg) \\ &- \Bigg(\Bigg(1 - \prod_{j=1}^{m}(1-T_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}}\Bigg) \times \Bigg(1 - (1-T_{m+1})^{\frac{\exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}}\Bigg)\Bigg), \\ &\prod_{j=1}^{m+1}(I_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}}, \prod_{j=1}^{m+1}(F_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}}\Bigg) \\ &= \Bigg(1 - \prod_{j=1}^{m+1}(1-T_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}}, \prod_{j=1}^{m+1}(I_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}}, \prod_{j=1}^{m+1}(F_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}}\Bigg) \end{aligned}$$

Therefore, Eq. (5) holds for n = m+1, hence the Eq. (5) holds for all positive integer by principle of mathematical induction. Hence, the proof of the theorem is completed. □

Example 3.2. Let$\alpha_1 = (0.6, 0.4, 0.5), \alpha_2 = (0.7, 0.3, 0.5), \alpha_3 = (0.8, 0.3, 0.4)$ and $\alpha_4 = (0.7, 0.4, 0.5)$ be the four SVN-numbers. Rank the four SVN-numbers using SSVNWA operator.

Solution: Here we have used the SSVNWA operator to aggregate the four SVN-numbers.

At first, we calculated the score values of four SVN-numbers using Eq. (1).

$S(\alpha_1) = 0.567, S(\alpha_2) = 0.633, S(\alpha_3) = 0.700, S(\alpha_4) = 0.600$

$\vartheta_1 = 1, \vartheta_2 = 0.567, \vartheta_3 = 0.359, \vartheta_4 = 0.251.$

To calculate $\exp(\vartheta_j/k)$ we take k =1, then $\exp(\vartheta_1/k) = 2.718, \exp(\vartheta_2/k) = 1.763, \exp(\vartheta_3/k) = 1.432, \exp(\vartheta_4/k) = 1.285$

and $\frac{exp(\vartheta_1/k)}{\sum_{j=1}^{4}\exp(\vartheta_j/k)} = 0.378, \frac{exp(\vartheta_2/k)}{\sum_{j=1}^{4}\exp(\vartheta_j/k)} = 0.245, \frac{exp(\vartheta_3/k)}{\sum_{j=1}^{4}\exp(\vartheta_j/k)} = 0.199, \frac{exp(\vartheta_4/k)}{\sum_{j=1}^{4}\exp(\vartheta_j/k)} = 0.178$

$$\begin{aligned} SSVNWA(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= \Big(\Big(1 - (1-0.6)^{0.378} \times (1-0.7)^{0.245} \times (1-0.8)^{0.199} \times (1-0.7)^{0.178}\Big), \\ &\Big((0.4)^{0.378} \times (0.3)^{0.245} \times (0.3)^{0.199} \times (0.4)^{0.178}\Big), \\ &\Big((0.5)^{0.378} \times (0.5)^{0.245} \times (0.4)^{0.199} \times (0.5)^{0.178}\Big)\Big) \\ &= (0.691, 0.352, 0.478) \end{aligned}$$

3.2.1. *Properties of $SSVNWA$ operator*

Property 1: Idem-potency

If $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha(say)$, then $SSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) = \alpha$

Proof. : Let $\alpha_j = \langle T_j, I_j, F_j \rangle, (j = 1, 2, 3..., n)$ and $\alpha = \langle T, I, F \rangle$.

Since all α_j are equal based on Theorem (1), we get

$SSVNWA(\alpha,\alpha,...,\alpha) = \left(1-(1-T)^{\sum_{j=1}^{n}\phi_k^j}, (I)^{\sum_{j=1}^{n}\phi_k^j}, (F)^{\sum_{j=1}^{n}\phi_k^j}\right) = \langle T, I, F\rangle = \alpha.$

Since, $\phi_k^j = \frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{n}\exp(\vartheta_j/k)}, k > 0$ *and* $\sum_{j=1}^{n}\phi_k^j = 1.$ □

Property 2: Monotonicity

Let $\alpha_j (j = 1, 2, ..., n)$ and $\beta_j (j = 1, 2, ..., n)$ be any two sets of SVN-numbers. If $\alpha_j \leq \beta_j$ for any j,

then $SSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \leq SSVNWA(\beta_1, \beta_2, ..., \beta_n)$.

Proof. Based on the Theorem (1), we get

$$SSVNWA(\alpha_1,\alpha_2,...,\alpha_n) = \left(1-\prod_{j=1}^{n}(1-T_{\alpha_j})^{\frac{\exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n}\exp(\vartheta_{\alpha_j}/k)}}, \prod_{j=1}^{n}(I_{\alpha_j})^{\frac{\exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n}\exp(\vartheta_{\alpha_j}/k)}}, \prod_{j=1}^{n}(F_{\alpha_j})^{\frac{\exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n}\exp(\vartheta_{\alpha_j}/k)}}\right)$$

and

$$SSVNWA(\beta_1,\beta_2,...,\beta_n) = \left(1-\prod_{j=1}^{n}(1-T_{\beta_j})^{\frac{\exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n}\exp(\vartheta_{\beta_j}/k)}}, \prod_{j=1}^{n}(I_{\beta_j})^{\frac{\exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n}\exp(\vartheta_{\beta_j}/k)}}, \prod_{j=1}^{n}(F_{\beta_j})^{\frac{\exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n}\exp(\vartheta_{\beta_j}/k)}}\right)$$

Since all $\alpha_j \leq \beta_j (j = 1, 2, ..., n)$. *Therefore,*

$$\begin{aligned}
T_{\alpha_j} \leq T_{\beta_j} \quad &\Rightarrow \quad (1-T_{\alpha_j}) \geq (1-T_{\beta_j}) \\
&\Rightarrow \quad (1-T_{\alpha_j})^{\frac{\exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n}\exp(\vartheta_{\alpha_j}/k)}} \geq (1-T_{\beta_j})^{\frac{\exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n}\exp(\vartheta_{\beta_j}/k)}} \\
&\Rightarrow \quad \prod_{j=1}^{n}(1-T_{\alpha_j})^{\frac{\exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n}\exp(\vartheta_{\alpha_j}/k)}} \geq \prod_{j=1}^{n}(1-T_{\beta_j})^{\frac{\exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n}\exp(\vartheta_{\beta_j}/k)}} \\
&\Rightarrow \quad \left(1-\prod_{j=1}^{n}(1-T_{\alpha_j})^{\frac{\exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n}\exp(\vartheta_{\alpha_j}/k)}}\right) \leq \left(1-\prod_{j=1}^{n}(1-T_{\beta_j})^{\frac{\exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n}\exp(\vartheta_{\beta_j}/k)}}\right)
\end{aligned}$$

Further,

$$\begin{aligned}
I_{\alpha_j} \geq I_{\beta_j} \quad &\Rightarrow \quad (I_{\alpha_j})^{\frac{\exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n}\exp(\vartheta_{\alpha_j}/k)}} \geq (I_{\beta_j})^{\frac{\exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n}\exp(\vartheta_{\beta_j}/k)}} \\
&\Rightarrow \quad \prod_{j=1}^{n}(I_{\alpha_j})^{\frac{\exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n}\exp(\vartheta_{\alpha_j}/k)}} \geq \prod_{j=1}^{n}(I_{\beta_j})^{\frac{\exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n}\exp(\vartheta_{\beta_j}/k)}}
\end{aligned}$$

Similarly, we have also

$$\prod_{j=1}^{n}(F_{\alpha_j})^{\frac{\exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n}\exp(\vartheta_{\alpha_j}/k)}} \geq (F_{\beta_j})^{\frac{\exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n}\exp(\vartheta_{\beta_j}/k)}}$$

Hence the proof is complete. □

Property 3: Boundedness

Let $\alpha_j (j = 1, 2, ..., n)$ be any set of SVN-number. If $\alpha^- = \min\{\alpha_j\}$ and $\alpha^+ = \max\{\alpha_j\}$, then $\alpha^- \leq SSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \leq \alpha^+$.

Proof. : Let $\alpha^+ = \max\{\alpha_1, \alpha_2, ..., \alpha_n\}$ *and* $\alpha^- = min\{\alpha_1, \alpha_2, ..., \alpha_n\}$. *According to properties 1 and 2, we have*

$SSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \geq SSVNWA(\alpha^-, \alpha^-, ..., \alpha^-) = \alpha^-$ *and*

$SSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \leq SSVNWA(\alpha^+, \alpha^+, ..., \alpha^+) = \alpha^+$

So, we have $\alpha^- \leq SSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \leq \alpha^+$.

Hence the proof is complete. □

Let $\alpha_j (j = 1, 2, ..., n)$ *be a collection of SVN-numbers. Then softmax single valued neutrosophic weighted geometric (SSVNWG) operator is a function from* $\alpha^n \to \alpha$ *such that*

$$SSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) = \oplus_{j=1}^{n}(\phi_k^j \alpha_j) = \otimes_{j=1}^{n} (\alpha_j)^{\phi_k^j} = (\alpha_1)^{\phi_k^1} \otimes (\alpha_2)^{\phi_k^2} \otimes ... \otimes (\alpha_n)^{\phi_k^n} \quad (6)$$

where $\phi_k^j = \frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{n}\exp(\vartheta_j/k)}, \vartheta_j = \begin{cases} \prod_{i=1}^{j-1} S_i, & \text{j} = 2, 3, ..., n \\ 1 & \text{j} = 1 \end{cases}$ and S_i is the score function of the SVN-number α_i.

Theorem 3.3. Let $\alpha_j (j = 1, 2, ..., n)$, be a collection of SVNNs, then aggregated value of SVN-numbers using the SSVNWG operation is also a SVN-number and

$$\begin{aligned} SSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \;=\; & \Bigg(\prod_{j=1}^{n}(T_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{n}\exp(\vartheta_j/k)}}, \\ & 1 - \prod_{j=1}^{n}(1 - I_j)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n}\exp(\vartheta_j/k)}}, 1 - \prod_{j=1}^{n}(1 - F_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{n}\exp(\vartheta_j/k)}}\Bigg) \end{aligned} \quad (7)$$

Proof. : We proof the above Theorem 2 by using mathematical induction.

For $n = 1$, *from the Eq. (7) we have*

$$\begin{aligned} SSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \;=\; & \Bigg(\prod_{j=1}^{1}(T_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{n}\exp(\vartheta_j/k)}}, \\ & 1 - \prod_{j=1}^{1}(1 - I_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{n}\exp(\vartheta_j/k)}}, 1 - \prod_{j=1}^{1}(1 - F_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{n}\exp(\vartheta_j/k)}}\Bigg) = \langle T_1, I_1, F_1\rangle \end{aligned}$$

Since, for $n = 1$, $\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)} = 1$. *Thus Eq.(7) holds for* $n = 1$. *Assume that the Eq. (7) holds for* $n = m$,

$$\begin{aligned} SSVNWG(\alpha_1, \alpha_2, ..., \alpha_m) \;=\; & \Bigg(\prod_{j=1}^{m}(T_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}}, \\ & 1 - \prod_{j=1}^{m}(1 - I_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}}, 1 - \prod_{j=1}^{m}(1 - F_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}}\Bigg) \end{aligned}$$

Now we have prove that the Eq. (7) hold for $n = m + 1$. Then

$$\begin{aligned}
SSVNWG(\alpha_1, \alpha_2, ..., \alpha_m, \alpha_{m+1}) &= (\alpha_1)^{\phi_k^1} \otimes (\alpha_2)^{\phi_k^2} \otimes ... \otimes (\alpha_m)^{\phi_k^m} \otimes (\alpha_{m+1})^{\phi_k^{m+1}} \\
&= \left(\prod_{j=1}^{m} (T_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}},\ 1 - \prod_{j=1}^{m} (1 - I_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}}, \right. \\
&\left. 1 - \prod_{j=1}^{m} (1 - F_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}} \right) \\
&= \left(\prod_{j=1}^{m} (T_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}} \times (T_{m+1})^{\frac{\exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}}, \right. \\
& \prod_{j=1}^{m} (I_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}} + (I_{m+1})^{\frac{\exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}} - \prod_{j=1}^{m} (I_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}} \\
& \times (I_{m+1})^{\frac{\exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}},\ \prod_{j=1}^{m} (F_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}} + (F_{m+1})^{\frac{\exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}} \\
&\left. - \prod_{j=1}^{m} (F_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}} \times (F_{m+1})^{\frac{\exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}} \right) \\
&= \left(\prod_{j=1}^{m+1} (T_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}},\ 1 - \prod_{j=1}^{m+1} (1 - I_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}}, \right. \\
&\left. 1 - \prod_{j=1}^{m+1} (1 - F_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}} \right)
\end{aligned}$$

Therefore, Eq. (7) holds for $n = m+1$, hence the Eq. (7) holds for all positive integer by principle of mathematical induction. Hence, the proof of the theorem is completed. □

Example 3.4. Let $\alpha_1 = (0.6, 0.4, 0.5), \alpha_2 = (0.7, 0.3, 0.5), \alpha_3 = (0.8, 0.3, 0.4)$ and $\alpha_4 = (0.7, 0.4, 0.5)$ be the four SVN-numbers. Rank the four SVN-numbers using $SSVNWG$ operator.

Solution: In the following, we use the $SSVNWG$ operator to aggregate these SVN-numbers. At first we have calculated the score values of four SVNNs using Eq. (1)

$S(\alpha_1) = 0.567, S(\alpha_2) = 0.633, S(\alpha_3) = 0.700, S(\alpha_4) = 0.600$

then $\vartheta_1 = 1, \vartheta_2 = 0.567, \vartheta_3 = 0.359, \vartheta_4 = 0.251$.

To calculate the $\exp(\vartheta_j/k)$ we take $k = 1$, then

$exp(\vartheta_1/k) = 2.718, exp(\vartheta_2/k) = 1.763, exp(\vartheta_3/k) = 1.432,$

$exp(\vartheta_4/k) = 1.285$ and

$$\frac{exp(\vartheta_1/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.378, \frac{exp(\vartheta_2/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.245,$$
$$\frac{exp(\vartheta_3/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.199, \frac{exp(\vartheta_4/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.178,$$

$$\begin{aligned} SSVNWG(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= \Big((0.6)^{0.378} \times (0.7)^{0.245} \times (0.8)^{0.199} \times (0.7)^{0.178}, \\ &\quad 1 - (1-0.4)^{0.378} \times (1-0.3)^{0.245} \times (1-0.3)^{0.199} \times (1-0.4)^{0.178}, \\ &\quad 1 - (1-0.5)^{0.378} \times (1-0.5)^{0.245} \times (1-0.4)^{0.199} \times (1-0.5)^{0.178} \Big) \\ &= (0.678, 0.357, 0.482). \end{aligned}$$

3.2.2. *Properties of $SSVNWG$ operator*

Property 1: Idem-potency

If $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha(say)$, then
the$SSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) = \alpha$

Proof. : Let $\alpha_j = \langle T_j, I_j, F_j \rangle, (j = 1, 2, 3..., n)$ and $\alpha = \langle T, I, F \rangle$.
Since all α_j are equal, based on Theorem (2), we get
$SSVNWG(\alpha, \alpha, ..., \alpha)$
$= \Big((T)^{\sum_{j=1}^{n} \phi_k^j}, 1 - (1-I)^{\sum_{j=1}^{n} \phi_k^j}, 1 - (1-F)^{\sum_{j=1}^{n} \phi_k^j} \Big)$
$= \langle T, I, F \rangle = \alpha$.
Since, $\phi_k^j = \frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)}, k > 0$ and $\sum_{j=1}^{n} \phi_k^j = 1$. Hence the proof is completed. □

Property 2: Monotonicity

Let $\alpha_j (j = 1, 2, ..., n)$ and $\beta_j (j = 1, 2, ..., n)$ be any two sets of SVN-numbers. If $\alpha_j \leq \beta_j$ for any j, then $SSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \leq SSVNWG(\beta_1, \beta_2, ..., \beta_n)$.

Proof. : *Based on the Theorem (2), we get*

$$\begin{aligned} SSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) &= \Bigg(\prod_{j=1}^{n} (T_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}, \\ &\quad (1 - \prod_{j=1}^{n} (1 - I_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}), (1 - \prod_{j=1}^{n} (1 - T_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}) \Bigg) \end{aligned}$$

and

$$SSVNWA(\beta_1, \beta_2, ..., \beta_n) = \Big(\prod_{j=1}^{n} (T_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}),$$

$$(1 - \prod_{j=1}^{n} (1 - I_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}, (1 - \prod_{j=1}^{n} (1 - F_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}) \Big)$$

Since all $\alpha_j \leq \beta_j (j = 1, 2, ..., n)$. *Therefore,*

$$T_{\alpha_j} \leq T_{\beta_j} \Rightarrow (T_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \leq (T_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}$$

$$\Rightarrow \prod_{j=1}^{n} (T_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \leq \prod_{j=1}^{n} (T_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}$$

Further,

$$I_{\alpha_j} \geq I_{\beta_j} \Rightarrow (1 - I_{\alpha_j}) \leq (1 - I_{\beta_j})$$

$$\Rightarrow (1 - I_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \leq (1 - I_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}$$

$$\Rightarrow \prod_{j=1}^{n} (1 - I_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \leq \prod_{j=1}^{n} (1 - I_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}$$

$$\Rightarrow \left(1 - \prod_{j=1}^{n} (1 - I_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \right) \geq \left(1 - \prod_{j=1}^{n} (1 - I_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}} \right)$$

Similarly, we have also

$$\Rightarrow \left(1 - \prod_{j=1}^{n} (1 - F_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \right) \geq \left(1 - \prod_{j=1}^{n} (1 - F_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}} \right)$$

.

Hence the proof is completed. □

Property 3: Boundedness

Let $\alpha_j (j = 1, 2, ..., n)$ be any set of SVN-number. If $\alpha^- = \min\{\alpha_j\}$ and $\alpha^+ = \max\{\alpha_j\}$, then $\alpha^- \leq SSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \leq \alpha^+$.

Proof. : *Let* $\alpha^+ = \max\{\alpha_1, \alpha_2, ..., \alpha_n\}$ *and* $\alpha^- = \min\{\alpha_1, \alpha_2, ..., \alpha_n\}$.

According to properties 1 and 2, we have

$SSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \geq SSVNWG(\alpha^-, \alpha^-, ..., \alpha^-) = \alpha^-$ *and*

$SSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \leq SSVNWG(\alpha^+, \alpha^+, ..., \alpha^+) = \alpha^+$

So we have $\alpha^- \leq SSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \leq \alpha^+$.

Hence the proof is completed. □

Let $\alpha_j (j = 1, 2, ..., n)$ be a collection of SVN-numbers. Then generalized softmax single valued neutrosophic weighted average (GSSVNWA) operator is a function $\alpha^n \to \alpha$ such that

$$GSSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) = \left(\oplus_{j=1}^{n} \phi_k^j \alpha_j^\lambda\right)^{\frac{1}{\lambda}} = \left(\phi_k^1 \alpha_1^\lambda \oplus \phi_k^2 \alpha_2^\lambda \oplus ... \oplus \phi_k^n \alpha_n^\lambda\right)^{\frac{1}{\lambda}} \tag{8}$$

Let $\alpha_j (j = 1, 2, ..., n)$ be a collection of SVN-numbers. Then generalized softmax single valued neutrosophic weighted geometric (GSSVNWG) operator is a function $\alpha^n \to \alpha$ such that

$$GSSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) = \frac{1}{\lambda}\left(\otimes_{j=1}^{n} (\lambda\phi_j)^{\phi_k^j}\right) = \frac{1}{\lambda}\left((\lambda\alpha_1)^{\phi_k^1} \otimes (\lambda\alpha_2)^{\phi_k^2} \otimes ... \otimes (\lambda\alpha_n)^{\phi_k^n}\right) \tag{9}$$

Theorem 3.5. Let $\alpha_j (j = 1, 2, ..., n)$, be a collection of SVN-numbers, then aggregated value of SVN-numbers using the GSSVNWA operation is also a SVNN, and

$$\begin{aligned} GSSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) = & \Bigg\langle \left(1 - \prod_{j=1}^{n} (1 - T_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} \exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, 1 - \left(1 - \prod_{j=1}^{n} \left(1 - (1 - I_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, \\ & 1 - \left(1 - \prod_{j=1}^{n} \left(1 - (1 - F_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}} \Bigg\rangle \end{aligned} \tag{10}$$

Proof. : *We proof the above theorem 3 by using mathematical induction.*
For n = 1, we have:

$$\begin{aligned} GSSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) = & \Bigg\langle \left(1 - \prod_{j=1}^{1} (1 - T_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{1} \exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, 1 - \left(1 - \prod_{j=1}^{1} \left(1 - (1 - I_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, \\ & 1 - \left(1 - \prod_{j=1}^{1} \left(1 - (1 - F_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}} \Bigg\rangle \\ = & \langle T_1, I_1, F_1 \rangle \end{aligned}$$

Since, for $n = 1$, $\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)} = 1$.
Thus Eq.(10) holds for n = 1, we assume that the Eq. (10) holds for n = m,

$$\begin{aligned} GSSVNWA(\alpha_1, \alpha_2, ..., \alpha_m) = & \Bigg\langle \left(1 - \prod_{j=1}^{m} (1 - T_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, 1 - \left(1 - \prod_{j=1}^{m} \left(1 - (1 - I_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, \\ & 1 - \left(1 - \prod_{j=1}^{m} \left(1 - (1 - F_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}} \Bigg\rangle \end{aligned}$$

Now, we prove that the Eq. (10) holds for $n = m+1$.

$$
\begin{aligned}
GSSVNWA(\alpha_1,\alpha_2,...,\alpha_m,\alpha_{m+1}) &= \left((\phi_k^1\alpha_1^\lambda \oplus \phi_k^2\alpha_2^\lambda \oplus ... \oplus \phi_k^m\alpha_m^\lambda) \oplus \phi_k^{m+1}\alpha_{m+1}^\lambda\right)^{\frac{1}{\lambda}} \\
&= \Bigg\langle \left(1-\prod_{j=1}^{m}(1-T_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}} + \left(1-(1-T_{m+1}^\lambda)^{\frac{exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m=+1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}} \\
&\quad - \left(1-\prod_{j=1}^{m}(1-T_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}} \times \left(1-(1-T_{m+1}^\lambda)^{\frac{exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m=+1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, \\
&\quad \left(1-\left(1-\prod_{j=1}^{m}\left(1-(1-I_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\right) \times \\
&\quad \left(1-\left(1-\left(1-(1-I_{m+1})^\lambda\right)^{\frac{exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\right), \\
&\quad \left(1-\left(1-\prod_{j=1}^{m}\left(1-(1-F_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\right) \times \\
&\quad \left(1-\left(1-\left(1-(1-F_{m+1})^\lambda\right)^{\frac{exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\right)\Bigg\rangle \\
&= \Bigg\langle \left(1-\prod_{j=1}^{m+1}(1-T_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m+1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, \\
&\quad 1-\left(1-\prod_{j=1}^{m+1}\left(1-(1-I_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m+1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, \\
&\quad 1-\left(1-\prod_{j=1}^{m+1}\left(1-(1-F_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m+1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\Bigg\rangle
\end{aligned}
$$

Therefore, Eq. (10) holds for $n = m+1$, hence the Eq. (10) holds for all positive integer by principle of mathematical induction. Hence the proof of the theorem is completed. □

Example 3.6. Let $\alpha_1 = (0.6, 0.4, 0.5), \alpha_2 = (0.7, 0.3, 0.5), \alpha_3 = (0.8, 0.3, 0.4)$ and $\alpha_4 = (0.7, 0.4, 0.5)$ be the four SVN-numbers. Rank the four SVN-numbers using the GSSVNWA operator.

Solution: In the following, we use the GSSVNWA operator to aggregate these SVN-numbers. At first we calculate the score values of four SVN-numbers using Eq. (1). $S(\alpha_1) = 0.567, S(\alpha_2) = 0.633, S(\alpha_3) = 0.700, S(\alpha_4) = 0.600$ then $\vartheta_1 = 1, \vartheta_2 = 0.567, \vartheta_3 = 0.359, \vartheta_4 = 0.251$. To calculate $\exp(\vartheta_j/k)$ we take $k = 1$, then

$$exp(\vartheta_1/k) = 2.718, exp(\vartheta_2/k) = 1.763, exp(\vartheta_3/k) = 1.432, exp(\vartheta_4/k) = 1.285$$

and

$$\frac{exp(\vartheta_1/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.378, \frac{exp(\vartheta_2/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.245,$$

$$\frac{exp(\vartheta_3/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.199, \frac{exp(\vartheta_4/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.178$$

Taking$\lambda = 1$ GSSVNWA reduces to SSVNWA and we get

$$GSSVNWA(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.691, 0.352, 0.478).$$

Taking $\lambda = 2$ we obtain

$$\begin{aligned} GSSVNWA(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= \Bigg(\left(1-\left(1-(0.6)^2\right)^{0.378} \times \left(1-(0.7)^2\right)^{0.245} \times \left(1-(0.8)^2\right)^{0.199} \times \left(1-(0.7)^2\right)^{0.178}\right)^{\frac{1}{2}}, \\ & 1-\Big(1-\left(1-(1-0.6)^2\right)^{0.378} \times \left(1-(1-0.3)^2\right)^{0.245} \times \\ & \left(1-(1-0.3)^2\right)^{0.199} \times \left(1-(1-0.4)^2\right)^{0.178}\Big)^{\frac{1}{2}}, \\ & 1-\Big(1-\left(1-(1-0.5)^2\right)^{0.378} \times \left(1-(1-0.5)^2\right)^{0.245} \times \\ & \left(1-(1-0.4)^2\right)^{0.199} \times \left(1-(1-0.5)^2\right)^{0.178}\Big)^{\frac{1}{2}} \Bigg) \\ &= (0.694, 0.401, 0.477) \end{aligned}$$

Taking $\lambda = 3$, we obtain

$$\begin{aligned} GSSVNWA(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= \Bigg(\left(1-\left(1-(0.6)^3\right)^{0.378} \times \left(1-(0.7)^3\right)^{0.245} \times \left(1-(0.8)^3\right)^{0.199} \times \left(1-(0.7)^3\right)^{0.178}\right)^{\frac{1}{3}}, \\ & 1-\Big(1-\left(1-(1-0.6)^3\right)^{0.378} \times \left(1-(1-0.3)^3\right)^{0.245} \times \\ & \left(1-(1-0.3)^3\right)^{0.199} \times \left(1-(1-0.4)^3\right)^{0.178}\Big)^{\frac{1}{3}}, \\ & 1-\Big(1-\left(1-(1-0.5)^3\right)^{0.378} \times \left(1-(1-0.5)^3\right)^{0.245} \times \\ & \left(1-(1-0.4)^3\right)^{0.199} \times \left(1-(1-0.5)^3\right)^{0.178}\Big)^{\frac{1}{3}} \Bigg) \\ &= (0.697, 0.392, 0.476) \end{aligned}$$

Similarly we can also checked for $\lambda = 4, 5, ...$ and so on

3.2.3. *Properties of $GSSVNWA$ operator*

Property 1: Idem-potency If $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$ (say), then the $GSSVNWAO(\alpha_1, \alpha_2, ..., \alpha_n) = \alpha$

Proof. *Let $\alpha_j = \langle T_j, I_j, F_j \rangle, (j = 1, 2, 3..., n)$ and $\alpha = \langle T, I, F \rangle$.*
Since all α_j are equal, based on Theorem (3), we get

$$\begin{aligned} SSVNWG(\alpha,\alpha,...,\alpha) &= \Bigg\langle \left(1-(1-T^{\lambda})^{\sum_{j=1}^{n}\phi_k^j}\right)^{\frac{1}{\lambda}}, 1-\left(1-\left(1-(1-I)^{\lambda}\right)^{\sum_{j=1}^{n}\phi_k^j}\right)^{\frac{1}{\lambda}}, \\ & \quad 1-\left(1-\left(1-(1-F)^{\lambda}\right)^{\sum_{j=1}^{n}\phi_k^j}\right)^{\frac{1}{\lambda}} \Bigg\rangle \\ &= \langle T, I, F\rangle = \alpha \end{aligned}$$

Since, $\phi_k^j = \frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)}, k > 0$ *and* $\sum_{j=1}^{n}\phi_k^j = 1$.
Hence the proof is completed. □

Property 2: Monotonicity

Let $\alpha_j (j = 1, 2, ..., n)$ and $\beta_j (j = 1, 2, ..., n)$ be any two SVN-numbers. If $\alpha_j \leq \beta_j$ for any j, then
$GSSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \leq GSSVNWA(\beta_1, \beta_2, ..., \beta_n)$.

Proof. : Based on the Theorem (3), we get

$$\begin{aligned} GSSVNWA(\alpha_1,\alpha_2,...,\alpha_n) &= \Bigg\langle \left(1-\prod_{j=1}^{n}(1-T_{\alpha_j}^{\lambda})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}}, \\ & \quad 1-\left(1-\prod_{j=1}^{n}\left(1-(1-I_{\alpha_j})^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}}, \\ & \quad 1-\left(1-\prod_{j=1}^{n}\left(1-(1-F_{\alpha_j})^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}} \end{aligned}$$

and

$$\begin{aligned} GSSVNWA(\beta_1,\beta_2,...,\beta_n) &= \Bigg\langle \left(1-\prod_{j=1}^{n}(1-T_{\beta_j}^{\lambda})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}}, \\ & \quad 1-\left(1-\prod_{j=1}^{n}\left(1-(1-I_{\beta_j})^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}}, \\ & \quad 1-\left(1-\prod_{j=1}^{n}\left(1-(1-F_{\beta_j})^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}} \Bigg\rangle \end{aligned}$$

Since all $\alpha_j \leq \beta_j (j = 1, 2, ..., n)$.
Therefore for $T_{\alpha_j} \leq T_{\beta_j}$ we have

$$\begin{aligned}
T_{\alpha_j}^{\lambda} &\leq T_{\beta_j}^{\lambda} \Rightarrow (1 - T_{\alpha_j}^{\lambda}) \geq (1 - T_{\beta_j}^{\lambda}) \\
&\Rightarrow (1 - T_{\alpha_j}^{\lambda})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \geq (1 - T_{\beta_j}^{\lambda})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}} \\
&\Rightarrow \prod_{j=1}^{n} (1 - T_{\alpha_j}^{\lambda})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \geq \prod_{j=1}^{n} (1 - T_{\beta_j}^{\lambda})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}} \\
&\Rightarrow \left(1 - \prod_{j=1}^{n} (1 - T_{\alpha_j}^{\lambda})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right) \leq \left(1 - \prod_{j=1}^{n} (1 - T_{\beta_j}^{\lambda})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right) \\
&\Rightarrow \left(1 - \prod_{j=1}^{n} (1 - T_{\alpha_j}^{\lambda})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}} \leq \left(1 - \prod_{j=1}^{n} (1 - T_{\beta_j}^{\lambda})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}}
\end{aligned}$$

Further for

$$\begin{aligned}
I_{\alpha_j} \geq I_{\beta_j} &\Rightarrow \left(1 - I_{\alpha_j}\right) \leq \left(1 - I_{\beta_j}\right) \\
&\Rightarrow \left(1 - I_{\alpha_j}\right)^{\lambda} \leq \left(1 - I_{\beta_j}\right)^{\lambda} \\
&\Rightarrow \left(1 - \left(1 - I_{\alpha_j}\right)^{\lambda}\right) \geq \left(1 - \left(1 - I_{\beta_j}\right)^{\lambda}\right) \\
&\Rightarrow \left(1 - \left(1 - I_{\alpha_j}\right)^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \geq \left(1 - \left(1 - I_{\beta_j}\right)^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}} \\
&\Rightarrow \prod_{j=1}^{n} \left(1 - \left(1 - I_{\alpha_j}\right)^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \geq \prod_{j=1}^{n} \left(1 - \left(1 - I_{\beta_j}\right)^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}} \\
&\Rightarrow \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - I_{\alpha_j}\right)^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}} \leq \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - I_{\beta_j}\right)^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}} \\
&\Rightarrow 1 - \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - I_{\alpha_j}\right)^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}} \geq 1 - \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - I_{\beta_j}\right)^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}}
\end{aligned}$$

Similarly, we can also show that

$$\Rightarrow 1 - \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - F_{\alpha_j}\right)^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}} \geq 1 - \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - F_{\beta_j}\right)^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}}.$$

Hence the proof is completed. □

Property 3: Boundedness

Let $\alpha_j (j = 1, 2, ..., n)$ be any set of SVNN. If $\alpha^- = \min\{\alpha_j\}$ and $\alpha^+ = \max\{\alpha_j\}$, then $\alpha^- \leq GSSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \leq \alpha^+$.

Proof. : Let $\alpha^+ = \max\{\alpha_1, \alpha_2, ..., \alpha_n\}$ and $\alpha^- = \min\{\alpha_1, \alpha_2, ..., \alpha_n\}$. According to properties 1 and 2, we have

$GSSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \geq GSSVNWA(\alpha^-, \alpha^-, ..., \alpha^-) = \alpha^-$ and

$GSSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \leq GSSVNWA(\alpha^+, \alpha^+, ..., \alpha^+) = \alpha^+$

So, we have $\alpha^- \leq GSSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \leq \alpha^+$.

Hence the proof is completed. □

Theorem 3.7. Let $\alpha_j (j = 1, 2, ..., n)$, be a collection of SVN-numbers. The aggregated value of SVN-numbers using the GSSVNWG operator is also a SVN-number and

$$GSSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) = \left\langle 1 - \left(1 - \prod_{j=1}^{n} \left(1 - (1 - T_j)^{\lambda}\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{j=1}^{n} (1 - I_j^{\lambda})^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{j=1}^{n} (1 - F_j^{\lambda})^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}} \right\rangle \quad (11)$$

Proof. : *We proof the above Theorem 4 by using mathematical induction.*
For n = 1, we have:

$$GSSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) = \left\langle 1 - \left(1 - \prod_{j=1}^{1} \left(1 - (1 - T_j)^{\lambda}\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{j=1}^{1} (1 - I_j^{\lambda})^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{j=1}^{1} (1 - F_j^{\lambda})^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}} \right\rangle = \langle T_1, I_1, F_1 \rangle$$

Since, for $n = 1$, $\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)} = 1$.
Thus Eq.(11) holds for n = 1. Assume that the Eq. (11) holds for n = m,

$$GSSVNWG(\alpha_1, \alpha_2, ..., \alpha_m) = \left\langle 1 - \left(1 - \prod_{j=1}^{m} \left(1 - (1 - T_j)^{\lambda}\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{j=1}^{m} (1 - I_j^{\lambda})^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{j=1}^{m} (1 - F_j^{\lambda})^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}} \right\rangle.$$

Now, we prove that the Eq. (12) holds for n = m + 1.

$$\begin{aligned}
GSSVNWG(\alpha_1,\alpha_2,...,\alpha_m,\alpha_{m+1}) &= \frac{1}{\lambda}\left((\lambda\alpha_1)^{\phi_k^1}\otimes(\lambda\alpha_2)^{\phi_k^2}\otimes...\otimes(\lambda\alpha_m)^{\phi_k^m}\otimes(\lambda\alpha_{m+1})^{\phi_k^{m+1}}\right)\\
&= \Bigg\langle\left(1-\left(1-\prod_{j=1}^{m}\left(1-(1-T_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\right)\\
&\quad\times\left(1-\left(1-\left(1-(1-T_{m+1})^\lambda\right)^{\frac{exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\right),\\
&\quad\left(1-\prod_{j=1}^{m}(1-I_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}+\left(1-(1-I_{m+1}^\lambda)^{\frac{exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\\
&\quad-\left(1-\prod_{j=1}^{m}(1-I_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\times\left(1-(1-I_{m+1}^\lambda)^{\frac{exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}},\\
&\quad\left(1-\prod_{j=1}^{m}(1-F_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}+\left(1-(1-F_{m+1}^\lambda)^{\frac{exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\\
&\quad-\left(1-\prod_{j=1}^{m}(1-F_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\times\left(1-(1-F_{m+1}^\lambda)^{\frac{exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\Bigg\rangle\\
&= \Bigg\langle 1-\left(1-\prod_{j=1}^{m+1}\left(1-(1-T_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m+1}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}},\\
&\quad\left(1-\prod_{j=1}^{m+1}(1-I_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m+1}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}},\\
&\quad\left(1-\prod_{j=1}^{m+1}(1-F_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m+1}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\Bigg\rangle
\end{aligned}$$

Therefore, Eq. (11) holds for n = m+1, hence the Eq. (11) holds for all positive integer by principle of mathematical induction. Hence the proof of the theorem is completed. □

Example 3.8. Let $\alpha_1 = (0.6, 0.4, 0.5), \alpha_2 = (0.7, 0.3, 0.5), \alpha_3 = (0.8, 0.3, 0.4)$ and $\alpha_4 = (0.7, 0.4, 0.5)$ be the four SVN-numbers. Rank the four SVN-numbers using GSSVNWG operator.

Solution : In the following, we use the GSSVNWG operator to aggregate these SVN-numbers. At first, we calculate the score values of four SVN-numbers using Eq. (1).

$S(\alpha_1) = 0.567, S(\alpha_2) = 0.633, S(\alpha_3) = 0.700, S(\alpha_4) = 0.600$, then

$\vartheta_1 = 1, \vartheta_2 = 0.567, \vartheta_3 = 0.359, \vartheta_4 = 0.251$.

To calculate $exp(\vartheta_j/k)$ we take k =1, then

$exp(\vartheta_1/k) = 2.718, exp(\vartheta_2/k) = 1.763, exp(\vartheta_3/k) = 1.432, exp(\vartheta_4/k) = 1.285$
and

$$\frac{exp(\vartheta_1/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.378, \frac{exp(\vartheta_2/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.245$$

,

$$\frac{exp(\vartheta_3/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.199, \frac{exp(\vartheta_4/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.178,$$

Taking $\lambda = 1$, GSSVNWG reduces to SSVNWG and we get $GSSVNWG(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.678, 0.357, 0.482)$.
Again for $\lambda = 2$, we obtain

$$\begin{aligned} GSSVNWG(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= \Bigg(1 - \Big(1 - \big(1 - (1 - 0.6)^2\big)^{0.378} \times \big(1 - (1 - 0.7)^2\big)^{0.245} \\ &\quad \times \big(1 - (1 - 0.8)^2\big)^{0.199} \times \big(1 - (1 - 0.7)^2\big)^{0.178}\Big)^{\frac{1}{2}}, \\ &\quad \Big(1 - \big(1 - (0.6)^2\big)^{0.378} \times \big(1 - (0.3)^2\big)^{0.245} \times \big(1 - (0.3)^2\big)^{0.199} \times \big(1 - (0.4)^2\big)^{0.178}\Big)^{\frac{1}{2}}, \\ &\quad \Big(1 - \big(1 - (0.5)^2\big)^{0.378} \times \big(1 - (0.5)^2\big)^{0.245} \times \big(1 - (0.4)^2\big)^{0.199} \times \big(1 - (0.5)^2\big)^{0.178}\Big)^{\frac{1}{2}}\Bigg) \\ &= (0.671, 0.464, 0.482). \end{aligned}$$

Taking $\lambda = 3$, we obtain

$$\begin{aligned} GSSVNWG(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= \Bigg(1 - \Big(1 - \big(1 - (1 - 0.6)^3\big)^{0.378} \times \big(1 - (1 - 0.7)^3\big)^{0.245} \\ &\quad \times \big(1 - (1 - 0.8)^3\big)^{0.199} \times \big(1 - (1 - 0.7)^3\big)^{0.178}\Big)^{\frac{1}{3}}, \\ &\quad \Big(1 - \big(1 - (0.6)^3\big)^{0.378} \times \big(1 - (0.3)^3\big)^{0.245} \times \big(1 - (0.3)^3\big)^{0.199} \times \big(1 - (0.4)^3\big)^{0.178}\Big)^{\frac{1}{3}}, \\ &\quad \Big(1 - \big(1 - (0.5)^3\big)^{0.378} \times \big(1 - (0.5)^3\big)^{0.245} \times \big(1 - (0.4)^3\big)^{0.199} \times \big(1 - (0.5)^3\big)^{0.178}\Big)^{\frac{1}{3}}\Bigg) \\ &= (0.662, 0.480, 0.484). \end{aligned}$$

Similarly, we can also show that for $\lambda = 4, 5, ...$ and so on.

3.2.4. *Properties of $GSSVNWG$ operator*

Property 1: Idem-potency

If $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha(say)$, then $GSSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) = \alpha$.

Proof. Let $\alpha_j = \langle T_j, I_j, F_j\rangle, (j = 1, 2, 3..., n)$ *and* $\alpha = \langle T, I, F\rangle$.
Since all α_j *are equal, based on Theorem (4), we get*

$$
\begin{aligned}
GSSVNWG(\alpha,\alpha,...,\alpha) &= \Bigg\langle 1-\left(1-\left(1-(1-T)^{\lambda}\right)^{\sum_{j=1}^{n}\phi_k^j}\right)^{\frac{1}{\lambda}}, \left(1-(1-I^{\lambda})^{\sum_{j=1}^{n}\phi_k^j}\right)^{\frac{1}{\lambda}}, \\
&\quad \left(1-(1-F^{\lambda})^{\sum_{j=1}^{n}\phi_k^j}\right)^{\frac{1}{\lambda}} \Bigg\rangle \\
&= \langle T, I, F\rangle = \alpha
\end{aligned}
$$

Since, $\phi_k^j = \frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)}, k > 0$ *and* $\sum_{j=1}^{n}\phi_k^j = 1$*. Hence the proof is completed.* □

2. Monotonicity:

Let $\alpha_j (j = 1, 2, ..., n)$ and $\beta_j (j = 1, 2, ..., n)$ be any two sets of SVN-numbers. If $\alpha_j \leq \beta_j$ for any j, then $GSSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \leq GSSVNWG(\beta_1, \beta_2, ..., \beta_n)$.

Proof. : Based on the Theorem (4), we get

$$
\begin{aligned}
GSSVNWG(\alpha_1,\alpha_2,...,\alpha_n) &= \left(1-\left(1-\prod_{j=1}^{n}\left(1-\left(1-T_{\alpha_j}\right)^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}}, \right. \\
&\quad \left(1-\prod_{j=1}^{n}(1-I_{\alpha_j}^{\lambda})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}}, \\
&\quad \left.\left(1-\prod_{j=1}^{n}(1-F_{\alpha_j}^{\lambda})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}} \right)
\end{aligned}
$$

and

$$
\begin{aligned}
GSSVNWG(\beta_1,\beta_2,...,\beta_n) &= \left(1-\left(1-\prod_{j=1}^{n}\left(1-\left(1-T_{\beta_j}\right)^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}}, \right. \\
&\quad \left(1-\prod_{j=1}^{n}(1-I_{\beta_j}^{\lambda})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}}, \\
&\quad \left.\left(1-\prod_{j=1}^{n}(1-F_{\beta_j}^{\lambda})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}} \right)
\end{aligned}
$$

Since all $\alpha_j \leq \beta_j (j = 1, 2, ..., n)$. Therefore,

$$\begin{aligned}
T_{\alpha_j} \leq T_{\beta_j} &\Rightarrow \left(1 - T_{\alpha_j}\right) \geq \left(1 - T_{\beta_j}\right), \\
&\Rightarrow \left(1 - T_{\alpha_j}\right)^\lambda \geq \left(1 - T_{\beta_j}\right)^\lambda, \\
&\Rightarrow \left(1 - \left(1 - T_{\alpha_j}\right)^\lambda\right) \leq \left(1 - \left(1 - T_{\beta_j}\right)^\lambda\right) \\
&\Rightarrow \left(1 - \left(1 - T_{\alpha_j}\right)^\lambda\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \leq \left(1 - \left(1 - T_{\beta_j}\right)^\lambda\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}} \\
&\Rightarrow \prod_{j=1}^{n} \left(1 - \left(1 - T_{\alpha_j}\right)^\lambda\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \leq \prod_{j=1}^{n} \left(1 - \left(1 - T_{\beta_j}\right)^\lambda\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}} \\
&\Rightarrow 1 - \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - T_{\alpha_j}\right)^\lambda\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}} \\
&\leq 1 - \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - T_{\beta_j}\right)^\lambda\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}}
\end{aligned}$$

Further,

$$\begin{aligned}
I_{\alpha_j} \geq I_{\beta_j} &\Rightarrow I_{\alpha_j}^\lambda \geq I_{\beta_j}^\lambda \Rightarrow \left(1 - I_{\alpha_j}^\lambda\right) \leq \left(1 - I_{\beta_j}^\lambda\right) \\
&\Rightarrow \left(1 - I_{\alpha_j}^\lambda\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \leq \left(1 - I_{\beta_j}^\lambda\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}} \\
&\Rightarrow \prod_{j=1}^{n} \left(1 - I_{\alpha_j}^\lambda\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \leq \prod_{j=1}^{n} \left(1 - I_{\beta_j}^\lambda\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}} \\
&\Rightarrow \left(1 - \prod_{j=1}^{n} \left(1 - I_{\alpha_j}^\lambda\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right) \geq \left(1 - \prod_{j=1}^{n} \left(1 - I_{\beta_j}^\lambda\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right) \\
&\Rightarrow \left(1 - \prod_{j=1}^{n} \left(1 - I_{\alpha_j}^\lambda\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}} \geq \left(1 - \prod_{j=1}^{n} \left(1 - I_{\beta_j}^\lambda\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}}
\end{aligned}$$

Similarly, we can also show that

$$\Rightarrow \left(1 - \prod_{j=1}^{n} \left(1 - F_{\alpha_j}^\lambda\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}} \geq \left(1 - \prod_{j=1}^{n} \left(1 - F_{\beta_j}^\lambda\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}}$$

Hence the proof is completed. □

Property 3: Boundedness

Let $\alpha_j (j = 1, 2, ..., n)$ be any set of SVN-numbers. If $\alpha^- = \min\{\alpha_j\}$ and $\alpha^+ = \max\{\alpha_j\}$, then $\alpha^- \leq GSSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \leq \alpha^+$.

Proof. : *Let* $\alpha^{+} = max\{\alpha_1, \alpha_2, ..., \alpha_n\}$ *and* $\alpha^{-} = min\{\alpha_1, \alpha_2, ..., \alpha_n\}$.
According to the Property 1 and 2, we have
$GSSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \geq GSSVNWG(\alpha^{-}, \alpha^{-}, ..., \alpha^{-}) = \alpha^{-}$
and $GSSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \leq GSSVNWG(\alpha^{+}, \alpha^{+}, ..., \alpha^{+}) = \alpha^{+}$
So, we have $\alpha^{-} \leq GSSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \leq \alpha^{+}$.
Hence the proof is completed. □

4. MADM under SVN environment based on proposed operators

The main goal of a MADM strategy is to find the one or more alternative which satisfies the objective of decision maker from a set of possible alternatives w.r.t significant attributes. Using the proposed aggregation operators a MADM strategy under SVN environment is considered. A MADM strategy is presented here ton show the application of proposed approach.

4.1. *Decision making approach based on proposed operators*

Let $\Psi = \{\psi_1, \psi_2, ..., \psi_m\}$and $C = \{C_1, C_2, ..., C_n\}$ be the possible set of alternatives and attributes respectively. Let $W = \{w_1, w_2, ..., w_n\}$ be the weight vector of attributes C_j (j = 1, 2, 3, . . . , n), where $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$. Now, we have described the steps of proposed MADM strategy by following algorithm.

Algorithm:
Step 1: Formulate the decision matrix
For MADM with SVN–number information, the rating values of the alternative $\psi_i(i = 1, 2, ..., m)$ on the basis of attribute $C_j(j = 1, 2, ..., n)$can be expressed in SVN–number as a_{ij} where (i = 1, 2, 3, . . . , m; j = 1, 2, 3, . . . , n).
The decision matrix is represented as follows:

$$[\ A_{ij}]_{m\times n} = \begin{array}{c} \\ \psi_1 \\ \psi_2 \\ \vdots \\ \psi_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \quad (12)$$

is called an decision making matrix.
Step 2: Compute the score matrix and ϑ_{ij} value matrix
Using Eq. (1) and Eq. (2), We calculate the score value of each alternative for different attribute and represent as matrix form as:

$$[\ S_{ij}]_{m\times n} = \begin{array}{c} \\ \psi_1 \\ \psi_2 \\ \vdots \\ \psi_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & \cdots & s_{mn} \end{pmatrix} \end{array} \tag{13}$$

and calculate ϑ_{ij} value using Eq. (3), represent as follows:

$$[\vartheta_{ij}]_{m\times n} = \begin{array}{c} \\ \psi_1 \\ \psi_2 \\ \vdots \\ \psi_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} \vartheta_{11} & \vartheta_{12} & \cdots & \vartheta_{1n} \\ \vartheta_{21} & \vartheta_{22} & \cdots & \vartheta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \vartheta_{m1} & \vartheta_{m2} & \cdots & \vartheta_{mn} \end{pmatrix} \end{array} \tag{14}$$

Step 3: Compute weighted matrix

We calculate weight of each alterntive for each attribute by the Eq. (3) and represent in matrix form as:

$$[\phi_k^{ij}]_{m\times n} = \begin{array}{c} \\ \psi_1 \\ \psi_2 \\ \vdots \\ \psi_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} \phi_k^{11} & \phi_k^{12} & \cdots & \phi_k^{1n} \\ \phi_k^{21} & \phi_k^{22} & \cdots & \phi_k^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_k^{m1} & \phi_k^{m2} & \cdots & \phi_k^{mn} \end{pmatrix} \end{array} \tag{15}$$

Where, $\phi_k^{ij} = \frac{exp(\vartheta_{ij}/k)}{\sum_{j=1}^{n} exp(\vartheta_{ij}/k)}, \vartheta_{ij} = \begin{cases} \prod_{i=1}^{j-1} S_i, & \text{j} = 2, 3, ..., n \\ 1 & \text{j} = 1 \end{cases}$, S_i is the score function of the SVNN α_i.

Step 4: Aggregate the all attributes

Using aggregation operators we aggragate the all attribute values for respective alternative and results are shown in table form as:

TABLE 1. The aggregated SVNNs and score values of aggregated SVNNs

Alternatives	Aggregated SVNNs	Score values
ψ_1	$\tilde{a_1}$	$\tilde{S_1}$
ψ_2	$\tilde{a_2}$	$\tilde{S_2}$
.	.	.
.	.	.
ψ_m	$\tilde{a_m}$	$\tilde{S_m}$

Step 5: Ranking the alternatives

Based on the score value (From Table 1) of alternative, we arranged the ranking odrer of alternatives using Eq. (1) and Eq. (2).

Step 6: End the algorithm.

5. **Numerical illustration**

Every year worldwide, many peoples are affected by various natural disasters. These disasters are Hurricanes and Tropical Storms, Drought, Wildfires, Floods, Earthquakes, Tornadoes, severe storms, etc. The most common thoughtful nature disaster is the Flood disaster. Flood disaster problems can handle by MADM strategy according to the given information Yu (2016). In Flood disaster control and mitigation, risk decision and evaluation are significant steps. According to our knowledge (Ya 2012), we have composed four essential attributes to evaluate the risk of Flood disaster, which are:

i) Disaster-inducing factors (C_1),

ii) Hazard-formative environment (C_2),

iii) Characters of hazard affected body (C_3), and

iv) Social disaster bearing capacity (C_4).

Apparently, these evaluation attribute are complicated and difficult to characterize quantitatively. We can handle this type of difficulties considering the attributes information by SVN set.

Let us assume that ψ_1, ψ_2, ψ_3, and ψ_4 are the four maritime cities in India. Our aim is to find the best city according to the four attributes. We expressed the appraisement informations of four cities according to the four attributes in terms of SVN-set. Now, we will solved this decision making problem using the proposed operators.

Step 1: Formulate the decision matrix

The appraisement informations of four cities consider by SVN-number according to the four attributes. Re-presentation of the decision matrix shown in Eq. (16) as given by:

$$[\,A]_{4\times 4} = \begin{matrix} & C_1 & C_2 & C_3 & C_4 \\ \psi_1 & \langle 0.6,0.4,0.3\rangle & \langle 0.7,0.3,0.4\rangle & \langle 0.8,0.4,0.6\rangle & \langle 0.6,0.2,0.4\rangle \\ \psi_2 & \langle 0.3,0.1,0.4\rangle & \langle 0.5,0.2,0.2\rangle & \langle 0.8,0.3,0.4\rangle & \langle 0.6,0.3,0.5\rangle \\ \psi_3 & \langle 0.6,0.3,0.5\rangle & \langle 0.7,0.3,0.5\rangle & \langle 0.8,0.3,0.5\rangle & \langle 0.4,0.3,0.2\rangle \\ \psi_4 & \langle 0.7,0.3,0.3\rangle & \langle 0.3,0.4,0.3\rangle & \langle 0.7,0.4,0.5\rangle & \langle 0.8,0.3,0.4\rangle \end{matrix} \tag{16}$$

Step 2: Compute the score matrix and ϑ_{ij} value matrix

Using Eq. (1), We calculate the score value of each alternative for different attribute and represent as matrix form as:

$$[S]_{4\times 4}=\begin{array}{c}\\ \psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{array}\begin{array}{c}\begin{array}{cccc}C_1 & C_2 & C_3 & C_4\end{array}\\ \begin{pmatrix}0.63 & 0.67 & 0.60 & 0.67\\ 0.60 & 0.70 & 0.70 & 0.60\\ 0.60 & 0.63 & 0.67 & 0.63\\ 0.70 & 0.53 & 0.60 & 0.70\end{pmatrix}\end{array} \tag{17}$$

and calculate ϑ_{ij} value using Eq. (4), represent as follows:

$$[\vartheta]_{4\times 4}=\begin{array}{c}\\ \psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{array}\begin{array}{c}\begin{array}{cccc}C_1 & C_2 & C_3 & C_4\end{array}\\ \begin{pmatrix}1 & 0.63 & 0.42 & 0.28\\ 1 & 0.60 & 0.42 & 0.29\\ 1 & 0.60 & 0.38 & 0.25\\ 1 & 0.70 & 0.37 & 0.22\end{pmatrix}\end{array} \tag{18}$$

Step 3: Compute ϕ_k^{ij} matrix (Let parameter $k = 1$)

We calculate the values of ϕ_k^{ij} for each alternative with respects to each attributes by the Eq. (3) and represent in matrix form as:

$$[\phi]_{4\times 4}=\begin{array}{c}\\ \psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{array}\begin{array}{c}\begin{array}{cccc}C_1 & C_2 & C_3 & C_4\end{array}\\ \begin{pmatrix}0.36 & 0.25 & 0.20 & 0.18\\ 0.34 & 0.24 & 0.21 & 0.18\\ 0.37 & 0.25 & 0.20 & 0.18\\ 0.37 & 0.27 & 0.20 & 0.17\end{pmatrix}\end{array} \tag{19}$$

Step 4: Aggregate the all attribute values of alternatives

Based on the $SSVNWA$ operator, the aggregated SVN-numbers and score values of Eq. (17) are shown in Table 2 (Parameter $k = 1$ fixed).

From Table 2, we find the riskiest city is ψ_2.

Based on the $SSVNWG$ operator, the aggregated SVN-numbers and corresponding score values of Eq. (16) are shown in Table 3(Parameter $k = 1$ fixed),

From Table 3, we find the riskiest city is ψ_2.

Based on the $GSSVNWA$ operator, the aggregated SVN-numbers and corresponding score values of Eq. (16) are shown in Table 4(Parameter $\lambda = 2, 5, 10$ and Parameter $k = 1$ fixed).

From Table 4, we find the riskiest city is ψ_4 for $\lambda = 2$ and ψ_2 for $\lambda = 5, 10$.

Based on the $GSSVNWG$ operator, the aggregated SVN-numbers and corresponding score values of Eq. (16) are shown in Table 5(Parameter $\lambda = 2, 5, 10$ and Parameter $k = 1$ fixed).

From Table 5, we find the riskiest city is ψ_1 for $\lambda = 2, 5, 10$.

Step 5: Ranking order of alternatives

TABLE 2. The aggregated SVN-numbers and score values based on SSVNWA operator

Alternatives	Aggregated SVNNs	Score values
ψ_1	$\langle 0.622, 0.332, 0.398 \rangle$	0.631
ψ_2	$\langle 0.546, 0.194, 0.362 \rangle$	**0.663**
ψ_3	$\langle 0.651, 0.300, 0.424 \rangle$	0.642
ψ_4	$\langle 0.652, 0.339, 0.345 \rangle$	0.656

TABLE 3. The aggregated SVNNs and score values based on SSVNWG operator

Alternatives	Aggregated SVNNs	Score values
ψ_1	$\langle 0.633, 0.340, 0.414 \rangle$	0.636
ψ_2	$\langle 0.489, 0.204, 0.368 \rangle$	**0.669**
ψ_3	$\langle 0.614, 0.300, 0.456 \rangle$	0.619
ψ_4	$\langle 0.568, 0.351, 0.365 \rangle$	0.617

TABLE 4. The aggregated SVNNs and score values based on GSSVNWA operator

value of λ	Alternatives	Aggregated SVNNs	Score values
$\lambda = 2$	ψ_1	$\langle 0.679, 0.327, 0.388 \rangle$	0.655
	ψ_2	$\langle 0.572, 0.189, 0.353 \rangle$	0.677
	ψ_3	$\langle 0.659, 0.300, 0.414 \rangle$	0.648
	ψ_4	$\langle 0.665, 0.245, 0.343 \rangle$	**0.692**
$\lambda = 5$	ψ_1	$\langle 0.689, 0.316, 0.372 \rangle$	0.667
	ψ_2	$\langle 0.632, 0.178, 0.330 \rangle$	**0.708**
	ψ_3	$\langle 0.680, 0.300, 0.378 \rangle$	0.643
	ψ_4	$\langle 0.692, 0.337, 0.336 \rangle$	0.673
$\lambda = 10$	ψ_1	$\langle 0.710, 0.294, 0.353 \rangle$	0.688
	ψ_2	$\langle 0.691, 0.163, 0.295 \rangle$	**0.744**
	ψ_3	$\langle 0.708, 0.300, 0.320 \rangle$	0.696
	ψ_4	$\langle 0.715, 0.330, 0.326 \rangle$	0.686

TABLE 5. The aggregated SVNNs and score values based on GSSVNWG operator

value of λ	Alternatives	Aggregated SVNNs	Score values
	ψ_1	$\langle 0.656, 0.347, 0.424\rangle$	**0.628**
	ψ_2	$\langle 0.464, 0.221, 0.382\rangle$	0.620
$\lambda = 2$	ψ_3	$\langle 0.598, 0.300, 0.465\rangle$	0.611
	ψ_4	$\langle 0.536, 0.353, 0.370\rangle$	0.604
	ψ_1	$\langle 0.638, 0.364, 0.463\rangle$	**0.603**
	ψ_2	$\langle 0.413, 0.253, 0.408\rangle$	0.584
$\lambda = 5$	ψ_3	$\langle 0.548, 0.300, 0.481\rangle$	0.589
	ψ_4	$\langle 0.451, 0.361, 0.395\rangle$	0.565
	ψ_1	$\langle 0.623, 0.378, 0.513\rangle$	**0.577**
	ψ_2	$\langle 0.369, 0.273, 0.433\rangle$	0.554
$\lambda = 10$	ψ_3	$\langle 0.492, 0.300, 0.490\rangle$	0.567
	ψ_4	$\langle 0.385, 0.373, 0.430\rangle$	0.527

According to the decreasing score value of alternatives $\psi_i (i = 1, 2, 3, 4)$ and based on the Table 2, Table 3, Table 4 and Table 5, the ranking order of alternatives is presented in Table 6.

TABLE 6. Ranking order of alternatives and riskiest city for various operators

Proposed operators	Ranking order of alternatives	riskiest city
$SSVNWA$	$\psi_2 \succ \psi_4 \succ \psi_3 \succ \psi_1$	ψ_2
$SSVNWG$	$\psi_2 \succ \psi_1 \succ \psi_3 \succ \psi_4$	ψ_2
$GSSVNWA,\ \lambda = 2$	$\psi_4 \succ \psi_2 \succ \psi_1 \succ \psi_3$	ψ_4
$\lambda = 5$	$\psi_2 \succ \psi_4 \succ \psi_1 \succ \psi_3$	ψ_2
$\lambda = 10$	$\psi_2 \succ \psi_3 \succ \psi_1 \succ \psi_4$	ψ_2
$GSSVNWG,\ \lambda = 2$	$\psi_1 \succ \psi_2 \succ \psi_3 \succ \psi_4$	ψ_1
$\lambda = 5$	$\psi_1 \succ \psi_3 \succ \psi_2 \succ \psi_4$	ψ_1
$\lambda = 10$	$\psi_1 \succ \psi_3 \succ \psi_2 \succ \psi_4$	ψ_1

Step 6: The procedure of proposed algorithm end here.

In the the numerical example we analysed FD-risk assessment problem. It easy to recognize that the neutrosophic set of information is expressed by SVN-number. Here we have been examined in details for the FD-risk assessment problem. Also the process of proposed strategies are reasonable for this problem. From the numerical example, we can say that it is comfortable to use the strategy to cope with the other risk assessment problems. Therefore the proposed decision making strategy has a deep practical value.

6. Comparative Analysis

A comparative study was constructed with other existing methods to show the validity of the proposed ranking method. The proposed method is compared to the other techniques such as Wei and Wei (2018), Nancy and Garg (2016), and Rong et al. (2020) SVN environments. In Table-7, we have presented a comparative analysis. By Wei and Wei (2018) method, the best alternative is ψ_2 and the worst one is ψ_1. According to the Nancy and Garg (2016) method, the best alternative is ψ_3, and the worst one is ψ_2. Again by the Rong et al. (2020) method, the best alternative is ψ_4, and the worst one is ψ_3. By our proposed method, the best alternative is ψ_2 and the worst one is ψ_1. From Table-7, it is clear that our proposed method gives better results than the other existing method.

TABLE 7. Comparative studies with other existence method

Proposed operators	Ranking order of alternatives	Best Alternatives
Wei and Wei (2018)	$\psi_2 \succ \psi_4 \succ \psi_3 \succ \psi_1$	ψ_2
Nancy and Garg (2016)	$\psi_3 \succ \psi_1 \succ \psi_4 \succ \psi_2$	ψ_3
Rong et al. (2020)	$\psi_4 \succ \psi_1 \succ \psi_2 \succ \psi_3$	ψ_4
Our Method	$\psi_2 \succ \psi_4 \succ \psi_1 \succ \psi_3$	ψ_2

7. Conclusions

In recent years, aggregation operators have become a popular research topic in decision-making problems. This paper presents some new aggregation operators for solving a real MADM problem under an SVN environment. Additionally, some different aggregation operators are developed, which are the softmax SVN weighted average (SVNWA) operator, softmax SVN weighted geometric (SVNWG) operator, generalized softmax SVN weighted average (GSSVNWA) operator, and generalized softmax SVN weighted geometric (GSSVNIFWG) operator. Then, we have presented some essential properties of these operators. Moreover, using the proposed operators, we have been built a MADM strategy under an SVN environment. Finally, we have illustrated one numerical example to express the usefulness and effectiveness of the proposed MADM technique. Also, we have presented a comparative analysis with other existing methods.

In the future, we will extend the proposed operators in interval neutrosophic set Wang 2005, neutrosophic cubic set (Ali 2016), and refined neutrosophic set (Smarandache 2013) environments. Also, we will try to apply the proposed operators to different realistic decision-making problems.

Compliance with ethical standards

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On Symbolic Plithogenic Algebraic Structures and Hyper Structures

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Abstract. The objective of this paper is to study symbolic Plithogenic algebraic structures and hyper structures. In particular, we study symbolic Plithogenic group, symbolic Plithogenic ring, symbolic Plithogenic hypergroup and symbolic Plithogenic canonical hypergroup.

Keywords: Plithogeny; Plithogenic; Plithogenic Set; Fuzzy Set; Intuitionistic Fuzzy Set; Neutrosophy; Neutrosophic Set; Plithogenic Group; Plithogenic Ring; Plithogenic Hypergroup; Plithogenic canonical Hypergroup.

1. **Introduction**

The concepts of Plithogeny, Plithogenic logic/set, Plithogenic probability and Plithogenic statistics were introduced by Smarandache in [26]. Plithogenic set/logic is an extension of the classical logic/set, fuzzy logic/set of Zadeh [37], intuitionistic fuzzy logic/set of Atanassov [11], neutrosophic logic/set of Smarandache [30] and quadruple neutrosophic logic/set of Smarandache [29]. Smarandache in [23], [25] and [28] introduced and studied symbolic Plithogenic algebraic structures and hyper structures. In [22], Merkepsi and Abobala studied symbolic 2-Plithogenic rings, in [10], Al-Basheer et al. studied symbolic 3-Plithogenic rings and in [17], Gayen et al. studied Plithogenic Hypersoft Subgroup. Also in [32], Taffach and Hatip studied Symbolic

2-Plithogenic Number Theory And Algebraic Equations, in [33], Taffach and Othman studied Symbolic 2-Plithogenic Modules over Symbolic 2-Plithogenic Rings and in [34], Taffach studied Symbolic 2-Plithogenic Vector Spaces. In [18], [24] and [27], applications of Plithogenic set/logic were presented. In the present paper, we study symbolic Plithogenic algebraic structures and hyper structures. In particular, we study symbolic Plithogenic group, symbolic Plithogenic ring, symbolic Plithogenic hypergroup and symbolic Plithogenic canonical hypergroup and present their basic properties.

2. Symbolic Plithogenic Set

A symbolic Plithogenic set SPX is defined by

$$SPX = \{(a, a_1P_1, a_2P_2, a_3P_3, \cdots, a_nP_n) : a, a_i \in \mathbb{R} \text{ or } \mathbb{C} \text{ or any AlgebraicStructure}\} \quad (1)$$

where P_i's are the Plithogenic parameters/variables. a is called the non-Plithogenic part of SPX, a_iP_i is called the Plithogenic part of SPX and a_i's are called the coefficients of P_is where $i = 1, 2, 3, \cdots, n$. For a positive integer k, P_i has the following properties :

$$P_i^k = P_i, \quad \forall i \text{ and } k \geq 2, \quad (2)$$

$$kP_i = P_i + P_i + P_i + \cdots + P_i \quad \text{[k summand]} \quad \forall i, \quad (3)$$

$$0P_i = 0 \quad \forall i, \quad (4)$$

$$P_i^{-1} = \frac{1}{P_i} \text{ does not exist } \forall i. \quad (5)$$

when $n = 1$, equation (1) reduces to

$$SPX = \{(a, a_1P_1) : a, a_1 \in \mathbb{R} \text{ or } \mathbb{C} \text{ or any AlgebraicStructure}\} \quad (6)$$

and SPX becomes the usual Neutrosophic set with $P_1 = I$.

When $n = 3$, equation (1) reduces to

$$SPX = \{(a, a_1P_1, a_2P_2, a_3P_3) : a, x_i \in \mathbb{R} \text{ or } \mathbb{C} \text{ or any AlgebraicStructure}\} \quad (7)$$

and SPX becomes the usual Neutrosophic Quadruple set with $P_1 = T$, $P_2 = I$ and $P_3 = F$.

When $n = 2$, equation (1) reduces to

$$SPX = \{(a, a_1P_1, a_2P_2) : a, a_i \in \mathbb{R} \text{ or } \mathbb{C} \text{ or any AlgebraicStructure}\} \quad (8)$$

which is called symbolic 2-Plithogenic set.

3. **Symbolic Plithogenic Algebraic Structure**

All the symbolic Plithogenic sets to be considered in this section and the section after will be symbolic 2-Plithogenic sets of the form given by equation (8) and we are going to assume throughout the prevalence order $P_1 > P_2$ so that

$$P_1P_1 = P_{\min\{1,1\}} = P_1, \tag{9}$$

$$P_2P_2 = P_{\min\{2,2\}} = P_2, \tag{10}$$

$$P_1P_2 = P_2P_1 = P_{\min\{1,2\}} = P_1. \tag{11}$$

Definition 3.1. Let $+$, $-$ and $.$ be the usual arithmetic operations of addition, subtraction and multiplication of numbers respectively and let k be a nonzero scalar. If $x = (a, a_1P_1, a_2P_2)$ and $y = (b, b_1P_1, b_2P_2)$ are arbitrary elements of the symbolic Plithogenic set SPX where $a, b, a_i, b_i \in \mathbb{R}$ or $\mathbb{C}$, then:

$$x \pm y = (a \pm b, (a_1 \pm b_1)P_1, (a_2 \pm b_2)P_2), \tag{12}$$

$$kx = (ka, ka_1P_1, ka_2P_2), \tag{13}$$

$$x.y = (ab, (ab_1 + a_1b + a_1b_1 + a_1b_2 + a_2b_1)P_1, (ab_2 + a_2b + a_2b_2)P_2). \tag{14}$$

When $k = 0$, then we have

$$0x = (0a, 0a_1P_1, 0a_2P_2) = (0, 0P_1, 0P_2) = (0, 0, 0). \tag{15}$$

Notation 3.2. In what follows next, we will use the symbols $SP\mathbb{N}$, $SP\mathbb{Z}$, $SP\mathbb{Q}$, $SP\mathbb{R}$ and $SP\mathbb{C}$ to denote the Plithogenic sets of natural, integer, rational, real and complex numbers respectively.

Example 3.3. $(SP\mathbb{Q}, .)$, $(SP\mathbb{R}, .)$ and $(SP\mathbb{C}, .)$ are symbolic Plithogenic groups.

Definition 3.4. Let $(X, *)$ be any algebraic structure and let SPX be the corresponding symbolic Plithogenic set. The couple $(SPX, *)$ is called a symbolic Plithogenic algebraic structure. SPX will be named according to the name of the underlying algebraic structure X. For instance if X is a group, SPX will be called a symbolic Plithogenic group, if X is a ring, SPX will be called a symbolic Plithogenic ring, if X is a hypergroup, SPX will be called a symbolic Plithogenic hypergroup and so on .

Theorem 3.5. *Let $(G, *)$ be a group and let SPG be the corresponding symbolic Plithogenic group. Then:*

(i) *$G \subset SPG$.*
(ii) *$(SPG, *)$ is a semigroup.*
(iii) *$(SPG, *)$ is not a group.*

Proof. (i) This follows from the definition of SPG.
(ii) Let $x = (a, a_1P_1, a_2P_2)$, $y = (b, b_1P_1, b_2P_2)$ and $z = (c, c_1P_1, c_2P_2)$ be arbitrary elements of SPG. Then:
$x * y = (ab, (ab_1 + a_1b + a_1b_1 + a_1b_2 + a_2b_1)P_1, (ab_2 + a_2b + a_2b_2)P_2) \in SPG$. Now,

$$\begin{aligned}(x * y) * z &= (abc, (abc_1 + ab_1c + a_1bc + a_1b_1c + a_1b_2c + a_2b_1c + ab_1c_1 + a_1bc_1 + a_1b_1c_1 \\ &\quad + a_1b_2c_1 + a_2b_1c_1 + ab_1c_2 + a_1bc_2 + a_1b_1c_2 + a_1b_2c_2 + a_2b_1c_2 + ab_2c_1 \\ &\quad + a_2bc_1 + a_2b_2c_1)P_1, (abc_2 + ab_2c + a_2bc + a_2b_2c + ab_2c_2 + a_2bc_2 + a_2b_2c_2)P_2) \\ x * (y * z) &= (abc, (abc_1 + ab_1c + ab_1c_1 + ab_1c_2 + ab_2c_1 + a_1bc + a_1bc_1 + a_1b_1c + a_1b_1c_1 \\ &\quad + a_1b_1c_2 + a_1b_2c_1 + a_1bc_2 + a_1b_2c + a_1b_2c_2 + a_2bc_1 + a_2b_1c + a_2b_1c_1 \\ &\quad + a_2b_1c_2 + a_2b_2c_1)P_1, (abc_2 + ab_2c + a_2bc + a_2b_2c + ab_2c_2 + a_2bc_2 + a_2b_2c_2)P_2) \\ &= x * (y * z).\end{aligned}$$

This shows that $(SPG, *)$ is a semigroup.
(iii) Since P_1^{-1} and P_2^{-1} do not exist, it follows that we cannot find x^{-1}, $\forall x \in SPG$. Hence, $(SPG, *)$ is not a group. □

Remark 3.6. If $(G, +)$ is a group, then the symbolic Plithogenic group $(SPG, +)$ is a group.

Example 3.7. $(SP\mathbb{Z}, +)$, $(SP\mathbb{Q}, +)$, $(SP\mathbb{R}, +)$ and $(SP\mathbb{C}, +)$ are abelian groups.

Theorem 3.8. *Every symbolic Plithogenic group $(SPG, .)$ has at least 2 nontrivial idempotent elements.*

Proof. Since $P_1P_1 = P_1, P_2P_2 = P_2$ in SPG, the required result follows. □

Theorem 3.9. *Let $(G, *)$ be a finite group of order n. Then $(SPG, *)$ is a finite symbolic Plithogenic group of order n^3.*

Example 3.10. Let $\mathbb{Z}_2$ be the group of integers modulo 2. Then

$$SP\mathbb{Z}_2 = \{(0, 0, 0), (1, 0, 0), (0, P_1, 0,), (0, 0, P_2), (0, P_1, P_2), (1, P_1, 0), (1, 0, P_2), (1, P_1, P_2)\}$$

is a symbolic Plithogenic group of integers modulo 2. The elements $(0, P_1, 0,), (0, 0, P_2)$ and $(1, P_1, P_2)$ of $SP\mathbb{Z}_2$ are nontrivial idempotent elements.

Definition 3.11. Let $\phi : SPG \to SPH$ be a mapping from the symbolic Plithogenic group $(SPG, *)$ to the symbolic Plithogenic group $(SPH, \star)$. ϕ is called a symbolic Plithogenic group homomorphism if the following conditions hold:

(i) $\phi(x * y) = \phi(x) \star \phi(y), \forall x, y \in SPG$,

(ii) $\phi(P_i) = P_i, i = 1, 2$.

The kernel of ϕ denoted by Kerϕ is defined by

$$\text{Ker}\phi = \{x \in SPG : \phi(x) = \text{identity element of } SPH\}.$$

Example 3.12. Let $(G, +)$ be a group and let $\phi : SPG \times SPG \to SPG$ be a mapping defined by

$$\phi(a, b) = a \ \ \forall (a, b) \in SPG \times SPG.$$

Then ϕ is a symbolic Plithogenic group homomorphism.

If $G = \mathbb{Z}_2$, then

$$\begin{aligned}\text{Ker}\phi \ = \ & \{((0,0,0),(0,0,0)),((0,0,0),(1,0,0)),((0,0,0),(0,P_1,0,)),((0,0,0),(0,0,P_2)),\\ & ((0,0,0),(0,P_1,P_2)),((0,0,0),(1,P_1,0),((0,0,0),(1,0,P_2)),((0,0,0),(1,P_1,P_2))\}\end{aligned}$$

which is a subgroup of $SP\mathbb{Z}_2 \times SP\mathbb{Z}_2$.

Example 3.13. Let $G = \mathbb{Z}$, let SPG be the corresponding symbolic Plithogenic group of integers and let $G(I)$ be the neutrosophic group of integers. If $\phi : SPG \to G(I)$ is a mapping defined by

$$\phi(x) = (a, (b + c)I), \forall x = (a, bP_1, cP_2) \in SPG,$$

then ϕ is a group homomorphism and Ker$\phi = \{(0, kP_1, -kP_2) : k \in \mathbb{Z}\}$ which is a subgroup of SPG.

Definition 3.14. Let $(R, +, .)$ be any ring. The triple $(SPR, +, .)$ is called a symbolic Plithogenic ring. If R is commutative with unity, so also is SPR.

Theorem 3.15. *Let $(R, +, .)$ be any ring. Then $(SPR, +, .)$ is a ring.*

Proof. Using Definition 3.1, it can easily be shown that $(SPR, +)$ is an abelian group and $(SPR, .)$ is a semigroup. Also, for arbitrary $x, y, z \in SPR$, it can be shown that $x(y + z) = xy + xz$ and $(y + z)x = yx + zx$. Hence, $(SPR, +, .)$ is a ring. □

Theorem 3.16. *Every symbolic Plithogenic ring $(SPR, +, .)$ has at least 2 nontrivial idempotent elements.*

Theorem 3.17. *Let $(R, +, .)$ be a finite ring of order n. Then $(SPR, +, .)$ is a finite symbolic Plithogenic ring of order n^3.*

Example 3.18. Let $\mathbb{Z}_2$ be the ring of integers modulo 2. Then

$$SP\mathbb{Z}_2 = \{(0,0,0),(1,0,0),(0,P_1,0,),(0,0,P_2),(0,P_1,P_2),(1,P_1,0),(1,0,P_2),(1,P_1,P_2)\}$$

is a symbolic Plithogenic ring of integers modulo 2.

Lemma 3.19. *Let $(SPR,+,.)$ be a symbolic Plithogenic ring and let $x = (a, a_1P_1, a_2P_2)$ and $y = (b, b_1P_1, b_2P_2)$ be any two nonzero elements of SPR.*

(a) *x is idempotent if and only if all the following hold:*
(i) *a is idempotent,*
(ii) *$a + a_2$ is idempotent and*
(iii) *$a + a_1 + a_2$ is idempotent.*
(b) *x and y are zero divisors if and only if all the following hold:*
(i) *a and b are zero divisors,*
(ii) *$a + a_2$ and $b + b_2$ are zero divisors and*
(iii) *$a + a_1 + a_2$ and $b + b_1 + b_2$ are zero divisors.*

Example 3.20. Let $SP\mathbb{Z}_6$ be the symbolic Plithogenic ring of integers modulo 6. Then

(i) $(1,3P_1,3P_2),(1,5P_1,3P_2),(3,5P_1,P_2)$ and $(4,P_1,5P_2)$ are idempotent elements.
(ii) $(2,P_1,P_2)$ and $(3,5P_1,P_2)$ are zero divisors.

Definition 3.21. Let $\phi : SPR \to SPS$ be a mapping from the symbolic Plithogenic ring $(SPR,+,.)$ to the symbolic Plithogenic ring $(SPS,+,.)$. ϕ is called a symbolic Plithogenic ring homomorphism if the following conditions hold:

(i) $\phi(x+y) = \phi(x) + \phi(y)$, $\forall x, y \in SPR$,
(ii) $\phi(xy) = \phi(x)\phi(y)$, $\forall x, y \in SPR$,
(iii) $\phi(P_i) = P_i, i = 1, 2$.

The kernel of ϕ denoted by $\text{Ker}\phi$ is defined by

$$\text{Ker}\phi = \{x \in SPR : \phi(x) = \text{identity element of } SPS\}.$$

Example 3.22. Let $(R,+,.)$ be a ring and let $\phi : SPR \times SPR \to SPR$ be a mapping defined by

$$\phi(a,b) = b \quad \forall (a,b) \in SPR \times SPR.$$

Then ϕ is a symbolic Plithogenic ring homomorphism.

If $R = \mathbb{Z}_2$, then

$$\begin{aligned}\text{Ker}\phi &= \{((0,0,0),(0,0,0)),((1,0,0),(0,0,0)),((0,P_1,0,),(0,0,0)),((0,0,P_2),(0,0,0)),\\ &\quad ((0,P_1,P_2),(0,0,0)),((1,P_1,0),(0,0,0)),((1,0,P_2),(0,0,0)),((1,P_1,P_2),(0,0,0))\}\end{aligned}$$

which is a subring of $SP\mathbb{Z}_2 \times SP\mathbb{Z}_2$.

Theorem 3.23. *Let $\psi : R \to S$ be a ring homomorphism and let $\phi : SPR \to SPS$ be a mapping from a symbolic Plithogenic ring SPR into a symbolic Plithogenic ring SPS defined by*

$$\phi(x) = (\psi(a), \psi(b)P_1, \psi(c)P_2), \quad \forall\ x = (a, bP_1, cP_2) \in SPR.$$

Then ϕ is a ring homomorphism.

Proof. Let $x = (a, bP_1, cP_2)$ and $y = (d, eP_1, fP_2)$ be two arbitrary elements in SPR. Then

$$\begin{aligned}
x + y &= (a + d, (b + e)P_1, (c + f)P_2),\\
xy &= (ad, (ae + bd + be + bf + ce)P_1, (af + cd + cf)P_2),\\
\phi(x) &= (\psi(a), \psi(b)P_1, \psi(c)P_2),\\
\phi(y) &= (\psi(d), \psi(e)P_1, \psi(f)P_2),\\
\therefore \quad \phi(x + y) &= (\psi(a + d), \psi(b + e)P_1, \psi(c + f)P_2),\\
&= (\psi(a) + \psi(d), \psi(b)P_1 + \psi(e)P_1, \psi(c)P_2 + \psi(f)P_2)\\
&= (\psi(a), \psi(b)P_1, \psi(c)P_2) + (\psi(d), \psi(e)P_1, \psi(f)P_2),\\
&= \phi(x) + \phi(y),\\
\phi(xy) &= (\psi(ad), \psi(ae + bd + be + bf + ce)P_1, \psi(af + cd + cf)P_2),\\
&= (\psi(a)\psi(d), (\psi(a)\psi(e) + \psi(b)\psi(d) + \psi(b)\psi(e) + \psi(b)\psi(f) + \psi(c)\psi(e))P_1,\\
&\quad (\psi(a)\psi(f) + \psi(c)\psi(d) + \psi(c)\psi(f))P_2),\\
&= [(\psi(a), \psi(b)P_1, \psi(c)P_2)][(\psi(d), \psi(e)P_1, \psi(f)P_2)],\\
&= \phi(x)\phi(y).
\end{aligned}$$

Accordingly, ϕ is a ring homomorphism. □

Example 3.24. Let $R = \mathbb{Z}_6$, $S = \mathbb{Z}_2$ and let $\psi : \mathbb{Z}_6 \to \mathbb{Z}_2$ be a ring homomorphism defined by $\psi(\bar{x}_6) = \bar{x}_2$. Let $\phi : SP\mathbb{Z}_6 \to SP\mathbb{Z}_2$ be a symbolic Plithogenic ring homomorphism defined by

$$\phi((a, bP_1, cP_2)) = (\psi(a), \psi(b)P_1, \psi(c)P_2), \forall (a, bP_1, cP_2) \in SP\mathbb{Z}_6.$$

Then, $\text{Ker}\psi = \{0, 2, 4\}$ and $\text{Ker}\phi = \{(i, jP_1, kP_2) : i, j, k = 0, 2, 4\}$.

4. Symbolic Plithogenic Algebraic Hyper Structure

Definition 4.1. Let H be a nonempty set and $* : H \times H \to \mathbb{P}^{\star}(H)$ be a hyperoperation. The couple $(H, *)$ is called a hypergroupoid.

For any two nonempty subsets A and B of H and $x \in H$, we define

$$\begin{aligned} A * B &= \bigcup_{a \in A, b \in B} a * b, \\ A * x &= A * \{x\} \quad \text{and} \\ x * B &= \{x\} * B. \end{aligned}$$

A hypergroupoid $(H, *)$ is called a semihypergroup if $\forall\ a, b, c \in H$ we have $(a * b) * c = a * (b * c)$, which means that

$$\bigcup_{u \in a*b} u * c = \bigcup_{v \in b*c} a * v.$$

A hypergroupoid $(H, *)$ is called a quasihypergroup if $\forall a \in H$ we have $a*H = H*a = H$. This condition is also called the reproduction axiom.

If a hypergroupoid $(H, *)$ is both a semihypergroup and a quasihypergroup, then it is called a hypergroup.

Example 4.2. (i) Let H be a nonempty set and let $x * y = H$, $\forall x, y \in H$. Then $(H, *)$ is a hypergroup called a total hypergroup.

(ii) Let $(H, .)$ be a group and let P be a nonempty subset of H. If $x * y = xPy$, $\forall\ x, y \in H$, then, $(H, *)$ is a hypergroup called a P-hypergroup.

(iii) Let $(H, .)$ be a group. If $x * y = < x, y >$, $\forall\ x, y \in H$, where $< x, y >$ is the subgroup generated by x and y, then $(H, *)$ is a hypergroup.

Definition 4.3. Let $(H, *)$ and $(K, \circ)$ be two hypergroups. A mapping $\phi : H \to K$, is called:

(i) an inclusion homomorphism if $\phi(x * y) \subseteq \phi(x) \circ \phi(y)$, $\forall\ x, y \in H$;

(ii) a good homomorphism if $\phi(x * y) = \phi(x) \circ \phi(y)$, $\forall\ x, y \in H$.

Definition 4.4. Let H be a nonempty set and let $+$ be a hyperoperation on H. The couple $(H, +)$ is called a canonical hypergroup if the following conditions hold:

(i) $x + y = y + x$, $\forall\ x, y \in H$,

(ii) $x + (y + z) = (x + y) + z$, $\forall\ x, y, z \in H$,

(iii) there exists a neutral element $0 \in H$ such that $x + 0 = \{x\} = 0 + x$, $\forall\ x \in H$,

(iv) for every $x \in H$, there exists a unique element $-x \in H$ such that $0 \in x + (-x) \cap (-x) + x$,

(v) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$, $\forall\ x, y, z \in H$.

Example 4.5. Let $H = \{0, a, b, c\}$ be a set and let $+$ be a hyperoperation on H defined in the Cayley table below.

$+$	0	a	b	c
0	0	a	b	c
a	a	$\{0, b\}$	$\{a, c\}$	b
b	b	$\{a, c\}$	$\{0, b\}$	a
c	c	b	a	0

Then $(H, +)$ is a canonical hypergroup.

Definition 4.6. Let $(H, +)$ and $(K, +)$ be two canonical hypergroups. A mapping $\phi : H \to K$ is called:

(a) a homomorphism if:
(i) $\phi(x + y) \subseteq \phi(x) + \phi(y)$, $\forall\ x, y \in H$ and
(ii) $\phi(0) = 0$.
(b) a good or strong homomorphism if:
(i) $\phi(x + y) = \phi(x) + \phi(y)$, $\forall\ x, y \in H$ and
(ii) $\phi(0) = 0$.

The kernel of ϕ denoted by Kerϕ is the set $\{x \in H : \phi(x) = 0\}$.

Definition 4.7. Let $(H, *)$ be any hypergroup. The couple $(SPH, *)$ is called a symbolic Plithogenic hypergroup. If $x = (a, a_1P_1, a_2P_2)$ and $y = (b, b_1P_1, b_2P_2)$ are any two elements of SPH, the composition of x and y in SPH denoted by $x * y$ is defined as

$$\begin{aligned} x * y &= \{(c, c_1P_1, c_2P_2) : c \in a * b, c_1 \in (a * b_1 \cup a_1 * b \cup a_1 * b_1 \cup a_1 * b_2 \cup a_2 * b_1)P_1), \\ &\quad c_2 \in (a * b_2 \cup a_2 * b \cup a_2 * b_2)P_2)\} \end{aligned} \tag{16}$$

Example 4.8. (i) Let $(H, *)$ be a total hypergroup. Then $(SPH, *)$ is a symbolic Plithogenic total hypergroup.
(ii) Let $(H, *)$ be a P-hypergroup. Then $(SPH, *)$ is a symbolic Plithogenic P-hypergroup.

Theorem 4.9. *Let $(H, *)$ be a hypergroup and let $(SPH, *)$ be the corresponding symbolic Plithogenic hypergroup. Then:*

(i) *$(SPH, *)$ is a semigroup.*
(ii) *$(SPH, *)$ generally is not a hypergroup.*

Proof. Let $x = (a, a_1P_1, a_2P_2)$, $y = (b, b_1P_1, b_2P_2)$ and $z = (c, c_1P_1, c_2P_2)$ be arbitrary elements of SPH.

(i)

$$\begin{aligned} x * y &= (a, a_1P_1, a_2P_2) * (b, b_1P_1, b_2P_2) \\ &= (a * b, (a * b_1 \cup a_1 * b \cup a_1 * b_1 \cup a_1 * b_2 \cup a_2 * b_1)P_1), \\ &\quad (a * b_2 \cup a_2 * b \cup a_2 * b_2)P_2) \\ &\subset SPH. \end{aligned}$$

This shows that $(SPH, *)$ is a groupoid. Next,

$$\begin{aligned} (x * y) * z &= [(a, a_1P_1, a_2P_2) * (b, b_1P_1, b_2P_2)] * (c, c_1P_1, c_2P_2) \\ &= [(a * b, (a * b_1 \cup a_1 * b \cup a_1 * b_1 \cup a_1 * b_2 \cup a_2 * b_1)P_1), \\ &\quad (a * b_2 \cup a_2 * b \cup a_2 * b_2)P_2)] * (c, c_1P_1, c_2P_2) \\ &= (a * b * c, (a * b * c_1 \cup a * b_1 * c \cup a_1 * b * c \cup a_1 * b_1 * c \cup a_1 * b_2 * c \\ &\quad \cup a_2 * b_1 * c \cup a * b_1 * c_1 \cup a_1 * b * c_1 \cup a_1 * b_1 * c_1 \cup a_1 * b_2 * c_1 \cup a_2 * b_1 * c_1 \\ &\quad \cup a * b_1 * c_2 \cup a_1 * b * c_2 \cup a_1 * b_1 * c_2 \cup a_1 * b_2 * c_2 \cup a_2 * b_1 * c_2 \cup a * b_2 * c_1 \\ &\quad \cup a_2 * b * c_1 \cup a_2 * b_2 * c_1)P_1, (a * b * c_2 \cup a * b_2 * c \cup a_2 * b * c \cup a_2 * b_2 * c \\ &\quad \cup a * b_2 * c_2 \cup a_2 * b * c_2 \cup a_2 * b_2 * c_2)P_2) \\ x * (y * z) &= (a * b * c, (a * b * c_1 \cup a * b_1 * c \cup a * b_1 * c_1 \cup a * b_1 * c_2 \cup a * b_2 * c_1 \cup a_1 * b * c \\ &\quad \cup a_1 * b * c_1 \cup a_1 * b_1 * c \cup a_1 * b_1 * c_1 \cup a_1 * b_1 * c_2 \cup a_1 * b_2 * c_1 \cup a_1 * b * c_2 \\ &\quad \cup a_1 * b_2 * c \cup a_1 * b_2 * c_2 \cup a_2 * b * c_1 \cup a_2 * b_1 * c \cup a_2 * b_1 * c_1 \cup a_2 * b_1 * c_2 \\ &\quad \cup a_2 * b_2 * c_1)P_1, (a * b * c_2 \cup a * b_2 * c \cup a_2 * b * c \cup a_2 * b_2 * c \cup a * b_2 * c_2 \\ &\quad \cup a_2 * b * c_2 \cup a_2 * b_2 * c_2)P_2 \\ &= x * (y * z). \end{aligned}$$

Accordingly, $(SPH, *)$ is a semigroup.
(ii) For all $x = (a, a_1P_1, a_2P_2)$ in SPH, it can be shown that $x * SPH \neq SPH \neq SPH * x$. This shows that reproduction axiom failed to hold in SPH. Hence, $(SPH, *)$ is not a hypergroup. □

Definition 4.10. Let $(SPH, *)$ and $(SPK, \circ)$ be any two symbolic Plithogenic hypergroups and let $\phi : SPH \to SPK$ be a mapping from SPH into SPK.

(a) ϕ is called a symbolic Plithogenic hypergroup homomorphism if the following conditions hold:
(i) $\phi(x * y) \subseteq \phi(x) \circ \phi(y), \ \forall x, y \in SPH$.
(ii) $\phi(P_i) = P_i$, for $i = 1, 2$.

(b) ϕ is called a symbolic Plithogenic good hypergroup homomorphism if the following conditions hold:

(i) $\phi(x * y) = \phi(x) \circ \phi(y),\ \forall x, y \in SPH.$

(ii) $\phi(P_i) = P_i$, for $i = 1, 2$.

Theorem 4.11. *Let $\psi : (H, *) \to (K, \circ)$ be a good hypergroup homomorphism from a hypergroup $(H, *)$ into a hypergroup $(K, \circ)$ and let $\phi : SPH \to SPK$ be a mapping from a symbolic Plithogenic hypergroup SPH into a symbolic Plithogenic hypergroup SPK defined by*

$$\phi(x) = (\psi(a), \psi(b)P_1, \psi(c)P_2), \quad \forall\ x = (a, bP_1, cP_2) \in SPH.$$

Then ϕ is a good hypergroup homomorphism.

Proof. Let $x = (a, bP_1, cP_2)$ and $y = (d, eP_1, fP_2)$ be two arbitrary elements in SPR. Then

$$\begin{aligned}
x * y &= (a * d, (a * e \cup b * d \cup b * e \cup b * f \cup c * e)P_1, (a * f \cup c * d \cup c * f)P_2), \\
\phi(x) &= (\psi(a), \psi(b)P_1, \psi(c)P_2), \\
\phi(y) &= (\psi(d), \psi(e)P_1, \psi(f)P_2), \\
\therefore\ \phi(x * y) &= (\psi(a * d), \psi((a * e \cup b * d \cup b * e \cup b * f \cup c * e)P_1, \psi(a * f \cup c * d \cup c * f)P_2)), \\
&= (\psi(a) \circ \psi(d), (\psi(a) \circ \psi(e) \cup \psi(b) \circ \psi(d) \cup \psi(b) \circ \psi(e) \cup \psi(b) \circ \psi(f) \cup \psi(c) \circ \psi(e))P_1, \\
&\quad (\psi(a) \circ \psi(f) \cup \psi(c) \circ \psi(d) \cup \psi(c) \circ \psi(f))P_2), \\
&= [(\psi(a), \psi(b)P_1, \psi(c)P_2)] \circ [(\psi(d), \psi(e)P_1, \psi(f)P_2)], \\
&= \phi(x) \circ \phi(y).
\end{aligned}$$

Accordingly, ϕ is a good hypergroup homomorphism. □

Definition 4.12. Let $(C, +)$ be any canonical hypergroup. The couple $(SPC, +)$ is called a symbolic Plithogenic canonical hypergroup. If $x = (a, a_1P_1, a_2P_2)$ and $y = (b, b_1P_1, b_2P_2)$ are any two elements of SPC, the composition of x and y in SPC denoted by $x + y$ is defined as

$$x + y = \{(c, c_1P_1, c_2P_2) : c \in a + b, c_1 \in a_1 + b_1, c_2 \in a_2 + b_2\}. \quad (17)$$

Theorem 4.13. *Let $(SPC, +)$ be a symbolic Plithogenic canonical hypergroup. Then $(SPC, +)$ is a canonical hypergroup.*

Proof. Let $x = (a, a_1P_1, a_2P_2)$, $y = (b, b_1P_1, b_2P_2)$ and $z = (c, c_1P_1, c_2P_2)$ be arbitrary elements of SPC. Then

$$\begin{aligned} x + y &= (a, a_1P_1, a_2P_2) + (b, b_1P_1, b_2P_2) \\ &= \{(u, u_1P_1, u_2P_2) : u \in a + b, u_1 \in a_1 + b_1, u_2 \in a_2 + b_2\} \\ &= \{(u, u_1P_1, u_2P_2) : u \in b + a, u_1 \in b_1 + a_1, u_2 \in b_2 + a_2\} \\ &= y + x. \end{aligned}$$

Next,

$$\begin{aligned} (x + y) + z &= ((a, a_1P_1, a_2P_2) + (b, b_1P_1, b_2P_2)) + (c, c_1P_1, c_2P_2) \\ &= \{(u, u_1P_1, u_2P_2) : u \in a + b, u_1 \in a_1 + b_1, u_2 \in a_2 + b_2\} + (c, c_1P_1, c_2P_2) \\ &= \{(u, u_1P_1, u_2P_2) : u \in a + b + c, u_1 \in a_1 + b_1 + c_1, u_2 \in a_2 + b_2 + c_2\} \\ &= \{(u, u_1P_1, u_2P_2) : u \in a + (b + c), u_1 \in a_1 + (b_1 + c_1), u_2 \in a_2 + (b_2 + c_2)\} \\ &= (a, a_1P_1, a_2P_2) + ((b, b_1P_1, b_2P_2) + (c, c_1P_1, c_2P_2)) \\ &= x + (y + z). \end{aligned}$$

Since SPC is a symbolic Plithogenic canonical hypergroup, it follows that $(0, 0P_1, 0P_2) = (0, 0, 0) \in SPC$ so that

$$\begin{aligned} x + (0, 0, 0) &= (a, a_1P_1, a_2P_2) + (0, 0, 0) \\ &= \{(u, u_1P_1, u_2P_2) : u \in a + 0, u_1 \in a_1 + 0, u_2 \in a_2 + 0\} \\ &= \{(u, u_1P_1, u_2P_2) : u \in \{a\}, u_1 \in \{a_1\}, u_2 \in \{a_2\}\} \\ &= \{(a, a_1P_1, a_2P_2)\} \\ &= \{x\} \quad \text{and similarly,} \\ (0, 0, 0) + x &= \{x\}. \end{aligned}$$

Also,

$$\begin{aligned} x + (-x) \cap (-x) + x &= [(a, a_1P_1, a_2P_2) + (-a, -a_1P_1, -a_2P_2)] \cap [(-a, -a_1P_1, -a_2P_2) \\ &\quad +(a, a_1P_1, a_2P_2)] \\ &= \{(u, u_1P_1, u_2P_2) : u \in a + (-a), u_1 \in a_1 + (-a_1), u_2 \in a_2 + (-a_2)\} \\ &\quad \cap\{(v, v_1P_1, v_2P_2) : v \in (-a) + a, v_1 \in (-a_1) + (a_1), v_2 \in (-a_2) + (a_2)\} \\ &= \{(u, u_1P_1, u_2P_2) : u \in \{0\}, u_1 \in \{0\}, u_2 \in \{0\}\} \\ &\quad \cap\{(v, v_1P_1, v_2P_2) : v \in \{0\}, v_1 \in \{0\}, v_2 \in \{0\}\} \\ \therefore\ (0, 0, 0) &\in x + (-x) \cap (-x) + x \end{aligned}$$

which shows that $-x$ is the unique inverse of x , $\forall\, x \in SPC$.

Lastly, suppose that $z \in x + y$, then

$$\begin{aligned}
(c, c_1P_1, c_2P_2) &\in (a, a_1P_1, a_2P_2) + (b, b_1P_1, b_2P_2) \\
&= \{(u, u_1P_1, u_2P_2) : u \in a + b, u_1 \in a_1 + b_1, u_2 \in a_2 + b_2\} \\
&= \{(u, u_1P_1, u_2P_2) : b \in -a + u, b_1 \in -a_1 + u_1, b_2 \in -a_2 + u_2\} \\
&= \{(b, b_1P_1, b_2P_2) : b \in -a + u, b_1 \in -a_1 + u_1, b_2 \in -a_2 + u_2\} \\
\therefore\ (b, b_1P_1, b_2P_2) &\in -(a, a_1P_1, a_2P_2) + (c, c_1P_1, c_2P_2) \\
\text{that is } y &\in -x + z \ \text{ and similarly,} \\
z &\in x + y \ \Rightarrow\ x \in z - y.
\end{aligned}$$

Accordingly, $(SPC, +)$ is a canonical hypergroup. □

Example 4.14. Let $\psi, \psi_1, \psi_2 : C_1 \to C_2$ be good canonical hypergroup homomorphisms and let SPC_1 and SPC_2 be two symbolic Plithogenic canonical hypergroups. If $\phi : SPC_1 \to SPC_2$ is a mapping defined by

$$\phi(x) = (\psi(a), \psi_1(a_1)P_1, \psi_2(a_2)P_2),\ \forall\ x = (a, a_1P_1, a_2P_2) \in SPC_1,$$

then ϕ is a good canonical hypergroup homomorphism and

$$\begin{aligned}
\text{Ker}\phi &= \{(a, a_1P_1, a_2P_2) \in SPC_1 : \phi((a, a_1P_1, a_2P_2)) = (0, 0P_1, 0P_2)\} \\
&= \{(a, a_1P_1, a_2P_2) \in SPC_1 : (\psi(a), \psi_1(a_1)P_1, \psi_2(a_2)P_2) = (0, 0P_1, 0P_2)\} \\
&= \{(a, a_1P_1, a_2P_2) \in SPC_1 : \psi(a) = 0, \psi_1(a_1) = 0, \psi_2(a_2) = 0\} \\
&= \{(a, a_1P_1, a_2P_2) \in SPC_1 : a \in \text{Ker}(\psi), a_1 \in \text{Ker}(\psi_1), a_2 \in \text{Ker}(\psi_2)\} \\
&= \{(\text{Ker}\psi, \text{Ker}\psi_1P_1, \text{Ker}\psi_2P_2)\}.
\end{aligned}$$

5. **Conclusion**

We have in this paper studied symbolic Plithogenic algebraic structures and hyper structures. In particular, we studied symbolic Plithogenic group, symbolic Plithogenic ring, symbolic Plithogenic hypergroup and symbolic Plithogenic canonical hypergroup, and we presented their basic properties.

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The Property (P) and New Fixed Point Results on Ordered Metric Spaces in Neutrosophic Theory

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Abstract. In this manuscript, We introduce some fixed point results for some contractive type mappings on complete ordered triangular neutrosophic metric spaces, and review many existing results in the literature. Furthermore, we use our results to obtain the property (P).

Keywords: the property (P), Contractive mapping, neutrosophic fixed Point,fixed point, neutrosophic topology, neutrosophic theory, neutrosophic cone metric space, complete ordered triangular neutrosophic metric space,.

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1. Introduction and Preliminaries

Zadeh [1] defined the concept of fuzzy set in 1965. After that, many authors have introduced and discussed several notions of generalizations of this fundamental concept. In the year 1968 Chang [3] initiated and study fuzzy topological spaces. In particular, the concept of intuitionistic fuzzy sets (IFSs for short) was first investigated by Atanassov [4]. This concept was extended and modified to intuitionistic L-fuzzy setting by Stoeva and Atanassov [5], which currently known by "intuitionistic L-topological spaces". Using the cocept of intuitionistic fuzzy sets, the concept of intuitionistic fuzzy topological space was introduced by Coker [6, 15, 16]. In diverse latest papers, F. Smarandache modified the concepts of intuitionistic fuzzy sets and different styles of sets to obtained neutrosophic sets (NSs for short) [7]. F. Smarandache and A. Al Shumrani obtained the concept of neutrosophic topology on the non-general and standard interval [8, 9]. Several authors was extended this principle with many applications (see [10, 19–23]). Recently, Alomari and Smarandache [11, 12] introduce and discussed the concepts of continuity in neutrosophic topology, neutrosophic closed and open sets in neutrosophic topological space, they also defined the notion of neutrosophic connectedness and neutrosophic mapping.

W. Al-Omeri et al, [13] introduce the concept of neutrosophic metric space. That is a generalization of intuitionistic fuzzy metric space due to Veeramani and George [17]. Zhang and Huang [18] focused on this new notion of cone metric space and they discussed some fixed point theorems for contractive type mappings. In 2019 wadei Al-Omeri et al. [13] introduced a new concept known by "neutrosophic cone metric space" which is generalized the corresponding concept of intuitionistic fuzzy metric space.

In intuitionistic fuzzy metric space, Bag et al [2] extended the concept of $(\emptyset, \Psi)$-weak contraction, then by using the altering distance function he proved some fixed point theorems. Metric fixed point and cone metric space results are played a remarkable role in the study of (Φ, Ψ)-weak contraction to neutrosophic cone metric space

The purpose of this paper is to introduce a new results about the property (P). In addition, some fixed point consequences with the aid of combine all the principles of these papers for a few contractive type mappings for such mappings in entire metric spaces on complete ordered triangular neutrosophic metric spaces.

2. Historical Background

In this part, we have studied some basic notions such as continuous t-norm, induced topology and neutrosophic cone metric space (NCMS, shortly) the which is defined as $\tau(\Sigma, \Xi)$. A sequence $\{u_m\}$ in an neutrosophic cone metric space $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is said to be Cauchy when every $z > 0$ and $\epsilon > 0$, there exists a natural number m_0 such that $M(u_m, u_n, z) > 1 - \epsilon$ and $N(u_m, u_n, z) < \epsilon$ for all $m, n \geq m_0$. Also, $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is said to be complete NCMS when each Cauchy sequence in NCMS is convergent with respect $\tau(\Sigma, \Xi)$.

Definition 2.1. [13] For any neutrosophic metric space $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$, the sequence $\{x_n\}$ is said to be neutrosophic cone contractive sequence if there exists $q \in (0, 1)$ such that

$$\frac{1}{\Xi(\epsilon_{1n+1}, \epsilon_{1n+2}, m)} - 1 \leq q(\frac{1}{\Xi(\epsilon_1, \epsilon_{1n+1}, m)} - 1)$$

,

$$\Theta(\epsilon_{1n+1}, \epsilon_{1n+2}, m) \leq q\Theta(\epsilon_1, \epsilon_{1n+1}, m) \text{ for every } n \in \Theta.$$

.

Definition 2.2. [13] Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a neutrosophic CMS and an identity mapping $k : \Sigma \to \Sigma$. Then k is said to be neutrosophic cone contractive if there exists $0 < q < 1$ such that that

$$\frac{1}{\Xi(k(\epsilon_1),k(\epsilon_2),m)}-1\leq q(\frac{1}{\Xi(\epsilon_1,\epsilon_2,m)}-1)$$

$$\Theta(k(\epsilon_1),k(\epsilon_2),m)\leq q\Theta(\epsilon_1,\epsilon_2,m)$$

for each $\epsilon_1,\epsilon_2\in\Sigma$ and $m\gg 0_\Theta$. The constant q is said to be contractive constant of k.

Definition 2.3. [14] For any neutrosophic CMS $(\Sigma,\Xi,\Theta,\bigotimes,\diamond)$ and the mappings $\mathcal{H},\mathcal{T}:\Sigma\to\Sigma$. Then the mapping $\mathcal{H}$ is called neutrosophic (Φ,Ψ)-weak contraction with respect to $\mathcal{T}$ if there exists an alternating distance function Φ and a function $\Psi:[0,\infty)\longrightarrow[0,\infty)$ with $\Psi(s)>0$ for $\Psi(s)=0$ and $s>0$ such that

$$\Phi(\frac{1}{\Xi(\mathcal{H}(\epsilon_1),\mathcal{H}(\epsilon_2),\mathcal{H}(\epsilon_3),m)}-1_\Theta)\leq\Psi(\frac{1}{\Xi(\mathcal{T}(\epsilon_1),\mathcal{T}(\epsilon_2),\mathcal{T}(\epsilon_3),m)}-1_\Theta). \tag{2.1}$$

hold for all $\epsilon_1,\epsilon_2,\epsilon_3\in\Xi$ and every $m\gg\Theta$. If $\mathcal{T}$ is the identity map, then $\mathcal{H}$ is called neutrosophic (Φ,Ψ)-weak contraction mapping.

Example 2.4. Let $\Sigma=[0,\infty)$ and $d(r,s)=|r-s|$. Define the self-map Γ on Σ and $\beta:\Sigma\times\Sigma\longrightarrow[0,\infty)$, respectively by the formulas $\Gamma_r=\sqrt{r}$, and $\beta(r,s)=exp(r-s)$, whenever $r\geq s$ and $\beta(r,s)=0$ whenever $r<s$ for all $r,s\in\Sigma$. Then Γ is β-admissible

Definition 2.5. [13] Let $(\Sigma,\Xi,\Theta,\bigotimes,\diamond)$ be a neutrosophic cone metric space. The cone metric (Σ,Ξ) is said to be triangular when

$$\frac{1}{\Xi(u,v,n)}-1\leq\frac{1}{\Xi(u,w,n)}-1+\frac{1}{\Xi(w,v,n)}-1$$

,

$$\Theta(u,v,n)\leq\Theta(u,w,n)+\Theta(w,v,n)\text{ for all }u,v,n\in\Sigma\text{ and }n>0$$

A self-map $\mathcal{H}:\Sigma\to\Sigma$ is said to be orbitally continuous at ϵ_1 when for every sequence $\{x(i)\}_{i\geq1}$ with $\mathcal{H}^{x(i)}\epsilon_1\to b$ for few $b\in\Sigma$, we have $\mathcal{H}^{x(i)+1}\to\mathcal{H}_b$. By [14], here $\mathcal{H}^{m+1}=\mathcal{H}(\mathcal{H}^m)$. Finally, we define the orbit of $\mathcal{H}$ at ϵ_1 by $O(\epsilon_1,\infty):=\{\epsilon_1,\mathcal{H}\epsilon_1,\mathcal{H}^2\epsilon_1,...,\mathcal{H}^n\epsilon_1,..\}$.

We say that $\mathcal{H}$ has the strongly similar property whilst $(\mathcal{H}^{n-1}y,\mathcal{H}^ny)\in\Sigma_\ll$ for each $n\geq1$ and $m\geq2$, where $y\in F(\mathcal{H}^m)$.

3. Existence result

In this part, we have studied some special mappings of discontinuity.

Theorem 3.1. *Let* $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ *be a complete ordered triangular NMS,* $\delta \in (0,1)$ *and* $\mathcal{H}$ *a self-map on* Σ *satisfying*

$$min\left\{\frac{[1-\Xi(\mathcal{H}u,\mathcal{H}v,t)]^2}{\Xi^2(\mathcal{H}u,\mathcal{H}v,t)}, \frac{[1-\Xi(u,v,t)][1-\Xi(\mathcal{H}u,\mathcal{H}v,t)]}{\Xi(u,v,t)\Xi(\mathcal{H}u,\mathcal{H}v,t)}, \frac{[1-\Xi(v,\mathcal{H}v,t)]^2}{\Xi^2(v,\mathcal{H}v,t)}\right\}$$
$$-min\left\{\frac{[1-\Xi(u,\mathcal{H}u,t)]^2}{\Xi^2(u,\mathcal{H}u,t)}, \frac{[1-\Xi(v,\mathcal{H}v,t)][1-\Xi(u,\mathcal{H}v,t)]}{\Xi(v,\mathcal{H}v,t)\Xi(u,\mathcal{H}v,t)}, \frac{[1-\Xi(v,\mathcal{H}u,t)]^2}{\Xi^2(v,\mathcal{H}u,t)}\right\}$$
$$\leq \delta\frac{[1-\Xi(u,\mathcal{H}u,t)][1-\Xi(v,\mathcal{H}v,t)]}{\Xi(u,\mathcal{H}u,t)\Xi(v,\mathcal{H}v,t)}.$$

Thus, for all $u, v \in \Sigma_{\ll}$*.if* $\mathcal{T}$ *has the strongly comparable property, then* $\mathcal{T}$ *has the property (P). Moreover,If there exists* $u_0 \in \Sigma$ *such that* $(\mathcal{H}^{m-1}u_0, \mathcal{H}^m u_0) \in \Sigma_{\ll}$ *for all* $m \geq 1$ *and* $\mathcal{H}$ *is orbitally continuous at* u_0*, then* $\mathcal{T}$ *has a fixed point.*

Proof. To prove that $\mathcal{H}$ has the property (P). Let $n \geq 2$ be given and $u \in T(\mathcal{H}_n)$. Since $\mathcal{H}$ has the strongly comparable property, we can put $x = \mathcal{H}^{m-1}u$ and $u = \mathcal{H}^m u$ in the condition. Then we have

$$min\left\{\frac{[1-\Xi(\mathcal{H}^m u,\mathcal{H}^{m+1}u,t)]^2}{\Xi^2(\mathcal{H}^m u,\mathcal{H}^{m+1}u,t)}, \frac{[1-\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)][1-\Xi(\mathcal{H}^m u,\mathcal{H}^{m+1}u,t)]}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)\Xi(\mathcal{H}^m u,\mathcal{H}^{m+1}u,t)}\right\}$$
$$\leq \delta\frac{[1-\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)][1-\Xi(\mathcal{H}^m u,\mathcal{H}^{m+1}u,t)]}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)\Xi(\mathcal{H}^m u,\mathcal{H}^{m+1}u,t)}.$$

Therefore,

$$min\left\{\frac{[1-\Xi(u,\mathcal{H}u,t)]^2}{\Xi^2(u,\mathcal{H}u,t)}, \frac{[1-\Xi(\mathcal{H}^{m-1}u,u,t)][1-\Xi(u,\mathcal{H}u,t)]}{\Xi(\mathcal{H}^{m-1}u,u,t)\Xi(u,\mathcal{H}u,t)}\right\},$$
$$\leq \delta\frac{[1-\Xi(\mathcal{H}^{m-1}u,u,t)][1-\Xi(u,\mathcal{H}u,t)]}{\Xi(\mathcal{H}^{m-1}u,u,t)\Xi(u,\mathcal{H}u,t)}.$$

and so we get two cases.

Case I.
$$\frac{[1-\Xi(u,\mathcal{H}u,t)]^2}{\Xi^2(u,\mathcal{H}u,t)} \leq \delta\frac{[1-\Xi(\mathcal{H}^{n-1}u,u,t)][1-\Xi(u,\mathcal{H}u,t)]}{\Xi(\mathcal{H}^{n-1}u,u,t)\Xi(u,\mathcal{H}u,t)}.$$

We claim that $\frac{1}{\Xi(u,\mathcal{H}u,t)} - 1 = 0$. If $\frac{1}{\Xi(u,\mathcal{H}u,t)} - 1 > 0$. Then $\frac{1}{\Xi(u,\mathcal{H}u,t)} - 1 = \frac{1}{\Xi(\mathcal{H}^m u,\mathcal{H}^{m+1}u,t)} - 1 \leq \delta\frac{1}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)} - 1$.

Again, by putting $x = \mathcal{H}^{m-2}u$ and $y = \mathcal{H}^{m-1}u$ in condition, we obtain

$$min\left\{\frac{[1-\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)]^2}{\Xi^2(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)}, \frac{[1-\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)][1-\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)]}{\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)}\right\}$$
$$\leq \delta\frac{[1-\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)][1-\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)]}{\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)}.$$

Again, we get two cases. Let

$$min\left\{\frac{[1-\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)]^2}{\Xi^2(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)} \leq \delta\frac{[1-\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)][1-\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)]}{\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)}\right\}.$$

If $\frac{1}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)} - 1 = 0$, then $\mathcal{H}^{m-1}u = u$ and so $u = \mathcal{H}^m u = \mathcal{H}u$. If $\frac{1}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)} - 1 > 0$, then $\frac{1}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)} - 1 \leq \delta\big[\frac{1}{\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)} - 1\big]$. Now, let

$$\frac{[1-\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)][1-\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^{m}u,t)]}{\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^{m}u,t)}$$
$$\leq \delta\frac{[1-\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)][1-\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^{m}u,t)]}{\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^{m}u,t)}.$$

In this case we should have $\frac{1}{\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)} - 1 = 0$ or $\frac{1}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^{m}u,t)} - 1 = 0$ (and so $u = \mathcal{H}u$), because if $\frac{1}{\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)} - 1 > 0$ and $\frac{1}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^{m}u,t)} - 1 > 0$, then we get $\delta \geq 1$ which is a contradiction. By continuing this process, we have

$$\frac{1}{\Xi(u,\mathcal{H}u,t)} - 1 = \frac{1}{\Xi(\mathcal{H}^{m}u,\mathcal{H}^{m+1}u,t)} - 1 \leq \delta\left[\frac{1}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^{m}u,t)} - 1\right]$$
$$\leq \delta^2\left[\frac{1}{\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)} - 1\right] \leq ... \leq \delta^m\left[\frac{1}{\Xi(u,\mathcal{H}u,t)} - 1\right]$$

which leads us to $\delta \geq 1$ which is a contradiction. Thus, in this case we obtain $\frac{1}{\Xi(u,\mathcal{H}u,t)} - 1$ and so $\mathcal{H}u = u$

Case II. $$\frac{[1-\Xi(\mathcal{H}^{m-1}u,u,t)][1-\Xi(u,\mathcal{H}u,t)]}{\Xi(\mathcal{H}^{m-1}u,u,t)\Xi(u,\mathcal{H}u,t)} \leq \delta\frac{[1-\Xi(\mathcal{H}^{m-1}u,u,t)][1-\Xi(u,\mathcal{H}u,t)]}{\Xi(\mathcal{H}^{m-1}u,u,t)\Xi(u,\mathcal{H}u,t)}.$$

In this case, we should have $\frac{1}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}u,t)} - 1 = 0$ or $\frac{1}{\Xi(u,\mathcal{H}u,t)} - 1 = 0$ (and so $u = \mathcal{H}u$) In fact, if $\frac{1}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}u,t)} - 1 > 0$ and $\frac{1}{\Xi(u,\mathcal{H}u,t)} - 1 > 0$, then $\delta \geq 1$ which is a contradiction. Therefore, we consequence that $T(\mathcal{H}m) \subseteq T(\mathcal{H})$. Therefore, $\mathcal{H}$ has the property

Now, define $u_{n+1} = \mathcal{H}u_n = \mathcal{H}^{n+1}u_0$ for all $n \geq 0$. If $u_{n0} = u_{n_0-1}$ for some natural number n_0, then $u_n = u_{n_0}$ for all $n \geq n_0$ and u_{n0} is a fixed point of $\mathcal{H}$. Suppose that $u_n \neq u_{n-1}$ for all $n \geq 1$. Now for each $n \geq 1$, by using the hypotheses, we can put $u = u_{n-1}$ and $y = u_n$ in the condition. Therefore we obtain

$$min\left\{\frac{[1-\Xi(u_m,u_{m+1},t)]^2}{\Xi^2(u_m,u_{m+1},t)}, \frac{[1-\Xi(u_{m-1},u_m,t)][1-\Xi(u_m,u_{m+1},t)]}{\Xi(u_{m-1},u_m,t)\Xi(u_m,u_{m+1},t)}\right\}$$
$$\leq \delta\frac{[1-\Xi(u_{m-1},u_m,t)][1-\Xi(u_m,u_{m+1},t)]}{\Xi(u_{m-1},u_m,t)\Xi(u_m,u_{m+1},t)}.$$

Since $\delta \leq 1$

$$min\left\{\frac{[1-\Xi(u_m,u_{m+1},t)]^2}{\Xi^2(u_m,u_{m+1},t)}, \frac{[1-\Xi(u_{m-1},u_m,t)][1-\Xi(u_m,u_{m+1},t)]}{\Xi(u_{m-1},u_m,t)\Xi(u_m,u_{m+1},t)}\right\}$$
$$= \frac{[1-\Xi(u_m,u_{m+1},t)]^2}{\Xi^2(u_m,u_{m+1},t)}.$$

Hence

$$\frac{1}{\Xi^2(u_m,u_{m+1},t)} - 1 \leq \delta\left(\frac{1}{\Xi^2(u_{m-1},u_m,t)} - 1\right).$$

By continuing this process we obtain

$$\frac{1}{\Xi^2(u_m,u_{m+1},t)} - 1 \leq \delta^m\left(\frac{1}{\Xi^2(u_0,u_1,t)} - 1\right),$$

for all $m \geq 1$. Thus for each natural number k we have

$$\frac{1}{\Xi^2(u_m, u_{m+k}, t)} - 1 \leq \sum_{i=m}^{m+k-1} \left(\frac{1}{\Xi^2(u_i, u_{i+1}, t)} - 1 \right) \leq \sum_{i=m}^{m+k-1} \delta^i \left(\frac{1}{\Xi^2(u_0, u_1, t)} - 1 \right)$$
$$\leq \frac{\delta^m}{1-\delta} \left(\frac{1}{\Xi^2(u_0, u_1, t)} - 1 \right).$$

Then, $\{u_m\}$ is a Cauchy sequence. If $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is a complete NMS, then there exists $v \in \Sigma$ such that $u_m \to v$. Since $\mathcal{H}$ is orbitally continuous, $u_{m+1} = \mathcal{H}_{u_m} \to \mathcal{H}_v$. This implies that $\mathcal{H}_v = v$. □

Theorem 3.2. *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a complete ordered triangular neutrosophic metric space, $c \in [0, 1)$, $b \geq 0$, n a nonnegative integer and $\mathcal{H}$ a selfmap on Σ satisfy the condition*

$$\frac{[1 - \Xi(\mathcal{H}^{n+1}u, \mathcal{H}^{n+2}v, t)]^2}{\Xi^2(\mathcal{H}^{n+1}u, \mathcal{H}^{n+2}v, t)} \leq c \frac{[1 - \Xi(\mathcal{H}^n u, \mathcal{H}^{n+1}u, t)][1 - \Xi(\mathcal{H}^{n+1}v, \mathcal{H}^{n+2}v, t)]}{\Xi(\mathcal{H}^n u, \mathcal{H}^{n+1}u, t)\Xi(\mathcal{H}^{n+1}v, \mathcal{H}^{n+2}v, t)}$$
$$+ b \frac{[1 - \Xi(\mathcal{H}^n u, \mathcal{H}^{n+2}v, t)][1 - \Xi(\mathcal{H}^{n+1}v, \mathcal{H}^{n+1}u, t)]}{\Xi(\mathcal{H}^n u, \mathcal{H}^{n+2}v, t)\Xi(\mathcal{H}^{n+1}v, \mathcal{H}^{n+1}u, t)}.$$

for all $u, v \in \Sigma_{\ll}$. Suppose that there exists $u_0 \in \Sigma$ such that $(\mathcal{H}^{m-1}x_0, \mathcal{H}^m u_0) \in \Sigma_{\ll}$ for all $m \geq 1$. If $\mathcal{H}$ is orbitally continuous at u_0 or $n = 0$, then $\mathcal{H}$ has a fixed point. Moreover, $\mathcal{H}$ has a unique fixed point whenever $b < 1$. If $\mathcal{H}$ has the strongly comparable property, then $\mathcal{H}$ has the property (P).

Proof. Define $u_1 = \mathcal{H}^{n+1}u_0$ and $u_{m+1} = \mathcal{H}x_m$ for all $m \geq 1$. Then

$$\frac{[1 - \Xi(u_1, u_2, t)]^2}{\Xi^2(u_1, u_2, t)} = \frac{[1 - \Xi(\mathcal{H}^{n+1}u_0, \mathcal{H}^{n+2}u_0, t)]^2}{\Xi^2(\mathcal{H}^{n+1}u_0, \mathcal{H}^{n+2}u_0, t)}$$
$$\leq c \frac{[1 - \Xi(\mathcal{H}^n u_0, \mathcal{H}^{n+1}u_0, t)][1 - \Xi(\mathcal{H}^{n+1}u_0, \mathcal{H}^{n+2}u_0, t)]}{\Xi(\mathcal{H}^n u_0, \mathcal{H}^{n+1}u_0, t)\Xi(\mathcal{H}^{n+1}u_0, \mathcal{H}^{n+2}u_0, t)}$$
$$+ b \frac{[1 - \Xi(\mathcal{H}^n u_0, \mathcal{H}^{n+2}u_0, t)][1 - \Xi(\mathcal{H}^{n+1}u_0, \mathcal{H}^{n+1}u_0, t)]}{\Xi(\mathcal{H}^n u_0, \mathcal{H}^{n+2}u_0, t)\Xi(\mathcal{H}^{n+1}u_0, \mathcal{H}^{n+1}u_0, t)}.$$
$$= c \frac{[1 - \Xi(\mathcal{H}^n u_0, \mathcal{H}^{n+1}u_0, t)][1 - \Xi(\mathcal{H}^{n+1}u_0, \mathcal{H}^{n+2}u_0, t)]}{\Xi(\mathcal{H}^n u_0, \mathcal{H}^{n+1}u_0, t)\Xi(\mathcal{H}^{n+1}u_0, \mathcal{H}^{n+2}u_0, t)}$$
$$= c \frac{[1 - \Xi(\mathcal{H}^n u_0, \mathcal{H}^{n+1}u_0, t)][1 - \Xi(u_1, u_2, t)]}{\Xi(\mathcal{H}^n u_0, \mathcal{H}^{n+1}u_0, t)\Xi(u_1, u_2, t)}.$$

If $\frac{1}{\Xi(u_1,u_2,t)}-1=0$, then $\mathcal{H}u_1=u_2=u_1$ and so $\mathcal{H}$ has a fixed point. If $\frac{1}{\Xi(u_1,u_2,t)}-1>0$, then $\frac{1}{\Xi(u_1,u_2,t)}-1\leq c\frac{1}{\Xi(\mathcal{H}^n u_0,u_1,t)}-1$. Similarly, we have.

$$\begin{aligned}\frac{[1-\Xi(u_2,u_3,t)]^2}{\Xi^2(u_2,u_3,t)} &= \frac{[1-\Xi(\mathcal{H}^{n+2}u_0,\mathcal{H}^{n+3}u_0,t)]^2}{\Xi^2(\mathcal{H}^{n+2}u_0,\mathcal{H}^{n+3}u_0,t)}\\
&\leq c\frac{[1-\Xi(\mathcal{H}^{n+1}u_0,\mathcal{H}^{n+2}u_0,t)][1-\Xi(\mathcal{H}^{n+2}u_0,\mathcal{H}^{n+3}u_0,t)]}{\Xi(\mathcal{H}^{n+1}u_0,\mathcal{H}^{n+2}u_0,t)\Xi(\mathcal{H}^{n+2}u_0,\mathcal{H}^{n+3}u_0,t)}\\
&+b\frac{[1-\Xi(\mathcal{H}^{n+1}u_0,\mathcal{H}^{n+3}u_0,t)][1-\Xi(\mathcal{H}^{n+2}u_0,\mathcal{H}^{n+2}u_0,t)]}{\Xi(\mathcal{H}^{n+1}u_0,\mathcal{H}^{n+3}u_0,t)\Xi(\mathcal{H}^{n+2}u_0,\mathcal{H}^{n+2}u_0,t)}.\\
&= c\frac{[1-\Xi(\mathcal{H}^{n+1}u_0,\mathcal{H}^{n+2}u_0,t)][1-\Xi(\mathcal{H}^{n+2}u_0,\mathcal{H}^{n+3}u_0,t)]}{\Xi(\mathcal{H}^{n+1}u_0,\mathcal{H}^{n+2}u_0,t)\Xi(\mathcal{H}^{n+2}u_0,\mathcal{H}^{n+3}u_0,t)}\\
&= c\frac{[1-\Xi(u_1,u_2,t)][1-\Xi(u_2,u_3,t)]}{\Xi(u_1,u_2,t)\Xi(u_2,u_3,t)}.\end{aligned}$$

If $\frac{1}{\Xi(u_1,u_2,t)}-1=0$, then $\mathcal{H}u_2=u_3=u_2$ and so $\mathcal{H}$ has a fixed point. If $\frac{1}{\Xi(u_2,u_3,t)}-1>0$, then $\frac{1}{\Xi(u_2,u_3,t)}-1\leq c\big[\frac{1}{\Xi(u_1,u_2,t)}-1\big]$ and so $\frac{1}{\Xi(u_2,u_3,t)}-1\leq c^2\big[\frac{1}{\Xi(\mathcal{H}^n u_0,u_1,t)}-1\big]$. By continuing this process we get that $\frac{1}{\Xi(u_n,u_{n+1},t)}-1\leq c^n\big[\frac{1}{\Xi(\mathcal{H}^n u_0,u_1,t)}-1\big]$ for all $m\geq 1$. This implies that $\{u_m\}$ is a Cauchy sequence. Since $(\Sigma,\Xi,\Theta,\bigotimes,\diamond)$ is a complete neutrosophic metric space, there exists $x\in\Sigma$ such that $u_m\to x$. If $\mathcal{H}$ is orbitally continuous, then $\mathcal{H}u_m\to\mathcal{H}x$. Hence, $\mathcal{H}x=x$.

If $n=0$, then for each $m\geq 2$ we have

$$\begin{aligned}\frac{[1-\Xi(\mathcal{H}x,\mathcal{H}^m u_0,t)]^2}{\Xi^2(\mathcal{H}x,\mathcal{H}^m u_0,t)} &\leq c\frac{[1-\Xi(x,\mathcal{H}x,t)][1-\Xi(\mathcal{H}u_{m-2},\mathcal{H}^2u_{m-2},t)]}{\Xi(x,\mathcal{H}x,t)\Xi(\mathcal{H}u_{m-2},\mathcal{H}^2u_{m-2},t)}\\
&+b\frac{[1-\Xi(x,\mathcal{H}^2u_{m-2},t)][1-\Xi(\mathcal{H}u_{m-2},\mathcal{H}x,t)]}{\Xi(x,\mathcal{H}^2u_{m-2},t)\Xi(\mathcal{H}u_{m-2},\mathcal{H}x,t)}.\end{aligned}$$

Since $u_m\to x$, we have

$$\frac{1}{\Xi(\mathcal{H}x,x,t)}-1\leq b\frac{[1-\Xi(x,x,t)][1-\Xi(x,\mathcal{H}x,t)]}{\Xi(x,x,t)\Xi(x,\mathcal{H}x,t)}=0$$

and so $\mathcal{H}x=x$. Now, we show that $\mathcal{H}$ has a unique fixed point whenever $b<1$. Let x and y be fixed points of $\mathcal{H}$. Then, we have

$$\begin{aligned}\left(\frac{1}{\Xi(x,y,t)}-1\right)^2 &= \left(\frac{1}{\Xi(\mathcal{H}^{n+1}x,\mathcal{H}^{n+2}y,t)}-1\right)^2\\
&\leq c\frac{[1-\Xi(\mathcal{H}^n x,\mathcal{H}^{n+1}x,t)][1-\Xi(\mathcal{H}^{n+1}y,\mathcal{H}^{n+2}y,t)]}{\Xi(\mathcal{H}^n x,\mathcal{H}^{n+1}x,t)\Xi(\mathcal{H}^{n+1}y,\mathcal{H}^{n+2}y,t)}\\
&+b\frac{[1-\Xi(\mathcal{H}^n x,\mathcal{H}^{n+2}y,t)][1-\Xi(\mathcal{H}^{n+1}y,\mathcal{H}^{n+1}x,t)]}{\Xi(\mathcal{H}^n x,\mathcal{H}^{n+2}y,t)\Xi(\mathcal{H}^{n+1}y,\mathcal{H}^{n+1}x,t)}=b\left(\frac{1}{\Xi(x,y,t)}-1\right)^2.\end{aligned}$$

Hence, $\frac{1}{\Xi(x,y,t)}-1=0$ because $b<1$. Thus, $x=y$ and so $\mathcal{H}$ has a unique fixed point. Finally, we prove that $\mathcal{H}$ has the property (P) whenever $\mathcal{H}$ has the strongly comparable property. Let $m2$ be given and $y\in T(\mathcal{H}m)$. We consider the following cases. **Case I**. $n=0$. In this case, we have

$$\begin{aligned}\left(\frac{1}{\Xi(y,\mathcal{H}y,t)}-1\right)^2 =&\left(\frac{1}{\Xi(\mathcal{H}(\mathcal{H}^{m-1}y),\mathcal{H}^2(\mathcal{H}^{m-1}y),t)}-1\right)^2\\ \leq& c\frac{[1-\Xi(\mathcal{H}^{m-1}y,\mathcal{H}^m y,t)][1-\Xi(\mathcal{H}^m y,\mathcal{H}^{m+1}y,t)]}{\Xi(\mathcal{H}^{m-1}y,\mathcal{H}^m y,t)\Xi(\mathcal{H}^m y,\mathcal{H}^{m+1}y,t)}\\ &+b\frac{[1-\Xi(\mathcal{H}^{m-1}y,\mathcal{H}^{m+1}y,t)][1-\Xi(\mathcal{H}^m y,\mathcal{H}^m y,t)]}{\Xi(\mathcal{H}^{m-1}y,\mathcal{H}^{m+1}y,t)\Xi(\mathcal{H}^m y,\mathcal{H}^m y,t)}\\ =&c\frac{[1-\Xi(\mathcal{H}^{m-1}y,y,t)][1-\Xi(y,\mathcal{H}y,t)]}{\Xi(\mathcal{H}^{m-1}y,y,t)\Xi(y,\mathcal{H}y,t)}.\end{aligned}$$

If $\frac{1}{\Xi(y,\mathcal{H}y,t)}-1=0$ then $\mathcal{H}y=y$. If $\frac{1}{\Xi(x,y,t)}-1>0$, then $\frac{1}{\Xi(\mathcal{H}^m y,\mathcal{H}^{m+1}y,t)}-1\leq c\left(\frac{1}{\Xi(\mathcal{H}^{m+1}y,\mathcal{H}^m y,t)}-1\right)$. By using a similar argument as in Theorem 3.1 and continuing the process, we obtain

$$\begin{aligned}&\frac{1}{\Xi(y,\mathcal{H}y,t)}-1=\frac{1}{\Xi(\mathcal{H}^m y,\mathcal{H}^{m+1}y,t)}-1\leq c\left[\frac{1}{\Xi(\mathcal{H}^{m-1}y,\mathcal{H}^m y,t)}-1\right]\\ &\leq c^2\left[\frac{1}{\Xi(\mathcal{H}^{m-2}y,\mathcal{H}^{m-1}y,t)}-1\right]\leq ...\leq c^m\left[\frac{1}{\Xi(y,\mathcal{H}y,t)}-1\right].\end{aligned}$$

Since $c<1$, $\mathcal{H}y=y$.

Case II. $n\geq 1$ and $m\leq n$. In this case, choose a natural number μ and an integer number $0\leq\nu<m$ such that $n+1=\mu m+\nu$. Then, we have $\mathcal{H}^m(\mathcal{H}^{m-\nu}y)=\mathcal{H}^{n+1}(\mathcal{H}^{n-\nu}y)=y$, and so

$$\begin{aligned}\left(\frac{1}{\Xi(y,\mathcal{H}y,t)}-1\right)^2 =&\left(\frac{1}{\Xi(\mathcal{H}^{n+1}(\mathcal{H}^{m-\nu}y),\mathcal{H}^{n+2}(\mathcal{H}^{m-\nu}y),t)}-1\right)^2\\ \leq& c\frac{[1-\Xi(\mathcal{H}^{n}(\mathcal{H}^{m-\nu}y),\mathcal{H}^{n+1}(\mathcal{H}^{m-\nu}y),t)][1-\Xi(\mathcal{H}^{n+1}(\mathcal{H}^{m-\nu}y),\mathcal{H}^{n+2}(\mathcal{H}^{m-\nu}y),t)]}{\Xi(\mathcal{H}^{n}(\mathcal{H}^{m-\nu}y),\mathcal{H}^{n+1}(\mathcal{H}^{m-\nu}y),t)\Xi(\mathcal{H}^{n+1}(\mathcal{H}^{m-\nu}y),\mathcal{H}^{n+2}(\mathcal{H}^{m-\nu}y),t)}\\ &+b\frac{[1-\Xi(\mathcal{H}^{n}(\mathcal{H}^{m-\nu}y),\mathcal{H}^{n+2}(\mathcal{H}^{m-\nu}y),t)][1-\Xi(\mathcal{H}^{n+1}(\mathcal{H}^{m-\nu}y),\mathcal{H}^{n+1}(\mathcal{H}^{m-\nu}y),t)]}{\Xi(\mathcal{H}^{n}(\mathcal{H}^{m-\nu}y),\mathcal{H}^{n+2}(\mathcal{H}^{m-\nu}y),t)\Xi(\mathcal{H}^{n+1}(\mathcal{H}^{m-\nu}y),\mathcal{H}^{n+1}(\mathcal{H}^{m-\nu}y),t)}\\ =&c\frac{[1-\Xi(\mathcal{H}^{m-1}y,y,t)][1-\Xi(y,\mathcal{H}y,t)]}{\Xi(\mathcal{H}^{m-1}y,y,t)\Xi(y,\mathcal{H}y,t)}.\end{aligned}$$

If $\frac{1}{\Xi(y,\mathcal{H}y,t)}-1=0$ then $\mathcal{H}y=y$. If $\frac{1}{\Xi(x,\mathcal{H}y,t)}-1>0$, then $\frac{1}{\Xi(\mathcal{H}^m y,\mathcal{H}^{m+1}y,t)}-1\leq c\left(\frac{1}{\Xi(\mathcal{H}^{m-1}y,\mathcal{H}^m y,t)}-1\right)$. By using a similar argument as in Theorem 3.1, we obtain

$$\begin{aligned}&\frac{1}{\Xi(y,\mathcal{H}y,t)}-1=\frac{1}{\Xi(\mathcal{H}^m y,\mathcal{H}^{m+1}y,t)}-1\leq c\left[\frac{1}{\Xi(\mathcal{H}^{m-1}y,\mathcal{H}^m y,t)}-1\right]\\ &\leq c^2\left[\frac{1}{\Xi(\mathcal{H}^{m-2}y,\mathcal{H}^{m-1}y,t)}-1\right]\leq ...\leq c^m\left[\frac{1}{\Xi(y,\mathcal{H}y,t)}-1\right].\end{aligned}$$

Since $c<1$, $\mathcal{H}y=y$. Thus, $T(\mathcal{H}^m)\subseteq T(\mathcal{H})$. Therefore, $\mathcal{H}$ has the property (P). □

Definition 3.3. Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a neutrosophic metric space and $\mathcal{H}$ a selfmap on Σ. Then, $\mathcal{H}$ is said to be a convex contraction of order 2 if there exist $r, s \in (01)$ and $t > 0$ with $r + s < 1$ such that

$$\frac{1}{\Xi(\mathcal{H}^2u, \mathcal{H}^2v, t)} - 1 \leq r\left[\frac{1}{\Xi(\mathcal{H}u, \mathcal{H}v, t)} - 1\right] + s\left[\frac{1}{\Xi(u, v, t)} - 1\right]$$

for all $u, v \in \Sigma$. Also, $\mathcal{H}$ is said to be a convex contraction of order 2 if there exist $r_1, r_2, s_1, s_2 \in (0, 1)$ with $r_1 + r_2 + s_1 + s_2 < 1$ such that

$$\frac{1}{\Xi(\mathcal{H}^2u, \mathcal{H}^2v, t)} - 1 \leq r_1\left[\frac{1}{\Xi(u, \mathcal{H}u, t)} - 1\right] + r_2\left[\frac{1}{\Xi(\mathcal{H}u, \mathcal{H}^2u, t)} - 1\right] + s_1\left[\frac{1}{\Xi(v, \mathcal{H}v, t)} - 1\right] + s_2\left[\frac{1}{\Xi(\mathcal{H}v, \mathcal{H}^2v, t)} - 1\right]$$

$\forall u, v \in \Sigma$

Theorem 3.4. *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a complete order triangular NMS, $r, s \in (0, 1)$ with $r + s < 1$ and $\mathcal{H}$ an orbitally continuous selfmap on Σ satisfy the condition*

$$\frac{1}{\Xi(\mathcal{H}^2u, \mathcal{H}^2v, t)} - 1 \leq r\left[\frac{1}{\Xi(\mathcal{H}u, \mathcal{H}v, t)} - 1\right] + s\left[\frac{1}{\Xi(u, v, t)} - 1\right]$$

for all $u, v \in \Sigma_{\ll}$, then $\mathcal{H}$ has a unique fixed point. Also, $T(\mathcal{H}) = T(\mathcal{H}^2)$.

Proof. Define $u_m = \mathcal{H}^m u_0$ for all $m \geq 1$, $y = \frac{1}{\Xi(\mathcal{H}u_0, \mathcal{H}^2u_0, t)} - 1 + \frac{1}{\Xi(u_0, \mathcal{H}u_0, t)} - 1$, and $\delta = r + s$. Thus $\frac{1}{\Xi(\mathcal{H}^2u_0, \mathcal{H}u_0, t)} - 1 \leq y$. Now, by using the assumption, we can put $u = \mathcal{H}u_0$ and $v = u_0$ in the condition. Thus, we obtain

$$\frac{1}{\Xi(\mathcal{H}^3u_0, \mathcal{H}^2u_0, t)} - 1 \leq r\left[\frac{1}{\Xi(\mathcal{H}^2u_0, \mathcal{H}u_0, t)} - 1\right] + s\left[\frac{1}{\Xi(u_0, \mathcal{H}u_0, t)} - 1\right] \leq \delta_y$$

Now, by putting $u = \mathcal{H}^2u_0$ and $v = \mathcal{H}u_0$ in the condition, we get

$$\begin{aligned}\frac{1}{\Xi(\mathcal{H}^4u_0, \mathcal{H}^3u_0, t)} - 1 &\leq r\left[\frac{1}{\Xi(\mathcal{H}^3u_0, \mathcal{H}^2u_0, t)} - 1\right] + s\left[\frac{1}{\Xi(\mathcal{H}^2u_0, u_0, t)} - 1\right] \\ &\leq r^2\left[\frac{1}{\Xi(\mathcal{H}^2u_0, \mathcal{H}u_0, t)} - 1\right] + rs\left[\frac{1}{\Xi(u_0, \mathcal{H}u_0, t)} - 1\right] + s\left[\frac{1}{\Xi(\mathcal{H}^2u_0, u_0, t)} - 1\right] \leq \delta^2 y.\end{aligned}$$

Again, by putting $u = \mathcal{H}^3u_0$ and $v = \mathcal{H}^2u_0$ in the condition, we obtain

$$\begin{aligned}\frac{1}{\Xi(\mathcal{H}^5u_0, \mathcal{H}^4u_0, t)} - 1 &\leq r\left[\frac{1}{\Xi(\mathcal{H}^4u_0, \mathcal{H}^3u_0, t)} - 1\right] + s\left[\frac{1}{\Xi(\mathcal{H}^3u_0, \mathcal{H}^2u_0, t)} - 1\right] \\ &\leq (r^3 + rs)\left[\frac{1}{\Xi(\mathcal{H}^2u_0, \mathcal{H}u_0, t)} - 1\right] + r^2s\left[\frac{1}{\Xi(u_0, \mathcal{H}u_0, t)} - 1\right] \\ &\quad + rs\left[\frac{1}{\Xi(\mathcal{H}^2u_0, \mathcal{H}u_0, t)} - 1\right] + s\left[\frac{1}{\Xi(u_0, \mathcal{H}u_0, t)} - 1\right] \\ &= (r^3 + 2rs)\left[\frac{1}{\Xi(\mathcal{H}^2u_0, \mathcal{H}u_0, t)} - 1\right] + (r^2s + s^2)\left[\frac{1}{\Xi(u_0, \mathcal{H}u_0, t)} - 1\right] \leq \delta^3 y.\end{aligned}$$

By continuing this process, we get $\frac{1}{\Xi(\mathcal{H}^{m+1}u_0,\mathcal{H}^m u_0,t)} - 1 \leq \delta^{m-1}y \ \forall m \geq 3$. This implies that

$$\frac{1}{\Xi^2(\mathcal{H}^n u_0, \mathcal{H}^m u_0, t)} - 1 \leq \sum_{i=n}^{m-1}\left(\frac{1}{\Xi^2(\mathcal{H}^i u_0, \mathcal{H}^{i+1}u_0, t)} - 1\right) \leq \sum_{i=n}^{m-1}\delta^{i-2}y \leq \frac{\delta^{i-2}}{1-\delta}y.$$

for all $m > n \geq 3$. Hence, $\{u_m\}$ is a Cauchy sequence. If there exists $x \in \Sigma$ such that $u_m \to x$. Then $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is a complete neutrosophic metric space. Since $\mathcal{H}$ is orbitally continuous, $\mathcal{H}u_m \to \mathcal{H}x$ and so $\mathcal{H}x = x$. Now, we show that $\mathcal{H}$ mapping has a unique fixed point. Let v and w be fixed points of $\mathcal{H}$. Then

$$\begin{aligned}\frac{1}{\Xi(v,w,t)} - 1 =& \frac{1}{\Xi(\mathcal{H}^2 v, \mathcal{H}^2 w, t)} - 1 \leq r\left[\frac{1}{\Xi(\mathcal{H}v, \mathcal{H}w, t)} - 1\right] + s\left[\frac{1}{\Xi(v,w,t)} - 1\right]\\ =&(r+s)\frac{1}{\Xi(v,w,t)} - 1\end{aligned}$$

Since $r + s < 1$, we get $\mathcal{H}v = v$. □

Theorem 3.5. *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a complete order triangular neutrosophic metric space, $r_1, r_2, s_1, s_2 \in (0,1)$ with $r_1 + r_2 + s_1 + s_2 < 1$ and $\mathcal{H}$ an orbitally continuous selfmap on Σ satisfy the condition*

$$\begin{aligned}\frac{1}{\Xi(\mathcal{H}^2 u, \mathcal{H}^2 v, t)} - 1 \leq r_1\left[\frac{1}{\Xi(u, \mathcal{H}u, t)} - 1\right] + r_2\left[\frac{1}{\Xi(\mathcal{H}u, \mathcal{H}^2 u, t)} - 1\right]\\ + s_1\left[\frac{1}{\Xi(v, \mathcal{H}v, t)} - 1\right] + s_2\left[\frac{1}{\Xi(\mathcal{H}v, \mathcal{H}^2 v, t)} - 1\right]\end{aligned}$$

$\forall u, v \in \Sigma_{\ll}$. If there exists $u_0 \in \Sigma$ such that $\mathcal{H}^{m-1}u_0, \mathcal{H}^m u_0) \in \Sigma_{\ll}$ $\forall m \geq 1$, then $\mathcal{H}$ has a unique fixed point. Also $T(\mathcal{H}) = T(\mathcal{H}^2)$.

Proof. Define $u_m = \mathcal{H}^m u_0$, for all $\forall m \geq 1$, and set $y = \frac{1}{\Xi(\mathcal{H}u_0, \mathcal{H}^2 u_0, t)} - 1 + \frac{1}{\Xi(u_0, \mathcal{H}u_0, t)} - 1$ Also, put $\delta = r_1 + r_2 + s_1$ and $\lambda = 1s_2$. We prove that

$$\frac{1}{\Xi(\mathcal{H}^{m+1}u_0, \mathcal{H}^m u_0, t)} - 1 \leq \left(\frac{\delta}{\lambda}\right)^{m-2} y$$

for all $m \geq 3$. Note that

$$\begin{aligned}\frac{1}{\Xi(\mathcal{H}^3 u_0, \mathcal{H}^2 u_0, t)} - 1 \leq& r_1\left[\frac{1}{\Xi(u_0, \mathcal{H}u_0, t)} - 1\right] + r_2\left[\frac{1}{\Xi(\mathcal{H}u_0, \mathcal{H}^2 u_0, t)} - 1\right]\\ &+ s_1\left[\frac{1}{\Xi(u_0, \mathcal{H}u_0, t)} - 1\right] + s_2\left[\frac{1}{\Xi(\mathcal{H}^3 u_0, \mathcal{H}^2 u_0, t)} - 1\right]\\ &\leq r_1 y + (r_1 + s_1)y + s_2\left[\frac{1}{\Xi(\mathcal{H}^3 u_0, \mathcal{H}^2 u_0, t)} - 1\right].\end{aligned}$$

Hence, $\frac{1}{\Xi(\mathcal{H}^3u_0,\mathcal{H}^2u_0,t)} - 1 \leq \big(\frac{\delta}{\lambda}\big)y$. Now, by using the assumption, we can put $u = \mathcal{H}u_0$ and $v = \mathcal{H}^2u_0$ in the condition. Thus, we obtain

$$\begin{aligned}\frac{1}{\Xi(\mathcal{H}^3u_0,\mathcal{H}^4u_0,t)} - 1 &\leq r_1\left[\frac{1}{\Xi(\mathcal{H}u_0,\mathcal{H}^2u_0,t)} - 1\right] + r_2\left[\frac{1}{\Xi(\mathcal{H}^2u_0,\mathcal{H}^3u_0,t)} - 1\right]\\ &+ s_1\left[\frac{1}{\Xi(\mathcal{H}^2u_0,\mathcal{H}^3u_0,t)} - 1\right] + s_2\left[\frac{1}{\Xi(\mathcal{H}^3u_0,\mathcal{H}^4u_0,t)} - 1\right]\\ &\leq r_1y + (r_1+s_1)\frac{r_1+r_2+s_1}{1-s_2}y + s_2\left[\frac{1}{\Xi(\mathcal{H}^3u_0,\mathcal{H}^4u_0,t)} - 1\right].\end{aligned}$$

Hence, $\frac{1}{\Xi(\mathcal{H}^3u_0,\mathcal{H}^4u_0,t)} - 1 \leq \big(\frac{\delta}{\lambda}\big)y$. Similarly, we have

$$\begin{aligned}\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^4u_0,t)} - 1 &\leq r_1\left[\frac{1}{\Xi(\mathcal{H}^3u_0,\mathcal{H}^2u_0,t)} - 1\right] + r_2\left[\frac{1}{\Xi(\mathcal{H}^4u_0,\mathcal{H}^3u_0,t)} - 1\right]\\ &+ s_1\left[\frac{1}{\Xi(\mathcal{H}^4u_0,\mathcal{H}^3u_0,t)} - 1\right] + s_2\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^4u_0,t)} - 1\right]\\ &\leq r_1\frac{r_1+r_2+s_1}{1-s_2}y + (r_2+s_1)\frac{r_1+r_2+s_1}{1-s_2}y + s_2\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^4u_0,t)} - 1\right].\end{aligned}$$

Hence, $\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^4u_0,t)} - 1 \leq \left(\frac{\delta}{\lambda}\right)^2 y$. Also, by using the assumption and putting $u = \mathcal{H}^3u_0$ and $v = \mathcal{H}^4u_0$ in the condition, we obtain

$$\begin{aligned}\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)} - 1 &\leq r_1\left[\frac{1}{\Xi(\mathcal{H}^3u_0,\mathcal{H}^4u_0,t)} - 1\right] + r_2\left[\frac{1}{\Xi(\mathcal{H}^4u_0,\mathcal{H}^5u_0,t)} - 1\right]\\ &+ s_1\left[\frac{1}{\Xi(\mathcal{H}^4u_0,\mathcal{H}^5u_0,t)} - 1\right] + s_2\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)} - 1\right]\\ &\leq r_1(\frac{\delta}{\lambda})y + (r_2+s_1)big(\frac{\delta}{\lambda})^2y + s_2\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)} - 1\right]\\ &= (\frac{\delta}{\lambda})^2\left[r_1(\frac{\lambda}{\delta})y + (r_2+s_1)y + (\frac{\lambda}{\delta})^2s_2\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)} - 1\right]\right]\\ &\leq (\frac{\delta}{\lambda})^2\left[r_1(\frac{\lambda}{\delta})y + (r_2+s_1)(\frac{\lambda}{\delta})y + (\frac{\lambda}{\delta})^2s_2\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)} - 1\right]\right]\\ &= (\frac{\delta}{\lambda})^2\left[(\frac{\lambda}{\delta})(r+1+r_2+s_1)v + (\frac{\lambda}{\delta})^2s_2\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)} - 1\right]\right]\\ &\leq (\frac{1}{\lambda})^2\left[(\frac{\lambda}{\delta})(r+1+r_2+s_1)^3v + (\lambda)^2s_2\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)} - 1\right]\right]\end{aligned}$$

which implies

$$(\frac{\lambda}{\delta})(\lambda)^3\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)} - 1\right] \leq 1-(s_2)^3\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)} - 1\right] \leq (\delta)^3y$$

Hence

$$\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)} - 1\right] \leq (\frac{\delta}{\lambda})^3y.$$

By continuing the process, we get $\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)}-1\right]\leq\left(\frac{\delta}{\lambda}\right)^{m-2}$ for all $m\geq 3$. This implies

$$\frac{1}{\Xi^2(\mathcal{H}^n u_0,\mathcal{H}^m u_0,t)}-1\leq\sum_{i=n}^{m-1}\left(\frac{1}{\Xi^2(\mathcal{H}^i u_0,\mathcal{H}^{i+1}u_0,t)}-1\right)\leq\sum_{i=n}^{m-1}\left(\frac{\delta}{\lambda}\right)^{i-2}y\leq\frac{\left(\frac{\delta}{\lambda}\right)^{m-2}}{1-\left(\frac{\delta}{\lambda}\right)}y,$$

for all $mn>m\geq 3$. Hence $\{u_m\}$ is a Cauchy sequence. Since $(\Sigma,\Xi,\Theta,\bigotimes,\diamond)$ is a complete neutrosophic metric space, there exists $x\in\Sigma$ such that $u_m\to x$. Since $\mathcal{H}$ is orbitally continuous, $\mathcal{H}u_m\to\mathcal{H}x$ and so $\mathcal{H}x=x$. Now, we show that $\mathcal{H}$ has a unique fixed point. Let v and w be fixed points of $\mathcal{H}$. Then,

$$\begin{aligned}\frac{1}{\Xi(v,w,t)}-1=&\frac{1}{\Xi(\mathcal{H}^2v,\mathcal{H}^2w,t)}-1\leq r_1\left[\frac{1}{\Xi(\mathcal{H}v,v,t)}-1\right]+r_2\left[\frac{1}{\Xi(\mathcal{H}v,\mathcal{H}^2v,t)}-1\right]\\&+s_1\left[\frac{1}{\Xi(w,\mathcal{H}w,t)}-1\right]+s_2\left[\frac{1}{\Xi(\mathcal{H}w,\mathcal{H}^2w,t)}-1\right]\end{aligned}$$

and so $v=w$. Now, we prove that $T(\mathcal{H})=T(\mathcal{H}^2)$. Let $v\in T(\mathcal{H}^2)$. Then, we have

$$\begin{aligned}\frac{1}{\Xi(v,\mathcal{H}v,t)}-1=&\frac{1}{\Xi(\mathcal{H}^2v,\mathcal{H}^2v,t)}-1\leq r_1\left[\frac{1}{\Xi(\mathcal{H}v,\mathcal{H}^2v,t)}-1\right]+r_2\left[\frac{1}{\Xi(\mathcal{H}^2v,\mathcal{H}^3v,t)}-1\right]\\&+s_1\left[\frac{1}{\Xi(v,\mathcal{H}v,t)}-1\right]+s_2\left[\frac{1}{\Xi(\mathcal{H}v,\mathcal{H}^2v,t)}-1\right]\\&=(r_1+r_2+s_1+s_2)\left[\frac{1}{\Xi(v,\mathcal{H}v,t)}-1\right].\end{aligned}$$

Since $r+s<1$, we get $\mathcal{H}v=v$ □

4. application

Example 4.1. Let $\Sigma=[0,\infty)$, be endowed with $d(u,v)=|u-v|$, $\Xi(u,v,m)=\frac{m}{m+d(u,v)}$ and $\Theta(u,v,m)=\frac{d(u,v)}{m+d(u,v)}$ for all $u,v\in\Sigma$ and $m\geq 0$. Define the selfmap $\mathcal{H}$ on Σ by $\mathcal{H}u=0$ whenever $0\leq u\leq 10$, $\mathcal{H}u=u10$ whenever $10\leq u\leq 11$ and $\mathcal{H}u=1.1$ whenever $u\geq 11$. Then by putting $\delta=\frac{1}{2}$. Therefore, the condition of Theorem 3.1 is satisfied for $\mathcal{H}$.

Example 4.2. Let $\Sigma=[0,\infty)$, be endowed with $d(u,v)=|u-v|$, $\Xi(u,v,m)=\frac{m}{m+d(u,v)}$ and $\Theta(u,v,m)=\frac{d(u,v)}{m+d(u,v)}$ for all $u,v\in\Sigma$ and $m\geq 0$. Define the selfmap $\mathcal{H}$ on Σ by $\mathcal{H}u=0$ whenever $0\leq u\leq 100$, $\mathcal{H}u=u100$ whenever $100\leq u\leq 100.1$ and $\mathcal{H}u=0.15$ whenever $u\geq 100.1$. Then by putting $\delta=\frac{1}{2}$ $n=0$. Therefore, the condition of Theorem 3.2 is satisfied for $\mathcal{H}$.

Example 4.3. Let $\Sigma=\{1,3,5\}$, be endowed with $d(u,v)=|u-v|$, $\Xi(u,v,m)=\frac{m}{m+d(u,v)}$ and $\Theta(u,v,m)=\frac{d(u,v)}{m+d(u,v)}$ for all $u,v\in\Sigma$ and $m\geq 0$. Define $\lll=\{(1,1),(3,3),(5,5)\}$ the selfmap $\mathcal{H}$ on Σ by $\mathcal{H}1=3,\mathcal{H}3=1,\mathcal{H}5=5$ Then, by putting $u_0=5$, $r=\frac{1}{2}$and $s=\frac{1}{4}$, we conclude that the condition of Theorem 3.4 is satisfied.

Example 4.4. Let $\Sigma = \{1, 3, 5\}$, be endowed with $d(u, v) = |u - v|$, $\Xi(u, v, m) = \frac{m}{m+d(u,v)}$ and $\Theta(u, v, m) = \frac{d(u,v)}{m+d(u,v)}$ for all $u, v \in \Sigma$ and $m \geq 0$. Define $\lll = \{(1, 1), (3, 3), (5, 5)\}$ the selfmap $\mathcal{H}$ on Σ by $\mathcal{H}1 = 3, \mathcal{H}3 = 1, \mathcal{H}5 = 5$ Then, by putting $u_0 = 5$, $r_1 = r_2 = s_1 = s_2 = \frac{1}{4}$, it is easy to verify that $\mathcal{H}$ satisfies the conditions of the last theorem 3.5, and so $\mathcal{H}$ has a unique solution.

Here the Cauchy sequence in neutrosophic metric space, complete neutrosophic metric space and complete ordered triangular neutrosophic metric spaces examples are introduced.

Example 4.5. Let $\Sigma = \frac{1}{n} : n \in N$ with the standard metric $d(\mu, \nu) = |\mu - \nu|$. For all $\mu, \nu \in \Sigma$ and $\alpha \in [0, \infty)$, be defined by

$$\Xi(\mu, \nu, \alpha) = \begin{cases} \dfrac{\alpha}{\alpha + d(\mu, \nu)}, & if \quad \alpha > 0, \\ 0, & if\ \alpha = 0 \end{cases}$$

$$\phi(\mu, \nu, \alpha) = \begin{cases} \dfrac{d(\mu, \nu)}{k\alpha + d(\mu, \nu)}, & if \quad \alpha > 0, k > 0 \\ 1, & if\ \alpha = 0 \end{cases}$$

$$\Upsilon(\mu, \nu, \alpha) = \frac{d(\mu, \nu)}{\alpha}\ if \quad \alpha > 0.$$

for all $\mu, \nu \in \Sigma$ and $\alpha > 0$. Then $(\Sigma, \Xi, \phi, \Upsilon, *, \diamondsuit)$ is called complete neutrosophic metric space on Σ, Here $*$ is defined by $\mu * \nu = \mu, \nu$ and $\diamondsuit$ is defined as $\mu \diamondsuit \nu = min\{1, \mu + \nu\}$. Define $\sigma(\mu) = \mu$, $\rho(\nu) = \nu$. Clearly $\sigma(\Sigma) \subseteq \rho(\Sigma)$, Also for $k = \frac{1}{3}$, we get

$$\Xi(\sigma(\mu), \rho(\nu), \frac{\alpha}{3}) = \frac{\frac{\alpha}{3}}{\frac{\alpha}{3} + d(\sigma(\mu), \rho(\nu))} \geq \frac{\alpha}{\alpha + \frac{d(\mu,\nu)}{3}} = \Xi(\sigma(\mu), \rho(\nu)).$$

Example 4.6. For $r > 0$, let $\Xi(y, r) = \frac{r}{r+\|y\|}$, $\phi(y, r) = \frac{\|y\|}{r+\|y\|}$, $\Upsilon(y, r) = \frac{\|y\|}{r}$. Then $(N, V, *, \diamondsuit)$ is an Neutrosophic norm space (NNS). Now,

$$\lim_{\mu,\nu \to \infty} \frac{r}{r+ \parallel y_\mu - y_\nu \parallel} = 1, \lim_{\mu,\nu \to \infty} \frac{\parallel y_\mu - y_\nu \parallel}{r+ \parallel y_\mu - y_\nu \parallel} = 0, \lim_{\mu,\nu \to \infty} \frac{\parallel y_\mu - y_\nu \parallel}{r} = 0.$$

$$\lim_{\mu,\nu \to \infty} \Xi(y_\mu - y_\nu, r) = 1, \lim_{\mu,\nu \to \infty} \phi(y_\mu - y_\nu, r) = 0, \lim_{\mu,\nu \to \infty} \Upsilon(y_\mu - y_\nu, r) = 0, as\, r \to \infty.$$

This shows that $\{y_\mu\}$ is a Cauchy sequence in the NNS $(N, V, *, \diamondsuit)$.

Example 4.7. Choose H as natural numbers set. Give the operations $*$ and $\diamondsuit$ as Triangular norms (TN) $\mu * \nu = max\{0, \mu + \nu - 1\}$ and Triangular conorms (TC) $\mu \diamondsuit \nu = \mu + \nu - \mu\nu$. for all $\mu, \nu \in H, \alpha > 0$

$$\Xi(\mu, \nu, \alpha) = \begin{cases} \dfrac{\mu}{\nu}, & if \quad \mu < \nu, \\ \dfrac{\nu}{\mu}, & if \quad \nu < \mu, \end{cases}$$

$$\Xi(\mu,\nu,\alpha) = \begin{cases} \dfrac{\nu-\mu}{y}, \ if \quad \mu x < \nu, \\ \dfrac{\mu-\nu}{x}, \ if \quad \nu < \mu, \end{cases}$$

$$\Xi(\mu,\nu,\alpha) = \begin{cases} \nu-\mu, \ if \quad \mu < \nu, \\ \mu-\nu, \ if \quad \nu < \mu, \end{cases}$$

Then, $(H, \mathcal{N}, *, \diamondsuit)$ is Neutrosophic metric space NMS such that $\mathcal{N} : H \times H \times R^{+} \longrightarrow [0,1]$.

5. Conclusion

In this article, I gave some results about the property (P). Moreover, I study and provide fixed point theorem for such mappings on complete ordered triangular neutrosophic metric spaces (NMS). Also stated and proved some results which extensions from the reference section of this paper of several results as in relevant items, as well as in the literature in general.

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7. Conflict of Interests

The authors declare that there is no conflict of interests regarding this manuscript.

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Selecting the Suitable Waste to Energy Technology for India Using MULTIMOORA Method under Pythagorean Neutrosophic Fuzzy Logic

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Abstract. Municipal solid waste (MSW) development has increased on a global scale, posing significant socio-economic and environmental challenges. Waste to energy conversion is strategic because it can determine the best waste-to-energy technology. Recognizing the significance of waste in energy selection, this research aims to select the best technology based on its eco-friendliness. A two-phase methodology is used for presenting a framework for waste to energy technology selection. The first phase involves the selection of criteria for this problem and the perspectives of decision makers. Ranking the selected criteria using a novel SWARA method, and ranking the technology of WtE with respect to selected criteria weights are obtained. In this paper, we investigate the WtE technology problem using the novel concept of the Pythagorean Neutrosophic fuzzy set (PNFS). This set is a hybrid of the Pythagorean and Neutrosophic fuzzy sets. In this paper, we propose the best WtE technology for India. We employ the MULTIMOORA method in the Pythagorean Neutrosophic fuzzy environment for this purpose. Sensitivity and comparison analyses are also performed to ensure the framework's robustness. The findings of this study are useful in ranking the technology, and as a result, waste management can replicate the proposed framework for waste disposal for their new platform.

Keywords: Pythagorean Fuzzy Set, Neutrosophic Fuzzy Set, Pythagorean Neutrosophic Fuzzy Set, Multi-Criteria Decision Making, MULTIMOORA, WtE Technology.

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1. Introduction

Real-world decision-making problems are typically so complex and organized that traditional decision-making techniques cannot be applied. Human decisions are represented as precise numbers in traditional decision-making techniques. In many real applications, however, the information may be incomplete, or the experts may be unable to assign precise statistical measures to the assessment. As some of the judging criteria are subjective and descriptive in nature, it is challenging for the decision maker to express his or her priorities using precise statistical measures [1]. Rather, decision-makers generally focus on making linguistic assessment processes in the decision matrix. Moreover, conventional decision-making strategies are far less capable of coping with the imprecise or ambiguous nature of linguistic evaluations [2].

The increasing industrialization, urbanisation, and changes in lifestyle that accompany the process of economic growth result in the generation of increasing amounts of waste, posing increased environmental threats. In recent years, technologies have been developed that not only aid in the generation of a significant amount of distributed energy but also in the reduction of waste for safe disposal. The Ministry is promoting all available technology options for establishing projects for the recovery of energy in the form of biogas, bio-CNG, and electricity from renewable agricultural, industrial, and urban wastes, such as municipal solid wastes, vegetable and other market wastes, slaughterhouse waste, agricultural residues, and industrial wastes and effluents [3].

Solid waste management (SWM) disposal is at an advanced stage in India. There is an imperative need to build facilities to treat and dispose of growing amounts of MSW. More than 90% of waste in India is thought to be dumped in an unsatisfactory fashion. Waste dumps are estimated to have occupied approximately 1400 km^2 in the past few years, and this figure is expected to rise in the future. Waste disposal that is properly engineered protects public health and preserves critical environmental resources such as groundwater, surface water, soil fertility, and air quality [4]. Globally, solid waste generation is steadily increasing. According to the World Bank [15], worldwide annual waste generation has been rapidly increasing. This significant increase in MSW generation is affected by a variety of factors like economic growth, rising population, technological growth, urban growth, and rural-to-urban migration, among others. Along with the increasing quantity of waste, the concentration of MSW is becoming more diverse and complicated as a result of the development of developing societies based on consumer-based lifestyles [15, 16]. There is an urgent need to develop WtE technology, which can significantly reduce waste while also protecting the environment and public health.

Real-world problems, such as decision-making problems, are complicated and involve ambiguity and fuzzy logic. This motivated Zadeh [5, 6] to develop fuzzy set theory as a means to describe and transform information that was not accurate, but rather imprecise. Fuzzy logic theory provides a mathematical foundation for capturing the uncertainty and risk related to human thought processes such as logic and understanding [7]. Because of the complexity of information and the ambiguity of the human mind, the membership function of the fuzzy set is not always sufficient to reveal the characteristics of things. To address this limitation of the fuzzy set, Atanassov [8] transformed it into an intuitionistic fuzzy set (IFS) by including a non-membership function and a hesitancy function. An IFS can describe things in three ways: superiority, inferiority, and hesitation, which are typically represented by intuitionistic fuzzy numbers (IFNs) [9]. Yager [10–12] recently proposed the concept of Pythagorean fuzzy set (PFS) as a different assessment feature to obtain more valuable information under imprecise and ambiguous environments, which is characterised by the membership and non-membership degree satisfying the condition that their square sum is not greater than 1. Zhang and Xu [13] developed the Pythagorean fuzzy number concept and provided a comprehensive computational model for PFS.

In 1998, Smarandache [14] proposed the concept of Neutrosophic set and Neutrosophic probability, as well as their logic, which has three distinct logic components: truthfulness, indeterminacy, and falsity. This concept also includes the concept of hesitation, which gives the research a significant impact in various research areas. In a Neutrosophic fuzzy set, truth membership is denoted by T, indeterminacy membership by I, and falsity membership by F. These are all independent, with a sum of $0 \leq T + I + F \leq 3$, while uncertainty in IFS is determined by the degree of membership and non-membership, the indeterminacy factor in Neutrosophic fuzzy sets is independent of the truth and falsity values. The uncertainty, falsity, and hesitation information of a real-life problem can be described using a Neutrosophic fuzzy number. In this paper, we use the proposed method to combine two sets, such as Pythagorean and Neutrosophic fuzzy sets, to provide a more reliable solution to the WtE problem. In addition, many researchers studied the WtE problem using various types of fuzzy sets in various MCDM methods.

Several studies have been conducted to propose criteria for selecting waste-to-energy technologies. Abdel-Basset et al. [17] proposed and defined some operational rules for an advanced type of Neutrosophic technique in a type 2 environment. Farooq et al. presented appropriate MSW waste-to-energy technologies [18]. Yap and Nixon [19] used multi-criteria decision making based on the analytical hierarchical process to evaluate and compare WtE technologies. Different types of waste to energy technology for MSW were evaluated by Atwadkar et al [20] for Kolhapur. A life cycle assessment (LCA) of WtE treatment plants for electricity generation was proposed by Ayodele [21]. Abdel-Basset et al. [22] discussed the creation of an evaluation strategy to help Egypt choose the best renewable energy sources. Beyene et al. [23] discussed the most recent updates on WtE technologies, which convert waste into electricity, hydrogen gas, and other chemical feedstocks while being environmentally friendly. Chiu et al. [24] investigated the feasibility of using microbial fuel cells to convert solid waste organics into energy under a variety of operational conditions. To locate sustainable photovoltaic farms, Abdel-Basset et al. [25] used a hybrid multi-criteria decision-making approach in a neutrosophic environment. Khan et al. [26] explored the effects of renewable electricity generation from waste. Kurbatova and Abu-Qdais [27] used AHP to evaluate various WtE options in order to select the most appropriate technology for the Moscow region. Malav et al. [28] discussed the difficulties associated with WtE projects in India. Reddy [29] examined MSW WtE conversion in India. Abdel-Basset et al. [30] evaluated the sustainable bioenergy production through a case study in Egypt and further suggested that converting municial wastes to biogas is the most suitable sustainable bioenergy technology.

Different WtE technologies were reviewed by many researchers. All of the studies reviewed were aimed at identifying the most feasible WtE options for various countries using MCDM methods that are rapidly developing waste management. As a result, the aim of the research was to create a general framework for selecting the most appropriate WtE technologies for India based on environmental, socioeconomic, and technological factors. To achieve this, we employ the MULTIMOORA method, which employs a Pythogorean Neutrosophic fuzzy set to select the best solution.

2. Preliminaries

Definition 2.1. [31–33] Let U be a universal set. Then, a Pythagorean fuzzy set P, which is a set of ordered pairs over U, is defined by the following:

$$P = \{(u, \phi(u), \gamma(u))|u \in U\}$$

where $\phi_P(u) : U \to [0, 1]$ and $\gamma_P(u) : U \to [0, 1]$ define the degree of membership and the degree of non-membership, respectively, of the element $u \in U$ to P, which is a subset of U, and for every $u \in U$:

$$0 \leq (\phi_P(u))^2 + (\gamma_P(u))^2 \leq 1$$

Suppose $(\phi_P(u))^2 + (\gamma_P(u))^2 \leq 1$ then there is a degree of indeterminacy of $u \in U$ to P defined by $\pi_P(u) : \sqrt{1 - [(\phi_P(u))^2 + (\gamma_P(u))^2]}$ and $\pi_P(u) \in [0, 1]$. In what follows, $(\phi_P(u))^2 + (\gamma_P(u))^2 + (\pi_P(u))^2 = 1$. Otherwise, $\pi_P(u) = 0$ whenever $(\phi_P(u))^2 + (\gamma_P(u))^2 = 1$.

Definition 2.2. [31–33]

Let U be a universal set. A Neutrosophic fuzzy set N on U is an object of the form:

$$N = \{(u, \phi_N(u), \Omega_N(u), \gamma_N(u)) : u \in U\}$$

where $\phi_N(u), \Omega_N(u), \gamma_N(u) \in [0, 1], 0 \leq \phi_N(u) + \Omega_N(u) + \gamma_N(u) \leq 3$ for all $u \in U$, $\phi_N(u)$ is the degree of membership, $\Omega_N(u)$ is the degree of indeterminacy and $\gamma_N(u)$ is the degree of non-membership. Here, $\phi_N(u)$ and $\gamma_N(u)$ are dependent component and $\Omega_N(u)$ is an independent components.

Definition 2.3. [31–33] Let U be a universal set. A Pythagorean Neutrosophic fuzzy set (PNFS) with T and F are dependent Neutrosophic components D on U is an object of the

Table 1. Pythagorean Neutrosophic fuzzy linguistic scale

Linguistic term	membership values	indeterminacy values	non-membership values
Extremely high (EH)	0.85	0.10	0.15
Very high (VH)	0.75	0.20	0.25
Medium High (MH)	0.65	0.30	0.35
Medium (M)	0.55	0.40	0.45
Medium Low (ML)	0.35	0.60	0.65
Very Low (VL)	0.25	0.70	0.75
Extremely Low (EL)	0.15	0.80	0.85

form

$$D = \{(u, \phi_N(u), \Omega_N(u), \gamma_N(u)) : u \in U\}$$

where $\phi_N(u), \Omega_N(u), \gamma_N(u) \in [0,1]$, $0 \leq (\phi_N(u))^2 + (\Omega_N(u))^2 + (\gamma_N(u))^2 \leq 2$, for all $u \in U$, $\phi_N(u)$ is the degree of membership, $\Omega_N(u)$ is the degree of indeterminacy and $\gamma_N(u)$ is the degree of non-membership. Here, $\phi_N(u)$ and $\gamma_N(u)$ are dependent component and $\Omega_N(u)$ is an independent components.

Definition 2.4. [32] The score function of the Pythagorean Neutrosophic fuzzy sets with dependent Pythogorean Neutrospohic components I and F are defined as:

$$S_D(u) = (T + (1 - I) + (1 - F))$$

with the condition $0 \leq (\phi_N(u))^2 + (\Omega_N(u))^2 + (\gamma_N(u))^2 \leq 2$.

Definition 2.5. Linguistic variable deals with many real-world decision-making problems which are more complex and uncertain. Many research studies have different linguistic variables with fuzzy numbers [34]. Here, the linguistic variables with Pythgorean Neutrosophic fuzzy number to evaluate the WtE technologies based on selected criteria and the linguistic scale is presented in Table 1.

3. Mathematical Methods

MULTIMOORA is one of the most proficient MCDM models that emerged from the important contributions of Brauers and Zavadskas [35]. The MOORA method is the ancestor of the MULTIMOORA method. It employs three methods: the ration system (RS), the reference point (RP), and the full multiplicative form (FMF). The MULTIMOORA method algorithm is as follows: [34, 36]

3.1. *Ration system approach:*

The overall significance of the i^{th}alternative is as follows:

$$X_i = x_i^+ - x_i^- \tag{1}$$

where

$$x_i^+ = \sum_{j \in B} a_{ij} \tag{2}$$

$$x_i^- = \sum_{j \in C} a_{ij} \tag{3}$$

where, x_i^+ and x_i^- denote the sum of the normalized performance ratings of the importances obtained on the basis of the benefit and cost criteria; a_{ij} denotes the normalized performance ratings. Here, B and C represents the benefit and cost criterion respectively; $i = 1, 2, 3, ..., s$ and $j = 1, 2, 3, ..., t$.

The normalized performance ratings are obtained as:

$$a_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{x} (x_{ij})^2}}$$

where x_{ij} is the performance rating of the i^{th} alternative to the j^{th} criterion. The compared alternatives are ranked in descending order based on their x_i values, with the alternative with the highest x_i value being the best ranked.

3.2. *The reference point approach:*

The optimization based on the reference point can be shown as below:

$$R_i = min_i(max_j W_j \times d(p_j - a_{ij})) \tag{4}$$

Here, R_i is the overall performance of the reference point approach and $d(p_j - a_{ij})$ is the distance between the reference point and the normalized matrix, which is multiplied by the criteria weights.Here, p_j represents the j^{th} coordinate of the reference point, as:

$$\begin{aligned} p_j &= max_i \; a_{ij}; j \in B \\ p_j &= min_i \; a_{ij}; j \in C \end{aligned} \tag{5}$$

The compared alternatives are ranked in ascending order based on their r_i values, and the alternative with the lowest value of r_i is the best ranked.

3.3. *Full multiplicative form:*

In this approach, the overall utility of the alternative can be calculated as follows:

$$F_i = \frac{B_i}{C_i} \tag{6}$$

where B_i is the product of the weighted performance ratings of the benefit criteria and C_i is the product of the weighted performance ratings of the alternative's cost criteria. The compared alternatives are ranked in descending order and the highest value of F_i is the best result.

The MULTIMOORA was used to determine the final ranking of alternatives. Generally, different parts of the MULTIMOORA approach provide different ranking orders. The dominance theory was proposed by Brauers and Zavadskas [37] in order to describe the ranks provided by three approaches of the MULTIMOORA and determine the final ranking values.

3.4. *SWARA weighting method*

Kresuliene et al. [38] propose the SWARA method, which assists experts in determining criteria weights. The SWARA method is described below: [34]

Step 1: Rank the criteria based on their importance.

Step 2: Determine its relative importance β_i.

Step 3: To calculate the coefficient value Γ_j, where $\Gamma_j = \beta_j + 1$.

Step 4: Calculate the initial weights ω_j, $\omega_j = \frac{\beta_{j-1}}{\beta}$.

Step 5: Obtain the final weight of the criteria W_j, where $W_j = \frac{\omega_j}{\sum \omega_j}$.

4. **Application**

Nowadays, the amount of waste is rapidly increasing day by day due to population growth and a wide range of technologies. The government authorities are implementing different kinds of disposal methods to reduce the waste, but they are still finding the best solution for this problem without any harmful impact on the environment and society. In the current situation, waste management is facing many difficulties in reducing the waste that comes from industries, houses, institutes, hospitals, etc. Therefore, we make it necessary to reduce the waste in a good manner and we make energy from that waste. Developing countries are focusing on advanced technologies to make energy from MSW wastes, which helps to find environmentally friendly energy while at the same time reducing the amount of waste. In this study, we proposed the MULIMOORA method to obtain the best WtE technologies for India using a Pythagorean Neutrosophic fuzzy set. For this, we chose four types of WtE technologies based on economic, environmental, social, and technology aspects.

Table 2. Weight values of the criteria

Criteria	β_j	$\Gamma_j = \beta_j + 1$	$\omega_j = \frac{\beta_{j-1}}{\beta}$	$W_j = \frac{\omega_j}{\sum \omega_j}$
Technology	0	1	1	0.392
Society	0.25	1.25	0.8	0.314
Environment	0.35	1.6	0.5	0.196
Economic	0.4	2	0.25	0.098

5. Numerical example

In this section, we discuss the WtE technology problem under the Pythagorean Neutrosophic fuzzy set using the MULTIMOORA method. Here, the experts evaluate this problem based on the selected criteria. The WtE technologies are M_1- chemical and mechanical methods; M_2-new trends in WtE; M_3-biochemical methods; and M_4-thermal conversion method. To solve this problem, experts evaluate the WtE technologies using the proposed method. The linguistic scale is used to form a decision matrix. We are now analyzing the problem using the suggested model.

5.1. *SWARA method:*

Using the SWARA method, we obtain the weight values of the criteria, which are shown in Table 2.

5.2. *MULTIMOORA method:*

The WtE technologies and the criteria are given below:

M_1 – Chemical and mechanical method

M_2 – New trends in WtE method

M_3 – Biochemical method

M_4 – Thermal conversion method

Table 3. The ratings of the WtE technology obtained from the expert

Criteria / WtE	C_1	C_2	C_3	C_4
M_1	(0.75, 0.40, 0.45)	(0.25, 0.70, 0.45)	(0.15, 0.30, 0.15)	(0.85, 0.70, 0.45)
M_2	(0.85, 0.40, 0.35)	(0.55, 0.80, 0.25)	(0.35, 0.20, 0.75)	(0.85, 0.60, 0.45)
M_3	(0.55, 0.20, 0.35)	(0.25, 0.40, 0.65)	(0.75, 0.70, 0.65)	(0.85, 0.20, 0.65)
M_4	(0.85, 0.30, 0.65)	(0.85, 0.60, 0.75)	(0.85, 0.30, 0.65)	(0.15, 0.40, 0.65)

Table 4. Decision matrix

	C_1	C_2	C_3	C_4
M_1	1.9	1.1	1.7	1.7
M_2	2.1	1.5	1.4	1.8
M_3	2.0	1.2	1.4	2.1
M_4	1.9	1.5	1.9	1.1

$$\begin{aligned} &C_1 - \text{Economic} \\ &C_2 - \text{Society} \\ &C_3 - \text{Environment} \\ &C_4 - \text{Technology} \end{aligned} \tag{7}$$

The experts then assess the WtE technology using the evaluation criteria they have chosen. Table 3 shows the expert evaluation results and which shows the ratings in the form of the PNFNs obtained as the result of the transformation of the linguistic variables from Table 1.

5.3. *Ratio system approach:*

The ranking results and the ranking order of the WtE technology were obtained on the basis of the RS approach. Using the PNFSs score function to create the decision matrix shown in Table 4, by applying Eqs. (2) and (3), we calculate the normalized decision matrix, which is shown in Table 5. The final ranking result of the RS approach is given in Table 6 using Eq. (1).

5.4. *Reference point approach*

By applying the procedure of RP, we obtained the weighted distance between the reference point and the normalized decision matrix using Eqs. (4), it is presented in Table 7. The reference point is calculated by applying Eq.(5) which are (0.4805, 0.5609, 0.5885, 0.5939) and the final ranking results are given in Table 8.

Table 5. Normalized decision matrix

	C_1	C_2	C_3	C_4
M_1	0.4805	0.4113	0.5266	0.5048
M_2	0.5311	0.5609	0.4337	0.5345
M_3	0.5058	0.4487	0.4337	0.5939
M_4	0.4805	0.5609	0.5885	0.3266

Table 6. The final ranking results for RS

WtE	Ranking values	Rank
M_1	0.9622	4
M_2	0.998	1
M_3	0.9705	3
M_4	0.9955	2

Table 7. Weighted distance between reference point and the normalized matrix

	C_1	C_2	C_3	C_4
M_1	0	0.0469	0.0121	0.0349
M_2	-0.0049	0	0.0303	0.0232
M_3	-0.0024	0.0352	0.0303	0
M_4	0	0	0	0.1047

Table 8. The final ranking results for RP

WtE	Ranking values	Rank
M_1	0.0469	3
M_2	0.0303	1
M_3	0.0352	2
M_4	0.1047	4

5.5. *Full multiplicative form:*

The ranking results and the ranking order of the WtE technology obtained on the basis of the FMF approach, by applying Eq (6), are shown in Table 10. For this, we first obtained the weighted normalized matrix, which is given in Table 9. The final ranking results are obtained using dominance theory, it as presented in Table 11.

Table 9. Weighted normalized decision matrix

	C_1	C_2	C_3	C_4
M_1	0.0470	0.1291	0.1032	0.1978
M_2	-0.0520	0.1761	0.0850	0.2095
M_3	-0.0495	0.1408	0.0850	0.2328
M_4	0.0470	0.1761	0.1153	0.1280

Table 10. The final ranking results for FMF

WtE	Ranking values	Rank
M_1	0.0560	2
M_2	0.0596	1
M_3	0.0545	3
M_4	0.0495	4

Table 11. The final ranking results of MULTIMOORA method

WtE	RA	RP	FMF	Final rank
M_1	4	3	2	2
M_2	1	1	1	1
M_3	3	2	3	3
M_4	2	4	4	4

From this Table 11, M_2- New trends in WtE technology is the most suitable and environment friendly method to convert the waste into energy.

6. **Comparison and sensitivity analysis**

6.1. *Comparison Analysis*

This section compares the proposed approach to a number of existing methods from the literature in order to demonstrate the method's efficiency and performance in comparison to those methods. The proposed methodology was compared to two existing techniques: the VIKOR [39] and the MOORA model [40]. These MCDM methods use the proposed criterion weights. The results of the ranking order comparison are shown in Table 12. The proposed ranking yields different results than the existing VIKOR and MOORA models. As a result, when compared to other MCDM models, the proposed approach yields more reliable findings.

Table 12. Comparison analysis results

WtE	VIKOR	Rank	MOORA	Rank	Proposed method
M_1	0.8010	4	0.3651	4	3
M_2	0	1	0.4248	1	1
M_3	0.4508	2	0.4160	2	2
M_4	0.5384	3	0.3762	3	4

Table 13. Weights in sensitivity analysis

WtE	Case 1	Case 2	Case 3
M_1	0.392	0.314	0.098
M_2	0.196	0.392	0.314
M_3	0.314	0.098	0.196
M_4	0.098	0.196	0.392

Table 14. Sensitivity analysis results-Case 1

WtE	RA	Rank	RP	Rank	FMF	Rank	DT
M_1	1.1787	3	0.0480	4	0.0559	2	2
M_2	1.3402	1	0.0388	2	0.0495	3	3
M_3	1.2615	2	0.0412	3	0.0396	4	4
M_4	1.1424	4	0.0388	1	0.1027	1	1

Table 15. Sensitivity analysis results-Case 2

WtE	RA	Rank	RP	Rank	FMF	Rank	DT
M_1	1.1787	3	0.0387	1	0.0401	4	4
M_2	1.3402	1	0.0412	2	0.0756	1	1
M_3	1.2615	2	0.0824	4	0.0744	2	2
M_4	1.1424	4	0.0697	3	0.0585	3	3

6.2. *Sensitivity analysis*

The sensitivity analysis of this model compares the results of three cases. Case 3 is the outcome of this study, and Cases 1 and 2 are the other outcomes discovered using different weights of the criteria. Sensitivity analysis shows that modifying the weights of the criteria has an effect on the ranking order.

7. Conclusion

The most widely used waste-to-energy technology for residual waste uses combustion to provide combined heat and power. Adopting maximum recycling with waste-to-energy in an integrated waste management system would significantly reduce dumping in India. Waste-to-energy technologies are available that can process unsegregated low-calorific value waste, and the industries are keen to exploit these technologies in India. Several waste-to-energy projects using combustion of un-segregated low-calorific value waste are currently being developed. Alternative thermal treatment processes to combustion include gasification, pyrolysis, production of refuse derived fuel and gas-plasma technology [4]. However, these WtE technologies have some drawbacks and in order to overcome this problem and selecting the best solution for waste to energy from new trends in WtE to achieve to reduce the sustainability factors resulting from the presence of many different indicators, this paper applies a hybrid multi-criteria decision-making approach under a Pythagorean Neutrosophic fuzzy environment.

Waste to energy (WtE) technologies have been identified as a promising solution for dealing with the problem of complexly composed and ever-increasing waste volumes in developed countries such as the European Union and the United States, among others. However, governments and policymakers continue to face significant challenges in selecting appropriate WtE technologies to design sustainable waste management systems for India. As a result, this study was carried out in order to propose a general systematic framework that can assist policymakers in identifying the most appropriate WtE technologies for designing waste management systems in India. In this paper, the score function of PNFS and the PNF-MULTIMOORA method based on it have been presented. The characteristic of each WtE technology is taken in the form of PNFNs. Based on the proposed method to find the best solution for this problem, New trends in WtE have been identified as the safest and most beneficial WtE technology in the current scenario, which is obtained from the proposed method. This method contributes to waste reduction while also producing energy, which will help with future energy demand issues.

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Synchronization of Time Delay Neutrosophic Stochastic System

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Abstract: This paper examines the asymptotic mean square solidness of a turbulent synchronization to time delay neutrosophic stochastic framework. The Lyapunov solidness hypothesis is utilized to plan a neutrosophic-based stochastic framework with time postpone that is supposed to be asymptotically mean square steady. The Takagi-Sugeno neutrosophic model had been made to make a neutrosophic spectator with a versatile refreshing system, as well as a neutrosophic spectator inside that presence. To communicate versatile update rules and control execution, direct framework disparity is utilized (LMI). A neutrosophic-based versatile control plot is assessed utilizing the idea of obscure. however, fixed boundary frameworks. The Lyapunov dependability hypothesis is utilized to foster the effectiveness of stochastic time delay tumultuous frameworks. For mathematical calculation, the Genesio-Tesi tumultuous framework is used. The neutrosophic fluffy framework yield is completed utilizing MATLAB. Hypothetical outcomes are approved by this recreation.

Keywords: Asymptotic mean square stability, T-S neutrosophic system, Chaos synchronization, Lyapunov function.

1 Introduction

Nonlinear dynamical frameworks that display perplexing and unusual way of behaving are known as tumultuous frameworks. The most intelligent qualities of turbulent frameworks are their touchy reliance on introductory circumstances and boundary varieties inside a given range[1-4]. Yamada and Fujisaka were quick to explore the synchronization of turbulent frameworks, trailed by Pecora and Carroll One way to deal with express delicate reliance on beginning circumstances is through turmoil syn- chronization The synchronization for turbulent frameworks has been far and wide to the degree, like summed up synchro- nization, stage synchronization , slack synchronization, projective synchronization , summed up projective syn- chronization and, surprisingly, hostile to synchronization [5-7]. The property of hostile to synchronization lay out a predominating peculiarity in even oscillators, in which the state vectors have similar outright qualities however inverse signs. At the point when synchronization and hostile to synchronization coincide, at the same time, in tumultuous frameworks, the synchronization is called half and half synchronization.

The hypotheses of vulnerability have equipped emphatically after the presentation of the fluffy set by zadeh and intuitionstic fluffy set where he presented the idea of parts capability of belongingness. Smarandache shows the possibility of a neutrosophic set. Neutrosophic set considers reality componentsthe indeterminacy parts capability, and the deception parts capability at the same time. Innovation of neutrosophic set plays a significant effect in science and designing examination area. In this ongoing age, it is for the most part utilized in direction (DM) issue and numerical demonstrating [8-12] . This supposition that is vital in a great deal of circumstances, for example, data combination when we attempt to join the information from various sensors. Neutrosophy was presented by Smarandache in 1995. ”It is a part of phi- losophy which concentrates on the beginning, nature and extent of neutralities, as well as their connections with various ideational spectra” . Neutrosophic set is a power general conventional structure which sums up the idea of the exemplary set, fluffy set, span esteemed fluffy set , intuitionistic fluffy set , and so forth [13-18]

Neutosophic fluffy based mayhem control has gotten a ton of interest as of late in an assortment of engineering applications. Versatile control configuration is an immediate mix of a control approach and some type of recursive framework recognizable proof, with the framework ID meaning to decide if the framework being controlled is direct or nonlinear. Just the qualities of a decent sort of model not set in stone for sys- tem ID, compelling parametric framework ID and parametric versatile control. In the hypothesis of questionable yet fixed boundary frameworks, versatile control configuration is contemplated and investigated [19-23].
This work proposes a neutosophic fluffy model for stochastic time-defer tumultuous frameworks with vulnerabilities in light of stochastic time-postpone turbulent frameworks with vulnerabilities. This technique is a deliberate plan procedure that guarantees the stochastic time-postpone tumultuous frameworks' worldwide mean square dependability. The versatile and supervi- siory control is resolved utilizing the Lyapunov capability to tune the regulator gain in view of the precalculated criticism control inputs. This paper is organized as follows. The issue is portrayed in Segment 2 alongside some fundamental data. The fundamental commitment of this study is referenced in Segment 3. The outline of the outcomes acquired in this paper is talked about in part 4.

2 Problem Statement and Preliminaries

Consider the stochastic time-delay chaotic system interms of T-S neutrosophic model with a time delay
$R^l : If Z_1(t_c)$ is M_1^l and $\dots$ and $Z_j(t_c)$ is M_j^l then

$$\begin{aligned} dx(t_c) &= [(A_{1l}(t_c)+\Delta A_{1l}(t_c))x(t_c)+(A_{2l}(t_c)+\Delta A_{2l}(t_c))x(t_c-\tau_c)]dt_c + \\ &\quad [(A_{3l}(t_c)+\Delta A_{3l}(t_c))x(t_c)+(A_{4l}(t_c)+\Delta A_{4l}(t_c))x(t_c-\tau_c)]dw(t_c) \\ y(t_c) &= Cx(t_c),\ l=1,\ 2,\ 3,\dots,r. \end{aligned} \tag{2.1}$$

where $x(t_c)\in\mathbb{R}^n$is state vector,$y(t_c)\in\mathbb{R}^m$ is output vector $x(t_c-\tau_c)$ is time delay state vector, $A_{1l},\ A_{2l},\ A_{3l}, A_{4l}$ and C are system matrices, $\Delta A_{1l},\ \Delta A_{2l},\ \Delta A_{3l}, \Delta A_{4l}$ and C are corresponding uncertainties. τ_c is consider as constant time delay. M_j^l be the fuzzy set, r be the number of fuzzy rule, $z(t_c)=[z_1(t_c),\ z_2(t_c),\dots\ z_j(t_c)]^T$ are premise variables associated with system states.

The center of gravity defuzzification method is used to get output of neutrosophic system.

$$\begin{array}{rcl} W_l &=& \langle t_c,\ \mu_{W_l},\ \sigma_{W_l},\ \gamma_{W_l}\rangle \\ dx &=& \langle t_c,\ \mu_{dx},\ \sigma_{W_{dx}},\ \gamma_{dx}\rangle \\ dt_c &=& \langle\ \mu_{dt_c},\ \sigma_{W_{dt_c}}\ \gamma_{dt_c}\rangle \\ dw(t_c) &=& \langle\ \mu_{dw},\ \sigma_{W_{dw}}\ \gamma_{dw}\rangle \\ y(t_c) &=& \langle \mu_{Cx(t_c)},\ \sigma_{W_x(t_c)},\ \gamma_{Cx(t_c)}\rangle \end{array} \tag{2.2}$$

where $\mu_{W_l}(z) = \prod_{i=1}^{j} \mu_{M_i^l}(z_j), \sigma_{W_l}(z) = \prod_{i=1}^{j} \sigma_{M_i^l}(z_j), \gamma_{W_l(z)} = \prod_{i=1}^{j} \gamma_{M_i^l(z_j)}$ and $\mu_{Cx(t_c)}, \sigma_{Cx(t_c)}, \gamma_{Cx(t_c)}$ are fuzzy matrices.

$$\begin{array}{rcl} \mu_{dx}(t_c) &=& \frac{\sum_{l=1}^{r} \mu_{W_l}(z)[(\mu_{A_{1l}}+\mu_{\Delta A_{1l}})(x(t_c))+(\mu_{A_{2l}}+\mu_{\Delta A_{2l}})(x(t_c-\tau_c))]\mu_{dt_c}+[(\mu_{A_{3l}}+\mu_{\Delta A_{3l}})x(t_c)+(\mu_{A_{4l}}+\mu_{\Delta A_{4l}})(x(t_c-\tau_c))]\mu_{dw(t_c)}}{\sum_{l=1}^{r} \mu_{W_l}(z)} \\ \sigma_{dx}(t_c) &=& \frac{\sum_{l=1}^{r} \sigma_{W_l}(z)[(\sigma_{A_{1l}}+\sigma_{\Delta A_{1l}})(x(t_c))+(\sigma_{A_{2l}}+\sigma_{\Delta A_{2l}})(x(t_c-\tau_c))]\sigma_{dt_c}+[(\sigma_{A_{3l}}+\sigma_{\Delta A_{3l}})x(t_c)+(\sigma_{A_{4l}}+\sigma_{\Delta A_{4l}})(x(t_c-\tau_c))]\sigma_{dw(t_c)}}{\sum_{l=1}^{r} \sigma_{W_l}(z)} \\ \gamma_{dx}(t_c) &=& 1 - \mu_{dx} t_c \\ \text{and} && \\ y(t_c) &=& Cx(t_c) \end{array} \tag{2.3}$$

$\mu_{M_i^l}(z_i), \sigma_{M_i^l}(z_i)$ and $\gamma_{M_i^l}(z_i)$ are the grades of components function and non components functions of M_i^l corresponding to z_i

$$\begin{array}{rcl} \mu_{M_l}(z_i) &=& \frac{\mu_{W_l}(z_i)}{\sum_{l=1}^{r} \mu_{W_l}(z_i)} \\ \sigma_{M_l}(z_i) &=& \frac{\sigma_{W_l}(z_i)}{\sum_{l=1}^{r} \sigma_{W_l}(z_i)} \\ \gamma_{M_l}(z_i) &=& \frac{\gamma_{W_l}(z_i)}{\sum_{l=1}^{r} \gamma_{W_l}(z_i)} \end{array} \tag{2.4}$$

Therefore the state of the neutrosophic system is given by

$$\begin{array}{rcl} \mu_{dx}(t_c) &=& \sum_{l=1}^{r} \mu_{M_l}(z_i)([(\mu_{A_{1l}} + \mu_{\Delta A_{1l}})(x(t_c)) + (\mu_{A_{2l}} + \mu_{\Delta A_{2l}})x(t_c - \tau_c)]\mu_{dt_c} + \\ && [(\mu_{A_{3l}} + \mu_{\Delta A_{3l}})(x(t_c)) + (\mu_{A_{4l}} + \mu_{\Delta A_{4l}})(x(t_c - \tau_c))]\mu_{dw(t_c)} \\ \sigma_{dx}(t_c) &=& \sum_{l=1}^{r} \sigma_{M_l}(z_i)([(\sigma_{A_{1l}} + \sigma_{\Delta A_{1l}})(x(t_c)) + (\sigma_{A_{2l}} + \sigma_{\Delta A_{2l}})x(t_c - \tau_c)]\sigma_{dt_c} + \\ && [(\mu_{A_{3l}} + \sigma_{\Delta A_{3l}})(x(t_c)) + (\sigma_{A_{4l}} + \sigma_{\Delta A_{4l}})(x(t_c - \tau_c))]\sigma_{dw(t_c)} \\ \gamma_{dx}(t_c) &=& 1 - \mu_{dx}(t_c) \\ \text{and} && \\ y(t_c) &=& Cx(t_c) \end{array} \tag{2.5}$$

Considering the assumptions

$$\mu_{\Delta A}(t_c)=\sum_{l=1}^{r}\mu_{M_l}(z_i)\mu_{\Delta A_{1l}}(t_c),\sigma_{\Delta A}(t_c)=\sum_{l=1}^{r}\sigma_{M_l}(z_i)\sigma_{\Delta A_{1l}}(t_c)\quad\text{and}\quad\gamma_{\Delta A}(t_c)=\sum_{l=1}^{r}\gamma_{M_l}(z_i)\gamma_{\Delta A_{1l}}(t_c)$$
$$\mu_{\Delta A_{d_1}}(t_c)=\sum_{l=1}^{r}\mu_{M_l}(z_i)\mu_{\Delta A_{2l}}(t_c),\sigma_{\Delta A_{d_1}}(t_c)=\sum_{l=1}^{r}\sigma_{M_l}(z_i)\sigma_{\Delta A_{2l}}(t_c)\quad\text{and}\quad\gamma_{\Delta A_{d_1}}(t_c)=\sum_{l=1}^{r}\gamma_{M_l}(z_i)\gamma_{\Delta A_{2l}}(t_c)$$
$$\mu_{\Delta A_{d_3}}(t_c)=\sum_{l=1}^{r}\mu_{M_l}(z_i)\mu_{\Delta A_{4l}}(t_c),\sigma_{\Delta A_{d_3}}(t_c)=\sum_{l=1}^{r}\sigma_{M_l}(z_i)\sigma_{\Delta A_{4l}}(t_c)\quad\text{and}\quad\gamma_{\Delta A_{d_3}}(t_c)=\sum_{l=1}^{r}\gamma_{M_l}(z_i)\gamma_{\Delta A_{4l}}(t_c)\tag{2.6}$$

Therefore the state of the neutrosophic systems is

$$\begin{aligned}
\mu_{dx}(t_c) &= [\sum_{l=1}^{r}\mu_{M_l}(z_i)(A_{1l}(x(t_c))+A_{2l}x(t_c-\tau_c))+\mu_{\Delta A}(t_c)x(t_c)+\mu_{\Delta A_{d_1}}x(t_c-\tau_c)]dt_c\\
&\quad+[\sum_{l=1}^{r}\mu_{M_l}(z_i)(A_{3l}(x(t_c))+A_{4l}x(t_c-\tau_c))+\mu_{\Delta A_{d_2}}(t_c)x(t_c)+\mu_{\Delta A_{d_3}}(t_c)x(t_c-\tau_c)]dw(t_c)\\
\sigma_{dx}(t_c) &= [\sum_{l=1}^{r}\sigma_{M_l}(z_i)(A_{1l}(x(t_c))+A_{2l}x(t_c-\tau_c))+\sigma_{\Delta A}(t_c)x(t_c)+\sigma_{\Delta A_{d_1}}x(t_c-\tau_c)]dt_c\\
&\quad+[\sum_{l=1}^{r}\sigma_{M_l}(z_i)(A_{3l}(x(t_c))+A_{4l}x(t_c-\tau_c))+\sigma_{\Delta A_{d_2}}(t_c)x(t_c)+\mu_{\Delta A_{d_3}}(t_c)x(t_c-\tau_c)]dw(t_c)\\
\gamma_{dx}(t_c) &= 1-\mu_{dx}(t_c)\\
\text{and}&\\
y(t_c) &= Cx(t_c)
\end{aligned}\tag{2.7}$$

Assume that the uncertainties are imposed on matching condition. Therefore there exist an uniformly continuous function $E_A(t_c)$ and$E_d(t_c)$ exist such that $\Delta A(t_c)=BE_A(t_c),\ \Delta A_d(t_c)=BE_d(t_c),\ B\ in\ \mathbb{R}^{n\times p}$ are known matrix. Therefore the uncertainties in equation (2.7) are represented as

$$\begin{aligned}
\Delta A(t_c)x(t_c) &= B\xi_1(x(t_c)t_c)\\
\Delta A_{d_1}(t_c)x(t_c-\tau_c) &= B\xi_2(x(t_c-\tau_c)t_c)\\
\Delta A_{d_2}(t_c)x(t_c) &= B\xi_3(x(t_c)t_c)\\
\Delta A_{d_3}(t_c)x(t_c-\tau_c) &= B\xi_4(x(t_c-\tau_c)t_c)
\end{aligned}\tag{2.8}$$

where $\xi_1,\ \xi_2,\ \xi_3,\ \xi_4$ are uncertainties, which are unknown, design of neutrosophic observer is required to estimate these uncertainties.

Therefore equation (2.7) becomes

$$\begin{aligned}
\mu_{dx}(t_c) &= [\sum_{l=1}^{r}\mu_l(z)(\mu A_{1l}(x(t_c)) + \mu A_{2l}x(t_c-\tau_c)) + \mu B\xi_1(x(t_c)t_c) + \mu B\xi_2(x(t_c-\tau_c)t_c)]dt_c \\
&\quad +[\sum_{l=1}^{r}\mu_l(z)(\mu A_{3l}(x(t_c)) + \mu A_{4l}x(t_c-\tau_c)) + \mu B\xi_3(x(t_c)t_c) + \mu B\xi_4(x(t_c-\tau_c)t_c)]\mu_{dw}(t_c) \\
\sigma_{dx}(t_c) &= [\sum_{l=1}^{r}\sigma_l(z)(\mu A_{1l}(x(t_c)) + \mu A_{2l}x(t_c-\tau_c)) + \sigma B\xi_1(x(t_c)t_c) + \sigma B\xi_2(x(t_c-\tau_c)t_c)]dt_c \\
&\quad +[\sum_{l=1}^{r}\mu_l(z)(\sigma A_{3l}(x(t_c)) + \sigma A_{4l}x(t_c-\tau_c)) + \sigma B\xi_3(x(t_c)t_c) + \sigma B\xi_4(x(t_c-\tau_c)t_c)]\sigma_{dw}(t_c) \\
\gamma_{dx}(t_c) &= 1-\mu_{dx}(t_c)\mu_{Al1}(x(t_c-\tau_c)) \\
\text{and} & \\
y(t_c) &= <\mu_y(t_c), \sigma_y(t_c), \gamma_y(t_c)>
\end{aligned} \tag{2.9}$$

where $\mu_y(t_c) = (\mu_c, \mu_x(t_c))$

Consider the following the neutrosophic system; that approximate the i^{th} components of the uncertainties $\xi_1(x(t_c)t_c),\ \xi_{li}$ as
$R^j : If x_1(t_c)$ is $\tilde{M}_1^j$ and ... and $x_n(t_c)$ is $\tilde{M}_n^j$, then $\hat{\xi}_{1i}$ is $\tilde{D}_{ij}, j = 1,\ 2,\ 3,\ 4\ldots\ q.$

The output of the neutrosophic inference is

$$\begin{aligned}
\mu(x/\theta_i) &= \frac{\sum_{j=1}^{q}\theta_{ij}(\prod_{h=1}^{n}\mu_{\tilde{M}_h^j}(x_h))}{\sum_{j=1}^{q}\prod_{h=1}^{n}\mu_{\tilde{M}_h^j}(x_h)} \\
\sigma(x/\theta_i) &= \frac{\sum_{j=1}^{q}\theta_{ij}(\prod_{h=1}^{n}\sigma_{\tilde{M}_h^j}(x_h))}{\sum_{j=1}^{q}\prod_{h=1}^{n}\sigma_{\tilde{M}_h^j}(x_h)} \\
\gamma(x/\theta_i) &= \frac{\sum_{j=1}^{q}\theta_{ij}(\prod_{h=1}^{n}\gamma_{\tilde{M}_h^j}(x_h))}{\sum_{j=1}^{q}\prod_{h=1}^{n}\gamma_{\tilde{M}_h^j}(x_h)} \\
\text{hence}\quad \hat{\xi}_{1i}(x/\theta_i) &= \theta_i^T\omega(x).
\end{aligned} \tag{2.10}$$

where $\theta_i = (\theta_{i1},\ \theta_{i2},\ \ldots,\ \theta_{iq})^T$ is an adjustable parameter vector.θ_{ij} is the center of $\tilde{D}_{ij}$ for $i = 1,\ 2,\ 3,\ \ldots,\ p;$ $j = 1,\ 2,\ 3,\ \ldots\ q,\ \omega(x)$ is the neutrosophic basic function.

The estimation of ξ_1 has the form $\hat{\xi}_{1i}(x/\theta) = \theta_i^T\omega(x), \theta \in \mathbb{R}^{p\times q}$.

The optimal parameter neutrosophic matrix θ^* is defined as $\theta^* =<\mu_{\theta*}, \sigma_{\theta*}, \gamma_{\theta*}>$

$$\mu_{\theta_*} = \text{arg min}_{\theta\in\Omega_\theta}(\sup\left\|\mu_{\hat{\xi}_1}(x/\theta) - \mu_{\hat{\xi}_1}(xt_ct_c)\right\|) \tag{2.11}$$

such that

$$\left\|\mu_{\hat{\xi}_1}(x/\theta^*) - \mu_{\hat{\xi}_1}(xt_c)\right\| \leq \mu_{\xi_1} \tag{2.12}$$

$$\sigma_{\theta_*} = \text{arg min}_{\theta\in\Omega_\theta}(\sup\left\|\sigma_{\hat{\xi}_1}(x/\theta) - \sigma_{\hat{\xi}_1}(x(t_c)t_c)\right\|)$$

such that

$$\left\|\sigma_{\hat{\xi}_1}(x/\theta^*) - \sigma_{\hat{\xi}_1}(xt_c)\right\| \leq \sigma_{\xi_1}$$

and

$$\gamma_{\theta^*}(t_c) = 1 - \mu_{\theta^*}(t_c)\mu_{Al1}(x(t_c - \tau_c))$$

where

$$\Omega_\theta = (\theta/\text{trace}(\theta^T\theta) < M_\theta^2), \tag{2.13}$$

here tr(.) is trace of the matrix, M_θ is designed constant, ξ_1 is unknown upper bound such that

$$\|\xi_2(x(t_c - \tau_c))t_c\| \leq \xi_2, \tag{2.14}$$

$\omega(x)$ is uniformly continuous then exist an Lipshiz constants

$$\|\omega(x) - \hat{\omega}(x)\| \leq \gamma \|x - \hat{x}\|. \tag{2.15}$$

Thus, the neutrosophic observer for a stochastic time- delay chaotic system (2.1) is
$R^l : If Z_1(t_c)$ is M_1^l and ... and $Z_j(t_c)$ is M_j^l then $d\hat{x}(t_c) =< \mu_{d\hat{x}}(t_c), \sigma_{d\hat{x}}(t_c), \gamma_{d\hat{x}}(t_c) >$ where

$$\begin{aligned}
\mu_{d\hat{x}}(t_c) &= [\mu A_{1l}\hat{x}(t_c) + \mu A_{2l}\hat{x}(t_c - \tau_c) + \mu L_l(y(t_c) - \hat{y}(t_c)) + \mu B(\hat{\xi_1}(\hat{x})/\theta) + \mu u_1(t_c) + \mu u_2(t_c)]dt_c \\
&\quad +[\mu A_{3l}\hat{x}(t_c) + \mu A_{4l}\hat{x}(t_c - \tau_c) + \mu L_l(y(t_c) - \hat{y}(t_c)) + \mu B(\hat{\xi_1}(\hat{x}/\theta) + \mu u_1(t_c) + \mu u_2 t_c]\mu_{dw}(t_c), \\
\sigma_{d\hat{x}}(t_c) &= [\sigma A_{1l}\hat{x}(t_c) + \sigma A_{2l}\hat{x}(t_c - \tau_c) + \sigma L_l(y(t_c) - \hat{y}(t_c)) + \sigma B(\hat{\xi_1}(\hat{x})/\theta) + \sigma u_1(t_c) + \mu u_2(t_c)]dt_c \\
&\quad +[\sigma A_{3l}\hat{x}(t_c) + \sigma A_{4l}\hat{x}(t_c - \tau_c) + \sigma L_l(y(t_c) - \hat{y}(t_c)) + \sigma B(\hat{\xi_1}(\hat{x}/\theta) + \sigma u_1(t_c) + \sigma u_2(t_c)]\sigma_{dw}(t_c), \\
\gamma(dc\hat{x(t_c)}) &= 1 - \mu_{d\hat{x}}(t_c)\mu_{A_{l1}}(x(t_c - \tau_c)) \\
\text{and} & \\
\hat{y}(t_c) &= C\hat{x}(t_c),\ l = 1,\ 2,\ 3, \ldots\ r.
\end{aligned} \tag{2.16}$$

where L_l is neutrosophic designed feedback gain matrices, u_1 and u_2 are neutrosophic supervisory control.

Therefore the output of the neutrosophic systems (2.16) is

$$\begin{aligned}
\mu_{d\hat{x}}(t_c) &= [\sum_{l=1}^{r} \mu_l(z)[\mu A_{1l}\hat{x}(t_c) + \mu A_{2l}\hat{x}(t_c - \tau_c) + \mu L_l(y(t_c) - \mu\hat{y}(t_c)) + \\
&\quad \mu B(\hat{\xi_1}(\hat{x}/\theta) + \mu u_1(t_c) + \mu u_2(t_c)]]dt_c + [\sum_{l=1}^{r} \mu_l(z)[\mu A_{3l}\hat{x}(t_c) + \mu A_{4l}\hat{x}(t_c - \tau_c) + \\
&\quad \mu L_l(y(t_c) - \hat{y}t_c) + B(\hat{\xi_1}(\hat{x}/\theta) + \mu u_1(t_c) + \mu u_2(t_c)]]\mu_{dw}(t_c), \\
\sigma_{d\hat{x}}(t_c) &= [\sum_{l=1}^{r} \sigma_l(z)[\sigma A_{1l}\hat{x}(t_c) + \sigma A_{2l}\hat{x}(t_c - \tau_c) + \sigma L_l(y(t_c) - \sigma\hat{y}(t_c)) + \\
&\quad \sigma B(\hat{\xi_1}(\hat{x}/\theta) + \sigma u_1(t_c) + \sigma u_2(t_c)]]dt_c + [\sum_{l=1}^{r} \sigma_l(z)[\sigma A_{3l}\hat{x}(t_c) + \sigma A_{4l}\hat{x}(t_c - \tau_c) + \\
&\quad \sigma L_l(y(t_c) - \hat{y}(t_c)) + B(\hat{\xi_1}(\hat{x}/\theta) + \sigma u_1 t_c + \sigma u_2 t_c]]\sigma_{dw} t_c, \\
\gamma_{d\hat{x}} t_c) &= 1 - \mu_{d\hat{x}} t_c \\
\text{and} & \\
\hat{y} t_c &= C\hat{x} t_c,\ l = 1,\ 2,\ 3 \ldots\ r.
\end{aligned} \tag{2.17}$$

The observation error is defined as

$$det_c = dxt_c - d\hat{x}t_c. \tag{2.18}$$

Therefore, the error dynamic is noted by

$$\begin{aligned}
\mu_{de}t_c &= [\sum_{l=1}^{r}\mu_l(z)[\mu(A_{1l}t_c-\mu L_lC)et_c+\mu A_{2l}e(t_c-\tau_c)]+\mu B(\xi_1(xt_ct_c)-\mu\hat{\xi}_1(\hat{x}/\theta)-\mu u_1t_c) \\
&+\mu B(\xi_1(x(t_c-\tau_c)t_c)-\mu u_2t_c)]dt+[\sum_{l=1}^{r}\mu_l(z)[\mu(A_{3l}t_c-\mu L_lC)et_c+\mu A_{4l}e(t_c-\tau_c)] \\
&-\mu B(\hat{\xi}_1(\hat{x}/\theta)+\mu u_1t_c)-\mu B(u_2t_c)]\mu_{dw}t_c. \\
\sigma_{de}t_c) &= [\sum_{l=1}^{r}\sigma_l(z)[\sigma(A_{1l}t_c-\sigma L_lC)et_c+\sigma A_{2l}e(t_c-\tau_c)]+\sigma B(\xi_1(xt_ct_c)-\sigma\hat{\xi}_1(\hat{x}/\theta)-\sigma u_1t_c) \\
&+\sigma B(\xi_1(x(t_c-\tau_c)t_c)-\sigma u_2t_c)]dt+[\sum_{l=1}^{r}\sigma_l(z)[\sigma(A_{3l}t_c-\sigma L_lC)et_c+\sigma A_{4l}e(t_c-\tau_c)] \\
&-\sigma B(\hat{\xi}_1(\hat{x}/\theta)+\sigma u_1t_c)-\sigma B(u_2t_c)]\sigma_{dw}t_c. \\
&\text{and} \\
\gamma_{de}t_c &= 1-\mu_{de}t_c
\end{aligned} \tag{2.19}$$

3 Main Result

Define the two state variables for stochastic time-delay chaotic system (2.6)

$$\begin{aligned}
\mu_ft_c &= \sum_{l=1}^{r}\mu_l(z)(\mu A_{1l}xt_c+\mu A_{2l}x(t_c-\tau_c))+\mu B\xi_1(xt_ct_c)+\mu B\xi_1(x(t_c-\tau_c)t_c) \\
\mu_gt_c &= \sum_{l=1}^{r}\mu_l(z)(\mu A_{3l}xt_c+\mu A_{4l}x(t_c-\tau_c))+\mu B\xi_3(xt_ct_c)+\mu B\xi_4(x(t_c-\tau_c)t_c). \\
\sigma_ft_c &= \sum_{l=1}^{r}\sigma_l(z)(\sigma A_{1l}xt_c+\sigma A_{2l}x(t_c-\tau_c))+\sigma B\xi_1(xt_ct_c)+\sigma B\xi_1(x(t_c-\tau_c)t_c) \\
\sigma_gt_c &= \sum_{l=1}^{r}\sigma_l(z)(\sigma A_{3l}xt_c+\sigma A_{4l}x(t_c-\tau_c))+\sigma B\xi_3(xt_ct_c)+\sigma B\xi_4(x(t_c-\tau_c)t_c). \\
\gamma_ft_c &= \sum_{l=1}^{r}\gamma_l(z)(\gamma A_{1l}xt_c+\gamma A_{2l}x(t_c-\tau_c))+\gamma B\xi_1(xt_ct_c)+\gamma B\xi_1(x(t_c-\tau_c)t_c) \\
\gamma_gt_c &= \sum_{l=1}^{r}\gamma_l(z)(\gamma A_{3l}xt_c+\gamma A_{4l}x(t_c-\tau_c))+\gamma B\xi_3(xt_ct_c)+\gamma B\xi_4(x(t_c-\tau_c)t_c).
\end{aligned} \tag{3.1}$$

Then the stochastic time delay chaotic system is

$$\begin{aligned}
\mu_xt_c-\mu_x(t_c-\tau_c) &= \int_{t-\tau_c}^{t}\mu dx(s)=\int_{t-\tau_c}^{t}\mu f(s)ds+\int_{t-\tau_c}^{t}\mu g(s)\mu_{dw}(s). \\
\sigma_xt_c-\sigma_x(t_c-\tau_c) &= \int_{t-\tau_c}^{t}\sigma dx(s)=\int_{t-\tau_c}^{t}\sigma f(s)ds+\int_{t-\tau_c}^{t}\sigma g(s)\sigma_{dw}(s). \\
\gamma_xt_c-\gamma_x(t_c-\tau_c) &= \int_{t-\tau_c}^{t}\gamma dx(s)=\int_{t-\tau_c}^{t}\gamma f(s)ds+\int_{t-\tau_c}^{t}\sigma g(s)\gamma_{dw}(s).
\end{aligned} \tag{3.2}$$

Theorem 3.1 Consider the stochastic time-delay chaotic system (2.1) and its corresponding neutrosophic observer (2.16). Suppose that positive definite matrices P, Q, S, R, D_0, D_1 and the feedback gain L_i, $i=1,\ 2,\ 3,\ 4,\dots,r$, such that $PB=C^T$and the following condition holds

$$\begin{aligned}
[(A_{1i}-L_iC)^TP+P(A_{1i}-L_iC)+R+D_0+D_1+PA_{2i}R^{-1}A_{2i}^T] &\le -Q_i, \\
\lambda_{\min}(Q_i) &\ge 2\gamma M_\theta\left\|C\right\|.
\end{aligned} \tag{3.3}$$

where $\lambda_{\min}$ denotes the minimum eigen value of a matrix.

Given the supervisory controls

$$u_1 = \hat{\xi}_1 \frac{B^T Pe}{\|B^T Pe\|},\ u_2 = \hat{\xi}_2 \frac{B^T Pe}{\|B^T Pe\|} \tag{3.4}$$

where $\hat{\xi}_1$ and $\hat{\xi}_2$ stand for estimators of ξ_1 and ξ_2.

The error dynamic is asymptotically stable in the mean square by applying

$$\begin{array}{rcl} \dot{\theta} & = & 2\eta\omega(\hat{x})B^T Pe, \\ \dot{\hat{\xi}}_1 & = & 2\eta_1 e^T PB, \\ \dot{\hat{\xi}}_2 & = & 2\eta_2 e^T PB. \end{array} \tag{3.5}$$

where $\eta,\ \eta_1,\ \eta_2$ denotes positive adaption constants.

Proof: Consider the Lyapunov-Krasovskii functional as follows:

$$Vt_c = e^T Pe + \frac{1}{2\eta} tr(\tilde{\theta}^T \tilde{\theta}) + \frac{1}{2\eta_1}\tilde{\xi}_1^2 + \frac{1}{2\eta_2}\tilde{\xi}_1^2 + \int_{t-\tau_c}^{t} e^T(\sigma)Se(\sigma)d\sigma, \tag{3.6}$$

where

$$\begin{array}{c} \tilde{\theta} = \theta^* - \theta, \\ \tilde{\xi}_1 = \xi_1 - \hat{\xi}_1, \\ \tilde{\xi}_2 = \xi_2 - \hat{\xi}_2, \end{array} \tag{3.7}$$

then its derivative can be obtained by $It\hat{o}$ formula that

$$\begin{array}{rcl} dvt_c & = & \mathbb{L}Vt_c dt + 2e^T Pgt_c dwt_c \\ \mathbb{L}Vt_c & = & V_t t_c + V_e t_c f t_c + \frac{1}{2}\text{trace}(g^T V_{ee} g) \end{array} \tag{3.8}$$

Therefore

$$\begin{array}{rcl} \mu_{\mathbb{L}V} t_c & = & 2e^T P([\sum_{i=1}^{r} \mu_i(z)[\mu(A_{1i}t_c - \mu L_i C)et_c + \mu A_{2i} e(t_c - \tau_c)] + \mu B(\xi_1(xt_c t_c) - \mu\hat{\xi}_1(\hat{x}/\theta) - \mu u_1 t_c) \\ & & +\mu B(\xi_2(x(t_c - \tau_c)t_c) - \mu u_2 t_c)]) + \mu e^T Se + \mu g^T Pg - \mu\frac{1}{\eta}\text{tr}(\tilde{\theta}^T \dot{\tilde{\theta}}) - \mu\frac{1}{\eta_1}\tilde{\xi}_1 \dot{\hat{\xi}}_1 \\ & & -\mu\frac{1}{\eta_2}\tilde{\xi}_2\dot{\hat{\xi}}_2 - \mu e^T(t_c - \tau_c)Se(t_c - \tau_c). \\ \sigma_{\mathbb{L}V} t_c & = & 2e^T P([\sum_{i=1}^{r} \sigma_i(z)[\sigma(A_{1i}t_c - \sigma L_i C)et_c + \sigma A_{2i} e(t_c - \tau_c)] + \sigma B(\xi_1(xt_c t_c) - \sigma\hat{\xi}_1(\hat{x}/\theta) - \sigma u_1 t_c) \\ & & +\sigma B(\xi_2(x(t_c - \tau_c)t_c) - \sigma u_2 t_c)]) + \sigma e^T Se + \sigma g^T Pg - \sigma\frac{1}{\eta}\text{tr}(\tilde{\theta}^T \dot{\tilde{\theta}}) - \sigma\frac{1}{\eta_1}\tilde{\xi}_1 \dot{\hat{\xi}}_1 \\ & & -\sigma\frac{1}{\eta_2}\tilde{\xi}_2\dot{\hat{\xi}}_2 - \sigma e^T(t_c - \tau_c)Se(t_c - \tau_c). \\ \gamma_{\mathbb{L}V} t_c & = & 1 - \mu_{\mathbb{L}V} t_c \end{array} \tag{3.9}$$

Now consider the relation

$$g^T Pg \leq e^T D_0 e + e^T(t_c - \tau_c)D_1 e(t_c - \tau_c). \tag{3.10}$$

Then

$$\begin{aligned}
\mu_{\mathbb{L}V} t_c &= 2e^T P([\sum_{i=1}^{r} \mu_i(z)[\mu(A_{1i}t_c - \mu L_i C)et_c + \mu A_{2i} e(t_c - \tau_c)] + \mu B(\xi_1(xt_ct_c) - \mu\hat{\xi}_1(\hat{x}/\theta) - \mu u_1 t_c) \\
&\quad + \mu B(\xi_2(x(t_c - \tau_c)t_c) - \mu u_2 t_c)]) + \mu e^T S e + \mu e^T D_0 e + \mu e^T (t_c - \tau_c) D_1 e(t_c - \tau_c) \\
&\quad \mu - \frac{1}{\eta}\text{tr}(\tilde{\theta}^T \dot{\tilde{\theta}}) - \mu\frac{1}{\eta_1}\tilde{\xi}_1\dot{\tilde{\xi}}_1 - \mu\frac{1}{\eta_2}\tilde{\xi}_2\dot{\tilde{\xi}}_2 - \mu e^T(t_c - \tau_c) S e(t_c - \tau_c) \\
&\leq \sum_{i=1}^{r} \mu_i [e^T P(A_{1i} -_i C)^T e + \mu e^T (A_{1i} -_i C) P e + \mu 2 e^T P A_{2i} e(t_c - \tau_c)] \\
&\quad + \mu 2 e^T P B(\xi_1(xt_ct_c) - \mu\hat{\xi}_1(\hat{x}/\theta)) \\
&\quad - \mu 2 e^T P B(u_1) + \mu 2 e^T P B(\xi_2(x(t_c - \tau_c)) - \mu u_2) + \mu e^T S e + \mu e^T D_0 e \\
&\quad + \mu e^T (t_c - \tau_c) D_1 e(t_c - \tau_c) - \frac{1}{\eta} tr(\tilde{\theta}^T \dot{\theta}) - \mu\frac{1}{\eta_1}\tilde{\xi}_1\dot{\hat{\xi}}_1 - \mu\frac{1}{\eta_2}\tilde{\xi}_2\dot{\hat{\xi}}_2 - \mu e^T (t_c - \tau_c) S e(t_c - \tau_c).
\end{aligned}$$

$$\begin{aligned}
\sigma_{\mathbb{L}V} t_c &= 2e^T P([\sum_{i=1}^{r} \sigma_i(z)[\sigma(A_{1i}t_c - \sigma L_i C)et_c + \sigma A_{2i} e(t_c - \tau_c)] + \sigma B(\xi_1(xt_ct_c) - \sigma\hat{\xi}_1(\hat{x}/\theta) - \sigma u_1 t_c) \\
&\quad + \sigma B(\xi_2(x(t_c - \tau_c)t_c) - \sigma u_2 t_c)]) + \sigma e^T S e + \sigma e^T D_0 e + \sigma e^T (t_c - \tau_c) D_1 e(t_c - \tau_c) \\
&\quad \sigma - \frac{1}{\eta}\text{tr}(\tilde{\theta}^T \dot{\tilde{\theta}}) - \sigma\frac{1}{\eta_1}\tilde{\xi}_1\dot{\tilde{\xi}}_1 - \sigma\frac{1}{\eta_2}\tilde{\xi}_2\dot{\tilde{\xi}}_2 - \sigma e^T(t_c - \tau_c) S e(t_c - \tau_c) \\
&\leq \sum_{i=1}^{r} \sigma_i [e^T P(A_{1i} - \sigma L_i C)^T e + \sigma e^T (A_{1i} - \sigma L_i C) P e + \sigma 2 e^T P A_{2i} e(t_c - \tau_c)] \\
&\quad + \sigma 2 e^T P B(\xi_1(xt_ct_c) - \sigma\hat{\xi}_1(\hat{x}/\theta)) \\
&\quad - \sigma 2 e^T P B(u_1) + \sigma 2 e^T P B(\xi_2(x(t_c - \tau_c)) - \sigma u_2) + \sigma e^T S e + \sigma e^T D_0 e \\
&\quad + \sigma e^T (t_c - \tau_c) D_1 e(t_c - \tau_c) - \frac{1}{\eta} tr(\tilde{\theta}^T \dot{\theta}) - \sigma\frac{1}{\eta_1}\tilde{\xi}_1\dot{\hat{\xi}}_1 - \sigma\frac{1}{\eta_2}\tilde{\xi}_2\dot{\hat{\xi}}_2 - \sigma e^T (t_c - \tau_c) S e(t_c - \tau_c). \\
\gamma_{\mathbb{L}V} t_c &= 1 - \gamma_{\mathbb{L}V} t_c
\end{aligned} \tag{3.11}$$

Now consider the relation

$$2e^T P A_{2i} e(t_c - \tau_c) \leq e^T P A_{2i} R^{-1} A_{2i}^T P e + e^T (t_c - \tau_c) R e(t_c - \tau_c). \tag{3.12}$$

Therefore the equation (3.11) becomes

$$\begin{aligned}
\mu_{\mathbb{L}V} t_c \;\leq\;& \sum_{i=1}^{r} \mu_i\, e^T[P(A_{1i}-L_iC)^T + (A_{1i}-L_iC)P + PA_{2i}R^{-1}A_{2i}^TP + S]e + e^T(t_c-\tau_c)Re(t_c-\tau_c)\\
&+2e^TPB(\xi_1(xt_ct_c) - \hat{\xi}_1(\hat{x}/\theta)) - 2e^TPB(u_1) + 2e^TPB(\xi_2(x(t_c-\tau_c)) - u_2)+\\
&e^TD_0e + e^T(t_c-\tau_c)D_1e(t_c-\tau_c) - \tfrac{1}{\eta}tr(\tilde{\theta}^T\dot{\theta}) - \tfrac{1}{\eta_1}\tilde{\xi}_1\dot{\hat{\xi}}_1 - \tfrac{1}{\eta_2}\tilde{\xi}_2\dot{\hat{\xi}}_2 - e^T(t_c-\tau_c)Se(t_c-\tau_c).\\
\leq\;& \sum_{i=1}^{r} \mu_i\, e^T[P(A_{1i}-L_iC)^T + (A_{1i}-L_iC)P + R + D_0 + D_1 + PA_{2i}R^{-1}A_{2i}^TP]e+\\
&+2e^TPB(\xi_1(xt_ct_c) - \hat{\xi}_1(\hat{x}/\theta)) - 2e^TPB(u_1) + 2e^TPB(\xi_2(x(t_c-\tau_c)t_c))\\
&-2e^TPBu_2 - \tfrac{1}{\eta}tr(\tilde{\theta}^T\dot{\theta}) - \tfrac{1}{\eta_1}\tilde{\xi}_1\dot{\hat{\xi}}_1 - \tfrac{1}{\eta_2}\tilde{\xi}_2\dot{\hat{\xi}}_2.\\
\sigma_{\mathbb{L}V} t_c \;\leq\;& \sum_{i=1}^{r} \sigma_i\, e^T[P(A_{1i}-L_iC)^T + (A_{1i}-L_iC)P + PA_{2i}R^{-1}A_{2i}^TP + S]e + e^T(t_c-\tau_c)Re(t_c-\tau_c)\\
&+2e^TPB(\xi_1(xt_ct_c) - \hat{\xi}_1(\hat{x}/\theta)) - 2e^TPB(u_1) + 2e^TPB(\xi_2(x(t_c-\tau_c)) - u_2)+\\
&e^TD_0e + e^T(t_c-\tau_c)D_1e(t_c-\tau_c) - \tfrac{1}{\eta}tr(\tilde{\theta}^T\dot{\theta}) - \tfrac{1}{\eta_1}\tilde{\xi}_1\dot{\hat{\xi}}_1 - \tfrac{1}{\eta_2}\tilde{\xi}_2\dot{\hat{\xi}}_2 - e^T(t_c-\tau_c)Se(t_c-\tau_c).\\
\leq\;& \sum_{i=1}^{r} \sigma_i\, e^T[P(A_{1i}-L_iC)^T + (A_{1i}-L_iC)P + R + D_0 + D_1 + PA_{2i}R^{-1}A_{2i}^TP]e+\\
&+2e^TPB(\xi_1(xt_ct_c) - \hat{\xi}_1(\hat{x}/\theta)) - 2e^TPB(u_1) + 2e^TPB(\xi_2(x(t_c-\tau_c)t_c))\\
&-2e^TPBu_2 - \tfrac{1}{\eta}tr(\tilde{\theta}^T\dot{\theta}) - \tfrac{1}{\eta_1}\tilde{\xi}_1\dot{\hat{\xi}}_1 - \tfrac{1}{\eta_2}\tilde{\xi}_2\dot{\hat{\xi}}_2.\\
\gamma_{\mathbb{L}V} t_c \;\leq\;& 1 - \mu_{\mathbb{L}V} t_c
\end{aligned} \tag{3.13}$$

Again we consider the relation

$$\begin{aligned}
\mu_{\xi_1}(xt_ct_c) - \mu\hat{\xi}_1(\hat{x}/\theta) &= \mu_{\xi_1}(xt_ct_c) - \mu_{\hat{\xi}_1}(\hat{x}/\theta) + \mu_{\hat{\xi}_1}(\hat{x}/\theta^*) - \mu_{\hat{\xi}_1}(\hat{x}/\theta^*) + \mu_{\hat{\xi}_1}(x/\theta^*) - \mu_{\hat{\xi}_1}(x/\theta^*)\\
&= \mu_{(\xi_1} - \mu_{\hat{\xi}_1}(\hat{x}/\theta^*)) + \mu_{\hat{\xi}_1}((\hat{x}/\theta^*) - \mu(\hat{x}/\theta)) + \mu_{\hat{\xi}_1}((x/\theta^*) - \mu(\hat{x}/\theta^*))\\
&= (\mu_{\xi_1} - \mu_{\hat{\xi}_1}(\hat{x}/\theta^*)) + \mu_{\tilde{\theta}^T\omega}(\hat{x}) + \mu_{\theta^{*T}(\omega}(x) - \mu_\omega(\hat{x})),\\
\sigma_{\xi_1}(xt_ct_c) - \sigma\hat{\xi}_1(\hat{x}/\theta) &= \sigma_{\xi_1}(xt_ct_c) - \sigma_{\hat{\xi}_1}(\hat{x}/\theta) + \sigma_{\hat{\xi}_1}(\hat{x}/\theta^*) - \sigma_{\hat{\xi}_1}(\hat{x}/\theta^*) + \sigma_{\hat{\xi}_1}(x/\theta^*) - \sigma_{\hat{\xi}_1}(x/\theta^*)\\
&= \sigma_{(\xi_1} - \sigma_{\hat{\xi}_1}(\hat{x}/\theta^*)) + \sigma_{\hat{\xi}_1}((\hat{x}/\theta^*) - \sigma(\hat{x}/\theta)) + \sigma_{\hat{\xi}_1}((x/\theta^*) - \sigma(\hat{x}/\theta^*))\\
&= (\sigma_{\xi_1} - \sigma_{\hat{\xi}_1}(\hat{x}/\theta^*)) + \sigma_{\tilde{\theta}^T\omega}(\hat{x}) + \sigma_{\theta^{*T}(\omega}(x) - \sigma_\omega(\hat{x})),
\end{aligned}$$

$$\gamma_{\xi_1}(xt_ct_c) - \gamma_{\hat{\xi}_1}(\hat{x}/\theta) = 1 - [\mu_{\xi_1}(xt_ct_c) - \mu\hat{\xi}_1(\hat{x}/\theta)] \tag{3.14}$$

and

$$P(A_{1i}-L_iC)^T + (A_{1i}-L_iC)P + R + D_0 + D_1 + PA_{2i}R^{-1}A_{2i}^TP \leq -Q_i. \tag{3.15}$$

Now, the equation (3.13) become

$$\begin{aligned}
\mu_{\mathbb{L}V} t_c &\leq \textstyle\sum_{i=1}^{r} \mu_i e^T(-Q_i)e + 2e^T PB(\xi_1(xt_ct_c) - \hat{\xi}_1(\hat{x}/\theta) + \hat{\xi}_1(\hat{x}/\theta^*) - \hat{\xi}_1(\hat{x}/\theta^*) + \hat{\xi}_1(x/\theta^*) \\
&\quad -\hat{\xi}_1(x/\theta^*)) - 2e^T PB(u_1) + 2e^T PB(\xi_2(x(t_c - \tau_c)t_c)) - 2e^T PBu_2 - \tfrac{1}{\eta} tr(\tilde{\theta}^T\dot{\theta}) - \tfrac{1}{\eta_1}\tilde{\xi}_1\dot{\hat{\xi}}_1 - \tfrac{1}{\eta_2}\tilde{\xi}_2\dot{\hat{\xi}}_2 \\
&\leq \textstyle\sum_{i=1}^{r} \mu_i e^T(-Q_i)e + 2e^T PB(\xi_1 - \hat{\xi}_1(\hat{x}/\theta^*)) + 2e^T PB(\tilde{\theta}^T\omega(\hat{x})) + 2e^T PB(\theta^{*T}(\omega(x) - \omega(\hat{x}))) \\
&\quad -2e^T PB(u_1) + 2e^T PB(\xi_2(x(t_c - \tau_c)t_c)) - 2e^T PBu_2 - \tfrac{1}{\eta} tr(\tilde{\theta}^T\dot{\theta}) - \tfrac{1}{\eta_1}\tilde{\xi}_1\dot{\hat{\xi}}_1 - \tfrac{1}{\eta_2}\tilde{\xi}_2\dot{\hat{\xi}}_2 \\
&\leq \textstyle\sum_{i=1}^{r} \mu_i e^T(-Q_i)e + 2\gamma\|e\|^2 \|C\| M_\theta + \tfrac{1}{\eta} tr[\tilde{\theta}^T(2\eta\omega\hat{x}e^T PB - \dot{\theta})] \\
&\quad +2e^T PB(\hat{\xi}_1 - u_1) + 2e^T PB(\hat{\xi}_2 - u_2) + \tfrac{1}{\eta_1}(2\eta_1 e^T PB - \dot{\hat{\xi}}_1) + \tfrac{1}{\eta_2}(2\eta_2 e^T PB - \dot{\hat{\xi}}_2)\tilde{\xi}_2 \\
\sigma_{LV} t_c &\leq \textstyle\sum_{i=1}^{r} \sigma_i e^T(-Q_i)e + 2\gamma\|e\|^2 \|C\| M_\theta + \tfrac{1}{\eta} tr[\tilde{\theta}^T(2\eta\omega\hat{x}e^T PB - \dot{\theta})] \\
&\quad +2e^T PB(\hat{\xi}_1 - u_1) + 2e^T PB(\hat{\xi}_2 - u_2) + \tfrac{1}{\eta_1}(2\eta_1 e^T PB - \dot{\hat{\xi}}_1) + \tfrac{1}{\eta_2}(2\eta_2 e^T PB - \dot{\hat{\xi}}_2)\tilde{\xi}_2 \\
&\text{and} \\
\gamma_{\mathbb{L}V} t_c &\geq 1 - [\mu_{\mathbb{L}V} t_c]
\end{aligned} \tag{3.16}$$

Now applying adaptive updating law (3.4) and (3.4) into (3.16), which yields

$$\mathbb{L}Vt_c \leq -e^T\beta e, \ \beta > 0 \tag{3.17}$$

Therefore

$$dVt_c = -e^T\beta e + 2e^T Pgt_c dwt_c \tag{3.18}$$

Taking expectation, then it follows that

$$\mathbb{E}[dVt_c] = \mathbb{E}[-e^T\beta e] + \mathbb{E}[2e^T Pgt_c dwt_c]. \tag{3.19}$$

Then it follows

$$dvt_c \leq -e^T\beta e. \tag{3.20}$$

Therefore $V \in L_\infty$, which indicate that $e, \ \tilde{\theta}, \ \tilde{\xi}_1, \ \tilde{\xi}_2, \ u_1, \ u_2 \in L_\infty$.

Integrating (3.20) from 0 to ∞ result in

$$\beta \int_0^\infty ee^T dt < V(0) - V(\infty) < \infty. \tag{3.21}$$

That mean that $e \in L_2$, applying the Lipschitz condition to the neutrosophic estimation lead to that $\dot{e} \in L_\infty$. Based on the Barbalat lemma, one may conclude that $e \to 0$ as $t \to \infty$.

Which is asymptotically stable in mean square.□

For neutrosophic observer, the output feedback control scheme is applied to stochastic chaotic time-delay.

The output feedback for a stochastic chaotic time-delay system is

$$
\begin{aligned}
\mu_{dx}t_c &= [\sum_{i=1}^{r}\mu_i(z)(\mu_{A_{1i}}+\Delta\mu_{A_{1i}})xt_c+(\mu_{A_{2i}}+\Delta\mu_{A_{2i})}(x(t_c-\tau_c))+\mu_{Bu}t_c]dt+\\
&\quad [\sum_{i=1}^{r}\mu_l(z)(\mu_{A_{3i}}+\Delta\mu_{A_{3i}})xt_c+(\mu_{A_{4i}}+\Delta\mu_{A_{4i}})(x(t_c-\tau_c))+\mu_{Bu}t_c]\mu_{dw}t_c\\
\sigma_{dx}t_c &= [\sum_{i=1}^{r}\sigma_i(z)(\sigma_{A_{1i}}+\Delta\sigma_{A_{1i}})xt_c+(\sigma_{A_{2i}}+\Delta\sigma_{A_{2i})}(x(t_c-\tau_c))+\sigma_{Bu}t_c]dt+\\
&\quad [\sum_{i=1}^{r}\sigma_l(z)(\sigma_{A_{3i}}+\Delta\sigma_{A_{3i}})xt_c+(\sigma_{A_{4i}}+\Delta\sigma_{A_{4i}})(x(t_c-\tau_c))+\sigma_{Bu}t_c]\sigma_{dw}t_c\\
\gamma_{dx} &= 1-\mu_{dx}t_c\\
&\text{and}\\
yt_c &= Cxt_c
\end{aligned}
\tag{3.22}
$$

and the corresponding neutrosophic observer is

$$
\begin{aligned}
\mu_{d\hat{x}}t_c &= [\sum_{i=1}^{r}\mu_i(z)(\mu_{\mu_{A_{1i}}}\hat{x}t_c+\mu_{A_{2i}}t_c)x(t_c-\tau_c)+\mu_{L_i}(yt_c-\hat{y}t_c)-\mu_{BK_i}\hat{x}t_c]dt\\
&\quad +[\sum_{i=1}^{r}\mu_i(z)(\mu_{A_{3i}}\hat{x}t_c+\mu_{A_{3i}}t_c)x(t_c-\tau_c)+\mu_{L_i}(yt_c-\hat{y}t_c)-\mu_{BK_i}\hat{x}t_c]\mu_{dw}t_c\\
\sigma_{d\hat{x}}t_c &= [\sum_{i=1}^{r}\sigma_i(z)(\sigma_{\mu_{A_{1i}}}\hat{x}t_c+\mu_{A_{2i}}t_c)x(t_c-\tau_c)+\mu_{L_i}(yt_c-\hat{y}t_c)-\mu_{BK_i}\hat{x}t_c]dt\\
&\quad +[\sum_{i=1}^{r}\mu_i(z)(\mu_{A_{3i}}\hat{x}t_c+\mu_{A_{3i}}t_c)x(t_c-\tau_c)+\mu_{L_i}(yt_c-\hat{y}t_c)-\mu_{BK_i}\hat{x}t_c]\mu_{dw}t_c\\
\gamma_{d\hat{x}}t_c &= 1-\mu_{dx}t_c\\
&\text{and}\\
\hat{y}t_c &= C\hat{x}t_c.
\end{aligned}
\tag{3.23}
$$

Now the system can be represented as

$$
\begin{aligned}
\mu_{dx}t_c &= \sum_{i=1}^{r}\mu_i(z)[\mu_{A_{1i}}xt_c+\mu_{A_{2i}}x(t_c-\tau_c)+\mu_{B\xi_1}(xt_ct_c)+\mu_{B\xi_2}(x(t_c-\tau_c)t_c)+\mu_B ut_c]dt\\
&\quad +\sum_{i=1}^{r}\mu_i(z)[\mu_{A_{3i}x}t_c+\mu_{A_{4i}x}(t_c-\tau_c)+\mu_{B\xi_1}(xt_ct_c)+\mu_{B\xi_2}(x(t_c-\tau_c)t_c)+\mu_B ut_c]\mu_{dw}t_c\\
\sigma_{dx}t_c &= \sum_{i=1}^{r}\sigma_i(z)[\sigma_{A_{1i}}xt_c+\sigma_{A_{2i}}x(t_c-\tau_c)+\sigma_{B\xi_1}(xt_ct_c)+\sigma_{B\xi_2}(x(t_c-\tau_c)t_c)+\sigma_B ut_c]dt\\
&\quad +\sum_{i=1}^{r}\sigma_i(z)[\mu_{A_{3i}x}t_c+\sigma_{A_{4i}x}(t_c-\tau_c)+\sigma_{B\xi_1}(xt_ct_c)+\sigma_{B\xi_2}(x(t_c-\tau_c)t_c)+\sigma_B ut_c]\sigma_{dw}t_c\\
\gamma_{dx}t_c &= 1-\mu_{dx}t_c\\
&\text{and}\\
yt_c &= Cxt_c.
\end{aligned}
\tag{3.24}
$$

Theorem 3.2 Let the neutrosophic controller in (3.24) be chosen as

$$
\begin{aligned}
\mu_{ut_c} &= \textstyle\sum_{l=1}^{r}\mu_i[-\mu_{K_i}-\mu_{L_i}\mu_M]\hat{x}-\mu_{\hat{\xi}_1}(\hat{x}/\theta)-u_1-u_2,\\
\sigma_{ut_c} &= \textstyle\sum_{l=1}^{r}\sigma_i[-\sigma_{K_i}-\sigma_{L_i}\sigma_M]\hat{x}-\sigma_{\hat{\xi}_1}(\hat{x}/\theta)-u_1-u_2,\\
\gamma_{ut_c} &= 1-\mu_u t_c
\end{aligned}
\tag{3.25}
$$

where M is a positive definite matrix and $K_i$is neutrosophic feedback gain. Suppose definitions of $\hat{\xi}_1(\hat{x}/\theta)$, u_1, u_2, and the stability conditions in theorem(1) hold. If the positive definite neutrosophic matrices M, W, U, V_0, V_1 and N_i for $i = 1,\ 2,\ 3, \ldots, r$ exist, the following conditions are satisfied

$$(A_{1i} - BK_i)^T M + M(A_{1i} - BK_i) + U + V_0 + V_1 + MA_{2i}U^{-1}A_{2i}^T M^T \leq -N_i \vee i \tag{3.26}$$

the observer-based control system is asymptotically mean square stable.
Proof: Consider the Lyapunov-Krasovskii functional as follows:

$$\begin{aligned}
Vt_c &= \hat{x}^T M\hat{x} + e^T Pe + \tfrac{1}{2\eta}\text{tr}(\tilde{\theta}^T\tilde{\theta}) + \tfrac{1}{2\eta}\tilde{\xi}_1^2 + \tfrac{1}{2\eta}\tilde{\xi}_2^2 + \\
&\quad \textstyle\int_{t-\tau_c}^{t} e^T(\sigma)Se(\sigma)d\sigma + \int_{t-\tau_c}^{t} \hat{x}^T(\sigma)S\hat{x}(\sigma)d\sigma, \\
\mu_{\mathbb{L}V}t_c &= [\sum_{i=1}^{r}\mu_i(2\hat{x}^T(\mu_{A_{1i}} - \mu_B\mu_{K_i)})\hat{x}t_c + 2\hat{x}^T\mu_{A_{2i}}\hat{x}(t_c-\tau_c)) + 2\hat{x}^T\mu_M\mu_{L_i}\mu_C e] \\
&\quad +[\sum_{i=1}^{r}\mu_i[2e^T\mu_P(\mu_{A_{1i}} - \mu_{L_iC)})et_c + 2e^T\mu_P\mu_{A_{2i}}e(t_c-\tau_c)]] + 2e^T\mu_P\mu_B\mu_{\xi_1} \\
&\quad +2e^T\mu_P\mu_B\mu_{\xi_2} + 2e^T\mu_P\mu_B ut_c + 2e^T\mu_P\mu_B\mu_{K_i}\hat{x} - \tfrac{1}{\eta}tr(\tilde{\theta}^T\dot{\theta}) - \tfrac{1}{\eta_1}\tilde{\xi}_1\dot{\hat{\xi}}_1 \\
&\quad -\tfrac{1}{\eta_2}\tilde{\xi}_2\dot{\hat{\xi}}_2 - e^Tt_cSet_c - e^T(t_c-\tau_c)\mu_S e(t_c-\tau_c) + \hat{x}^T\mu_W\hat{x} - \hat{x}(t_c-\tau_c)^T\mu_W\hat{x}(t_c-\tau_c) + \\
&\quad g^Tt_c\mu_w gt_c + g^Tt_c\mu_P gt_c \\
\sigma_{\mathbb{L}V}t_c &= [\sum_{i=1}^{r}\sigma_i(2\hat{x}^T(\sigma_{A_{1i}} - \sigma_B\sigma_{K_i)})\hat{x}t_c + 2\hat{x}^T\sigma_{A_{2i}}\hat{x}(t_c-\tau_c)) + 2\hat{x}^T\sigma_M\sigma_{L_i}\sigma_C e] \\
&\quad +[\sum_{i=1}^{r}\sigma_i[2e^T\sigma_P(\sigma_{A_{1i}} - \sigma_{L_iC)})et_c + 2e^T\sigma_P\sigma_{A_{2i}}e(t_c-\tau_c)]] + 2e^T\sigma_P\sigma_B\sigma_{\xi_1} \\
&\quad +2e^T\sigma_P\sigma_B\sigma_{\xi_2} + 2e^T\sigma_P\sigma_B ut_c + 2e^T\sigma_P\sigma_B\sigma_{K_i}\hat{x} - \tfrac{1}{\eta}tr(\tilde{\theta}^T\dot{\theta}) - \tfrac{1}{\eta_1}\tilde{\xi}_1\dot{\hat{\xi}}_1 \\
&\quad -\tfrac{1}{\eta_2}\tilde{\xi}_2\dot{\hat{\xi}}_2 - e^Tt_cSet_c - e^T(t_c-\tau_c)\sigma_S e(t_c-\tau_c) + \hat{x}^T\sigma_W\hat{x} - \hat{x}(t_c-\tau_c)^T\sigma_W\hat{x}(t_c-\tau_c) + \\
&\quad g^Tt_c\sigma_w gt_c + g^Tt_c\sigma_P gt_c \\
\gamma_{\mathbb{L}V}t_c &= 1 - \mu_{LV}t_c
\end{aligned} \tag{3.27}$$

Now consider the relation

$$\begin{aligned}
g^Tt_cWgt_c &\leq \hat{x}^TV_0\hat{x} + \hat{x}^T(t_c-\tau_c)V_1\hat{x}(t_c-\tau_c) \\
g^Tt_cPgt_c &\leq e^TD_0e + e^T(t_c-\tau_c)D_1e(t_c-\tau_c)
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
\mu_{\mathbb{L}V}t_c &\leq \sum_{i=1}^{r}\mu_i et_c^T[(\mu_{A_{1i}} - \mu_{L_i}\mu_C)^T\mu_P + \mu_P(\mu_{A_{1i}} - \mu_{L_i}\mu_C) + \mu_R + \mu_{D_0} + \mu_{D_1} + \mu_P\mu_{A_{2i}R^{-1}}\mu_{A_{2i}^T}\mu_P]et_c \\
&\quad +2e^T\mu_P\mu_B\xi_1 + 2e^T\mu_P\mu_B\xi_2 + 2e^TPBut_c - \tfrac{1}{\eta}tr(\tilde{\theta}^T\dot{\theta}) - \tfrac{1}{\eta_1}\tilde{\xi}_1\dot{\hat{\xi}}_1 - \tfrac{1}{\eta_2}\tilde{\xi}_2\dot{\hat{\xi}}_2 \\
&\quad +\sum_{i=1}^{r}\mu_i 2\hat{x}^T\mu_M\mu_{L_i}\mu_C e + \sum_{i=1}^{r}\mu_i\hat{x}^T[(\mu_{A_{1i}} - \mu_B\mu_{K_i})^T\mu_M \\
&\quad +\mu_M(\mu_{A_{1i}} - \mu_B\mu_{K_i}) + \mu_M\mu_{A_{2i}}\mu_{U^{-1}}\mu_{A_{2i}^T}\mu_M + \mu_U + \mu_{V_0} + \mu_{V_1}]\hat{x}t_c \\
\sigma_{\mathbb{L}V}t_c &\leq \sum_{i=1}^{r}\sigma_i et_c^T[(\sigma_{A_{1i}} - \sigma_{L_i}\sigma_C)^T\sigma_P + \sigma_P(\sigma_{A_{1i}} - \sigma_{L_i}\sigma_C) + \sigma_R + \sigma_{D_0} + \sigma_{D_1} + \sigma_P\sigma_{A_{2i}R^{-1}}\sigma_{A_{2i}^T}\sigma_P]et_c \\
&\quad +2e^T\sigma_P\sigma_B\xi_1 + 2e^T\sigma_P\sigma_B\xi_2 + 2e^TPBut_c - \tfrac{1}{\eta}tr(\tilde{\theta}^T\dot{\theta}) - \tfrac{1}{\eta_1}\tilde{\xi}_1\dot{\hat{\xi}}_1 - \tfrac{1}{\eta_2}\tilde{\xi}_2\dot{\hat{\xi}}_2 \\
&\quad +\sum_{i=1}^{r}\sigma_i 2\hat{x}^T\sigma_M\sigma_{L_i}\sigma_C e + \sum_{i=1}^{r}\sigma_i\hat{x}^T[(\sigma_{A_{1i}} - \sigma_B\sigma_{K_i})^T\sigma_M \\
&\quad +\sigma_M(\sigma_{A_{1i}} - \sigma_B\sigma_{K_i}) + \sigma_M\sigma_{A_{2i}}\sigma_{U^{-1}}\sigma_{A_{2i}^T}\sigma_M + \sigma_U + \sigma_{V_0} + \sigma_{V_1}]\hat{x}t_c
\end{aligned}$$

$$\gamma_{\mathbb{L}V}t_c \quad = \quad 1-[\mu_{\mathbb{L}V}t_c] \tag{3.29}$$

Now we put

$$\begin{aligned} \mu_u t_c &= \textstyle\sum_{l=1}^{r}\mu_i[-\mu_{K_i}-\mu_{L_i}\mu_M]\hat{x}-\hat{\xi}_1(\hat{x}/\theta)-u_1-u_2 \\ \sigma_u t_c &= \textstyle\sum_{l=1}^{r}\sigma_i[-\sigma_{K_i}-\sigma_{L_i}\sigma_M]\hat{x}-\hat{\xi}_1(\hat{x}/\theta)-u_1-u_2 \\ \gamma_u t_c &= 1-\mu_u t_c \end{aligned} \tag{3.30}$$

Therefore

$$\begin{aligned} \mu_{\mathbb{L}V}t_c \leq\ & \sum_{i=1}^{r}\sigma_i e t_c^T[(\mu_{A_{1i}}-\mu_{L_i}C)^T\mu_P+\mu_P(\mu_{A_{1i}}-\mu_{L_i}\mu_C)+\mu_R+\mu_{D_0}+\mu_{D_1}+\mu_P\mu_{A_{2i}}\mu_{R^{-1}}\mu_{A_{2i}^T}\mu_P]et_c \\ & +2e^T\mu_P\mu_B(\xi_1-\mu_{\hat{\xi}_1}(\hat{x}/\theta)-u_1)2e^T\mu_P\mu_B\mu_{(\xi_2}-u_2-\tfrac{1}{\eta}tr(\tilde{\theta}^T\dot{\theta})-\tfrac{1}{\mu_{\eta_1}}\tilde{\xi}_1\mu_{\dot{\hat{\xi}}_1}-\tfrac{1}{\mu_{\eta_2}}\tilde{\xi}_2\mu_{\dot{\hat{\xi}}2} \\ & +\sum_{i=1}^{r}\mu_i\hat{x}^T[(\mu_{A_{1i}}-\mu_B\mu_{K_i})^T\mu_M+\mu_M(\mu_{A_{1i}}-\mu_{Bi})+\mu_M\mu_{A_{2i}}\mu_{U^{-1}}\mu_{A_{2i}^T}\mu_M+U+V_0+V_1]\hat{x}t_c \\ \sigma_{\mathbb{L}V}t_c \leq\ & \sum_{i=1}^{r}\sigma_i e t_c^T[(\sigma_{A_{1i}}-\sigma_{L_i}C)^T\sigma_P+\sigma_P(\sigma_{A_{1i}}-\sigma_{L_i}\sigma_C)+\sigma_R+\sigma_{D_0}+\sigma_{D_1}+\sigma_P\sigma_{A_{2i}}\sigma_{R^{-1}}\sigma_{A_{2i}^T}\sigma_P]et_c+ \\ & 2e^T\sigma_P\sigma_B(\xi_1-\sigma_{\hat{\xi}_1}(\hat{x}/\theta)-u_1)2e^T\sigma_P\sigma_B\sigma_{(\xi_2}-u_2-\tfrac{1}{\eta}tr(\tilde{\theta}^T\dot{\theta})-\tfrac{1}{\sigma_{\eta_1}}\tilde{\xi}_1\sigma_{\dot{\hat{\xi}}_1}-\tfrac{1}{\sigma_{\eta_2}}\tilde{\xi}_2\sigma_{\dot{\hat{\xi}}2} \\ & +\sum_{i=1}^{r}\sigma_i\hat{x}^T[(\sigma_{A_{1i}}-\sigma_B\sigma_{K_i})^T\sigma_M+\sigma_M(\sigma_{A_{1i}}-\sigma_{Bi})+\sigma_M\sigma_{A_{2i}}\sigma_{U^{-1}}\sigma_{A_{2i}^T}\sigma_M+U+V_0+V_1]\hat{x}t_c \\ \gamma_{\mathbb{L}V}t_c \leq\ & 1-\mu_{LV}t_c \end{aligned} \tag{3.31}$$

adopting the results from the theorem 1 yields

$$\begin{aligned} \mu_{\mathbb{L}V}t_c &\leq -\beta e^T e+\sum_{i=1}^{r}\mu_i\hat{x}^T[(\mu_{A_{1i}}-\mu_B\mu_{K_i})^T\mu_M+\mu_M(\mu_{A_{1i}}-\mu_B\mu_{K_i}) \\ &\quad +\mu_M\mu_{A_{2i}}\mu_{U^{-1}}\mu_{A_{2i}^T}\mu_M+\mu_U+\mu V_0+\mu_{V_1}]\hat{x}t_c \\ \sigma_{\mathbb{L}V}t_c &\leq -\beta e^T e+\sum_{i=1}^{r}\sigma_i\hat{x}^T[(\sigma_{A_{1i}}-\sigma_B\sigma_{K_i})^T\sigma_M+\sigma_M(\sigma_{A_{1i}}-\sigma_B\sigma_{K_i}) \\ &\quad +\sigma_M\sigma_{A_{2i}}\sigma_{U^{-1}}\sigma_{A_{2i}^T}\sigma_M+\sigma_U+\sigma V_0+\sigma_{V_1}]\hat{x}t_c \\ \gamma_{\mathbb{L}V}t_c &\geq 1-\mu_{\mathbb{L}V}t_c \end{aligned} \tag{3.32}$$

Given the stability conditions (3.26), it follows that

$$\begin{aligned} \mathbb{L}Vt_c &\leq -\beta e^T e-\textstyle\sum_{i=1}^{r}\mu_i\lambda_{min}(N_i)\left\|\hat{x}\right\|^2, \\ &\leq \beta e^T e-\alpha\hat{x}^T\hat{x},\ \alpha>0 \end{aligned} \tag{3.33}$$

Therefore

$$dVt_c=-e^T\beta e-\alpha\hat{x}^T\hat{x}+2e^TPgt_c dwt_c \tag{3.34}$$

Taking expectation, then it follows that

$$\begin{aligned} \mathbb{E}[dVt_c] &= \mathbb{E}[(-e^T\beta e-\alpha\hat{x}^T\hat{x})]+\mathbb{E}[2e^TPgt_c dwt_c] \\ dvt_c &\leq -e^T\beta e-\alpha\hat{x}^T\hat{x} \end{aligned}$$

Using the Barbalat lemma, both e and $\hat{x}$ will eventually approach zero.□

As a general rule, a tumultuous framework has unsound fixed focuses or temperamental circles. The synchronization of the framework typically centers around fostering a control technique that powers framework directions to join to unsound fixed places or circles.

The stochastic time postpone tumultuous framework is planned as the following reference model.

The neutrosophic reference model for stochastic time-delay chaotic system is

$$\begin{aligned}
\mu_{dx_m}t_c &= [\sum_{i=1}^{r}\mu_i(\mu_{\overline{A}_{1i}}x_mt_c+\mu_{\overline{A}_{2i}}x_m(t_c-\tau_c)+\mu_{\overline{B}_i}rt_c)]dt \\
&\quad +[\sum_{i=1}^{r}\mu_i(\mu_{\overline{A_{3i}}}x_mt_c+\mu_{\overline{A}_{4i}}x_m(t_c-\tau_c)+\mu_{\overline{B}_i}rt_c)]\mu_{dw}t_c, \\
\sigma_{dx_m}t_c &= [\sum_{i=1}^{r}\sigma_i(\sigma_{\overline{A}_{1i}}x_mt_c+\sigma_{\overline{A}_{2i}}x_m(t_c-\tau_c)+\sigma_{\overline{B}_i}rt_c)]dt \\
&\quad +[\sum_{i=1}^{r}\sigma_i(\sigma_{\overline{A_{3i}}}x_mt_c+\sigma_{\overline{A}_{4i}}x_m(t_c-\tau_c)+\sigma_{\overline{B}_i}rt_c)]\sigma_{dw}t_c, \\
\gamma_{dx_m}t_c &= 1-\mu_{dx_m}(t_c-\tau_c) \\
\text{and} & \\
y_mt_c &= Cx_mt_c
\end{aligned} \tag{3.35}$$

where$\overline{A}_{1i}=A_{1i}-BK_i,\ \overline{A}_{2i}=A_{2i}\overline{A}_{3i}=A_{3i}-BK_i,\ \overline{A}_{4i}=A_{4i},\ \overline{B}_i=BK_{mi},K_{mi}$ is a known real matrix and rt_c is reference input.

The observer for tracking control is

$$\begin{aligned}
\mu_{d\hat{x}}t_c &= \sum_{i=1}^{r}[(\mu_{A_{1i}}-\mu_B\mu_{K_i})\hat{x}t_c+\mu_{A_{2i}}\hat{x}(t_c-\tau_c)+\mu_{L_i}(yt_c-\hat{y}t_c)+\mu_B\mu_{K_{mi}}rt_c]dt \\
&\quad +\sum_{i=1}^{r}[(\mu_{A_{3i}}-\mu_B\mu_{K_i})\hat{x}t_c+\mu_{A_{4i}}\hat{x}(t_c-\tau_c)+\mu_{L_i}(yt_c-\hat{y}t_c)+\mu_B\mu_{K_{mi}}rt_c]\mu_{dw}t_c \\
\sigma_{d\hat{x}}t_c &= \sum_{i=1}^{r}[(\sigma_{A_{1i}}-\sigma_B\sigma_{K_i})\hat{x}t_c+\sigma_{A_{2i}}\hat{x}(t_c-\tau_c)+\sigma_{L_i}(yt_c-\hat{y}t_c)+\sigma_B\sigma_{K_{mi}}rt_c]dt \\
&\quad +\sum_{i=1}^{r}[(\sigma_{A_{3i}}-\sigma_B\sigma_{K_i})\hat{x}t_c+\sigma_{A_{4i}}\hat{x}(t_c-\tau_c)+\sigma_{L_i}(yt_c-\hat{y}t_c)+\sigma_B\sigma_{K_{mi}}rt_c]\sigma_{dw}t_c \\
\gamma_{d\hat{x}}t_c &= 1-\mu_{d\hat{x}}t_c \\
\hat{y}t_c &= C\hat{x}t_c.
\end{aligned} \tag{3.36}$$

Therefore the neutrosophic reference model can be written as

$$\begin{aligned}
\mu_{dx_m}t_c &= \sum_{i=1}^{r}\sigma_i[(\mu_{A_{1i}}-\mu_B\mu_{K_i})x_mt_c+\mu_{A_{2i}}x_m(t_c-\tau_c)+\mu_B\mu_{K_{mi}}rt_c]dt \\
&\quad +\sum_{i=1}^{r}\mu_i[(\mu_{A_{3i}}-\mu_B\mu_{K_i})x_mt_c+\mu_{A_{4i}}x_m(t_c-\tau_c)+\mu_B\mu_{K_{mi}}rt_c]\mu_{dw}t_c
\end{aligned} \tag{3.37}$$

$$\begin{aligned}
\sigma_{dx_m}t_c &= \sum_{i=1}^{r}\sigma_i[(\sigma_{A_{1i}}-\sigma_B\sigma_{K_i})x_mt_c+\sigma_{A_{2i}}x_m(t_c-\tau_c)+\mu_B\sigma_{K_{mi}}rt_c]dt \\
&\quad +\sum_{i=1}^{r}\sigma_i[(\sigma_{A_{3i}}-\sigma_B\sigma_{K_i})x_mt_c+\sigma_{A_{4i}}x_m(t_c-\tau_c)+\sigma_B\sigma_{K_{mi}}rt_c]\sigma_{dw}t_c
\end{aligned} \tag{3.38}$$

$$\begin{aligned} \gamma_{dx_m} t_c &= 1 - \mu_{dx_m} t_c \quad (3.39) \\ \text{and} & \\ y_m t_c &= C x_m t_c. \quad (3.40) \end{aligned}$$

Define the error vectors $et_c = xt_c - \hat{x}_m t_c, \ \overline{x}t_c = x_m t_c - \hat{x}t_c$ The error dynamics system of et_c and $\overline{x}$ is

$$\begin{aligned} \mu_{de} t_c &= \sum_{i=1}^{r} \mu_i[(\mu_{A_{1i}} - \mu_{L_i}\mu_C)et_c + \mu_{A_{2i}} e(t_c - \tau_c) + \mu_B \xi_2 - \mu_B \mu_{K_{mi}} rt_c + \mu_B ut_c]dt \\ &+ \sum_{i=1}^{r} \mu_i[(\mu_{A_{3i}} - \mu L_i \mu_C)et_c + \mu_{A_{4i}} e(t_c - \tau_c) + \mu_B \xi_2 - \mu_B \mu_{K_{mi}} rt_c + \mu_B ut_c]\mu_{dw} t_c \\ \sigma_{de} t_c &= \sum_{i=1}^{r} \sigma_i[(\sigma_{A_{1i}} - \sigma_{L_i}\sigma_C)et_c + \sigma_{A_{2i}} e(t_c - \tau_c) + \sigma_B \xi_2 - \sigma_B \sigma_{K_{mi}} rt_c + \sigma_B ut_c]dt \\ &+ \sum_{i=1}^{r} \sigma_i[(\sigma_{A_{3i}} - \sigma L_i \sigma_C)et_c + \sigma_{A_{4i}} e(t_c - \tau_c) + \sigma_B \xi_2 - \sigma_B \sigma_{K_{mi}} rt_c + \sigma_B ut_c]\sigma_{dw} t_c \\ \gamma_{de} t_c &= 1 - \mu_{de} t_c \\ \text{and} & \quad (3.41) \\ \mu_{d\overline{x}} &= \sum_{i=1}^{r} \mu_i[(\mu_{A_{1i}} - \mu_B \mu_{K_i})\overline{x}t_c + \mu A_{2i} \overline{x}(t_c - \tau_c) - \mu_{L_i}\mu_{C\overline{x}} t_c]dt \\ &+ \sum_{i=1}^{r} \mu_i[(\mu_{A_{3i}} - \mu_B \mu_{K_i})\overline{x}t_c + \mu_{A_{4i}} \overline{x}(t_c - \tau_c) - \mu_{L_i}\mu_{C\overline{x}} t_c]\mu_{dw} t_c \\ \sigma_{d\overline{x}} &= \sum_{i=1}^{r} \sigma_i[(\sigma_{A_{1i}} - \sigma_B \sigma_{K_i})\overline{x}t_c + \sigma A_{2i} \overline{x}(t_c - \tau_c) - \sigma_{L_i}\sigma_{C\overline{x}} t_c]dt \\ &+ \sum_{i=1}^{r} \sigma_i[(\sigma_{A_{3i}} - \sigma_B \sigma_{K_i})\overline{x}t_c + \sigma_{A_{4i}} \overline{x}(t_c - \tau_c) - \sigma_{L_i}\sigma_{C\overline{x}} t_c]\sigma_{dw} t_c \\ \mu_{d\overline{x}} &= 1 - \mu_{d\overline{x}} \end{aligned}$$

Theorem 3.3 Suppose that the fuzzy controller in (3.24) is

$$\begin{aligned} \mu_u t_c &= \textstyle\sum_{i=1}^{r} \mu_i(-\mu_{K_i}\hat{x} - \mu_{Li}\mu_M \hat{x} + \mu_{Kmi} rt_c) - \hat{\xi}_1(\hat{x}/\theta) - u_1 - u_2, \\ \sigma_u t_c &= \textstyle\sum_{i=1}^{r} \sigma_i(-\sigma_{K_i}\hat{x} - \sigma_{Li}\sigma_M \hat{x} + \sigma_{Kmi} rt_c) - \hat{\xi}_1(\hat{x}/\theta) - u_1 - u_2, \quad (3.42) \\ \mu_u t_c &= 1 - \mu_u t_c \end{aligned}$$

and the stability conditions addressed in Theorems 3.1 and 3.2 hold. Then, the mean square asymptotic stabilities of the closed-loop systems (3.41) are guaranteed.

Proof: The Lyapunov $-$ Krasovskii functional candidate is chosen as

$$\begin{aligned} Vt_c &= \overline{x}^T M \overline{x} + e^T P e + \tfrac{1}{2\eta} tr(\tilde{\theta^T} \tilde{\theta}) + \tfrac{1}{2\eta} \tilde{\xi}_1^2 + \tfrac{1}{2\eta} \tilde{\xi}_2^2 + \\ & \int_{t-\tau_c}^{t} e^T(\sigma) S e(\sigma) d\sigma + \int_{t-\tau_c}^{t} \overline{x}^T(\sigma) S \overline{x}(\sigma) d\sigma \quad (3.43) \end{aligned}$$

Suppose the derivative of (3.43) as

$$\begin{aligned} \mu_{\mathbb{L}V} t_c \leq & \sum_{i=1}^{r} \mu_i \, e^T[(\mu_{A_{1i}} - \mu_{L_i}\mu_C)^T \mu_P + \mu_P(\mu_{A_{1i}} - \mu_{L_i}\mu_C) + \mu_{D_0} + \mu_{D_1} + \mu_R + \mu_P \mu_{A_{2i}} \mu_{R^{-1}} \mu_{A_{2i}}^T \mu_P]e \\ & + 2e^T \mu_P \mu_B \xi_1 + 2e^T\_P \mu_B \xi_2 + \sum_{i=1}^{r} 2e^T \mu_P \mu_B \mu_{Ki} \hat{x} + \sum_{i=1}^{r} 2e^T PB(-K_i)\hat{x} - 2e^T \mu_P \mu_B \hat{\xi}_1(\hat{x}/\theta) \\ & - 2e^T \mu_P \mu_B u_1 - 2e^T \mu_P \mu_B u_2 - \sum_{i=1}^{r} \mu_i 2\overline{x}^T \mu_M \mu_{Li} c\overline{x} + \sum_{i=1}^{r} \mu_i(-J)\overline{x}^T \overline{x} \end{aligned}$$

$$\begin{aligned}\sigma_{\mathbb{L}V}t_c &\leq \sum_{i=1}^{r}\sigma_i\, e^T[(\sigma_{A_{1i}}-\sigma_{L_i}\sigma_C)^T\sigma_P+\sigma_P(\sigma_{A_{1i}}-\sigma_{L_i}\sigma_C)+\sigma_{D_0}+\sigma_{D_1}+\sigma_R+\sigma_P\sigma_{A_{2i}}\sigma_{R^{-1}}\sigma_{A_{2i}}^T\sigma_P]e\\ &+2e^T\sigma_P\sigma_B\xi_1+2e^T\_P\sigma_B\xi_2+\sum_{i=1}^{r}2e^T\sigma_P\sigma_B\sigma_{Ki}\hat{x}+\sum_{i=1}^{r}2e^TPB(-K_i)\hat{x}-2e^T\sigma_P\sigma_B\hat{\xi}_1(\hat{x}/\theta)\\ &-2e^T\sigma_P\sigma_Bu_1-2e^T\sigma_P\sigma_Bu_2-\sum_{i=1}^{r}\sigma_i2\overline{x}^T\sigma_M\sigma_{Li}c\overline{x}+\sum_{i=1}^{r}\sigma_i(-J)\overline{x}^T\overline{x}\\ \gamma_{\mathbb{L}V}t_c &\geq 1-\mu_{\mathbb{L}V}t_c\end{aligned} \tag{3.44}$$

adopting the results from the theorem 1 and theorem 2 and Given the stability conditions (3.42), it follows that

$$\mathbb{L}Vt_c \leq \beta e^Te-\delta\overline{x}^T\overline{x},\ \delta>0 \tag{3.45}$$

Therefore

$$dVt_c=-e^T\beta e-\delta\overline{x}^T\overline{x}+2[e^TP+\overline{x}^TM]gt_cdwt_c \tag{3.46}$$

Taking expectation, then it follows that

$$\begin{aligned}\mathbb{E}[dVt_c] &= \mathbb{E}[(-e^T\beta e-\delta\overline{x}^T\overline{x})]+\mathbb{E}[2[e^TP+\overline{x}^TM]Pgt_cdwt_c]\\ &\leq -e^T\beta e-\delta\overline{x}^T\overline{x}\end{aligned}$$

Using the Barbalat lemma, both β and $\overline{x}$ will eventually approach zero.

Which is asymptotically mean square stable.

4 Numerical Simulations

Gensio-Tesi is a three-dimensional autonomous chaotic system. The system is defined by the following set of differential equations:

$$\begin{aligned}\dot{x}_1 &= -x_3-x_2\\ \dot{x}_2 &= x_1+ax_2\\ \dot{x}_3 &= b+x_3(x_1-c)\end{aligned} \tag{4.1}$$

where a, b, and c are system parameters. The Gensio-Tesi system has been studied extensively in the field of chaos theory and has been used as a benchmark system for testing various chaos analysis methods.

The Gensio Tesi chaotic delay system is a nonlinear dynamic system that exhibits chaotic behavior. The mathematical representation of the Gensio Tesi chaotic delay system is given by the following set of delay differential equations:

$$\begin{aligned}\dot{x}_1 &= -ax_1+\beta x_2(t_c-\tau_c)\\ \dot{x}_2 &= \gamma x_1+\delta x_2(t_c-\tau_c)+\epsilon x_3(t_c-\tau_c)\\ \dot{x}_3 &= \mu x_3+vx_1x_2\end{aligned} \tag{4.2}$$

The Gensio Tesi chaotic delay system has been applied in various fields such as secure communication, chaos control, and image encryption. Its complex and unpredictable behavior makes it a useful tool in these applications.

The system exhibits chaotic behavior under certain parameter regimes, which means that small perturbations in the initial conditions can lead to vastly different outcomes over time. The delay term introduces a time lag in the system's response, while the stochastic term adds random fluctuations to the dynamics. The system

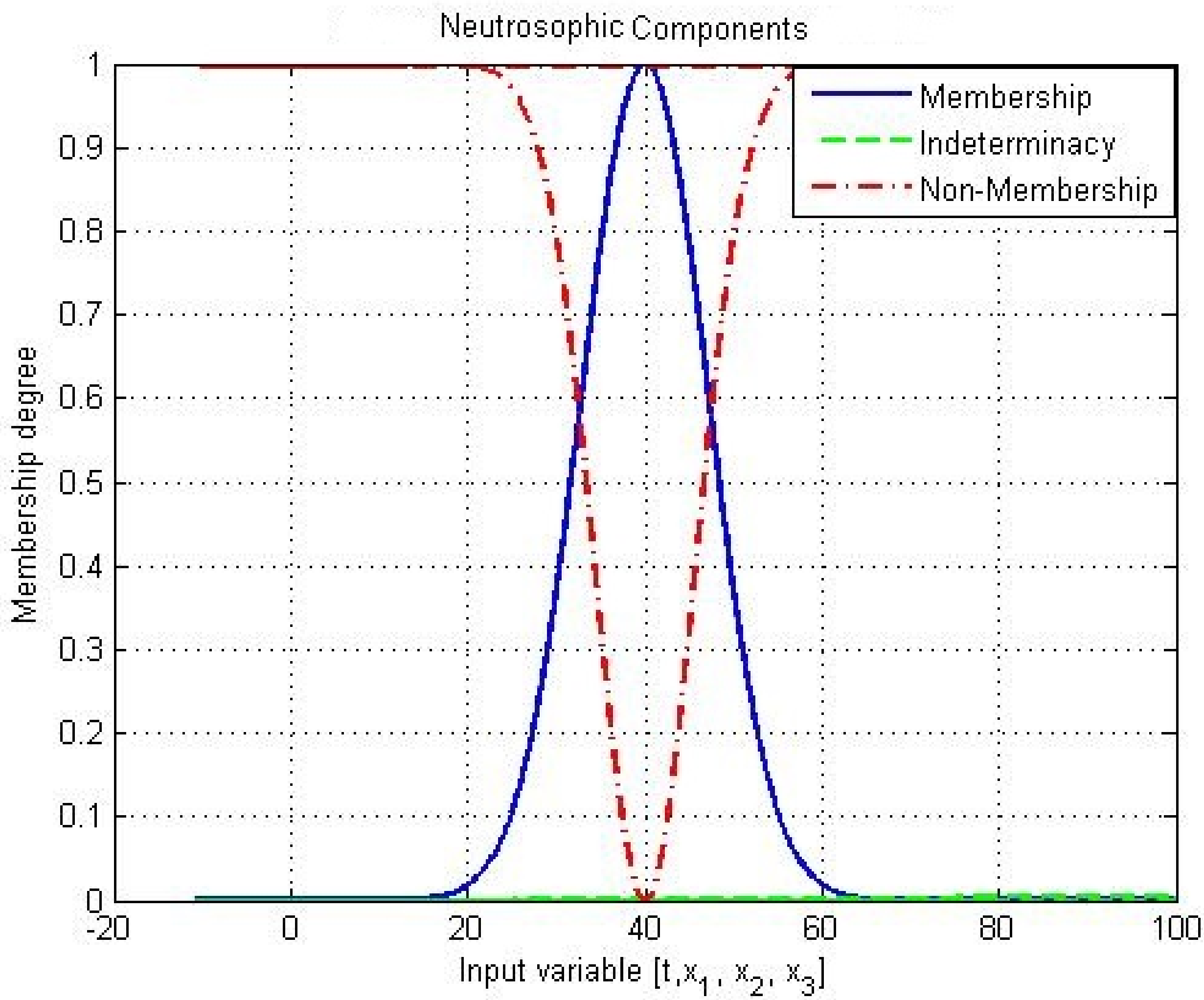

Figure 1: Neutrosophic components Function for Time$t_c and state variables$x$_1, x_2, x_3$

is defined by the following set of differential equations:

$$
\begin{aligned}
dx_1 t_c &= [-ax_1 + \beta x_2(t_c - \tau_c)]dt + \sigma_1[-ax_1]dwt_c \\
dx_2 t_c &= [\gamma x_1 + \delta x_2(t_c - \tau_c) + \epsilon x_3(t_c - \tau_c)]dt + \sigma_2[\delta x_2(t_c - \tau_c)]dwt_c \\
dx_2 t_c &= [\mu x_3 + v x_1 x_2]dt + \sigma_3[x_1 x_2]dwt_c
\end{aligned} \tag{4.3}
$$

The system (4.3) is the stochastic differential equation that gives the Gensio Tesi chaotic delay system. Here $\sigma_1, \sigma_2, \sigma_3$ are the noise parameter. For this problem these parameter values are vary inbetween 0 and 1. wt_c is the *wiener.*

The MATLAB is used for numerical simulation. Calculation is performed using the exponential fuzzy components function.

Figure 1 shows neutrosophic components function for time$t_c and state variables$x$_1, x_2, x_3$.

Figure 2 shows neutrosophic components function the state variables x_1 without time t.

Figure 3 portraits neutrosophic components function the state variables x_2 without time t.

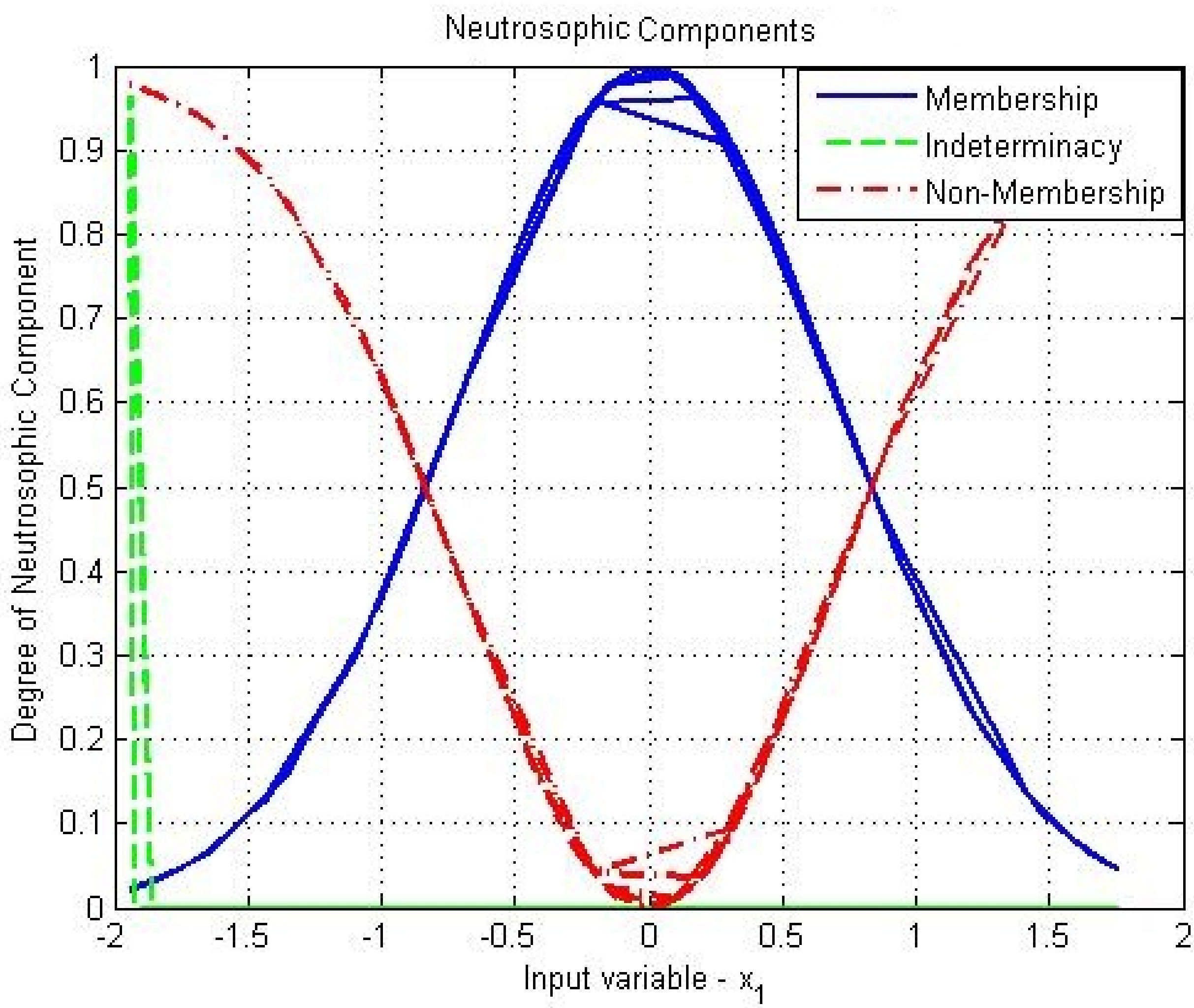

Figure 2: Neutrosophic components Function for state variables $x1$

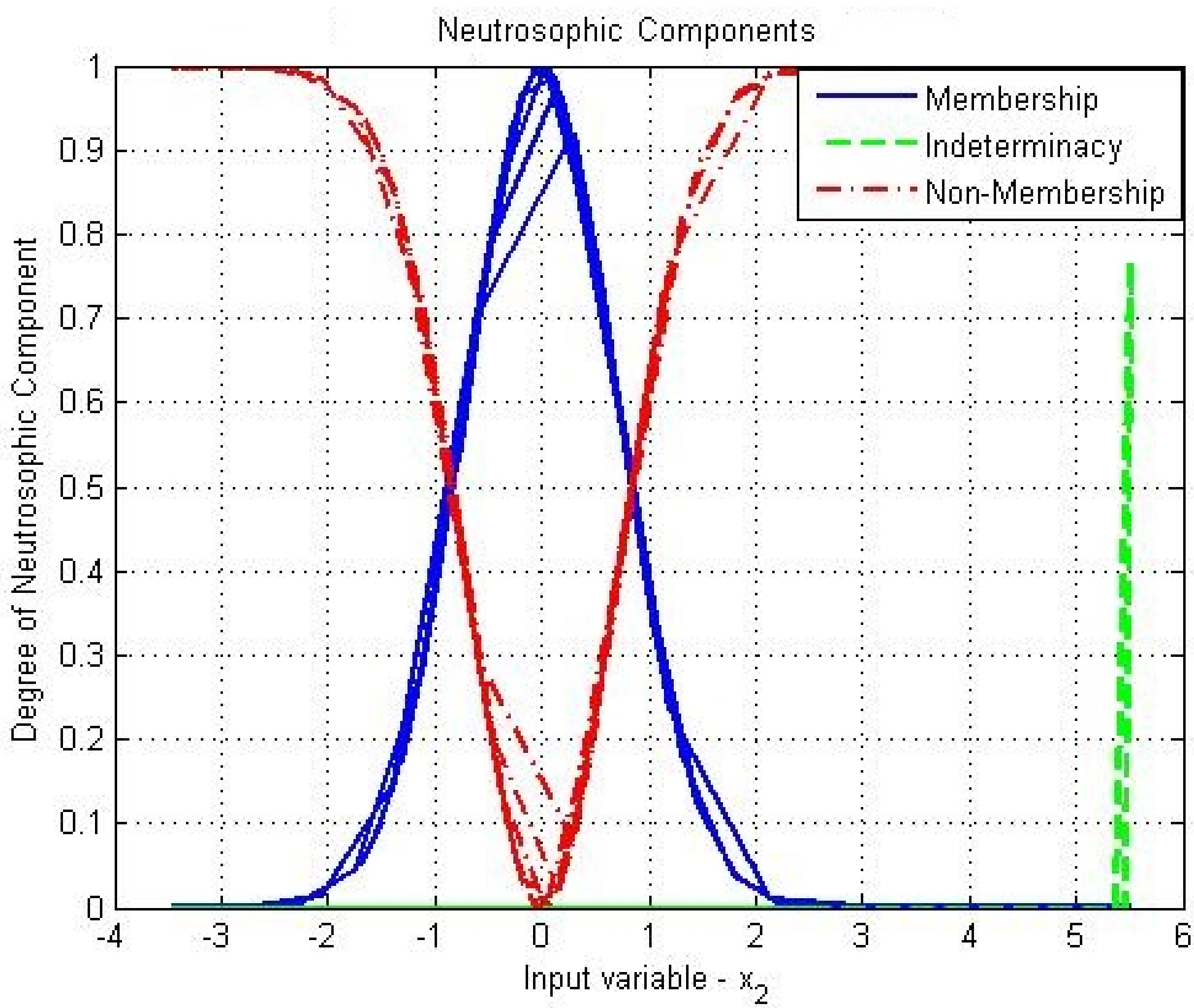

Figure 3: Neutrosophic components Function for state variables $x2$

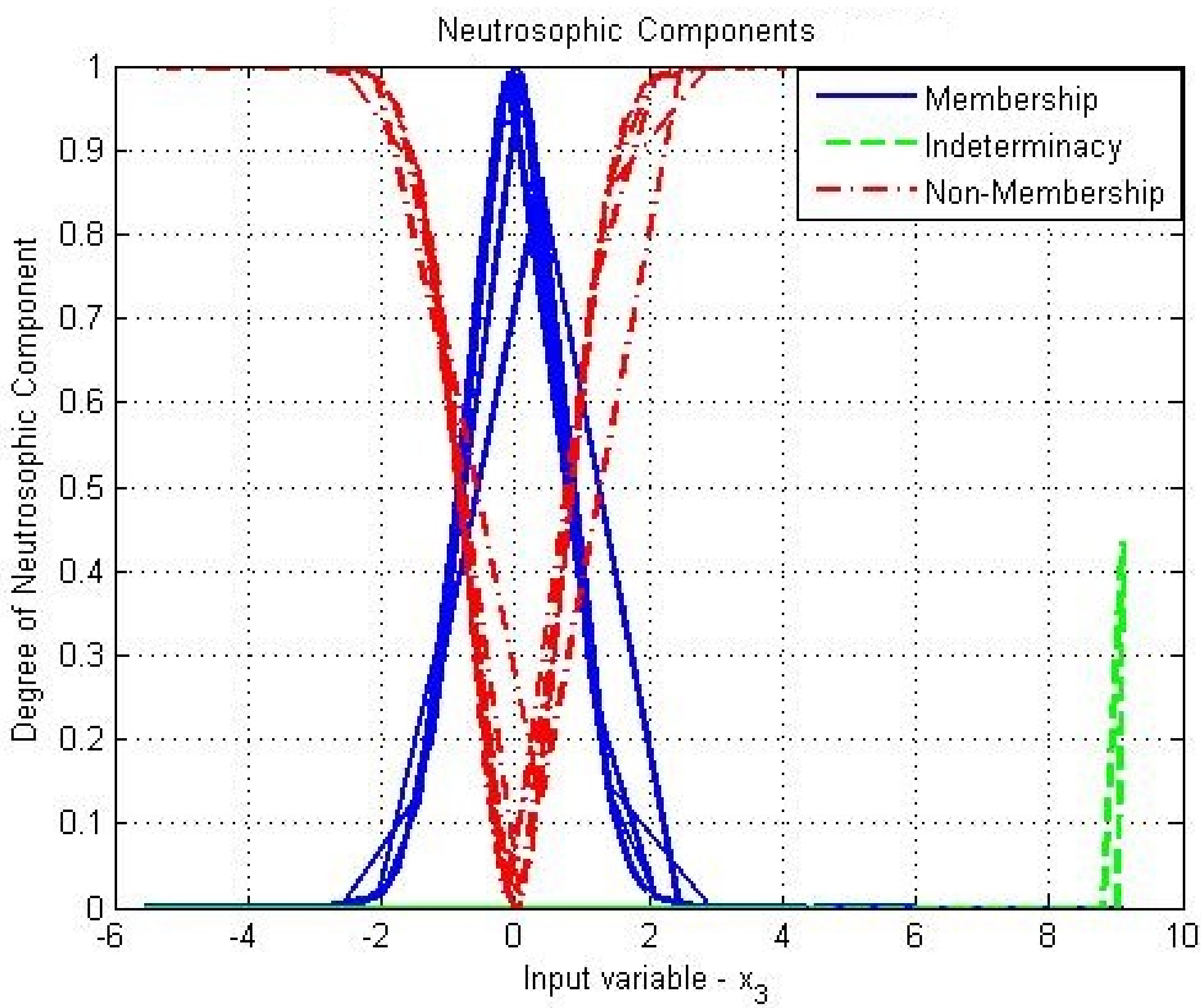

Figure 4: Neutrosophic components Function for state variables $x3$

Figure 4 portraits neutrosophic components function the state variables x_3 without time t.

5 Conclusion

This work proposes a neutrosophic T-S stochastic turbulent framework with time delay using an eyewitness based approach. A fluffy versatile administrative control strategy has been utilized to survey the asymptotic mean square soundness of tumultuous frameworks. To infer the neutrosophic versatile update regulation and control execution, direct lattice imbalance is utilized (LMI). Since the Lyapunov examples are not required for these computations, the versatile administrative control engineering is extremely proficient and advantageous for acquiring asymptotic mean square synchronization. For mathematical reenactment, the Genisio Tesi turbulent defer framework is utilized. The given hypothetical outcome is approved by the mathematical reenactment. The article is quick to address stochastic tumultuous frameworks with time delays and neutrosophic fuzzy.

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Application Of Neutrosophic Sets Based On Score Function in Medical Diagnosis

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Abstract. The process of determining which illness or disease is to blame for a person's symptoms and indicators is known as medical diagnosis. Most frequently, it is referred to as analysis with the clinical environment implied. The data needed for finding is normally gathered from a clinical trial and actual assessment of the individual looking for medical consideration. This paper's major objective is to identify a methodical strategy for decision-making problems that involves choosing the appropriate choices and qualities for a neutrosophic score function utilising neutrosophic topology. Additionally, we use a neutrosophic topological space based on attributes and alternatives combined with graphical representation to apply a neutrosophic scoring function to medical diagnosis problems.

Keywords: Neutrosophic set, Neutrosophic topology, Neutrosophic score function.

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1. Introduction

Zadeh [41] as part of logic and set hypothesis was the first to introduce the concept of a fuzzy set between intervals in mathematics. Chang's [10] general topology framework, that utilisesfuzzy topological space, was created with a fuzzy set. Adlassnig [6] used fuzzy set theory to formalise medical interactions and fuzzy logic to create a framework for automated analysis. This theory has been used in the areas of artificial intelligence, probability, science, control structures, and financial concerns [16, 20, 26].

In 1983, Atanassov [7] developed an intuitionistic fuzzy set with membership and non-membership values. Coker [14] created intuitionistic fuzzy topological spaces from intuitionistic fuzzy sets. De et al. [15] were the first to develop the applications of intuitionistic fuzzy sets in medical diagnosis. Several researchers [8,17,27] investigated intuitionistic fuzzy sets in medical diagnostics further.

Smarandache [23, 24] offered the notions of neutrosophy and neutrosophic set at the beginning of the 21^{th} century and has a wide range of consistent applications in computer science, information systems, applied mathematics, artificial intelligence, mechanics, medicine, dynamic, management science, and electrical & electronics, etc [1–4,36,37]. Salama and Alblowi, [21,22] in 2012, developed neutrosophic set and neutrosophic crisp set in a neutrosophic topological space. Recently, Vadivel and authors [29,30,33–35] presented various open sets and mappings in neutrosophic topological spaces. Smarandache [24] described the single valued Neutrosophic set on three portions (T-Truth, F-Falsehood, I-Indeterminacy) Neutrosophic sets, which Wang et al. [38] worked on. In decision making problems, Majumdar and Samanta [18] described various similarity measures of single valued neutrosophic sets. Several researchers have recently proposed numerous similarity measures and single-valued neutrosophic sets in medical diagnostics [5, 9, 11–13, 19, 28, 39, 40]. Vadivel and authors [31, 32] discussed an applications using neutrosophic score function in mobile networking and material selection problems.

2. Preliminaries

Definition 2.1. [21] Let T be a non-empty set. A neutrosophic set (briefly, $N_s eus$) L is an object having the form $L = \{\langle t, \mu_L(t), \sigma_L(t), \nu_L(t)\rangle : t \in T\}$ where $\mu_L, \sigma_L, \nu_L \to [0, 1]$ denote the degree of membership, indeterminacy, non-membership functions respectively of each element $t \in T$ to the $N_s eus$ L and $0 \le \mu_L(t) + \sigma_L(t) + \nu_L(t) \le 3$ for each $t \in T$.

Definition 2.2. [21] Let T be a non-empty set & the $N_s eus$'s L & K in the form $L = \{\langle t, \mu_L(t), \sigma_L(t), \nu_L(t)\rangle : t \in T\}$, $K = \{\langle t, \mu_K(t), \sigma_K(t), \nu_K(t)\rangle : t \in T\}$, then

(i) $0_{N_s} = \langle t, 0, 0, 1\rangle$ and $1_{N_s} = \langle t, 1, 1, 0\rangle$,
(ii) $L \subseteq K$ iff $\mu_L(t) \le \mu_K(t)$, $\sigma_L(t) \le \sigma_K(t)$ & $\nu_L(t) \ge \nu_K(t) : t \in T$,
(iii) $L = K$ iff $L \subseteq K$ and $K \subseteq L$,
(iv) $1_{N_s} - L = \{\langle t, \nu_L(t), 1 - \sigma_L(t), \mu_L(t)\rangle : t \in T\} = L^c$,
(v) $L \cup K = \{\langle t, \max(\mu_L(t), \mu_K(t)), \max(\sigma_L(t), \sigma_K(t)), \min(\nu_L(t), \nu_K(t))\rangle : t \in T\}$,
(vi) $L \cap K = \{\langle t, \min(\mu_L(t), \mu_K(t)), \min(\sigma_L(t), \sigma_K(t)), \max(\nu_L(t), \nu_K(t))\rangle : t \in T\}$.

Definition 2.3. [21] A neutrosophic topology (briefly, $N_s euty$) on a non-empty set T is a family Γ_{N_s} of neutrosophic subsets of T satisfying

(i) $0_{N_s}, 1_{N_s} \in \Gamma_{N_s}$.
(ii) $L_1 \cap L_2 \in \Gamma_{N_s}$ for any $L_1, L_2 \in \Gamma_{N_s}$.
(iii) $\bigcup L_x \in \Gamma_{N_s}, \forall\ L_x : x \in T \subseteq \Gamma_{N_s}$.

Then (T, Γ_{N_s}) is called a neutrosophic topological space (briefly, $N_s eutysp$) in T. The Γ_{N_s} elements are called neutrosophic open sets (briefly, $N_s euos$) in T. A $N_s eus$ C_{N_s} is called a neutrosophic closed sets (briefly, $N_s eucs$) iff its complement $C^c_{N_s}$ is $N_s euos$.

Definition 2.4. [25] The Neutrosophic Score Function (briefly, $N_s euScFu$) on $s : L \to [0, 1]$ is defined by

$$s(\mu_L, \sigma_L, \nu_L) = \frac{2 + \mu_L - \sigma_L - \nu_L}{3}$$

that represents the average of positiveness of the neutrosophic components μ_L, σ_L, ν_L.

3. Neutrosophic Score Function

In this section, we present a neutrosophic score function based on methodical approach for decision-making problem with neutrosophic information. The following essential steps are proposed the precise way to deal with select the proper attributes and alternative in the decision-making situation.

Step 1: Problem field selection:

Consider multi-attribute decision making problems with m attributes $At_1, At_2, \cdots, At_m$ and n alternatives $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$ and p attributes $\xi_1, \xi_2, \cdots, \xi_p$, $(n \le p)$.

	Γ_1	Γ_2	.	.	.	Γ_n
At_1	(b_{11})	(b_{12})	.	.	.	(b_{1n})
At_2	(b_{21})	(b_{22})	.	.	.	(b_{2n})
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
At_m	(b_{m1})	(b_{m2})	.	.	.	(b_{mn})

	At_1	At_2	.	.	.	At_m
ξ_1	(e_{11})	(e_{12})	.	.	.	(e_{1m})
ξ_2	(e_{21})	(e_{22})	.	.	.	(e_{2m})
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
ξ_p	(e_{p1})	(e_{p2})	.	.	.	(e_{pm})

Here all the attributes b_{ij} and e_{ki} are neutrosophic numbers, where $(i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, p)$.

Step 2: Form neutrosophic topologies for Γ_j and ξ_k:

(i) $\Gamma^*_j = \Gamma \cup \Gamma^* \cup \Gamma^{**}$, where $\Gamma = \{1_{N_s}, 0_{N_s}, b_{1j}, b_{2j}, \cdots b_{mj}\}$, $\Gamma^* = \{b_{1j} \cup b_{2j}, b_{2j} \cup b_{3j}, \cdot, b_{m-1j} \cup b_{mj}\}$ and $\Gamma^{**} = \{b_{1j} \cap b_{2j}, b_{2j} \cap b_{3j}, \cdot, b_{m-1j} \cap b_{mj}\}$.

(ii) $\xi^*_k = \xi \cup \xi^* \cup \xi^{**}$, where $\xi = \{1_{N_s}, 0_{N_s}, e_{k1}, e_{k2}, \cdots e_{km}\}$, $\xi^* = \{e_{k1} \cup e_{k2}, e_{k2} \cup e_{k3}, \cdot, e_{km-1} \cup e_{km}\}$ and $\xi^{**} = \{e_{k1} \cap e_{k2}, e_{k2} \cap e_{k3}, \cdot, e_{km-1} \cap e_{km}\}$.

Step 3: Find neutrosophic score functions:

Neutrosophic score functions (shortly, $N_s euScFu$) of $\Gamma, \Gamma^*, \Gamma^{**}, \xi, \xi^*, \xi^{**}, \Gamma_j$ and ξ_k are defined as follows.

(i) $N_s euScFu(\Gamma) = \frac{1}{3(m+2)} \left[\sum_{i=1}^{m+2} [2 + \mu_i - \sigma_i - \nu_i]\right]$, $N_s euScFu(\Gamma^*) = \frac{1}{3q} \left[\sum_{i=1}^{q} [2 + \mu_i - \sigma_i - \nu_i]\right]$, where q is the number of element of Γ^* and

$N_seuScFu(\Gamma^{**}) = \frac{1}{3r}\left[\sum_{i=1}^{r}[2+\mu_i-\sigma_i-\nu_i]\right]$, where r is the number of element of Γ^{**}. For $j = 1, 2, \cdots, n$,

$$N_seuScFu(\Gamma_j) = \begin{cases} N_seuScFu(\Gamma) & \text{if } N_seuScFu(\Gamma^*) = 0; N_seuScFu(\Gamma^{**}) = 0 \\ \frac{1}{2}\left[N_seuScFu(\Gamma) + N_seuScFu(\Gamma^*)\right] & \text{if } N_seuScFu(\Gamma^{**}) = 0 \\ \frac{1}{3}\left[N_seuScFu(\Gamma) + N_seuScFu(\Gamma^*) + N_seuScFu(\Gamma^{**})\right] & \text{otherwise} \end{cases}$$

(ii) $N_seuScFu(\xi) = \frac{1}{3(m+2)}\left[\sum_{i=1}^{m+2}[2+\mu_i-\sigma_i-\nu_i]\right]$, $N_seuScFu(\xi^*) = \frac{1}{3s}\left[\sum_{i=1}^{s}[2+\mu_i-\sigma_i-\nu_i]\right]$, where s is the number of element of ξ^* and $N_seuScFu(\xi^{**}) = \frac{1}{3t}\left[\sum_{i=1}^{t}[2+\mu_i-\sigma_i-\nu_i]\right]$, where t is the number of element of ξ^{**}. For $k = 1, 2, \cdots, p$,

$$N_seuScFu(\xi_k) = \begin{cases} N_seuScFu(\xi) & \text{if } N_seuScFu(\xi^*) = 0; N_seuScFu(\xi^{**}) = 0 \\ \frac{1}{2}\left[N_seuScFu(\xi) + N_seuScFu(\xi^*)\right] & \text{if } N_seuScFu(\xi^{**}) = 0 \\ \frac{1}{3}\left[N_seuScFu(\xi) + N_seuScFu(\xi^*) + N_seuScFu(\xi^{**})\right] & \text{otherwise} \end{cases}$$

Step 4: Final Decision

Arrange neutrosophic score values for the alternatives $\Gamma_1 \leq \Gamma_2 \leq \cdots \leq \Gamma_n$ and the attributes $\xi_1 \leq \xi_2 \leq \cdots \leq \xi_p$. Choose the attribute ξ_p for the alternative Γ_1 and ξ_{p-1} for the alternative Γ_2 etc. If $n < p$, then ignore ξ_k, where $k = 1, 2, \cdots, n-p$.

4. Numerical Example

Medical diagnosis has increased volume of data accessible to doctors from new medical innovations and includes vulnerabilities. In medical diagnosis, very difficult task is the way toward classifying different set of symptoms under a single name of an illness. In this part, we exemplify a medical diagnosis problem for effectiveness and applicability of above proposed approach.

Step 1: Problem field selection:

Consider the following tables giving informations when consulted physicians about five patients, Patient 1 (shortly, Pat_1), Patient 2 (shortly, Pat_2), Patient 3 (shortly, Pat_3), Patient 4 (shortly, Pat_4) and Patient 5 (shortly, Pat_5) and symptoms are Weight gain (shortly, Wg), Tiredness (shortly, Td), Myalgia (shortly, Ml), Swelling of legs (shortly, Sl) and Mensus Problem (shortly, Mp). We need to find the patient and to find the disease such as Lymphedema, Insomnia, Hypothyroidism, Menarche, Arthritis of the patient. The data in Table 1 and Table

2 are explained by the membership, the indeterminacy and the non-membership functions of the patients and diseases respectively.

Symptoms \ Patients	Pat_1	Pat_2	Pat_3	Pat_4	Pat_5
Wg	(0.9,0.1,0)	(0.8,0,0.2)	(0,0.1,0.9)	(0.1,0,0.7)	(0.3,0.2,0.5)
Td	(0,0.3,0.7)	(0.1,0.2,0.7)	(0.8,0.1,0.2)	(0.1,0.1,0.8)	(0.6,0.5,0.3)
Ml	(0.3,0.1,0.6)	(0.8,0,0.3)	(0.3,0.1,0.6)	(0.2,0.1,0.6)	(0.3,0.4,0.4)
Sl	(0.9,0,0.1)	(0.4,0.2,0.5)	(0.2,0.2,0.7)	(0.4,0.2,0.5)	(0.4,0.6,0.3)
Mp	(0.2,0.1,0.7)	(0.3,0.2,0.5)	(0.4,0.3,0.2)	(0.9,0,0.1)	(0.7,0.4,0.5)

TABLE 1. Neutrosophic values for patients

Disease \ Symptoms	Wg	Td	Ml	Sl	Mp
Lymphedema	(0,0.2,0.8)	(0.2,0.2,0.1)	(0.7,0.2,0.1)	(0.9,0,0.1)	(0.2,0.6,0.4)
Insomnia	(0,0.1,0.9)	(0.9,0,0.1)	(0.2,0,0.8)	(0.2,0.4,0.1)	(0.2,0.1,0.7)
Hypothyroidism	(0.9,0.1,0.1)	(0.1,0.1,0.8)	(0,0.1,0.9)	(0.1,0.4,0.3)	(0.2,0.6,0.4)
Menarche	(0.6,0.3,0.1)	(0.1,0.1,0.8)	(0.2,0.4,0.1)	(0.2,0.5,0.3)	(0.9,0,0.2)
Arthritis	(0,0.1,0.8)	(0.1,0.4,0.6)	(0.9,0.1,0.1)	(0.1,0.3,0.5)	(0.3,0.1,0.6)

TABLE 2. Neutrosophic values for diseases

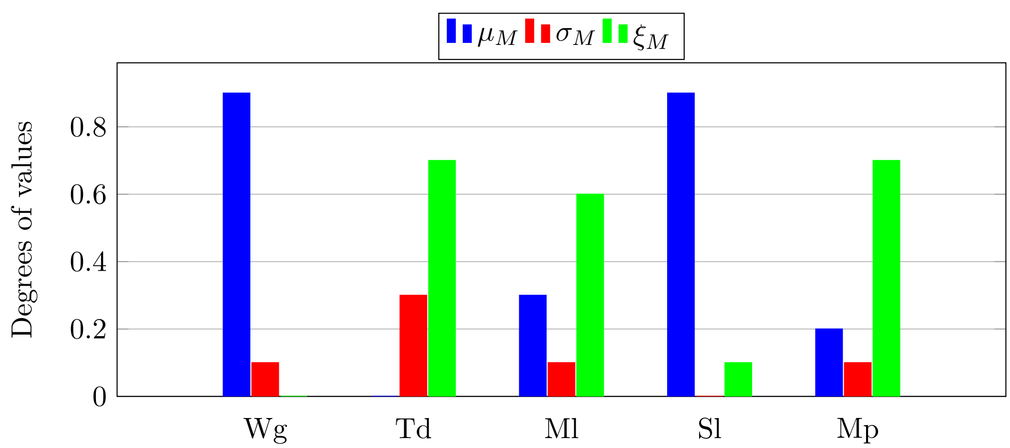

FIGURE 1. Neutrosophic values for Patient 1

Step 2: Form neutrosophic topologies for (Γ_j) and (ξ_k):

(i) $\Gamma_1^* = \Gamma \cup \Gamma^* \cup \Gamma^{**}$, where

$\Gamma = \{(0,0,1),(1,1,0),(0.9,0.1,0),(0,0.3,0.7),(0.3,0.1,0.6),(0.9,0,0.1),(0.2,0.1,0.7)\}$,

$\Gamma^* = \{(0.9,0.3,0),(0.3,0.3,0.6),(0.9,0.3,0.1),(0.2,0.3,0.7),(0.9,0.1,0.1)\}$ and

$\Gamma^{**} = \{(0,0.1,0.7),(0,0,0.7),(0.3,0,0.6),(0.2,0,\ 0.7)\}$.

(ii) $\Gamma_2^* = \Gamma \cup \Gamma^* \cup \Gamma^{**}$, where

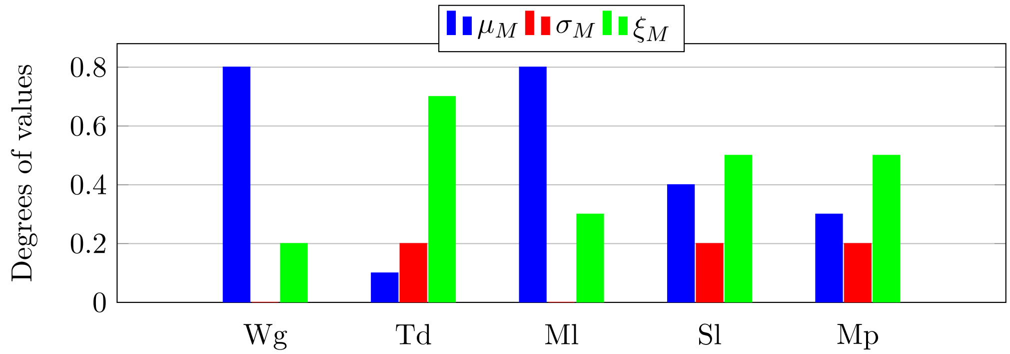

FIGURE 2. Neutrosophic values for Patient 2

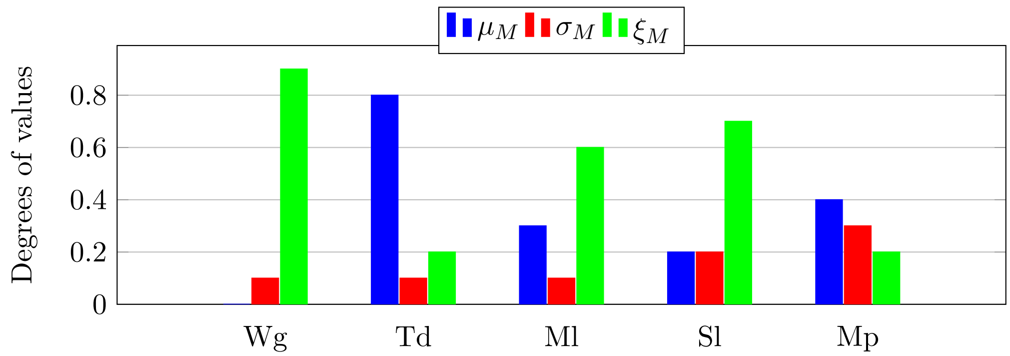

FIGURE 3. Neutrosophic values for Patient 3

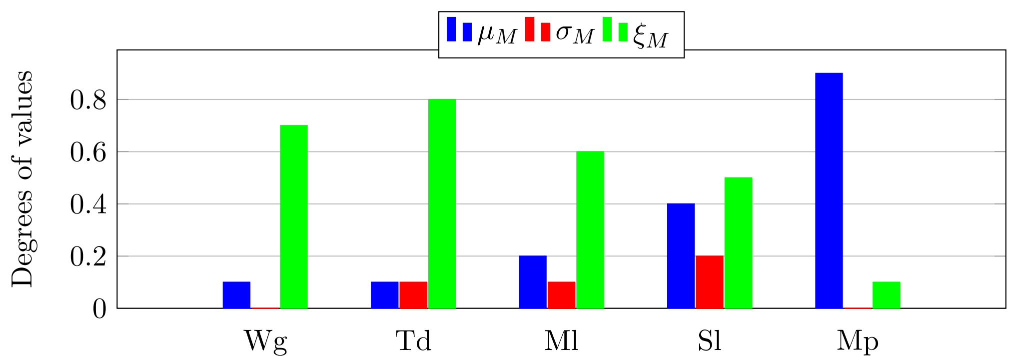

FIGURE 4. Neutrosophic values for Patient 4

$\Gamma = \{(0,0,1),(1,1,0),(0.8,0,0.2),(0.1,0.2,0.7),(0.8,0,0.2),(0.4,0.2,0.5),(0.3,0.2,0.5)\}$,

$\Gamma^* = \{(0.8,0.2,0.2),(0.8,0.2,0.3)\}$ and

$\Gamma^{**} = \{(0.1,0,0.7),(0.4,0,0.5),(0.3,0,0.5)\}$.

(iii) $\Gamma_3^* = \Gamma \cup \Gamma^* \cup \Gamma^{**}$, where

$\Gamma = \{(0,0,1),(1,1,0),(0,0.1,0.9),(0.8,0.1,0.2),(0.3,0.1,0.6),(0.2,0.2,0.7),(0.4,0.3,0.2)\}$,

$\Gamma^* = \{(0.8,0.2,0.2),(0.8,0.3,0.2),(0.3,0.2,0.6)\}$ and

$\Gamma^{**} = \{(0.2,0.1,0.7),(0.4,0.1,0.2)\}$.

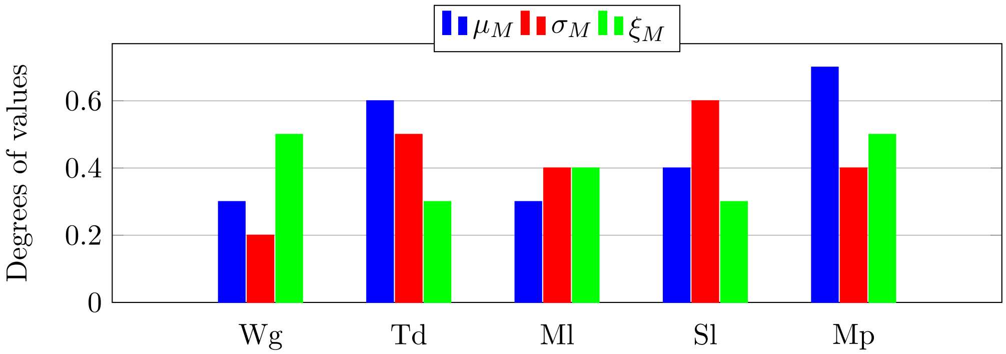

FIGURE 5. Neutrosophic values for Patient 5

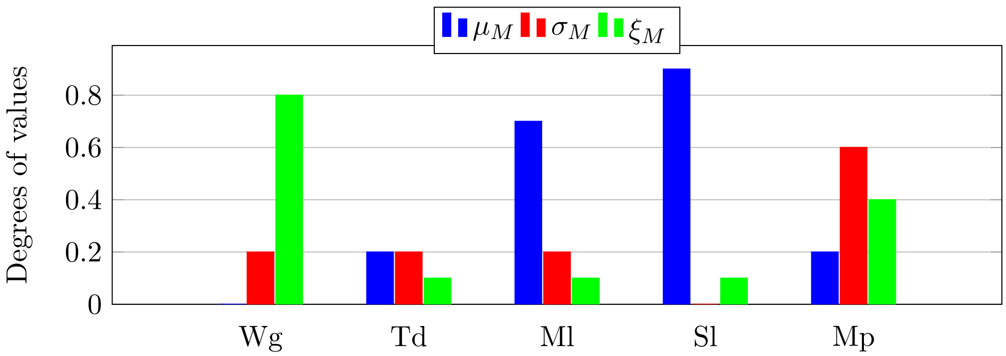

FIGURE 6. Neutrosophic values for Lymphedema

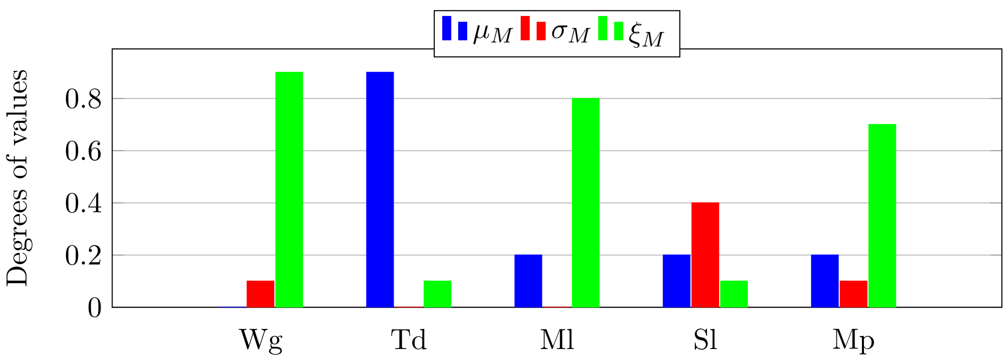

FIGURE 7. Neutrosophic values for Insomnia

(iv) $\Gamma_4^* = \Gamma \cup \Gamma^* \cup \Gamma^{**}$, where

$\Gamma = \{(0,0,1),(1,1,0),(0.1,0,0.7),(0.1,0.1,0.8),(0.2,0.1,0.6),(0.4,0.2,0.5),(0.9,0,0.1)\}$,

$\Gamma^* = \{(0.1,0.1,0.7),(0.9,0.1,0.1),(0.9,0.2,0.1)\}$ and

$\Gamma^{**} = \{(0.1,0,0.8),(0.2,0,0.6),(0.4,0,0.5)\}$.

(v) $\Gamma_5^* = \Gamma \cup \Gamma^* \cup \Gamma^{**}$, where

$\Gamma = \{(0,0,1),(1,1,0),(0.3,0.2,0.5),(0.6,0.5,0.3),(0.3,0.4,0.4),(0.4,0.6,0.3),(0.7,0.4,0.5)\}$,

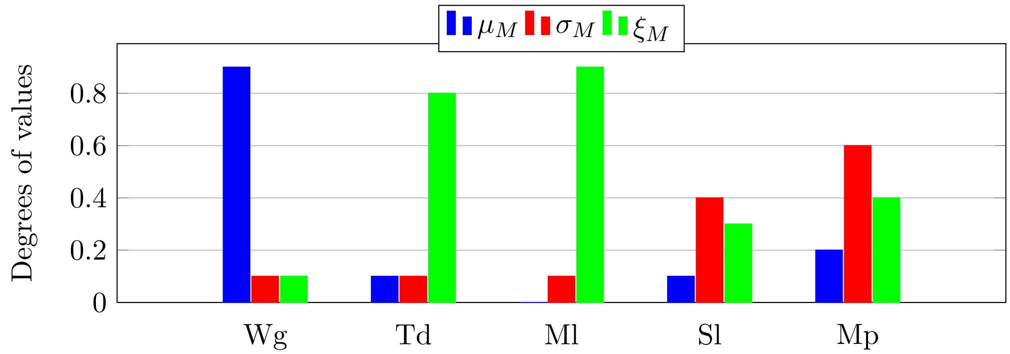

FIGURE 8. Neutrosophic values for Hypothyroidism

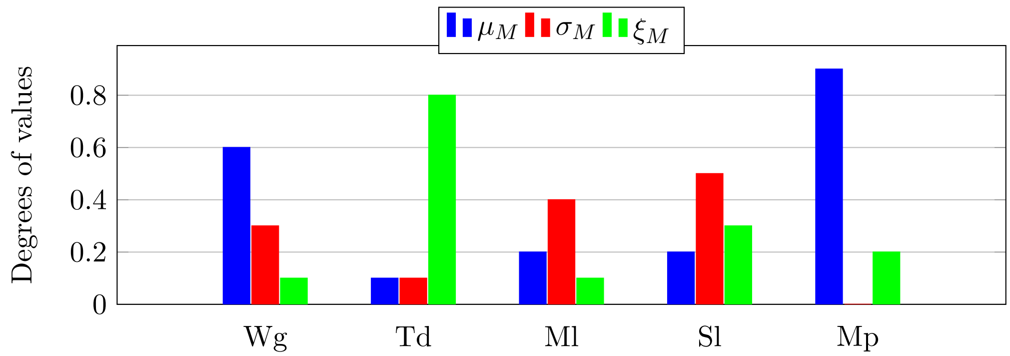

FIGURE 9. Neutrosophic values for Menarche

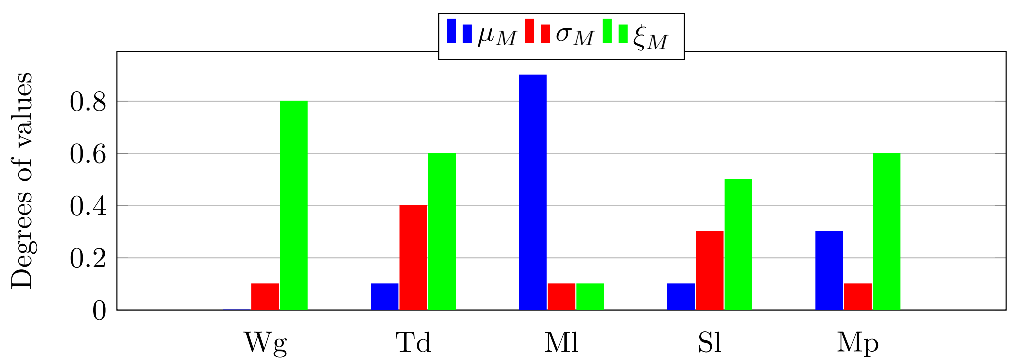

FIGURE 10. Neutrosophic values for Arthritis

$\Gamma^* = \{(0.6, 0.6, 0.3), (0.7, 0.5, 0.3), (0.7, 0.4, 0.4), (0.7, 0.6, 0.3)\}$ and

$\Gamma^{**} = \{(0.4, 0.5, 0.3), (0.6, 0.4, 0.5), (0.3, 0.4, 0.5), (0.4, 0.4, 0.5)\}$.

(i) $\xi_1^* = \xi \cup \xi^* \cup \xi^{**}$, where

$\xi = \{(0, 0, 1), (1, 1, 0), (0, 0.2, 0.8), (0.2, 0.2, 0.1), (0.7, 0.2, 0.1), (0.9, 0, 0.1), (0.2, 0.6, 0.4)\}$,

$\xi^* = \{(0.9, 0.2, 0.1), (0.2, 0.6, 0.1), (0.7, 0.6, 0.1), (0.9, 0.6, 0.1)\}$ and

$\xi^{**} = \{(0, 0, 0.8), (0.2, 0.2, 0.1), (0.2, 0, 0.1), (0.2, 0.2, 0.4), (0.7, 0, 0.1), (0.2, 0, 0.4)\}$.

(ii) $\xi_2^* = \xi \cup \xi^* \cup \xi^{**}$, where

$\xi = \{(0,0,1),(1,1,0),(0,0.1,0.9),(0.9,0,0.1),(0.2,0,0.8),(0.2,0.4,0.1),(0.2,0.1,0.7)\}$,

$\xi^* = \{(0.9,0.1,0.1),(0.2,0.1,0.8),(0.9,0.4,0.1)\}$ and

$\xi^{**} = \{(0,0,0.9),(0.2,0,0.1),(0.2,0,0.7)\}$.

(iii) $\xi_3^* = \xi \cup \xi^* \cup \xi^{**}$, where

$\xi = \{(0,0,1),(1,1,0),(0.9,0.1,0.1),(0.1,0.1,0.8),(0,0.1,0.9),(0.1,0.4,0.3),(0.2,0.6,0.4)\}$,

$\xi^* = \{(0.9,0.4,0.1),(0.9,0.6,0.1),(0.2,0.6,0.3)\}$ and

$\xi^{**} = \{(0.1,0.1,0.3),(0.2,0.1,0.4),(0,0.1,0.9),(0.1,0.4,0.4)\}$.

(iv) $\xi_4^* = \xi \cup \xi^* \cup \xi^{**}$, where

$\xi = \{(0,0,1),(1,1,0),(0.6,0.3,0.1),(0.1,0.1,0.8),(0.2,0.4,0.1),(0.2,0.5,0.3),(0.9,0,0.2)\}$,

$\xi^* = \{(0.6,0.4,0.1),(0.6,0.5,0.1),(0.9,0.3,0.1),(0.9,0.1,0.2),(0.2,0.5,0.1),(0.9,0.4,0.1),(0.9,0.5,0.2)\}$ and

$\xi^{**} = \{(0.2,0.3,0.1),(0.2,0.3,0.3),(0.6,0,0.2),(0.1,0,0.8),(0.2,0.4,0.3),(0.2,0.4,0.2)\}$.

(v) $\xi_5^* = \xi \cup \xi^* \cup \xi^{**}$, where

$\xi = \{(0,0,1),(1,1,0),(0,0.1,0.8),(0.1,0.4,0.6),(0.9,0.1,0.1),(0.1,0.3,0.5),(0.3,0.1,0.6)\}$,

$\xi^* = \{(0.9,0.4,0.1),(0.1,0.4,0.5),(0.3,0.4,0.6),(0.9,0.3,0.1),(0.3,0.3,0.5)\}$ and

$\xi^{**} = \{(0.1,0.1,0.6),(0.1,0.3,0.6),(0.1,0.1,0.5)\}$.

Step 3: Find neutrosophic score functions:

(i) $N_s euScFu(\Gamma) = 0.6$, $N_s euScFu(\Gamma^*) = 0.6933$ and $N_s euScFu(\Gamma^{**}) = 0.475$.

$N_s euScFu(\Gamma_1) = 0.5894$.

(ii) $N_s euScFu(\Gamma) = 0.6$, $N_s euScFu(\Gamma^*) = 0.7833$ and $N_s euScFu(\Gamma^{**}) = 0.5666$.

$N_s euScFu(\Gamma_2) = 0.6499$.

(iii) $N_s euScFu(\Gamma) = 0.5381$, $N_s euScFu(\Gamma^*) = 0.6888$ and $N_s euScFu(\Gamma^{**}) = 0.5833$.

$N_s euScFu(\Gamma_3) = 0.6034$.

(iv) $N_s euScFu(\Gamma) = 0.5524$, $N_s euScFu(\Gamma^*) = 0.7333$ and $N_s euScFu(\Gamma^{**}) = 0.5333$.

$N_s euScFu(\Gamma_4) = 0.6063$.

(v) $N_s euScFu(\Gamma) = 0.5533$, $N_s euScFu(\Gamma^*) = 0.6083$ and $N_s euScFu(\Gamma^{**}) = 0.5166$.

$N_s euScFu(\Gamma_5) = 0.5527$.

(i) $N_s euScFu(\xi) = 0.5857$, $N_s euScFu(\xi^*) = 0.6917$ and $N_s euScFu(\xi^{**}) = 0.5857$.

$N_s euScFu(\xi_1) = 0.6332$.

(ii) $N_s euScFu(\xi) = 0.5381$, $N_s euScFu(\xi^*) = 0.7111$ and $N_s euScFu(\xi^{**}) = 0.5222$.

$N_s euScFu(\xi_2) = 0.5905$.

(iii) $N_s euScFu(\xi) = 0.5$, $N_s euScFu(\xi^*) = 0.6555$ and $N_s euScFu(\xi^{**}) = 0.475$.

$N_s euScFu(\xi_3) = 0.5435$.

(iv) $N_s euScFu(\xi) = 0.5809$, $N_s euScFu(\xi^*) = 0.7666$ and $N_s euScFu(\xi^{**}) = 0.5888$.

$N_s euScFu(\xi_4) = 0.6454$.

(v) $N_s euScFu(\xi) = 0.5143$, $N_s euScFu(\xi^*) = 0.5933$ and $N_s euScFu(\xi^{**}) = 0.4555$.

$N_s euScFu(\xi_5) = 0.5210.$

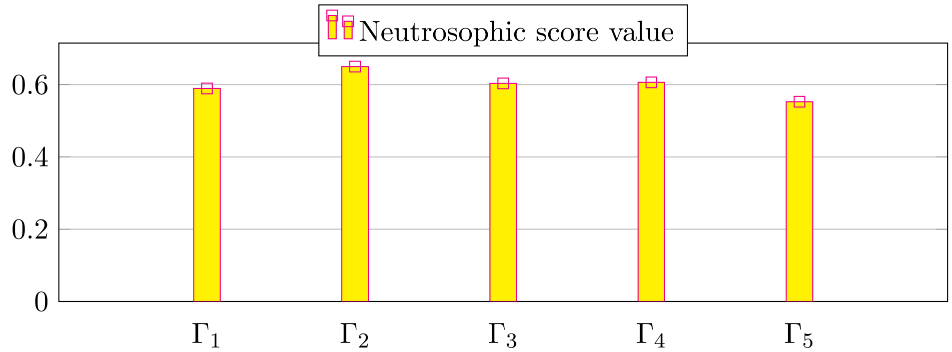

FIGURE 11. Neutrosophic score values for Patients

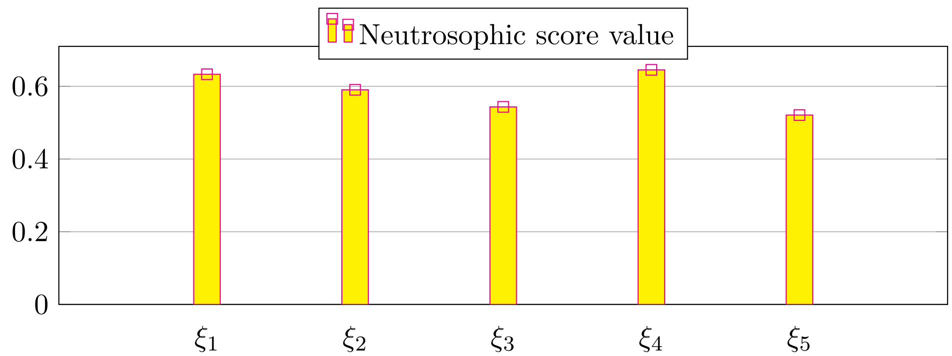

FIGURE 12. Neutrosophic score values for Diseases

Step 4: Final Decision:

Arrange neutrosophic score values for the alternatives Γ_1, Γ_2, Γ_3, Γ_4, Γ_5 and the attributes ξ_1, ξ_2, ξ_3, ξ_4, ξ_5 in ascending order. We get the following sequences $\Gamma_5 \leq \Gamma_1 \leq \Gamma_3 \leq \Gamma_4 \leq \Gamma_2$ and $\xi_5 \leq \xi_3 \leq \xi_2 \leq \xi_1 \leq \xi_4$. Thus the Pat_5 suffers from Menarche, the Pat_1 suffers from Lymphedema, the Pat_3 suffers from Insomnia, the Pat_4 suffers from Hypothyroidism and the Pat_2 suffers from Arthritis.

5. Conclusions

In this numerical example, we found out that the patients suffering from a diseases in the form of neutrosophic set by using neutrosophic score functions. This will help to find out the correct attributes and alternative in any field environment problems.

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A contemporary postulates on resolvable sets and functions

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Abstract. In this article, a new class of sets namely neutrosophic resolvable sets in neutrosophic topological space have been introduced. We present the neutrosophic resolvable functions between neutrosophic topological spaces by neutrosophic resolvables sets. Also we examine the characteristics of neutrosophic resolvable sets and neutrosophic resolvable functions with the existing sets.

Keywords: Neutrosophic resolvable sets, Neutrosophic continuous functions, Neutrosophic contra continuous functions, Neutrosophic resolvable functions.

1. Introduction

Zadeh presented the notion of fuzzy sets with membership functions in 1965 [15]. This concept is succesfully used to handle uncertainty in real life where each element has a membership function. In 1986, Attanassov proposed vague intuitionistic fuzzy sets which are characterised by the membership function and the non-member function. [2]. But in real world we have to handle the indeterminancy and incompleteness. For this purpose, Smarandache introduced the neutrosophic set theory to solve many practical problems in the real world [5]. Topology has been a vital part of research in recent years. In this context, Salama and Albowi succesfully applied the neutrosophic sets in the topological space called as neutrosophic topological space in 2012 [12].

The concept of resolvable sets in topological space was presented by Kuratowski in 1966 [6]. Maximilian Ganster gave the concept of pre-open sets and resolvable spaces in 1987 [7]. Resolvable spaces and irresolvable spaces were explored by Chandan Chakkopadhyay and Chhanda Bandyopadhyay in 1993 [4]. In 2017 neutrosophic resolvable, neutrosophic irresolvable, neutrosophic open hereditarily irresolvable spaces and maximally neutrosophic irresolvable spaces were studied by Caldas, Maximilian Ganster, Dhavaseelan, and Jafari through neutrosophic topological spaces [3]. In 2017, Thangaraj initiated to introduce the idea of resolvable sets and their functions in fuzzy topological spaces. Additionally, they discussed the conditions for fuzzy topological spaces to become a fuzzy Baire spaces obtained by fuzzy resolvable sets. [8]. Thangaraj and Lokeshwari studied irresolvable sets and open hereditarily irresolvable spaces in fuzzy topological spaces [9]. Also, they discussed resolvable sets and functions in fuzzy hyperconnected spaces [10]. In 2020, fuzzy resolvable functions were briefly inspected by Thangaraj and Senthil. [11]. These studies motivated us to ideate a new notion of sets namely neutrosophic resolvable sets in neutrosophic topological spaces. Also we define the new functions namely neutrosophic resolvable functions between neutrosophic topological spaces $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$ and $\mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$. Finally the properties of neutrosophic resolvable sets and functions are discussed by theorems and suitable examples.

Through out this paper, neutrosophic topological space [simply nts] is denoted by $\mathcal{N}(\mathcal{X}, \pi)$.

2. Preliminaries

Definition 2.1. [1] Let $\mathcal{X}$ be a nonempty fixed set. A neutrosophic set $\mathcal{P}$ is an object having the form $\mathcal{P} = \{< r, \mu_{\mathcal{P}}(r), \nu_{\mathcal{P}}(r), \omega_{\mathcal{P}}(r) >; r \in \mathcal{X}\}$ where $\mu_{\mathcal{P}}(r)$, $\nu_{\mathcal{P}}(r)$ and $\omega_{\mathcal{P}}(r)$ represent the degree of true, indeterminancy and false membership functions respectively of each element $r \in \mathcal{X}$ to the set $\mathcal{P}$.

Definition 2.2. [1] Let $\mathcal{P} = \{< r, \mu_{\mathcal{P}}(r), \nu_{\mathcal{P}}(r), \omega_{\mathcal{P}}(r) >, r \in \mathcal{X}\}$ be a neutrosophic sets, then the complement of $\mathcal{P}$ can be defined as by the following three kinds,
$C_1 : C[\mathcal{P}] = \{< r, 1_{\mathcal{N}} - \mu_{\mathcal{P}}(r), 1_{\mathcal{N}} - \nu_{\mathcal{P}}(r), 1_{\mathcal{N}} - \omega_{\mathcal{P}}(r) >; r \in \mathcal{X}\}$
$C_2 : C[\mathcal{P}] = \{< r, \omega_{\mathcal{P}}(r), \nu_{\mathcal{P}}(r), \mu_{\mathcal{P}}(r) >; r \in \mathcal{X}\}$
$C_3 : C[\mathcal{P}] = \{< r, \omega_{\mathcal{P}}(r), 1_{\mathcal{N}} - \nu_{\mathcal{P}}(r), \mu_{\mathcal{P}}(r) >; r \in \mathcal{X}\}$
Throughout this paper, the examples are derived using C_1.

proposition 2.3. *[1] For any neutrosophic set $\mathcal{P}$ in $\mathcal{N}(\mathcal{X}, \pi)$, the following conditions are hold*

(i) $0_{\mathcal{N}} \leq 0_{\mathcal{N}}, 0_{\mathcal{N}} \leq \mathcal{P}$
(ii) $1_{\mathcal{N}} \leq 1_{\mathcal{N}}, \mathcal{P} \leq 1_{\mathcal{N}}$

Definition 2.4. [12] Let $\mathcal{X}$ be a non empty set and the neutrosophic sets $\mathcal{P}$ and $\mathcal{Q}$ in the form $\mathcal{P} = \{< r, \mu_{\mathcal{P}}(r), \nu_{\mathcal{P}}(r), \omega_{\mathcal{P}}(r) >, r \in \mathcal{X}\}$ and $\mathcal{Q} = \{< r, \mu_{\mathcal{Q}}(r), \nu_{\mathcal{Q}}(r), \omega_{\mathcal{Q}}(r) >; r \in \mathcal{X}\}$, then $\mathcal{P}$ is the subset of $\mathcal{Q}$ i.e., $\mathcal{P} \subseteq \mathcal{Q}$ is defined by the following two ways:

(i) $\mathcal{P} \leq \mathcal{Q} \Leftrightarrow \mu_{\mathcal{P}}(r) \leq \mu_{\mathcal{Q}}(r), \nu_{\mathcal{P}}(r) \leq \nu_{\mathcal{Q}}(r), \omega_{\mathcal{P}}(r) \geq \omega_{\mathcal{Q}}(r); \forall r \in \mathcal{X}$
(ii) $\mathcal{P} \leq \mathcal{Q} \Leftrightarrow \mu_{\mathcal{P}}(r) \leq \mu_{\mathcal{Q}}(r), \nu_{\mathcal{P}}(r) \geq \nu_{\mathcal{Q}}(r), \omega_{\mathcal{P}}(r) \geq \omega_{\mathcal{Q}}(r); \forall r \in \mathcal{X}$

Definition 2.5. [12] Let $\mathcal{X}$ be a non-empty set and π be a collection of all neutrosophic subsets of $\mathcal{X}$. Then π is said to be neutrosophic topology on $\mathcal{X}$ if the following conditions are hold.

(i) $0_{\mathcal{N}}$, $1_{\mathcal{N}} \in \pi$

(ii) $\bigcup \mathcal{P}_i \in \pi, \forall \{\mathcal{P}_i; i \in \pi\} \leq \pi$

(iii) $\mathcal{P}_1 \cap \mathcal{P}_2 \in \pi$, for any $\mathcal{P}_1$, $\mathcal{P}_2 \in \pi$

Then the pair $\mathcal{N}(\mathcal{X}, \pi)$ is called as neutrosophic topological space. The elements of $\mathcal{N}(\mathcal{X}, \pi)$ are called neutrosophic open sets. A neutrosophic set is said to be neutrosophic closed if its complement is neutrosophic open.

Definition 2.6. [1] Let $\mathcal{X}$ be a *nts* and $\mathcal{P} = \{< r, \mu_{\mathcal{P}}(r), \nu_{\mathcal{P}}(r), \omega_{\mathcal{P}}(r) >; r \in \mathcal{X}\}$ be a neutrosophic set in $\mathcal{X}$. Then the neutrosophic interior and neutrosophic closure of $\mathcal{P}$ are defined as,

$$\mathcal{N}int[\mathcal{P}] = \bigcup\{H : H \text{ is a neutrosophic open set in } \mathcal{X} \text{ and } H \leq \mathcal{P}\}$$

$$\mathcal{N}cl[\mathcal{P}] = \bigcap\{K : K \text{ is a neutrosophic closed set in } \mathcal{X} \text{ and } \mathcal{P} \leq K\}$$

It follows that,

(i) $\mathcal{P}$ is a neutrosophic closed set if and only if $\mathcal{N}cl[\mathcal{P}] = \mathcal{P}$

(ii) $\mathcal{P}$ is a neutrosophic open set if and only if $\mathcal{N}int[\mathcal{P}] = \mathcal{P}$

Definition 2.7. [1] A neutrosophic subset $\mathcal{P}$ in a *nts* $\mathcal{N}(\mathcal{X}, \pi)$ is called as

(i) neutrosophic semi open set if $\mathcal{P} \leq \mathcal{N}cl[\mathcal{N}int[\mathcal{P}]]$

(ii) neutrosophic pre open set if $\mathcal{P} \leq \mathcal{N}int[\mathcal{N}cl[\mathcal{P}]]$

(iii) neutrosophic semi pre open set if $\mathcal{P} \leq \mathcal{N}cl[\mathcal{N}int[\mathcal{N}cl[\mathcal{P}]]]$

(iv) neutrosophic regular open set if $\mathcal{P} = \mathcal{N}int\mathcal{N}cl[\mathcal{P}]$

Definition 2.8. [5] A neutrosophic subset $\mathcal{P}$ in a *nts* $\mathcal{N}(\mathcal{X}, \pi)$ is called

(i) neutrosophic dense if $\mathcal{N}cl[\mathcal{P}] = 1_{\mathcal{N}}$

(ii) neutrosophic nowhere dense $\mathcal{N}int\mathcal{N}cl[\mathcal{P}] = 0_{\mathcal{N}}$

(iii) neutrosophic somewhere dense if $\mathcal{N}int[\mathcal{N}cl[\mathcal{P}]] \neq 0_{\mathcal{N}}$

Definition 2.9. [5] Let $\mathcal{N}(\mathcal{X}, \pi)$ be a *nts* and $\mathcal{P}_1$, $\mathcal{P}_2$ are neutrosophic subsets in $\mathcal{X}$, then the following conditions are hold.

(i) $\mathcal{N}int[\mathcal{P}_1] \leq \mathcal{P}_1$

(ii) $\mathcal{N}cl[\mathcal{P}_1] \geq \mathcal{P}_1$

(iii) $\mathcal{P}_1 \leq \mathcal{P}_2 \implies \mathcal{N}int[\mathcal{P}_1] \leq \mathcal{N}int[\mathcal{P}_2]$

(iv) $\mathcal{P}_1 \leq \mathcal{P}_2 \implies \mathcal{N}cl[\mathcal{P}_1] \leq \mathcal{N}cl[\mathcal{P}_2]$

(v) $\mathcal{N}int[\mathcal{N}int[\mathcal{P}_1]] = \mathcal{N}int[\mathcal{P}_1]$

(vi) $\mathcal{N}cl[\mathcal{N}cl[\mathcal{P}_1]] = \mathcal{N}cl[\mathcal{P}_1]$

(vii) $\mathcal{N}int[\mathcal{P}_1 \wedge \mathcal{P}_2] = \mathcal{N}int[\mathcal{P}_1] \wedge \mathcal{N}int[\mathcal{P}_2]$

(viii) $\mathcal{N}cl[\mathcal{P}_1 \vee \mathcal{P}_2] = \mathcal{N}cl[\mathcal{P}_1] \vee \mathcal{N}cl[\mathcal{P}_2]$

(ix) $\mathcal{N}int[0_\mathcal{N}] = 0_\mathcal{N}$

(x) $\mathcal{N}int[1_\mathcal{N}] = 1_\mathcal{N}$

(xi) $\mathcal{N}cl[0_\mathcal{N}] = 0_\mathcal{N}$

(xii) $\mathcal{N}cl[1_\mathcal{N}] = 1_\mathcal{N}$

(xiii) $\mathcal{P}_1 \leq \mathcal{P}_2 \implies C[\mathcal{P}_2] \leq C[\mathcal{P}_1]$

(xiv) $\mathcal{N}cl[\mathcal{P}_1 \wedge \mathcal{P}_2] \leq \mathcal{N}cl[\mathcal{P}_1] \wedge \mathcal{N}cl[\mathcal{P}_2]$

(xv) $\mathcal{N}int[\mathcal{P}_1 \wedge \mathcal{P}_2] \geq \mathcal{N}int[\mathcal{P}_1] \wedge \mathcal{N}int[\mathcal{P}_2]$

Definition 2.10. [14] Let $\mathcal{N}(\mathcal{X}, \pi_\mathcal{X})$ and $\mathcal{N}(\mathcal{Y}, \pi_\mathcal{Y})$ be any two neutrosophic topological space. Then the function $f : [\mathcal{X}, \pi_\mathcal{X}] \to [\mathcal{Y}, \pi_\mathcal{Y}]$ is called a

(i) Neutrosophic continuous function if $f^{-1}[\mathcal{P}]$ is neutrosophic open in $\mathcal{N}(\mathcal{X}, \pi_\mathcal{X})$, for each neutrosophic open set $\mathcal{P}$ in $\mathcal{N}(\mathcal{Y}, \pi_\mathcal{Y})$

(ii) Neutrosophic contra continuous function if $f^{-1}[\mathcal{P}]$ is neutrosophic closed in $\mathcal{N}(\mathcal{X}, \pi_\mathcal{X})$, for each neutrosophic open set $\mathcal{P}$ in $\mathcal{N}(\mathcal{Y}, \pi_\mathcal{Y})$.

3. **Neutrosophic resolvable sets**

Definition 3.1. A neutrosophic set $\mathcal{P}$ is said to be neutrosophic resolvable set in neutrosophic topological space $\mathcal{N}(\mathcal{X}, \pi)$, if $\{\mathcal{N}cl[\mathcal{Q} \wedge \mathcal{P}] \wedge \mathcal{N}cl[\mathcal{Q} \wedge C[\mathcal{P}]]\}$ is neutrosophic nowhere dense in $\mathcal{N}(\mathcal{X}, \pi)$ for each neutrosophic closed set $\mathcal{Q}$ in $\mathcal{N}(\mathcal{X}, \pi)$. i.e.,$\mathcal{N}int\mathcal{N}cl\{\mathcal{N}cl[\mathcal{Q} \wedge \mathcal{P}] \wedge \mathcal{N}cl[\mathcal{Q} \wedge C[\mathcal{P}]]\} = 0_\mathcal{N}$, where $C[\mathcal{Q}] \in \pi$.

Example 3.2. Consider $\mathcal{X} = \{\mu, \nu\}$ and the neutrosophic sets $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$ in $\mathcal{X}$ as follows

$\mathcal{P}_1$=$\{< \mu, 0.4, 0.3, 0.6 >, < \nu, 0.5, 0.3, 0.2 >; \mu, \nu \in \mathcal{X}\}$

$\mathcal{P}_2$=$\{< \mu, 0.3, 0.4, 0.3 >, < \nu, 0.6, 0.5, 0.2 >; \mu, \nu \in \mathcal{X}\}$

$\mathcal{P}_3$=$\{< \mu, 0.4, 0.4, 0.3 >, < \nu, 0.6, 0.5, 0.2 >; \mu, \nu \in \mathcal{X}\}$

$\mathcal{P}_4$=$\{< \mu, 0.3, 0.3, 0.6 >, < \nu, 0.5, 0.3, 0.2 >; \mu, \nu \in \mathcal{X}\}$ Then $\pi = \{0_\mathcal{N}, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4, 1_\mathcal{N}\}$ is a nts. Now, $\pi^C = \{0_\mathcal{N}, C[\mathcal{P}_1], C[\mathcal{P}_2], C[\mathcal{P}_3], C[\mathcal{P}_4], 1_\mathcal{N}\}$

Let A=$\{< \mu, 0.2, 0.1, 0.7 >, < \nu, 0.3, 0.1, 0.3 >; \mu, \nu \in \mathcal{X}\}$. Now,

$\mathcal{N}int\mathcal{N}cl\{\mathcal{N}cl[C[\mathcal{P}_1] \wedge A] \wedge \mathcal{N}cl[C[\mathcal{P}_1] \wedge C[A]]\} = 0_\mathcal{N}$

$\mathcal{N}int\mathcal{N}cl\{\mathcal{N}cl[C[\mathcal{P}_2] \wedge A] \wedge \mathcal{N}cl[C[\mathcal{P}_2] \wedge C[A]]\} = 0_\mathcal{N}$

$\mathcal{N}int\mathcal{N}cl\{\mathcal{N}cl[C[\mathcal{P}_3] \wedge A] \wedge \mathcal{N}cl[C[\mathcal{P}_3] \wedge C[A]]\} = 0_\mathcal{N}$

$\mathcal{N}int\mathcal{N}cl\{\mathcal{N}cl[C[\mathcal{P}_4] \wedge A] \wedge \mathcal{N}cl[C[\mathcal{P}_4] \wedge C[A]]\} = 0_\mathcal{N}$.

Therefore A is a neutrosophic resolvable set in $\mathcal{N}(\mathcal{X}, \pi)$.

Example 3.3. Consider $\mathcal{X} = \{\mu\}$ and consider the neutrosophic sets $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$ and R in $\mathcal{X}$ as follows:
$\mathcal{Q}_1 = \{< \mu, 0.4, 0.5, 0.6 >, \mu \in \mathcal{X}\}$
$\mathcal{Q}_2 = \{< \mu, 0.3, 0.4, 0.8 >, \mu \in \mathcal{X}\}$
$\mathcal{Q}_3 = \{< \mu, 0.4, 0.5, 0.8 >, \mu \in \mathcal{X}\}$
Then $[\mathcal{X}, \pi] = \{0_{\mathcal{N}}, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, 1_{\mathcal{N}}\}$ is a nts. Here $\pi^C = \{0_{\mathcal{N}}, C[\mathcal{Q}_1], C[\mathcal{Q}_2], C[\mathcal{Q}_3], 1_{\mathcal{N}}\}$. Take $R = \{< \mu, 0.6, 0.6, 0.4 >, \mu \in x\}$. Now,
$\mathcal{N}int\mathcal{N}cl[\mathcal{N}cl[C[\mathcal{Q}_1] \wedge R] \wedge \mathcal{N}cl[C[\mathcal{Q}_1] \wedge C[R]]] = \mathcal{Q}_1 \neq 0_{\mathcal{N}}$
$\mathcal{N}int\mathcal{N}cl[\mathcal{N}cl[C[\mathcal{Q}_2] \wedge R] \wedge \mathcal{N}cl[C[\mathcal{Q}_2] \wedge C[R]]] = \mathcal{Q}_1 \neq 0_{\mathcal{N}}$
$\mathcal{N}int\mathcal{N}cl[\mathcal{N}cl[C[\mathcal{Q}_3] \wedge R] \wedge \mathcal{N}cl[C[\mathcal{Q}_3] \wedge C[R]]] = 0_{\mathcal{N}}$
This implies R is not a neutrosophic resolvable set in $\mathcal{N}(\mathcal{X}, \pi)$.

Remark 3.4. In a nts $\mathcal{N}(\mathcal{X}, \pi)$, every neutrosophic resolvable set need not be a neutrosophic open set. In example 3.2, A is a neutrosophic resolvable set but not neutrosophic open set.

proposition 3.5. *In a nts $\mathcal{N}(\mathcal{X}, \pi)$, if $\mathcal{P}$ is a neutrosophic resolvable set, then $\mathcal{N}int\mathcal{N}cl[\mathcal{P} \wedge C[\mathcal{P}] \wedge \mathcal{Q}] = 0_{\mathcal{N}}$ for each neutrosophic closed set $\mathcal{Q}$ in $\mathcal{N}(\mathcal{X}, \pi)$.*

Proof. Let $\mathcal{P}$ be a neutrosophic resolvable set in $\mathcal{N}(\mathcal{X}, \pi)$, then for each neutrosophic closed set $\mathcal{Q}$, we have $\mathcal{N}int\mathcal{N}cl\{\mathcal{N}cl[\mathcal{Q} \wedge \mathcal{P}] \wedge \mathcal{N}cl[\mathcal{Q} \wedge C[\mathcal{P}]]]\} = 0_{\mathcal{N}}$, Using 3.1, $\mathcal{N}cl[\mathcal{Q} \wedge \mathcal{P}] \wedge \mathcal{N}cl[\mathcal{Q} \wedge C[\mathcal{P}]] \geq \mathcal{N}cl[[\mathcal{Q} \wedge \mathcal{P}] \wedge [\mathcal{Q} \wedge C[\mathcal{P}]] \geq \mathcal{N}cl[\mathcal{Q} \wedge \mathcal{P} \wedge C[\mathcal{P}]]$. Now, $\mathcal{N}int\mathcal{N}cl\{\mathcal{N}cl[\mathcal{Q} \wedge \mathcal{P}] \wedge \mathcal{N}cl[\mathcal{Q} \wedge C[\mathcal{P}]]\} \geq \mathcal{N}int\mathcal{N}cl[\mathcal{Q} \wedge \mathcal{P} \wedge C[\mathcal{P}]]$. This implies $0_{\mathcal{N}} \geq \mathcal{N}int\mathcal{N}cl[\mathcal{Q} \wedge \mathcal{P} \wedge C[\mathcal{P}]]$. Hence $\mathcal{N}int\mathcal{N}cl[\mathcal{Q} \wedge \mathcal{P} \wedge C[\mathcal{P}]] = 0_{\mathcal{N}}$ in $\mathcal{N}(\mathcal{X}, \pi)$. □

proposition 3.6. *In a nts $\mathcal{N}(\mathcal{X}, \pi)$, if $\mathcal{P}$ is a neutrosophic resolvable set in $\mathcal{N}(\mathcal{X}, \pi)$, then $\mathcal{N}cl[\mathcal{P} \vee C[\mathcal{P}] \vee R] = 1_{\mathcal{N}}$ for each neutrosophic open set R in $\mathcal{N}(\mathcal{X}, \pi)$.*

Proof. Let $\mathcal{P}$ be a neutrosophic resolvable set in $\mathcal{N}(\mathcal{X}, \pi)$ using proposition 3.5, we have $\mathcal{N}int\mathcal{N}cl[\mathcal{P} \wedge C[\mathcal{P}] \wedge \mathcal{Q}] = 0_{\mathcal{N}}$ for each neutrosophic closed set $\mathcal{Q}$ in $\mathcal{N}(\mathcal{X}, \pi)$. Then $C[\mathcal{N}int\mathcal{N}cl[\mathcal{P} \wedge C[\mathcal{P}] \wedge \mathcal{Q}]] = 1_{\mathcal{N}}$. We know that $\mathcal{N}cl\mathcal{N}int[C[\mathcal{P}] \vee \mathcal{P} \vee C[\mathcal{Q}]] \leq \mathcal{N}cl[C[\mathcal{P}] \vee \mathcal{P} \vee C[\mathcal{Q}]]$. This implies $1_{\mathcal{N}} \leq \mathcal{N}cl[C[\mathcal{P}]] \vee [\mathcal{P}] \vee C[\mathcal{Q}]$ and put $C[\mathcal{Q}] = R$. Then we have $\mathcal{N}cl[\mathcal{P} \vee C[\mathcal{P}] \vee R] = 1_{\mathcal{N}}$ for each neutrosophic open set R in $\mathcal{N}(\mathcal{X}, \pi)$. □

proposition 3.7. *Let $\mathcal{N}(\mathcal{X}, \pi)$ be a nts, then the neutrosophic interior of a neutrosophic closed set is neutrosophic regular open.*

Proof. Let $\mathcal{Q}$ be a neutrosophic closed set in $\mathcal{N}(\mathcal{X}, \pi)$ and take $R = \mathcal{N}int\mathcal{Q}$. Therefore $R \leq \mathcal{Q}$. Since $\mathcal{Q}$ is closed set in $\mathcal{N}(\mathcal{X}, \pi)$. $\mathcal{N}cl[R] \leq \mathcal{N}cl[\mathcal{Q}] = \mathcal{Q}$. This implies $\mathcal{N}int\mathcal{N}cl[R] \leq$

$\mathcal{N}int[\mathcal{Q}] = R$. We have $R \leq \mathcal{N}cl[R]$. Now $\mathcal{N}int[R] \leq \mathcal{N}int\mathcal{N}cl[R]$. This implies $R \leq \mathcal{N}int\mathcal{N}cl[R]$ gives $\mathcal{N}int\mathcal{N}cl[R] = R$. Therefore R is a neutrosophic regular open set. □

proposition 3.8. *In a nts$\mathcal{N}(\mathcal{X},\pi)$, if $\mathcal{P}$ is a neutrosophic resolvable set, then there exists a neutrosophic regular open set R in $\mathcal{N}(\mathcal{X},\pi)$ such that $R \leq \mathcal{N}cl[\mathcal{P} \vee C[\mathcal{P}]]$.*

Proof. Let $\mathcal{P}$ be a neutrosophic resolvable set in $\mathcal{N}(\mathcal{X},\pi)$. Using proposition 3.5, we have $\mathcal{N}int\mathcal{N}cl[\mathcal{P} \wedge C[\mathcal{P}] \wedge \mathcal{Q}] = 0_{\mathcal{N}}$ for each closed set $\mathcal{Q}$ in $\mathcal{N}(\mathcal{X},\pi)$.$\mathcal{N}int[\mathcal{P} \wedge C[\mathcal{P}] \wedge \mathcal{Q}] \leq \mathcal{N}int\mathcal{N}cl[\mathcal{P} \wedge C[\mathcal{P}] \wedge \mathcal{Q}] = 0_{\mathcal{N}} \implies \mathcal{N}int[\mathcal{P} \wedge C[\mathcal{P}] \wedge \mathcal{Q}] = 0_{\mathcal{N}}$ in $\mathcal{N}(\mathcal{X},\pi) \implies \mathcal{N}int[\mathcal{P} \wedge C[\mathcal{P}]] \wedge \mathcal{N}int[\mathcal{Q}] = 0_{\mathcal{N}} \implies \mathcal{N}int\mathcal{Q} \leq C[\mathcal{N}int[\mathcal{P} \wedge C[\mathcal{P}]]] = \mathcal{N}cl[C[\mathcal{P}] \vee \mathcal{P}]$ in $\mathcal{N}(\mathcal{X},\pi)$. Since $\mathcal{Q}$ is a neutrosophic closed set. Using Proposition 3.7 , $\mathcal{N}int\mathcal{Q}$ is a neutrosophic regular open in $\mathcal{N}(\mathcal{X},\pi)$. Put $\mathcal{N}int\mathcal{Q} = R$. Hence if $\mathcal{P}$ is a neutrosophic resolvable set in $\mathcal{N}(\mathcal{X},\pi)$, there exist a neutrosophic regular open set R in $\mathcal{N}(\mathcal{X},\pi)$ such that $R \leq \mathcal{N}cl[\mathcal{P} \vee C[\mathcal{P}]]$. □

proposition 3.9. *In a nts$\mathcal{N}(\mathcal{X},\pi)$ if $\mathcal{P}$ is neutrosophic resolvable set in $\mathcal{N}(\mathcal{X},\pi)$, then $C[\mathcal{P}]$ is also a neutrosophic resolvable set in $\mathcal{N}(\mathcal{X},\pi)$.*

Proof. Let $\mathcal{P}$ be a neutrosophic resolvable set in $\mathcal{N}(\mathcal{X},\pi)$ then $\mathcal{N}int\mathcal{N}cl\{[\mathcal{N}cl[\mathcal{Q} \wedge \mathcal{P}] \wedge \mathcal{N}cl[\mathcal{Q} \wedge C[\mathcal{P}]]]\} = 0_{\mathcal{N}}$ for each neutrosophic closed set $\mathcal{Q}$ in $\mathcal{N}(\mathcal{X},\pi)$. For the set $C[\mathcal{P}]$, $\mathcal{N}int\mathcal{N}cl\{[\mathcal{N}cl[\mathcal{Q} \wedge C[\mathcal{P}]] \wedge \mathcal{N}cl[\mathcal{Q} \wedge C[C[\mathcal{P}]]]]\} = 0_{\mathcal{N}} \implies \mathcal{N}int\mathcal{N}cl[\mathcal{N}cl[\mathcal{Q} \wedge C[\mathcal{P}]] \wedge \mathcal{N}cl[\mathcal{Q} \wedge \mathcal{P}]] = 0_{\mathcal{N}}$. Hence $C[\mathcal{P}]$ is also a neutrosophic resolvable set in $\mathcal{N}(\mathcal{X},\pi)$. □

proposition 3.10. *In a nts $\mathcal{N}(\mathcal{X},\pi)$, if $\mathcal{P}$ is a neutrosophic closed set with $\mathcal{N}int[\mathcal{P}] = 0_{\mathcal{N}}$, then $\mathcal{P}$ is a neutrosophic resolvable set in $\mathcal{N}(\mathcal{X},\pi)$.*

Proof. Let $\mathcal{P}$ be a neutrosophic closed set and $\mathcal{N}int[\mathcal{P}] = 0_{\mathcal{N}}$ in $\mathcal{N}(\mathcal{X},\pi)$. For a neutrosophic closed set $\mathcal{Q}$ in $\mathcal{N}(\mathcal{X},\pi)$, we have $\mathcal{N}cl[\mathcal{Q} \wedge \mathcal{P}] \wedge \mathcal{N}cl[\mathcal{Q} \wedge C[\mathcal{P}]] \leq \mathcal{N}cl[\mathcal{P}] \wedge \mathcal{N}cl[C[\mathcal{P}]] \leq \mathcal{P}$. Since $\mathcal{P}$ is a neutrosophic closed set in $\mathcal{N}(\mathcal{X},\pi)$. Thus $\mathcal{N}int\mathcal{N}cl[\mathcal{N}cl[\mathcal{Q} \wedge \mathcal{P}] \wedge \mathcal{N}cl[\mathcal{Q} \wedge C[\mathcal{P}]]] \leq \mathcal{N}int\mathcal{N}cl[\mathcal{P}] = \mathcal{N}int[\mathcal{P}] = 0_{\mathcal{N}}$. Hence $\mathcal{P}$ is a resolvable set in $\mathcal{N}(\mathcal{X},\pi)$. □

proposition 3.11. *Let $\mathcal{N}(\mathcal{X},\pi)$ be the nts. If $\mathcal{P}$ is a neutrosophic open set and neutrosophic dense in $\mathcal{N}(\mathcal{X},\pi)$, then $\mathcal{P}$ is a neutrosophic resolvable set in $\mathcal{N}(\mathcal{X},\pi)$.*

Proof. Let $\mathcal{P}$ be a neutrosophic open set and neutrosophic dense in $\mathcal{N}(\mathcal{X},\pi)$. Then $C[\mathcal{P}]$ is neutrosophic closed and $\mathcal{N}cl[\mathcal{P}] = 1_{\mathcal{N}}$. For a neutrosophic closed set $\mathcal{Q}$, we have $\mathcal{N}cl[\mathcal{Q} \wedge \mathcal{P}] \wedge \mathcal{N}cl[\mathcal{Q} \wedge C[\mathcal{P}]] \leq \mathcal{N}cl[\mathcal{Q}] \wedge \mathcal{N}cl[\mathcal{P}] \wedge \mathcal{N}cl[\mathcal{Q}] \wedge \mathcal{N}cl[C[\mathcal{P}]] \implies \mathcal{Q} \wedge C[\mathcal{P}]$. Since $\mathcal{N}cl[\mathcal{P}] = 1_{\mathcal{N}}$, $\mathcal{Q}$ and $C[\mathcal{P}]$ are neutrosophic closed set in $\mathcal{N}(\mathcal{X},\pi)$. This implies $\mathcal{N}int\mathcal{N}cl[\mathcal{N}cl[\mathcal{Q} \wedge \mathcal{P}] \wedge$

$\mathcal{N}cl[\mathcal{Q} \wedge C[\mathcal{P}]]] \leq \mathcal{N}int\mathcal{N}cl[\mathcal{Q} \wedge C[\mathcal{P}]] \leq \mathcal{N}int[\mathcal{N}cl[\mathcal{Q}] \wedge \mathcal{N}cl[C[\mathcal{P}]]]$. By computation, we have $\mathcal{N}int\mathcal{N}cl[\mathcal{N}cl[\mathcal{Q} \wedge \mathcal{P}] \wedge \mathcal{N}cl[\mathcal{Q} \wedge C[\mathcal{P}]]] = 0_{\mathcal{N}}$. Hence $\mathcal{P}$ is a neutrosophic resolvable set in $\mathcal{N}(\mathcal{X}, \pi)$. □

proposition 3.12. *If $\mathcal{P}$ is a neutrosophic open and neutrosophic dense in a nts $\mathcal{N}(\mathcal{X}, \pi)$, then $C[\mathcal{P}]$ is neutrosophic resolvable set in $\mathcal{N}(\mathcal{X}, \pi)$.*

Proof. Let $\mathcal{P}$ be a neutrosophic open set and neutrosophic dense set in $\mathcal{N}(\mathcal{X}, \pi)$. This implies $C[\mathcal{P}]$ is a neutrosophic closed set and $\mathcal{N}cl[\mathcal{P}] = 1_{\mathcal{N}}$. Then $C[\mathcal{N}cl[\mathcal{P}]] = C[1_{\mathcal{N}}] \implies \mathcal{N}int[C[\mathcal{P}]] = 0_{\mathcal{N}}$ Using Proposition 3.10, $C[\mathcal{P}]$ is neutrosophic resolvable set in $\mathcal{N}(\mathcal{X}, \pi)$. □

proposition 3.13. *Let $\mathcal{N}(\mathcal{X}, \pi)$ be a nts. If $\mathcal{P}$ is a neutrosophic resolvable set in $\mathcal{N}(\mathcal{X}, \pi)$, then $\mathcal{N}int[\mathcal{P} \wedge C[\mathcal{P}]] \leq \mathcal{N}cl[C[\mathcal{Q}]]$ for each neutrosophic closed set $\mathcal{Q}$ in $\mathcal{N}(\mathcal{X}, \pi)$.*

Proof. Let $\mathcal{P}$ be a neutrosophic resolvable set in $\mathcal{N}(\mathcal{X}, \pi)$. Using Proposition 3.5 $\mathcal{N}int\mathcal{N}cl[\mathcal{P} \wedge C[\mathcal{P}] \wedge \mathcal{Q}] = 0_{\mathcal{N}}$ in $\mathcal{N}(\mathcal{X}, \pi)$. We know that, $\mathcal{N}int[\mathcal{P} \wedge C[\mathcal{P}] \wedge \mathcal{Q}] \leq \mathcal{N}int\mathcal{N}cl[\mathcal{P} \wedge C[\mathcal{P}] \wedge \mathcal{Q}] = 0_{\mathcal{N}}$ $\implies \mathcal{N}int[\mathcal{P} \wedge C[\mathcal{P}] \wedge \mathcal{N}int[\mathcal{Q}]] = 0_{\mathcal{N}} \implies \mathcal{N}int[\mathcal{P} \wedge C[\mathcal{P}]] \leq C[\mathcal{N}int\mathcal{Q}] = \mathcal{N}cl[C[\mathcal{Q}]]$ in $\mathcal{N}(\mathcal{X}, \pi)$. □

4. **Neutrosophic resolvable functions**

Definition 4.1. Let $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$ and $\mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$ be any two neutrosophic topological spaces. A function $R : \mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}}) \to \mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$ is called as neutrosophic resolvable functions if $R^{-1}[\mathcal{P}]$ is neutrosophic resolvable set in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$ for each neutrosophic open set $\mathcal{P}$ in $\mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$.

Example 4.2. Let $\mathcal{X} = \{\mu, \nu, \omega\}$ and consider the neutrosophic sets as follows,
$U_1 = \{< \mu, 0.6, 0.7, 0.2 >, < \nu, 0.7, 0.8, 0.2 >, < \omega, 1.0, 0.6, 0.3 >; \mu, \nu, \omega \in \mathcal{X}\}$
$U_2 = \{< \mu, 0.5, 0.6, 0.3 >, < \nu, 0.8, 0.7, 0.4 >, < \omega, 1.0, 0.8, 0.4 >; \mu, \nu, \omega \in \mathcal{X}\}$
$U_3 = \{< \mu, 0.5, 0.6, 0.3 >, < \nu, 0.7, 0.7, 0.4 >, < \omega, 1.0, 0.6, 0.4 >; \mu, \nu, \omega \in \mathcal{X}\}$
$U_4 = \{< \mu, 0.6, 0.7, 0.2 >, < \nu, 0.8, 0.8, 0.2 >, < \omega, 1.0, 0.8, 0.3 >; \mu, \nu, \omega \in \mathcal{X}\}$
$S = \{< \mu, 0.3, 0.2, 0.8 >, < \nu, 0.1, 0.2, 0.9 >, < \omega, 0.0, 0.3, 0.8 >; \mu, \nu, \omega \in \mathcal{X}\}$
$T = \{< \mu, 0.0, 0.3, 0.8 >, < \nu, 0.3, 0.2, 0.8 >, < \omega, 0.1, 0.2, 0.4 >; \mu, \nu, \omega \in \mathcal{X}\}$
Then $\pi_{\mathcal{X}} = \{0_{\mathcal{N}}, U_1, U_2, U_3, U_4, 1_{\mathcal{N}}\}$, $\pi_{\mathcal{Y}} = \{0_{\mathcal{N}}, T, 1_{\mathcal{N}}\}$ are the nts. Therefore
$\pi^C_{\mathcal{X}} = \{0_{\mathcal{N}}, C[U_1], C[U_2], C[U_3], C[U_4], 1_{\mathcal{N}}\}$, $\pi^C_{\mathcal{Y}} = \{0_{\mathcal{N}}, C[T], 1_{\mathcal{N}}\}$. Now,
$\mathcal{N}int\mathcal{N}cl[\mathcal{N}cl[C[U_1] \wedge S] \wedge \mathcal{N}cl[C[U_1] \wedge C[S]]] = 0_{\mathcal{N}}$, $\mathcal{N}int\mathcal{N}cl[\mathcal{N}cl[C[U_2] \wedge S] \wedge \mathcal{N}cl[C[U_2] \wedge C[S]]] = 0_{\mathcal{N}}$, $\mathcal{N}int\mathcal{N}cl[\mathcal{N}cl[C[U_3] \wedge S] \wedge \mathcal{N}cl[C[U_3] \wedge C[S]]] = 0_{\mathcal{N}}$, $\mathcal{N}int\mathcal{N}cl[\mathcal{N}cl[C[U_4] \wedge S] \wedge \mathcal{N}cl[C[U_4] \wedge C[S]]] = 0_{\mathcal{N}}$.
This implies S is a neutrosophic resolvable set. Now we define a function $R : \mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}}) \to$

$\mathcal{N}(\mathcal{Y},\pi_{\mathcal{Y}})$ by $R(\mu)=\nu$, $R(\nu)=\omega$ and $R(\omega)=\mu$. By computation $R^{-1}[T]=S$, for each neutrosophic open set T in $\mathcal{N}(\mathcal{Y},\pi_{\mathcal{Y}})$. Thus, S is neutrosophic resolvable set. Hence R is a neutrosophic resolvable function.

Example 4.3. Consider the set $\mathcal{X}=\{\mu,\nu,\omega\}$ and the neutrosophic sets V_1, V_2, V_3, V_4 and V_5 are defined in $\mathcal{X}$ as follows:-

$$V_1 = \{<\mu,0.0,0.0,0.5>,<\nu,0.1,0.1,0.6>,<\omega,0.2,0.1,0.5>,\mu,\nu,\omega\in\mathcal{X}\}$$
$$V_2 = \{<\mu,1.0,0.9,0.0>,<\nu,0.9,0.8,0.1>,<\omega,0.8,0.7,0.5>,\mu,\nu,\omega\in\mathcal{X}\}$$
$$V_3 = \{<\mu,0.0,0.0,0.4>,<\nu,0.1,0.1,0.5>,<\omega,0.1,0.1,0.4>,\mu,\nu,\omega\in\mathcal{X}\}$$
$$V_4 = \{<\mu,0.2,0.2,0.4>,<\nu,0.3,0.3,0.5>,<\omega,0.3,0.2,0.5>,\mu,\nu,\omega\in\mathcal{X}\}$$
$$V_5 = \{<\mu,0.3,0.2,0.5>,<\nu,0.2,0.2,0.5>,<\omega,0.3,0.3,0.5>,\mu,\nu,\omega\in\mathcal{X}\}$$

Then $\pi_{\mathcal{X}}=\{0_{\mathcal{N}},V_1,V_2,1_{\mathcal{N}}\}$, $\pi_{\mathcal{Y}}=\{0_{\mathcal{N}},V_5,1_{\mathcal{N}}\}$ are two neutrosophic topological spaces. Now we define a function $R:\mathcal{N}(\mathcal{X},\pi_{\mathcal{X}})\to\mathcal{N}(\mathcal{Y},\pi_{\mathcal{Y}})$ by $R(\mu)=\nu$, $R(\nu)=\mu$ and $R(\omega)=\mu$. Now $\implies \mathcal{N}int\mathcal{N}cl[\mathcal{N}cl[C[V_1]\wedge V_3]\wedge\mathcal{N}cl[C[V_1]\wedge C[V_3]]]=0_{\mathcal{N}} \implies \mathcal{N}int\mathcal{N}cl[\mathcal{N}cl[C[V_2]\wedge V_3]\wedge \mathcal{N}cl[C[V_2]\wedge C[V_3]]]=0_{\mathcal{N}}$. This implies V_3 is a neutrosophic resolvable set in $\mathcal{N}(\mathcal{X},\pi_{\mathcal{X}})$. But $\mathcal{N}int\mathcal{N}cl[\mathcal{N}cl[C[V_1]\wedge V_4]]\wedge\mathcal{N}cl[C[V_1]\wedge C[V_4]]=V_1\neq 0_{\mathcal{N}}$. Therefore V_4 is not a neutrosophic resolvable set in $\mathcal{N}(\mathcal{X},\pi_{\mathcal{X}})$. By computation, we have $R^{-1}[V_5]=V_4\neq V_3$ for a non empty neutrosophic open set V_5 in $\mathcal{N}(\mathcal{Y},\pi_{\mathcal{Y}})$. Therefore the function $R:\mathcal{N}(\mathcal{X},\pi_{\mathcal{X}})\to\mathcal{N}(\mathcal{Y},\pi_{\mathcal{Y}})$ is not a neutrosophic resolvable function. Since V_4 is not a neutrosophic resolvable set.

proposition 4.4. *If a function $R:\mathcal{N}(\mathcal{X},\pi_{\mathcal{X}})\to\mathcal{N}(\mathcal{Y},\pi_{\mathcal{Y}})$ is a neutrosophic resolvable function, then for any neutrosophic open set $\mathcal{P}$ in $\mathcal{N}(\mathcal{Y},\pi_{\mathcal{Y}})$*

(a) *$\mathcal{N}int\mathcal{N}cl[\mathcal{Q}\wedge R^{-1}[\mathcal{P}\wedge C[\mathcal{P}]]]=0_{\mathcal{N}}$ in $\mathcal{N}(\mathcal{X},\pi)$ for each neutrosophic closed set $\mathcal{Q}$, where $C[\mathcal{Q}]\in\pi_{\mathcal{X}}$.*

(b) *For the neutrosophic closed set $\mathcal{Q}$, $\mathcal{N}int[\mathcal{Q}\wedge R^{-1}[\mathcal{P}\wedge C[\mathcal{P}]]]=0_{\mathcal{N}}$.*

Proof. (a) Let $R:\mathcal{N}(\mathcal{X},\pi_{\mathcal{X}})\to\mathcal{N}(\mathcal{Y},\pi_{\mathcal{Y}})$ be the neutrosophic resolvable function. Then for the neutrosophic open set $0_{\mathcal{N}}\neq\mathcal{P}$ in $\mathcal{N}(\mathcal{Y},\pi_{\mathcal{Y}})$, there exist the neutrosophic resolvable set $R^{-1}[\mathcal{P}]$ in $\mathcal{N}(\mathcal{X},\pi_{\mathcal{X}})$. Using the definition of resolvable set, we have $\mathcal{N}int\mathcal{N}cl[\mathcal{N}cl[\mathcal{Q}\wedge R^{-1}[\mathcal{P}]]\wedge \mathcal{N}cl[\mathcal{Q}\wedge C[R^{-1}[\mathcal{P}]]]]=0_{\mathcal{N}}$ in $\mathcal{N}(\mathcal{X},\pi_{\mathcal{X}})$. $\mathcal{N}cl[[\mathcal{Q}\wedge R^{-1}[\mathcal{P}]]\wedge[\mathcal{Q}\wedge C[R^{-1}[\mathcal{P}]]]]\leq\mathcal{N}cl[\mathcal{Q}\wedge R^{-1}[\mathcal{P}]]\wedge\mathcal{N}cl[\mathcal{Q}\wedge C[R^{-1}[\mathcal{P}]]] \implies \mathcal{N}int\mathcal{N}cl[\mathcal{Q}\wedge R^{-1}[\mathcal{P}]\wedge C[R^{-1}[\mathcal{P}]]]=0_{\mathcal{N}}$ in $\mathcal{N}(\mathcal{X},\pi_{\mathcal{X}})$. Therefore $\mathcal{Q}\wedge R^{-1}[\mathcal{P}]\wedge C[R^{-1}[\mathcal{P}]]$ is the neutrosophic nowhere dense set in $\mathcal{N}(\mathcal{X},\pi_{\mathcal{X}})$.
(b) Using (a), we have $\mathcal{N}int\mathcal{N}cl[\mathcal{Q}\wedge R^{-1}[\mathcal{P}\wedge C[\mathcal{P}]]]=0_{\mathcal{N}}$, in $\mathcal{N}(\mathcal{X},\pi_{\mathcal{X}})$ for the neutrosophic open set $\mathcal{P}$ in $\mathcal{N}(\mathcal{Y},\pi_{\mathcal{Y}})$. Now, $\mathcal{N}int[\mathcal{Q}\wedge R^{-1}[\mathcal{P}\wedge C[\mathcal{P}]]]\leq\mathcal{N}int\mathcal{N}cl[\mathcal{Q}\wedge R^{-1}[\mathcal{P}\wedge C[\mathcal{P}]]]=0_{\mathcal{N}}$ $\implies \mathcal{N}int[\mathcal{Q}\wedge R^{-1}[\mathcal{P}\wedge C[\mathcal{P}]]]=0_{\mathcal{N}}$ in $\mathcal{N}(\mathcal{X},\pi_{\mathcal{X}})$. □

proposition 4.5. *If a function $R : \mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}}) \rightarrow \mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$ is the neutrosophic resolvable function then $\mathcal{N}int[R^{-1}[\mathcal{P} \wedge C[\mathcal{P}]]] \leq \mathcal{N}cl[C[\mathcal{Q}]]$ for the neutrosophic open set $\mathcal{P}$ in $\mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$, and $\mathcal{Q}$ is the neutrosophic closed set in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$.*

Proof. Let us take a neutrosophic resolvable function $R : \mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}}) \rightarrow \mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$. Then for any neutrosophic open set $\mathcal{P}$ in $\mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$ using the proposition 4.4 (b), we have $\mathcal{N}int[\mathcal{Q} \wedge R^{-1}[\mathcal{P}] \wedge C[R^{-1}[\mathcal{P}]]] = 0_{\mathcal{N}}$, here $\mathcal{Q}$ is the neutrosophic closed set in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$. Now, $\mathcal{N}int[\mathcal{Q} \wedge R^{-1}[\mathcal{P}] \wedge C[R^{-1}[\mathcal{P}]]] = \mathcal{N}int[\mathcal{Q}] \wedge \mathcal{N}int[R^{-1}[\mathcal{P}] \wedge C[R^{-1}[\mathcal{P}]]] = 0_{\mathcal{N}} \implies \mathcal{N}int[R^{-1}[\mathcal{P}] \wedge C[R^{-1}[\mathcal{P}]]] \leq C[\mathcal{N}int[\mathcal{Q}]] = \mathcal{N}cl[C[\mathcal{Q}]]$. □

proposition 4.6. *If a function $R : \mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}}) \rightarrow \mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$ is the neutrosophic resolvable function from the neutrosophic topological space $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$ to nts $\mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$, then for any neutrosophic open set $\mathcal{P}$ in $\mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$.*

(a) *there exist a regular open set S in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$ such that $\mathcal{N}cl[R^{-1}[\mathcal{P} \vee C[\mathcal{P}]]] \geq S$.*

(b) *$\mathcal{N}int\mathcal{N}cl[\mathcal{P} \vee C[\mathcal{P}]] \neq 0_{\mathcal{N}}$ in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$.*

Proof. (a) Let R be a neutrosophic resolvable function from *nts* $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$ into *nts* $\mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$. Then for any neutrosophic open set $\mathcal{P}$ in $\mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$, using the Proposition 4.4 (b), we have $\mathcal{N}int[\mathcal{Q} \wedge R^{-1}[\mathcal{P}] \wedge C[R^{-1}[\mathcal{P}]]] = 0_{\mathcal{N}}$ in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$, where $\mathcal{Q}$ is the neutrosophic closed set. This implies $\mathcal{N}int[\mathcal{Q}] \wedge \mathcal{N}int[R^{-1}[\mathcal{P}] \wedge C[R^{-1}[\mathcal{P}]]] = 0_{\mathcal{N}} \implies \mathcal{N}int[\mathcal{Q}] \leq C[\mathcal{N}int[R^{-1}[\mathcal{P}]] \wedge C[R^{-1}[\mathcal{P}]]] = \mathcal{N}cl[C[R^{-1}[\mathcal{P}]] \vee R^{-1}[\mathcal{P}]]$. Since $\mathcal{Q}$ is the neutrosophic closed set. Using 3.7, $\mathcal{N}int[\mathcal{Q}]$ is the neutrosophic regular open set in $\mathcal{N}(\mathcal{X}, \pi)$. Put $R = \mathcal{N}int[\mathcal{Q}]$. Then $\mathcal{N}cl[R^{-1}[C[\mathcal{P}] \wedge \mathcal{P}]] \geq R$, for a neutrosophic regular open set R in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$.

(b) Since every neutrosophic regular open set is neutrosophic open in a *nts* $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$. Therefore the neutrosophic regular open set S is neutrosophic open in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$. Using (a) $S \leq \mathcal{N}cl[R^{-1}[[C[\mathcal{P}]] \vee [\mathcal{P}]]] \neq 0_{\mathcal{N}} \implies \mathcal{N}intS = S \leq \mathcal{N}int\mathcal{N}cl[R^{-1}[C[\mathcal{P}] \vee [\mathcal{P}]]] \neq 0_{\mathcal{N}}$ in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$. □

proposition 4.7. *Let $R : \mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}}) \rightarrow \mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$ be a neutrosophic resolvable function, then for any neutrosophic open set $\mathcal{P}$ in $\mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$, there exists a neutrosophic regular closed set R in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$ such that $S \geq \mathcal{N}int[R^{-1}[\mathcal{P}] \wedge C[R^{-1}[\mathcal{P}]]]$.*

Proof. Consider $R : \mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}}) \rightarrow \mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$ be a neutrosophic resolvable function. Using the proposition 4.5, for any neutrosophic open set $\mathcal{P}$ in $\mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$. $\mathcal{N}cl[C[\mathcal{Q}]] \geq \mathcal{N}int[R^{-1}[\mathcal{P} \wedge C[\mathcal{P}]]]$, here $\mathcal{Q}$ is the neutrosophic closed set in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$. Therefore $C[\mathcal{Q}]$ is the neutrosophic open set. Put $S = C[\mathcal{Q}]$. This implies, $\mathcal{N}cl[S]$ is the neutrosophic regular closed set in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$. Hence $S \geq \mathcal{N}int[R^{-1}[\mathcal{P}] \wedge C[R^{-1}[\mathcal{P}]]]$. □

proposition 4.8. *If $R : \mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}}) \to \mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$ is the neutrosophic resolvable function, then $\mathcal{N}cl[R^{-1}[\mathcal{P} \vee C[\mathcal{P}]] \vee S] = 1_{\mathcal{N}}$ in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$ for the neutrosophic open set $\mathcal{P}$ in $\mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$ and $S \in \mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$.*

Proof. Let $R : \mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}}) \to \mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$ be the neutrosophic resolvable function. By Proposition 4.5 $\mathcal{N}int\mathcal{N}cl[\mathcal{Q} \wedge R^{-1}[\mathcal{P}] \wedge C[R^{-1}[\mathcal{P}]]] = 0_{\mathcal{N}}$ for the neutrosophic open set $\mathcal{P}$ and neutrosophic closed set $\mathcal{Q}$ in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$. Then $C[\mathcal{N}int\mathcal{N}cl[\mathcal{Q} \wedge R^{-1}[\mathcal{P}] \wedge C[R^{-1}[\mathcal{P}]]]] = 1_{\mathcal{N}}$ $\implies \mathcal{N}cl\mathcal{N}int[C[\mathcal{Q}] \vee C[R^{-1}[\mathcal{P}]] \vee R^{-1}[\mathcal{P}]] = 1_{\mathcal{N}}$. Since $C[C[\mathcal{P}]] = \mathcal{P}$. Now, $\mathcal{N}cl\mathcal{N}int[C[\mathcal{Q}] \vee C[R^{-1}[\mathcal{P}]] \vee R^{-1}[\mathcal{P}]] \leq \mathcal{N}cl[C[\mathcal{Q}] \vee C[R^{-1}[\mathcal{P}]] \vee R^{-1}[\mathcal{P}]] \implies \mathcal{N}cl[C[\mathcal{Q}] \vee R^{-1}[\mathcal{P} \vee C[\mathcal{P}]]] = 1_{\mathcal{N}}$. Put $S = C[\mathcal{Q}]$ then we have $\mathcal{N}cl[S \vee R^{-1}[\mathcal{P} \vee C[\mathcal{P}]]] = 1_{\mathcal{N}}$ in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$ where S is the neutrosophic open set in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$. □

proposition 4.9. *If the function $R : \mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}}) \to \mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$ is the neutrosophic resolvable function and $\mathcal{P}$ is the neutrosophic open set in $\mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$, then $C[R^{-1}[\mathcal{P}]]$ is also neutrosophic resolvable set in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$.*

Proof. Let $\mathcal{P}$ be a neutrosophic open set in $\mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$ and R be a neutrosophic resolvable function from a *nts* $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$ into a *nts*$[\mathcal{Y}, \pi_{\mathcal{Y}}]$. This implies $R^{-1}[\mathcal{P}]$ is the neutrosophic resolvable set $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$, using Proposition 3.9 $C[R^{-1}[\mathcal{P}]]$ is the neutrosophic resolvable set in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$. □

proposition 4.10. *Let $R_1 : \mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}}) \to \mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$ be the neutrosophic resolvable function and $R_2 : \mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}}) \to \mathcal{N}(\mathcal{Z}, \pi_{\mathcal{Z}})$ be the neutrosophic continuous function, then $R_2 \circ R_1 : [\mathcal{X}, \pi_{\mathcal{X}}] \to [\mathcal{Z}, \pi_{\mathcal{Z}}]$ is the neutrosophic resolvable function.*

Proof. Let $0_{\mathcal{N}} \neq \mathcal{P}$ be a neutrosophic open set in $\mathcal{N}(\mathcal{Z}, \pi_{\mathcal{Z}})$. Since R_2 is the neutrosophic continuous function from $\mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$ into $\mathcal{N}(\mathcal{Z}, \pi_{\mathcal{Z}})$. This implies $R^{-1}[\mathcal{P}]$ is the neutrosophic open set in $\mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$. Since R_1 is the neutrosophic resolvable set from $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$ into $\mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$. Therefore $R_1^{-1}[R_2^{-1}[\mathcal{P}]]$ is the neutrosophic resolvable set in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$. Thus $[R_2 \circ R_1]^{-1}[\mathcal{P}]$ is the neutrosophic resolvable set in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$, for the neutrosophic resolvable function from $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$ into $\mathcal{N}(\mathcal{Z}, \pi_{\mathcal{Z}})$. □

proposition 4.11. *If $R : \mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}}) \to \mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$ is the neutrosophic contra continuous function from the nts $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$ into the nts $\mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$, and if $\mathcal{N}int[\mathcal{Q}] = 0_{\mathcal{N}}$, for each neutrosophic closed set $\mathcal{Q}$ in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$, then $R : \mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}}) \to \mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}}))$ is the neutrosophic resolvable function.*

Proof. Let $R : \mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}}) \to \mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$ be the neutrosophic contra continuous function. Take $\mathcal{P}$ be the neutrosophic open set in $\mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$. This implies $R^{-1}[\mathcal{P}]$ is the neutrosophic closed set in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$. Using the hypothesis, $\mathcal{N}int[R^{-1}[\mathcal{P}]] = 0_{\mathcal{N}}$ in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$. For the neutrosophic closed set $\mathcal{Q}$ in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$. $\mathcal{N}cl[\mathcal{Q} \wedge [R^{-1}][\mathcal{P}]] \wedge \mathcal{N}cl[\mathcal{Q} \wedge C[R^{-1}[\mathcal{P}]]] \leq \mathcal{N}cl[R^{-1}[\mathcal{P}]] \wedge \mathcal{N}cl[C[R^{-1}[\mathcal{P}]]]= \mathcal{N}cl[R^{-1}[\mathcal{P}]] \wedge C[\mathcal{N}int[R^{-1}[\mathcal{P}]]]= R^{-1}[\mathcal{P}]$. Since $R^{-1}[\mathcal{P}]$ is neutrosophic closed set in $\mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}})$. Therefore $\mathcal{N}cl[\mathcal{Q} \wedge R^{-1}[\mathcal{P}]] \wedge \mathcal{N}cl[\mathcal{Q} \wedge C[R^{-1}[\mathcal{P}]]] \leq R^{-1}[\mathcal{P}]$. Now, $\mathcal{N}int\mathcal{N}cl\{[\mathcal{N}cl[\mathcal{Q} \wedge R^{-1}[\mathcal{P}] \wedge \mathcal{N}cl[\mathcal{Q} \wedge C[R^{-1}[\mathcal{P}]]]\} \leq \mathcal{N}int\mathcal{N}cl[R^{-1}[\mathcal{P}]] = \mathcal{N}int[R^{-1}[\mathcal{P}]] = 0_{\mathcal{N}}$. This implies $R^{-1}[\mathcal{P}]$ is the neutrosophic resolvable set in $\mathcal{N}(\mathcal{X}, \pi)$. Hence $R : \mathcal{N}(\mathcal{X}, \pi_{\mathcal{X}}) \to \mathcal{N}(\mathcal{Y}, \pi_{\mathcal{Y}})$ is the neuttosophic resolvable function. □

5. Conclusion

Neutrosophic resolvable sets and neutrosphic resolvable functions were introduced in this article. The characteristics of such sets are closely examined and studied to arrive at a solution. As a result, the concept of neutrosophic resolvable sets has been generalized according to areas of research.

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A Contemporary approach on Generalised NB Closed Sets in Neutrosophic Binary Topological Spaces

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Abstract. The paper, concentrated on the introduction of neutrosophic binary αgs-closed sets and neutrosophic binary αgs-open sets in Neutrosophic binary topological Space. The properties of neutrosophic binary αgs-closed sets and neutrosophic binary αgs-open sets have been studied. Furthermore, we examined its relationship with other framed neutrosophic binary sets. Also, some of the theorems were contemplated and the contrary part have been analyzed using examples.

Keywords: Neutrosophic Binary αgs-closed sets, Neutrosophic Binary αgs-open sets.

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1. **Introduction**

Neutrosophic Topological Space was initiated and formulated by A.A.Salama [14] in 2012 by using Smarandache [10] neutrosophic sets which was introduced in 2002, a generalisation of intuitionistic fuzzy sets. The neutrosophic sets consists of the degree of truth membership, degree of indeterminancy and the degree of false membership which are self-supporting and defines uncertainity. Neutrosophic α closed sets were introduced by I.Arockiarani [1] in 2017. In 2016, P.Ishwarya [12] defined the neutrosophic semi open and closed sets in neutrosophic topological space. Also, they studied the neutrosophic semi interior and closure operators. Neutrosophic α-generalized closed sets was established by R. Dhavaseelan et al., V.K.Shanthi [15] in 2018 initiated neutrosophic generalised semi closed sets. Furthermore, neutrosophic αgs closed sets were initiated by V.Banu Priya [2] in 2019. S.N.Jothi [5] in 2011 introduced the topology between two universal sets which is defined to be binary topology. The binary topology is a binary structure from $\dot{\mathcal{U}}$ to $\dot{\mathcal{V}}$ which consists of ordered pairs $(\dot{S}, \dot{Q})$ where $A \subseteq \dot{\mathcal{U}}$ and $B \subseteq \dot{\mathcal{V}}$. In continuation, S.S.Surekha, J.Elekiah and G.Sindhu [16] in 2022 introduced Neutrosophic Binary Topological Space which consists of two universal sets and each universal set contain its own truth, indeterminancy and false membership values. Also, in 2022, S.S.Surekha and G.sindhu [17] formulated binary αgs closed sets in binary topological space. In this article, we defined Neutrosophic Binary regular, semi and α open and closed sets. Also, Neutrosophic Binary αgs closed sets was defined and some characteristics have been framed.

2. Motivation

Neutrosophic Topological Space was formulated by A.A.Salama [14] in 2012. The neutrosophic sets consists of the degree of truth membership, degree of indeterminancy and the degree of false membership. S.N.Jothi [5] in 2011 introduced the binary topology. Later, S.S.Surekha, J.Elekiah and G.Sindhu [16] in 2022 introduced Neutrosophic Binary Topological Space consisting of two universal sets and each has its own truth, indeterminancy and false membership values. Also, in 2022, S.S.Surekha and G.sindhu [17] formulated binary αgs-closed sets in binary topological space. Also, neutrosophic αgs closed sets were initiated by V.Banu Priya [2] in 2019. Hence these implications motivated the researcher to investigate the role of neutrosophic binary αgs closed and open sets in neutrosophic binary topological spaces.

3. Preliminaries

Definition 3.1. [16] A Neutrosophic binary topology is a binary structure consisting of two universal sets $\dot{\mathcal{U}}$ and $\dot{\mathcal{V}}$ where $\mathcal{MN} \subseteq P(\dot{\mathcal{U}}) \times P(\dot{\mathcal{V}})$ and it satisfies the following conditions:

(1) $(\dot{0}_{\dot{\mathcal{U}}}, \dot{0}_{\dot{\mathcal{V}}}) \in \mathcal{MN}$ and $(\dot{1}_{\dot{\mathcal{U}}}, \dot{1}_{\dot{\mathcal{V}}}) \in \mathcal{MN}$.
(2) $(\dot{S}_1 \cap \dot{Q}_2, \dot{S}_1 \cap \dot{Q}_2) \in \mathcal{MN}$ whenever $(\dot{S}_1, \dot{Q}_1) \in \mathcal{MN}$ and $(\dot{S}_2, \dot{Q}_2) \in \mathcal{MN}$.
(3) If $(\dot{S}_\alpha, \dot{Q}_\alpha)_{\alpha \in S}$ is a family of members of $\mathcal{MN}$, then $(\cup_{\alpha \in S}\dot{S}_\alpha, \cup_{\alpha \in S}\dot{Q}_\alpha) \in \mathcal{MN}$.

The triplet $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$ is called Neutrosophic Binary Topological space.

Definition 3.2. [16] $(0_{\dot{\mathcal{U}}}, 0_{\dot{\mathcal{V}}})$ can be defined as

(0_1) $0_{\dot{\mathcal{U}}} = \{< \dot{U}, 0, 0, 1 >: \dot{u} \in \dot{\mathcal{U}}\}$, $0_{\dot{\mathcal{V}}} = \{< \dot{V}, 0, 0, 1 >: \dot{v} \in \dot{\mathcal{V}}\}$
(0_2) $0_{\dot{\mathcal{U}}} = \{< \dot{U}, 0, 1, 1 >: \dot{u} \in \dot{\mathcal{U}}\}$, $0_{\dot{\mathcal{V}}} = \{< \dot{V}, 0, 1, 1 >: \dot{v} \in \dot{\mathcal{V}}\}$
(0_3) $0_{\dot{\mathcal{U}}} = \{< \dot{U}, 0, 1, 0 >: \dot{u} \in \dot{\mathcal{U}}\}$, $0_{\dot{\mathcal{V}}} = \{< \dot{V}, 0, 1, 0 >: \dot{v} \in \dot{\mathcal{V}}\}$

(0_4) $0_{\dot{\mathcal{U}}} = \{< \dot{U}, 0, 0, 0 >: \dot{u} \in \dot{\mathcal{U}}\}$, $0_{\dot{\mathcal{V}}} = \{< \dot{V}, 0, 0, 0 >: \sqsubseteq \in \dot{\mathcal{V}}\}$

$(1_{\dot{\mathcal{U}}}, 1_{\dot{\mathcal{V}}})$ can be defined as
(1_1) $1_{\dot{\mathcal{U}}} = \{< \dot{U}, 1, 0, 0 >: \dot{u} \in \dot{\mathcal{U}}\}$, $1_{\dot{\mathcal{V}}} = \{< \dot{V}, 1, 0, 0 >: \dot{v} \in \dot{\mathcal{V}}\}$
(1_2) $1_{\dot{\mathcal{U}}} = \{< \dot{U}, 1, 0, 1 >: \dot{u} \in \dot{\mathcal{U}}\}$, $1_{\dot{\mathcal{V}}} = \{< \dot{V}, 1, 0, 1 >: \dot{v} \in \dot{\mathcal{V}}\}$
(1_3) $1_{\dot{\mathcal{U}}} = \{< \dot{U}, 1, 1, 0 >: \dot{u} \in \dot{\mathcal{U}}\}$, $1_{\dot{\mathcal{V}}} = \{< \dot{V}, 1, 1, 0 >: \dot{v} \in \dot{\mathcal{V}}\}$
(1_4) $1_{\dot{\mathcal{U}}} = \{< \dot{U}, 1, 1, 1 >: \dot{u} \in \dot{\mathcal{U}}\}$, $1_{\dot{\mathcal{V}}} = \{< \dot{V}, 1, 1, 1 >: \dot{v} \in \dot{\mathcal{V}}\}$

Definition 3.3. [16] Let $(\dot{S}, \dot{Q}) = \{< \mu_S, \sigma_S, \gamma_S >, < \mu_Q, \sigma_Q, \gamma_Q >\}$ be a neutrosophic binary set on $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$, then the complement of the set $\tilde{C}(\dot{S}, \dot{Q})$ may be defined as

$$(\tilde{C_1}) \quad \tilde{C}(\dot{S}, \dot{Q}) = \{\dot{U}, < 1 - \mu_{\dot{S}}(\dot{U}), \sigma_{\dot{S}}(\dot{U}), 1 - \gamma_{\dot{S}}(\dot{U}) >: \dot{u} \in \dot{U}, < \dot{V}, 1 - \mu_{\dot{Q}}(\dot{V}), \sigma_{\dot{Q}}(\dot{V}), 1 - \gamma_{\dot{Q}}(\dot{V}) >: \dot{v} \in \dot{V}\}$$

$$(\tilde{C_2}) \quad \tilde{C}(\dot{S}, \dot{Q}) = \{\dot{U}, < \gamma_{\dot{S}}(\dot{U}), \sigma_{\dot{S}}(\dot{U}), \mu_{\dot{S}}(\dot{U}) >: \dot{u} \in \dot{U}, < \dot{V}, \gamma_{\dot{Q}}(\dot{V}), \sigma_{\dot{Q}}(\dot{V}), \mu_{\dot{Q}}(\dot{V}) >: \dot{v} \in \dot{V}\}$$

$$(\tilde{C_3}) \quad \tilde{C}(\dot{S}, \dot{Q}) = \{\dot{U}, < \gamma_{\dot{S}}(\dot{U}), 1 - \sigma_{\dot{S}}(\dot{U}), \mu_{\dot{S}}(\dot{U}) >: \dot{u} \in \dot{U}, < \dot{V}, \gamma_{\dot{Q}}(\dot{V}), 1 - \sigma_{\dot{Q}}(\dot{V}), \mu_{\dot{Q}}(\dot{V}) >: \dot{v} \in \dot{V}\}$$

Definition 3.4. [16] Let $(\dot{S}, \dot{Q})$ and $(\dot{T}, \dot{R})$ be two neutrosophic binary sets.
Then $(\dot{S}, \dot{Q}) \subseteq (\dot{T}, \dot{R})$ can be defined as

(1) $(\dot{S}, \dot{Q}) \subseteq (\dot{T}, \dot{R}) \iff \mu_{\dot{S}}(\dot{U}) \leq \mu_{\dot{T}}(\dot{U}), \sigma_{\dot{S}}(\dot{U}) \leq \sigma_{\dot{T}}(\dot{U}), \gamma_{\dot{S}}(\dot{U}) \geq \gamma_{\dot{T}}(\dot{U}) \forall \dot{u} \in \dot{U}$
$\mu_{\dot{Q}}(\dot{V}) \leq \mu_{\dot{R}}(\dot{V}), \sigma_{\dot{Q}}(\dot{V}) \leq \sigma_{\dot{R}}(\dot{V}), \gamma_{\dot{Q}}(\dot{V}) \geq \gamma_{\dot{R}}(\dot{V}) \forall \dot{v} \in \dot{V}$
(2) $(\dot{S}, \dot{Q}) \subseteq (\dot{T}, \dot{R}) \iff \mu_{\dot{S}}(\dot{U}) \leq \mu_{\dot{T}}(\dot{U}), \sigma_{\dot{S}}(\dot{U}) \geq \sigma_{\dot{T}}(\dot{U}), \gamma_{\dot{S}}(\dot{U}) \geq \gamma_{\dot{T}}(\dot{U}) \forall \dot{u} \in \dot{U}$
$\mu_{\dot{Q}}(\dot{V}) \leq \mu_{\dot{R}}(\dot{V}), \sigma_{\dot{Q}}(\dot{V}) \geq \sigma_{\dot{R}}(\dot{V}), \gamma_{\dot{Q}}(\dot{V}) \geq \gamma_{\dot{R}}(\dot{V}) \forall \dot{v} \in \dot{V}$

Definition 3.5. [16] Let $(\dot{S}, \dot{Q})$ and $(\dot{T}, \dot{R})$ be two neutrosophic binary sets.
(1) $(\dot{S}, \dot{Q}) \cap (\dot{T}, \dot{R})$ can be defined as

$$(\dot{S}, \dot{Q}) \cap (\dot{T}, \dot{R}) = \{ < \dot{U}, \mu_{\dot{S}}(\dot{U}) \wedge \mu_{\dot{T}}(\dot{U}), \sigma_{\dot{S}}(\dot{U}) \wedge \sigma_{\dot{T}}(\dot{U}), \gamma_{\dot{S}}(\dot{U}) \vee \gamma_{\dot{T}}(\dot{U}) > < \dot{V}, \mu_{\dot{Q}}(\dot{V}) \wedge \mu_{\dot{R}}(\dot{V}), \sigma_{\dot{Q}}(\dot{V}) \wedge \sigma_{\dot{R}}(\dot{V}), \gamma_{\dot{Q}}(\dot{V}) \vee \gamma_{\dot{R}}(\dot{V}) >\}$$

$$(\dot{S}, \dot{Q}) \cap (\dot{T}, \dot{R}) = \{ < \dot{U}, \mu_{\dot{S}}(\dot{U}) \wedge \mu_{\dot{T}}(\dot{U}), \sigma_{\dot{S}}(\dot{U}) \vee \sigma_{\dot{T}}(\dot{U}), \gamma_{\dot{S}}(\dot{U}) \vee \gamma_{\dot{T}}(\dot{U}) > < \dot{V}, \mu_{\dot{Q}}(\dot{V}) \wedge \mu_{\dot{R}}(\dot{V}), \sigma_{\dot{Q}}(\dot{V}) \vee \sigma_{\dot{R}}(\dot{V}), \gamma_{\dot{Q}}(\dot{V}) \vee \gamma_{\dot{R}}(\dot{V}) >\}$$

(2) $(\dot{S}, \dot{Q}) \cup (\dot{T}, \dot{R})$ can be defined as

$$(\dot{S}, \dot{Q}) \cup (\dot{T}, \dot{R}) = \{ < \dot{U}, \mu_{\dot{S}}(\dot{U}) \vee \mu_{\dot{T}}(\dot{U}), \sigma_{\dot{S}}(\dot{U}) \vee \sigma_{\dot{T}}(\dot{U}), \gamma_{\dot{S}}(\dot{U}) \wedge \gamma_{\dot{T}}(\dot{U}) > < \dot{V}, \mu_{\dot{Q}}(\dot{V}) \vee \mu_{\dot{R}}(\dot{V}), \sigma_{\dot{Q}}(\dot{V}) \vee \sigma_{\dot{R}}(\dot{V}), \gamma_{\dot{Q}}(\dot{V}) \wedge \gamma_{\dot{R}}(\dot{V}) >\}$$

$$(\dot{S},\dot{Q})\cap(\dot{T},\dot{R})=\{<\dot{U},\mu_{\dot{S}}(\dot{U})\vee\mu_{\dot{T}}(\dot{U}),\sigma_{\dot{S}}(\dot{U})\wedge\sigma_{\dot{T}}(\dot{U}),\gamma_{\dot{S}}(\dot{U})\wedge\gamma_{\dot{T}}(\dot{U})> \\ <\dot{V},\mu_{\dot{Q}}(\dot{V})\vee\mu_{\dot{R}}(\dot{V}),\sigma_{\dot{Q}}(\dot{V})\wedge\sigma_{\dot{R}}(\dot{V}),\gamma_{\dot{Q}}(\dot{V})\wedge\gamma_{\dot{R}}(\dot{V})>\}$$

Definition 3.6. [16] Let $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$ be a Neutrosophic Binary Topological Space. Then,
$(\dot{S},\dot{Q})^{1^\star_N}=\cap\{S_\alpha:(S_\alpha,Q_\alpha)$ is neutrosophic binary closed and $(\dot{S},\dot{Q})\subseteq(S_\alpha,Q_\alpha)\}$
$(\dot{S},\dot{Q})^{2^\star_N}=\cap\{Q_\alpha:(S_\alpha,Q_\alpha)$ is neutrosophic binary closed and $(\dot{S},\dot{Q})\subseteq(S_\alpha,Q_\alpha)\}$.
The ordered pair $((\dot{S},\dot{Q})^{1^\star_N},(\dot{S},\dot{Q})^{2^\star_N})$ is called the neutrosophic binary closure of $(\dot{S},\dot{Q})$.

Definition 3.7. [16] Let $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$ be a Neutrosophic Binary Topological Space. Then,
$(\dot{S},\dot{Q})^{1^0_N}=\cup\{S_\alpha:(S_\alpha,Q_\alpha)$ is neutrosophic binary open and $(S_\alpha,Q_\alpha)\subseteq(\dot{S},\dot{Q})\}$
$(\dot{S},\dot{Q})^{2^0_N}=\cup\{Q_\alpha:(S_\alpha,Q_\alpha)$ is neutrosophic binary open and $(S_\alpha,Q_\alpha)\subseteq(\dot{S},\dot{Q})\}$.
The ordered pair $((\dot{S},\dot{Q})^{1^0_N},(\dot{S},\dot{Q})^{2^0_N})$ is called the neutrosophic binary interior of $(\dot{S},\dot{Q})$.

Definition 3.8. Let $(\dot{\mathcal{U}},\tau_N)$ be a Neutrosophic topological space. Then the subset A is said to be

(1) neutrosophic α open [1] if $A\subseteq Nint(Ncl(Nint(A)))$.
(2) neutrosophic semi open [12] if $A\subseteq Ncl(Nint(A))$.

Definition 3.9. [1] Let $(\dot{\mathcal{U}},\tau_N)$ be a Neutrosophic Topological Space. Let A be the subset of neutrosophic topological space. The intersection of all the neutrosophic α closed sets which contains A is called the neutrosophic α closure of A and is denoted by $N\alpha cl(A)$.

Definition 3.10. [2] Let $(\dot{\mathcal{U}},\tau_N)$ be a Neutrosophic topological space. Then the subset A is said to be neutrosophic αgs closed if $N\alpha cl(A)\subseteq U$ whenever $A\subseteq U$ and U is neutrosophic semi open.

4. Neutrosophic Binary αgs-closed sets

Definition 4.1. Let $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$ be a Neutrosophic Binary Topological Space. Then $(\dot{S},\dot{Q})$ is called

(1) Neutrosophic binary regular open if $(\dot{S},\dot{Q})=\mathcal{N}^{\mathbf{b}}int(\mathcal{N}^{\mathbf{b}}cl(\dot{S},\dot{Q}))$ and the complement of neutrosophic binary regular open sets are called as neutrosophic binary regular closed (shortly $\mathcal{N}^{\mathbf{b}}$-regular closed).
(2) Neutrosophic binary semiopen if $(\dot{S},\dot{Q})\subseteq\mathcal{N}^{\mathbf{b}}cl(\mathcal{N}^{\mathbf{b}}int(\dot{S},\dot{Q}))$ and the complement of neutrosophic binary semi open sets are called as neutrosophic binary semi closed sets (shortly $\mathcal{N}^{\mathbf{b}}$-semi closed).

(3) Neutrosophic binary α open if $(\dot{S},\dot{Q}) \subseteq \mathcal{N}^{\mathbf{b}}int(\mathcal{N}^{\mathbf{b}}cl(\mathcal{N}^{\mathbf{b}}int(\dot{S},\dot{Q})))$ and the complement of neutrosophic binary α open sets are called as neutrosophic binary α closed sets (shortly $\mathcal{N}^{\mathbf{b}}$-α closed).

Definition 4.2. Let $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$ be a Neutrosophic Binary Topological Space. Then,
$(\dot{S},\dot{T})^{1\star_N}_{\alpha} = \cap\{S_\alpha : (S_\alpha,T_\alpha)$ is neutrosophic binary α closed and $(\dot{S},\dot{T}) \subseteq (S_\alpha,T_\alpha)\}$
$(\dot{S},\dot{T})^{2\star_N}_{\alpha} = \cap\{T_\alpha : (S_\alpha,T_\alpha)$ is neutrosophic binary α closed and $(\dot{S},\dot{T}) \subseteq (S_\alpha,T_\alpha)\}$.
The ordered pair $((\dot{S},\dot{T})^{1\star_N}_{\alpha},(\dot{S},\dot{T})^{2\star_N}_{\alpha})$ is called the neutrosophic binary α closure of $(\dot{S},\dot{Q})$ and is denoted by $\mathcal{N}^{\mathbf{b}}\alpha cl(\dot{S},\dot{Q})$.

Definition 4.3. Let $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$ be a Neutrosophic Binary Topological Space. Then,
$(\dot{S},\dot{T})^{1 0_N}_{\alpha} = \cup\{S_\alpha : (S_\alpha,B_\alpha)$ is neutrosophic binary α open and $(S_\alpha,T_\alpha) \subseteq (\dot{S},\dot{T})\}$
$(\dot{S},\dot{T})^{2 0_N}_{\alpha} = \cup\{T_\alpha : (S_\alpha,T_\alpha)$ is neutrosophic binary α open and $(S_\alpha,T_\alpha) \subseteq (\dot{S},\dot{T})\}$.
The ordered pair $((\dot{S},\dot{T})^{1 0_N}_{\alpha},(\dot{S},\dot{T})^{2 0_N}_{\alpha})$ is called the neutrosophic binary α interior of $(\dot{S},\dot{T})$ and it is denoted by $\mathcal{N}^{\mathbf{b}}\alpha int(\dot{S},\dot{T})$.

Example 4.4. Let $\dot{\mathcal{U}} = \{a_1,a_2,a_3\}$ and $\dot{\mathcal{V}} = \{b_1,b_2,b_3\}$. Let

$$(A_1,A_2) = \{<\dot{\mathcal{U}},(O.4,0.5,0.2),(0.3,0.2,0.1),(0.9,0.6,0.8)>,$$
$$<\dot{\mathcal{V}},(0.2,0.4,0.5),(0.1,0.1,0.2),(0.6,0.5,0.8)>\}$$
$$(B_1,B_2) = \{<\dot{\mathcal{U}},(0.5,0.6,0.2),(0.4,0.3,0.1),(0.7,0.6,0.7)>,$$
$$<\dot{\mathcal{V}},(0.3,0.5,0.4),(0.3,0.2,0.1),(0.7,0.5,0.6)>\}.$$

The Neutrosophic Binary Topological space is given by $\mathcal{M}_N = \{(0_{\dot{\mathcal{U}}},0_{\dot{\mathcal{V}}}),(1_{\dot{\mathcal{U}}},1_{\dot{\mathcal{V}}}),(A_1,A_2),(B_1,B_2)\}$. Let

$$(C_1,C_2) = \{<\dot{\mathcal{U}},(0.5,0.6,0.1),(0.4,0.3,0.1),(0.9,0.8,0.5)>,$$
$$<\dot{\mathcal{V}},(0.3,0.4,0.5),(0.9,0.3,0.1),(0.7,0.6,0.7)>\}.$$

Clearly, (C_1,C_2) is Neutrosophic Binary semi open. Also, it is Neutrosophic Binary alpha open.

Theorem 4.5. *In a neutrosophic binary topological space* $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$,

(1) *Every* $\mathcal{N}^{\mathbf{b}}$*-regular closed sets are* $\mathcal{N}^{\mathbf{b}}$ *closed sets.*
(2) *Every* $\mathcal{N}^{\mathbf{b}}$*-semi closed sets are* $\mathcal{N}^{\mathbf{b}}\alpha$ *closed sets.*

Proof. (1) Since, $(\dot{S},\dot{Q})$ is Neutrosophic Binary-regular closed set, we have $(\dot{S},\dot{Q}) = \mathcal{N}^{\mathbf{b}}cl(\mathcal{N}^{\mathbf{b}}int(\dot{S},\dot{Q}))$. Obviously, $(\dot{S},\dot{Q})$ is Neutrosophic Binary closed set.
(2) Since, $(\dot{S},\dot{Q})$ is Neutrosophic Binary Semi closed, we have $\mathcal{N}^{\mathbf{b}}int(\mathcal{N}^{\mathbf{b}}cl(\dot{S},\dot{Q})) \subseteq (\dot{S},\dot{Q})$. This implies, $\mathcal{N}^{\mathbf{b}}cl(\mathcal{N}^{\mathbf{b}}int(\mathcal{N}^{\mathbf{b}}cl(\dot{S},\dot{Q}))) \subseteq \mathcal{N}^{\mathbf{b}}cl(\dot{S},\dot{Q})$. Since, every neutrosophic binary semi closed sets are closed, we have $(\dot{S},\dot{Q}) = \mathcal{N}^{\mathbf{b}}cl(\dot{S},\dot{Q})$. Therefore,

$\mathcal{N}^{\mathbf{b}}cl(\mathcal{N}^{\mathbf{b}}int(\mathcal{N}^{\mathbf{b}}cl(\dot{S},\dot{Q}))) \subseteq (\dot{S},\dot{Q})$. Hence, $(\dot{S},\dot{Q})$ is neutrosophic binary α closed in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$. □

Remark 4.6. Since, every $\mathcal{N}^{\mathbf{b}}$-semi closed sets are $\mathcal{N}^{\mathbf{b}}\alpha$ closed sets, it is obvious that $\mathcal{N}^{\mathbf{b}}scl(\dot{S},\dot{Q}) \subseteq \mathcal{N}^{\mathbf{b}}\alpha cl(\dot{S},\dot{Q})$.

Definition 4.7. Let $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$ be a Neutrosophic Binary Topological Space. Let $(\dot{S},\dot{T}) \subseteq (\dot{\mathcal{U}},\dot{\mathcal{V}})$. Then $(\dot{S},\dot{T})$ is called a Neutrosophic Binary α generalised semiclosed set (shortly $\mathcal{N}^{\mathbf{b}}\alpha gs$-closed set) if $\mathcal{N}^{\mathbf{b}}\alpha cl(\dot{S},\dot{T}) \subseteq (P,V)$ whenever (P,V) is Neutrosophic Binary Semiopen.

Theorem 4.8. *The union of two $\mathcal{N}^{\mathbf{b}}\alpha gs$-closed set is also a $\mathcal{N}^{\mathbf{b}}\alpha gs$-closed set.*

Proof. Let $(\dot{S},\dot{Q})$ and $(\dot{T},\dot{R})$ be two $\mathcal{N}^{\mathbf{b}}$-αgs closed set in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$. Then by definition 4.7, we have $\mathcal{N}^{\mathbf{b}}\alpha cl(\dot{S},\dot{Q}) \subseteq (\dot{P},\dot{V})$ whenever $(\dot{S},\dot{Q}) \subseteq (\dot{P},\dot{V})$ and $(\dot{P},\dot{V})$ is $\mathcal{N}^{\mathbf{b}}$ semiopen. Also, $\mathcal{N}^{\mathbf{b}}\alpha cl(\dot{T},\dot{R}) \subseteq (\dot{P},\dot{V})$ whenever $(\dot{T},\dot{R}) \subseteq (\dot{P},\dot{V})$ and $(\dot{P},\dot{V})$ is $\mathcal{N}^{\mathbf{b}}$ semiopen. This implies $\mathcal{N}^{\mathbf{b}}\alpha cl(\dot{S},\dot{Q}) \cup \mathcal{N}^{\mathbf{b}}\alpha cl(\dot{T},\dot{R}) \subseteq (\dot{P},\dot{V}) \implies \mathcal{N}^{\mathbf{b}}\alpha cl[(\dot{S},\dot{Q}) \cup (\dot{T},\dot{R})] \subseteq (\dot{P},\dot{V})$. Therefore, $(\dot{S},\dot{Q}) \cup (\dot{T},\dot{R})$ is a $\mathcal{N}^{\mathbf{b}}$-αgs closed set in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$. □

Remark 4.9. The intersection of two $\mathcal{N}^{\mathbf{b}}\alpha gs$-closed sets need not be a $\mathcal{N}^{\mathbf{b}}\alpha gs$-closed set.

It is demonstrated by the following example.

Example 4.10. Let $\dot{\mathcal{U}} = \{a_1,a_2,a_3\}$ and $\dot{\mathcal{V}} = \{b_1,b_2,b_3\}$ be the universe. Let $\mathcal{M}_N = \{(0_{\dot{\mathcal{U}}},0_{\dot{\mathcal{V}}}),(1_{\dot{\mathcal{U}}},1_{\dot{\mathcal{V}}}),(A_1,A_2),(B_1,B_2),(C_1,C_2),(D_1,D_2)\}$ be the neutrosophic binary topological space. Here

$$(A_1,A_2) = \{ < \dot{\mathcal{U}},(0.4,0.5,0.2),(0.3,0.5,0.1),(0.9,0.6,0.8) >, < \dot{\mathcal{V}},(0.2,0.5,0.5),(0.1,0.5,0.2),(0.6,0.5,0.8) >\}$$
$$(B_1,B_2) = \{ < \dot{\mathcal{U}},(0.5,0.6,0.2),(0.4,0.5,0.1),(0.7,0.6,0.7) >, < \dot{\mathcal{V}},(0.3,0.5,0.4),(0.3,0.5,0.1),(0.7,0.5,0.6) >\}$$
$$(C_1,C_2) = \{ < \dot{\mathcal{U}},(0.5,0.5,0.2),(0.4,0.5,0.1),(0.9,0.6,0.7) >, < \dot{\mathcal{V}},(0.3,0.5,0.4),(0.3,0.5,0.1),(0.7,0.5,0.6) >\}$$
$$(D_1,D_2) = \{ < \dot{\mathcal{U}},(0.4,0.6,0.2),(0.3,0.5,0.1),(0.7,0.6,0.8) >, < \dot{\mathcal{V}},(0.2,0.5,0.5),(0.1,0.5,0.2),(0.6,0.5,0.8) >\}$$

$$Let\ (\dot{S},\dot{Q}) = \{ <\dot{\mathcal{U}}, (0.3,0.4,0.2),(0.3,0.1,0.1),(0.6,0.4,0.9)>, <\dot{\mathcal{V}},(0.2,0.3,0.6),(0.1,0.1,0.3),(0.4,0.4,0.9)>\}\ and$$

$$(\dot{T},\dot{R}) = \{ <\dot{\mathcal{U}}, (0.4,0.5,0.2),(0.3,0.5,0.1),(0.9,0.4,0.8)>, <\dot{\mathcal{V}},(0.2,0.5,0.5),(0.1,0.5,0.2),(0.6,0.5,0.8)>\}$$

be two $\mathcal{N}^{\mathbf{b}}\alpha gs$-closed sets in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$. The intersection of the two subsets

$$(\dot{S},\dot{Q})\cap(\dot{T},\dot{R}) = \{ <\dot{\mathcal{U}}, (0.3,0.5,0.2),(0.3,0.5,0.1),(0.6,0.4,0.9)>, <\dot{\mathcal{V}},(0.2,0.5,0.6),(0.1,0.5,0.3),(0.4,0.5,0.9)>\}$$

which is not $\mathcal{N}^{\mathbf{b}}\alpha gs$-closed set in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$.

Theorem 4.11. *In a neutrosophic binary topological space $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$, every $\mathcal{N}^{\mathbf{b}}$-closed sets are $\mathcal{N}^{\mathbf{b}}\alpha gs$-closed set.*

Proof. Let $(\dot{S},\dot{Q})$ be a $\mathcal{N}^{\mathbf{b}}$-closed set in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$. Let us consider a neutrosophic binary set $(\dot{S},\dot{Q})\subseteq(\dot{P},\dot{V})$ where $(\dot{P},\dot{V})$ is neutrosophic binary semiopen in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$. Since $\mathcal{N}^{\mathbf{b}}\alpha cl(\dot{S},\dot{Q})\subseteq\mathcal{N}^{\mathbf{b}}cl(\dot{S},\dot{Q})$ and also $(\dot{S},\dot{Q})$ is neutrosophic binary closed set, we have $\mathcal{N}^{\mathbf{b}}\alpha cl(\dot{S},\dot{Q})\subseteq\mathcal{N}^{\mathbf{b}}cl(\dot{S},\dot{Q})=(\dot{S},\dot{Q})\subseteq(\dot{P},\dot{V})$ which implies $\mathcal{N}^{\mathbf{b}}\alpha cl(\dot{S},\dot{Q})\subseteq(\dot{P},\dot{V})$ whenever $(\dot{P},\dot{V})$ is neutrosophic binary semiopen. Hence $(\dot{S},\dot{Q})$ is $\mathcal{N}^{\mathbf{b}}\alpha gs$-closed set. □

Remark 4.12. The converse of the above theorem need not be true as illustrated by the following example.

Example 4.13. Let $\dot{\mathcal{U}}=\{a_1,a_2\}$ and $\dot{\mathcal{V}}=\{b_1,b_2\}$ be the universe.
Let $\mathcal{MN}=\{(0_{\dot{\mathcal{U}}},0_{\dot{\mathcal{V}}}),(1_{\dot{\mathcal{U}}},1_{\dot{\mathcal{V}}}),(V,W)\}$ be the neutrosophic binary topological space.
$(V,W)=\{<\dot{\mathcal{U}},(0.7,0.5,0.3),(0.6,0.5,0.4)>,<\dot{\mathcal{V}},(0.6,0.5,0.4),(0.7,0.5,0.3)>\}$.
$(\dot{S},\dot{Q})=\{<\dot{\mathcal{U}},(0.2,0.5,0.8),(0.3,0.5,0.7)>,<\dot{\mathcal{V}},(0.3,0.5,0.8),(0.2,0.4,0.7)>\}$ is $\mathcal{N}^{\mathbf{b}}$-αgs closed set but not $\mathcal{N}^{\mathbf{b}}$ closed set in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$.

Theorem 4.14. *Every $\mathcal{N}^{\mathbf{b}}$-Regular closed set in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$ is $\mathcal{N}^{\mathbf{b}}$-αgs closed set in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$.*

Proof. Let $(\dot{S},\dot{Q})$ be a $\mathcal{N}^{\mathbf{b}}$-regular closed set in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$. By theorem 4.5, every $\mathcal{N}^{\mathbf{b}}$-regular closed set is $\mathcal{N}^{\mathbf{b}}$-closed set in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$. This implies, $(\dot{S},\dot{Q})$ is $\mathcal{N}^{\mathbf{b}}$-closed set. Also, by theorem 4.11, we have $(\dot{S},\dot{Q})$ is $\mathcal{N}^{\mathbf{b}}$-αgs-closed set in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$. □

Remark 4.15. The converse of the above theorem need not be true as proved in the following example.

Example 4.16. Let $\dot{\mathcal{U}} = \{a_1, a_2\}$ and $\dot{\mathcal{V}} = \{b_1, b_2\}$ be the universe.
Let $\mathcal{MN} = \{(0_{\dot{\mathcal{U}}}, 0_{\dot{\mathcal{V}}}), (1_{\dot{\mathcal{U}}}, 1_{\dot{\mathcal{V}}}), (V, W)\}$ be the neutrosophic binary topological space.
$(V, W) = \{< \dot{\mathcal{U}}, (0.6, 0.5, 0.4), (0.6, 0.5, 0.4) >, < \dot{\mathcal{V}}, 0.7, 0.5, 0.3), (0.8, 0.5, 0.4)\}$.
Let $(\dot{S}, \dot{Q}) = \{< \dot{\mathcal{U}}, (0.3, 0.5, 0.7), (0.2, 0.5, 0.8) >, < \dot{\mathcal{V}}, (0.4, 0.5, 0.8), (0.3, 0.5, 0.7) >\}$ is $\mathcal{N}^{\mathbf{b}}$-αgs closed but not $\mathcal{N}^{\mathbf{b}}$ regular closed in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$.

Theorem 4.17. *Every $\mathcal{N}^{\mathbf{b}}$-αgs closed set is $\mathcal{N}^{\mathbf{b}}$-semi closed in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$.*

Proof. Let $(\dot{S}, \dot{Q})$ be a $\mathcal{N}^{\mathbf{b}}$-αgs closed set in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$. By remark 4.6, we have $\mathcal{N}^{\mathbf{b}}scl(\dot{S}, \dot{Q}) \subseteq \mathcal{N}^{\mathbf{b}}\alpha cl(\dot{S}, \dot{Q})$. Since $(\dot{S}, \dot{Q})$ is $\mathcal{N}^{\mathbf{b}}$-αgs closed set, we have $\mathcal{N}^{\mathbf{b}}\alpha cl(\dot{S}, \dot{Q}) \subseteq (\dot{P}, \dot{V})$, where $(\dot{P}, \dot{V})$ is $\mathcal{N}^{\mathbf{b}}$- semiopen. This implies $\mathcal{N}^{\mathbf{b}}Scl(\dot{S}, \dot{Q}) \subseteq (\dot{P}, \dot{V})$. Hence, $(\dot{S}, \dot{Q})$ is $\mathcal{N}^{\mathbf{b}}$-semi closed. □

Remark 4.18. The converse of the above theorem need not be true as illustrated in the following example.

Example 4.19. Let $\dot{\mathcal{U}} = \{a_1, a_2\}$ and $\dot{\mathcal{V}} = \{b_1, b_2\}$ be the universe.
Let $\mathcal{MN} = \{(0_{\dot{\mathcal{U}}}, 0_{\dot{\mathcal{V}}}), (1_{\dot{\mathcal{U}}}, 1_{\dot{\mathcal{V}}}), (V, W)\}$ be the neutrosophic binary topological space where $(V, W) = \{< \dot{\mathcal{U}}, (0.6, 0.5, 0.2), (0.3, 0.5, 0.2) >, < \dot{\mathcal{V}}, (0.7, 0.5, 0.1), (0.3, 0.5, 0.3) > \}$.
Let $(\dot{S}, \dot{Q}) = \{< \dot{\mathcal{U}}, (0.1, 0.5, 0.7), (0.1, 0.5, 0.5) >, < \dot{\mathcal{V}}, (0.2, 0.5, 0.6), (0.2, 0.5, 0.7) >\}$ is $\mathcal{N}^{\mathbf{b}}$ semi closed but not $\mathcal{N}^{\mathbf{b}}$-αgs closed set in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$.

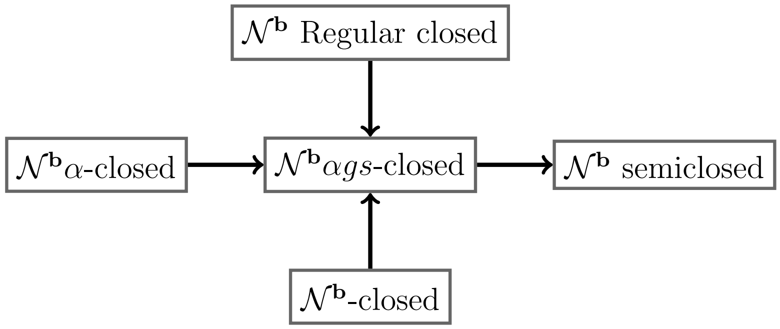
Theorem 4.20. *Every $\mathcal{N}^{\mathbf{b}}$-α closed set is $\mathcal{N}^{\mathbf{b}}$-αgs closed set in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$.*

Proof. Let $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$ be a neutrosophic binary topological space. Let $(\dot{S}, \dot{Q})$ be a $\mathcal{N}^{\mathbf{b}}$-α closed set in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$. Let $(\dot{S}, \dot{Q}) \subseteq (\dot{P}, \dot{V})$ where $(\dot{P}, \dot{V})$ is $\mathcal{N}^{\mathbf{b}}$ semiopen. Since $(\dot{S}, \dot{Q})$ is $\mathcal{N}^{\mathbf{b}}$- α closed set, we have $\mathcal{N}^{\mathbf{b}}$-$\alpha cl(\dot{S}, \dot{Q}) = (\dot{S}, \dot{Q}) \subseteq (\dot{P}, \dot{V})$ which implies $\mathcal{N}^{\mathbf{b}}$-$\alpha cl(\dot{S}, \dot{Q}) \subseteq (\dot{P}, \dot{V})$. Hence $(\dot{S}, \dot{Q})$ is $\mathcal{N}^{\mathbf{b}}$- αgs closed set in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$. □

Remark 4.21. The converse of the above theorem need not be true as seen in the following example.

Example 4.22. Let $\dot{\mathcal{U}} = \{a_1, a_2\}$ and $\dot{\mathcal{V}} = \{b_1, b_2\}$ be the universe.
Let $\mathcal{MN} = \{(0_{\dot{\mathcal{U}}}, 0_{\dot{\mathcal{V}}}), (1_{\dot{\mathcal{U}}}, 1_{\dot{\mathcal{V}}}), (V_1, W_1), (V_2, W_2)\}$ be the neutrosophic binary topological space. Here
$(V_1, W_1) = \{< \dot{\mathcal{U}}, (0.4, 0.5, 0.5), (0.3, 0.5, 0.6) >, < \dot{\mathcal{V}}, (0.3, 0.5, 0.5), (0.4, 0.5, 0.7) >\}$
$(V_2, W_2) = \{< \dot{\mathcal{U}}, (0.3, 0.5, 0.6), (0.2, 0.5, 0.7) >, < \dot{\mathcal{V}}, (0.2, 0.5, 0.6), (0.3, 0.5, 0.7) >\}$.
$(\dot{S}, \dot{Q}) = \{< \dot{\mathcal{U}}, (0.8, 0.5, 0.1), (0.8, 0.5, 0.1) >, < \dot{\mathcal{V}}, (0.7, 0.5, 0.2), (0.6, 0.5, 0.1) >\}$ is $\mathcal{N}^{\mathbf{b}}$-αgs closed set but not $\mathcal{N}^{\mathbf{b}}\alpha$ closed set in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$.

Remark 4.23. The following diagram shows the above implications.



5. **Neutrosophic Binary αgs open sets**

Definition 5.1. A neutrosophic binary set $(\dot{S}, \dot{T})$ in a neutrosophic binary topological space $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$ is said to be neutrosophic binary αgs-open sets (shortly $\mathcal{N}^{\mathbf{b}}$-αgs open) if the complement $(\dot{S}, \dot{Q})^c$ is $\mathcal{N}^{\mathbf{b}}$-αgs closed in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$.

Theorem 5.2. *In a neutrosophic binary topological space $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$, every $\mathcal{N}^{\mathbf{b}}$-open sets are $\mathcal{N}^{\mathbf{b}}\alpha gs$-open sets.*

Proof. Let $(\dot{S}, \dot{Q})$ be the $\mathcal{N}^{\mathbf{b}}$-open set in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$. Then, $(\dot{S}, \dot{Q})^c$ is $\mathcal{N}^{\mathbf{b}}$-closed set in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$. Then by theorem 4.11,$(\dot{S}, \dot{Q})^c$ is $\mathcal{N}^{\mathbf{b}}\alpha gs$-closed set, which implies $(\dot{S}, \dot{Q})$ is $\mathcal{N}^{\mathbf{b}}\alpha gs$-open set in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$. □

Remark 5.3. The converse of the above theorem is not true as illustrated in the following example.

Example 5.4. Let $\dot{\mathcal{U}} = \{a_1, a_2\}$ and $\dot{\mathcal{V}} = \{b_1, b_2\}$. The neutrosophic binary topological space is given by $\mathcal{MN} = \{(0_{\dot{\mathcal{U}}}, 0_{\dot{\mathcal{V}}}), (1_{\dot{\mathcal{U}}}, 1_{\dot{\mathcal{V}}}), (V, W)\}$ where
$(V, W) = \{< \dot{\mathcal{U}}, (0.8, 0.5, 0.2), (0.6, 0.5, 0.4) >, < \dot{\mathcal{V}}, (0.7, 0.5, 0.1), (0.6, 0.5, 0.3) >\}$.
Consider the neutrosophic binary set
$(\dot{S}, \dot{Q}) = \{< \dot{\mathcal{U}}, (0.9, 0.5, 0.1), (0.7, 0.5, 0.3) >, < \dot{\mathcal{V}}, (0.8, 0.5, 0.1), (0.8, 0.5, 0.3) >\}$. Here, $(\dot{S}, \dot{Q})^c$ is neutrosophic binary *alphags* closed set set. This implies that $(\dot{S}, \dot{Q})$ is neutrosophic binary αgs open set. But $(\dot{S}, \dot{Q})$ is not the neutrosophic binary open set.

Theorem 5.5. *In a neutrosophic binary topological space $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$, every $\mathcal{N}^{\mathbf{b}}$- regular open sets are $\mathcal{N}^{\mathbf{b}}\alpha gs$-open sets.*

Proof. Let $(\dot{S}, \dot{Q})$ be the $\mathcal{N}^{\mathbf{b}}$- regular open set in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$. Then, $(\dot{S}, \dot{Q})^c$ is $\mathcal{N}^{\mathbf{b}}$-regular closed set in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$. Then by theorem 4.14, $(\dot{S}, \dot{Q})^c$ is $\mathcal{N}^{\mathbf{b}}\alpha gs$-closed set, which implies $(\dot{S}, \dot{Q})$ is $\mathcal{N}^{\mathbf{b}}\alpha gs$-open set in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$. □

Remark 5.6. The converse of the above theorem is not true as seen in the following example.

Example 5.7. Let $\dot{\mathcal{U}} = \{a_1, a_2\}$ and $\dot{\mathcal{V}} = \{b_1, b_2\}$. The neutrosophic binary topological space is given by $\mathcal{MN} = \{(0_{\dot{\mathcal{U}}}, 0_{\dot{\mathcal{V}}}), (1_{\dot{\mathcal{U}}}, 1_{\dot{\mathcal{V}}}), (V, W)\}$ where
$(V, W) = \{< \dot{\mathcal{U}}, (0.6, 0.5, 0.4), (0.6, 0.5, 0.4) >, < \dot{\mathcal{V}}, (0.7, 0.5, 0.3), (0.6, 0.5, 0.4) >\}$.
Consider the neutrosophic binary set
$(\dot{S}, \dot{Q}) = \{< \dot{\mathcal{U}}, (0.7, 0.5, 0.3), (0.8, 0.5, 0.2) >, < \dot{\mathcal{V}}, (0.8, 0.5, 0.3), (0.8, 0.5, 0.2) >\}$. Here, $(\dot{S}, \dot{Q})^c$ is neutrosophic binary αgs closed set set. This implies that $(\dot{S}, \dot{Q})$ is neutrosophic binary αgs open set. But $(\dot{S}, \dot{Q})$ is not the neutrosophic binary regular open set.

Theorem 5.8. *In a neutrosophic binary topological space* $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$*, every* $\mathcal{N}^{\mathbf{b}}\alpha$*- open sets are* $\mathcal{N}^{\mathbf{b}}\alpha gs$*-open sets.*

Proof. Let $(\dot{S}, \dot{Q})$ be the $\mathcal{N}^{\mathbf{b}}\alpha$- open set in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$. Then, $(\dot{S}, \dot{Q})^c$ is $\mathcal{N}^{\mathbf{b}}\alpha$-closed set in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$. Then by theorem 4.20, $(\dot{S}, \dot{Q})^c$ is $\mathcal{N}^{\mathbf{b}}\alpha gs$-closed set, which implies $(\dot{S}, \dot{Q})$ is $\mathcal{N}^{\mathbf{b}}\alpha gs$-open set in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$. □

Remark 5.9. The converse of the above theorem is not true as seen in the example 4.22.

Theorem 5.10. *In a neutrosophic binary topological space* $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$*, every* $\mathcal{N}^{\mathbf{b}}\alpha gs$*-open sets are* $\mathcal{N}^{\mathbf{b}}$ *semi open sets.*

Proof. Let $(\dot{S}, \dot{Q})$ be the $\mathcal{N}^{\mathbf{b}}\alpha gs$- open set in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$. Then, $(\dot{S}, \dot{Q})^c$ is $\mathcal{N}^{\mathbf{b}}\alpha gs$-closed set in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$. Then by theorem 4.17, $(\dot{S}, \dot{Q})^c$ is $\mathcal{N}^{\mathbf{b}}$ semi-closed set, which implies $(\dot{S}, \dot{Q})$ is $\mathcal{N}^{\mathbf{b}}$- semi open set in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$. □

Remark 5.11. The converse of the above theorem is not true as seen in the example 4.19.

Theorem 5.12. *A neutrosophic binary set* $(\dot{S}, \dot{Q})$ *of a neutrosophic binary topological space* $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$ *is* $\mathcal{N}^{\mathbf{b}}\alpha gs$*-open set if and only if* $(\dot{E}, \dot{F}) \subseteq \mathcal{N}^{\mathbf{b}}\alpha\text{-}int(\dot{S}, \dot{Q})$ *whenever* $(\dot{E}, \dot{F})$ *is* $\mathcal{N}^{\mathbf{b}}$*-semi closed in* $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$ *and* $(\dot{E}, \dot{F}) \subseteq (\dot{S}, \dot{Q})$.

Proof. Necessary Part:
Let $(\dot{S}, \dot{Q})$ be a $\mathcal{N}^{\mathbf{b}}$-αgs open set in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$.
Let $(\dot{E}, \dot{F})$ be a $\mathcal{N}^{\mathbf{b}}$-αgs closed set and also $(\dot{E}, \dot{F}) \subseteq (\dot{S}, \dot{Q})$. This implies, $(\dot{E}, \dot{F})^c$ is $\mathcal{N}^{\mathbf{b}}$-αgs open set in $(\dot{\mathcal{U}}, \dot{\mathcal{V}}, \mathcal{MN})$ and $(\dot{S}, \dot{Q})^c \subseteq (\dot{E}, \dot{F})^c$. Since, $(\dot{S}, \dot{Q})^c$ is $\mathcal{N}^{\mathbf{b}}$-αgs closed set, we have $\mathcal{N}^{\mathbf{b}}$-$\alpha cl(\dot{S}, \dot{Q})^c \subseteq (\dot{E}, \dot{F})^c$. Hence, $(\dot{E}, \dot{F}) \subseteq \mathcal{N}^{\mathbf{b}}$-$\alpha int(\dot{S}, \dot{Q})$.

Sufficient Part:
Let $(\dot{E},\dot{F}) \subseteq \mathcal{N}^{\mathbf{b}}\text{-}\alpha int(\dot{S},\dot{Q})$.
This implies, $\mathcal{N}^{\mathbf{b}}\text{-}\alpha cl(\dot{S},\dot{Q})^c \subseteq (\dot{E},\dot{F})^c$ whenever $(\dot{E},\dot{F})^c$ is $\mathcal{N}^{\mathbf{b}}$-semi open. Therefore, $(\dot{S},\dot{Q})^c$ is $\mathcal{N}^{\mathbf{b}}\text{-}\alpha gs$ closed set in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$. Hence, $(\dot{S},\dot{Q})$ is $\mathcal{N}^{\mathbf{b}}\text{-}\alpha gs$ open set in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$. □

Theorem 5.13. *Let $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$ be a Neutrosophic Binary Topological Space. Then for every $\mathcal{N}^{\mathbf{b}}\text{-}\alpha gs$ open set $(\dot{S},\dot{Q})$ in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$ and for every Neutrosophic Binary set $(\dot{T},\dot{R})$ in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$, $\mathcal{N}^{\mathbf{b}}\alpha int(\dot{S},\dot{Q}) \subseteq (\dot{T},\dot{R}) \subseteq (\dot{S},\dot{Q}) \implies (\dot{T},\dot{R})$ is a $\mathcal{N}^{\mathbf{b}}\text{-}\alpha gs$ open set in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$.*

Proof. By hypothesis, $(\dot{S},\dot{Q})^c \subseteq (\dot{T},\dot{R})^c \subseteq (\mathcal{N}^{\mathbf{b}}\alpha int(\dot{S},\dot{Q}))^c$. Let $(\dot{T},\dot{R})^c \subseteq (\dot{P},\dot{V})$ and $(\dot{P},\dot{V})$ be a $\mathcal{N}^{\mathbf{b}}$-semiopen set in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$. Since $(\dot{S},\dot{Q})^c \subseteq (\dot{T},\dot{R})^c$, we have $(\dot{S},\dot{Q})^c \subseteq (\dot{P},\dot{V})$. But $(\dot{S},\dot{Q})^c$ is $\mathcal{N}^{\mathbf{b}}\text{-}\alpha gs$ closed. Therefore, $\mathcal{N}^{\mathbf{b}}\alpha cl((\dot{S},\dot{Q})^c) \subseteq (\dot{P},\dot{V})$. Also, $(\dot{T},\dot{R})^c \subseteq (\mathcal{N}^{\mathbf{b}}\alpha int(\dot{S},\dot{Q}))^c = \mathcal{N}^{\mathbf{b}}\alpha cl((\dot{S},\dot{Q})^c)$.That implies,
$\mathcal{N}^{\mathbf{b}}\alpha cl((\dot{T},\dot{R})^c) \subseteq \mathcal{N}^{\mathbf{b}}\alpha cl((\dot{S},\dot{Q})^c) \subseteq (\dot{P},\dot{V})$. Therefore, $(\dot{T},\dot{R})^c$ is $\mathcal{N}^{\mathbf{b}}\text{-}\alpha gs$ closed set in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$. Hence $(\dot{T},\dot{R})$ is $\mathcal{N}^{\mathbf{b}}\text{-}\alpha gs$ open set in $(\dot{\mathcal{U}},\dot{\mathcal{V}},\mathcal{MN})$. □

6. Conclusion

The Neutrosophic Binary αgs closed and open sets were introduced in this article. Also,its relationship with other sets in Neutrosophic Binary Topological Spaces were analyzed. The characteristics of such sets are closely examined and studied with the examples. In future, the decision making problems in real life will be analyzed using the neutrosophic binary sets.

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Choice of Suitable Referral Hospital to Improve the Financial Result of Hospital Operations and Quality of Patient Care under a Neutrosophic Environment

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Abstract: The referral cooperation hospital choice has been studied to better rationalize the allocation of healthcare assets and enhance the efficacy of resource utilization. Choosing hospitals to work within a collaborative referral arrangement is a crucial step in the patient-referral process. A referral cooperative hospital is chosen after careful consideration of these aspects to guarantee that patients will get the kind of detailed treatment that is appropriate for their condition. The concept of multi-criteria decision-making (MCDM) is used due to various criteria. The VIKOR method is the MCDM method used in this paper to rank the referral cooperation hospitals. The VIKOR method is integrated with the single-valued neutrosophic set to overcome uncertain information. The single-valued neutrosophic VIKOR method is applied to a case study in Egypt. We achieved quality of care as the best criterion from the eleven criteria used.

Keywords: Neutrosophic Set, VIKOR Method, Healthcare, Referral Cooperative Hospital, Patient Care.

1. Introduction

As the world's population becomes older, the healthcare system will be put to the test in new ways. Establishing a trustworthy healthcare system to improve human health has long been an important and engaging issue. Budgetary restraints prevent hospitals from purchasing the necessary number of high-priced yet in-demand medical devices, leading to inequitable distribution of these devices and extended wait periods for patients. As a result, several nations are adopting the medical consortium model to boost the efficacy of their healthcare systems and increase the effectiveness with which their healthcare resources are being use[1], [2].

Providing patients with access to specialized treatment and resources that may be lacking at other hospitals or medical practices, referral cooperative hospitals play a crucial role in the healthcare system. These facilities also aid in ensuring that patients get the best treatment possible by connecting them with other facilities or practices that better suit their requirements.

Hospitals are carefully chosen throughout the referral cooperative selection process to form a cooperative connection for the referral of patients. This step is essential for providing treatment that consistently exceeds patients' expectations. Care quality, service availability, location, insurance coverage, cost, communication and coordination, and cultural competence are just a few of the variables considered when choosing a referral cooperative hospital[3], [4].

Choosing a referral cooperative hospital is a multi-step procedure that must consider a wide range of considerations. The quality of care offered by the facility is a crucial consideration. The hospital's reputation, accreditation, results, and patient happiness all play a role. Referrals tend to choose hospitals with a solid reputation for offering excellent treatment.

Consideration must also be given to the accessibility of the hospital's services. Diagnostic and imaging services, as well as specialized care in fields like cardiology, cancer, and neurology, should all be available at the hospital. The hospital should also have enough beds and medical personnel to care for its patients[5], [6].

When deciding on a referral cooperative hospital, it is also crucial to take location into account. Many patients prefer to get care at a hospital that is conveniently situated near the facility or clinic that is recommended to them. This may also aid improve inter-hospital communication and cooperation. When choosing a referral cooperative hospital, insurance coverage is an important consideration. For the patient's convenience, the hospital should take the same insurance plans as the facility or clinic recommending them. Patients' financial contributions may be kept to a minimum and they can get the treatment they need with fewer issues if this is implemented[7], [8].

When choosing a referral cooperative hospital, cost should be considered. The hospital's price policy should be made clear to patients, and the cost of treatment should be affordable. This may make it easier for people to get the treatment they need without having to worry about how to pay for it. When deciding on a referral cooperative hospital, it is also important to think about how well you and the facility can communicate and work together. The hospital's communication and coordination systems, including the transmission of patient information, should be efficient and timely. This may aid in ensuring that patients get the treatment they need with little disruption to their schedules[9], [10].

Overcoming the conceptual ambiguity associated with human expert judgments has been accomplished via the application of neutrosophic logic[11], [12].

The Neutrosophic Set generalizes the intuitionistic set, classical set, fuzzy set, paraconsistent set, dialetheist set, paradoxist set, and tautological set, whereas the Neutrosophic Logic generalizes fuzzy logic based on Neutrosophy[13], [14].

The usual problem selection is the greatest suit for the Neutrosophic representation of uncertainty and indeterminacy[15], [16].

2. Referral Cooperative Hospitals

A referral cooperative hospital has partnered with another facility or clinic to facilitate patient referrals. This kind of partnership is often formed to guarantee top-notch treatment for patients by connecting them with clinics or hospitals that can provide the right services for them.

Hospitals that are part of referral cooperatives have earned a stellar reputation within their communities and are well-equipped to satisfy the diverse demands of their patients. These hospitals may feature specialized technology and personnel to treat patients with certain conditions, such as those requiring cardiology, cancer, or neurological treatment[17], [18].

In most cases, a doctor at one facility will send a patient to another facility that is part of a referral cooperative to get specialized care. The cooperative hospital receives the patient, assesses their condition, and treats them accordingly. To ensure that patients get the best possible treatment, the referring hospital or medical practice must communicate and coordinates with the referral cooperative hospital[19], [20].

3. Insurance Coverage of Referral Cooperative Hospitals

Most hospitals that participate in referral networks have systems in place to assist patients with understanding their insurance coverage for treatment received at an out-of-network facility. The following are examples of procedures:

Hospitals participating in a referral network may coordinate with insurance providers to get prior approval for treating patients who are not in their networks. The patient's out-of-pocket expenses may be reduced if their insurance covers the necessary medical procedures.

Claims help Referral hospitals may also help patients through the insurance claims process. This may improve the likelihood that the claims will be paid in full and that the patient will get the most out of their insurance[21], [22].

Hospitals participating in referral networks may provide patients with access to financial counselors who can answer questions about insurance and other healthcare financing concerns. Payment plan negotiations and applications for government aid are examples of the kinds of services that may fall under this category.

To better serve their patients, hospitals in referral networks may try to negotiate in-network status with insurance providers.

Patients should check with their insurance providers and learn about any out-of-pocket charges before seeking treatment at a referral cooperative hospital. Patients should contact their insurance companies to learn more about out-of-network coverage and whether prior permission is needed for any upcoming medical procedures. If a patient has questions regarding their health insurance, they should ask their doctor or the hospital that made the recommendation[23], [24].

4. Neutrosophic VIKOR method

Opricovic recommended the use of VIKOR. VIKOR is a compromise ranking approach that chooses many options when competing criteria are present. The compromise answer is the best approximation to the optimal answer[25]–[27]. The VIKOR method is integrated with the single-valued neutrosophic set to select the best referral cooperative hospital with a set of criteria. Figure 1 shows the proposed framework.

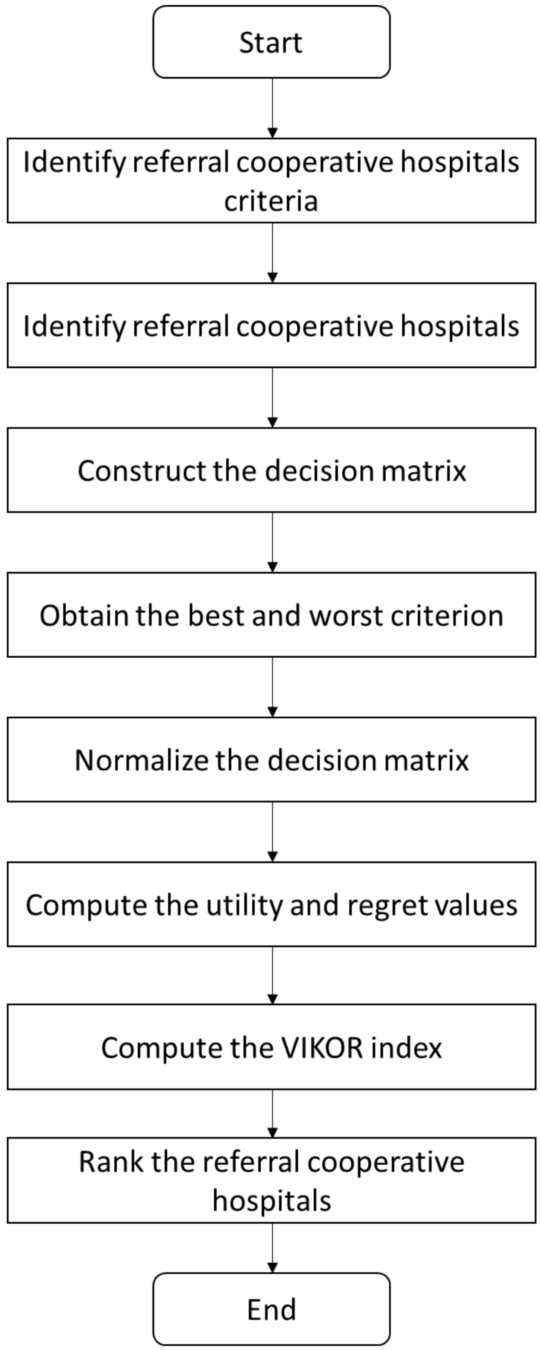

Figure 1. Steps of the neutrosophic VIKOR method.

A) Construct the decision matrix

B) Obtain the best and worst criterion

The ideal best and worst criterion are computed by:

$$y_j^+ = \max_i y_{ij} \tag{1}$$

$$y_j^- = \min_i y_{ij} \tag{2}$$

C) Normalize the decision matrix

$$r_{ij} = \frac{y_{ij} - y_j^-}{y_j^+ - y_j^-} \tag{3}$$

D) Compute the utility and regret values

The utility measures and regret measures are computed by using Manhattan and Chebyshev distances as:

$$U_i = \sum_{j=1}^{n} \left(w_j \frac{r_j^+ - r_{ij}}{r_j^+ - r_j^-} \right) \tag{4}$$

$$G_i = \max_j \left(w_j \frac{r_j^+ - r_{ij}}{r_j^+ - r_j^-} \right) \tag{5}$$

E) Compute the VIKOR index

$$V_i = \frac{U_i - U_i^-}{U_i^- + U_i^+} + (1 - p) \frac{G_i - G_i^-}{G_i^- + G_i^+} \tag{6}$$

Where $p = 0.5$

5. Case Study

This section introduces the selection of referral cooperative hospitals in Egypt. This study used eight hospitals to be ranked and eleven criteria.

When choosing hospitals to work with on patient referrals, it is important to find a balance between the needs of both parties. To guarantee high-quality treatment and patient happiness, this method seeks institutions that satisfy criteria and features.

Referral hospitals are chosen using the following criteria and factors:

Consideration should be given to the hospital's reputation for providing high-quality treatment. The hospital's reputation, accreditation, results, and patient happiness all play a role.

The hospital should provide all the resources and tools required to care for its patients. This comprises diagnostic and imaging services in addition to specialized care in areas like cardiology, cancer, and neurology.

Hospitals that are geographically near to the referral hospital or medical practice are chosen so that patients may spend less time traveling.

For the patient's convenience, the hospital should accept the same insurance plans as the facility making the referral.

The hospital's price policy should be made clear to patients, and the cost of treatment should be affordable.

The hospital should have efficient communication and coordination mechanisms in place to guarantee that all patients get consistent, high-quality treatment.

The hospital's ability to offer treatment that is culturally competent and respectful of the needs of all patients is essential.

When choosing a referral cooperative hospital, it is important to take all these considerations into account to provide patients with the best chance of receiving treatment that is up to par with their standards.

Table 1. Referral cooperative hospital data.

	RCHC$_1$	RCHC$_2$	RCHC$_3$	RCHC$_4$	RCHC$_5$	RCHC$_6$	RCHC$_7$	RCHC$_8$	RCHC$_9$	RCHC$_{10}$	RCHC$_{11}$
RCH 1	(0.9,0.15,0.15)	(0.35,0.70,0.60)	(0.8,0.25,0.20)	(0.9,0.15,0.15)	(0.75,0.30,0.20)	(0.8,0.25,0.20)	(0.9,0.15,0.15)	(0.75,0.30,0.20)	(0.9,0.15,0.15)	(0.8,0.25,0.20)	(0.9,0.15,0.15)
RCH 2	(0.8,0.25,0.20)	(0.75,0.30,0.20)	(0.8,0.25,0.20)	(0.35,0.70,0.60)	(0.35,0.70,0.60)	(0.75,0.30,0.20)	(0.35,0.70,0.60)	(0.15,0.90,0.80)	(0.8,0.25,0.20)	(0.15,0.90,0.80)	(0.8,0.25,0.20)
RCH 3	(0.75,0.30,0.20)	(0.15,0.90,0.80)	(0.9,0.15,0.15)	(0.35,0.70,0.60)	(0.9,0.15,0.15)	(0.15,0.90,0.80)	(0.9,0.15,0.15)	(0.8,0.25,0.20)	(0.35,0.70,0.60)	(0.9,0.15,0.15)	(0.8,0.25,0.20)
RCH 4	(0.8,0.25,0.20)	(0.75,0.30,0.20)	(0.15,0.90,0.80)	(0.8,0.25,0.20)	(0.15,0.90,0.80)	(0.8,0.25,0.20)	(0.15,0.90,0.80)	(0.75,0.30,0.20)	(0.15,0.90,0.80)	(0.15,0.90,0.80)	(0.75,0.30,0.20)
RCH 5	(0.9,0.15,0.15)	(0.15,0.90,0.80)	(0.15,0.90,0.80)	(0.9,0.15,0.15)	(0.15,0.90,0.80)	(0.75,0.30,0.20)	(0.9,0.15,0.15)	(0.75,0.30,0.20)	(0.15,0.90,0.80)	(0.9,0.15,0.15)	(0.35,0.70,0.60)
RCH 6	(0.75,0.30,0.20)	(0.75,0.30,0.20)	(0.8,0.25,0.20)	(0.15,0.90,0.80)	(0.8,0.25,0.20)	(0.15,0.90,0.80)	(0.8,0.25,0.20)	(0.8,0.25,0.20)	(0.15,0.90,0.80)	(0.75,0.30,0.20)	(0.8,0.25,0.20)
RCH 7	(0.8,0.25,0.20)	(0.35,0.70,0.60)	(0.9,0.15,0.15)	(0.35,0.70,0.60)	(0.15,0.90,0.80)	(0.9,0.15,0.15)	(0.15,0.90,0.80)	(0.9,0.15,0.15)	(0.35,0.70,0.60)	(0.9,0.15,0.15)	(0.75,0.30,0.20)
RCH 8	(0.9,0.15,0.15)	(0.75,0.30,0.20)	(0.8,0.25,0.20)	(0.35,0.70,0.60)	(0.75,0.30,0.20)	(0.8,0.25,0.20)	(0.35,0.70,0.60)	(0.75,0.30,0.20)	(0.35,0.70,0.60)	(0.8,0.25,0.20)	(0.9,0.15,0.15)

Experts and decision-makers used the single-valued neutrosophic numbers to evaluate the criteria and referral cooperative hospitals as shown in Table 1. Then by using Equation. (3), the data are normalized as shown in Table 2 based on the best and worst criterion. Then compute the utility and regret measures by using Equations. (4-5). Then compute the VIKOR index by using Equation. (6) as shown in Figure 2.

Table 2. Referral cooperative hospital normalization data.

	RCHC$_1$	RCHC$_2$	RCHC$_3$	RCHC$_4$	RCHC$_5$	RCHC$_6$	RCHC$_7$	RCHC$_8$	RCHC$_9$	RCHC$_{10}$	RCHC$_{11}$
RCH$_1$	0	0.027344	0.010675	0	0.014308	0.081122	0	0.014308	0	0.081122	0
RCH$_2$	0.072549	0	0.010675	0.07322	0.063363	0.076852	0.07322	0.087891	0.011811	0	0.016382
RCH$_3$	0.101563	0.041016	0	0.07322	0	0	0	0.010221	0.07322	0.091796	0.016382
RCH$_4$	0.072549	0	0.091796	0.011811	0.087891	0.081122	0.101563	0.014308	0.101563	0	0.022934
RCH$_5$	0	0.041016	0.091796	0	0.087891	0.076852	0	0.014308	0.101563	0.091796	0.101563
RCH$_6$	0.101563	0	0.010675	0.101563	0.010221	0	0.011811	0.010221	0.101563	0.076852	0.016382

RCH_7	0.072549	0.027344	0	0.07322	0.087891	0.091796	0.101563	0	0.07322	0.091796	0.022934
RCH_8	0	0	0.010675	0.07322	0.014308	0.081122	0.07322	0.014308	0.07322	0.081122	0

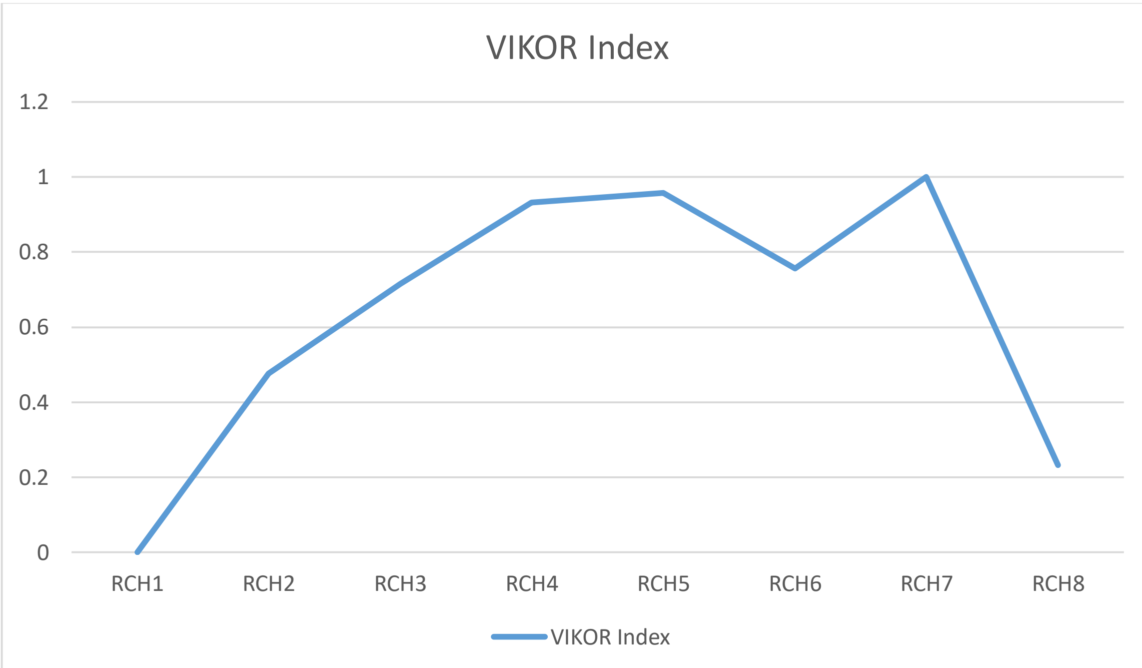

Figure 2. The referral cooperative hospital ranking.

There are many challenges of referral cooperative hospitals, benefits, and drawbacks. Care coordination between many providers and various locations is a significant difficulty for referral cooperative hospitals. Cultural, linguistic, and technological barriers may make this process more difficult than it must be. Satisfaction of Patients: Patients may be unsatisfied with the referral procedure if they find it cumbersome or if they must wait an extended period before they get treatment. Referral cooperative hospitals may be costlier than choices, depending on the complexity and rarity of the patient's disease. While hospitals in a referral cooperative may be selected for their specialization, patients may have doubts about the quality of treatment they would get there. Care at a referral cooperative hospital may not be covered by insurance if it is located outside of the patient's plan's coverage area.

Advantages of Hospitals Working Together to Refer Patients:

Referral cooperative hospitals provide access to specialized treatment that may be unavailable at general medical facilities. Care of a higher level is provided by the referral cooperative hospitals, which have earned a solid reputation for excellence in patient care and provide a full range of specialized medical services. Patients who are treated in hospitals that are members of a referral cooperative may have better results than those treated elsewhere, especially if they have a complicated or uncommon ailment. Better patient outcomes and care coordination are possible when diverse clinicians and institutions work together, as is encouraged by referral cooperative hospitals.

Consequences of Hospitals Working Together to Refer Patients:

Referral cooperative hospitals may charge more than average, depending on the complexity or rarity of the patient's illness. Patients may have to travel farther to reach the referral cooperative hospitals, which may be difficult and time-consuming. Appointments and operations at referral cooperation hospitals may have lengthier wait periods, especially during times of high demand. Care at a referral cooperative hospital may not be covered by insurance if it is located outside of the patient's plan's coverage area. Referrals to certain medical facilities might make patients feel as if they have fewer choices in where to get treatment.

6. Conclusions

Choosing a hospital for a referral cooperative is a crucial step that must consider several variables. Care quality, service availability, location, insurance coverage, cost, communication and coordination, and cultural competence are just a few of the variables considered when choosing a referral cooperative hospital. Hospitals that can deliver high-quality, all-encompassing care that is up to patients' requirements and expectations are given preference when establishing referral connections. However, the process of choosing a referral cooperative hospital is difficult, and there is always the chance that something may go wrong in terms of communication or coordination. This paper used the single-valued neutrosophic set with a MCDM methodology to overcome the uncertain data. The VIKOR method is the MCDM method used to rank the hospitals for a referral cooperative. This method is applied to Egypt as a case study.

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Common Fixed Point Theorems for Occasionally Weakly Compatible Mappings in Neutrosophic Cone Metric Spaces

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Abstract. The idea of Neutrosophic Cone Metric Space is introduced in this study. In order to illustrate fixed point, the idea of occassionally weakly compatible is also used.

Keywords: Fixed point; Neutrosophic Metric Space; Compatible Mappings; Occasionally weakly compatible mapping.

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1. **Introduction**

As it lays the groundwork for modern mathematics, classical set theory progresses through numerous extensions. Zadeh [32] first suggested the idea of Fuzzy Sets (FS) in 1965. Fuzzy set theory was crucially applied in all branches of science and engineering. The idea of Fuzzy Meric Space (FMS) was first put forth in 1975 by Kramosil and Michalek[14]. This important characteristic of assigning graded membership polarised the academics, prompting them to develop different analyses and applications for various types of fuzzy metric spaces. George and Veeramani [4] reconstructed FMS using triangular criteria. Following then, other researchers explored the properties of FMS and produced numerous fixed point results.

Intuitionistic Fuzzy Sets, which expanded fuzzy set theory to include the idea of non-membership grade, were introduced by Atanassov [1] in 1983. Since then, a lot of work has been put into coming to new findings and extending existing ideas to the intuitionistic fuzzy environment. Park [17] developed intuitionistic fuzzy metric space (IFMS), and several fixed point findings based on the concept of IFS were published. Alaca et al. [2] and other researchers have developed several fixed point theorems in FMS and IFMS. Tarkan Oner et al. [16] developed the idea of fuzzy cone metric space. By Priyobartal et al. [18], several fixed point outcomes in fuzzy cone metric space were studied. Neutrosophy is an extension of the intuitionistic fuzzy set presented by Florentin Smarandache [20] in 1998. It holds that there exists a continuum-power spectrum of neutralities between a notion and its opponent.

According to this theory, there is a continuum-power spectrum of neutralities that might exist between a concept and its adversary. Neutralities were added to the Intuitionistic Fuzzy Set by neutrosophy, which energised the scientific community, and the field is now thriving with countless investigations, analyses, computing techniques, and applications. Neutrosophic metric space was established by Kirisci et al. [15] in 2019 as an expansion of intuitionistic fuzzy metric space that produces fixed point theorms in complete neutrosophic metric space. In Neutrosophic Metric Spaces(NMS), Sowndrarajan, Jeyaraman, and Florentin Smarandache demonstrated fixed point findings for contraction theorems.

This paper introduces the idea of Neutrosophic Cone Metric Space (NCMS) and explains its key components.. On Neutrosophic Cone Metric Space, the Banach contraction theorem and a few fixed point results are presented and demonstrated. Furthermore, by utilising the idea of occasionally weakly compatible on two self mappings, fixed point results on NCMS have been demonstrated.

2. **Preliminaries**

Definition 2.1. Think about a non-empty set that presummably serves as a common fixed point of mappings $\mathfrak{G} : \daleth \times \daleth \to \daleth$ and $\mathfrak{F} : \daleth \to \daleth$ if $\varrho = \mathfrak{F}(\varrho) = \mathfrak{G}(\varrho, \varrho)$.

Definition 2.2. If the mappings $\mathfrak{G} : \daleth \times \daleth \to \daleth$ and $\mathfrak{F} : \daleth \to \daleth$ in a nonempty set are considered to be commutative, then $\mathfrak{F}(\mathfrak{G}(\varrho, \varsigma)) = \mathfrak{G}(\mathfrak{F}(\varrho), \mathfrak{F}(\varsigma))$ for all $\varrho, \varsigma \in \daleth$.

Definition 2.3. Consider a non-empty set $\daleth$ and $\mathfrak{G}$, $\mathfrak{F}$ self-maps of $\daleth$. ϱ serves as a coincidence point of $\mathfrak{G}$ and $\mathfrak{F}$ if and only if $\mathfrak{G}(\varrho) = \mathfrak{F}(\varrho)$ where $\varrho \in \daleth$. Then $\mathfrak{w} = \mathfrak{G}(\varrho) = \mathfrak{F}(\varrho)$ is referred to as a point of coincidence of $\mathfrak{G}$ and $\mathfrak{F}$.

Definition 2.4. Let $\mathfrak{G}$ and $\mathfrak{F}$ represents two set's self- maps of a set $\daleth$. $\mathfrak{G}$ and $\mathfrak{F}$ are referred to be occasionally weakly compatible if and only if a point is made $\varrho \in \daleth$ which is an instance of coincidence point of $\mathfrak{G}$ and $\mathfrak{F}$, where $\mathfrak{G}$ and $\mathfrak{F}$ commute.

Definition 2.5. Consider a cone metric space $(\urcorner, d)$ Next, for each $\mathfrak{n}_1 \gg 0$ and $\mathfrak{n}_2 \gg 0$, $\mathfrak{n}_1, \mathfrak{n}_2 \in \mathfrak{E}$, a thing exists $\mathfrak{n} \gg 0$, $\mathfrak{n} \in \mathfrak{E}$ like that $\mathfrak{n} \ll \mathfrak{n}_1$ and $\mathfrak{n} \ll \mathfrak{n}_2$.

Lemma 2.6. *Assume that $\urcorner$ is a collection of $\mathfrak{G}, \mathfrak{F}$'s occasionally weakly compatible self maps. If $\mathfrak{G}$ and $\mathfrak{F}$ share a unique fixed point $\mathfrak{w}$, then $\mathfrak{G}$ and $\mathfrak{F}$ share a special coincidence $\mathfrak{w} = \mathfrak{G}(\varrho) = \mathfrak{F}(\varrho)$.*

Definition 2.7. Let $* : [0,1] \times [0,1] \to [0,1]$ where it meets the following requirements be a continuous t-norm [CTN]:

(i) $*$ is commutative and associative,
(ii) $*$ is continuous,
(iii) $\varepsilon_1 * 1 = \varepsilon_1$ for all $\varepsilon_1 \in [0,1]$,
(iv) $\varepsilon_1 * \varepsilon_2 \leq \varepsilon_3 * \varepsilon_4$ whenever $\varepsilon_1 \leq \varepsilon_3$ and $\varepsilon_2 \leq \varepsilon_4$, for every $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0,1]$.

Definition 2.8. Let $\diamond : [0,1] \times [0,1] \to [0,1]$ where it meets the following requirements be a continuous t-conorm [CTC]:

(i) $\diamond$ is commutative and associative,
(ii) $\diamond$ is continuous,
(iii) $\varepsilon_1 \diamond 0 = \varepsilon_1$ for all $\varepsilon_1 \in [0,1]$,
(iv) $\varepsilon_1 \diamond \varepsilon_2 \leq \varepsilon_3 \diamond \varepsilon_4$ whenever $\varepsilon_1 \leq \varepsilon_3$ and $\varepsilon_2 \leq \varepsilon_4$, for each $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0,1]$.

3. Neutrosophic Cone Metric Spaces

Definition 3.1. It is claimed that a 6-tuple $(\urcorner, \Xi, \Theta, \Upsilon, *, \diamond)$ is a Neutrosophic Cone Metric Space, a cone of E is P, $\urcorner$ can be any non empty set, $*$ be a neutrosophic CTN, $\diamond$ be a neutrosophic CTC and Ξ, Θ and Υ are neutrosophic sets on $\urcorner^2 \times int(\mathfrak{P})$ where it meets the criteria listed below:
for every $\varrho, \varsigma, \delta, \omega \in \urcorner, \alpha, \mu \in int(\mathfrak{P})$.

(i) $0 \leq \Xi(\varrho, \varsigma, \alpha) \leq 1;\ 0 \leq \Theta(\varrho, \varsigma, \alpha) \leq 1;\ 0 \leq \Upsilon(\varrho, \varsigma, \alpha) \leq 1;$
(ii) $\Xi(\varrho, \varsigma, \alpha) + \Theta(\varrho, \varsigma, \alpha) + \Upsilon(\varrho, \varsigma, \alpha) \leq 3;$
(iii) $\Xi(\varrho, \varsigma, \alpha) > 0;$
(iv) $\Xi(\varrho, \varsigma, \alpha) = 1$ if and only if $\varrho = \varsigma$;
(v) $\Xi(\varrho, \varsigma, \alpha) = \Xi(\varsigma, \varrho, \alpha);$
(vi) $\Xi(\varrho, \varsigma, \alpha) * \Xi(\varsigma, \delta, \mu) \leq \Xi(\varrho, \delta, \alpha + \mu)$, for all $\alpha, \mu > 0$;
(vii) $\Xi(\varrho, \varsigma, \cdot) : int(\mathfrak{P}) \to (0,1]$ is neutrosophic continuous;
(viii) $\lim_{\alpha \to \infty} \Xi(\varrho, \varsigma, \alpha) = 1$ for all $\alpha > 0$;
(ix) $\Theta(\varrho, \varsigma, \alpha) < 1;$
(x) $\Theta(\varrho, \varsigma, \alpha) = 0$ if and only if $\varrho = \varsigma$;
(xi) $\Theta(\varrho, \varsigma, \alpha) = \Theta(\varsigma, \varrho, \alpha);$

(xii) $\Theta(\varrho,\varsigma,\alpha) \diamond \Theta(\varsigma,\delta,\mu) \geq \Theta(\varrho,\delta,\alpha+\mu)$, for all $\alpha,\mu > 0$;

(xiii) $\Theta(\varrho,\varsigma,\cdot) : int(\mathfrak{P}) \to (0,1]$ is neutrosophic continuous;

(xiv) $\lim_{\alpha\to\infty} \Theta(\varrho,\varsigma,\alpha) = 0$ for all $\alpha > 0$;

(xv) $\Upsilon(\varrho,\varsigma,\alpha) < 1$;

(xvi) $\Upsilon(\varrho,\varsigma,\alpha) = 0$ if and only if $\varrho = \varsigma$;

(xvii) $\Upsilon(\varrho,\varsigma,\alpha) = \Upsilon(\varsigma,\varrho,\alpha)$;

(xviii) $\Upsilon(\varrho,\varsigma,\alpha) \diamond \Upsilon(\varsigma,\delta,\mu) \geq \Upsilon(\varrho,\delta,\alpha+\mu)$, for all $\alpha,\mu > 0$;

(xix) $\Upsilon(\varrho,\varsigma,\cdot) : int(\mathfrak{P}) \to (0,1]$ is neutrosophic continuous;

(xx) $\lim_{\alpha\to\infty} \Upsilon(\varrho,\varsigma,\alpha) = 0$ for all $\alpha > 0$;

(xxi) If $\alpha \leq 0$ then $\Xi(\varrho,\varsigma,\alpha) = 0; \Theta(\varrho,\varsigma,\alpha) = 1; \Upsilon(\varrho,\varsigma,\alpha) = 1$.

Then, (Ξ,Θ,Υ) is referred to as a NCMS on $\urcorner$. The mappings Ξ,Θ and Υ represents degree of closedness, neturalness and non-closedness between ϱ and ς in relation to α respectively.

Example 3.2. Consider a metric space $(\urcorner, d)$. Let $\mathfrak{E} = \mathfrak{R}$ and $\mathfrak{P} = [0,\infty)$. Define $\omega * \sigma = \min\{\omega,\sigma\}$ and $\omega \diamond \sigma = \max\{\omega,\sigma\}$, then every neutrosophic metric spaces became an NCMS.

Example 3.3. $\mathfrak{P}$ could be an any cone, $\urcorner = N$. Define $\omega * \sigma = \min\{\omega,\sigma\}$ and $\omega \diamond \sigma = \max\{\omega,\sigma\}$, $\Xi,\Theta,\Upsilon : \urcorner^2 \times int(\mathfrak{P}) \to [0,1]$ defined by

$$\Xi(\varrho,\varsigma,\alpha) = \begin{cases} \frac{\varrho}{\varsigma}, & \text{if } \varrho \leq \varsigma \\ \frac{\varsigma}{\varrho}, & \text{if } \varsigma \leq \varrho \end{cases} \quad \Theta(\varrho,\varsigma,\alpha) = \begin{cases} \frac{\varsigma-\varrho}{\varsigma}, & \text{if } \varrho \leq \varsigma \\ \frac{\varrho-\varsigma}{\varrho}, & \text{if } \varsigma \leq \varrho \end{cases} \quad \Upsilon(\varrho,\varsigma,\alpha) = \begin{cases} \frac{\varsigma-\varrho}{\varrho}, & \text{if } \varrho \leq \varsigma \\ \frac{\varrho-\varsigma}{\varsigma}, & \text{if } \varsigma \leq \varrho \end{cases}$$

for all $\varrho,\varsigma \in \urcorner$ and $\alpha \gg 0$. Then $(\urcorner,\Xi,\Theta,\Upsilon,*,\diamond)$ be a NCMS.

Example 3.4. Assume $\mathfrak{E} = \mathfrak{R}^2$. Then $\mathfrak{P} = \{(\rho_1,\rho_2) : \rho_1,\rho_2 \geq 0\} \subset \mathfrak{E}$ with normal constant $\mathfrak{P} = 1$, let $\mathfrak{P}$ be a normal cone assume $\urcorner = \mathfrak{R}$, $\omega * \sigma = \min\{\omega,\sigma\}$, $\omega \diamond \sigma = \max\{\omega,\sigma\}$ and $\Xi,\Theta,\Upsilon : \urcorner^2 \times int(\mathfrak{P}) \to [0,1]$ defined by $\Xi(\varrho,\varsigma,\alpha) = \frac{1}{e^{\frac{|\varrho-\varsigma|}{\|\alpha\|}}}, \Theta(\varrho,\varsigma,\alpha) = \frac{e^{\frac{|\varrho-\varsigma|}{\|\alpha\|}}-1}{e^{\frac{|\varrho-\varsigma|}{\|\alpha\|}}}$ and $\Upsilon(\varrho,\varsigma,\alpha) = e^{\frac{|\varrho-\varsigma|}{\|\alpha\|}} - 1$, for each $\varrho,\varsigma \in \urcorner$ and $\alpha \gg 0$. Then $(\urcorner,\Xi,\Theta,\Upsilon,*,\diamond)$ is a NCMS.

Definition 3.5. Consider a NCMS $(\urcorner,\Xi,\Theta,\Upsilon,*,\diamond)$. For $\alpha \gg 0$, the open ball $\mathcal{O}(\varrho,\mathfrak{r},\alpha)$ where ϱ is its center and $\mathfrak{r} \in (0,1)$ is its radius as $\mathcal{O}(\varrho,\mathfrak{r},\alpha) = \{\varsigma \in \urcorner : \Xi(\varrho,\varsigma,\alpha) > 1-\mathfrak{r}, \Theta(\varrho,\varsigma,\alpha) < \mathfrak{r}$ and $\Upsilon(\varrho,\varsigma,\alpha) < \mathfrak{r}\}$.

Definition 3.6. Consider a NCMS $(\urcorner,\Xi,\Theta,\Upsilon,*,\diamond)$. Let $\varrho \in \urcorner$ and $\{\varrho_n\}$ be a sequence in $\urcorner$. Then $\{\varrho_n\}$ suppose converges to ϱ if for any $\alpha \gg 0$ and $\mathfrak{r} \in (0,1)$ are present, a natural integer n_0 exists such that $\Xi(\varrho_n,\varrho,\alpha) > 1-\mathfrak{r}, \Theta(\varrho_n,\varrho,\alpha) < \mathfrak{r}$ and $\Upsilon(\varrho_n,\varrho,\alpha) < \mathfrak{r}$ for all $n > n_0$. Then $\lim_{n\to\infty} \varrho_n = \varrho$ or $\varrho_n \to \varrho$ as $n \to \infty$.

Definition 3.7. Consider a NCMS $(\urcorner,\Xi,\Theta,\Upsilon,*,\diamond)$. Let $\varrho \in \urcorner$ and $\{\varrho_n\}$ be a sequence in $\urcorner$. If for any $0 < \varepsilon < 1$ and any $\alpha \gg 0$ a natural number n_0 like that exists and $\Xi(\varrho_n,\varrho_m,\alpha) >$

$1-\varepsilon, \Theta(\varrho_n, \varrho_m, \alpha) < \varepsilon$ and $\Upsilon(\varrho_n, \varrho_m, \alpha) < \varepsilon$ for each $n, m > n_0$ then $\{\varrho_n\}$ referred to as a Cauchy sequence.

Definition 3.8. Consider a NCMS $(\daleth, \Xi, \Theta, \Upsilon, *, \diamond)$. One calls $\daleth$ complete if each and every Cauchy sequence converges.

Definition 3.9. Consider a NCMS $(\daleth, \Xi, \Theta, \Upsilon, *, \diamond)$. We refer to a subset Φ of $\daleth$ as $\mathfrak{FC}$ - bounded assuming if $\alpha \gg 0$ and $\mathfrak{r} \in (0,1)$ are present like that $\Xi(\varrho, \varsigma, \alpha) > 1-\mathfrak{r}, \Theta(\varrho, \varsigma, \alpha) < \mathfrak{r}$ and $\Upsilon(\varrho, \varsigma, \alpha) < \mathfrak{r}$ for each $\varrho, \varsigma \in \Phi$.

Definition 3.10. Consider a NCMS $(\daleth, \Xi, \Theta, \Upsilon, *, \diamond)$ and $\mathfrak{h} : \daleth \to \daleth$ be a self mapping. Then neutrosophic cone contractive is the name given to $\mathfrak{h}$ and assuming there is $\rho \in (0,1)$ like that $\dfrac{1}{\Xi(\mathfrak{h}(\varrho), \mathfrak{h}(\varsigma), \alpha)} - 1 \leq \rho\left(\dfrac{1}{\Xi(\varrho, \varsigma, \alpha)} - 1\right), \Theta(\mathfrak{h}(\varrho), \mathfrak{h}(\varsigma), \alpha) \leq \rho\Theta(\varrho, \varsigma, \alpha)$ and $\Upsilon(\mathfrak{h}(\varrho), \mathfrak{h}(\varsigma), \alpha) \leq \rho\Upsilon(\varrho, \varsigma, \alpha)$ for each $\varrho, \varsigma \in \daleth$ and $\alpha \gg 0$. ρ is referred to as the $\mathfrak{h}$ contractive constant.

Lemma 3.11. *Consider any two points $\varrho, \varsigma \in \daleth$ and $\rho \in (0,1)$ such that $\Xi(\varrho, \varsigma, \rho\alpha) \geq \Xi(\varrho, \varsigma, \alpha), \Theta(\varrho, \varsigma, \rho\alpha) \leq \Theta(\varrho, \varsigma, \alpha)$ and $\Upsilon(\varrho, \varsigma, \rho\alpha) \leq \Upsilon(\varrho, \varsigma, \alpha)$. Then $\varrho = \varsigma$.*

Theorem 3.12. *Consider a NCMS $(\daleth, \Xi, \Theta, \Upsilon, *, \diamond)$. Define $\tau = \{\Phi \subseteq \daleth : \varrho \in \Phi$ iff a thing exists $\mathfrak{r} \in (0,1)$ and $\alpha \gg 0$ like that $\mathcal{O}(\varrho, \mathfrak{r}, \alpha) \subset \Phi\}$, which is a topology on $\daleth$.*

Proof. Let $\varrho \in \phi$. Hence $\phi = \mathcal{O}(\varrho, \mathfrak{r}, \alpha) \subset \phi$ and $\phi \in \tau$.
Since for any $\varrho \in \daleth$, and $\mathfrak{r} \in (0,1)$, $\alpha \gg 0$ are present then $\mathcal{O}(\varrho, \mathfrak{r}, \alpha) \subset \daleth$, then $\daleth \in \tau$. Let $\Phi, \mathcal{O} \in \tau$ and $\varrho \in \Phi \cap \mathcal{O}$, then $\varrho \in \Phi$ and $\varrho \in \mathcal{O}$ so a thing exists $\alpha_1 \gg 0, \alpha_2 \gg 0$ and $\mathfrak{r}_1, \mathfrak{r}_2 \in (0,1)$ such that $\mathcal{O}(\varrho, \mathfrak{r}_1, \alpha_1) \subset \Phi$ and $\mathcal{O}(\varrho, \mathfrak{r}_2, \alpha_2) \subset \mathcal{O}$.
From Definition (2.5), for $\alpha_1 \gg 0, \alpha_2 \gg 0$, a thing exists $\alpha \gg 0$ such that $\alpha \gg \alpha_1, \alpha \gg \alpha_2$ and take $\mathfrak{r} = \min\{\mathfrak{r}_1, \mathfrak{r}_2\}$. Then $\mathcal{O}(\varrho, \mathfrak{r}, \alpha) \subset \mathcal{O}(\varrho, \mathfrak{r}_1, \alpha_1) \cap \mathcal{O}(\varrho, \mathfrak{r}_2, \alpha_2) \subset \Phi \cap \mathcal{O}$. Hence $\Phi \cap \mathcal{O} \in \tau$. Let $\Phi_j \in \mathfrak{r}$ for every $j \in I$ and $\varrho \in \cup_{j\in I}\Phi_j$. Afterwards, there is $i_0 \in I$ similar to $\varrho \in \Phi_{j_0}$. So, there is $\alpha \gg 0$ and $\mathfrak{r} \in (0,1)$ like that $\mathcal{O}(\varrho, \mathfrak{r}, \alpha) \subset \Phi_{j_0}$. Since $\Phi_{j_0} \subset \cup_{j\in I}\Phi_j, \mathcal{O}(\varrho, \mathfrak{r}, \alpha) \subset \cup_{j\in I}\Phi_j$. Thus $\cup_{j\in I}\Phi_j \in \tau$. Hence, τ is therefore a topology on $\daleth$.

Theorem 3.13. *Consider a NCMS $(\daleth, \Xi, \Theta, \Upsilon, *, \diamond)$. Then $(\daleth, \tau)$ is Hausdorff.*

Proof. Consider a NCMS $(\daleth, \Xi, \Theta, \Upsilon, *, \diamond)$. Let ϱ, ς be the two separate points of $\daleth$. Then $0 < \Xi(\varrho, \varsigma, \alpha) < 1, 0 < \Theta(\varrho, \varsigma, \alpha) < 1$ and $0 < \Upsilon(\varrho, \varsigma, \alpha) < 1$. Assume $\Xi(\varrho, \varsigma, \alpha) = \mathfrak{r}_1$, $\Theta(\varrho, \varsigma, \alpha) = \mathfrak{r}_2$ and $\Upsilon(\varrho, \varsigma, \alpha) = \mathfrak{r}_3$ and $\mathfrak{r} = \max\{\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3\}$. Then for each $\mathfrak{r}_0 \in (\mathfrak{r}, 1)$, there is $\mathfrak{r}_4$, $\mathfrak{r}_5$ and $\mathfrak{r}_6$ such that $\mathfrak{r}_4 * \mathfrak{r}_4 \geq \mathfrak{r}_0$, $(1-\mathfrak{r}_5) \diamond (1-\mathfrak{r}_5) \leq (1-\mathfrak{r}_0)$ and $(1-\mathfrak{r}_6) \diamond (1-\mathfrak{r}_6) \leq (1-\mathfrak{r}_0)$. Assume $\mathfrak{r}_7 = \max\{\mathfrak{r}_4, \mathfrak{r}_5, \mathfrak{r}_6\}$. Think about open balls $\mathcal{O}(\varrho, 1-\mathfrak{r}_7, \frac{\alpha}{2})$ and $\mathcal{O}(\varsigma, 1-\mathfrak{r}_7, \frac{\alpha}{2})$. Then obviously $\mathcal{O}(\varrho, 1-\mathfrak{r}_7, \frac{\alpha}{2}) \cap \mathcal{O}(\varsigma, 1-\mathfrak{r}_7, \frac{\alpha}{2}) = \emptyset$. Assume that $\mathcal{O}(\varrho, 1-\mathfrak{r}_7, \frac{\alpha}{2}) \cap \mathcal{O}(\varsigma, 1-\mathfrak{r}_7, \frac{\alpha}{2}) \neq \emptyset$.

Then there is $\nu \in \mathcal{O}(\varrho, 1-\mathfrak{r}_7, \frac{\alpha}{2}) \cap \mathcal{O}(\varsigma, 1-\mathfrak{r}_7, \frac{\alpha}{2})$.

$$\begin{aligned}
\mathfrak{r}_1 &= \Xi(\varrho,\varsigma,\alpha)\\
&\geq \Xi(\varrho,\nu,\frac{\alpha}{2}) * \Xi(\nu,\varsigma,\frac{\alpha}{2}) \geq \mathfrak{r}_7 * \mathfrak{r}_7 \geq \mathfrak{r}_4 * \mathfrak{r}_4 \geq \mathfrak{r}_0 > \mathfrak{r}_1,\\
\mathfrak{r}_2 &= \Theta(\varrho,\varsigma,\alpha)\\
&\leq \Theta(\varrho,\nu,\frac{\alpha}{2}) \diamond \Theta(\nu,\varsigma,\frac{\alpha}{2}) \leq (1-\mathfrak{r}_7) \diamond (1-\mathfrak{r}_7) \leq (1-\mathfrak{r}_5) \diamond (1-\mathfrak{r}_5) \leq (1-\mathfrak{r}_0) < \mathfrak{r}_2 \text{ and}\\
\mathfrak{r}_3 &= \Upsilon(\varrho,\varsigma,\alpha)\\
&\leq \Upsilon(\varrho,\nu,\frac{\alpha}{2}) \diamond \Upsilon(\nu,\varsigma,\frac{\alpha}{2}) \leq (1-\mathfrak{r}_7) \diamond (1-\mathfrak{r}_7) \leq (1-\mathfrak{r}_6) \diamond (1-\mathfrak{r}_6) \leq (1-\mathfrak{r}_0) < \mathfrak{r}_3,
\end{aligned}$$

which contradicts itself. Hence, $(\daleth, \Xi, \Theta, \Upsilon, *, \diamond)$ is Hausdorff.

Theorem 3.14. *Consider a NCMS $(\daleth, \Xi, \Theta, \Upsilon, *, \diamond)$, $\varrho \in \daleth$ and (ϱ_n) be an $\daleth$ sequence. Then (ϱ_n) converges to ϱ if, then, just $\Xi(\varrho_n, \varrho, \alpha) \to 1, \Theta(\varrho_n, \varrho, \alpha) \to 0$ and $\Upsilon(\varrho_n, \varrho, \alpha) \to 0$ as $n \to \infty$, for every $\alpha \gg 0$.*

Proof. Assume that $(\varrho_n) \to \varrho$. Then, for each $\alpha \gg 0$ and $\mathfrak{r} \in (0,1)$, there is a natural number n_0 such that $\Xi(\varrho_n, \varrho, \alpha) > 1-\mathfrak{r}, \Theta(\varrho_n, \varrho, \alpha) < \mathfrak{r}$ and $\Upsilon(\varrho_n, \varrho, \alpha) < \mathfrak{r}$, for all $n \gg n_0$. We have $1-\Xi(\varrho_n, \varrho, \alpha) < \mathfrak{r}, \Theta(\varrho_n, \varrho, \alpha) < \mathfrak{r}$ and $\Upsilon(\varrho_n, \varrho, \alpha) < \mathfrak{r}$. Hence $\Xi(\varrho_n, \varrho, \alpha) \to 1, \Theta(\varrho_n, \varrho, \alpha) \to 0$ and $\Upsilon(\varrho_n, \varrho, \alpha) \to 0$ as $n \to \infty$.

However, suppose that $\Xi(\varrho_n, \varrho, \alpha) \to 1$ as $n \to \infty$. Then, there exists a natural integer n_0 such that for each $\alpha \gg 0$ and $\mathfrak{r} \in (0,1)$, $1-\Xi(\varrho_n, \varrho, \alpha) < \mathfrak{r}, \Theta(\varrho_n, \varrho, \alpha) < \mathfrak{r}$ and $\Upsilon(\varrho_n, \varrho, \alpha) < \mathfrak{r}$ for each $n \geq n_0$. Hence, $\Xi(\varrho_n, \varrho, \alpha) > 1-\mathfrak{r}, \Theta(\varrho_n, \varrho, \alpha) < \mathfrak{r}$ and $\Upsilon(\varrho_n, \varrho, \alpha) < \mathfrak{r}$ for each $n \geq n_0$. Hence $\varrho_n \to \varrho$ as $n \to \infty$.

4. Main Results

Theorem 4.1. *Consider a complete NCMS $(\daleth, \Xi, \Theta, \Upsilon, *, \diamond)$ in which neutrosophic cone contractive sequences are Cauchy. Let $\mathfrak{F} : \daleth \to \daleth$ be a neutrosophic cone contractive mapping, the contractice constant is ρ. Then $\mathfrak{F}$ has a distinct fixed point.*

Proof. Consider $\varrho \in \daleth$ and let $\varrho_n = \mathfrak{F}^n(\varrho), n \in \mathbb{N}$. For $\alpha \gg 0$, we have

$$\begin{gathered}
\frac{1}{\Xi(\mathfrak{F}(\varrho), \mathfrak{F}^2(\varrho), \alpha)} - 1 \leq \rho\left(\frac{1}{\Xi(\varrho, \varrho_1, \alpha)} - 1\right),\\
\Theta(\mathfrak{F}(\varrho), \mathfrak{F}^2(\varrho), \alpha) \leq \rho\Theta(\varrho, \varrho_1, \alpha)\\
\Upsilon(\mathfrak{F}(\varrho), \mathfrak{F}^2(\varrho), \alpha) \leq \rho\Upsilon(\varrho, \varrho_1, \alpha)
\end{gathered}$$

and by induction

$$\frac{1}{\Xi(\varrho_{n+1},\varrho_{n+2},\alpha)}-1\leq\rho\left(\frac{1}{\Xi(\varrho_n,\varrho_{n+1},\alpha)}-1\right),$$
$$\Theta(\varrho_{n+1},\varrho_{n+2},\alpha)\leq\rho\Theta(\varrho_n,\varrho_{n+1},\alpha)$$
$$\Upsilon(\varrho_{n+1},\varrho_{n+2},\alpha)\leq\rho\Upsilon(\varrho_n,\varrho_{n+1},\alpha),\ \text{for all } n\in\mathbb{N}$$

Then (ϱ_n) is a neutrosophic contractive Cauchy sequence which converges to ς where $\varsigma\in\daleth$. Theorem (3.14), gives us

$$\frac{1}{\Xi(\mathfrak{F}(\varsigma),\mathfrak{F}(\varrho_n),\alpha)}-1\leq\rho\left(\frac{1}{\Xi(\varsigma,\varrho_n,\alpha)}-1\right)\to 1,$$
$$\Theta(\mathfrak{F}(\varsigma),\mathfrak{F}(\varrho_n),\alpha)\leq\rho\Theta(\varsigma,\varrho_n,\alpha)\to 0 \text{ and}$$
$$\Upsilon(\mathfrak{F}(\varsigma),\mathfrak{F}(\varrho_n),\alpha)\leq\rho\Upsilon(\varsigma,\varrho_n,\alpha)\to 0 \text{ as } n\to\infty.$$

Then for every $\alpha\gg 0$,
$\lim_{n\to\infty}\Xi(\mathfrak{F}(\varsigma),\mathfrak{F}(\varrho_n),\alpha)=1$, $\lim_{n\to\infty}\Theta(\mathfrak{F}(\varsigma),\mathfrak{F}(\varrho_n),\alpha)=0$ and $\lim_{n\to\infty}\Upsilon(\mathfrak{F}(\varsigma),\mathfrak{F}(\varrho_n),\alpha)=0$ and hence $\lim_{n\to\infty}\mathfrak{F}(\varrho_n)=\mathfrak{F}(\varsigma)$.
Now, we prove uniqueness. Assume $\mathfrak{F}(\nu)=\nu$ for some $\nu\in\mathcal{V}$. For $\alpha\gg 0$, we have

$$\begin{aligned}\frac{1}{\Xi(\varsigma,\nu,\alpha)}-1&=\frac{1}{\Xi(\mathfrak{F}(\varsigma),\mathfrak{F}(\nu),\alpha)}-1\leq\rho\left(\frac{1}{\Xi(\varsigma,\nu,\alpha)}-1\right)\\&=\rho\left(\frac{1}{\Xi(\mathfrak{F}(\varsigma),\mathfrak{F}(\nu),\alpha)}-1\right)\leq\rho^2\left(\frac{1}{\Xi(\varsigma,\nu,\alpha)}-1\right)\\&\leq\cdots\leq\rho^n\left(\frac{1}{\Xi(\varsigma,\nu,\alpha)}-1\right)\to 1 \text{ as } n\to\infty,\\ \Theta(\varsigma,\nu,\alpha)&=\Theta(\mathfrak{F}(\varsigma),\mathfrak{F}(\nu),\alpha)\leq\rho\Theta(\varsigma,\nu,\alpha)\\&=\rho\Theta(\mathfrak{F}(\varsigma),\mathfrak{F}(\nu),\alpha)\leq\rho^2\Theta(\varsigma,\nu,\alpha)\\&\leq\cdots\leq\rho^n\Theta(\varsigma,\nu,\alpha)\to 0 \text{ as } n\to\infty,\\ \Upsilon(\varsigma,\nu,\alpha)&=\Upsilon(\mathfrak{F}(\varsigma),\mathfrak{F}(\nu),\alpha)\leq\rho\Upsilon(\varsigma,\nu,\alpha)\\&=\rho\Upsilon(\mathfrak{F}(\varsigma),\mathfrak{F}(\nu),\alpha)\leq\rho^2\Upsilon(\varsigma,\nu,\alpha)\\&\leq\cdots\leq\rho^n\Upsilon(\varsigma,\nu,\alpha)\to 0 \text{ as } n\to\infty,\end{aligned}$$

Hence $\Xi(\varsigma,\nu,\alpha)=1,\Theta(\varsigma,\nu,\alpha)=0$ and $\Upsilon(\varsigma,\nu,\alpha)=0$ and $\varsigma=\nu$.

Theorem 4.2. *Consider a complete NCMS* $(\daleth,\Xi,\Theta,\Upsilon,*,\diamond)$*, and let* $\mathcal{P},\mathcal{R},\mathcal{Q}$ *and* $\mathcal{S}$ *be self-mappings of* $\daleth$*. Let* $\{\mathcal{P},\mathcal{Q}\}$ *and* $\{\mathcal{R},\mathcal{S}\}$ *be Occasionally Weakly Compatible (OWC) pairings.*

Assuming there is a $\rho \in (0,1)$ *such that*

$$\Xi(\mathcal{P}(\varrho),\mathcal{R}(\varsigma),\rho(\alpha)) \geq \min\left\{\begin{array}{ll}\Xi(\mathcal{Q}(\varrho),\mathcal{S}(\varsigma),\alpha), & \Xi(\mathcal{Q}(\varrho),\mathcal{P}(\varrho),\alpha),\\ \Xi(\mathcal{R}(\varsigma),\mathcal{S}(\varsigma),\alpha), & \Xi(\mathcal{P}(\varrho),\mathcal{S}(\varsigma),\alpha),\\ \Xi(\mathcal{R}(\varsigma),\mathcal{Q}(\varrho),\alpha) & \end{array}\right\},$$

$$\Theta(\mathcal{P}(\varrho),\mathcal{R}(\varsigma),\rho(\alpha)) \leq \max\left\{\begin{array}{ll}\Theta(\mathcal{Q}(\varrho),\mathcal{S}(\varsigma),\alpha), & \Theta(\mathcal{Q}(\varrho),\mathcal{P}(\varrho),\alpha),\\ \Theta(\mathcal{R}(\varsigma),\mathcal{S}(\varsigma),\alpha), & \Theta(\mathcal{P}(\varrho),\mathcal{S}(\varsigma),\alpha),\\ \Theta(\mathcal{R}(\varsigma),\mathcal{Q}(\varrho),\alpha) & \end{array}\right\},$$

$$\Upsilon(\mathcal{P}(\varrho),\mathcal{R}(\varsigma),\rho(\alpha)) \leq \max\left\{\begin{array}{ll}\Upsilon(\mathcal{Q}(\varrho),\mathcal{S}(\varsigma),\alpha), & \Upsilon(\mathcal{Q}(\varrho),\mathcal{P}(\varrho),\alpha),\\ \Upsilon(\mathcal{R}(\varsigma),\mathcal{S}(\varsigma),\alpha), & \Upsilon(\mathcal{P}(\varrho),\mathcal{S}(\varsigma),\alpha),\\ \Upsilon(\mathcal{R}(\varsigma),\mathcal{Q}(\varrho),\alpha) & \end{array}\right\} \quad (4.2.1)$$

for each $\varrho,\varsigma \in \daleth$ *and for each* $\alpha \gg 0$*, afterward there is a special point* $\omega \in \daleth$ *like that* $\mathcal{P}(\omega) = \mathcal{Q}(\omega) = \omega$ *and a unique point* $\nu \in \daleth$ *such that* $\mathcal{R}(\nu) = \mathcal{S}(\nu) = \nu$*. Moreover* $\nu = \omega$*, hence* $\mathcal{P},\mathcal{R},\mathcal{Q}$ *and* $\mathcal{S}$ *have a singular shared fixed point.*

Proof. Consider $\{\mathcal{P},\mathcal{Q}\}$ and $\{\mathcal{R},\mathcal{S}\}$ which are OWC pairings, consequently points $\varrho,\varsigma \in \daleth$ is such that $\mathcal{P}(\varrho) = \mathcal{Q}(\varrho)$ and $\mathcal{R}(\varsigma) = \mathcal{S}(\varsigma)$. We claim that $\mathcal{P}(\varrho) = \mathcal{R}(\varsigma)$.
By inequality (4.2.1),

$$\begin{aligned}\Xi(\mathcal{P}(\varrho),\mathcal{R}(\varsigma),\rho(\alpha)) &\geq \min\left\{\begin{array}{ll}\Xi(\mathcal{Q}(\varrho),\mathcal{S}(\varsigma),\alpha), & \Xi(\mathcal{Q}(\varrho),\mathcal{P}(\varrho),\alpha),\\ \Xi(\mathcal{R}(\varsigma),\mathcal{S}(\varsigma),\alpha), & \Xi(\mathcal{P}(\varrho),\mathcal{S}(\varsigma),\alpha),\\ \Xi(\mathcal{R}(\varsigma),\mathcal{Q}(\varrho),\alpha) & \end{array}\right\}\\ &= \min\left\{\begin{array}{ll}\Xi(\mathcal{P}(\varrho),\mathcal{R}(\varsigma),\alpha), & \Xi(\mathcal{P}(\varrho),\mathcal{P}(\varrho),\alpha),\\ \Xi(\mathcal{R}(\varsigma),\mathcal{R}(\varsigma),\alpha), & \Xi(\mathcal{P}(\varrho),\mathcal{R}(\varsigma),\alpha),\\ \Xi(\mathcal{R}(\varsigma),\mathcal{P}(\varrho),\alpha) & \end{array}\right\}\\ &= \Xi(\mathcal{P}(\varrho),\mathcal{R}(\varsigma),\alpha),\\ \Theta(\mathcal{P}(\varrho),\mathcal{R}(\varsigma),\rho(\alpha)) &\leq \max\left\{\begin{array}{ll}\Theta(\mathcal{Q}(\varrho),\mathcal{S}(\varsigma),\alpha), & \Theta(\mathcal{Q}(\varrho),\mathcal{P}(\varrho),\alpha),\\ \Theta(\mathcal{R}(\varsigma),\mathcal{S}(\varsigma),\alpha), & \Theta(\mathcal{P}(\varrho),\mathcal{S}(\varsigma),\alpha),\\ \Theta(\mathcal{R}(\varsigma),\mathcal{Q}(\varrho),\alpha) & \end{array}\right\}\\ &= \max\left\{\begin{array}{ll}\Theta(\mathcal{P}(\varrho),\mathcal{R}(\varsigma),\alpha), & \Theta(\mathcal{P}(\varrho),\mathcal{P}(\varrho),\alpha),\\ \Theta(\mathcal{R}(\varsigma),\mathcal{R}(\varsigma),\alpha), & \Theta(\mathcal{P}(\varrho),\mathcal{R}(\varsigma),\alpha),\\ \Theta(\mathcal{R}(\varsigma),\mathcal{P}(\varrho),\alpha) & \end{array}\right\}\\ &= \Theta(\mathcal{P}(\varrho),\mathcal{R}(\varsigma),\alpha)\\ \Upsilon(\mathcal{P}(\varrho),\mathcal{R}(\varsigma),\rho(\alpha)) &\leq \max\left\{\begin{array}{ll}\Upsilon(\mathcal{Q}(\varrho),\mathcal{S}(\varsigma),\alpha), & \Upsilon(\mathcal{Q}(\varrho),\mathcal{P}(\varrho),\alpha),\\ \Upsilon(\mathcal{R}(\varsigma),\mathcal{S}(\varsigma),\alpha), & \Upsilon(\mathcal{P}(\varrho),\mathcal{S}(\varsigma),\alpha),\\ \Upsilon(\mathcal{R}(\varsigma),\mathcal{Q}(\varrho),\alpha) & \end{array}\right\}\end{aligned}$$

$$= \max\left\{\begin{array}{ll} \Upsilon(\mathcal{P}(\varrho),\mathcal{R}(\varsigma),\alpha), & \Upsilon(\mathcal{P}(\varrho),\mathcal{P}(\varrho),\alpha), \\ \Upsilon(\mathcal{R}(\varsigma),\mathcal{R}(\varsigma),\alpha), & \Upsilon(\mathcal{P}(\varrho),\mathcal{R}(\varsigma),\alpha), \\ \Upsilon(\mathcal{R}(\varsigma),\mathcal{P}(\varrho),\alpha) & \end{array}\right\}$$
$$= \Upsilon(\mathcal{P}(\varrho),\mathcal{R}(\varsigma),\alpha)$$

By Lemma (3.11), $\mathcal{P}(\varrho) = \mathcal{R}(\varsigma)$, i.e., $\mathcal{P}(\varrho) = \mathcal{Q}(\varrho) = \mathcal{R}(\varsigma) = \mathcal{S}(\varsigma)$.
Supposing there is a point ν that is different $\mathcal{P}(\nu) = \mathcal{Q}(\nu)$ then by (4.2.1), we have $\mathcal{P}(\nu) = \mathcal{Q}(\nu) = \mathcal{R}(\varsigma) = \mathcal{S}(\varsigma)$, so $\mathcal{P}(\varrho) = \mathcal{P}(\nu)$ and $\omega = \mathcal{P}(\varrho) = \mathcal{Q}(\varrho)$ is the special place where $\mathcal{P}$ and $\mathcal{Q}$ coincide.
By Lemma (3.11), the only fixed point between $\mathcal{P}$ and $\mathcal{Q}$ is ω. Likewise, there is a special point $\nu \in \urcorner$ like that $\nu = \mathcal{R}(\nu) = \mathcal{S}(\nu)$. Assume that $\omega \neq \nu$, we have

$$\begin{aligned}
\Xi(\omega,\nu,\rho(\alpha)) &= \Xi(\mathcal{P}(\omega),\mathcal{R}(\nu),\rho(\alpha)) \\
&\geq \min\left\{\begin{array}{ll} \Xi(\mathcal{Q}(\omega),\mathcal{S}(\nu),\alpha), & \Xi(\mathcal{Q}(\omega),\mathcal{P}(\nu),\alpha), \\ \Xi(\mathcal{R}(\nu),\mathcal{S}(\nu),\alpha), & \Xi(\mathcal{P}(\omega),\mathcal{S}(\nu),\alpha), \\ \Xi(\mathcal{R}(\nu),\mathcal{Q}(\omega),\alpha) & \end{array}\right\} \\
&= \min\Big\{\Xi(\omega,\nu,\alpha),\Xi(\omega,\nu,\alpha),\Xi(\nu,\nu,\alpha),\Xi(\omega,\nu,\alpha),\Xi(\nu,\omega,\alpha)\Big\} \\
&= \Xi(\omega,\nu,\alpha), \\
\Theta(\omega,\nu,\rho(\alpha)) &= \Theta(\mathcal{P}(\omega),\mathcal{R}(\nu),\rho(\alpha)) \\
&\leq \max\left\{\begin{array}{ll} \Theta(\mathcal{Q}(\omega),\mathcal{S}(\nu),\alpha), & \Theta(\mathcal{Q}(\omega),\mathcal{P}(\nu),\alpha), \\ \Theta(\mathcal{R}(\nu),\mathcal{S}(\nu),\alpha), & \Theta(\mathcal{P}(\omega),\mathcal{S}(\nu),\alpha), \\ \Theta(\mathcal{R}(\nu),\mathcal{Q}(\omega),\alpha) & \end{array}\right\} \\
&= \max\Big\{\Theta(\omega,\nu,\alpha),\Theta(\omega,\nu,\alpha),\Theta(\nu,\nu,\alpha),\Theta(\omega,\nu,\alpha),\Theta(\nu,\omega,\alpha)\Big\} \\
&= \Theta(\omega,\nu,\alpha), \\
\Upsilon(\omega,\nu,\rho(\alpha)) &= \Upsilon(\mathcal{P}(\omega),\mathcal{R}(\nu),\rho(\alpha)) \\
&\leq \max\left\{\begin{array}{ll} \Upsilon(\mathcal{Q}(\omega),\mathcal{S}(\nu),\alpha), & \Upsilon(\mathcal{Q}(\omega),\mathcal{P}(\nu),\alpha), \\ \Upsilon(\mathcal{R}(\nu),\mathcal{S}(\nu),\alpha), & \Upsilon(\mathcal{P}(\omega),\mathcal{S}(\nu),\alpha), \\ \Upsilon(\mathcal{R}(\nu),\mathcal{Q}(\omega),\alpha) & \end{array}\right\} \\
&= \max\Big\{\Upsilon(\omega,\nu,\alpha),\Upsilon(\omega,\nu,\alpha),\Upsilon(\nu,\nu,\alpha),\Upsilon(\omega,\nu,\alpha),\Upsilon(\nu,\omega,\alpha)\Big\} \\
&= \Upsilon(\omega,\nu,\alpha).
\end{aligned}$$

Hence, we have $\nu = \omega$ by Lemma (3.11), a common fixed point of $\mathcal{P},\mathcal{R},\mathcal{Q}$ and $\mathcal{S}$ is ν. (4.2.1) states that the fixed point's uniqueness is true.

Theorem 4.3. *Consider a complete NCMS* $(\daleth, \Xi, \Theta, \Upsilon, *, \diamond)$ *and let* $\mathcal{P}, \mathcal{R}, \mathcal{Q}$ *and* $\mathcal{S}$ *be self-mappings of* $\daleth$. *Let* $\{\mathcal{P}, \mathcal{Q}\}$ *and* $\{\mathcal{R}, \mathcal{S}\}$ *be OWC pairings. If* $\rho \in (0,1)$ *exists in a way that*

$$\Xi(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)) \geq \chi\left[\min\left\{\begin{array}{ll} \Xi(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha), & \Xi(\mathcal{Q}(\varrho), \mathcal{P}(\varrho), \alpha), \\ \Xi(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha), & \Xi(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha), \\ \Xi(\mathcal{R}(\varsigma), \mathcal{Q}(\varrho), \alpha) & \end{array}\right\}\right],$$

$$\Theta(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)) \leq \psi\left[\max\left\{\begin{array}{ll} \Theta(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha), & \Theta(\mathcal{Q}(\varrho), \mathcal{P}(\varrho), \alpha), \\ \Theta(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha), & \Theta(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha), \\ \Theta(\mathcal{R}(\varsigma), \mathcal{Q}(\varrho), \alpha) & \end{array}\right\}\right],$$

$$\Upsilon(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)) \leq \phi\left[\max\left\{\begin{array}{ll} \Upsilon(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha), & \Upsilon(\mathcal{Q}(\varrho), \mathcal{P}(\varrho), \alpha), \\ \Upsilon(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha), & \Upsilon(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha), \\ \Upsilon(\mathcal{R}(\varsigma), \mathcal{Q}(\varrho), \alpha) & \end{array}\right\}\right],$$

for each $\varrho, \varsigma \in \daleth$ *and* $\chi, \psi, \phi : [0,1] \to [0,1]$, *such that* $\chi(\alpha) > \alpha, \psi(\alpha) < \alpha, \phi(\alpha) < \alpha$ *for all* $0 \ll \alpha < 1$, *therefore*$\mathcal{P}, \mathcal{R}, \mathcal{Q}$ *and* $\mathcal{S}$ *have a special shared fixed point.*

Proof. Theorem (4.2) leads to the theorem's proof.

Theorem 4.4. *Consider a complete NCMS* $(\daleth, \Xi, \Theta, \Upsilon, *, \diamond)$ *and let* $\mathcal{P}, \mathcal{R}, \mathcal{Q}$ *and* $\mathcal{S}$ *be self-mappings of* $\daleth$. *Let* $\{\mathcal{P}, \mathcal{Q}\}$ *and* $\{\mathcal{R}, \mathcal{S}\}$ *be OWC pairings. If there is a* $\rho \in (0,1)$ *such that*

$$\Xi(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)) \geq \chi\left\{\begin{array}{ll} \Xi(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha), & \Xi(\mathcal{Q}(\varrho), \mathcal{P}(\varrho), \alpha), \\ \Xi(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha), & \Xi(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha), \\ \Xi(\mathcal{R}(\varsigma), \mathcal{Q}(\varrho), \alpha) & \end{array}\right\},$$

$$\Theta(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)) \leq \psi\left\{\begin{array}{ll} \Theta(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha), & \Theta(\mathcal{Q}(\varrho), \mathcal{P}(\varrho), \alpha), \\ \Theta(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha), & \Theta(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha), \\ \Theta(\mathcal{R}(\varsigma), \mathcal{Q}(\varrho), \alpha) & \end{array}\right\},$$

$$\Upsilon(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)) \leq \phi\left\{\begin{array}{ll} \Upsilon(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha), & \Upsilon(\mathcal{Q}(\varrho), \mathcal{P}(\varrho), \alpha), \\ \Upsilon(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha), & \Upsilon(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha), \\ \Upsilon(\mathcal{R}(\varsigma), \mathcal{Q}(\varrho), \alpha) & \end{array}\right\} \quad (4.4.1)$$

for each $\varrho, \varsigma \in \daleth$ *and* $\chi, \psi, \phi : [0,1]^5 \to [0,1]$, *such that* $\chi(\alpha, 1, 1, \alpha, \alpha) > \alpha, \psi(\alpha, 0, 0, \alpha, \alpha) < \alpha, \phi(\alpha, 0, 0, \alpha, \alpha) < \alpha$ *for all* $0 \ll \alpha < 1$, *then* $\mathcal{P}, \mathcal{R}, \mathcal{Q}$ *and* $\mathcal{S}$ *have a special shared fixed point.*

Proof. Consider $\{\mathcal{P}, \mathcal{Q}\}$ and $\{\mathcal{R}, \mathcal{S}\}$ which are OWC pairings, there are points $\varrho, \varsigma \in \daleth$ such that $\mathcal{P}(\varrho) = \mathcal{Q}(\varrho)$ and $\mathcal{R}(\varsigma) = \mathcal{S}(\varsigma)$.

We show that $\mathcal{P}(\varrho) = \mathcal{R}(\varsigma)$. By inequality (4.4.1), we have

$$\begin{aligned}
\Xi(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)) &\geq \chi \left\{ \begin{array}{ll} \Xi(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha), & \Xi(\mathcal{Q}(\varrho), \mathcal{P}(\varrho), \alpha), \\ \Xi(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha), & \Xi(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha), \\ \Xi(\mathcal{R}(\varsigma), \mathcal{Q}(\varrho), \alpha) & \end{array} \right\} \\
&= \chi \left\{ \begin{array}{ll} \Xi(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha), & \Xi(\mathcal{P}(\varrho), \mathcal{P}(\varrho), \alpha), \\ \Xi(\mathcal{R}(\varsigma), \mathcal{R}(\varsigma), \alpha), & \Xi(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha), \\ \Xi(\mathcal{R}(\varsigma), \mathcal{P}(\varrho), \alpha) & \end{array} \right\} \\
&= \chi \Big\{ \Xi(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha), 1, 1, \Xi(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha), \Xi(\mathcal{R}(\varsigma), \mathcal{P}(\varrho), \alpha) \Big\} \\
&\geq \Xi(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha), \\
\Theta(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)) &\leq \psi \left\{ \begin{array}{ll} \Theta(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha), & \Theta(\mathcal{Q}(\varrho), \mathcal{P}(\varrho), \alpha), \\ \Theta(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha), & \Theta(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha), \\ \Theta(\mathcal{R}(\varsigma), \mathcal{Q}(\varrho), \alpha) & \end{array} \right\} \\
&= \psi \left\{ \begin{array}{ll} \Theta(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha), & \Theta(\mathcal{P}(\varrho), \mathcal{P}(\varrho), \alpha), \\ \Theta(\mathcal{R}(\varsigma), \mathcal{R}(\varsigma), \alpha), & \Theta(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha), \\ \Theta(\mathcal{R}(\varsigma), \mathcal{P}(\varrho), \alpha) & \end{array} \right\} \\
&= \psi \Big\{ \Theta(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha), 0, 0, \Theta(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha), \Theta(\mathcal{R}(\varsigma), \mathcal{P}(\varrho), \alpha) \Big\} \\
&\leq \Theta(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha), \\
\Upsilon(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)) &\leq \phi \left\{ \begin{array}{ll} \Upsilon(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha), & \Upsilon(\mathcal{Q}(\varrho), \mathcal{P}(\varrho), \alpha), \\ \Upsilon(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha), & \Upsilon(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha), \\ \Upsilon(\mathcal{R}(\varsigma), \mathcal{Q}(\varrho), \alpha) & \end{array} \right\} \\
&= \phi \left\{ \begin{array}{ll} \Upsilon(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha), & \Upsilon(\mathcal{P}(\varrho), \mathcal{P}(\varrho), \alpha), \\ \Upsilon(\mathcal{R}(\varsigma), \mathcal{R}(\varsigma), \alpha), & \Upsilon(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha), \\ \Upsilon(\mathcal{R}(\varsigma), \mathcal{P}(\varrho), \alpha) & \end{array} \right\} \\
&= \phi \Big\{ \Upsilon(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha), 0, 0, \Upsilon(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha), \Upsilon(\mathcal{R}(\varsigma), \mathcal{P}(\varrho), \alpha) \Big\} \\
&\leq \Upsilon(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha),
\end{aligned}$$

which is a contradiction, hence $\mathcal{P}(\varrho) = \mathcal{R}(\varsigma)$. That is $\mathcal{P}(\varrho) = \mathcal{Q}(\varrho) = \mathcal{R}(\varsigma) = \mathcal{S}(\varsigma)$.
Assume that there is a point ν such that $\mathcal{P}(\nu) = \mathcal{Q}(\nu)$, then by (4.4.1) $\mathcal{P}(\nu) = \mathcal{Q}(\nu) = \mathcal{R}(\varsigma) = \mathcal{S}(\varsigma)$, so $\mathcal{P}(\varrho) = \mathcal{P}(\nu)$ and $\omega = \mathcal{P}(\varrho) = \mathcal{S}(\varrho)$ is the special place where $\mathcal{P}$ and $\mathcal{Q}$ coincide. From Lemma (2.6), ω is the sole fixed point that connects $\mathcal{P}$ and $\mathcal{Q}$. Likewise, there is a speical point $\nu \in \daleth$ like that $\nu = \mathcal{R}(\nu) = \mathcal{S}(\nu)$. Thus a common fixed point between $\mathcal{P}, \mathcal{R}, \mathcal{Q}$ and $\mathcal{S}$ is ν. (4.4.1) states that the fixed point's uniqueness holds.

Theorem 4.5. *Consider a complete NCMS* $(\daleth, \Xi, \Theta, \Upsilon, *, \diamond)$ *and let* $\mathcal{P}, \mathcal{R}, \mathcal{Q}$ *and* $\mathcal{S}$ *be self - mappings of* $\daleth$. *Consider* $\{\mathcal{P}, \mathcal{Q}\}$ *and* $\{\mathcal{R}, \mathcal{S}\}$ *which are OWC pairings. If point* $\rho \in (0, 1)$

exists, then for every $\varrho, \varsigma \in \daleth$ *and* $\alpha \gg 0$ *satisfying*

$$\Xi\big(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)\big) \geq \Xi(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) * \Xi(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) * \Xi(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha) * \Xi(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha)$$
$$\Theta\big(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)\big) \leq \Theta(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) \diamond \Theta(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) \diamond \Theta(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha) \diamond \Theta(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha)$$
$$\Upsilon\big(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)\big) \leq \Upsilon(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) \diamond \Upsilon(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) \diamond \Upsilon(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha) \diamond \Upsilon(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha) \tag{4.5.1}$$

then $\mathcal{P}, \mathcal{R}, \mathcal{Q}$ *and* $\mathcal{S}$ *have a special shared fixed point.*

Proof. Consider $\{\mathcal{P}, \mathcal{Q}\}$ and $\{\mathcal{R}, \mathcal{S}\}$ which are OWC pairings. There are points $\varrho, \varsigma \in \daleth$ such that $\mathcal{P}(\varrho) = \mathcal{Q}(\varrho)$ and $\mathcal{R}(\varsigma) = \mathcal{S}(\varsigma)$. We claim that $\mathcal{P}(\varrho) = \mathcal{R}(\varsigma)$.
By inequality (4.5.1), we have

$$\begin{aligned}
\Xi\big(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)\big) &\geq \Xi(\mathcal{Q}(\varrho), \mathcal{R}(\varsigma), \alpha) * \Xi(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) * \Xi(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha) * \Xi(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha) \\
&= \Xi(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha) * \Xi(\mathcal{P}(\varrho), \mathcal{P}(\varrho), \alpha) * \Xi(\mathcal{R}(\varsigma), \mathcal{R}(\varsigma), \alpha) * \Xi(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha) \\
&\geq \Xi(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha) * 1 * 1 * \Xi(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha) \\
&= \Xi(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha) \\
\Theta\big(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)\big) &\leq \Theta(\mathcal{Q}(\varrho), \mathcal{R}(\varsigma), \alpha) \diamond \Theta(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) \diamond \Theta(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha) \diamond \Theta(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha) \\
&= \Theta(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha) \diamond \Theta(\mathcal{P}(\varrho), \mathcal{P}(\varrho), \alpha) \diamond \Theta(\mathcal{R}(\varsigma), \mathcal{R}(\varsigma), \alpha) \diamond \Theta(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha) \\
&\leq \Theta(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha) \diamond 0 \diamond 0 \diamond \Theta(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha) \\
&\leq \Theta(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha) \\
\Upsilon\big(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)\big) &\leq \Upsilon(\mathcal{Q}(\varrho), \mathcal{R}(\varsigma), \alpha) \diamond \Upsilon(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) \diamond \Upsilon(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha) \diamond \Upsilon(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha) \\
&= \Upsilon(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha) \diamond \Upsilon(\mathcal{P}(\varrho), \mathcal{P}(\varrho), \alpha) \diamond \Upsilon(\mathcal{R}(\varsigma), \mathcal{R}(\varsigma), \alpha) \diamond \Upsilon(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha) \\
&\leq \Upsilon(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha) \diamond 0 \diamond 0 \diamond \Upsilon(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha) \\
&\leq \Upsilon(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \alpha)
\end{aligned}$$

By Lemma (3.11), we have $\mathcal{P}(\varrho) = \mathcal{R}(\varsigma)$ i.e., $\mathcal{P}(\varrho) = \mathcal{Q}(\varrho) = \mathcal{R}(\varsigma) = \mathcal{S}(\varsigma)$. Assume that there is a another point ν such that $\mathcal{P}(\nu) = \mathcal{R}(\nu)$ then by (4.5.1), we have $\mathcal{P}(\nu) = \mathcal{Q}(\nu) = \mathcal{R}(\varsigma) = \mathcal{S}(\varsigma)$, so $\mathcal{P}(\varrho) = \mathcal{P}(\nu)$ and $\omega = \mathcal{P}(\varrho) = \mathcal{Q}(\varrho)$ is the one and only place where $\mathcal{P}$ and $\mathcal{Q}$ coincide.
Similarly, there is a special aspect $\omega \in \daleth$ like that $\omega = \mathcal{R}(\omega) = \mathcal{S}(\omega)$.
The common fixed point between $\mathcal{P}, \mathcal{R}, \mathcal{Q}$ and $\mathcal{S}$ is ω.

Theorem 4.6. *Consider a complete NCMS* $(\daleth, \Xi, \Theta, \Upsilon, *, \diamond)$ *and let* $\mathcal{P}, \mathcal{R}, \mathcal{Q}$ *and* $\mathcal{S}$ *be self-mappings of* $\daleth$. *Let* $\{\mathcal{P}, \mathcal{Q}\}$ *and* $\{\mathcal{R}, \mathcal{S}\}$ *be OWC pairings. If there is a* $\rho \in (0, 1)$ *for each*

$\varrho, \varsigma \in \daleth$ and $\alpha \gg 0$ satisfying

$$\Xi\big(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)\big) \geq \Xi(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) * \Xi(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) * \Xi(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha) * \Xi(\mathcal{R}(\varsigma), \mathcal{Q}(\varrho), 2\alpha) * \Xi(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha)$$

$$\Theta\big(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)\big) \leq \Theta(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) \diamond \Theta(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) \diamond \Theta(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha) \diamond \Theta(\mathcal{R}(\varsigma), \mathcal{Q}(\varrho), 2\alpha) \diamond \Theta(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha)$$

$$\Upsilon\big(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)\big) \leq \Upsilon(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) \diamond \Upsilon(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) \diamond \Upsilon(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha) \diamond \Upsilon(\mathcal{R}(\varsigma), \mathcal{Q}(\varrho), 2\alpha) \diamond \Upsilon(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha)$$

then there is a singular common fixed point between $\mathcal{P}, \mathcal{R}, \mathcal{Q}$ and $\mathcal{S}$.

Proof. We have

$$\begin{aligned}
\Xi\big(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)\big) &\geq \Xi(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) * \Xi(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) * \Xi(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha) * \Xi(\mathcal{R}(\varsigma), \mathcal{Q}(\varrho), 2\alpha) * \\
&\quad \Xi(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha) \\
&\geq \Xi(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) * \Xi(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) * \Xi(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha) * \Xi(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) * \\
&\quad \Xi(\mathcal{S}(\varsigma), \mathcal{R}(\varsigma), \alpha) * \Xi(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha) \\
&\geq \Xi(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) * \Xi(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) * \Xi(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha) * \Xi(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha) \\
\Theta\big(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)\big) &\leq \Theta(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) \diamond \Theta(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) \diamond \Theta(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha) \diamond \Theta(\mathcal{R}(\varsigma), \mathcal{Q}(\varrho), 2\alpha) \diamond \\
&\quad \Theta(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha) \\
&\leq \Theta(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) \diamond \Theta(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) \diamond \Theta(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha) \diamond \Theta(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) \diamond \\
&\quad \Theta(\mathcal{S}(\varsigma), \mathcal{R}(\varsigma), \alpha) \diamond \Theta(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha) \\
&\leq \Theta(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) \diamond \Theta(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) \diamond \Theta(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha) \diamond \Theta(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha) \\
\Upsilon\big(\mathcal{P}(\varrho), \mathcal{R}(\varsigma), \rho(\alpha)\big) &\leq \Upsilon(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) \diamond \Upsilon(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) \diamond \Upsilon(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha) \diamond \Upsilon(\mathcal{R}(\varsigma), \mathcal{Q}(\varrho), 2\alpha) \diamond \\
&\quad \Upsilon(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha) \\
&\leq \Upsilon(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) \diamond \Upsilon(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) \diamond \Upsilon(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha) \diamond \Upsilon(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) \diamond \\
&\quad \Upsilon(\mathcal{S}(\varsigma), \mathcal{R}(\varsigma), \alpha) \diamond \Upsilon(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha) \\
&\leq \Upsilon(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) \diamond \Upsilon(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) \diamond \Upsilon(\mathcal{R}(\varsigma), \mathcal{S}(\varsigma), \alpha) \diamond \Upsilon(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha)
\end{aligned}$$

and hence from Theorem (4.5) there is a shared fixed point for $\mathcal{P}, \mathcal{R}, \mathcal{Q}$ and $\mathcal{S}$.

Corollary 4.7. *Consider a complete NCMS $(\daleth, \Xi, \Theta, \Upsilon, *, \diamond)$. Then continuous self-mappings $\mathcal{Q}$ and $\mathcal{S}$ of $\daleth$ possess a shared fixed point in $\daleth$ if and only if a self - mapping $\mathcal{P}$ of $\daleth$ exists so that that the aforementioned requirements are met.*

(1) $\mathcal{P}\daleth \subset \mathcal{S}\daleth \cap \mathcal{Q}\daleth$

(2) the pairs $\{\mathcal{P}, \mathcal{Q}\}$ and $\{\mathcal{P}, \mathcal{S}\}$ are weakly compatible,

(3) a point has been made $\rho \in (0,1)$ such that for every $\varrho, \varsigma \in \daleth$ and $\alpha \gg 0$,

$$\Xi\big(\mathcal{P}(\varrho), \mathcal{P}(\varsigma), \rho(\alpha)\big) \geq \Xi(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) * \Xi(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) * \Xi(\mathcal{P}(\varsigma), \mathcal{S}(\varsigma), \alpha) * \Xi(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha),$$

$$\Theta\big(\mathcal{P}(\varrho), \mathcal{P}(\varsigma), \rho(\alpha)\big) \leq \Theta(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) \diamond \Theta(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) \diamond \Theta(\mathcal{P}(\varsigma), \mathcal{S}(\varsigma), \alpha) \diamond \Theta(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha),$$

$$\Upsilon\big(\mathcal{P}(\varrho), \mathcal{P}(\varsigma), \rho(\alpha)\big) \leq \Upsilon(\mathcal{Q}(\varrho), \mathcal{S}(\varsigma), \alpha) \diamond \Upsilon(\mathcal{P}(\varrho), \mathcal{Q}(\varrho), \alpha) \diamond \Upsilon(\mathcal{P}(\varsigma), \mathcal{S}(\varsigma), \alpha) \diamond \Upsilon(\mathcal{P}(\varrho), \mathcal{S}(\varsigma), \alpha)$$

Fixed point in common between $\mathcal{P}, \mathcal{Q}$ and $\mathcal{S}$ is distinct.

Proof. Since compatibility also implies OWC , Theorem (4.6) leads to the conclusion.

Theorem 4.8. *Consider a complete NCMS $(\daleth, \Xi, \Theta, \Upsilon, *, \diamond)$ and let $\mathcal{P}$ and $\mathcal{Q}$ be self -mappings of $\daleth$. Let the $\mathcal{P}$ and $\mathcal{Q}$ are OWC . If there is a point $\rho \in (0,1)$ for every $\varrho, \varsigma \in \daleth$ and $\alpha \gg 0$*

$$\Xi\big(\mathcal{Q}(\varrho), \mathcal{Q}(\varsigma), \rho(\alpha)\big) \geq \alpha\Xi(\mathcal{P}(\varrho), \mathcal{P}(\varsigma), \alpha) + \beta \min\Big\{\Xi(\mathcal{P}(\varrho), \mathcal{P}(\varsigma), \alpha), \Xi(\mathcal{Q}(\varrho), \mathcal{P}(\varrho), \alpha), \Xi(\mathcal{Q}(\varsigma), \mathcal{P}(\varsigma), \alpha)\Big\},$$

$$\Theta\big(\mathcal{Q}(\varrho), \mathcal{Q}(\varsigma), \rho(\alpha)\big) \leq \alpha\Theta(\mathcal{P}(\varrho), \mathcal{P}(\varsigma), \alpha) + \beta \max\Big\{\Theta(\mathcal{P}(\varrho), \mathcal{P}(\varsigma), \alpha), \Theta(\mathcal{Q}(\varrho), \mathcal{P}(\varrho), \alpha), \Theta(\mathcal{Q}(\varsigma), \mathcal{P}(\varsigma), \alpha)\Big\},$$

$$\Upsilon\big(\mathcal{Q}(\varrho), \mathcal{Q}(\varsigma), \rho(\alpha)\big) \leq \alpha\Upsilon(\mathcal{P}(\varrho), \mathcal{P}(\varsigma), \alpha) + \beta \max\Big\{\Upsilon(\mathcal{P}(\varrho), \mathcal{P}(\varsigma), \alpha), \Upsilon(\mathcal{Q}(\varrho), \mathcal{P}(\varrho), \alpha), \Upsilon(\mathcal{Q}(\varsigma), \mathcal{P}(\varsigma), \alpha)\Big\} \tag{4.8.1}$$

for all $\varrho, \varsigma \in \daleth$, where $\alpha, \beta > 0, \alpha + \beta > 1$. $\mathcal{P}$ and $\mathcal{Q}$ share a distinct common fixed point.

Proof. Consider $\{\mathcal{P}, \mathcal{Q}\}$ which are OWC pair, so that there is a point $\varrho \in \daleth$ such that $\mathcal{P}(\varrho) = \mathcal{Q}(\varrho)$. Consider the possibiity of another point $\varsigma \in \daleth$ for which $\mathcal{P}(\varsigma) = \mathcal{Q}(\varsigma)$. We claim that $\mathcal{Q}(\varrho) = \mathcal{Q}(\varsigma)$. By inequality (4.8.1), we have

$$\begin{aligned}
\Xi\big(\mathcal{Q}(\varrho), \mathcal{Q}(\varsigma), \rho(\alpha)\big) &\geq \alpha\Xi(\mathcal{P}(\varrho), \mathcal{P}(\varsigma), \alpha) + \beta \min\Big\{\Xi(\mathcal{P}(\varrho), \mathcal{P}(\varsigma), \alpha), \Xi(\mathcal{Q}(\varrho), \mathcal{P}(\varrho), \alpha), \Xi(\mathcal{Q}(\varsigma), \mathcal{P}(\varsigma), \alpha)\Big\} \\
&= \alpha\Xi(\mathcal{Q}(\varrho), \mathcal{Q}(\varsigma), \alpha) + \beta \min\Big\{\Xi(\mathcal{Q}(\varrho), \mathcal{Q}(\varsigma), \alpha), \Xi(\mathcal{Q}(\varrho), \mathcal{Q}(\varrho), \alpha), \Xi(\mathcal{Q}(\varsigma), \mathcal{Q}(\varsigma), \alpha)\Big\} \\
&= (\alpha + \beta)\Xi(\mathcal{Q}(\varrho), \mathcal{Q}(\varsigma), \alpha), \\
\Theta\big(\mathcal{Q}(\varrho), \mathcal{Q}(\varsigma), \rho(\alpha)\big) &\leq \alpha\Theta(\mathcal{P}(\varrho), \mathcal{P}(\varsigma), \alpha) + \beta \max\Big\{\Theta(\mathcal{P}(\varrho), \mathcal{P}(\varsigma), \alpha), \Theta(\mathcal{Q}(\varrho), \mathcal{P}(\varrho), \alpha), \Theta(\mathcal{Q}(\varsigma), \mathcal{P}(\varsigma), \alpha)\Big\} \\
&= \alpha\Theta(\mathcal{Q}(\varrho), \mathcal{Q}(\varsigma), \alpha) + \beta \max\Big\{\Theta(\mathcal{Q}(\varrho), \mathcal{Q}(\varsigma), \alpha), \Theta(\mathcal{Q}(\varrho), \mathcal{Q}(\varrho), \alpha), \Theta(\mathcal{Q}(\varsigma), \mathcal{Q}(\varsigma), \alpha)\Big\} \\
&= (\alpha + \beta)\Theta(\mathcal{Q}(\varrho), \mathcal{Q}(\varsigma), \alpha), \\
\Upsilon\big(\mathcal{Q}(\varrho), \mathcal{Q}(\varsigma), \rho(\alpha)\big) &\leq \alpha\Upsilon(\mathcal{P}(\varrho), \mathcal{P}(\varsigma), \alpha) + \beta \max\Big\{\Upsilon(\mathcal{P}(\varrho), \mathcal{P}(\varsigma), \alpha), \Upsilon(\mathcal{Q}(\varrho), \mathcal{P}(\varrho), \alpha), \Upsilon(\mathcal{Q}(\varsigma), \mathcal{P}(\varsigma), \alpha)\Big\} \\
&= \alpha\Upsilon(\mathcal{Q}(\varrho), \mathcal{Q}(\varsigma), \alpha) + \beta \max\Big\{\Upsilon(\mathcal{Q}(\varrho), \mathcal{Q}(\varsigma), \alpha), \Upsilon(\mathcal{Q}(\varrho), \mathcal{Q}(\varrho), \alpha), \Upsilon(\mathcal{Q}(\varsigma), \mathcal{Q}(\varsigma), \alpha)\Big\} \\
&= (\alpha + \beta)\Upsilon(\mathcal{Q}(\varrho), \mathcal{Q}(\varsigma), \alpha)
\end{aligned}$$

a contradiction, since $(\alpha + \beta) > 1$. Therefore $\mathcal{Q}(\varrho) = \mathcal{Q}(\varsigma)$. Therefore $\mathcal{P}(\varrho) = \mathcal{P}(\varsigma)$ and $\mathcal{P}(\varrho)$ is unique. $\mathcal{P}$ and $\mathcal{Q}$ have a distinct fixed point from lemma (2.6).

Example 4.9. Let $\Sigma = [0,1]$ and let $\Xi(\varrho, \varsigma, \alpha) = \frac{\alpha}{\alpha + |\varrho - \varsigma|}$, $\Theta(\varrho, \varsigma, \alpha) = \frac{|\varrho - \varsigma|}{\alpha + |\varrho - \varsigma|}$ and $\Upsilon(\varrho, \varsigma, \alpha) = \frac{|\varrho - \varsigma|}{\alpha}$ are neutrosophic metric on Σ. Define self mappings $\mathcal{P}$ and $\mathcal{Q}$ on Σ as follows $\mathcal{P}(\varrho) = \frac{1-\varrho}{3}$ and $\mathcal{Q}(\varrho) = \frac{\sqrt{5-4(1-2\varrho)^2}-1}{4}$. Clearly $\mathcal{P}$ and $\mathcal{Q}$ are OWC maps. Also, $\mathcal{P}$ and $\mathcal{Q}$ are satisfy all the conditions of Theorem 4.8. The self maps $\mathcal{P}$ and $\mathcal{Q}$ have coincidence points $\varrho = 1, \frac{1}{4}$ and the common fixed point $\varrho = \frac{1}{4}$.

5. Conclusion

Using the idea of contractive conditions and OWC , we have demonstrated that there is a common fixed point for four self mappings, three self mappings and two self mappings in a complete NCMS.

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Application Of Neutrosophic Sets Based On Neutrosophic Negative Score Function in Medical Diagnosis

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Abstract. Medical diagnosis is the process of determining which illness or disease is causing an individual's symptoms and warning signs. It is most commonly referred to as analysis, with the clinical environment implied. The evidence required for discovery is typically acquired from a clinical study and an examination of the individual seeking medical treatment. The major purpose of this research is to use topology to establish a methodical technique for decision making difficulties in order to select the appropriate attributes and alternatives for neutrosophic negative score function. In addition, we use a neutrosophic topological space based on attributes and alternatives, as well as graphical representation, to apply a neutrosophic negative score function in medical diagnosis problems.

Keywords: Neutrosophic set, Neutrosophic topology, Neutrosophic topological spaces, Neutrosophic negative score function.

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1. Introduction

Zadeh [41] as part of logic and set hypothesis was the first to introduce the concept of a fuzzy set between intervals in mathematics. Chang's [10] general topology framework, that utilisesfuzzy topological space, was created with a fuzzy set. Adlassnig [6] used fuzzy set theory to formalise medical interactions and fuzzy logic to create a framework for automated analysis. This theory has been used in the areas of artificial intelligence, probability, science, control structures, and financial concerns [16, 20, 26].

In 1983, Atanassov [7] developed an intuitionistic fuzzy set with membership and non-membership values. Coker [14] created intuitionistic fuzzy topological spaces from intuitionistic fuzzy sets. De et al. [15] were the first to develop the applications of intuitionistic fuzzy sets in medical diagnosis. Several researchers [8,17,27] investigated intuitionistic fuzzy sets in medical diagnostics further.

Smarandache [23, 24] offered the notions of neutrosophy and neutrosophic set at the beginning of the 21^{th} century and has a wide range of consistent applications in computer science, information systems, applied mathematics, artificial intelligence, mechanics, medicine, dynamic, management science, and electrical & electronics, etc [1–4, 36, 37]. Salama and Alblowi, [21, 22] in 2012, developed neutrosophic set and neutrosophic crisp set in a neutrosophic topological space. Recently, Vadivel and authors [29, 30, 33–35] presented various open sets and mappings in neutrosophic topological spaces. Smarandache [24] described the single valued Neutrosophic set on three portions (T-Truth, F-Falsehood, I-Indeterminacy) Neutrosophic sets, which Wang et al. [38] worked on. In decision making problems, Majumdar and Samanta [18] described various similarity measures of single valued neutrosophic sets. Several researchers have recently proposed numerous similarity measures and single-valued neutrosophic sets in medical diagnostics [5, 9, 11–13, 19, 28, 39, 40]. Vadivel and authors [31, 32] discussed an applications using neutrosophic score function in mobile networking and material selection problems.

The methodical strategy for decision making issues to identify the appropriate qualities and alternatives for neutrosophic negative score function by employing topology is defined in this work. In addition, we use neutrosophic topological spaces to apply a neutrosophic negative score function in medical diagnosis problems based on their features and alternatives.

2. Preliminaries

Definition 2.1. [21] Let T be a non-empty set. A neutrosophic set (briefly, N_seus) L is an object having the form $L = \{\langle t, \mu_L(t), \sigma_L(t), \nu_L(t)\rangle : t \in T\}$ where $\mu_L, \sigma_L, \nu_L \to [0, 1]$ denote the degree of membership, indeterminacy, non-membership functions respectively of each element $t \in T$ to the N_seus L and $0 \leq \mu_L(t) + \sigma_L(t) + \nu_L(t) \leq 3$ for each $t \in T$.

Definition 2.2. [21] Let T be a non-empty set & the N_seus's L & K in the form $L = \{\langle t, \mu_L(t), \sigma_L(t), \nu_L(t)\rangle : t \in T\}$, $K = \{\langle t, \mu_K(t), \sigma_K(t), \nu_K(t)\rangle : t \in T\}$, then

(i) $0_{N_s} = \langle t, 0, 0, 1\rangle$ and $1_{N_s} = \langle t, 1, 1, 0\rangle$,

(ii) $L \subseteq K$ iff $\mu_L(t) \leq \mu_K(t)$, $\sigma_L(t) \leq \sigma_K(t)$ & $\nu_L(t) \geq \nu_K(t) : t \in T$,

(iii) $L = K$ iff $L \subseteq K$ and $K \subseteq L$,

(iv) $1_{N_s} - L = \{\langle t, \nu_L(t), 1 - \sigma_L(t), \mu_L(t)\rangle : t \in T\} = L^c$,

(v) $L \cup K = \{\langle t, \max(\mu_L(t), \mu_K(t)), \max(\sigma_L(t), \sigma_K(t)), \min(\nu_L(t), \nu_K(t))\rangle : t \in T\}$,

(vi) $L \cap K = \{\langle t, \min(\mu_L(t), \mu_K(t)), \min(\sigma_L(t), \sigma_K(t)), \max(\nu_L(t), \nu_K(t))\rangle : t \in T\}$.

Definition 2.3. [21] A neutrosophic topology (briefly, N_seuty) on a non-empty set T is a family Γ_{N_s} of neutrosophic subsets of T satisfying

(i) $0_{N_s}, 1_{N_s} \in \Gamma_{N_s}$.
(ii) $L_1 \cap L_2 \in \Gamma_{N_s}$ for any $L_1, L_2 \in \Gamma_{N_s}$.
(iii) $\bigcup L_x \in \Gamma_{N_s}, \forall \; L_x : x \in T \subseteq \Gamma_{N_s}$.

Then (T, Γ_{N_s}) is called a neutrosophic topological space (briefly, $N_seutysp$) in T. The Γ_{N_s} elements are called neutrosophic open sets (briefly, N_seuos) in T. A N_seus C_{N_s} is called a neutrosophic closed sets (briefly, N_seucs) iff its complement $C^c_{N_s}$ is N_seuos.

Definition 2.4. [25] The Neutrosophic Negative Score Function (briefly, $N_seuNeScFu$) on $s : L \to [0, 1]$ is defined by

$$s(\mu_L, \sigma_L, \nu_L) = \frac{1 - \mu_L + \sigma_L + \nu_L}{3}$$

that represents the average of positiveness of the neutrosophic components μ_L, σ_L, ν_L.

3. Neutrosophic Negative Score Function

In this section, we provide a neutrosophic scoring function that is based on a methodical approach to solving a decision-making problem with neutrosophic information. In the decision-making situation, the following vital stages are recommended as the precise technique to deal with selecting the appropriate qualities and alternative based on neutrosophic negative score function.

Step 1: Problem field selection:

Consider multi-attribute decision making problems with m attributes $At_1, At_2, \cdots, At_m$ and n alternatives $\tau_1, \tau_2, \cdots, \tau_n$ and p attributes $\nu_1, \nu_2, \cdots, \nu_p$, $(n \leq p)$.

	τ_1	τ_2	.	.	.	τ_n
At_1	(b_{11})	(b_{12})	.	.	.	(b_{1n})
At_2	(b_{21})	(b_{22})	.	.	.	(b_{2n})
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
At_m	(b_{m1})	(b_{m2})	.	.	.	(b_{mn})

	At_1	At_2	.	.	.	At_m
ν_1	(e_{11})	(e_{12})	.	.	.	(e_{1m})
ν_2	(e_{21})	(e_{22})	.	.	.	(e_{2m})
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
ν_p	(e_{p1})	(e_{p2})	.	.	.	(e_{pm})

Here all the attributes b_{ij} and e_{ki} are neutrosophic numbers, where $(i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, p)$.

Step 2: From neutrosophic topologies for τ_j and ν_k:

(i) $\tau^*_j = \tau \cup \tau^* \cup \tau^{**}$, where $\tau = \{1_{N_s}, 0_{N_s}, b_{1j}, b_{2j}, \cdots b_{mj}\}$, $\tau^* = \{b_{1j} \cup b_{2j}, b_{2j} \cup b_{3j}, \cdot, b_{m-1j} \cup b_{mj}\}$ and $\tau^{**} = \{b_{1j} \cap b_{2j}, b_{2j} \cap b_{3j}, \cdot, b_{m-1j} \cap b_{mj}\}$.

(ii) $\nu_j^* = \nu \cup \nu^* \cup \nu^{**}$, where $\nu = \{1_{N_s}, 0_{N_s}, e_{k1}, e_{k2}, \cdots e_{km}\}$, $\nu^* = \{e_{k1} \cup e_{k2}, e_{k2} \cup e_{k3}, \cdot, e_{km-1} \cup e_{km}\}$ and $\nu^{**} = \{e_{k1} \cap e_{k2}, e_{k2} \cap e_{k3}, \cdot, e_{km-1} \cap e_{km}\}$.

Step 3: Find neutrosophic negative score functions:

Neutrosophic negative score functions (shortly, $N_s euNeScFu$) of $\tau, \tau^*, \tau^{**}, \nu, \nu^*, \nu^{**}, \tau_j$ and ν_k are defined as follows.

(i) $N_s euNeScFu(\tau) = \frac{1}{3(m+2)} \left[\sum_{i=1}^{m+2} [1 - \mu_i + \sigma_i + \gamma_i]\right]$, $N_s euNeScFu(\tau^*) = \frac{1}{3q} \left[\sum_{i=1}^{q} [1 - \mu_i + \sigma_i + \gamma_i]\right]$, where q is the number of element of τ^* and $N_s euNeScFu(\tau^{**}) = \frac{1}{3r} \left[\sum_{i=1}^{r} [1 - \mu_i + \sigma_i + \gamma_i]\right]$, where r is the number of element of τ^{**}. For $j = 1, 2, \cdots, n$,

$$N_s euNeScFu(\tau_j) = \begin{cases} N_s euNeScFu(\tau) & \text{if } N_s euNeScFu(\tau^*) = 0; N_s euNeScFu(\tau^{**}) = 0 \\ \frac{1}{2}\left[N_s euNeScFu(\tau) + N_s euNeScFu(\tau^*)\right] & \text{if } N_s euNeScFu(\tau^{**}) = 0 \\ \frac{1}{3}\left[N_s euNeScFu(\tau) + N_s euNeScFu(\tau^*) + N_s euNeScFu(\tau^{**})\right] & \text{otherwise} \end{cases}$$

(ii) $N_s euNeScFu(\nu) = \frac{1}{3(m+2)} \left[\sum_{i=1}^{m+2} [1 - \mu_i + \sigma_i + \gamma_i]\right]$, $N_s euNeScFu(\nu^*) = \frac{1}{3s} \left[\sum_{i=1}^{s} [1 - \mu_i + \sigma_i + \gamma_i]\right]$, where s is the number of element of ν^* and $N_s euNeScFu(\nu^{**}) = \frac{1}{3t} \left[\sum_{i=1}^{t} [1 - \mu_i + \sigma_i + \gamma_i]\right]$, where t is the number of element of ν^{**}. For $k = 1, 2, \cdots, p$,

$$N_s euNeScFu(\nu_k) = \begin{cases} N_s euNeScFu(\nu) & \text{if } N_s euNeScFu(\nu^*) = 0; N_s euNeScFu(\nu^{**}) = 0 \\ \frac{1}{2}\left[N_s euNeScFu(\nu) + N_s euNeScFu(\nu^*)\right] & \text{if } N_s euNeScFu(\nu^{**}) = 0 \\ \frac{1}{3}\left[N_s euNeScFu(\nu) + N_s euNeScFu(\nu^*) + N_s euNeScFu(\nu^{**})\right] & \text{otherwise} \end{cases}$$

Step 4: Final Decision

Arrange neutrosophic negative score values for the alternatives $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n$ and the attributes $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_p$. Choose the attribute ν_p for the alternative τ_1 and ν_{p-1} for the alternative τ_2 etc. If $n < p$, then ignore ν_k, where $k = 1, 2, \cdots, n - p$.

4. Numerical Example

Medical diagnosis has expanded the number of data available to medical professionals as a result of new medical advancements, which also includes vulnerabilities. The path towards grouping multiple sets of symptoms under a single term of a disease is a particularly tough issue in medical diagnosis. In this section, we demonstrate the usefulness and application of the neutrosophic negative score function technique to a medical diagnosis problem.

Step 1: Problem field selection:

Consider the following tables giving informations when consulted physicians about five patients, Patient 1 (shortly, Pat_1), Patient 2 (shortly, Pat_2), Patient 3 (shortly, Pat_3), Patient 4 (shortly, Pat_4) and Patient 5 (shortly, Pat_5) and symptoms are Weight gain (shortly, Wg), Tiredness (shortly, Td), Myalgia (shortly, Ml), Swelling of legs (shortly, Sl) and Mensus Problem (shortly, Mp). We need to find the patient and to find the disease such as Lymphedema, Insomnia, Hypothyroidism, Menarche, Arthritis of the patient. The data in Table 1 and Table 2 are explained by the membership, the indeterminacy and the non-membership functions of the patients and diseases respectively.

Symptoms \ Patients	Pat_1	Pat_2	Pat_3	Pat_4	Pat_5
Wg	(0.9,0.1,0)	(0.8,0,0.2)	(0,0.1,0.9)	(0.1,0,0.7)	(0.3,0.2,0.5)
Td	(0,0.3,0.7)	(0.1,0.2,0.7)	(0.8,0.1,0.2)	(0.1,0.1,0.8)	(0.6,0.5,0.3)
Ml	(0.3,0.1,0.6)	(0.8,0,0.3)	(0.3,0.1,0.6)	(0.2,0.1,0.6)	(0.3,0.4,0.4)
Sl	(0.9,0,0.1)	(0.4,0.2,0.5)	(0.2,0.2,0.7)	(0.4,0.2,0.5)	(0.4,0.6,0.3)
Mp	(0.2,0.1,0.7)	(0.3,0.2,0.5)	(0.4,0.3,0.2)	(0.9,0,0.1)	(0.7,0.4,0.5)

TABLE 1. Neutrosophic values for patients

Disease \ Symptoms	Wg	Td	Ml	Sl	Mp
Lymphedema	(0,0.2,0.8)	(0.2,0.2,0.1)	(0.7,0.2,0.1)	(0.9,0,0.1)	(0.2,0.6,0.4)
Insomnia	(0,0.1,0.9)	(0.9,0,0.1)	(0.2,0,0.8)	(0.2,0.4,0.1)	(0.2,0.1,0.7)
Hypothyroidism	(0.9,0.1,0.1)	(0.1,0.1,0.8)	(0,0.1,0.9)	(0.1,0.4,0.3)	(0.2,0.6,0.4)
Menarche	(0.6,0.3,0.1)	(0.1,0.1,0.8)	(0.2,0.4,0.1)	(0.2,0.5,0.3)	(0.9,0,0.2)
Arthritis	(0,0.1,0.8)	(0.1,0.4,0.6)	(0.9,0.1,0.1)	(0.1,0.3,0.5)	(0.3,0.1,0.6)

TABLE 2. Neutrosophic values for diseases

Step 2: From neutrosophic topologies for (τ_j) and (ν_k):

(i) $\tau_1^* = \tau \cup \tau^* \cup \tau^{**}$, where

$\tau = \{(0,0,1),(1,1,0),(0.9,0.1,0),(0,0.3,0.7),(0.3,0.1,0.6),(0.9,0,0.1),(0.2,0.1,0.7)\}$,

$\tau^* = \{(0.9,0.3,0),(0.3,0.3,0.6),(0.9,0.3,0.1),(0.2,0.3,0.7),(0.9,0.1,0.1)\}$ and

$\tau^{**} = \{(0,0.1,0.7),(0,0,0.7),(0.3,0,0.6),(0.2,0,\ 0.7)\}$.

(ii) $\tau_2^* = \tau \cup \tau^* \cup \tau^{**}$, where

$\tau = \{(0,0,1),(1,1,0),(0.8,0,0.2),(0.1,0.2,0.7),(0.8,0,0.2),(0.4,0.2,0.5),(0.3,0.2,0.5)\}$,

$\tau^* = \{(0.8,0.2,0.2),(0.8,0.2,0.3)\}$ and

$\tau^{**} = \{(0.1,0,0.7),(0.4,0,0.5),(0.3,0,0.5)\}$.

(iii) $\tau_3^* = \tau \cup \tau^* \cup \tau^{**}$, where

$\tau = \{(0,0,1),(1,1,0),(0,0.1,0.9),(0.8,0.1,0.2),(0.3,0.1,0.6),(0.2,0.2,0.7),(0.4,0.3,0.2)\}$,

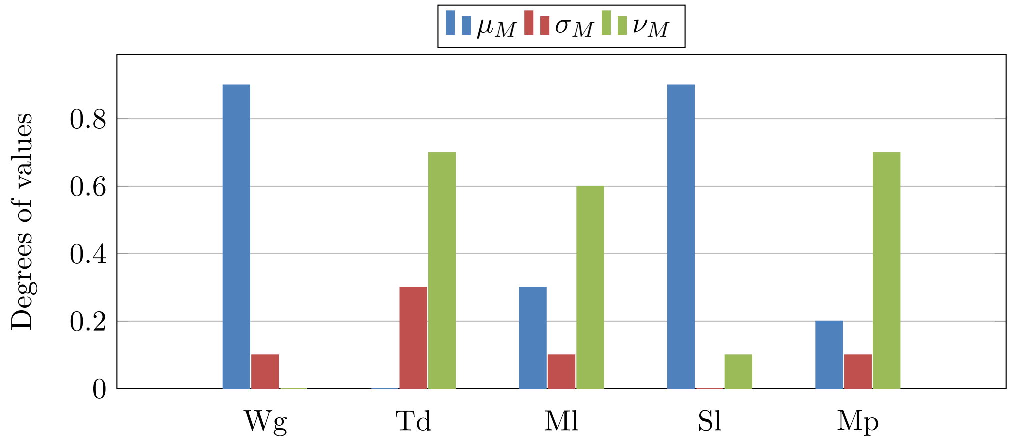

FIGURE 1. Neutrosophic values for Patient 1

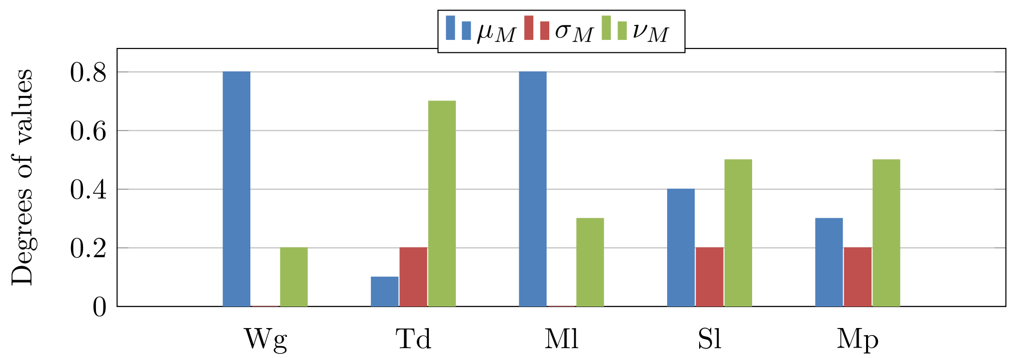

FIGURE 2. Neutrosophic values for Patient 2

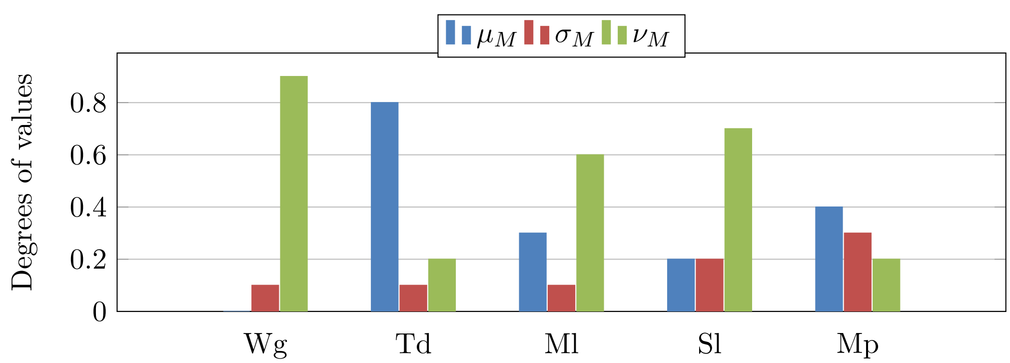

FIGURE 3. Neutrosophic values for Patient 3

$\tau^* = \{(0.8, 0.2, 0.2), (0.8, 0.3, 0.2), (0.3, 0.2, 0.6)\}$ and

$\tau^{**} = h\{(0.2, 0.1, 0.7), (0.4, 0.1, 0.2)\}$.

(iv) $\tau_4^* = \tau \cup \tau^* \cup \tau^{**}$, where

$\tau = \{(0, 0, 1), (1, 1, 0), (0.1, 0, 0.7), (0.1, 0.1, 0.8), (0.2, 0.1, 0.6), (0.4, 0.2, 0.5), (0.9, 0, 0.1)\}$,

$\tau^* = \{(0.1, 0.1, 0.7), (0.9, 0.1, 0.1), (0.9, 0.2, 0.1)\}$ and

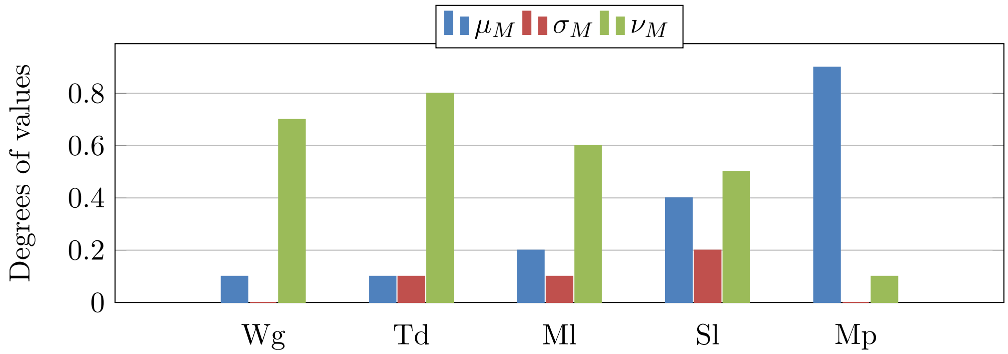

FIGURE 4. Neutrosophic values for Patient 4

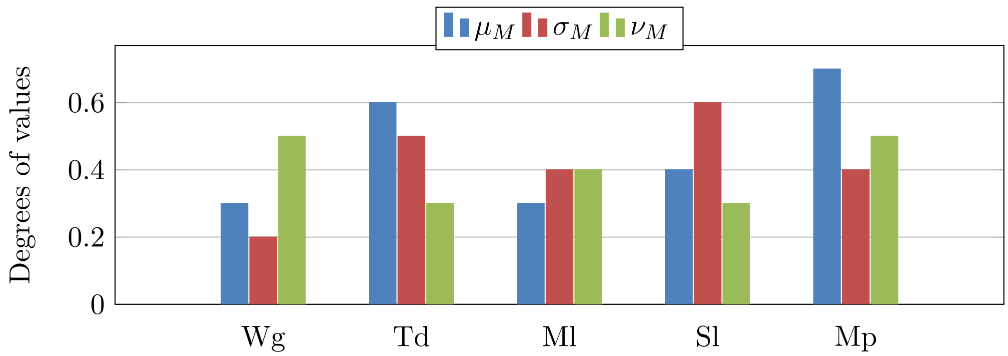

FIGURE 5. Neutrosophic values for Patient 5

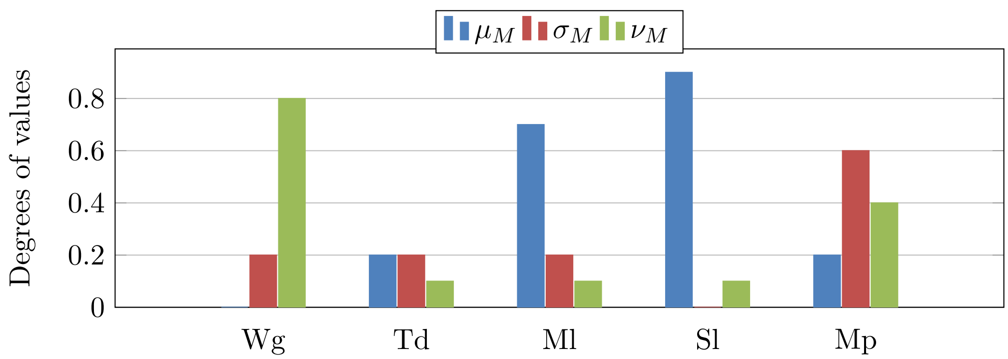

FIGURE 6. Neutrosophic values for Lymphedema

$\tau^{**} = \{(0.1, 0, 0.8), (0.2, 0, 0.6), (0.4, 0, 0.5)\}$.

(v) $\tau_5^* = \tau \cup \tau^* \cup \tau^{**}$, where $\tau = \{(0, 0, 1), (1, 1, 0), (0.3, 0.2, 0.5), (0.6, 0.5, 0.3), (0.3, 0.4, 0.4), (0.4, 0.6, 0.3), (0.7, 0.4, 0.5)\}$,

$\tau^* = \{(0.6, 0.6, 0.3), (0.7, 0.5, 0.3), (0.7, 0.4, 0.4), (0.7, 0.6, 0.3)\}$ and

$\tau^{**} = \{(0.4, 0.5, 0.3), (0.6, 0.4, 0.5), (0.3, 0.4, 0.5), (0.4, 0.4, 0.5)\}$.

(i) $\nu_1^* = \nu \cup \nu^* \cup \nu^{**}$, where

$\nu = \{(0, 0, 1), (1, 1, 0), (0, 0.2, 0.8), (0.2, 0.2, 0.1), (0.7, 0.2, 0.1), (0.9, 0, 0.1), (0.2, 0.6, 0.4)\}$,

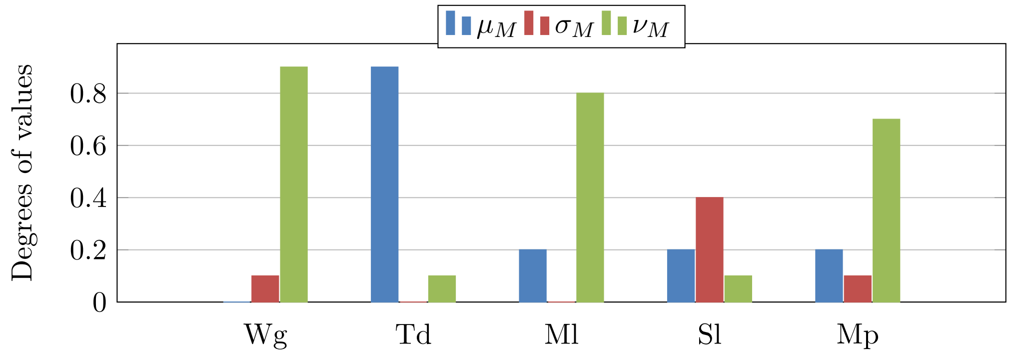

FIGURE 7. Neutrosophic values for Insomnia

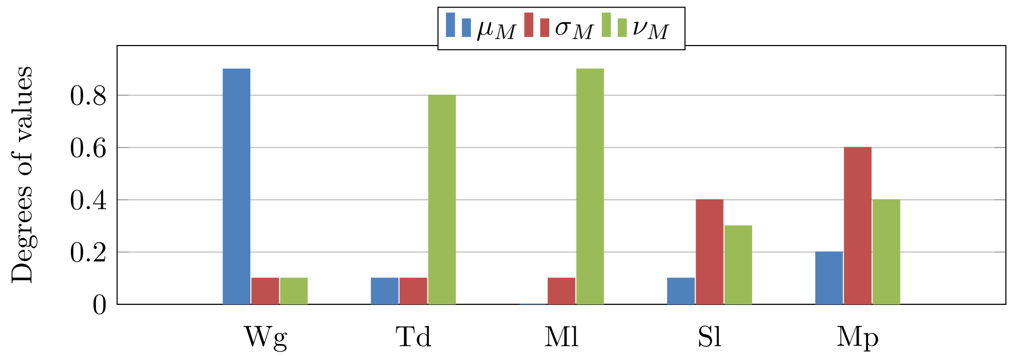

FIGURE 8. Neutrosophic values for Hypothyroidism

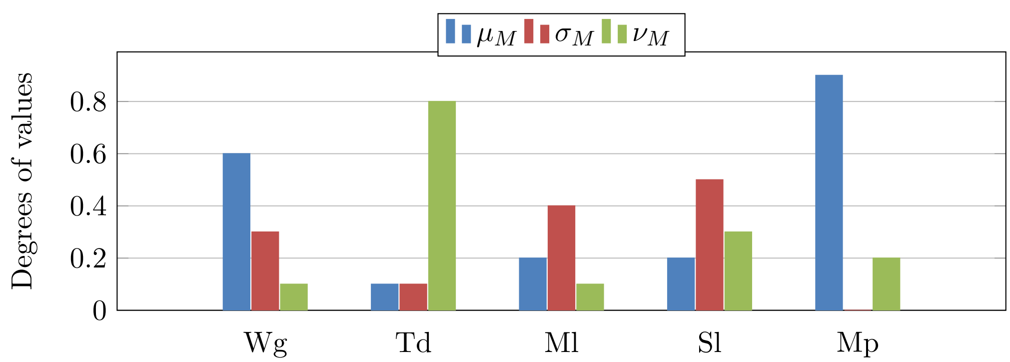

FIGURE 9. Neutrosophic values for Menarche

$\nu^* = \{(0.9, 0.2, 0.1), (0.2, 0.6, 0.1), (0.7, 0.6, 0.1), (0.9, 0.6, 0.1)\}$ and

$\nu^{**} = \{(0, 0, 0.8), (0.2, 0.2, 0.1), (0.2, 0, 0.1), (0.2, 0.2, 0.4), (0.7, 0, 0.1), (0.2, 0, 0.4)\}$.

(ii) $\nu_2^* = \nu \cup \nu^* \cup \nu^{**}$, where

$\nu = \{(0, 0, 1), (1, 1, 0), (0, 0.1, 0.9), (0.9, 0, 0.1), (0.2, 0, 0.8), (0.2, 0.4, 0.1), (0.2, 0.1, 0.7)\}$,

$\nu^* = \{(0.9, 0.1, 0.1), (0.2, 0.1, 0.8), (0.9, 0.4, 0.1)\}$ and

$\nu^{**} = \{(0, 0, 0.9), (0.2, 0, 0.1), (0.2, 0, 0.7)\}$.

(iii) $\nu_3^* = \nu \cup \nu^* \cup \nu^{**}$, where

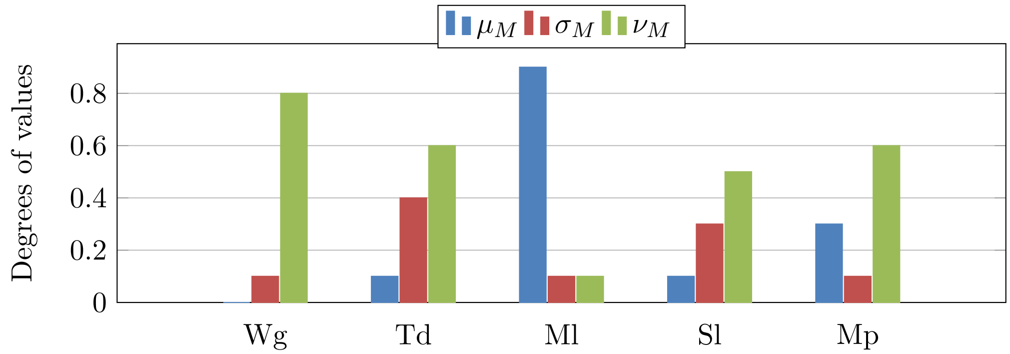

FIGURE 10. Neutrosophic values for Arthritis

$\nu = \{(0, 0, 1), (1, 1, 0), (0.9, 0.1, 0.1), (0.1, 0.1, 0.8), (0, 0.1, 0.9), (0.1, 0.4, 0.3), (0.2, 0.6, 0.4)\}$,

$\nu^* = \{(0.9, 0.4, 0.1), (0.9, 0.6, 0.1), (0.2, 0.6, 0.3)\}$ and

$\nu^{**} = \{(0.1, 0.1, 0.3), (0.2, 0.1, 0.4), (0, 0.1, 0.9), (0.1, 0.4, 0.4)\}$.

(iv) $\nu_4^* = \nu \cup \nu^* \cup \nu^{**}$, where

$\nu = \{(0, 0, 1), (1, 1, 0), (0.6, 0.3, 0.1), (0.1, 0.1, 0.8), (0.2, 0.4, 0.1), (0.2, 0.5, 0.3), (0.9, 0, 0.2)\}$,

$\nu^* = \{(0.6, 0.4, 0.1), (0.6, 0.5, 0.1), (0.9, 0.3, 0.1), (0.9, 0.1, 0.2), (0.2, 0.5, 0.1), (0.9, 0.4, 0.1), (0.9, 0.5, 0.2)\}$ and

$\nu^{**} = \{(0.2, 0.3, 0.1), (0.2, 0.3, 0.3), (0.6, 0, 0.2), (0.1, 0, 0.8), (0.2, 0.4, 0.3), (0.2, 0.4, 0.2)\}$.

(v) $\nu_5^* = \nu \cup \nu^* \cup \nu^{**}$, where

$\nu = \{(0, 0, 1), (1, 1, 0), (0, 0.1, 0.8), (0.1, 0.4, 0.6), (0.9, 0.1, 0.1), (0.1, 0.3, 0.5), (0.3, 0.1, 0.6)\}$,

$\nu^* = \{(0.9, 0.4, 0.1), (0.1, 0.4, 0.5), (0.3, 0.4, 0.6), (0.9, 0.3, 0.1), (0.3, 0.3, 0.5)\}$ and

$\nu^{**} = \{(0.1, 0.1, 0.6), (0.1, 0.3, 0.6), (0.1, 0.1, 0.5)\}$.

Step 3: Find neutrosophic negative score functions:

(i) $N_seuNeScFu(\tau) = 0.4$, $N_seuNeScFu(\tau^*) = 0.3067$ and $N_seuNeScFu(\tau^{**}) = 0.525$.

$$N_seuNeScFu(\tau_1) = 0.4106.$$

(ii) $N_seuNeScFu(\tau) = 0.4$, $N_seuNeScFu(\tau^*) = 0.2167$ and $N_seuNeScFu(\tau^{**}) = 0.4334$.

$$N_seuNeScFu(\tau_2) = 0.3501.$$

(iii) $N_seuNeScFu(\tau) = 0.4619$, $N_seuNeScFu(\tau^*) = 0.3112$ and $N_seuNeScFu(\tau^{**}) = 0.4167$.

$$N_seuNeScFu(\tau_3) = 0.3966.$$

(iv) $N_seuNeScFu(\tau) = 0.4476$, $N_seuNeScFu(\tau^*) = 0.2667$ and $N_seuNeScFu(\tau^{**}) = 0.4667$.

$$N_seuNeScFu(\tau_4) = 0.3937.$$

(v) $N_seuNeScFu(\tau) = 0.4467$, $N_seuNeScFu(\tau^*) = 0.3917$ and $N_seuNeScFu(\tau^{**}) = 0.4834$.

$$N_s euNeScFu(\tau_5) = 0.4473.$$

(i) $N_s euNeScFu(\nu) = 0.4143$, $N_s euNeScFu(\nu^*) = 0.3083$ and $N_s euNeScFu(\nu^{**}) = 0.4143$.

$$N_s euNeScFu(\nu_1) = 0.3668.$$

(ii) $N_s euNeScFu(\nu) = 0.4619$, $N_s euNeScFu(\nu^*) = 0.2889$ and $N_s euNeScFu(\nu^{**}) = 0.4778$.

$$N_s euNeScFu(\nu_2) = 0.4095.$$

(iii) $N_s euNeScFu(\nu) = 0.5$, $N_s euNeScFu(\nu^*) = 0.3445$ and $N_s euNeScFu(\nu^{**}) = 0.525$.

$$N_s euNeScFu(\nu_3) = 0.4565.$$

(iv) $N_s euNeScFu(\nu) = 0.4191$, $N_s euNeScFu(\nu^*) = 0.2334$ and $N_s euNeScFu(\nu^{**}) = 0.4112$.

$$N_s euNeScFu(\nu_4) = 0.3546.$$

(v) $N_s euNeScFu(\nu) = 0.4857$, $N_s euNeScFu(\nu^*) = 0.4067$ and $N_s euNeScFu(\nu^{**}) = 0.5445$.

$$N_s euNeScFu(\nu_5) = 0.479.$$

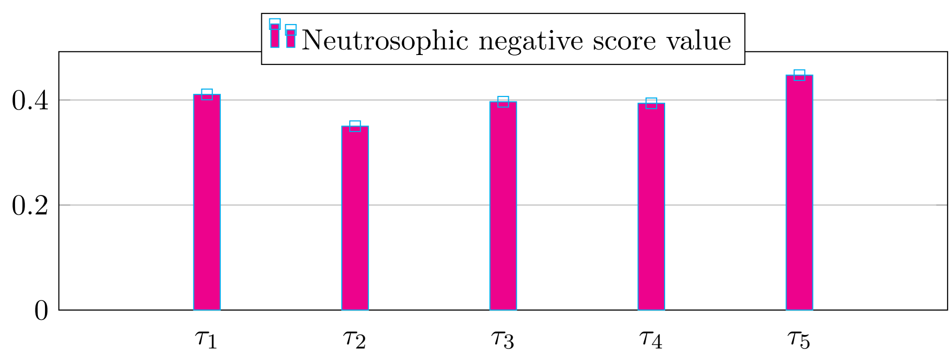

FIGURE 11. Neutrosophic negative score values for Patients

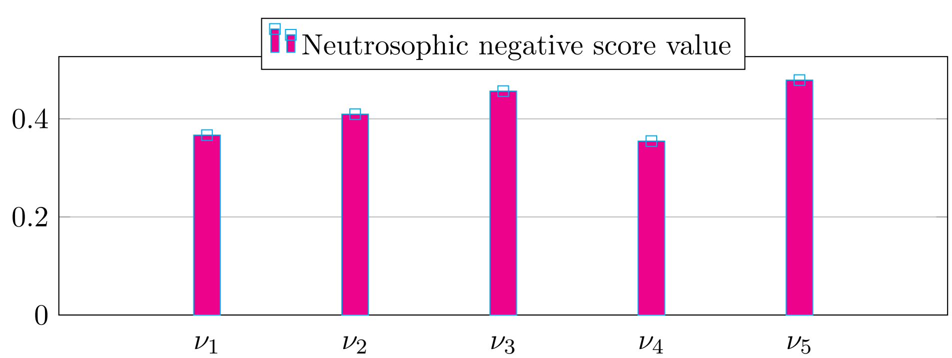

FIGURE 12. Neutrosophic negative score values for Diseases

Step 4: Final Decision:

Arrange neutrosophic negative score values for the alternatives τ_1, τ_2, τ_3, τ_4, τ_5 and the attributes ν_1, ν_2, ν_3, ν_4, ν_5 in descending order. We get the following sequences $\tau_5 \leq \tau_1 \leq \tau_3 \leq \tau_4 \leq \tau_2$ and $\nu_5 \leq \nu_3 \leq \nu_2 \leq \nu_1 \leq \nu_4$. Thus the Pat_5 suffers from Menarche, the Pat_1 suffers from Lymphedema, the Pat_3 suffers from Insomnia, the Pat_4 suffers from Hypothyroidism and the Pat_2 suffers from Arthritis.

5. Conclusions

One of the research areas in general fuzzy topological spaces dealing with the concept of vagueness is neutrosophic topological space. This study established the neutrosophic negative scoring function and the technique based on qualities, as well as alternatives to its real-world application. In addition, the medical diagnosis decision-making problem employing qualities and choices on the neutrosophic score function. This theory can be developed and applied to other general topology study areas such as rough topology, digital topology, image processing, neural networks, and so on. Furthermore, the neutrosophic accuracy function and neutrosophic certainty function are based on applications that can be carried out in future studies.

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State of Art of Plithogeny Multi Criteria Decision Making Methods

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Abstract

Plithogenic sets coined by Smarandache in the year 2018 has unveiled new research opportunities in the field of Multi criteria decision making (MCDM). The contributions and developments of new decision making approaches based on plithogeny is gaining high momentum presently. The theoretical conceptualization of different phenomenon with plithogenic sets are also applied in designing optimal solutions to the decision making problems. This review paper presents the applications of plithogenic MCDM from the year 2018 to till date in almost all the spheres of decision making scenario. The literature works of the researchers presented in this paper will certainly portray the compatibility and flexibility of plithogenic sets, operators and other decision making tools. Though the time span considered for counting on the plithogeny based works is short, the applications of plithogenic sets are growing many in number and also plithogeny based theories are amplifying in a speedy manner. This has motivated the authors to investigate on the proliferation of plithogeny applications in decision making. This review paper has focused on the dimensions of different fields to which plithogeny is applied, new plithogeny based theories, extension of plithogeny, plithogenic based operators and measures. In addition to it the data on the publications of plithogeny based articles and interests of researchers are also presented as a part of this review work. The overall impact of plithogeny in the arena of decision making science and on the researchers of the same field is well sketched in this paper with the intention and hope of inspiring plithogenic researchers.

Keywords: Plithogeny, Plithogenic MCDM, Applications

1. Introduction

Multi criteria decision making (MCDM) is a growing research area which attracts many researchers to develop new decision making techniques to accomplish the objective of finding optimal solutions. MCDM otherwise called as Multi attribute decision making (MADM) is a convoluted process entailing alternatives, criteria and feasible methods of deriving solutions. The primary aim of every decision making problem is to identify ideal alternative that highly fulfill all the criteria to a significant extent. MCDM methods are applied in different fields such as supply chain management, Education, Internet of Things, COVID-19 epidemic, Material selection for various manufacturing industries, Renewable Energy Development, Business and Banking sector, Planning, Medical, Agriculture ,construction and logistic.

Decision making under deterministic environment is not possible always as the decision making data is based on decision maker's opinions and perspectives. This happens as MCDM problems do not deal only with quantitative data but also with qualitative data. At certain instances, the decision matrix comprises of linguistic representations are handled by fuzzy MCDM methods. The theory of MCDM is integrated with fuzzy sets introduced by Zadeh [1] to handle imprecise and vague data. The fuzzy MCDM models are extended to intuitionistic fuzzy MCDM. In an intuitionistic MCDM the data representations are made using intuitionistic sets. Atanssov [2] introduced intuitionistic sets comprising membership and non-membership values. Intuitionistic MCDM methods are extended to neutrosophic MCDM to handle indeterminacy. Smarandache [3] introduced the theory of neutrosophy to handle the condition of indeterminacy. MCDM methods discussed under the environments of fuzzy, intuitionistic and neutrosophic are more compatible in making precise decisions. But still to develop a more comprehensive genre of MCDM models Smarandache developed [4] the theory of Plithogeny. Plithogenic based MCDM are gaining more impetus at recent times as these MCDM models are more adaptive to any kind of decision making models based on the characteristics of crisp, fuzzy, intuitionistic and neutrosophic. The compatibility and comprehensive nature of plithogenic sets has motivated the authors to investigate on the plithogenic based MCDM. This review article intends to answer the following questions (i) What are the theories developed based on Plithogeny? (ii) How far the theory of plithogeny is associated with MCDM? (iii) What are the significant applications of Plithogenic MCDM? (iv)To what extent the theory of plithogeny has gained the interest of the researchers in the recent years?

The remaining contents are structured as follows: section 2 comprises the overview of the MCDM methods; section 3 elucidates on the origin and development of Plithogeny and the conceptualization of Plithogenic based theories; section 4 sketches out the applications of the Plithogeny based MCDM; section 5 presents the analysis of plithogenic publication and the last section concludes the review work with future directions

2. Overview of MCDM

The theoretical arguments of Multi criteria decision making is an integral part of Decision theory. MCDM is otherwise termed as MCDA where the latter focuses on analysis. A MCDM problem generally begins with the formulation of a primary decision making matrix with alternatives and criteria. The alternatives are referred as the options, the criteria as the characteristic features and the value of the matrix indicates the satisfactory extent of the criteria by the alternatives. The number of MCDM methods that currently exist are many in number but the process of finding optimal solution to the decision making problems follows certain steps in common.

(i) Formulation of Decision Making matrix with finite number of alternatives and criteria of the form

$$D_M(T) = \begin{pmatrix} t_{11} & t_{12} & \dots. & \dots. & t_{1j} \\ t_{21} & t_{22} & \dots. & \dots. & t_{2j} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ t_{i1} & t_{i2} & \dots & \dots & t_{ij} \end{pmatrix}; \; i=1,2\dots m \, , \, j=1,2\dots n$$

The above matrix has m alternatives and n criteria where the values of m and n ranges from 1 to I and 1to j respectively

(ii) Computing the criterion weights Wj. At some cases the criterion weights are assumed to be equal but in some cases the criterion weights are computed using preferences and relative importance.

(iii) Normalization of the matrix

This is a very essential step in every decision making process. As the values of the decision matrix are of different ranges, the values are normalized and scaled down to the range between 0 and 1. Different methods of normalization such as linear sum, linear max-min, linear max and VIKOR are used.

(iv) Finding the weighted normalized matrix

The weighted normalized matrix is obtained by multiplying the weight vector of the criteria with the normalized matrix.

(v) Ranking of the alternatives based on the relative score values.

The alternatives are ranked after computing the score values of the alternatives. The modality of calculating the score values differs with respect to the methods. The graphical representation of any MCDM framework in general is presented in Fig.1

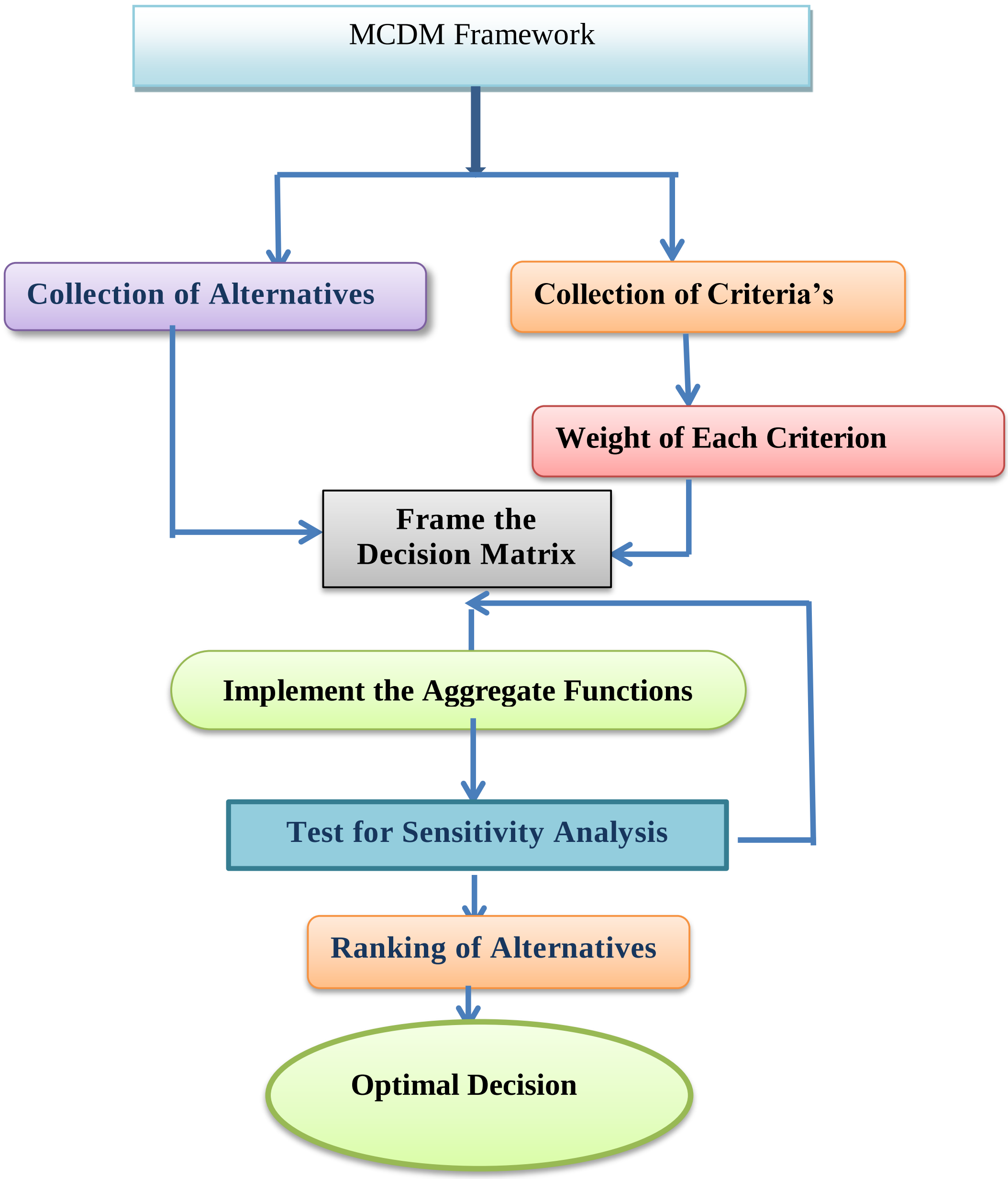

Fig.1. MCDM Framework

The MCDM methods are classified based on the following classifiers as follows

(i) Number of decision makers

(ii) Number of alternatives

(iii) Nature of the criteria

(iv) Nature of the values in the decision making matrix

(v) Goals of the decision making problem

(vi) Criterion weights computation

(vii) Calculation of the score values of the alternatives

(viii) Nature of the decision making environment

Based on the above described classifiers, many numbers of MCDM methods are developed especially to find the criterion weights and to rank the alternatives. The MCDM methods that are formulated by the researchers are presented as follows in Table 1.

Table 1 Chronological Development of MCDM Methods

S.No	MCDM Methods	Authors	Year
1	Taxonomy Method	Adanson	1763
2	Weighted Product Model (WPM)	Bridgeman	1922
3	Simple Additive Weighting (SAW)	Fish burn et al.,	1967
4	Weighted Sum Model (WSM)	L. A. Zadeh	1963
5	Multi Attribute Utility Theory (MAUT)	P.C. Fishburn, Keeney , Raiffa	1965, 1976
6	Multi-Attribute Utility Analysis (MAUA)	P.C. Fishburn	1965
7	Elimination and Choice Translating Reality (ELECTRE)	Benayoun Roy	1968
8	Multi-Attribute Utility Analysis (MAUA)	R.L. Keeney, H.R. Raiffa	1969
9	Analytic Hierarchy Process (AHP)	Thomas Saaty	1970
10	Decision-Making Trial and Evaluation Laboratory (DEMATEL)	Fonetla and Gabus	1971
11	QUALItative FLEXible (QUALIFLEX)	Paelinck , Jacquet Lagreze	1975
12	ORESTE	Roubens	1980
13	Preference Ranking Organization Method for Enrichment Evaluations (PROMETHEE)	J. P. Brans , P. Vicke	1982
14	Evaluation of Mixed Data (EVAMIX)	Voogd , Martel and Matarazzo	1982
15	Grey relational analysis(GRA)	Deng	1982
16	REGIME	Hinloopen, Nijkamp, and Rietveld	1983
17	Simple Multi-Attribute Rating Technique (SMART)	Winterfeldt & Edwards	1986
18	Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS)	S. Opricovic	1990

19	Measuring Attractiveness by a Categorical Based Evaluation Technique (MACBETH)	Banae Costa , Vansnick	1990
20	EXPROM I & II (Extension of the PROMETHEE)	Diakoulaki , Koumoutsos	1991
21	TODIM	Gomes, Lima	1992
22	Ashby	Ashby	1992
23	Complex Proportional Assessment (COPRAS)	Zavadskas , Kaklauskas and Sarka	1994
24	The Criteria Importance Through Inter criteria Correlation (CRITIC)	Diakoulaki, Mavrotas, and Papayannakis	1995
25	Analytic Network Process (ANP)	Saaty T. T	1996
26	PAMSSEM I & II	Martel, Kiss, and Rousseau	1996
27	Multi criteria Optimization and Compromise Solution (VlseKriterijumska Optimizacija I Kompromisno Resenje) (VIKOR)	S. Opricovic	1998
28	superiority and inferiority ranking method (SIR)	Xu	2001
29	Multi-Objective Optimization by Ratio Analysis Method (MOORA)	Brauers , Zavadskas	2004, 2006
30	Case Based Reasoning (CBR)	Li, Sun, Kolodner	2008
31	Preference selection index (PSI)	Maniya , Bhatt	2010
32	Additive Ratio Assessment (ARAS)	Zavadskas ,Turskis	2010
33	Stepwise Weight Assessment Ratio Analysis (SWARA)	Kersuliene, Zavadskas, and Turskis	2010
34	Data Envelopment Analysis (DEA)	Thanassoulis, Kortelainen, and Allen	2012
35	Weighted Aggregates Sum Product Assessment (WASPAS)	Zavadskas, Turskis, Antucheviciene, and Zakarevicius in	2012
36	Kemeny Median Indicator Ranks Accordance (KEMIRA)	Krylovas, Zavadskas, Kosareva, and Dadelo	2014
37	Evaluation based on Distance from Average Solution (EDAS)	Keshavarz Ghorabaee, Zavadskas, Olfat, and Turskis	2015
38	Multi-Attributive Border Approximation area Comparison (MABAC)	Pamucar and Cirovic	2015
39	Best Worst Method (BWM)	Rezaei	2015
40	Integrated Determination of Objective CRIteria Weights (IDOCRIW)	Zavadskas and Podvezko	2016
41	PIvot Pairwise RElative Criteria Importance	Stanujkic et al.,	2017

	Assessment (PIPRECIA)		
42	Full Consistency Method (FUCOM)	Pamucar	2018
43	MultiAtributive Ideal-Real Comparative Analysis (MAIRCA)	D.S. Pamucar	2018
44	COmbined COmpromise SOlution (CoCoSo)	Morteza Y., et al	2019
45	Method based on the removal effects of criteria (MEREC)	Keshavarz-Ghorabaee et al	2021

The significant characteristics of the most commonly applied MCDM methods are measured by the nature of the method, attribute dependency and facilitation in handling qualitative and quantitative values. In general the methods are of compensatory in nature. Table 2 sketches out the core characteristics of the MCDM methods.

Table 2 Characteristics of MCDM methods

MCDM Methods	**Characteristics**		
	Compensatory Method	**Attributes Dependency**	**Conversion of Qualitative to Quantitative**
Weighted Sum Model (WSM)	✓	X	✓
Weighted Product Model (WPM)	✓	X	✓
Analytic Hierarchy Process (AHP)	✓	X	✓
Data Envelopment Analysis (DEA)	-	✓	✓
Analytic Network Process (ANP)	✓	✓	✓
ELimination and Choice Translating REality (ELECTRE)	✓	✓	✓
Multicriteria Optimization and Compromise Solution (VIKOR)	✓	X	✓
Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS)	✓	X	✓
Preference Ranking Organization Method for Enrichment Evaluations (PROMETHEE)	✓	✓	✓
Best Worst Method (BWM)	-	-	-

Case Based Reasoning (CBR)	-	-	-
Multi Attribute Utility Theory (MAUT)	✓	X	✓
DEcision-Making Trial and Evaluation Laboratory (DEMATEL)	✓	✓	✓
COPRAS	✓	X	✓
Preference selection index (PSI)	-	-	-
SMART	✓	✓	✓
REGIME	✓	X	X
ORESTE	✓	X	X
MOORA	✓	X	✓
QUALIFLEX	✓	X	X
SIR	✓	X	✓
EVAMIX	✓	X	X
ARAS	✓	X	✓
Taxonomy Method	✓	X	✓
MACBETH	✓	X	✓
WASPAS	✓	X	✓
SWARA	✓	X	-
MAIRCA	-	-	-
CRITIC	✓	✓	✓
FUCOM	-	-	-
TODIM	✓	X	✓
IDOCRIW	✓	X	✓
EDAS	✓	X	✓
PAMSSEM I & II	✓	✓	✓
EXPROM I & II	✓	✓	✓
MABAC	✓	X	✓
KEMIRA	✓	-	✓
Grey relational analysis (GRA)	-	-	-
Method based on the removal effects of	-	✓	✓

criteria (MEREC)			
COmbined COmpromise SOlution (CoCoSo)	✓	X	✓
Simple Additive Weighting (SAW)	-	X	✓

Hence the MCDM methods that are formulated by the researchers have high utility in making optimal decisions, there are few limitations. The existence of speculations about the decision making scenario in different perspectives are quite inevitable. As every decision making circumstances are inscribed with uncertainty, ambiguity and indeterminacy, it is quite natural and essential to extend the crisp decision making methods to fuzzy, intuitionistic and neutrosophic environments.

Table: 3 Pioneers of MCDM methods in Different Decision Making Environments

MCDM methods	Fuzzy MCDM	Intuitionistic MCDM	Neutrosophic MCDM	Plithogenic MCDM
AHP	Van Laarhoven and Pedrycs [5]	Jian Wu Hai-bin Huang, Qing-wei Cao [6]	Nouran M. Radwan, M. Badr Senousy [7]	Mohamed Abdel-Basset [8]
BWM	Guo Sen, Haoran Zhao [9]	Mou Qiong, Xu Zeshui, Huchang Liao[10]	Vafadarnikjoo, Amin ,Madjid, Tavana [11]	Mohamed Grida [12]
FUCOM	Galina Ilieva [13]	Arunodaya Raj Mishra, Abhishek Kumar Garg [14]	Fatih Yiğit [15]	S.Sudha & Nivetha Martin [120]
MAIRCA	Boral et al[16]	Fatih Ecer [17]	Dragan et al [18]	A.Ozcil et al[19]
TOPSIS	Chen. Or Lai et al [20]	Deepa Joshi, Sanjay Kumar[21]	A Elhassouny[22]	M. Abdel-Basset & Rehab Mohamed[23]
CRITIC	Kahraman et al [24]	Quan-Song Qi [25]	Esra Aytaç Adalı, Tayfun Öztaş [26]	Abdel-Basset et al .,[23] Korucuk, Demir, Karamasa, & Stević.,[27]
MABAC	Liang WZ, Zhao GY, Wu H, Dai B. [28]	Jia F et al,[29] Mengwei Zhao & Guiwu Wei et al [30]	Sahin R, Altun F [31]	Florentin Smarandache et al.,[32]
MACBETH	Dhouib [33]	Mustafa said Yurtyapan Erdal Aydemir [34]	Irvanizam et al.,[35]	S.Sudha & Nivetha Martin [121]
MEREC	Mohamad Shahiir Saidin et al.,[36]	Ibrahim M. Hezam et al.,[37]	-	Sudha.S , Edwin Deepak F.X , Nivetha Martin [124]

PIPRECIA	Stević et al.[123]	-	-	Alptekin Ulutas et al [123]

3. Origin and Development of Plithogeny

Smarandache is the founding father of the theory of Plithogeny. The Plithogenic sets are introduced as the generalization of crisp sets,fuzzy sets, intuitionistic sets, neutrosophic sets [38]. A plithogenic set is a quintuple of the form (P,a,V,d,c), where P is the set, a is the attribute, V is the set of attribute values, d is the degree of appurtenance and c is the degree of contradiction. The plithogenic sets are a boon to the field of decision making as it deals with attributes. These sets are very comprehensive in nature as it facilitates in accommodating multi expert's opinion and several attributes values with respective d and c values.

Smarandache [4] also developed Plithogenic based theories of probability, statistics. The notion of Plithogenic logic is also formulated as a generalization to multi valued logic [39]. Villacrés et al.,[40] applied Plithogenic logic in determining the occupational health risks. George [41] applied Plithogenic sets and logic in information analysis. Smarandache [42] has also made the extensions of neutrosophic over set/under set/off set to Plithogenic. Smarandache [43,44] has extended neutrosophic statistics to Plithogenic statistics. Smarandache and Guo [45] developed neutrosophic based plithogenic optimization. The plithogenic statistics is considered to be the most generalized form of the statistics. Castro Sánchez et al., [46] have applied the neutrosophic and plithogenic statistical concepts in making decisions on developing educational field. Prem Kumar Singh [47] has applied Plithogenic sets in multivariate data analysis.

Smarandache [48,49] was the pioneer of Hypersoft sets and Plithogenic Hypersoft sets which are the extensions of soft sets. Shawkat et al.,[50] introduced Plithogenic soft sets. Smarandache [51] has also introduced new types of soft sets such as indeterm soft set, indeterm Hypersoft set, Tree soft sets. Rana et al.,[52] introduced plithogenic whole hypersoft set and generalized plithogenic whole hypersoft set. Dhivya and Arockia Lancy [53,54] have developed Near plithogenic hypersoft sets and discussed the properties of the sets. Nivetha Martin and Smarandache [55,56] have introduced combined plithogenic hypersoft sets and extended plithogenic Hypersoft sets with dual dominant attributes. Shazia Rana et al., [57] together designed Plithogenic Subjective Hyper-Super-Soft Matrices with different levels of ranking

Vasantha and Smarandache [58] developed Plithogenic graphs. Sultana et al., [59] applied plithogenic graphs in analysing the spread of corona virus. Bharathi [60] introduced plithogenic product fuzzy graphs and studied its applications in social networks. Prem kumar singh [61-63] has discoursed on single valued plithogenic graph, single valued neutrosophic plithogenic graph and Intuitionistic plithogenic graph. Smarandache [64] has also developed the notion of n-Super Hyper Graph and Plithogenic n-Super Hyper Graph. Smarandache and Nivetha Martin [65] have developed concentric plithogenic hypergraph based on Plithogenic Hypersoft sets.

Researchers have also explored plithogenic based algebraic structures. Smarandache [66-68] laid the foundation of Plithogenic algebraic structures. Gayen et al., [69] plithogenic Hypersoft subgroups. Basumatary et al., [70] investigated the topological properties based on *plithogenic neutrosophic Hypersoft.* Taffach, & Hatip [71] presented a brief review of Symbolic 2-Plithogenic Algebraic Structures. Taffach et al.,[72] discoursed on Plithogenic rings. Taffach & Nader Mahmoud [73] discussed on the fusion of Symbolic Plithogenic Sets and Vector Spaces. Priyadharshini & Nirmala Irudayam[74] have explored plithogenic based topological spaces and their properties. Merkepci & Abobala [75] evolved the theory of Symbolic 2-Plithogenic Rings. Al-Basheer et al., [76] elicited on Symbolic 3-Plithogenic Rings and their Algebraic Properties. Smarandache [66] presented an overall view of the plithogenic algebraic structures and symbolic plithogenic algebraic structures. Taffach et al.,[77] have explored Plithogenic Number Theory and algebraic equations. Khaldi.,[78] formulated algorithms for solving algebraic equations with Symbolic 2-Plithogenic numbers.

Plithogenic numbers is also investigated by several researchers. Nivetha et al.,[79] have introduced Plithogenic numbers. Followed by Noel Batista Hernández et al.,[80] have applied plithogenic numbers in assessing competency of the students. Raúl Comas Rodríguez et al.,[81] have applied in evaluation of education and society. Zuñiga et al.,[82] have used plithogenic numbers in study of the soil attributes

Nivetha Martin et al.,[79]introduced Plithogenic Sociogram. Sudha, Nivetha Martin , Florentin Smarandache [83] introduced extended plithogenic sets and applied the same in plithogenic sociogram. The plithogenic sociogram approaches are the extensions of neutrosophic sociogram approaches. These are used as alternatives of MCDM methods. Nivetha Martin and Florentine Smarandache [84] have introduced the theory of Plithogenic Cognitive Maps to make optimal decisions. Sujatha et al.,[85] have applied Plithogenic Cognitive Maps in making analysis of the novel corona virus. Nivetha Martin et al.,[86] have developed the concept of new plithogenic sub cognitive maps with mediating effects. Priya and Nivetha Martin [87] introduced Induced Plithogenic Cognitive Maps with Combined Connection Matrix. Priya, Martin, & Kishore [88] have applied PCM in the field of behaviour modification.

Priyadharshini et al.,[89] have developed plithogenic cubic sets and have explored their properties with suitable illustrations. Prem kumar Singh [90] has introduced complex plithogenic set. The Plithogenic sets are applied to other physical fields. Smarandache [91] has coined Physical plithogenic sets. Within a very short span of time of the conceptualization of Plithogeny, the plithogenic sets are widely applied in almost all the domains of mathematics.

4. Applications of Plithogeny based MCDM

This section presents the applications of Plithogenic sets in the arena of decision making. The plithogenic representations, plithogenic operators are integrated with decision making elements to make optimal decisions. The plithogenic concepts that are presented in the section [3] are applied in making decisions based on multi attributes. The plithogenic based decision making are used in solving ranking based problem. It is also used in making assessments and evaluation study. The plithogenic environment is extensively applied in solving various problems. Abdel-Basset et al.,[8] in green supply chain management, Abdel-Basset et al.,[92] in making evaluations of hospital administration, Tayal et al., [93] in ranking of products, Rously et al., [94] in prioritizing. Gómez et al.,[95] in evaluating strategies of promoting education, Fernández et al.,[96] in selecting investment projects, Moncayo et al.,[97] in defining strategies, Öztaş et al.,[98] in performance evaluation, Pai, & Prabhu Gaonkar [99] in making risk assessments on evidential reasoning. Rehab Mohamed et al.,[12] in evaluating the performances of IoT based supply chain. M. Abdel-Basset et al.,[32] in supplier selection. Abdel-Basset et al.,[100] in financial performances. In the above mentioned plithogenic based decision making models, the plithogenic representations are used to represent data.

Some of the MCDM methods presented in Table 4, are also discussed under plithogeny. The applications of the extended plithogenic based MCDM methods are described as follows. Ansari and Kant [101] have applied plithogenic based neutrosophic Analytical Hierarchy Process in handling a decision-making problem on supply chain. TOPSIS is one of the most preferred MCDM method. Sankar et al.,[102] applied Plithogenic TOPSIS in modelling COVID- 19 problem. Mohamed Abdel-Basset & Rehab [23] used TOPSIS-CRITIC in supply chain management. Nivetha Martin [103] employed TOPSIS-SWARA in food processing technology. Abdullah Ozcil et al. [19] used plithogenic MAIRCA in building novel decision making model. Sudha & Nivetha Martin [104] developed TOPSIS integrated Plithogenic Cognitive Maps in making optimal decisions on the problems based on the evaluation of teachers performance. Korucuk et al.,[27] formulated plithogenic CRITIC decision model to optimize logistics based problems. Sudha & Nivetha Martin [105] applied the integrated Plithogenic CRITIC-MAIRCA in making decisions on feasible livestock feeding stuffs. Wang et al., [106] devised multi attribute group decision making using Plithogenic VIKOR with linguistic representations. Wang et al.,[107] used rough numbers in framing plithogenic neutrosophic based decision making model. Ulutaş & Topal [108] used Plithogenic PIPRECIA in dealing logistics problem. Abdel-Basset et al.,[109] designed plithogenic best-worst method in making decisions on supply chain. Sudha & Nivetha Martin [110] made comparative analysis of the efficiency of plithogenic and neutrosophic best and worst method.

Shio Gai Quek et al.,[111] used Plithogenic entropy measures in solving a multi-attribute decision making. Gomathy et al.,[112] applied plithogenic sets in making decisions on health dynamics. Abdel-Basset et al., [8] used quality functions in devising hybrid decision making models in solving supply chain management problems. Priyadharshini and Nirmala Irudayam [113] used refined and single valued plithogenic neutrosophic sets in solving MCDM problems. Nivetha Martin et al [114]., used PROMTHEE Plithogenic Pythagorean Hypergraphic Approach in Smart Materials Selection. Walid Abdelfattah [115] developed plithogenic DEA in assessment based problems.

Inaddition to the applications of the Plithogenic based MCDM methods in making optimal decisions. Florentin Smarandache and Nivetha Martin[116] have used Plithogenic n- Super Hypergraph in making Novel Multi -Attribute Decision Making. Muhammad Rayees Ahmad et al., [117] used Plithogenic Hypersoft Sets with Fuzzy Neutrosophic representations in framing novel decision making model. Dhivya & Arokia Lancy [118] constructed a multi attribute decision-making model with Heronian Mean Aggregation Operators using the representations of near plithogenic neutrosophic hypersoft representations. Sudha and Nivetha Martin [119] applied combined Plithogenic Hypersoft sets in making optimal decisions on business analytics tools. The Table 4 presents the applications of the above discussed Plithogenic based MCDM methods.

Table:4 Applications of the Plithogenic MCDM methods

Plithogenic MCDM Methods	**Problem Specification**
Plithogenic TOPSIS	• Quality Function Deployment for Selecting Supply Chain Sustainability Metrics • COVID-19 pandemic problem
Plithogenic VIKOR	Hospital medical care systems
Plithogenic AHP	Eco-innovation practices in supply chain
Plithogenic TOPSIS-CRITIC	Risk management in sustainable supply chain
Plithogenic MAIRCA	Constructing innovative decision making model
Plithogenic BWM & VIKOR	Evaluating the Performance of IoT Based Supply Chain
Plithogenic AHP,TOPSIS & VIKOR	Financial performance evaluation of manufacturing industries.
Plithogenic MABAC, BWM	Supplier Selection problem
Plithogenic SWARA-TOPSIS	Food Processing Methods with Different Normalization Techniques
PROMTHEE Plithogenic Pythagorean Hypergraph	Smart Materials Selection

Plithogenic Best-Worst method	Supply chain ,Evaluation of Teaching & Learning process
Plithogenic CRITIC-MAIRCA	Feasible Livestock Feeding Stuffs
TOPSIS-Plithogenic Cognitive Maps	Performance of teachers being evaluated
Plithogenic CRITIC	Optimize logistics problems
Plithogenic VIKOR	Group decision making problem
Plithogenic PIPRECIA	Logistics Selection problem
Plithogenic FUCOM-MAIRCA	sustainable factors and suppliers for transforming business sectors to Green Globe Creators

Analysis on Publications

This section presents the analysis of the publications of the articles based on Plithogenic MCDM. The list of journals in which the article are published are presented in Table 5.

Table : 5 List of the Journals

Name of the Journal	Number
Neutrosophic Sets and Systems (NSS).	27
Infinite Study	4
International Journal of Neutrosophic Science (IJNS)	17
AIP Conference Proceedings	1
MDPI- Symmetry	3
Intelligent and Fuzzy Techniques- Smart and Innovative Solutions	2
In Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics. Academic Press	1
International Journal of Sustainable Engineering	1
Acta Scientific Computer Sciences	1
Multimedia Tools and Applications	1
Artificial intelligence in medicine	2
Octogon Mathematical Magazine	1
International Journal of Creative Research Thoughts (IJCRT)	1
International Journal of Mechanical Engineering	1
Indian Journal of Natural Sciences (IJONS)	1
Journal of Intelligent & Fuzzy Systems,	1
Journal of Fuzzy Extension and Applications	3
AIP Publishing LLC	3
Advances in Decision Making	1
Journal of Cleaner Production	1
International Conference on Intelligent and fuzzy systems	1
Mathematics-MDPI	1
Computers, Materials & Continua., Tech Press Science.	1
Risk Management	1
Artificial intelligence in medicine.	1
International Journal of Fuzzy Systems	1
Sustainability	1
IGI Global - Optimization and Decision-Making in the Renewable Energy	1

Industry(Book)	
Stochastic Modeling & Applications	1
Journal of Ambient Intell Humaniz Comput	1
Indian Journal of Science and Technology	1
Optimization Theory Based on Neutrosophic and Plithogenic Sets	1
Research gate	1
International Journal of Fuzzy Logic and Intelligent Systems	1
IJSRM- International Journal of Scientific Research and Management	1
Galoitica Journal Of Mathematical Structures And Applications (GJMSA)	2
REVISTA INVESTIGACION OPERACIONAL	1
Current Advances in Mechanical Engineering	1
International Research Journal of Modernization in Engineering Technology and Science	1

5. Conclusion and Future Directions

This review paper has presented an extensive overview of the philosophy of plithogeny and its applications in decision making. The ongoing research works in the plithogenic field is an actual substantiation for the effectiveness of plithogenic principles in making optimal decisions. The theoretical developments of plithogeny constructed by Smarandache are articulated by the plithogenic researchers in terms of excellent applications. The applications of plithogeny have smoothened the hurdles of determining solutions to the problems of varied kinds. The manifestations of plithogeny embedded with realistic applications in various fields are also bountiful. The recent developments and extensions of plithogenic sets are the instances of the comprehensives of plithogenic sets. As the researches and contributions of plithogenic sets are persisting with vibrancies, the opportunities of integrating plithogeny with other facets of decision making models are in manifold. The construction of new kinds of sets at recent times shall also be dealt with plithogeny. The contributions of Plithogenic sets are scaled up within a very short span of time and certainly, it is expected to reach a huge number in the coming decades.

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Neutrosophic General Machine: A Group-Based Study

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Abstract. Interval-valued neutrosophic sets have been shown to provide a limited platform for computational complexity, but neutrosophic sets are suitable for it. The neutrosophic setsare a suitable mechanism for interpreting the philosophical problems of real-life, but not for scientific problems because it is difficult to consolidate. This study aims to develop the notion of neutrosophic single-valued general machine over a finite group, which is knows as "group neutrosophic general machine", for simplicity, GNGM. After that, we present the notions of max-min GNGM, single-valued neutrosophic subgroup (SVNSG) and single-valued neutrosophic normal subgroup (SVNNSG). We show that if there exists a homomorphism between two GNGMs $\mathcal{M}_1$ and $\mathcal{M}_2$ and M is a single-valued neutrosophic fundamental (SVNF) of $\mathcal{M}_2$, then $f^{-1}(M)$ is a SVNF of $\mathcal{M}_1$. Also, if M is a singe-valued neutrosophic kernel (SVNK) of $\mathcal{M}_2$, then $f^{-1}(M)$ is a SVNK of $\mathcal{M}_1$. Moreover, we prove that if M is a SVNK and N is a SVNF of $\mathcal{M}$, then $M * N$ is a SVNF of $\mathcal{M}$. In addition, we show that if M and N are SVNK of $\mathcal{M}$, then $M * N$ is a SVNK of $\mathcal{M}$.

Keywords: Neutrosophic set, Automata, Intuitionistic set, Submachine, General fuzzy automata

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1. Introduction

The concept of 'fuzzy' together with some other notions in mathematics and other areas were fuzzified by Zadeh [16] in 1965. Within this real, among the first investigations was the concept of fuzzy automaton suggested by Wee [15] and Santos [8]. Doostfatemeh and Kremer [4] introduced the concept of general fuzzy automata.

An intuitionistic fuzzy set can be viewed as an alternative approach when available information is not sufficient to define the impreciseness by the conventional fuzzy set. Subsequently, as a generalization, the concept of intuitionistic general fuzzy automata has been introduced and studied by Shamsizadeh and Zahedi [10]. For more details see the recent literature as [2,3,5,9,11,12].

Neutrosophy deals with origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophy is the basis of neutrosophic sets (derivative of neutrosophy). A neutrosophic set is a general framework which generalizes the concept of fuzzy set, interval valued fuzzy set, and intuitionistic fuzzy set. Wang et al. [14] introduced single valued neutrosophic sets which is a neutrosophic set defined in the range [0, 1]. Wang et.al: [13] introduced the notion of interval-valued neutrosophic sets. Interval-valued neutrosophic sets have been shown that fuzzy sets provides limited platform for computational complexity but neutrosophic sets is suitable for it. The neutrosophic sets is an appropriate mechanism for interpreting real-life philosophical problems but not for scientific problems since it is difficult to consolidate.

The concept of interval neutrosophic finite state machine was introduced by Tahir Mahmood [7]. In 2019 [6] Kavikumar introduced the notion of neutrosophic general fuzzy automata.

The present paper is organized as follows: Section 2 encompasses preliminary information pertaining to the content of the paper. In Section 3 we present the notion of neutrosophic single-valued general automata over a finite group which is knows as "group neutrosophic general machine" (GNGM). Moreover, we give the notions of max-min GNGM, single-valued neutrosophic subgroup (SVNSG) and single-valued neutrosophic normal subgroup (SVNNSG). We prove that if there exists a homomorphism between two GNGMs $\mathcal{M}_1$ and $\mathcal{M}_2$ and M be a single-valued neutrosophic fundamental (SVNF) of $\mathcal{M}_2$, then $f^{-1}(M)$ is a SVNF of $\mathcal{M}_1$. Also, if M is a singe-valued neutrosophic kernel (SVNK) of $\mathcal{M}_2$, then $f^{-1}(M)$ is a SVNK of $\mathcal{M}_1$. Moreover, we prove that if M is a SVNK and N is a SVNF of $\mathcal{M}$, then $M * N$ is a SVNF of $\mathcal{M}$. In addition, we show that if M and N are SVNK of $\mathcal{M}$, then $M * N$ is a SVNK of $\mathcal{M}$.

2. Preliminaries

In this section, some concepts and definitions related to single-valued neutrosophy and automata are introduced.

Definition 2.1. [4] A general fuzzy automaton (GFA) is considered as: $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$, where (i) Q is a finite set of states, $Q = \{q_1, q_2, ..., q_n\}$, (ii) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, ..., a_m\}$, (iii) $\tilde{R}$ is the set of fuzzy start states, $\tilde{R} \subseteq \tilde{P}(Q)$, (iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, ..., b_k\}$, (v) $\omega : Q \to Z$ is the output function, (vi) $\tilde{\delta} : (Q \times [0, 1]) \times \Sigma \times Q \to [0, 1]$ is the augmented transition function. (vii) Function $F_1 : [0, 1] \times [0, 1] \to [0, 1]$ is called membership assignment function. (viii) $F_2 : [0, 1]^* \to [0, 1]$, is called multi-membership resolution function.

Definition 2.2. Let Σ be a space of points (objects), with a generic element in Σ denoted by x. A neutrosophic set A in Σ is characterized by a truth-membership function T_A, an indeterminacy-membership function I_A and a falsity-membership function F_A. $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$. That is $T_A : \Sigma \to]0^-, 1^+[$, $I_A : \Sigma \to]0^-, 1^+[$, $F_A : \Sigma \to]0^-, 1^+[$. There is no restriction on the sum of $T_A(x), I_A(x)$ and $F_A(x)$, so $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

Definition 2.3. Single-valued neutrosophic set is the immediate results of neutrosophic set if it is defined over standard unit interval $[0, 1]$ instead of the non-standard unit interval $]0^-, 1^+[$. A single-valued neutrosophic subset (SVNS) A of Q is defined by $SVNS(A) = \{(x, T_A(x), I_A(x), F_A(x)) \big| x \in \Sigma\}$, where $T_A(x), I_A(x), F_A(x) : \Sigma \to [0, 1]$ such that $0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3$.

Definition 2.4. [1] The support of a single-valued neutrosophic set $A = \{(x, T_A(x), I_A(x), F_A(x)) \big| x \in X\}$ is denoted by $supp(A)$, defined by $supp(A) = \{x \in A \big| T_A(x) \neq 0, I_A(x) \neq 0, F_A(x) \neq 0\}$. The support of a single-valued neutrosophic set is a crisp set.

3. Single-Valued Neutrosophic General Machine Over a Finite Group

Definition 3.1. A group single-valued neutrosophic general machine (GNGM) is a seven-tuple machine $\mathcal{M} = (Q, *, \Sigma, \tilde{R}, \tilde{\delta}, E_1, E_2)$, such that $(Q, *)$ is a finite group and

1. Q is called the set of states,
2. Σ is a finite set of input symbols,
3. $\tilde{R} \subseteq \tilde{P}(Q)$ is the set of single-valued neutrosophic initial states,
4. $\tilde{\delta} : (Q \times [0, 1] \times [0, 1] \times [0, 1]) \times \Sigma \times Q \to [0, 1] \times [0, 1] \times [0, 1]$ is the single-valued neutrosophic augmented transition function,
5. $E_1 = (E_1^T, E_1^I, E_1^F)$, where $E_1^T : [0, 1] \times [0, 1] \to [0, 1]$ is a t-norm and it is called the truth-membership assignment function. $E_1^T(T, T_\delta)$ is motivated by two parameters T and T_δ, where T is the truth-membership value of a predecessor and T_δ is the truth-membership value of the transition. Also, $E_1^I : [0, 1] \times [0, 1] \to [0, 1]$ is a t-norm and it is called the indeterminacy-membership function. $E_1^I(I, I_\delta)$ is motivated by two parameters I and I_δ, where I is the indeterminacy-membership value of a predecessor and I_δ is the indeterminacy-membership value of the transition. Moreover, $E_1^F : [0, 1] \times [0, 1] \to [0, 1]$ is a t-conorm and it is called the falsity-membership function. $E_1^F(F, F_\delta)$ is motivated by two parameters F and F_δ, where F is the falsity-membership value of a predecessor and F_δ is the falsity-membership value of the transition.

   In this definition, the process that takes place upon the transition from the state q_i to

q_j on an input a_k is represented by:

$$T^{t+1}(q_j) = \tilde{\delta}_1((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j) = E_1^T(T^t(q_i), \delta_1(q_i, a_k, q_j)),$$
$$I^{t+1}(q_j) = \tilde{\delta}_2((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j) = E_1^I(I^t(q_i), \delta_2(q_i, a_k, q_j)),$$
$$F^{t+1}(q_j) = \tilde{\delta}_3((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j) = E_1^F(T^t(q_i), \delta_3(q_i, a_k, q_j)),$$

where

$$\tilde{\delta}((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j) = (\tilde{\delta}_1((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j), \tilde{\delta}_2((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j), \tilde{\delta}_3((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j)),$$

and

$$\delta(q_i, a_k, q_j) = (\delta_1(q_i, a_k, q_j), \delta_2(q_i, a_k, q_j), \delta_3(q_i, a_k, q_j)).$$

6. $E_2 = (E_2^T, E_2^I, E_2^F)$, where $E_2^T : [0,1]^* \to [0,1]$ is a T-conorm and it is called multi-truth-membership function, $E_2^I : [0,1]^* \to [0,1]$ is a T-conorm and it is called multi-indetermincy-membership function, $E_2^F : [0,1]^* \to [0,1]$ is a T-norm and it is called multi-falsity-membership function.

Example 3.2. Let the GNGM $\mathcal{M} = (Q, *, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ such that $Q = \{q_0, q_1, q_2\}, \Sigma = \{a\}$, $\tilde{R} = \{(q_0, 0.4, 0.7, 0.3)\}$ and δ is defined as follows:

$$\delta(q_0, a, q_0) = (0.6, 0.7, 1), \qquad \delta(q_0, a, q_1) = (0.7, 0.5, 0.5),$$
$$\delta(q_0, a, q_2) = (0.9, 0.7, 0.4), \qquad \delta(q_1, a, q_1) = (0.4, 0.5, 0.2),$$
$$\delta(q_1, a, q_2) = (0.3, 0.7, 0.6), \qquad \delta(q_2, a, q_0) = (0.7, 0.9, 0.6),$$
$$\delta(q_2, a, q_1) = (0.7, 1, 1), \qquad \delta(q_2, a, q_2) = (0.6, 0.9, 0.5).$$

Now, we can consider E_1 as follows:

1. $E_1^T = T \wedge T_\delta,\ E_1^I = I \wedge I_\delta,\ E_1^F = F \vee F_\delta$,

$$T^{t+1}(q_m) = \bigvee_{i=1}^{n} E_1^T(T^t(q_i), \delta_1(q_i, a_k, q_m)),$$
$$I^{t+1}(q_m) = \bigvee_{i=1}^{n} E_1^I(I^t(q_i), \delta_2(q_i, a_k, q_m)),$$
$$F^{t+1}(q_m) = \bigwedge_{i=1}^{n} E_3^F(F^t(q_i), \delta_3(q_i, a_k, q_m)),$$

2. $E_1^T = T.T_\delta,\ E_1^I = I.I_\delta,\ E_1^F = F + F_\delta - F.F_\delta,$

$$T^{t+1}(q_m) = \bigvee_{i=1}^{n} E_1^T(T^t(q_i), \delta_1(q_i, a_k, q_m)),$$
$$I^{t+1}(q_m) = \bigvee_{i=1}^{n} E_1^I(I^t(q_i), \delta_2(q_i, a_k, q_m)),$$
$$F^{t+1}(q_m) = \bigwedge_{i=1}^{n} E_3^F(F^t(q_i), \delta_3(q_i, a_k, q_m)),$$

3. $E_1^T = T \wedge T_\delta,\ E_1^I = I \wedge I_\delta,\ E_1^F = F \vee F_\delta,$

$$T^{t+1}(q_m) = T_p(T_p(T^t(q_i), \delta_1(q_i, a_k, q_m))),$$
$$I^{t+1}(q_m) = T_p(T_p(I^t(q_i), \delta_2(q_i, a_k, q_m)),$$
$$F^{t+1}(q_m)) = S_p(S_p E_3^F(F^t(q_i), \delta_3(q_i, a_k, q_m))),$$

where T_p is the product t-norm and S_p is the product t-conorm.
If we choose the case 1, then we have

$$T^{t_1}(q_0) = E_1^T(T^{t_0}(q_0), \delta_1(q_0, a, q_0)) = 0.4 \wedge 0.6 = 0.4,$$
$$I^{t_1}(q_0) = E_1^I(I^{t_0}(q_0), \delta_2(q_0, a, q_0)) = 0.7 \wedge 0.7 = 0.7,$$
$$F^{t_1}(q_0) = E_1^F(F^{t_0}(q_0), \delta_3(q_0, a, q_0)) = 0.3 \vee 1 = 1,$$
$$T^{t_1}(q_1) = E_1^T(T^{t_0}(q_0), \delta_1(q_0, a, q_1)) = 0.4 \wedge 0.7 = 0.4,$$
$$I^{t_1}(q_1) = E_1^I(I^{t_0}(q_0), \delta_2(q_0, a, q_1)) = 0.7 \wedge 0.5 = 0.5,$$
$$F^{t_1}(q_1) = E_1^F(F^{t_0}(q_0), \delta_3(q_0, a, q_1)) = 0.3 \vee 0.5 = 0.5,$$
$$T^{t_1}(q_2) = E_1^T(T^{t_0}(q_0), \delta_1(q_0, a, q_2)) = 0.4 \wedge 0.9 = 0.4,$$
$$I^{t_1}(q_2) = E_1^I(I^{t_0}(q_0), \delta_2(q_0, a, q_2)) = 0.7 \wedge 0.7 = 0.7,$$
$$F^{t_1}(q_2) = E_1^F(F^{t_0}(q_0), \delta_3(q_0, a, q_2)) = 0.3 \vee 0.4 = 0.4,$$
$$T^{t_2}(q_0) = E_1^T(T^{t_1}(q_0), \delta_1(q_0, a, q_0)) \vee E_1^T(T^{t_1}(q_2), \delta_1(q_2, a, q_0)) = (0.4 \wedge 0.6) \vee (0.4 \wedge 0.7) = 0.4,$$
$$I^{t_2}(q_0) = E_1^I(I^{t_1}(q_0), \delta_2(q_0, a, q_0)) \vee E_1^I(I^{t_1}(q_2), \delta_2(q_2, a, q_0)) = (0.7 \wedge 0.7) \vee (0.7 \wedge 0.9) = 0.7,$$
$$F^{t_2}(q_0) = E_1^F(F^{t_1}(q_0), \delta_3(q_0, a, q_0)) \wedge E_1^F(F^{t_1}(q_2), \delta_3(q_2, a, q_0)) = (1 \vee 1) \wedge (0.4 \vee 0.6) = 0.6,$$
$$T^{t_2}(q_1) = E_1^T(T^{t_1}(q_0), \delta_1(q_0, a, q_1)) \vee E_1^T(T^{t_1}(q_1), \delta_1(q_1, a, q_1)) \vee E_1^T(T^{t_1}(q_2), \delta_1(q_2, a, q_1))$$
$$= (0.4 \wedge 0.7) \vee (0.4 \wedge 0.4) \vee (0.4 \wedge 0.7) = 0.4,$$

$$
\begin{aligned}
I^{t_2}(q_1) &= E_1^I(I^{t_1}(q_0), \delta_2(q_0,a,q_1)) \vee E_1^I(I^{t_1}(q_1), \delta_2(q_1,a,q_1)) \vee E_1^I(I^{t_1}(q_2), \delta_2(q_2,a,q_1)) \\
&= (0.7 \wedge 0.5) \vee (0.5 \wedge 0.5) \vee (0.7 \wedge 1) = 0.7, \\
F^{t_2}(q_1) &= E_1^F(F^{t_1}(q_0), \delta_3(q_0,a,q_1)) \wedge E_1^F(F^{t_1}(q_1), \delta_3(q_1,a,q_1)) \wedge E_1^F(F^{t_1}(q_2), \delta_3(q_2,a,q_1)) \\
&= (1 \vee 0.5) \wedge (0.5 \vee 0.2) \wedge (0.4 \vee 1) = 0.5, \\
T^{t_2}(q_2) &= E_1^T(T^{t_1}(q_0), \delta_1(q_0,a,q_2)) \vee E_1^T(T^{t_1}(q_1), \delta_1(q_1,a,q_2)) \vee E_1^T(T^{t_1}(q_2), \delta_1(q_2,a,q_2)) \\
&= (0.4 \wedge 0.9) \vee (0.4 \wedge 0.3) \vee (0.4 \wedge 0.6) = 0.4, \\
I^{t_2}(q_2) &= E_1^I(I^{t_1}(q_0), \delta_2(q_0,a,q_2)) \vee E_1^I(I^{t_1}(q_1), \delta_2(q_1,a,q_2)) \vee E_1^I(I^{t_1}(q_2), \delta_2(q_2,a,q_2)) \\
&= (0.7 \wedge 0.7) \vee (0.5 \wedge 0.7) \vee (0.7 \wedge 0.9) = 0.7, \\
F^{t_2}(q_2) &= E_1^F(F^{t_1}(q_0), \delta_3(q_0,a,q_2)) \wedge E_1^F(F^{t_1}(q_1), \delta_3(q_1,a,q_2)) \wedge E_1^F(F^{t_1}(q_2), \delta_3(q_2,a,q_2)) \\
&= (1 \vee 0.4) \wedge (0.5 \vee 0.6) \wedge (0.4 \vee 0.5) = 0.5.
\end{aligned}
$$

Clearly, we can see that there are three simultaneous transition to the action states q_0, q_1 and q_2 at time t_2.

Definition 3.3. Let $\mathcal{M} = (Q, *, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be a GNGM. We define max-min GNGM of the form $\mathcal{M} = (Q, *, \Sigma, \tilde{\delta}^*, \tilde{R}, E_1, E_2)$ such that $\tilde{\delta}^* : Q_{act} \times \Sigma^* \times Q \to [0,1] \times [0,1] \times [0,1]$, where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), ...\}$ and for every $i \geq 0$,

$$
\tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), \Lambda, p) = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{otherwise} \end{cases},
$$

$$
\tilde{\delta}_2^*((q, T^t(q), I^t(q), F^t(q)), \Lambda, p) = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{otherwise} \end{cases},
$$

$$
\tilde{\delta}_3^*((q, T^t(q), I^t(q), F^t(q)), \Lambda, p) = \begin{cases} 0 & \text{if p=q} \\ 1 & \text{otherwise} \end{cases},
$$

and for every $i \geq 1, \tilde{\delta}^*((q, T^t(q), I^t(q), F^t(q)), a, p) = \tilde{\delta}((q, T^t(q), I^t(q), F^t(q)), a, p)$ and recursively,

$$
\begin{aligned}
&\tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), a_1a_2...a_n, p) = \vee\{\tilde{\delta}_1((q, T^t(q), I^t(q), F^t(q)), a_1, p_1) \wedge ... \\
&\quad \wedge \tilde{\delta}_1((p_{n-1}, T^t(p_{n-1}), I^t(p_{n-1}), F^t(p_{n-1})), a_n, p) \big| p_1 \in Q_{act}(t_1), ..., p_{n-1} \in Q_{act}(t_{n-1})\},
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{\delta}_2^*((q, T^t(q), I^t(q), F^t(q)), a_1a_2...a_n, p) = \vee\{\tilde{\delta}_2((q, T^t(q), I^t(q), F^t(q)), a_1, p_1) \wedge ... \\
&\quad \wedge \tilde{\delta}_2((p_{n-1}, T^t(p_{n-1}), I^t(p_{n-1}), F^t(p_{n-1})), a_n, p) \big| p_1 \in Q_{act}(t_1), ..., p_{n-1} \in Q_{act}(t_{n-1})\},
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{\delta}_3^*((q, T^t(q), I^t(q), F^t(q)), a_1a_2...a_n, p) = \wedge\{\tilde{\delta}_3((q, T^t(q), I^t(q), F^t(q)), a_1, p_1) \vee ... \\
&\quad \vee \tilde{\delta}_3((p_{n-1}, T^t(p_{n-1}), I^t(p_{n-1}), F^t(p_{n-1})), a_n, p) \big| p_1 \in Q_{act}(t_1), ..., p_{n-1} \in Q_{act}(t_{n-1})\},
\end{aligned}
$$

in which $a_i \in \Sigma$, for all $1 \leq i \leq n$ and assuming that the entered in put at time t_i is a_i, for $1 \leq i \leq n-1$.

In the rest of paper, instead of max-min GNGM we say that GNGM.

Definition 3.4. A SVNS N of Q is called a single-valued neutrosophic subgroup (SVNSG) of Q if the following properties hold:

(1) $T_N(p * q) \geq T_N(p) \wedge T_N(q)$,
(2) $I_N(p * q) \geq I_N(p) \wedge I_N(q)$,
(3) $F_N(p * q) \leq F_N(p) \vee F_N(q)$,
(4) $(T_N(p), I_N(p), F_N(p)) = (T_N(p^{-1}), I_N(p^{-1}), F_N(p^{-1}))$,

for every $p, q \in Q$.

Definition 3.5. A SVNSG N of Q is called a single-valued neutrosophic normal subgroup (SVNNSG) of Q if for every $p, q \in Q$, $T_N(p * q * p^{-1}) \geq T_N(q), I_N(p * q * p^{-1}) \geq I_N(q), F_N(p * q * p^{-1}) \leq F_N(q)$.

Definition 3.6. Let N and M be two SVNS of Q. Then product $N * M$ is defined as follows: $N * M(p) = (T_{N*M}(p), I_{N*M}(p), F_{N*M}(p))$, where

$$T_{N*M}(p) = \vee\{T_N(r) \wedge T_M(s) \big| r, s \in Q, p = r * s\},$$
$$I_{N*M}(p) = \vee\{I_N(r) \wedge I_M(s) \big| r, s \in Q, p = r * s\},$$
$$F_{N*M}(p) = \wedge\{F_N(r) \vee F_M(s) \big| r, s \in Q, p = r * s\},$$

for every $p \in Q$.

Definition 3.7. Let N and M be two SVNSG of Q such that $N \subseteq M$. Then N is called SVNNSG of M if for every $p, q \in Q$, we have

$$T_N(p * q * p^{-1}) \geq T_N(q) \wedge T_M(p),$$
$$I_N(p * q * p^{-1}) \geq I_N(q) \wedge I_M(p),$$
$$F_N(p * q * p^{-1}) \leq F_N(q) \vee F_M(p).$$

Let N and M be two SVNSG of Q such that N is a SVNNSG of M. Then $supp(N)$ is a SVNNSG of $supp(M)$, too.

Definition 3.8. Let N and M be two SVNS of the groups Q and P, respectively. Let $f : Q \to P$ be a group homomorphism. Then the SVNSs $f(N)$ of P and $f^{-1}(M)$ of Q are defined as follows:

$$f(N)(p) = \begin{cases} (T_{f(N)}(p), I_{f(N)}(p), F_{f(N)}(p)) & \text{if } f^{-1}(P) \neq \emptyset \\ (0,0,1) & \text{if } f^{-1}(P) = \emptyset \end{cases}, \tag{1}$$

where $T_{f(N)}(p) = \vee\{T_N(q) \big| q \in Q \text{and } f(q) = p\}$, $I_{f(N)}(p) = \vee\{I_N(q) \big| q \in Q \text{and } f(q) = p\}$ and $F_{f(N)}(p) = \wedge\{F_N(q) \big| q \in Q \text{and } f(q) = p\}$, for every $p \in P$. Also, $f^{-1}(M)(q) = (T_M(f(q)), I_M(f(q)), F_M(f(q)))$, for every $q \in Q$.

Definition 3.9. Let M and N be two SVNS of the groups Q and P, respectively. A function $f : Q \to P$ is called a weak homomorphism from M into N if the following conditions hold:

(1) f is an epimorphism,
(2) $f(M) \subseteq N$.

If f is an isomorphism from Q onto P, then we say that f is a weak isomorphism from M to N.

Definition 3.10. Let $\mathcal{M}_i = (Q_i, *, \Sigma_i, \tilde{R}_i, \tilde{\delta}^i, E_1, E_2), i = 1, 2$, be two GNGM. A pair of functions (f, g), where $f : Q_1 \to Q_2$ and $g : \Sigma_1 \to \Sigma_2$ is called a homomorphism from $\mathcal{M}_1$ into $\mathcal{M}_2$ and written $(f, g) : \mathcal{M}_1 \to \mathcal{M}_2$ if the following conditions hold:

(1) f is a group homomorphism,
(2) $\tilde{\delta}_1^1((q, T^t(q), I^t(q), F^t(q)), a, p) \leq \tilde{\delta}_1^2((f(q), T^t(f(q)), I^t(f(q)), F^t(f(q))), g(a), f(q))$,
(3) $\tilde{\delta}_2^1((q, T^t(q), I^t(q), F^t(q)), a, p) \leq \tilde{\delta}_2^2((f(q), T^t(f(q)), I^t(f(q)), F^t(f(q))), g(a), f(q))$,
(4) $\tilde{\delta}_3^1((q, T^t(q), I^t(q), F^t(q)), a, p) \geq \tilde{\delta}_3^2((f(q), T^t(f(q)), I^t(f(q)), F^t(f(q))), g(a), f(q))$,

for every $p, q \in Q$ and $a \in \Sigma_i$.

The pair (f, g) is called a strong homomorphism from $\mathcal{M}_1$ into $\mathcal{M}_2$ if it satisfies 1 of Definition 3.10, and the following conditions holds:

(1) $\tilde{\delta}_1^2((f(q), T^t(f(q)), I^t(f(q)), F^t(f(q))), g(a), f(p)) =$
$\vee\{\tilde{\delta}_1^1((q, T^t(q), I^t(q), F^t(q)), a, r) \big| f(r) = f(p), r \in Q_1\}$,
(2) $\tilde{\delta}_2^2((f(q), T^t(f(q)), I^t(f(q)), F^t(f(q))), g(a), f(p)) =$
$\vee\{\tilde{\delta}_2^1((q, T^t(q), I^t(q), F^t(q)), a, r) \big| f(r) = f(p), r \in Q_1\}$,
(3) $\tilde{\delta}_3^2((f(q), T^t(f(q)), I^t(f(q)), F^t(f(q))), g(a), f(p)) =$
$\wedge\{\tilde{\delta}_3^1((q, T^t(q), I^t(q), F^t(q)), a, r) \big| f(r) = f(p), r \in Q_1\}$,

for every $p, q \in Q$, $a \in \Sigma_i$ and $z \in Z$.

In Definition 3.10, if $\Sigma_1 = \Sigma_2$ and g is the identity map of Σ_1, then we say that f is a homomorphism (strong homomorphism) from $\mathcal{M}_1$ into $\mathcal{M}_2$ and we write $f : \mathcal{M}_1 \to \mathcal{M}_2$.

Definition 3.11. Let $\mathcal{M} = (Q, *, \Sigma, \tilde{R}, \tilde{\delta}, E_1, E_2)$ be a GNGM. A SVNS M of Q is called a single-valued neutrosophic kernel (SVNK) of δ if the following conditions hold:

(1) M is a SVNS of Q,
(2) $T_M(p) \geq \delta_1(q, a, p) \wedge T_M(q)$,
(3) $I_M(p) \geq \delta_2(q, a, p) \wedge I_M(q)$,
(4) $F_M(p) \leq \delta_3(q, a, p) \vee F_M(q)$,

for every $p, q \in Q$ and $a \in \Sigma$.

Theorem 3.12. *Let $\mathcal{M}$ be a GNGM. A SVNNSG M of Q is a SVNK of $\mathcal{M}$ if and only if the following conditions hold:*

(1) $T_M(p * r^{-1}) \geq$
$\tilde{\delta}_1^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), x, p) \wedge \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \wedge T_M(k)$,

(2) $I_M(p * r^{-1}) \geq$
$\tilde{\delta}_2^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), x, p) \wedge \tilde{\delta}_2^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \wedge I_M(k)$,

(3) $F_M(p * r^{-1}) \leq$
$\tilde{\delta}_3^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), x, p) \vee \tilde{\delta}_3^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \vee F_M(k)$,

for every $p, q, k, r \in Q$ and $x \in \Sigma^*$.

Proof. Let M be a SVNK of $\mathcal{M}$. We prove the claim by induction on $|x| = n$. Let $n = 0$. Then $x = \Lambda$. If $p = q * k$ and $r = q$. So,

$$\tilde{\delta}_1^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), \Lambda, p) \wedge \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), \Lambda, r) \wedge T_M(k) \leq T_M(k) \leq T_M(q*k*q^{-1}),$$

$$\tilde{\delta}_2^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), \Lambda, p) \wedge \tilde{\delta}_2^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), \Lambda, r) \wedge I_M(k) \leq I_M(k) \leq I_M(q*k*q^{-1}),$$

$$\tilde{\delta}_3^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), \Lambda, p) \vee \tilde{\delta}_3^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), \Lambda, r) \vee F_M(k) \geq F_M(k) \geq F_M(q*k*q^{-1}).$$

If $p \neq q * k$ or $r \neq q$. Then

$$\tilde{\delta}_1^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), \Lambda, p) \wedge \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), \Lambda, r) \wedge T_M(k) = 0 \leq T_M(p*r^{-1}),$$

$$\tilde{\delta}_2^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), \Lambda, p) \wedge \tilde{\delta}_2^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), \Lambda, r) \wedge I_M(k) = 0 \leq I_M(p*r^{-1}),$$

$$\tilde{\delta}_3^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), \Lambda, p) \vee \tilde{\delta}_3^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), \Lambda, r) \vee F_M(k) = 1 \geq F_M(p*r^{-1}).$$

So, the result holds for $n = 0$. Now, let the claim holds for every $y \in \Sigma^*$ such that $|y| = n-1$ and $n > 0$. Let $x \in \Sigma^*, x = ya, y \in \Sigma^*, a \in \Sigma, |y| = n-1$ and $n > 0$. Then

$$\begin{aligned}
\tilde{\delta}_1^*((q*k, & T^t(q*k), I^t(q*k), F^t(q*k)), x, p) \wedge \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \wedge T_M(k) \\
&= (\vee\{\tilde{\delta}_1^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), y, u) \\
&\wedge \tilde{\delta}_1^*((u, T^{t+n-1}(u), I^{t+n-1}(u), F^{t+n-1}(u)), a, p) \big| u \in Q\}) \\
&\wedge (\vee\{\tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), y, v) \\
&\wedge \tilde{\delta}_1^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r) \big| v \in Q\}) \wedge T_M(k) \\
&\leq \vee\{\vee\{\tilde{\delta}_1^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), y, u) \\
&\wedge \tilde{\delta}_1^*((u, T^{t+n-1}(u), I^{t+n-1}(u), F^{t+n-1}(u)), a, p) \\
&\wedge \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), y, v) \\
&\wedge \tilde{\delta}_1^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r) \wedge T_M(k) \big| u \in Q\} \big| v \in Q\} \\
&\leq \vee\{\vee\{T_M(u*v^{-1}) \wedge \tilde{\delta}_1^*((u, T^{t+n-1}(u), I^{t+n-1}(u), F^{t+n-1}(u)), a, p) \\
&\wedge \tilde{\delta}_1^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r) \big| u \in Q\} \big| v \in Q\} \\
&\leq \vee\{\vee\{T_M(v^{-1}*u) \\
&\wedge \tilde{\delta}_1^*((v*v^{-1}*u, T^{t+n-1}(v*v^{-1}*u), I^{t+n-1}(v*v^{-1}*u), F^{t+n-1}(v*v^{-1}*u)), a, p) \\
&\wedge \tilde{\delta}_1^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r) \big| u \in Q\} \big| v \in Q\} \\
&\leq T_M(p*r^{-1}),
\end{aligned}$$

also,

$$\begin{aligned}
\tilde{\delta}_2^*((q*k, & T^t(q*k), I^t(q*k), F^t(q*k)), x, p) \wedge \tilde{\delta}_2^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \wedge I_M(k) \\
&= (\vee\{\tilde{\delta}_2^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), y, u) \\
&\wedge \tilde{\delta}_2^*((u, T^{t+n-1}(u), I^{t+n-1}(u), F^{t+n-1}(u)), a, p) \big| u \in Q\}) \\
&\wedge (\vee\{\tilde{\delta}_2^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), y, v) \\
&\wedge \tilde{\delta}_2^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r) \big| v \in Q\}) \wedge I_M(k) \\
&\leq \vee\{\vee\{\tilde{\delta}_2^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), y, u) \\
&\wedge \tilde{\delta}_2^*((u, T^{t+n-1}(u), I^{t+n-1}(u), F^{t+n-1}(u)), a, p)
\end{aligned}$$

$$\wedge\, \tilde{\delta}_2^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), y, v)$$
$$\wedge\, \tilde{\delta}_2^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r)$$
$$\wedge\, I_M(k)\big| u \in Q\}\big| v \in Q\}$$
$$\leq \vee\{\vee\{I_M(u * v^{-1})$$
$$\wedge\, \tilde{\delta}_2^*((u, T^{t+n-1}(u), I^{t+n-1}(u), F^{t+n-1}(u)), a, p)$$
$$\wedge\, \tilde{\delta}_2^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r)\big| u \in Q\}\big| v \in Q\}$$
$$\leq \vee\{\vee\{I_M(v^{-1} * u)$$
$$\wedge\, \tilde{\delta}_2^*((v * v^{-1} * u, T^{t+n-1}(v * v^{-1} * u), I^{t+n-1}(v * v^{-1} * u), F^{t+n-1}(v * v^{-1} * u)), a, p)$$
$$\wedge\, \tilde{\delta}_2^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r)\big| u \in Q\}\big| v \in Q\}$$
$$\leq I_M(p * r^{-1}),$$

and

$$\tilde{\delta}_3^*((q * k, T^t(q * k), I^t(q * k), F^t(q * k)), x, p) \vee \tilde{\delta}_3^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \vee F_M(k)$$
$$= (\wedge\{\tilde{\delta}_3^*((q * k, T^t(q * k), I^t(q * k), F^t(q * k)), y, u)$$
$$\vee\, \tilde{\delta}_3^*((u, T^{t+n-1}(u), I^{t+n-1}(u), F^{t+n-1}(u)), a, p)\big| u \in Q\})$$
$$\vee\, (\wedge\{\tilde{\delta}_3^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), y, v)$$
$$\vee\, \tilde{\delta}_3^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r)\big| v \in Q\}) \vee F_M(k)$$
$$\geq \wedge\{\wedge\{\tilde{\delta}_3^*((q * k, T^t(q * k), I^t(q * k), F^t(q * k)), y, u)$$
$$\vee\, \tilde{\delta}_3^*((u, T^{t+n-1}(u), I^{t+n-1}(u), F^{t+n-1}(u)), a, p)$$
$$\vee\, \tilde{\delta}_3^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), y, v)$$
$$\vee\, \tilde{\delta}_3^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r)$$
$$\vee\, F_M(k)\big| u \in Q\}\big| v \in Q\}$$
$$\geq \wedge\{\wedge\{F_M(u * v^{-1})$$
$$\vee\, \tilde{\delta}_3^*((u, T^{t+n-1}(u), I^{t+n-1}(u), F^{t+n-1}(u)), a, p)$$
$$\vee\, \tilde{\delta}_3^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r)\big| u \in Q\}\big| v \in Q\}$$
$$\geq \wedge\{\wedge\{F_M(v^{-1} * u)$$
$$\vee\, \tilde{\delta}_3^*((v * v^{-1} * u, T^{t+n-1}(v * v^{-1} * u), I^{t+n-1}(v * v^{-1} * u), F^{t+n-1}(v * v^{-1} * u)), a, p)$$
$$\vee\, \tilde{\delta}_3^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r)\big| u \in Q\}\big| v \in Q\}$$
$$\geq F_M(p * r^{-1}),$$

□

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Definition 3.13. Let $\mathcal{M}$ be a GNGM and M be a SVNSM of Q. M is called single-valued neutrosophic fundamental (SVNF) of $\mathcal{M}$ if the following hold:

(1) M is a SVNSG of Q,

(2)

$$T_M(p) \geq \tilde{\delta}_1((q, T^t(q), I^t(q), F^t(q)), x, p) \wedge T_M(q)$$
$$I_M(p) \geq \tilde{\delta}_2((q, T^t(q), I^t(q), F^t(q)), x, p) \wedge I_M(q)$$
$$F_M(p) \leq \tilde{\delta}_3((q, T^t(q), I^t(q), F^t(q)), x, p) \vee F_M(q),$$

for every $p, q \in Q$ and $x \in \Sigma$.

Theorem 3.14. *Let $\mathcal{M}$ be a GNGM and M be a SVNSM of Q. Then M is a SVNF of $\mathcal{M}$ if and only if the following hold:*

$$T_M(p) \geq \tilde{\delta}^*_1((q, T^t(q), I^t(q), F^t(q)), x, p) \wedge T_M(q)$$
$$I_M(p) \geq \tilde{\delta}^*_2((q, T^t(q), I^t(q), F^t(q)), x, p) \wedge I_M(q)$$
$$F_M(p) \leq \tilde{\delta}^*_3((q, T^t(q), I^t(q), F^t(q)), x, p) \vee F_M(q),$$

for every $p, q \in Q$ and $x \in \Sigma^*$.

Proof. The proof is similar to proof of Theorem 3.12. □

Theorem 3.15. *Let $\mathcal{M} = (Q, *, \Sigma, \tilde{R}, \tilde{\delta}, E_1, E_2)$ and $\mathcal{M}' = (Q', *, \Sigma, \tilde{R}', \tilde{\delta}', E_1, E_2)$ be two GNGM. Let f be a homomorphism from Q into Q'. If M is a SVNF of $\mathcal{M}'$, then $f^{-1}(M)$ is a SVNF of $\mathcal{M}$.*

Proof. The proof is straightforward. □

Theorem 3.16. *Let $\mathcal{M} = (Q, *, \Sigma, \tilde{R}, \tilde{\delta}, E_1, E_2)$ and $\mathcal{M}' = (Q', *, \Sigma, \tilde{R}', \tilde{\delta}', E_1, E_2)$ be two GNGM. Let f be a homomorphism from Q into Q'. If M is a SVNK of $\mathcal{M}'$, then $f^{-1}(M)$ is a SVNK of $\mathcal{M}$.*

Theorem 3.17. *Let $\mathcal{M} = (Q, *, \Sigma, \tilde{R}, \tilde{\delta}, E_1, E_2)$ and $\mathcal{M}' = (Q', *, \Sigma, \tilde{R}', \tilde{\delta}', E_1, E_2)$ be two GNGM. Let f be a strong homomorphism from $\mathcal{M}$ into $\mathcal{M}'$. If M is a SVNK of $\mathcal{M}$, then $f(M)$ is a SVNK of $\mathcal{M}'$.*

Proof. Let M be a SVNK of $\mathcal{M}$. Then M is a SVNNSG of Q. So, $f(M)$ is a SVNNSG of Q', since f is an epimorphism from Q onto Q'. Let $p', q', r', k' \in Q'$ and $x \in \Sigma$. Then

$$\begin{aligned}
&\tilde{\delta}_1'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p_1)\\
&\quad \wedge \tilde{\delta}_1'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \wedge T_{f(M)}(k_1)\\
&\quad = \tilde{\delta}_1'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p) \wedge \tilde{\delta}_1'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1)\\
&\quad \wedge (\vee\{T_M(k) \big| k \in Q, f(k) = k_1\})\\
&\quad = \vee\{\tilde{\delta}_1'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p)\\
&\quad \wedge \tilde{\delta}_1'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \wedge T_M(k) \big| k \in Q, f(k) = k_1\},
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\delta}_2'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p_1)\\
&\quad \wedge \tilde{\delta}_2'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \wedge I_{f(M)}(k_1)\\
&\quad = \tilde{\delta}_2'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p) \wedge \tilde{\delta}_2'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1)\\
&\quad \wedge (\vee\{I_M(k) \big| k \in Q, f(k) = k_1\})\\
&\quad = \vee\{\tilde{\delta}_2'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p)\\
&\quad \wedge \tilde{\delta}_2'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \wedge I_M(k) \big| k \in Q, f(k) = k_1\},
\end{aligned}$$

also,

$$\begin{aligned}
&\tilde{\delta}_3'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p_1)\\
&\quad \vee \tilde{\delta}_3'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \vee F_{f(M)}(k_1)\\
&\quad = \tilde{\delta}_3'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p) \vee \tilde{\delta}_3'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1)\\
&\quad \vee (\wedge\{F_M(k) \big| k \in Q, f(k) = k_1\}) \wedge \{\tilde{\delta}_3'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p)\\
&\quad \vee \tilde{\delta}_3'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \vee F_M(k) \big| k \in Q, f(k) = k_1\}.
\end{aligned}$$

Now, suppose that $p, q, r, k \in Q$ be such that $f(p) = p_1,\ f(q) = q_1,\ f(r) = r_1$ and $f(k) = k_1$. Then

$$\tilde{\delta}_1'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p_1)$$
$$\wedge\, \tilde{\delta}_1'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \wedge T_{f(M)}(k_1)$$
$$= \tilde{\delta}_1'^*((f(q * k), T^t(f(q * k)), I^t(f(q * k)), F^t(f(q * k))), x, f(p))$$
$$\wedge\, \tilde{\delta}_1'^*((f(q), T^{t'}(f(q)), I^{t'}(f(q)), F^{t'}(f(q))), x, f(r)) \wedge T_M(k)$$
$$= \vee\{\tilde{\delta}_1^*((q * k, T^t(q * k), I^t(q * k), F^t(q * k)), x, h) \big| f(h) = f(p), h \in Q\}$$
$$\wedge \vee\{\tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, l) \big| f(l) = f(r), l \in Q\} \wedge T_M(k)$$
$$= \vee\{\vee\{\tilde{\delta}_1^*((q * k, T^t(q * k), I^t(q * k), F^t(q * k)), x, h) \wedge \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, l)$$
$$\wedge\, T_M(k) \big| f(h) = f(p), f(l) = f(r), l, h \in Q\}\}$$
$$\leq \vee\{\vee\{T_M(h * l^{-1}) \big| h \in Q, f(h) = f(p) \big| l \in Q, f(l) = f(r)\}\}$$
$$\leq T(f(M))(p_1 * r_1^{-1}).$$

Similarly,

$$I(f(M))(p_1 * r_1^{-1}) \geq \tilde{\delta}_2'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p_1)$$
$$\wedge\, \tilde{\delta}_2'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \wedge I_{f(M)}(k),$$

and

$$F(f(M))(p_1 * r_1^{-1}) \leq \tilde{\delta}_3'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p_1)$$
$$\vee\, \tilde{\delta}_3'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \vee F_{f(M)}(k).$$

Hence, $f(M)$ is a SVNK of $\mathcal{M}'$. □

Theorem 3.18. *Let $\mathcal{M}_i = (Q_i, *, \Sigma, \tilde{R}^i, \tilde{\delta}^i, E_1, E_2), i = 1, 2$ be two GNGM and f be a strong homomorphism from $\mathcal{M}_1$ onto $\mathcal{M}_2$. If M is a SVNF of $\mathcal{M}_1$, then $f(M)$ is a SVNF of $\mathcal{M}_2$.*

Proof. The proof is similar to the proof of Theorem 3.17. □

In the rest of paper for SVNGA $\mathcal{M} = (Q, *, \Sigma, \tilde{R}, \tilde{\delta}, E_1, E_2)$ the following conditions are satisfied:

$$\begin{aligned}
\tilde{\delta}_1^*((p*q, T^t(p*q), I^t(p*q), F^t(p*q)), x, r) &\leq \tilde{\delta}_1^*((p, T^{t'}(p), I^{t'}(p), F^{t'}(p)), x, k) \\
&\quad \wedge \tilde{\delta}_1^*((q, T^{t''}(q), I^{t''}(q), F^{t''}(q)), x, k), \\
\tilde{\delta}_2^*((p*q, T^t(p*q), I^t(p*q), F^t(p*q)), x, r) &\leq \tilde{\delta}_2^*((p, T^{t'}(p), I^{t'}(p), F^{t'}(p)), x, k) \\
&\quad \wedge \tilde{\delta}_2^*((q, T^{t''}(q), I^{t''}(q), F^{t''}(q)), x, k), \\
\tilde{\delta}_3^*((p*q, T^t(p*q), I^t(p*q), F^t(p*q)), x, r) &\geq \tilde{\delta}_3^*((p, T^{t'}(p), I^{t'}(p), F^{t'}(p)), x, k) \\
&\quad \vee \tilde{\delta}_3^*((q, T^{t''}(q), I^{t''}(q), F^{t''}(q)), x, k),
\end{aligned}$$

for every $p, q, r, k \in Q$ and $x \in \Sigma^*$. Also, e will denote the identity element of the group $(Q, *)$.

Theorem 3.19. *Let $\mathcal{M} = (Q, *, \Sigma, \tilde{R}, \tilde{\delta}, E_1, E_2)$ be a GNGM and M be a SVNK of $\mathcal{M}$. Then M is a SVNF of $\mathcal{M}$ if and only if*

1. $T_M(p) \geq \tilde{\delta}_1^*((e, T^t(e), I^t(e), F^t(e)), x, p) \wedge T_M(e)$,
2. $I_M(p) \geq \tilde{\delta}_2^*((e, T^t(e), I^t(e), F^t(e)), x, p) \wedge I_M(e)$,
3. $F_M(p) \leq \tilde{\delta}_3^*((e, T^t(e), I^t(e), F^t(e)), x, p) \vee F_M(e)$,

for every $p \in Q$ and $x \in \Sigma^*$

Proof. Let conditions 1, 2 and 3 are satisfied. Then

$$\begin{aligned}
T_M(p) &= T_M(p * r^{-1} * r) \\
&\geq T_M(p * r^{-1}) \wedge T_M(r) \\
&\geq \tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), x, p) \wedge \tilde{\delta}_1^*((e, T^{t'}(e), I^{t'}(e), F^{t'}(e)), x, r) \wedge T_M(q) \wedge T_M(r) \\
&\geq \tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), x, p) \wedge \tilde{\delta}_1^*((e, T^{t'}(e), I^{t'}(e), F^{t'}(e)), x, r) \wedge T_M(e) \wedge T_M(q) \\
&= \tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), x, p) \wedge T_M(q).
\end{aligned}$$

Since, $T_M(q * q^{-1}) \geq T_M(q) \wedge T_M(q^{-1}) = T_M(q)$, then $T_M(e) \geq T_M(q)$, also

$$\begin{aligned}
\tilde{\delta}_1^*((e, T^{t'}(e), I^{t'}(e), F^{t'}(e)), x, r) &\wedge \tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), x, r) \\
&\geq \tilde{\delta}_1^*((e*q, T^{t''}(e*q), I^{t''}(e*q), F^{t''}(e*q)), x, r),
\end{aligned}$$

so, $\tilde{\delta}_1^*((e, T^{t'}(e), I^{t'}(e), F^{t'}(e)), x, r) \geq \tilde{\delta}_1^*((e*q, T^{t''}(e*q), I^{t''}(e*q), F^{t''}(e*q)), x, r)$. On the other hand,

$$\begin{aligned} I_M(p) &= I_M(p * r^{-1} * r) \\ &\geq I_M(p * r^{-1}) \wedge I_M(r) \\ &\geq \tilde{\delta}_2^*((q, T^t(q), I^t(q), F^t(q)), x, p) \wedge \tilde{\delta}_2^*((e, T^{t'}(e), I^{t'}(e), F^{t'}(e)), x, r) \wedge I_M(q) \wedge I_M(r) \\ &\geq \tilde{\delta}_2^*((q, T^t(q), I^t(q), F^t(q)), x, p) \wedge \tilde{\delta}_2^*((e, T^{t'}(e), I^{t'}(e), F^{t'}(e)), x, r) \wedge I_M(e) \wedge I_M(q) \\ &\geq \tilde{\delta}_2^*((q, T^t(q), I^t(q), F^t(q)), x, p) \wedge I_M(q). \end{aligned}$$

Also,

$$\begin{aligned} F_M(p) &= F_M(p * r^{-1} * r) \\ &\leq F_M(p * r^{-1}) \vee F_M(r) \\ &\leq \tilde{\delta}_3^*((q, T^t(q), I^t(q), F^t(q)), x, p) \vee \tilde{\delta}_3^*((e, T^{t'}(e), I^{t'}(e), F^{t'}(e)), x, r) \vee F_M(q) \vee F_M(e) \\ &\leq \tilde{\delta}_3^*((q, T^t(q), I^t(q), F^t(q)), x, p) \vee F_M(q). \end{aligned}$$

Hence, M is a SVNF of $\mathcal{M}$. The converse is clear. □

Corollary 3.20. *If M is a SVNK and N is a SVNK of $\mathcal{M}$ such that $M \subseteq N$ and $(T_M(e), I_M(e), F_M(e)) = (T_N(e), I_N(e), F_N(e))$, then M is a SVNF of $\mathcal{M}$.*

Theorem 3.21. *Let M be a SVNK and N be a SVNF of $\mathcal{M}$. Then $M * N$ is a SVNF of $\mathcal{M}$.*

Proof. Since M is a SVNNSG and N is a SVNSG of Q, then $M * N$ is a SVNSG of Q and $M * N = N * M$. Now, we have

$$\begin{aligned} T_{M*N}(p) \geq T_M(p * r^{-1}) \wedge T_N(r) &\geq \tilde{\delta}_1^*((a*b, T^t(a*b), I^t(a*b), F^t(a*b)), x, p) \\ &\wedge \tilde{\delta}_1^*((a, T^{t'}(a), I^{t'}(a), F^{t'}(a)), x, r) \wedge T_M(b) \\ &\wedge \tilde{\delta}_1^*((a, T^{t'}(a), I^{t'}(a), F^{t'}(a)), x, r) \wedge T_N(a) \\ &= \tilde{\delta}_1^*((a*b, T^{t'}(a*b), I^{t'}(a*b), F^{t'}(a*b)), x, p) \\ &\wedge T_M(b) \wedge T_N(a), \end{aligned}$$

In addition, $I_{M*N}(p) \geq \tilde{\delta}_2^*((a*b, T^t(a*b), I^t(a*b), F^t(a*b)), x, p) \wedge I_M(b) \wedge I_N(a)$, since $\tilde{\delta}_2^*((a*b, T^t(a*b), I^t(a*b), F^t(a*b)), x, p) \leq \tilde{\delta}_2^*((a, T^{t'}(a), I^{t'}(a), F^{t'}(a)), x, r)$. Also,

$$\begin{aligned} F_{M*N}(p) &\leq F_M(p * r^{-1}) \vee F_N(r) \\ &\leq \tilde{\delta}_3^*((a*b, T^t(a*b), I^t(a*b), F^t(a*b)), x, p) \vee \tilde{\delta}_3^*((a, T^{t'}(a), I^{t'}(a), F^{t'}(a)), x, r) \\ &\vee F_M(b) \ \vee \tilde{\delta}_3^*((a, T^{t'}(a), I^{t'}(a), F^{t'}(a)), x, r) \vee F_N(a) \\ &\tilde{\delta}_3^*((a*b, T^t(a*b), I^t(a*b), F^t(a*b)), x, p) \vee F_M(b) \vee F_N(a), \end{aligned}$$

for every $a, b, p \in Q$ and $x \in \Sigma^*$. So, for every $p, q \in Q$ and $x \in \Sigma^*$

$$\begin{aligned}T_{M*N}(p) &\geq \vee\{\tilde{\delta}_1^*((a*b, T^t(a*b), I^t(a*b), F^t(a*b)), x, p) \wedge T_M(b) \wedge T_N(a) \big| a, b \in Q, a*b = q\}\\ &= \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, p) \wedge (\vee\{T_M(b) \wedge T_N(a) \big| a, b \in Q, a*b = q\})\\ &= \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, p) \wedge T_{M*N}(q),\end{aligned}$$

clearly, $I_{M*N}(p) \geq \tilde{\delta}_2^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, p) \wedge I_{M*N}(q)$ and

$$F_{M*N}(p) \leq \tilde{\delta}_3^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, p) \vee F_{M*N}(q).$$

Therefore, $M*N$ is a SVNF of $\tilde{\mathcal{A}}$. □

Theorem 3.22. *If M and N are two SVNKs of $\tilde{\mathcal{M}}$, then $M*N$ is a SVNK of $\tilde{\mathcal{M}}$.*

Proof. Let M and N be two SVNNSGs of Q. Then $M*N$ is a SVNNSG of Q and $M*N = N*M$. Now, we have

$$\begin{aligned}T_{M*N}(p*r^{-1}) &\geq T_M(p*q^{-1}) \wedge T_N(q*r^{-1})\\ &\geq \tilde{\delta}_1^*((a*b*c, T^t(a*b*c), I^t(a*b*c), F^t(a*b*c)), x, p)\\ &\quad \wedge \tilde{\delta}_1^*((a*b, T^{t'}(a*b), I^{t'}(a*b), F^{t'}(a*b)), x, p) \wedge T_M(c)\\ &\quad \wedge \tilde{\delta}_1^*((a*b, T^{t'}(a*b), I^{t'}(a*b), F^{t'}(a*b)), x, r)\\ &\quad \wedge \tilde{\delta}_1^*((a, T^{t''}(a), I^{t''}(a), F^{t''}(a)), x, r) \wedge I_M(c) \wedge I_N(c)\\ &= \tilde{\delta}_1^*((a*b*c, T^t(a*b*c), I^t(a*b*c), F^t(a*b*c)), x, p)\\ &\quad \wedge \tilde{\delta}_1^*((a, T^{t''}(a), I^{t''}(a), F^{t''}(a)), x, r) \wedge T_M(c) \wedge T_N(b),\end{aligned}$$

and

$$\begin{aligned}I_{M*N}(p*r^{-1}) &\geq I_M(p*q^{-1}) \wedge I_N(q*r^{-1})\\ &\geq \tilde{\delta}_2^*((a*b*c, T^t(a*b*c), I^t(a*b*c), F^t(a*b*c)), x, p)\\ &\quad \wedge \tilde{\delta}_2^*((a*b, T^{t'}(a*b), I^{t'}(a*b), F^{t'}(a*b)), x, p) \wedge I_M(c)\\ &\quad \wedge \tilde{\delta}_2^*((a*b, T^{t'}(a*b), I^{t'}(a*b), F^{t'}(a*b)), x, r)\\ &\quad \wedge \tilde{\delta}_2^*((a, T^{t''}(a), I^{t''}(a), F^{t''}(a)), x, r) \wedge T_M(c) \wedge T_N(c)\\ &= \tilde{\delta}_2^*((a*b*c, T^t(a*b*c), I^t(a*b*c), F^t(a*b*c)), x, p)\\ &\quad \wedge \tilde{\delta}_2^*((a, T^{t''}(a), I^{t''}(a), F^{t''}(a)), x, r) \wedge I_M(c) \wedge I_N(b),\end{aligned}$$

also,

$$\begin{aligned}F_{M*N}(p*r^{-1}) &\leq \tilde{\delta}_3^*((a*b*c, T^t(a*b*c), I^t(a*b*c), F^t(a*b*c)), x, p)\\ &\quad \vee \tilde{\delta}_3^*((a, T^{t''}(a), I^{t''}(a), F^{t''}(a)), x, r) \vee F_M(c) \vee F_N(b),\end{aligned}$$

for every $a, b, c, p, r, q \in Q$ and $x \in \Sigma^*$.

Now, we have

$$\begin{aligned} T_{M*N}(p * r^{-1}) \geq \vee\{&\tilde{\delta}_1^*((a*b*c, T^t(a*b*c), I^t(a*b*c), F^t(a*b*c)), x, p) \\ &\wedge \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \wedge T_M(c) \wedge T_N(b) \big| b, c \in Q, b*c = k\} \\ &= (\tilde{\delta}_1^*((q*k, T^{t''}(q*k), I^{t''}(q*k), F^{t''}(q*k)), x, p) \\ &\wedge \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \\ &\wedge \vee\{T_M(c) \wedge T_N(b) \big| b, c \in Q, b*c = k\}) \\ &= \tilde{\delta}_1^*((q*k, T^{t''}(q*k), I^{t''}(q*k), F^{t''}(q*k)), x, p) \\ &\wedge \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \wedge T_{M*N}(k), \end{aligned}$$

and

$$\begin{aligned} I_{M*N}(p * r^{-1}) \geq\ &\tilde{\delta}_2^*((q*k, T^{t''}(q*k), I^{t''}(q*k), F^{t''}(q*k)), x, p) \\ &\wedge \tilde{\delta}_2^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \wedge I_{M*N}(k), \end{aligned}$$

also,

$$\begin{aligned} F_{M*N}(p * r^{-1}) \leq\ &\tilde{\delta}_3^*((q*k, T^{t''}(q*k), I^{t''}(q*k), F^{t''}(q*k)), x, p) \\ &\vee \tilde{\delta}_3^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \vee F_{M*N}(k), \end{aligned}$$

Hence, $M * N$ is a SVNK of $\tilde{\mathcal{M}}$. □

Let M be a SVNK of $\tilde{\mathcal{M}}$. Let

$L(M) = \{N \big| N$ is a SVNK of $\tilde{\mathcal{M}}$ such that $(T_M(e), I_M(e), F_M(e)) = (T_N(e), I_N(e), F_N(e))\}$.

Theorem 3.23. *$(L(M), \subseteq)$ is a lattice.*

Proof. Let $B, N \in L(M)$. Then $B \cap N \in L(M)$ and $B \cap N$ is infimum of B and N, for relation $\subseteq$. By Theorem 3.22, $B * N \in L(M)$ and $B * N$ is the supremum of B and N for the relation $\subseteq$. Therefore, $(T(M), \subseteq)$ is a lattice. □

4. Conclusion

In this note, we presented the notion of neutrosophic single-valued general machine over a finite group, which is known as a ”group neutrosophic general machine”, for simplicity, GNGM. Also, we presented the notions of max-min GNGM, single-valued neutrosophic subgroup (SVNSG) and single-valued neutrosophic normal subgroup (SVNNSG). Moreover, we proved that if M is a SVNK and N is a SVNF of $\mathcal{M}$, then $M * N$ is a SVNF of $\mathcal{M}$.

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Interior and closure in anti-minimal and anti-biminimal spaces in the frame of anti-topology

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Abstract. The main goal of this project is to analyze the structure of anti-biminimal spaces through the lens of the notions of interior and closure. Anti-biminimal spaces can be considered as generalizations of anti-bitopological spaces that have been introduced and studied earlier. On the other hand, they can be viewed as counterparts of biminimal spaces. Finally, they are spaces equipped with two anti-minimal structures. Thus, we show some basic results on the latter. In particular, we refer to the concepts of density, nowhere density and rarity in anti-minimal spaces.

In general, anti-topology is a structure κ in which at least one classical axiom is totally false. In this paper, we consider the first axiom. Hence, $\emptyset$ and X do not belong to κ.

Keywords: anti-biminimal space, biminimal space, anti-topology

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1. Introduction

We know that topology on some non-empty universe X is defined as a family $\tau \subseteq P(X)$ that is closed under finite intersections and arbitrary unions. The elements of this family of subsets are called *open* sets. Moreover, we always assume that $\emptyset$ and X are open too. This definition can be considered as a generalization of the idea of open intervals on real line or open balls on real plane. Using this concept mathematicians defined many notions that are used in other branches of mathematics. In particular, continuity is very important in analysis.

Many modern topologists try to reconstruct typical topological notions (like continuity, density, nowhere density, compactness, connectedness, separation etc.) in weaker or just different frameworks than the one mentioned above. For example, we can assume that closure of our family under finite intersections is not necessary and thus we obtain supra topological spaces (see [13]). Instead of this, we can eliminate necessity of closure under unions (to get infra topological spaces, see [3] and [14]). One can reject both these conditions: this leads to minimal structures, like in [15]. We can even assume that X may not be open. These are weak structures (see [6]), where the only requirement is that $\emptyset$ is open. If weak structure is closed under arbitrary unions then it can be considered as a generalized topological space in the sense of Császár (see [5]). Finally, we can assume that τ is arbitrary (we can use the notion of generalized weak structure here), see [1] and [7].

We have different approaches too. The whole direction of our present research is based on Smarandache's suggestion that we should investigate anti-algebras. They are based on the idea that some classical requirements (like closure under finite intersections in case of topologies) are *forbidden*. Initial papers on this concept are e.g. [17], [18], [19] and [20].

In general, Smarandache invented six new types of topologies recently (that is, in the years 2019 - 2022). These are: refined neutrosophic topology, refined neutrosophic crisp topology, neutro-topology, anti-topology, super-hyper topology and neutrosophic super-hyper-topology. The last two are based on the idea of the n-th power set of a given non-empty set (be it classical or neutrosophic). As for the neutro-topological structures, they are based on the assumption that at least one of the classical topological axioms is partially true, partially indeterminate, and partially false. They have been studied e.g. by Şahin et al. in [16]. There was also another paper by Khaklary and Chandra Ray, see [10].

Refined structures are characterized by the fact that truth, falsity and indeterminacy can be split into arbitrarily many subcomponents (depending on applications and needs). This leads to the idea of refined fuzzy, refined intuitionistic fuzzy and then refined neutrosophic set.

Finally, we have the idea of anti-topology. Anti-topology means a topology where at least one of its classical axioms is totally false. In particular, it is possible that any non-trivial intersection or union of the elements of τ is *beyond* τ . Moreover, we may assume $\emptyset$ and X are never open. Such anti-topologies are anti-chains of sets. These spaces have been already studied e.g. in [21], [22]. Moreover, anti-bitopological spaces equipped with two anti-topologies have been investigated in [9].

As for the present paper, it is devoted to the initial study of anti-minimal and anti-weak structures. Anti-biminimal structures are presented too.

2. **Basic notions**

Let us introduce several basic notions that will be used extensively throughout the paper. The first two are somewhat classical, the next two refer to the general program of anti-algebra.

By "non-trivial" we mean intersection or union that engages at least two different sets.

Then we say that such a family is "anti-closed" under these operations.

Definition 2.1. (compare [15]). Assume that X is a non-empty universe and $\mathfrak{m} \subseteq P(X)$. If $\emptyset, X \in \mathfrak{m}$ then we say that $\mathfrak{m}$ is a *minimal structure* on X. We say that an ordered pair $(X, \mathfrak{m})$ is a *minimal structure space.*

Clearly, every topology on X is a minimal structure too. In particular, anti-discrete topology is the simplest example of minimal structure.

Definition 2.2. (compare [4]). Assume that X is a non-empty universe and $\mathfrak{m}^1$, $\mathfrak{m}^2$ are two minimal structures on X. Then we say that an ordered pair $(X, \mathfrak{m}^1, \mathfrak{m}^2)$ is a *biminimal structure space.*

Remark 2.3. If the structures mentioned in the last definition are weak structures (and not necessarily minimal), then the whole space is called *biweak structure.* Such spaces have been investigated e.g. in [11].

As it was announced earlier, the next two definitions can be considered as negations of the former two.

Definition 2.4. Assume that X is a non-empty universe and $\kappa \subseteq P(X)$. If $\emptyset, X \notin \kappa$, then we say that κ is an *anti-minimal structure* on X. We say that an ordered pair (X, κ) is an *anti-minimal structure space.* The elements of κ are called κ-*open* sets and their complements are κ-*closed.* The set of all κ-closed sets is denoted with κ_{Cl}.

In the light of our earlier considerations, anti-minimal structure is an example of anti-topology (since the first axiom of topology is totally false).

Definition 2.5. Assume that X is a non-empty universe and κ^1, κ^2 are two anti-minimal structures on X. Then we say that an ordered pair (X, κ^1, κ^2) is an *anti-biminimal structure space.*

One can easily give many examples of anti-minimal and anti-biminimal structures. Some of them will be presented throughout the paper. Clearly, anti-minimal structures can be closed under some operations (like non-empty intersections or unions that do not lead to X). They can be anti-closed under these operations too.

We can define closure and interior in terms of anti-minimal structures. Both the definitions below are standard.

Definition 2.6. Assume that (X, κ) is an anti-minimal structure space and $A \subseteq X$. We say that κ-*interior* of A is the following set: $\kappa Int(A) = \bigcup\{B \in \kappa; B \subseteq A\}$.

Definition 2.7. Assume that (X, κ) is an anti-minimal structure space and $A \subseteq X$. We say that κ-*closure* of A is the following set: $\kappa Cl(A) = \bigcap\{B \in \kappa_{Cl}; A \subseteq B\}$.

Remark 2.8. Note that anti-minimal structures have one interesting property. Due to the fact that $\emptyset$ and X are never κ-open (nor κ-closed) we can define four non-trivial sets:

(1) $\kappa Ynt(A) = \bigcap\{B \in \kappa; B \subseteq A\}$ (*subinterior* of A).
(2) $\kappa Kl(A) = \bigcup\{B \in \kappa_{Cl}; A \subseteq B\}$ (*superclosure* of A).
(3) $\kappa Ent(A) = \bigcup\{B \in \kappa; B \subseteq A\}$ (*superinterior* of A).
(4) $\kappa Gl(A) = \bigcap\{B \in \kappa_{Cl}; A \subseteq B\}$ (*subclosure* of A).

This will be analyzed in our further research. Clearly, similar operators can be defined even for topological spaces but with some more or less additional assumptions (like e.g. "the intersection of all *non-empty* open sets contained in A"), while in anti-minimal structures they are more natural. Note (for example) that if our space is *not* closed under unions, then $\kappa Ent(A)$ may be different than $\bigcup \kappa$ (because the union of all open sets need not to be open).

The following properties of κ-interior and κ-closure are true just because they are true for any generalized weak structure (and anti-minimal structure is a generalized weak one, without any doubt). The reader can compare this e.g. with [1].

Lemma 2.9. *Assume that* (X, κ) *is an anti-minimal structure space and* $A, B \subseteq X$*. Then:*

(1) $\kappa Int(A) \subseteq A$.
(2) *If* $A \in \kappa$*, then* $\kappa Int(A) = A$.
(3) *If* $A \subseteq B$*, then* $\kappa Int(A) \subseteq \kappa Int(B)$.
(4) $\kappa Int(\kappa Int(A)) = \kappa Int(A)$.
(5) $A \subseteq \kappa Cl(A)$.
(6) *If* $A \in \kappa_{Cl}$*, then* $\kappa Cl(A) = A$.
(7) *If* $A \subseteq B$*, then* $\kappa Cl(A) \subseteq \kappa Cl(B)$.
(8) $\kappa Cl(\kappa Cl(A)) = \kappa Cl(A)$.
(9) $-\kappa Int(A) = \kappa Cl(-A)$.
(10) $\kappa Int(-A) = -\kappa Cl(A)$.
(11) $x \in \kappa Int(A)$ *if and only if there is* $U \in \kappa$ *such that* $x \in U \subseteq A$.
(12) $x \in \kappa Cl(A)$ *if and only if* $U \cap A \neq \emptyset$ *for any* $U \in \kappa$ *such that* $x \in U$.
(13) $\kappa Int(A \cap B) \subseteq \kappa Int(A) \cap \kappa Int(B)$.

Remark 2.10. Note that the converses of Lemma 2.9 (2), (6) need not to be true. For example, let $X = \{a, b, c, d\}$ and $\kappa = \{\{a, b\}, \{b, c\}, \{c, d\}\}$. Now let $A = \{a, b, c\}$. Then $\kappa Int(A) = \{a, b\} \cup \{b, c\} = A \notin \kappa$. Besides, this particular κ is an example of anti-topological space. This is because it is anti-closed under unions and intersections (any union and intersection of two different sets from κ leads beyond κ). Moreover, in the definition of anti-topological space (see [21]) we always assume that $\emptyset, X \notin \kappa$. Hence, each anti-topological space is an anti-minimal structure too.

Remark 2.11. The converse of Lemma 2.9 (13) may not be true. Consider $X = \{1, 2, 3, 4, 5\}$ with $\kappa = \{\{1, 3\}, \{2\}, \{3, 4\}\}$, where $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$. One can easily check that $\kappa Int(A) \cap \kappa Int(B) = A \cap B = \{2, 3\} \nsubseteq \{2\} = \kappa Int(\{2, 3\}) = \kappa Int(A \cap B)$.

3. On density and related notions

We can briefly discuss the concepts of density, nowhere density and rarity (leaving more detailed theorems for the further research).

Definition 3.1. Let (X, κ) be an anti-minimal structure and $A \subseteq X$. Then we say that A is:

(1) κ-*dense* if and only if $\kappa Cl(A) = X$.
(2) κ-*nowhere dense* if and only if $\kappa Int(\kappa Cl(A)) = \emptyset$.
(3) Strongly κ-*nowhere dense* if and only if for any κ-open set B there is some κ-open $U \subseteq B$ such that $A \cap U = \emptyset$.
(4) κ-*rare* if and only if $\kappa Int(A) = \emptyset$.

Remark 3.2. Note that if A is κ-dense in an anti-minimal space, then it is equivalent with saying that the set $Z = \{C \in \kappa_{Cl}; A \subseteq C\}$ is empty. Note that X is never κ-open. Analogously, A is κ-rare in anti-minimal space if and only if the set $J = \{C \in \kappa; C \subseteq A\}$ is empty.

We can prove equivalent characterization of κ-dense sets.

Theorem 3.3. *Let (X, κ) be an anti-minimal structure. Let $A \subseteq X$. Then A is κ-dense if and only if $A \cap B \neq \emptyset$ for any $B \in \kappa$.*

Proof. ($\Rightarrow$). We have $\kappa Cl(A) = X$ and $B \in \kappa$. Assume that $A \cap B = \emptyset$. Now let $x \in B$ (there must be some $x \in B$ because B is non-empty as a member of κ). Hence $x \in X = \kappa Cl(A)$. By Lemma 2.9 (13) we obtain $A \cap B \neq \emptyset$.

($\Leftarrow$). Let $A \cap B \neq \emptyset$ for any $B \in \kappa$. Suppose that A is not κ-dense. Then $\kappa Cl(A) \neq X$. Hence, for some $D \in \kappa_{Cl}$, $A \subseteq D$. But $D \neq X$. Then $-D = X \setminus D \in \kappa$ and $A \cap (-D) = \emptyset$. This is contradiction. □

The theorem above can be proved for generalized weak structures too. However, we should assume that B is non-empty (while in anti-minimal structures it is clear by the very definition of κ).

Remark 3.4. Consider $X = \{1, 2, 3, 4\}$ and $\kappa = \{\{1, 2\}, \{2, 3\}, \{2, 3, 4\}, \{1, 3, 4\}\}$. Then $\kappa_{Cl} = \{\{3, 4\}, \{1, 4\}, \{1\}, \{2\}\}$. Hence, $A = \{1, 2\}$ and $B = \{2, 3\}$ are both κ-open and κ-dense. However, their intersection, that is $A \cap B = \{2\}$ is not κ-dense. This is because $\{2\}$ is κ-closed, so its κ-closure is just $\{2\}$ itself.

Note that the remark above shows us that the situation is different than e.g. in topological spaces. Recall that in topological spaces we can prove that if A, B are dense and at least B is open, then their intersection is dense too. However, we can prove the following lemma and theorem.

Lemma 3.5. *Let (X, κ) be an anti-minimal structure that is closed under non-empty finite intersections. If A is κ-dense, then $\kappa Cl(U) = \kappa Cl(A \cap U)$ for any $U \in \kappa$.*

Proof. Clearly, $U \cap A \subseteq U$. Now we use Lemma 2.9 (8) to say that $\kappa Cl(U \cap A) \subseteq \kappa Cl(U)$. Let $x \in \kappa Cl(U)$. Now it must be that $W \cap U \neq \emptyset$ for any $W \in \kappa$ such that $x \in W$ (that is, for any κ-open neighborhood of x). Because of the closure of our κ under non-empty finite intersections, $W \cap U \in \kappa$. Hence, by Lemma 2.9 (13), $(W \cap U) \cap A \neq \emptyset$. Thus, $x \in \kappa Cl(U \cap A)$. □

Theorem 3.6. *Let (X, κ) be an anti-minimal space that is closed under non-empty finite intersections. Suppose that $A, B \subseteq X$ are both κ-dense and B is κ-open. Then $A \cap B$ is κ-dense.*

Proof. We already know that if A is κ-dense, then $A \cap U \neq \emptyset$ for any $U \in \kappa$. Now let $V \in \kappa$. There is some $x \in V$. But $x \in X = \kappa Cl(B)$. Hence, $B \cap V \neq \emptyset$. Moreover, $B \cap V \in \kappa$ (because of the closure under non-empty finite intersections). Hence, $A \cap (B \cap V) \neq \emptyset$. But this means that $A \cap B$ is κ-dense. Note that we can write $(A \cap B) \cap V \neq \emptyset$ to emphasize this fact. □

The fact that we distinguish between nowhere density and strong nowhere density is important. In topological spaces these two notions are equivalent but not here.

On the one hand we can prove the following theorem.

Theorem 3.7. *Every κ-strongly nowhere dense set in anti-minimal structure is κ-nowhere dense too.*

Proof. Suppose that A is κ-strongly nowhere dense but $\kappa Int(\kappa Cl(A)) \neq \emptyset$. Then there is some $x \in \kappa Int(\kappa Cl(A))$. In particular, it means that $x \in \kappa Cl(A)$. Then for any $V \in \kappa$ such that $x \in V$, $V \cap A \neq \emptyset$. However, from the property of κ-strongly nowhere density of A we infer that for any $B \in \kappa$ we can find $U \in \kappa$, $U \subseteq B$ such that $A \cap U = \emptyset$. Thus we obtain contradiction. □

However, the converse is not necessarily true.

Example 3.8. Let $X = \{a, b, c, d\}$, $\kappa = \{\{a, b\}, \{b, c\}, \{c, d\}\}$. Then $\kappa_{Cl} = \{\{c, d\}, \{a, d\}, \{a, b\}\}$. Take $A = \{a, d\}$. Its κ-closure is just $\{a, d\} = A$ but then we see that $\kappa Int(A) = \emptyset$ (there are no κ-open sets contained in A). However, A it not strongly nowhere dense. Take $B = \{a, b\}$. We see that $A \cap B = \{a\} \neq \emptyset$.

4. More on anti-biminimal structures

Now let us concentrate on anti-biminimal structures. We would like to determine some specific class of subsets that will be the object of our investigation. Let us assume that i and j will be always interpreted as elements of $\{1, 2\}$. However, first let us express some basic facts.

Remark 4.1. Note that if A and B are anti-minimal structures on some universe X, then $A \cap B$ and $A \cup B$ are anti-minimal structures too.

If A is anti-minimal structure and B is minimal structure, then $A \cup B$ is a minimal structure, while $A \cap B$ is an anti-minimal structure.

Definition 4.2. Let (X, κ^1, κ^2) be an anti-biminimal structure space. Assume that $A \subseteq X$. We say that A is $\kappa^i\kappa^j$-*closed* set if and only if $A = \kappa^i Cl(\kappa^j Cl(A))$. The complement of $\kappa^i\kappa^j$-closed set is called $\kappa^i\kappa^j$-*open*.

Lemma 4.3. *Let (X, κ^1, κ^2) be an anti-biminimal structure space. Then A is an $\kappa^i\kappa^j$-open subset of X if and only if $A = \kappa^i Int(\kappa^j Int(A))$.*

Proof. Assume that A is $\kappa^i\kappa^j$-open. It means that $-A$ is $\kappa^i\kappa^j$-closed. Hence, $\kappa^i Cl(\kappa^j Cl(-A)) = -A$. However, by virtue of the general properties of κ-interior, $\kappa^i Cl(\kappa^j Cl(-A)) = -\kappa^i Int(-\kappa^j Cl(-A)) = -\kappa^i Int(\kappa^j Int(-(-A))) = -\kappa^i Int(\kappa^j Int(A)) = -A$. But then $A = \kappa^i Int(\kappa^j Int(A))$. Now we can repeat the whole reasoning in the opposite direction to obtain the expected result: namely, that $-A$ is $\kappa^i\kappa^j$-closed and thus A is $\kappa^i\kappa^j$-open. □

As we could see, the lemma above is based on the general properties of interior and closure in arbitrary generalized weak structures rather, than on the specific properties of anti-biminimal structure. The same can be said about the next lemma:

Lemma 4.4. *Let (X, κ^1, κ^2) be an anti-biminimal structure space. Assume that $A, B \subseteq X$ are $\kappa^1\kappa^2$-closed subsets of X. Then $A \cap B$ is $\kappa^1\kappa^2$-closed too.*

Proof. By the assumption, $\kappa^1 Cl(\kappa^2 Cl(A)) = A$ and $\kappa^1 Cl(\kappa^2 Cl(B)) = B$. Clearly, $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Thus $\kappa^1 Cl(\kappa^2 Cl(A \cap B)) \subseteq \kappa^1 Cl(\kappa^2 Cl(A))$ and $\kappa^1 Cl(\kappa^2 Cl(A \cap B)) \subseteq$

$\kappa^1 Cl(\kappa^2 Cl(B))$. Then $\kappa^1 Cl(\kappa^2 Cl(A \cap B)) \subseteq \kappa^1 Cl(\kappa^2 Cl(A)) \cap \kappa^1 Cl(\kappa^2 Cl(B)) = A \cap B$. However, on the other hand, $A \cap B \subseteq \kappa^1 Cl(\kappa^2 Cl(A \cap B))$ (by the very definition of closure). Thus, $A \cap B \subseteq \kappa^1 Cl(\kappa^2 Cl(A \cap B)) = A \cap B$ and due to this reason $A \cap B$ is $\kappa^1\kappa^2$-closed. □

Remark 4.5. Assume that (X, κ^1, κ^2) is an anti-biminimal structure on X and $A, B \subseteq X$ are two $\kappa^1\kappa^2$-closed sets. Then $A \cup B$ does not need to be $\kappa^1\kappa^2$-closed. For example, take $X = \{a, b, c, d, e\}$, $\kappa^1 = \{\{c, d, e\}, \{a, b, e\}, \{a, b, c, d\}\}$ and $\kappa^2 = \{\{c, d, e\}, \{a, b, e\}, \{a, b, d, e\}\}$. Then $\kappa^1_{Cl} = \{\{a, b\}, \{c, d\}, \{e\}\}$ and $\kappa^2_{Cl} = \{\{a, b\}, \{c, d\}, \{c\}\}$. Now take $A = \{a, b\}$ and $B = \{c, d\}$. They are both $\kappa^1\kappa^2$-closed. Let us check their union, that is $\{a, b, c, d\}$. We see that $\kappa^1 Cl(\kappa^2 Cl(\{a, b, c, d\})) = \kappa^1 Cl(\bigcap\{A \in \kappa^2_{Cl}; \{a, b, c, d\} \subseteq A\}) = \kappa^1 Cl(\bigcap \emptyset) = \kappa^1 Cl(X) = \bigcap\{A \in \kappa^1_{Cl}; X \subseteq A\} = \bigcap \emptyset = X \neq \{a, b, c, d\}$.

Besides, let us calculate directly $\kappa^1 Cl(\kappa^2 Cl(A \cap B)) = \kappa^1 Cl(\kappa^2 Cl(\emptyset)) = \kappa^1 Cl(\bigcap\{A \in \kappa^2_{Cl}; \emptyset \subseteq A\}) = \kappa^1 Cl(\{a, b\} \cap \{c, d\} \cap \{c\}) = \kappa^1 Cl(\emptyset) = \bigcap\{A \in \kappa^1_{Cl}; \emptyset \subseteq A\} = \{a, b\} \cap \{c, d\} \cap \{e\} = \emptyset$.

Now, both from the general lemma and from this direct calculation, we know that $\emptyset$ is $\kappa^1\kappa^2$-closed in this particular anti-biminimal structure. However, it is clear (by the very definition of anti-biminimal and anti-minimal structure as such) that $\emptyset \notin \kappa^1_{Cl} \cap \kappa^2_{Cl}$. Hence, we see that $\kappa^1 \cap \kappa^2$ is not identical with the set of all $\kappa^1\kappa^2$-closed sets. This leads us to the next lemma.

Lemma 4.6. *Let (X, κ^1, κ^2) be an anti-biminimal space. Let $A \in \kappa^1_{Cl} \cap \kappa^2_{Cl}$. Then A is $\kappa^1\kappa^2$-closed.*

Proof. $A \in \kappa^2_{Cl}$, hence $\kappa^2 Cl(A) = A$. But $A \in \kappa^1_{Cl}$ too, hence $\kappa^1 Cl(\kappa^2 Cl(A)) = \kappa^1 Cl(A) = A$. □

The converse need not to be true, as we could already seen. However, we should not think that empty set is the only possible counter-example.

Example 4.7. Let (X, κ^1, κ^2) be an anti-biminimal space where $X = \{a, b, c, d, e\}$, $\kappa^1 = \{\{a, d\}, \{e\}\}$ and $\kappa^2 = \{\{a, e\}, \{d\}\}$. Now $\kappa^1_{Cl} = \{\{b, c, e\}, \{a, b, c, d\}\}$ and $\kappa^2_{Cl} = \{\{b, c, d\}, \{a, b, c, e\}\}$. Now take $A = \{b, c\}$. We see that $\kappa^2 Cl(A) = \{b, c, d\} \cap \{a, b, c, e\} = \{b, c\}$. Then $\kappa^1 Cl(\kappa^2 Cl(A)) = \kappa^1 Cl(A) = \{b, c, e\} \cap \{a, b, c, d\} = \{b, c\} = A$. Hence, A is $\kappa^1\kappa^2$-closed but $A \notin \kappa^1_{Cl} \cap \kappa^2_{Cl}$. In fact, A is not even in $\kappa^1_{Cl} \cup \kappa^2_{Cl}$.

Remark 4.8. Let us think about the example above again. Take $A = \{a, b, d\}$. On the one hand, $\kappa^1 Int(A) = \{a, d\}$ and then $\kappa^2 Int(\{a, d\}) = \{d\}$. On the other hand, $\kappa^2 Int(A) = \{d\}$ and $\kappa^1 Int(\{d\}) = \emptyset$. This shows us that in general $\kappa^1 Int(\kappa^2 Int(A))$ may not be identical with $\kappa^2 Int(\kappa^1 Int(A))$.

However, the fact mentioned in the remark above should not be confused with the following theorem:

Theorem 4.9. *Assume that* (X, κ^1, κ^2) *is an anti-biminimal space. Then* $A \subseteq X$ *is* $\kappa^1\kappa^2$*-open if and only if it is* $\kappa^2\kappa^1$*-open.*

Proof. Let A be $\kappa^1\kappa^2$-open. Then $\kappa^1 Int(\kappa^2 Int(A)) = A$. However, it must be that $\kappa^2 Int(A)) = A$. Assume the contrary. Clearly, it means that $\kappa^2 Int(A) \subseteq A$ and $\kappa^2 Int(A) \neq A$. Thus there is some $x \in A$ such that $x \notin \kappa^2 Int(A)$. Suppose that $x \in \kappa^1 Int(\kappa 2 Int(A))$. But $\kappa^1 Int(\kappa^2 Int(A)) \subseteq \kappa^2 Int(A)$, so we obtain contradiction. Thus $x \notin \kappa^1 Int(\kappa^2 Int(A)) = A$. Hence $x \notin A$. Contradiction.

Now, if we already know that $\kappa^2 Int(A) = A$, then $\kappa^2 Int(\kappa^1 Int(\kappa^2 Int(A))) = \kappa^2 Int(A) = A$. But on the left side we have (by the assumption) $\kappa^2 Int(\kappa^1 Int(A))$. Finally, $\kappa^2 Int(\kappa^1 Int(A)) = A$. Hence A is $\kappa^1\kappa^2$-open.

Clearly, the other direction of the proof is similar.

Note that this proof would be true for any generalized weak structure: we did not use the fact that $\emptyset$ and X are not open in κ^1 and κ^2. □

Analogously, we have:

Theorem 4.10. *Assume that* (X, κ^1, κ^2) *is an anti-biminimal space. Then* $A \subseteq X$ *is* $\kappa^1\kappa^2$*-closed if and only if it is* $\kappa^2\kappa^1$*-closed.*

Proof. Assume that $\kappa^1 Cl(\kappa^2 Cl(A)) = A$. Then $\kappa^2 Cl(A)) = A$. Suppose the contrary. It means that $A \subseteq \kappa^2 Cl(A)$ but $\kappa^2 Cl(A) \neq A$. Hence there is some $x \in \kappa^2 Cl(A)$ such that $x \notin A$. Suppose that $x \in \kappa^1 Cl(\kappa^2 Cl(A))$. But $\kappa^2 Cl(A) \subseteq \kappa^1 Cl(\kappa^2 Cl(A)) = A$ which gives us that $x \in A$ and this is contradiction.

Now we see that $\kappa^2 Cl(\kappa^1 Cl(\kappa^2 Cl(A))) = \kappa^2 Cl(A) = A$. But on the left side we have $\kappa^2 Cl(\kappa^1(A))$, so finally we get $\kappa^2 Cl(\kappa^1(A)) = A$ which means that A is $\kappa^2\kappa^1$-closed. □

As for the empty set, we may prove the following theorem.

Theorem 4.11. *Let* (X, κ^1, κ^2) *be an anti-biminimal space. Then* $\emptyset$ *is* $\kappa^1\kappa^2$*-closed if and only if* $\bigcap \kappa^1_{Cl} = \emptyset$ *and* $\bigcap \kappa^2_{Cl} = \emptyset$.

Proof. $(\Rightarrow)$. Assume that $\bigcap \kappa^2_{Cl} = L \neq \emptyset$. Now let us calculate: $\kappa^1 Cl(\kappa^2 Cl(\emptyset)) = \kappa^1 Cl(\bigcap\{A \in \kappa^2_{Cl}; \emptyset \subseteq A\}) = \kappa^1 Cl(\bigcap\{A \in \kappa^2_{Cl}\}) = \kappa^1 Cl(\bigcap \kappa^2_{Cl}) = \kappa^1 Cl(L)$. However, if L is non-empty, as we assumed, then its κ^1-closure must be non-empty too. Finally, $\kappa^1 Cl(\kappa^2 Cl(\emptyset)) \neq \emptyset$, so $\emptyset$ is not $\kappa^1\kappa^2$-closed.

Analogously, assume that $\bigcap \kappa^1_{Cl} = K \neq \emptyset$. If $\kappa^2 Cl(\emptyset) \neq \emptyset$, then $\kappa^1 Cl(\kappa^2 Cl(\emptyset))$ is non-empty (as a κ^1-closure of non-empty set). However, even if $\kappa^2 Cl(\emptyset) = \emptyset$, then $\kappa^1 Cl(\emptyset) = \bigcap\{A \in \kappa^1; \emptyset \subseteq A\} = \bigcap\{A \in \kappa^1_{Cl}\} = \bigcap \kappa^1_{Cl} = K \neq \emptyset$. Again, in both cases $\emptyset$ is not $\kappa^1\kappa^2$-closed.

($\Leftarrow$).

Assume that $\bigcap \kappa^1_{Cl} = \emptyset$ and $\bigcap \kappa^2_{Cl} = \emptyset$. Let us calculate $\kappa^1 Cl(\kappa^2 Cl(\emptyset)) = \kappa^1 Cl(\bigcap\{A \in \kappa^2_{Cl}; \emptyset \subseteq A\}) = \kappa^1 Cl(\bigcap\{A \in \kappa^2_{Cl}\}) = \kappa^1 Cl(\bigcap \kappa^2_{Cl}) = \kappa^1 Cl(\emptyset) = \bigcap\{A \in \kappa^1_{Cl}; \emptyset \subseteq A\} = \bigcap\{A \in \kappa^1_{Cl}\} = \bigcap \kappa^1_{Cl} = \emptyset$. □

Remark 4.12. Clearly the left side of this theorem can be reformulated as: "X is $\kappa^1\kappa^2$-open".

The theorem above can be illustrated.

Example 4.13. Let (X, κ^1, κ^2) be an anti-biminimal space where $X = \{a, b, c, d, e\}$, $\kappa^1 = \{\{c, d, e\}, \{a, b, e\}\}$, $\kappa^2 = \{\{a, d, e\}, \{a, b, e\}$. Hence $\kappa^1_{Cl} = \{\{a, b\}, \{c, d\}\}$ and $\kappa^2_{Cl} = \{\{b, c\}, \{c, d\}\}$. Clearly, $\bigcap \kappa^2_{Cl} = \{c\} \neq \emptyset$. Now $\kappa^1 Cl(\kappa^2 Cl(\emptyset)) = \kappa^1 Cl(\bigcap \kappa^2_{Cl}) = \kappa^1 Cl(\{c\}) = \bigcap\{A \in \kappa^1_{Cl}; \{c\} \subseteq A\} = \{c, d\} \neq \emptyset$.

This was an example of the situation where $\emptyset$ was not $\kappa^1\kappa^2$-closed. Now take the same universe and $\kappa^1 = \{\{a\}, \{b, c, d, e\}\}$ and $\kappa^2 = \{\{a, d, e\}, \{a, b, c\}\}$. Now $\kappa^1_{Cl} = \kappa^1$ and $\kappa^2_{Cl} = \{\{b, c\}, \{d, e\}\}$. In both these minimal structures the intersection of all closed sets is empty. Now one can check that $\kappa^1 Cl(\kappa^2 Cl(\emptyset)) = \emptyset$ just repeating the reasoning presented in ($\Leftarrow$) part of the last theorem.

Let us go back to the notion of interior. We prove the following lemma.

Lemma 4.14. *Let (X, κ^1, κ^2) be an anti-biminimal structure space. Assume that $A, B \subseteq X$ are $\kappa^1\kappa^2$-open subsets of X. Then $A \cup B$ is $\kappa^1\kappa^2$-open too.*

Proof. Assume that both A and B are $\kappa^1\kappa^2$-open. Hence, $\kappa^1 Int(\kappa^2 Int(A)) = A$ and analogously $\kappa^1 Int(\kappa^2 Int(B)) = B$. Clearly, $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Hence, $\kappa^1 Int(\kappa^2 Int(A)) \subseteq \kappa^1 Int(\kappa^2 Int(A \cup B))$ and $\kappa^1 Int(\kappa^2 Int(B)) \subseteq \kappa^1 Int(\kappa^2 Int(A \cup B))$. Thus, $A \cup B = \kappa^1 Int(\kappa^2 Int(A)) \cup \kappa^1 Int(\kappa^2 Int(B)) \subseteq \kappa^1 Int(\kappa^2(A \cup B))$. However, on the other hand, $\kappa^1 Int(\kappa^2 Int(A \cup B)) \subseteq A \cup B$. Finally, $\kappa^1 Int(\kappa^2 Int(A \cup B)) = A \cup B$. Thus, $A \cup B$ is $\kappa^1\kappa^2$-open. □

As for the whole universe, we have a theorem analogous to Theorem 4.11.

Theorem 4.15. *Let (X, κ^1, κ^2) be an anti-biminimal space. Then X is $\kappa^1\kappa^2$-open if and only if $\bigcup \kappa^1 = X$ and $\bigcup \kappa^2 = X$.*

Proof. ($\Rightarrow$). Assume that $\bigcup \kappa^2 = M \neq X$. Now let us calculate: $\kappa^1 Int(\kappa^2 Int(X)) = \kappa^1 Int(\bigcup\{A \in \kappa^2; A \subseteq X\}) = \kappa^1 Int(\bigcup \kappa^2) = \kappa^1 Int(M)$. However, it M is properly contained in X, then its κ^1-interior must be properly contained in X too. Finally, $\kappa^1 Int(\kappa^2 Int(X)) \neq X$, so X is not $\kappa^1\kappa^2$-open.

Assume now that $\bigcup \kappa^1 = N \neq X$. If $\kappa^2 Int(X) = M \neq X$, then $\kappa^1 Int(\kappa^2 Int(X)) = \kappa^1 Int(M) \neq X$ (being contained in M). However, even if $\kappa^2 Int(X) = X$, then $\kappa^1 Int(X) = \bigcup\{A \in \kappa^1; A \subseteq X\} = \bigcup \kappa^1 = N \neq X$. In both cases X is not $\kappa^1\kappa^2$-open.

($\Leftarrow$). Suppose that $\bigcup \kappa^1 = X$ and $\bigcup \kappa^2 = X$. Let us calculate $\kappa^1 Int(\kappa^2 Int(X)) = \kappa^1 Int(\bigcup\{A \in \kappa^2; A \subseteq X\}) = \kappa^1 Int(\bigcup \kappa^2) = \kappa^1 Int(X) = \bigcup\{A \in \kappa^1; A \subseteq X\} = \bigcup\{A \in \kappa^1\} = \bigcup \kappa^1 = X$. □

This was direct proof but it was enough to use Theorem 4.11, Remark 4.12 and the fact that $\bigcup \kappa = -\bigcap \kappa_{Cl}$.

5. Conclusion

In this paper we presented anti-minimal and anti-biminimal spaces. We proved some initial claims about these structures. Now our idea is to analyze the notion of nowhere density and to introduce the idea of continuous functions (in both frameworks). Moreover, we think that it would be valuable to analyze those somewhat untypical operators that have been mentioned in Remark 2.8.

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