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# Neutrosophic Sets and Systems

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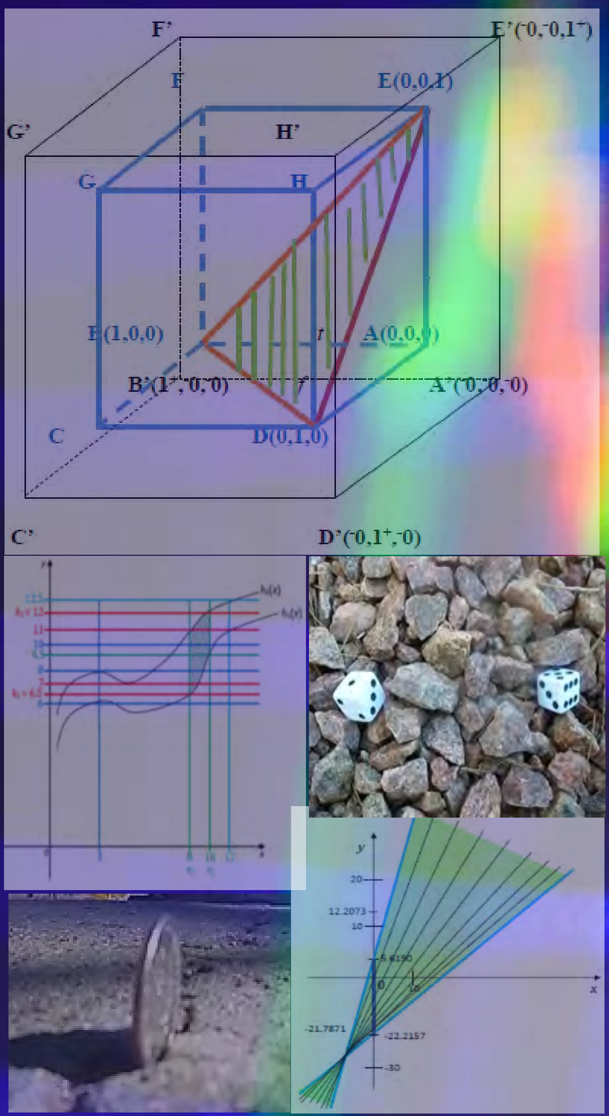


<A> <neutA> <antiA>

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# Neutrosophic Sets and Systems

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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

*Neutrosophy* is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

*Neutrosophic Set* and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth ($T$), a degree of indeterminacy ($I$), and a degree of falsity ($F$), where $T, I, F$ are standard or non-standard subsets of $]^-0, 1^+[$.

*Neutrosophic Probability* is a generalization of the classical probability and imprecise probability.

*Neutrosophic Statistics* is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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**Google Dictionaries** have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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## Contents

NSS

*Neutrosophic Sets and Systems, Vol. 58, 2023*

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# The neutrosophic numerical integration and MATLAB

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**Abstract:** In this paper, the idea of nitro has been introduced to numerical integrals, where we studied neutrosophic numerical integrals by submitting the neutrosophic trapezoidal method and estimation of error of the neutrosophic trapezoidal method, in addition to supporting examples for that and verified using MATLAB.

**Keywords:** neutrosophic numerical integrals; estimation of error; the neutrosophic trapezoidal method.

---

## 1. Introduction

In contrast to the current logics, Smarandache suggested the Neutrosophic Logic to describe a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, and contradiction. Smarandache introduced the concept of neutrosophy as a new school of philosophy [4]. He presented the definition of the standard form of neutrosophic real number, the Neutrosophic statistics [3-5-6], and professor Smarandache entered the concept of preliminary calculus of the differential and integral calculus, where he introduced for the first time the notions of neutrosophic mereo-limit, mereo-continuity, mereoderivative, and mereo-integral [1]. A number of studies in the area of integration and differentiation were given by Y. Alhasan [10-11-16], also he presented the definition of the concept of neutrosophic complex numbers and its properties [2-9]. The AH isometry was used to study many structures such as conic sections, real analysis concepts, and geometrical surfaces [13-14-15].

The calculation of area, size, and arc length is one of integration's most crucial uses in daily life. We encounter things in our world that are ill-defined and partially indeterminate.

There are four sections of paper. first section, which also includes a study of neutrosophic science, serves as an introduction. In the second section, a few definitions and theories of the neutrosophic integrals are offered. The 3th section frames neutrosophic numerical integrals and

MATLAB, in which the neutrosophic trapezoidal method and estimation of error of the neutrosophic trapezoidal method were studied. In 4th section, a conclusion to the paper is given.

## 2. Preliminaries

### 2.1. Neutrosophic integration by substitution method [16]

**Definition2.1.1**

$Let \quad f: D_f \subseteq R \rightarrow R_f \cup \{I\}$, to evaluate $\int f(x)dx$

Put: $x = g(u) \quad \Rightarrow \quad dx = \acute{g}(u)du$

By substitution, we get:

$$\int f(x)dx = \int f(u)\acute{g}(u)du$$

then we can directly integral it.

**Theorme2.1.1:**

If $\int f(x, I)dx = \varphi(x, I) \quad then,$

$$\int f\big((a + bI)x + c + dI\big)\, dx = \left(\frac{1}{a} - \frac{b}{a(a + b)} I\right) \varphi\big((a + bI)x + c + dI\big) + C$$

where $C$ is an indeterminate real constant, $a \neq 0$ , $a \neq -b$ $and$ $b, c, d$ are real numbers, while $I$ = indeterminacy.

## 3. Methods of the neutrosophic numerical integration

### 3.1 The neutrosophic trapezoidal method

Let:

$$\int_{a+bI}^{c+dI} f(x, I)\, dx$$

Where $f(x, I)$ is neutrosophic function and $a, b, c, d$ are real numbers, while $I =$ indeterminacy ($I \in [0,1]$).

we divide the interval $[a + bI, c + dI]$ into $n$ equal parts:

$$a + bI = x_0 + I\acute{x}_0 \ , x_1 + I\acute{x}_1 \ , x_2 + I\acute{x}_2, \dots, x_{n-1} + I\acute{x}_{n-1}, x_n + I\acute{x}_n = c + dI$$

such that the length of each sub interval is:

$$h_I = \frac{c + dI - (a + bI)}{n}$$

$$= \frac{c - a + (d - b)I}{n} = \Delta x_I$$

Let $y_0 + I\acute{y}_0 \ , y_1 + I\acute{y}_1 \ , y_2 + I\acute{y}_2, \dots, y_{n-1} + I\acute{y}_{n-1}, y_n + I\acute{y}_n$ be $y$ –coordinate, whereas:

$$y_0 + I\acute{y}_0 = f(x_0 + I\acute{x}_0), y_1 + I\acute{y}_1 = f(\ x_1 + I\acute{x}_1), y_2 + I\acute{y}_2 = f(x_2 + I\acute{x}_2), \dots, y_{n-1} + I\acute{y}_{n-1} = f(\ x_{n-1} + I\acute{x}_{n-1}), y_n + I\acute{y}_n = f(x_n + I\acute{x}_n)$$

Then the area between the two lines $a + bI, c + dI$, the curve of $f(x, I)$ and the x-axis equal the sum of upright trapezoidal which are bounded from up by arc of the neutrosophic function, Where the area of the first neutrosophic trapezoidal is:

$$\left(\frac{y_0 + I\acute{y}_0 + y_1 + I\acute{y}_1}{2}\right) h_I$$

and the area of the second neutrosophic trapezoidal is:

$$\left(\frac{y_1 + I\acute{y}_1 + y_2 + I\acute{y}_2}{2}\right) h_I$$

And the area of the last neutrosophic trapezoidal is:

$$\left(\frac{y_{n-1} + I\acute{y}_{n-1} + y_n + I\acute{y}_n}{2}\right) h_I$$

Hence:

$$\int_{a+bI}^{c+dI} f(x,I)\,dx = \left(\frac{y_0 + I\acute{y}_0 + y_1 + I\acute{y}_1}{2}\right) h_I + \left(\frac{y_1 + I\acute{y}_1 + y_2 + I\acute{y}_2}{2}\right) h_I + .. + \left(\frac{y_{n-1} + I\acute{y}_{n-1} + y_n + I\acute{y}_n}{2}\right) h_I$$

$$= \frac{h_I}{2}[y_0 + I\acute{y}_0 + 2y_1 + 2I\acute{y}_1 + \cdots + 2y_{n-1} + 2I\acute{y}_{n-1} + y_n + I\acute{y}_n]$$

$$= \frac{h_I}{2}[y_0 + I\acute{y}_0 + 2(y_1 + I\acute{y}_1 + \cdots + y_{n-1} + I\acute{y}_{n-1}) + y_n + I\acute{y}_n]$$

**3.1.1 Estimation of error**

$$E_I \le \frac{1}{12} {h_I}^3 Max\left|\acute{\acute{f}}(x,I)\right| \quad ; \quad a + bI \le x \le c + dI$$

Then the error of the step (subinterval) is:

$$E_I \le \frac{n}{12} {h_I}^3 Max\left|\acute{\acute{f}}(x,I)\right| \quad ; \quad a + bI \le x \le c + dI$$

We have:

$$h_I = \frac{c + dI - (a + bI)}{n} \quad \Rightarrow \quad n = \frac{c + dI - (a + bI)}{h_I}$$

Then the estimation of error of the neutrosophic trapezoidal method is:

$$E \le \frac{c + dI - (a + bI)}{12} {h_I}^2 Max\left|\acute{\acute{f}}(x,I)\right| \quad ; \quad a + bI \le x \le c + dI$$

**Example 3.1**
Evaluate

$$\int_{0+0I}^{1+I} \frac{dx}{1 + x^2}$$

by trapezoidal where $n = 4$, then calculate the occurred error.

**Solution:**

$$h_I = \frac{c - a + (d - b)I}{n} = \frac{1 + I}{4} = 0.25 + 0.25I$$

Then we will divide the interval $[0, 1+I]$ into four subintervals of length: $h_I = 0.25 + 0.25I$

| $(x, I)$ | $0$ | $0.25(1+I)$ | $0.5(1+I)$ | $0.75(1+I)$ | $1+I$ |
|---|---|---|---|---|---|
| $f(x, I)$ | $1$ | $0.9412 - 0.1412I$ | $0.8 - 0.3I$ | $0.64 - 0.3323I$ | $0.5 - 0.3I$ |

where:

- $f(0) = 1$
- $f\big(0.25(1+I)\big) = \frac{1}{1+\left(0.25(1+I)\right)^2} = \frac{1}{1.0625+0.1875I} = 0.9412 - 0.1412I$
- $f\big(0.5(1+I)\big) = \frac{1}{1+\left(0.5(1+I)\right)^2} = \frac{1}{1.25+0.75I} = 0.8 - 0.3I$
- $f\big(0.75(1+I)\big) = \frac{1}{1+\left(0.75(1+I)\right)^2} = \frac{1}{1.5625+1.6875I} = 0.64 - 0.3323I$
- $f(1+I) = \frac{1}{1+(1+I)^2} = \frac{1}{2+3I} = 0.5 - 0.3I$

then:

$$\int_{0+0I}^{1+I} \frac{dx}{1+x^2} = \frac{h_I}{2}[y_0 + I\acute{y}_0 + 2(y_1 + I\acute{y}_1 + y_2 + I\acute{y}_2 + y_3 + I\acute{y}_3) + y_4 + I\acute{y}_4]$$

$$= \frac{0.25+0.25I}{2}[1 + 2(0.9412 - 0.1412I + 0.8 - 0.3I + 0.64 - 0.3323I) + 0.5 - 0.3I]$$

$$= \frac{0.25+0.25I}{2}[6.2624 - 1.847I]$$

$$= (0.25 + 0.25I)[3.1312 - 0.9235I] = 0.7828 + 0.32105I$$

To finding the occurred error:

$$f(x, I) = \frac{1}{1+x^2} \quad \Rightarrow \quad \acute{f}(x, I) = \frac{-2x}{(1+x^2)^2}$$

$$\Rightarrow \acute{\acute{f}}(x, I) = \frac{-2(1+x^2)^2 + 6x^2(1+x^2)}{(1+x^2)^4} = \frac{-2 - 2x^2 + 8x^2}{(1+x^2)^3} = \frac{6x^2 - 2}{(1+x^2)^3}$$

$$\acute{\acute{f}}(0) = -2$$

$$\acute{\acute{f}}(1+I) = \frac{22I}{8+117I} = \frac{22}{125}I$$

$$\left|\acute{\acute{f}}(0)\right| > \left|\acute{\acute{f}}(1+I)\right| \quad on \quad I = [0,1]$$

$$\Rightarrow \qquad Max\left|\acute{\acute{f}}(x, I)\right| = \left|\acute{\acute{f}}(0)\right| = |-2| = 2$$

then:

$$E_I \le \frac{c+dI-(a+bI)}{12} {h_I}^2 \quad Max\left|\acute{\acute{f}}(x,I)\right| \quad ; \quad a+bI \le x \le c+dI$$

$$E_I \le \frac{1+I}{12}(0.25+0.25I)^2 \quad |-2| \quad ; \quad 0 \le x \le 1+I$$

$$E_I \le \frac{0.0625+0.4375I}{12}(2) \approx 0.01042+0.07292I \qquad (1)$$

Note:
If we put $I=0$ in (1), we get: $E_I \le 0.01042$ (the same result without indeterminacy).

**Example 3.2**
Evaluate

$$\int_{0.5+0.5I}^{1+I} xe^x \, dx$$

by trapezoidal where $h_I = 0.1+0.1I$, then calculate the occurred error.

**Solution:**

$$h_I = 0.1+0.1I$$

| $(x,I)$ | $0.5+0.5I$ | $0.6+0.6I$ | $0.7+0.7I$ | $0.8+0.8I$ | $0.9+0.9I$ | $1+I$ |
|---|---|---|---|---|---|---|
| $f(x,I)$ | $0.8244 + 1.894I$ | $1.09326 + 2.89086I$ | $1.40966 + 2.89086I$ | $1.7804 + 6.1444I$ | $2.21364 + 8.67564I$ | $2.7183 + 12.0599I$ |

where:

- $f(0.5+0.5I) = (0.5+0.5I)e^{0.5+0.5I}$

$$= (0.5+0.5I)\left(\sqrt{e}+I\left[e-\sqrt{e}\right]\right)$$

$$= (0.5+0.5I)(1.6487+1.0696I)$$

$$\approx 0.8244+1.894I$$

- $f(0.6+0.6I) = (0.6+0.6I)e^{(0.6+0.6I)}$

$$= (0.6+0.6I)(e^{0.6}+I[e^{1.2}-e^{0.6}])$$

$$= (0.6+0.6I)(1.8221+1.498I)$$

$$\approx 1.09326+2.89086I$$

- $f(0.7+0.7I) = (0.7+0.7I)e^{(0.7+0.7I)}$

$$= (0.7+0.7I)(e^{0.7}+I[e^{1.4}-e^{0.7}])$$

$$= (0.7+0.7I)(2.0138+2.0414I)$$

$$\approx 1.40966 + 2.89086I$$

- $f(0.8 + 0.8I) = (0.8 + 0.8I)e^{(0.8+0.8I)}$

$$= (0.8 + 0.8I)(e^{0.8} + I[e^{1.6} - e^{0.8}])$$

$$= (0.8 + 0.8I)(2.2255 + 2.7275I)$$

$$\approx 1.7804 + 6.1444I$$

- $f(0.9 + 0.9I) = (0.9 + 0.9I)e^{(0.9+0.9I)}$

$$= (0.9 + 0.9I)(e^{0.9} + I[e^{1.8} - e^{0.9}])$$

$$= (0.9 + 0.9I)(2.4596 + 3.5900I)$$

$$\approx 2.21364 + 8.67564I$$

- $f(1 + I) = (1 + I)e^{(1+I)}$

$$= (1 + I)(e + I[e^2 - e])$$

$$= (1 + I)(2.7183 + 4.6708I)$$

$$\approx 2.7183 + 12.0599I$$

then:

$$\int_{0.5+0.5I}^{1+I} xe^x \, dx = \frac{h_I}{2}[y_0 + I\acute{y}_0 + 2(y_1 + I\acute{y}_1 + y_2 + I\acute{y}_2 + y_3 + I\acute{y}_3 + y_4 + I\acute{y}_4) + y_5 + I\acute{y}_5]$$

$$= \frac{0.1 + 0.1I}{2}[0.8244 + 1.894I + 2(1.09326 + 2.89086I + 1.40966 + 4.26762I + 1.7804 + 6.1444I + 2.21364 + 8.67564I) + 2.7183 + 12.0599I]$$

$$= \frac{0.1 + 0.1I}{2}[16.53662 + 57.91094I]$$

$$= 0.826831 + 0.826831I + 2.895547I + 2.895547I$$

$$= 0.826831 + 6.6179251I \qquad (1)$$

to finding the occurred error:

$$f(x, I) = xe^x \quad \Longrightarrow \quad \acute{f}(x, I) = (1 + x)e^x$$

$$\Longrightarrow \quad \acute{\acute{f}}(x, I) = (2 + x)e^x$$

$$\acute{\acute{f}}(0.5 + 0.5I) = (2.5 + 0.5I)e^{0.5+0.5I} = (2.5 + 0.5I)(e^{0.5} + I[e - e^{0.5}])$$

$$= (2.5 + 0.5I)(1.64872 + 1.06956I)$$

$$= 4.1218 + 4.1218I + 0.53478I + 0.53478I$$

$$= 4.1218 + 5.19136I$$

$$\acute{\acute{f}}(1+I) = (3+I)e^{1+I} = (3+I)(e + I[e^2 - e])$$

$$= (3+I)(2.7183 + 4.6708I)$$

$$= 8.1549 + 8.1549I + 4.6708I + 4.6708I$$

$$= 8.1549 + 17.4965I$$

$$\left|\acute{\acute{f}}(1+I)\right| > \left|\acute{\acute{f}}(0.5+0.5I)\right| \quad on \quad I = [0,1]$$

$$\Rightarrow \qquad Max\left|\acute{\acute{f}}(x,I)\right| = \left|\acute{\acute{f}}(1+I)\right| = |8.1549 + 17.4965I|$$

$$= |8.1549| + I[|25.6514| - |8.1549|]$$

$$= 8.1549 + 17.4965I$$

then:

$$E_I \le \frac{c + dI - (a + bI)}{12} {h_I}^2 \ \ Max\left|\acute{\acute{f}}(x,I)\right| \quad ; \quad a + bI \le x \le c + dI$$

$$E_I \le \frac{1 + I - (0.5 + 0.5I)}{12}(0.1 + 0I)^2 \ \ (8.1549 + 17.4965I) \quad ; \quad 0.5 + 0.5I \le x \le 1 + I$$

$$E_I \le \frac{0.5 + 0.5I}{12}(0.081549 + 0.174965I) \approx 0.00398 + 0.01856I \qquad (*)$$

**3.1.2 The neutrosophic numerical integration by using MATLAB**

**If we go back to the previous example 3.1 and evaluate**

$$\int_{0+0I}^{1+I} \frac{dx}{1+x^2} = 0.7828 + 0.32105I$$

by trapezoidal, where $h_I = 0.1 + 0.1I$, by using MATLAB

solution:

- for $I = 1$ by substitution in (2), we find:

$$\int_{0+0I}^{1+I} \frac{dx}{1+x^2} = 0.7828 + 0.32105(1) = 1.10385$$

Let's now use MATLAB (for $I = 1$):

```
a=0;
>> b=2;
>> h=0.5;
```

```
>> f=0;
>> for x=0.5:(h):1.5;
f=(f+(1/(1+x^2)));
end
>> Am=(h/2)*((1/(1+a^2))+(2*f)+(1/(1+b^2)))

Am =

1.1038
```

- for $I = 0$ by substitution in (2), we find:

$$\int_{0+0I}^{1+I} \frac{dx}{1+x^2} = 0.7828 + 0.32105(0) = 0.7828$$

Let's now use MATLAB (for $I = 0$):

```
a=0;
b=1;
h=0.25;
f=0;
for x=0.25:(h):0.75;
f=(f+(1/(1+x^2)));
end
Am=(h/2)*((1/(1+a^2))+(2*f)+(1/(1+b^2)))

Am =

0.7828
```

If we go back to the previous example 3.2 and evaluate

$$\int_{0.5+0.5I}^{1+I} xe^x \, dx$$

by trapezoidal, where $h_I = 0.1 + 0.1I$, by using MATLAB

solution:

- for $I = 1$ by substitution in (1), we find:

$$\int_{1}^{2} xe^x \, dx = 0.826831 + 6.6179251(1) = 7.4447561$$

Let's now use MATLAB (for $I = 1$):

```
>> a=1;
b=2;
h=0.2;
f=0;
for x=1.2:(h):1.8;
f=(f+(x*(exp(x))));
```

```
end
>> Am=(h/2)*((a*(exp(a))+2*f+(b*(exp(b)))))

Am =

7.4448
```

➢ for $I = 0$ by substitution in (1), we find:

$$\int_{0.5}^{1} xe^{x}\,dx = 0.826831 + 6.6179251(0) = \boxed{0.826831}$$

Let's now use MATLAB (for $I = 0$):

```
a=0.5;
b=1;
h=0.1;
f=0;
for x=0.6:(h):0.9;
f=(f+(x*(exp(x))));
end
>> Am=(h/2)*((a*(exp(a))+2*f+(b*(exp(b)))))

Am =

0.8268
```

We note that we got the same results, noting that the numerical integration is approximate and the error estimate is studied.

## 4. Conclusions

This essay expands on the writings we previously wrote on neutrosophic integrals. Integrals play a significant role in daily life since they make several mathematical operations possible in the actual world, this is why we decided to explore the neutrosophic numerical integration, where we introduced the concept of neutrosophic numerical integration through the trapezoidal method, and the estimation of error study. Some examples of this were also solved and the correctness of the results was confirmed using MATLAB. In addition, this study is also regarded as being significant for neutrosophic integral applications.

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# A Novel of Domination in Neutrosophic Over Graphs

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**Abstract:** The degree of data uncertainty may be measured using a variety of mathematical techniques, but neutrosophic logic is a potent instrument for analysis when compared to fuzzy and intuitionistic fuzzy logics. This article introduces the novel idea of "dominance in neutrosophic over graph (NOG)s" and discusses some of the intriguing characteristics of complete, complete bipartite, and neutrosophic over bridge domination. With the relevant instances, the characterization of neutrosophic over domination set and neutrosophic over minimal domination set is developed, and their dominance numbers are examined.

**Keywords:** Neutrosophic **o**ver domination set, Neutrosophic **o**ver minimum domination set, Neutrosophic **o**ver domination number.

**Introduction:**

Graph theory has various uses in a range of fields, including operations research, physics, chemistry, economics, genetic engineering, and computer science, among others. A classical graph cannot accurately represent uncertain issues since there are two choices for each vertex or edge: it is either in the graph or it is not. A generalized form of the classical set known as a fuzzy set [18] is one in which objects have membership degrees ranging from zero to one. On fuzzy graphs, more work has previously been done. Zadeh invented the fuzzy set and presented the degree of membership in 1965. The intuitionistic fuzzy set and the degree of falsity (F) were both proposed by Atanassov [1] in 1983. The neutrosophic set (NS) of components (T, I, F) and the degree of indeterminacy (I) were created by Smarandache [10,11,12,13] in 1995. Prem Kumar [14,15,16] developed three different forms of lattices for neutrosophic graphs. Three sets of neutrosophic over/off/under were introduced by Smarandache [8, 9] and their use in nursing research approaches was discussed. This inspired the creation of the notion of dominance number [17] in the realm of NOG. Later, Narmada Devi worked on a novel neutrosophic off graph and minimum dominance using NOGs and NO- top graphs [2, 3, 4, 5, 6, 7].

Smarandache has created a NS [15]. It is a generalization of intuitionistic fuzzy sets and fuzzy sets. With some ambiguity, consistency, and partial knowledge that is used in daily life, it is a potent instrument to influence. The three elements of membership (M), indeterminacy (I), and non-membership (N) functions are dealt with by NSs. It is a really useful programme for dealing with challenges in everyday life. However, it only applies to three attribute values. Advanced research suggests that in order to increase accuracy, the measurement of data uncertainty needs to be handled with greater attribute value. This is true across many scientific disciplines, including biology, physics, information technology, networks, decision-making analysis, etc.

Recently, several writers have focused on dealing with multi-valued attribute data sets based on Smarandache Plithogenic set [7, 8]. It is regarded as one of the important sets that depict the contradictory multi-valued properties. Finding some of the valuable patterns from data with lithogenic properties and its graphical visualisation, which were inspired by several recent work in this subject, presents a challenge.

## 1. PRELIMINARIES:

**Definition 2.1** [8] Let $V$ be a given non-empty set. Then the set $A$ is said to be Fuzzy set on $V$ such that its membership function $\mu_A : V \to [0,1]$ for each $x \in V$.

**Definition 2.2** [5] Let $V$ be a given non-empty set. A NS $A$ in $V$ is characterized by a truth membership function $T_A(x)$, an indeterminate membership function $I_A(x)$ and a false membership function $F_A(x)$. The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are fuzzy sets on V. That is, $T_A, I_A, F_A : V \to [0, 1]$ and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

**Definition 2.3** [6] Let $\mathcal{U}$ be the universe of discourse and the NS $A \subset \mathcal{U}$. A N**eutrosophic over set** $A$ is defined as $A = \{(x, \langle T(x), I(x), F(x)\rangle), x \in \mathcal{U}$, where $T, I, F : V \to [0, \Omega]$ that is, $0 < 1 < \Omega$ and $\Omega$ is said to be a over limit, $T(x), I(x), F(x) \in [0, \Omega]$ such that no element has neutrosophic components of <0, and there is at least one element that has at least one neutrosophic component of >1.

**Definition 2.4** [6] Let $G^* = (V, E)$ be a crisp graph with $V$ is vertex and $E$ is edge. A **NOG** is a pair $G = (A, B)$ of $G^*$ where $A$ is a neutrosophic over set in $V$ and $B$ is a neutrosophic over set in $E \subseteq V \times V$ such that

$$T_B(xy) \leq min\{T_A(x), T_A(y)\},$$

$$I_B(xy) \leq min\{I_A(x), I_A(y)\},$$

$$F_B(xy) \geq max\{F_A(x), F_A(y)\}, \text{ for all } xy \in E.$$

Here $A$ and $B$ are said to be neutrosophic over vertex set and neutrosophic over edge set of $G$, respectively.

## 2. VARIOUS TYPES OF NOGs

**Definition 3.1** Let $G = (A, B)$ be a **NOG** in $G^*$. $G$ is said to be **complete** if:

$$\mathcal{T}_B(xy) = min\{\mathcal{T}_A(x), \mathcal{T}_A(y)\},$$

$$\mathcal{I}_B(xy) = min\{\mathcal{I}_A(x), \mathcal{I}_A(y)\},$$

$$\mathcal{F}_B(xy) = max\{\mathcal{F}_A(x), \mathcal{F}_A(y)\}, \text{ for all } (x, y) \in E.$$

**Example 3.1:**

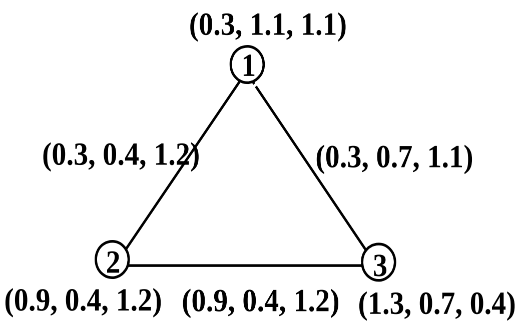


Figure 1: Complete NOG.

**Definition 3.2** A NOG $G = (A, B)$ in $G^*$ is a neutrosophic over bipartite (**NOBipar**) if the set $V$ can be partitioned into two non-empty over sets $V_1$ and $V_2$ such that $\mathcal{T}_B(xy) = \mathcal{I}_B(xy) = \mathcal{F}_B(xy) = 0$, for all $(x, y) \in E_1$ or $(x, y) \in E_2$.

A **NOBipar** graph is a **complete-NOBipar** if $\mathcal{T}_B(xy) = min\{\mathcal{T}_A(x), \mathcal{T}_A(y)\}$, $\mathcal{I}_B(xy) = min\{\mathcal{I}_A(x), \mathcal{I}_A(y)\}$, $\mathcal{F}_B(xy) = max\{\mathcal{F}_A(x), \mathcal{F}_A(y)\}$, for all $(x, y) \in E$.

A complete-**NOBipar** graph is a **star** NOG if either $|V_1| = 1$ or $|V_2| = 1$.

**Example 3.2:**

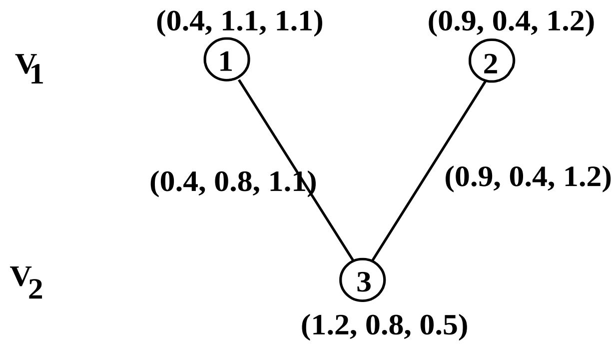


Figure 2: Complete Bi-par and Star NOG.

**Definition 3.3** Let $G = (A, B)$ be a NOG in $G^*$. Then

a) The real number $L$ is said to be the $\mathcal{T}$-order, if $L = \sum_{u \in V} \mathcal{T}_A(u)$.

b) The real number $M$ is said to be the $\mathcal{I}$-order, if $M = \sum_{u \in V} \mathcal{I}_A(u)$.

c) The real number $N$ is said to be the $\mathcal{F}$-order, if $N = \sum_{u \in V} \mathcal{F}_A(u)$.

d) A real number is said to be the order of a NOG, if it is equal to order $= \langle L, M, N \rangle$.

**Example 3.3:**

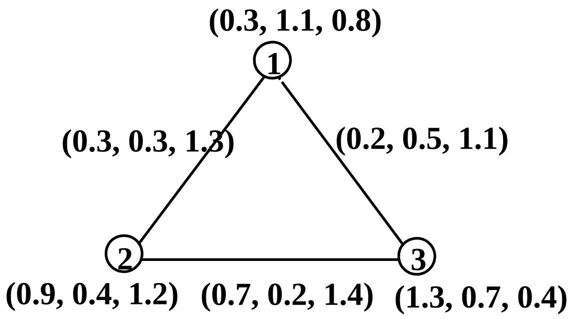


Figure 3: NOG.

In this above Figure 3 of NOG, the $\mathcal{T}$-order, $L = 2.5$, $\mathcal{I}$-order, $M = 2.2$, $\mathcal{F}$-order, $N = 2.4$ and the order is <2.5, 2.2, 2.4>.

**Definition 3.4** Let $v_0, v_n$ be two given vertices in a NOG $G = (A, B)$ such that $n \in \mathbb{N}$. Then

a) A **$\mathcal{T}$ -path** is an sequence of distinct vertices $P : v_0, v_1, \dots, v_n$ of length $n$ from $v_0$ to $v_n$ if $\mathcal{T}_B(v_i v_{i+1}) > 0$, for $i = 0,1, \dots, n-1$.

The **strength** of that $\mathcal{T}$-path is $min_{i=0}^{n-1}\{\mathcal{T}_B(v_i v_{i+1})\}$ and denoted by $\mu_G(P)_{\mathcal{T}}$

b) A **$\mathcal{I}$-path** is an sequence of distinct vertices $P : v_0, v_1, \dots, v_n$ of length $n$ from $v_0$ to $v_n$ if $\mathcal{I}_B(v_i v_{i+1}) > 0$, for $i = 0,1, \dots, n-1$.

The **strength** of that $\mathcal{I}$-path is $min_{i=0}^{n-1}\{\mathcal{I}_B(v_i v_{i+1})\}$ and denoted by $\mu_G(P)_{\mathcal{I}}$.

c) A **$\mathcal{F}$-path** is an sequence of distinct vertices $P : v_0, v_1, \dots, v_n$ of length $n$ from $v_0$ to $v_n$ if $\mathcal{F}_B(v_i v_{i+1}) > 0$, for $i = 0,1, \dots, n-1$.

The **strength** of that $\mathcal{F}$-path is $min_{i=0}^{n-1}\{\mathcal{F}_B(v_i v_{i+1})\}$ and denoted by $\mu_G(P)_{\mathcal{F}}$.

d) A **path** is an sequence of distinct vertices $P: v_0, v_1, \dots, v_n$ of length $n$ from $v_0$ to $v_n$ if it be $\mathcal{T}$-path, $\mathcal{I}$-path, $\mathcal{F}$-path, simultaneously.

The **strength** of that path is $min\{\mu_G(P)_{\mathcal{T}}, \mu_G(P)_{\mathcal{I}}, \mu_G(P)_{\mathcal{F}}\}$ and denoted by $\mu_G(P)$.

**Example 3.4:**

In the figure 3 of NOG, the various types of path from 1 - 2 are $P_1$ : $1 \longrightarrow 2$ and $P_2$ : $1 \longrightarrow 3 \longrightarrow 2$ which are $\mathcal{T}$ -path, $\mathcal{I}$-path, $\mathcal{F}$-path respectively. $\mu_G(P_1)$= (0.3, 0.3, 1.3), $\mu_G(P_2)$= (0.2, 0.2, 1.1).

**Definition 3.5**

Let $v_i, v_j$ be two given vertices in a NOG $G = (A, B)$ such that $i > j$ and $i, j \in \mathbb{N}$. Then

a) The **$\mathcal{T}$-strength** between $v_i$ and $v_j$ is $max\{\mu_G(P)_{\mathcal{T}}\}$ and denoted by $\mu_G^{\infty}(v_i, v_j)_{\mathcal{T}}$

b) The **$\mathcal{I}$-strength** between $v_i$ and $v_j$ is $max\{\mu_G(P)_{\mathcal{I}}\}$ and denoted by $\mu_G^{\infty}(v_i, v_j)_{\mathcal{I}}$

c) The **$\mathcal{F}$-strength** between $v_i$ and $v_j$ is $\max\{\mu_G(P)_{\mathcal{F}}\}$ and denoted by $\mu_G^{\infty}(v_i, v_j)_{\mathcal{F}}$

d) The **strength** between $v_i$ and $v_j$ is $max\{\mu_G^{\infty}(v_i, v_j)_{\mathcal{T}}, \mu_G^{\infty}(v_i, v_j)_{\mathcal{I}}, \mu_G^{\infty}(v_i, v_j)_{\mathcal{F}}\}$ and denoted by $\mu_G^{\infty}(v_i, v_j)$.

**Example 3.5**

In the Figure 3 of NOG, the various type of paths from 1 to 2 are $P_1$ : $1 \longrightarrow 2$ and $P_2$ : $1 \longrightarrow 3 \longrightarrow 2$ which are $\mathcal{T}$ -path, $\mathcal{I}$-path, $\mathcal{F}$-path respectively. $\mu_G(P_1)$= (0.3, 0.3, 1.3), $\mu_G(P_2)$= (0.2, 0.2, 1.1). Then the $\mathcal{T}$ -strength, $\mathcal{I}$-strength, $\mathcal{F}$-strength are 0.3, 0.3,1.3 and the strength = 1.3.

**Example 3.6**

Consider $G = (A, B)$ is a NOG on $G^*$ as Figure 4. Various paths of length 3 from $v_1$ to $v_2$ are explored.

$P_1 : v_1 \longrightarrow v_3 \longrightarrow v_4 \longrightarrow v_2$

$P_2 : v_1 \longrightarrow v_3 \longrightarrow v_5 \longrightarrow v_2$

$P_3 : v_1 \longrightarrow v_4 \longrightarrow v_3 \longrightarrow v_2$

$P_4 : v_1 \longrightarrow v_4 \longrightarrow v_5 \longrightarrow v_2$

$P_5 : v_1 \longrightarrow v_5 \longrightarrow v_3 \longrightarrow v_2$

$P_6 : v_1 \longrightarrow v_5 \longrightarrow v_4 \longrightarrow v_2$, respectively.

And from following graph, we find the strength of that paths as follows.

For $P_1$ : $\mu_G(P_1)_{\mathcal{T}}$ = 0.3, $\mu_G(P_1)_{\mathcal{I}}$ = 0.3, $\mu_G(P_1)_{\mathcal{F}}$ = 1.2 and $\mu_G(P_1)$ = 0.3

For $P_2$ : $\mu_G(P_2)_{\mathcal{T}}$ = 0.2 , $\mu_G(P_2)_{\mathcal{I}}$ = 0.5, $\mu_G(P_2)_{\mathcal{F}}$ = 1.2 and $\mu_G(P_2)$= 0.2

For $P_3$ : $\mu_G(P_3)_{\mathcal{T}}$ = 0.3 , $\mu_G(P_3)_{\mathcal{I}}$ = 0.4, $\mu_G(P_3)_{\mathcal{F}}$ = 1.2 and $\mu_G(P_3)$= 0.3

For $P_4$: $\mu_G(P_4)_{\mathcal{T}}$ = 0.4 , $\mu_G(P_4)_{\mathcal{I}}$ = 0.6, $\mu_G(P_4)_{\mathcal{F}}$ = 1.2 and $\mu_G(P_4)$= 0.4

For $P_5$ : $\mu_G(P_5)_{\mathcal{T}}$ = 0.2, $\mu_G(P_5)_{\mathcal{I}}$ = 0.5, $\mu_G(P_5)_{\mathcal{F}}$ = 1.2 and $\mu_G(P_5)$ = 0.2

For $P_6$ : $\mu_G(P_6)_{\mathcal{T}}$ = 0.5 , $\mu_G(P_6)_{\mathcal{I}}$ = 0.3, $\mu_G(P_6)_{\mathcal{F}}$ = 1.2 and $\mu_G(P_6)$= 0.3

Finally, we discuss about the strength between two vertices $v_1$ and $v_2$.

$\mu_G^{\infty}(v_1, v_2)_{\mathcal{T}}$ = 0.5, $\mu_G^{\infty}(v_1, v_2)_{\mathcal{I}}$ = 0.6, $\mu_G^{\infty}(v_1, v_2)_{\mathcal{F}}$ = 1.2, $\mu_G^{\infty}(v_1, v_2)$ = 1.2.

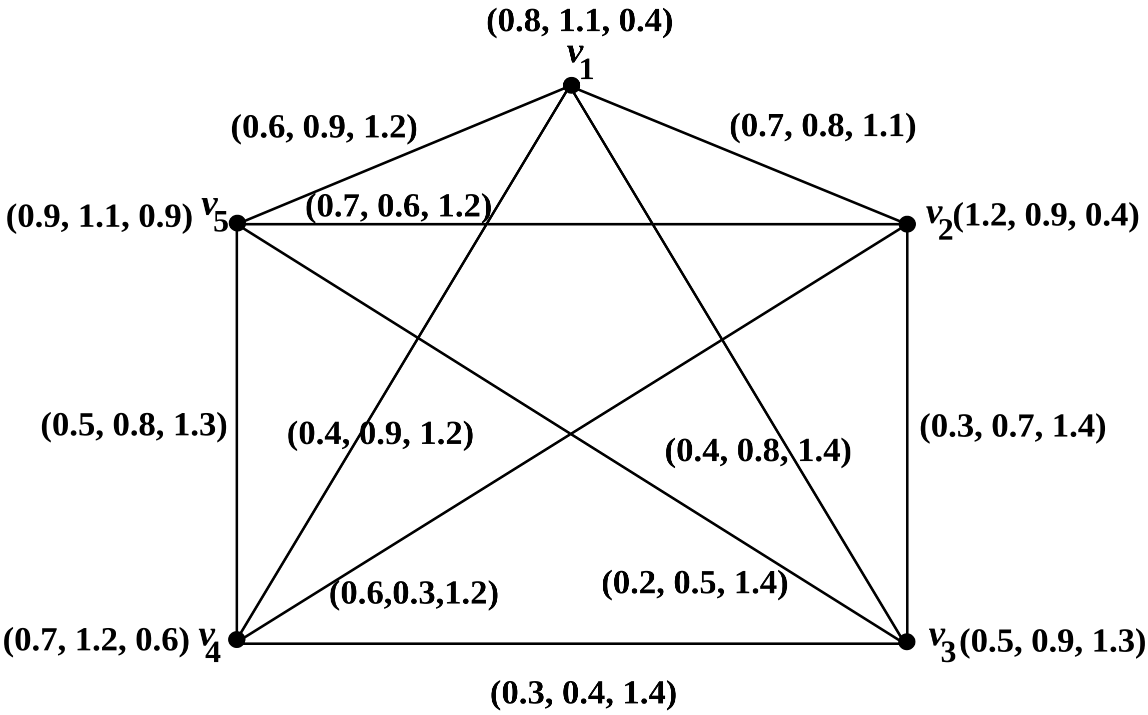


**Figure 4 NOG**

**Definition 3.6** An edge $xy$ in a NOG $G = (A, B)$ is said to be the

a) $\mathcal{T}$**-bridge**, if the strengths of each $\mathcal{T}$ -path $P$ from $x$ to $y$ excluding $xy$ were less than $\mathcal{T}_B(xy)$.

b) $\mathcal{I}$**- bridge**, if the strengths of each $\mathcal{I}$-path $P$ from $x$ to $y$ excluding $xy$ were less than $\mathcal{I}_B(xy)$.

c) **$\mathcal{F}$- bridge**, if the strengths of each $\mathcal{F}$-path $P$ from $x$ to $y$ excluding $xy$ were less than $\mathcal{F}_B(xy)$.

d) **Bridge**, if it is either $\mathcal{T}$ -bridge, $\mathcal{I}$-bridge, $\mathcal{F}$-bridge.

**Example 3.7** In the figure 3 of NOG, the edges 12 and 23 are $\mathcal{T}$-bridge and $\mathcal{F}$-bridge respectively

a) The edges 12 and 13 are $\mathcal{I}$-bridge and

b) The edges 12, 13 and 23 is bridge

**Notation:** $\mu^{\infty}_{G-\{xy\}}(x,y)$ is the strength between $x$ and $y$ obtained from $G$ by deleting the edge $xy$.

Thus, we notate for $\mathcal{T}$-strength, $\mathcal{I}$-strength, $\mathcal{F}$-strength as $\mu^{\infty}_{G-\{xy\}}(x,y)_{\mathcal{T}}$, $\mu^{\infty}_{G-\{xy\}}(x,y)_{\mathcal{I}}$ and $\mu^{\infty}_{G-\{xy\}}(x,y)_{\mathcal{F}}$ respectively.

**Definition 3.7** An edge $xy$ in $G = (A,B)$ is said to be

a) **$\mathcal{T}$-effective**, if $\mathcal{T}_B(xy) > \mu^{\infty}_{G-\{xy\}}(x,y)_{\mathcal{T}}$.

b) **$\mathcal{I}$-effective**, if $\mathcal{I}_B(xy) > \mu^{\infty}_{G-\{xy\}}(x,y)_{\mathcal{I}}$.

c) **$\mathcal{F}$- effective**, if $\mathcal{F}_B(xy) > \mu^{\infty}_{G-\{xy\}}(x,y)_{\mathcal{F}}$.

d) **Effective** if it is either of $\mathcal{T}$-effective, $\mathcal{I}$-effective and $\mathcal{F}$-effective.

**Example 3.8**

Consider $G = (A,B)$ is a NOG. In the below Table 1, effectiveness of edges of G is established.

| Edges | $\mathcal{T}$-effective | $\mathcal{I}$-effective | $\mathcal{F}$-effective | Effective |
|---|---|---|---|---|
| $v_1v_2$ | √ | √ | × | √ |
| $v_1v_3$ | √ | √ | √ | √ |
| $v_1v_4$ | × | √ | × | √ |
| $v_1v_5$ | × | √ | × | √ |

| | | | | |
|---|---|---|---|---|
| $v_2v_3$ | × | × | √ | √ |
| $v_2v_4$ | √ | × | × | √ |
| $v_2v_5$ | √ | × | × | √ |
| $v_3v_4$ | × | × | √ | √ |
| $v_3v_5$ | × | × | √ | √ |
| $v_4v_5$ | × | × | × | × |

**Table 1 Effectiveness of edges**

For example, $v_4v_5$ have neither of $\mathcal{T}$- effective, $\mathcal{I}$- effective, $\mathcal{F}$- effective and effective property. $v_1v_2$ have $\mathcal{T}$-effective and $\mathcal{I}$-effective property.

Therefore, it is an effective edge. The collection of edges $\{v_1v_2, v_1v_3, v_2v_4, v_2v_5\}$, $\{v_1v_2, v_1v_3, v_1v_4, v_1v_5\}$, $\{v_1v_3, v_2v_3, v_3v_4, v_3v_5\}$, $\{v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_2v_3, v_2v_4, v_2v_5, v_3v_4, v_3v_5\}$ have $\mathcal{T}$-effective, $\mathcal{I}$-effective, $\mathcal{F}$-effective and effective property, respectively.

## 3. DOMINATION ON NOGs.

**Definition 4.1**

a) Let $G = (A, B)$ be a NOG, let $x, y \in V$, we say $x$ **dominates** $y$ in $G$ if there exist an $\mathcal{T}$-effective edge between them.

b) A subset $S$ of $V$ is said to be a **$\mathcal{T}$-effective dominating** set in $G$ if for every $v \in V - S$, there exists $u \in S$ such that $u$ dominates $v$ as $\mathcal{T}$-eff.

c) The **$\mathcal{T}$-weight of** $x$ is defined by $w(x)_{\mathcal{T}} = \mathcal{T}_A(x) + \frac{\sum_{xy \text{ is a } \mathcal{T}-eff\ edge} \mathcal{T}_B(xy)}{\sum_{xy \text{ is a } edge} \mathcal{T}_B(xy)}$.

d) For any $S \subseteq V$, **$\mathcal{T}$-weight of** $S$ is defined by $w(S)_{\mathcal{T}} = \sum_{u \in S}(w(u)_{\mathcal{T}})$.

e) Let $A$ be the set of all $\mathcal{T}$-effective dominating sets in $G$. The **$\mathcal{T}$-domination number** of $G$ is given by $\gamma(G)_{\mathcal{T}} = \min_{D\in U}(w(D)_{\mathcal{T}})$. Then the $\mathcal{T}$-effective dominating set that correspond to $\gamma(G)_{\mathcal{T}}$ is known as **$\mathcal{T}$-dominating set**.

f) Further, in the similar manner we define $\mathcal{I}$-dominating set and $\mathcal{F}$-dominating set of $G$, respectively.

**Note:** If $\sum_{xy \text{ is a edge}} \mathcal{T}_B(xy)$ equals 0, for some $x \in V$. Then $\frac{\sum_{xy \text{ is a } \mathcal{T}-eff\ edge} \mathcal{T}_B(xy)}{\sum_{xy \text{ is a edge}} \mathcal{T}_B(xy)}$ equal with 0.

**Definition 4.2**

a) We say $x$ **dominates** $y$ in $G$, if there exist an effective edge between them.

b) A subset $S$ of $V$ is said to be the **effective dominating** set in $G$, if for every $v \in V - S$, there exists $u \in S$ such that $u$ dominates $v$ as eff.

c) The **weight of S** is defined by $w(S) = \min\{w(D)_T, w(D)_I, w(D)_F\}$.

d) Let $A$ be the set of all eff dominating sets in $G$. The **domination number** of $G$ is given by $\gamma(G) = \min_{D\in U}(w(D))$. Then the effective dominating set that correspond to $\gamma(G)$ is known as **dominating set**.

## 4. PROPERTIES OF EFFECTIVE EDGES IN NOGs

**Proposition 5.1**

Let $G = (A, B)$ be a complete NOG in $G^*$ which has exactly one path between two given vertices and has

a) $\mathcal{T}$-strength. Then $\gamma(G)_{\mathcal{T}} \leq \min_{u\in V} \mathcal{T}_A(u) + 2$.

b) $\mathcal{I}$- strength. Then $\gamma(G)_{\mathcal{I}} \leq \min_{u\in V} \mathcal{I}_A(u) + 2$.

c) $\mathcal{F}$- strength. Then $\gamma(G)_{\mathcal{F}} \leq \min_{u\in V} \mathcal{F}_A(u)$+ 2.

d) strength. Then $\gamma(G) \leq \min_{u\in V}(\mathcal{T}_A(u), \mathcal{I}_A(u), \mathcal{F}_A(u)) + 2$.

**Proof**

(a) Let $G = (A, B)$ be a NOG on $G^{*}$. Then the $\mathcal{T}$-strength of path $P$ from $u$ to $v$ will be $\mathcal{T}_A(u) \wedge \ldots \wedge \mathcal{T}_A(v) \leq \mathcal{T}_A(u) \wedge \mathcal{T}_A(v) = \mathcal{T}_B(uv)$. So $\mu_G^{\infty}(u,v)_{\mathcal{T}} < \mathcal{T}_B(uv)$. $uv$ is a path from $u$ to $v$ such that $\mathcal{T}_B(uv) = \mathcal{T}_A(u) \wedge \mathcal{T}_A(v)$. Therefore $\mu_G^{\infty}(u,v)_{\mathcal{T}} \geq \mathcal{T}_B(uv)$. Hence $\mu_G^{\infty}(u,v)_{\mathcal{T}} = \mathcal{T}_B(uv)$. Then $\mathcal{T}_B(uv) > \mu_{G-\{xy\}}^{\infty}(u,v)_{\mathcal{T}}$ which means the edge $uv$ is $\mathcal{T}$-effective. Therefore, all edges are $\mathcal{T}$-effective and each vertex is adjacent to all other vertices. So $D = \{u\}$ will be the $\mathcal{T}$-effective dominating set and $\sum_{xy \text{ is a } \mathcal{T}-eff \text{ edge}} \mathcal{T}_B(xy) = \sum_{xy \text{ is a edge}} \mathcal{T}_B(xy)$ for each $u \in V$. The result follows.

Similarly, we can prove for (b), (c) and (d).

**Proposition 5.2** Let $G = (A, B)$ be any **complete-NOBipar** graph in $G^{*}$ which has exactly one path between two given vertices and has

a) $\mathcal{T}$-strength. Then $\gamma(G)_{\mathcal{T}}$ is either $\mathcal{T}_A(u) + 1$ or $\min_{u \in V_1, v \in V_2}(\mathcal{T}_A(u) + \mathcal{T}_A(v)) + 2$.

b) $\mathcal{I}$-strength. Then $\gamma(G)_{\mathcal{I}}$ is either $\mathcal{I}_A(u) + 1$ or $\min_{u \in V_1, v \in V_2}(\mathcal{I}_A(u) + \mathcal{I}_A(v)) + 2$.

c) $\mathcal{F}$- strength. Then $\gamma(G)_{\mathcal{F}}$ is either $\mathcal{F}_A(u) + 1$ or $\min_{u \in V_1, v \in V_2}(\mathcal{F}_A(u) + \mathcal{F}_A(v)) + 2$.

d) Then $\gamma(G)$ is either $\min(\mathcal{T}_A(u), I_A(u), \mathcal{F}_A(u)) + 1$ or $\min_{u \in V_1, v \in V_2}(\mathcal{T}_A(u) + \mathcal{T}_A(v), \mathcal{I}_A(u) + \mathcal{I}_A(v), \mathcal{F}_A(u) + \mathcal{F}_A(v)) + 2$. for all $u, v \in V$.

**Proof**

(a) Let $G = (A, B)$ be any **complete-NOBipar** graph on $G^{*}$ which has exactly one path and has $\mathcal{T}$-strength between two given vertices. By the proof of Proposition 5.1, all the edges are $\mathcal{T}$-effective.

**Case (i):** Consider, if $G$ is the star NOG with $V = \{u, v_1, v_2, \ldots v_n\}$ in which $u$ and $v_i$ are the center and the leaves of G, for $1 \leq i \leq n$, respectively. Then $\{u\}$ is the $\mathcal{T}$-dominating set of $G$. Hence $\gamma(G)_T = T_A(u) + 1$.

**Case (ii):** Let both of $V_1$ and $V_2$ include more than one vertex. Every vertex in $V_1$ is dominated by every vertex in $V_2$, as $\mathcal{T}$-effective and conversely every vertex in $V_2$ is dominated by every vertex in $V_1$, as $\mathcal{T}$-effective. Thus the $\mathcal{T}$-effective dominating sets are $V_1$ and $V_2$ and any set containing 2 vertices one in $V_1$ and the other in $V_2$. Therefore, $\gamma(G)_{\mathcal{T}} = \min_{u \in V_1, v \in V_2} (\mathcal{T}_A(u) + \mathcal{T}_A(v)) + 2$. The result follows.

Similarly, we can prove for (b), (c) and (d).

**Proposition 5.3** Let $G = (A, B)$ be a NOG in $G^*$. Then $xy \in E$ is a

a) $\mathcal{T}$-effective edge ⇔ $xy$ is a $\mathcal{T}$-bridge.

b) $\mathcal{I}$- effective edge ⇔ $xy$ is a $\mathcal{I}$-bridge.

c) $\mathcal{F}$-effective edge ⇔ $xy$ is a $\mathcal{F}$-bridge.

d) Effective edge ⇔ $xy$ is a bridge.

**Proof**

(a) ⇔ Suppose $xy$ is a $\mathcal{T}$-effective edge.

⇔ By Definition 3.7(a), $\mathcal{T}_B(xy) > \mu^{\infty}_{G-\{xy\}}(x, y)_{\mathcal{T}}$.

⇔ Since, $\mathcal{T}_B(xy) = \mu^{\infty}_{G}(x, y)_{\mathcal{T}}$. Therefore $\mu^{\infty}_{G}(x, y)_{\mathcal{T}} > \mu^{\infty}_{G-\{xy\}}(x, y)_{\mathcal{T}}$.

⇔ $xy$ is a $\mathcal{T}$-bridge. Therefore, the result follows.

Similarly, we can prove for (b), (c) and (d).

## 5. CONCLUSION:

Both theoretical studies and practical applications for the idea of dominance in graphs are fairly extensive. In this study, we used strength of path to construct the NOG dominance number and explain it with relevant instances. Numerous applications of expert systems, image processing, computer networks, and social systems can make use of the NOG notion. Additionally, we look at several noteworthy characteristics of the product NOGs with the dominance number, and the suggested ideas are explained with useful examples.

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# Neutrosophic Inference System (NIS) in Power Electrical Transformers, Adapted the MIL-STD-1629A

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## Abstract:

This study is concerned with creating a neutrosophic inference system (NIS) and developing its mathematical concepts, as well as determining the most critical problems in the power electrical transformers. Studying their potential failure mode and effects analysis and then, analyzing the risk assessment and management. New insight and novel techniques have been presented to interpret the failure modes (i.e. severity, occurrence, and detection). This paper presents a novel operator called ANOR which is used for the first time to combine the (IF-Then) inference rules. The neutrosophic inference system using failure mode effect analysis is a modern tool for studying the reliability of electrical power transformers. Also, this study suggested some modifications in the standard MIL-STD-1629A. this article presents and for the first time, a new inferencing sixty-three neutrosophic rules in which their biasing is categorized into three types: truth state, indeterminacy state, and falsity state.

**Keywords:** Neutrosophic Failure Mode Effect Analysis (NFMEA); Neutrosophic Risk Priority Number (NRPN); Neutrosophic Inference Analysis NFMEA; type -1- Mamdani inference system; MIL-STD-1629A.

## 1. Introduction

It is well-known that all inference systems techniques such as the type-1 and type-2 Mamdani system and type-1 and type-2 Sugeno system were created in the 1970s. In 1975, Prof. E. Mamdani built one of the first fuzzy systems to control a steam engine and boiler combination, it is still implemented in fuzzy environments as an active tool for analyzing fuzzy control systems, we can define the fuzzy controllers simply as it came in literature, they are very simple conceptually because they are composed of input, processing, and output stages. As a special case, if we focused on the electrical power transformer for the reasons that:

1- The power transformers have an active role in the efficiency and reliability of power transmission lines,

2- The electrical power transformer is one of the most expensive network equipment.

3- It is important to know when the electrical power transformer facing danger because it contains a great quantity of oil in contact with high-voltage elements.

Therefore, the objective is to study the failure modes and their effect analysis for it by giving up the well-known fuzzy approach and replacing it with a neutrosophic inference system (NIS). Before moving forward with this new approach (i.e. putting the mathematical fundamentals of NIS), some of the basic concepts are present in the upcoming sections. Till this moment and before this study, the traditional and fuzzy methods are being taken up to analyze the parameters, and criteria which increase the risk of fire and explosion in case of abnormal circumstances or technical failures using the traditional and fuzzy control systems with the general goal of improving the reliability of the system. Many researchers published dozens of articles in this field including but not limited to [1-6].

To describe the use of potential failure mode and effects analysis in the neutrosophic environments (NFMEA), we need to determine the inputs of the neutrosophic inference system (NIS) that will be the mathematical tool for risk assessment and management, the inputs of the NIS in the electrical power transformers (EPT) that have been considered in this paper are [7-10]:

1- Active Part which consists of the Core (i.e. having the function of concentrating the magnetic flux), the Windings (i.e. the function of the windings is to carry current. In addition to dielectric stress and thermal requirements the windings have to withstand mechanical forces that may cause windings replacement).

2- Insulation system which consists of the solid insulation and the transformer oil.

3- Some transformer components which is known as accessories are Bushings, Tap Changer, cooling system, Tank, Mechanical structure includes (clamping, coil blocking and lead support), and Winding Connections that are between windings, tap leads, and to bushings).

4- Protection which includes the Buchholz protection, pressure relief, valve circuity, sure protection, and tap changer pressure relief.

At the experts, these parts are well known, it should be studied the severity, occurrence, and detection of each problem caused in one of the previously mentioned parts of (EPT). Again, the neutrosophic inference system (NIS) consists of an input stage, a processing stage, and an output stage, the processing stage invokes each appropriate rule and generates results for each rule, then combines the results of the rules. The output stage converts the combined result back into a specific control output value.

The membership functions that should be used in the neutrosophic inference system, and how to classify the (If-then), (If- and- then), (If-or-then), and (If- anor-then) statements will be discussed with details in the forthcoming sections.

## 2. A Comparison Between Conventional FMEA, Fuzzy FMEA and Neutrosophic FMEA

NASA agency in 1963 adopted the FMEA as a formal system analysis methodology for their reliability requirements. Then, it was adopted and implemented by Ford Motor in 1977 [8] simultaneously with the military standard procedures for performing a failure mode, effects, and critical analysis which has been released by the USA Department of Defense on June 12, 1977. Since then, it has become a powerful tool extensively used for risk and reliability analysis of systems in a wide range of industries, including automotive, construction, aerospace, nuclear, and electro-technical [5].

Reliability and quality assurance have been of increasing concern in various industries in recent years. The evidence of this argumentation is that there are many different standards developed for failure modes and effects analysis (FMEA) application in various industries, and the most popular standards are:

- SAE-J-1739 [state ref.], /great for the ground vehicle community.
- AIAG's [state ref.], /a reference manual to be used by suppliers to Chrysler LLC, Ford Motor Company, and General Motors Corporation.
- MIL-STD-1629A [state ref.] / drafted by the United States Department of Defense.
- IEC 60812 [state ref.] / guidance to how these techniques may be applied to achieve various reliability program objectives.
- BSEN 60812 [state ref.]/ the European adoption of the IEC 60812.

The above standards are dedicated for conventional FME, a typical standard will outline Severity, Occurrence, and Detection rating scales as well as examples of an FMEA spreadsheet layout. Also, a glossary will be included that defines all terms used in the FMEA. The rating scales and the layout of the data can be different between standards, but the processes and definitions remain similar. However, in general, FMEA is a systematic, proactive method for evaluating a process to identify where and how it might fail and to assess the relative impact of different failures, in order to identify the parts of the process that are must in need of repair and maintenance [2].

Even though the conventional FMEA is probably the most popular tool for reliability and failure mode analysis in electrical power transformers, there are some limitations in it since it is difficult or even impossible for experts to precisely evaluate the three risk factors S, O, and D, since the risk factors are often expressed in a linguistic way (such as 'likely', 'important', 'very high', 'catastrophic', 'marginal', 'minor'…etc.). as well as, in traditional FMEA methodology, the three risk factors are assumed to have the same importance [11-16]. However, it is observed that many operative and management experts give more preference to the "fault detection factor".

The neutrosophic failure mode effect analysis (NFMEA) is proposed in this study, which will be more general in its aspects than the fuzzy failure mode effect analysis (FFMEA) since the latter concentrates on the creation of membership functions for the antecedents and consequences of rules, while the (NFMEA) is more accurate in classifying the cases of Severity into three main portions (membership function supports the Truth Side/ MFT, membership function supports the Indeterminate Side/ MFI, and membership function supports the Falsity Side/ MFF), same talking goes for Occurrence and Detections. In this manner, the membership functions that were used in FFMEA will be hugely different from what they will be used in NFMEA. In this study we adopt the strategy that the relations between MFT, MFI, and MFF are simply represented as:

$$MFF = 1 - MFT, MFI = MFT \cap \text{MFF} \;\dots\dots\dots\dots\dots\dots. (1)$$

It means from a philosophical point of view that the behavior of the truth membership function for any neutrosophic object (i.e. number, element, variable, function…etc.) has the inverse behavior of the falsity membership function for the same object, while the indeterminacy membership function for that object is exactly the intersection of the two membership functions MFT and MFF because the indeterminacy case is neither truth nor false but it is swinging between them this means the indeterminacy may contain some truth combined with some falsity. The following mathematical definition is a great representation of the severity criteria, Occurrence criteria, and Detection criteria that would be used to formulate the rules in NFMEA.

## 2.1 Definition [7]

Let $A_0$ be the set of all neutrosophic non-liner functions $g(x)$ that are neutrosophically less than or equal to $z$, i.e. $A_0 = \{x_i \in FN_m,\ g(x) < _N z\}$. The membership functions of $g(x)$ and $anti(g(x))$ are:

$$\mu_{A_o}\big(g(x)\big) = \begin{cases} 1 & 0 \le g(x) \le z \\ \left(e^{\frac{-1}{d_o}(g(x)-z)} + e^{\frac{-1}{d_o}(\text{anti }(g(x))-z)} - 1\right), & z < g(x) \le z - d_o \ln 0.5 \end{cases} \quad (2)$$

$$\mu_{A_o}(\text{anti}(\,g(x))) = \begin{cases} 0 & 0 \le g(x) \le z \\ \left(1 - e^{\frac{-1}{d_o}(\text{anti }(g(x))-z)} - e^{\frac{-1}{d_o}(g(x)-z)}\right), & z - d_o \ln 0.5 \le g(x) \le z + d_o \end{cases} \quad (3)$$

It is clear that $\mu_{A_o}(\text{neut}(\,g(x)))$ consists from the intersection of the following functions:

$$e^{\frac{-1}{d_o}(g(x)-z)} \quad \& \quad 1 - e^{\frac{-1}{d_o}(\text{anti}(g(x))-z)} \quad (4)$$

$$\mu_{A_o}(\text{neut}(\,g(x))) = \begin{cases} 1 - e^{\frac{-1}{d_o}(\text{anti}(\,g(x))-z)} & z \le g(x) \le z - d_o \ln 0.5 \\ e^{\frac{-1}{d_o}(g(x)-z)} & z - d_o \ln 0.5 < g(x) \le z + d_o \end{cases} \quad (5)$$

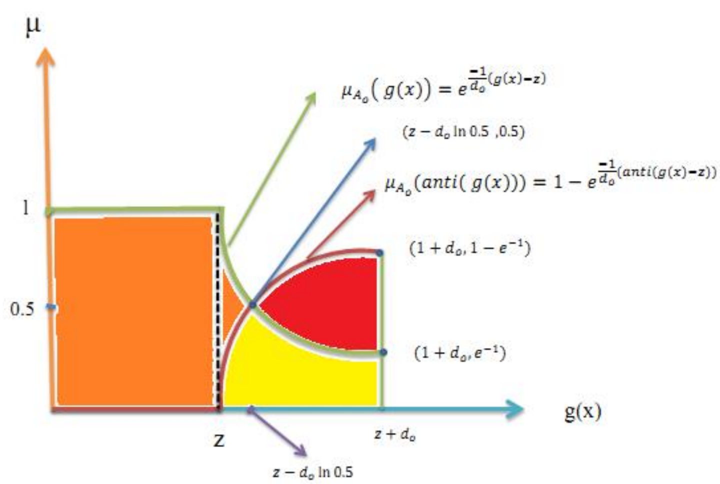



Figure 1.1: The orange color means the region covered by $\mu_{A_o}\big(g(x)\big)$, the red color means the region covered by $\mu_{A_o}(\text{anti}(\,g(x)))$, and the yellow color means the region covered by $\mu_{A_o}(\text{neut}(\,g(x)))$.

## 3. The construction of Rules in the Neutrosophic Inference Systems

It is an important issue to talk about the IF-Then statements in fuzzy rule sets before presenting our modification on these fuzzy rules to transfer it from fuzzy environments to neutrosophic environments.

We mentioned in the introduction section, one of the stages in any inference system is the processing stage which consists of a collection of logic rules in the frame of IF_THEN phrases. However, the IF-portion is called the antecedent, while the THEN-portion is called the consequent. Regular fuzzy control systems have dozens of rules.

Consider a rule for the severity input in the electrical power transformer:

| |
|---|
| IF (the primary function can be done) and (urgent repair is required) THEN (the risk severity on the power transformer is minor) |

The above fuzzy inference rules no longer meet the need, more sophisticated rules should be formulated from the neutrosophic perspective.

Moreover, the fuzzy operators (AND, OR, NOT) that combines several antecedents should be extended to include new operator.

### 3.1 Creation of the Neutrosophic Inference Rules

The truth, indeterminacy, and falsity for neutrosophic terms such as (about, near, close to, approximately, very slightly, too, extremely, somewhat) have spectrums of severity due to the definitions can vary considerably among different implementations.

The following three tables illustrate neutrosophic rules in which the neutrosophic inference systems can be in new apparel:

Table (1): Neutrosophic Rules for Severity Inputs of the Power Transformer

| | |
|---|---|
| Truth | IF the primary function can be done AND un-urgent repair is required THEN the risk severity on the power transformer is minor-minor |
| Indeterminacy | IF the primary function can be done AND urgent repair is required THEN the risk severity on the power transformer is rather-minor |
| Falsity | IF the primary function can be done AND very-urgent repair is required THEN the risk severity on the power transformer is minor |
| Truth | IF there is a few reduction in the ability to implement the primary function THEN the risk severity on the power transformer is a mini-marginal. |
| Indeterminacy | IF there is a normal reduction in the ability to implement the primary function THEN the risk severity on the power transformer is a rather-marginal. |
| Falsity | IF there is an extremely reduction in the ability to implement the primary function THEN the risk severity on the power transformer is a marginal. |
| Truth | IF the problem does not cause a loss of primary function THEN the state of the system is not critical |
| Indeterminacy | IF the problem causes a partial loss of primary function THEN the state of the system is rather critical |
| Falsity | IF the problem causes a totally loss of primary function THEN the state of the system is critical |
| Truth | IF the system becomes partially inoperative THEN the state is not catastrophic |
| Indeterminacy | IF the system becomes inoperative THEN the state is rather catastrophic |
| Falsity | IF the system becomes completely inoperative THEN the state completely catastrophic |

Table (2): Neutrosophic Rules for Occurrence Inputs of the Power Transformer

| | |
|---|---|
| Truth1 | IF a single failure mode probability of occurrence is less than 0.001 THEN the probability of the risk occurrence on the power transformer is extremely unlikely. |
| Truth2 | IF a single failure mode probability of occurrence is less than 0.01 THEN the probability of the risk occurrence on the power transformer is unlikely. |
| Indeterminate1 | IF a single failure mode probability of occurrence is less than 0.1 THEN the probability of the risk occurrence on the power transformer is occasional. |

| | |
|---|---|
| Indeterminate2 | IF a single failure mode probability of occurrence is less than 0.2 THEN the probability of the risk occurrence on the power transformer is reasonably probable. |
| Falsity1 | IF a single failure mode probability of occurrence is greater than 0.2 THEN the probability of the risk occurrence on the power transformer is sometimes frequent. |
| Falsity2 | IF a single failure mode probability of occurrence is greater than 0.3 THEN the probability of the risk occurrence on the power transformer is permanently frequent. |

Table (3): Neutrosophic Rules for Detection Inputs of the Power Transformer

| | |
|---|---|
| Truth1 | IF the problem has been well identified THEN the problem will be completely fixed before the electricity services reach to the customer. |
| Truth2 | IF the problem has been fairly identified THEN the problem will be fixed before the electricity services reach to the customer. |
| Indeterminate1 | IF the problem has been well detected AND rough identification THEN the problem will be nearly fixed before the electricity services reach to the customer. |
| Indeterminate2 | IF the problem has been fairly detected THEN the problem will cause a delay in reaching the electricity services to the customer. |
| Falsity1 | IF the problem has been roughly detected THEN the problem will cause a temporary pause in the system. |
| Falsity2 | IF the system needs to complementary test THEN the problem will cause a pause in the system. |

Regarded table (3), it is worth mentioning that the term ''identification'' indicates that the source or the location of the defect or the fault has been determined by the test. while the term "Detection" indicates that the defect or the fault exists.

Again, if we intend to combine several antecedents using the traditional operators that used to be in traditional or in fuzzy inference systems which are (AND, OR, NOT), but in neutrosophic theory, we need to establish new operators called (ANOR, NOT ANOR), the operator (ANOR) is neither "AND" nor "OR", but it is the value between them, the following neutrosophic operators are considered in banding the neutrosophic statements:

- AND operator: it means to select the minimum weight of all combined antecedents.
- OR operator: it means to select the maximum weight of all combined antecedents.
- Not operator: it is the resulting value of subtracting 2 minus the (truth membership function plus indeterminacy membership function) to give the complementary function.
- ANOR operator: it is resulting from the equation $\frac{AND\ operator\ result + OR\ operator\ result}{2}$ that is mean (the minimum weight of all combined antecedents plus the maximum weight of all combined antecedents divided by two).
- Not ANOR: it is the value resulting from the formula $1 - \frac{AND\ operator\ result + OR\ operator\ result}{2}$, which is exactly the complement of the operator ANOR.

The above neutrosophic operators have been created by the Ph.D. student (Ahemd K. Essa) and first appeared in this dissertation similar to the proposed neutrosophic inference system that was presented newly in this work (i.e. field of research that did not fathomless yet).

In the fuzzy inference systems, the most popular shapes of membership functions are the triangular, trapezoidal, and bell curves, but the shape is generally less important than the number of curves and their placement, from three to seven curves are generally appropriate to cover the required range of an input value in fuzzy inference system. The neutrosophic inference system will differ in taking the curves, we will depend on the truth membership function for those antecedents (i.e. inputs) and consequents (i.e. outputs) that belong to the truth spectrum, indeterminate membership functions are dedicated to those antecedents and consequents that represent the spectrum of indeterminacy, similarly, the falsity membership functions have been specified for those antecedents and consequences that represent the spectrum of falsity. So, the shape of the membership functions in the neutrosophic inference systems are as important as number of curves and their placements.

## 4. Suggestions to Modify MIL-STD-1629A

The generality and speedily used standard is MIL-STD-1629A. with more than four decades of years' usage and improvements, it has been utilized in various industries for failure mode, effects, and criticality analysis (FMECA). The objective of a FMCA is to identify all modes of failure within a system design, its first purpose is the early identification of all catastrophic and critical failure possibilities so they can be eliminated or minimized through design correction at the earliest possible time [16].

## 4.1 Pioneering Neutrosophic Thoughts in Modifying MIL-STD-1629A

To replace the severity classification categories that were stated in section 4.4.3, page 9 of MIL-STD-1629A, issued on 24 NOVEMBER 1980 into new categories in the perspective of neutrosophic theory, we can customize the following components with their appropriate membership function:

| Neutrosophic Bias | IF + Antecedent Statement | AND + Antecedent Statement | THEN + Consequence Statement | The membership function concerning to the neutrosophic bias |
|---|---|---|---|---|
| Truth | IF the primary function can be done | And un-urgent repair is required | Then the risk severity on the power transformer is minor-minor | All statements in this row are adhering to the truth membership function represented by (6) |
| indeterminacy | IF the primary function can be done | And urgent repair is required | Then the risk severity on the power transformer is rather-minor | The last two statements in this row are adhering to the Indeterminacy membership function represented by (7) |
| falsity | IF the primary function can be done | And very-urgent repair is required | Then the risk severity on the power transformer is minor | The last two statements in this row are adhering to the falsity membership function represented by (8) |
| Truth | IF there is a few reduction in the ability to implement the primary function | ……………………. | Then the risk severity on the power transformer is mini-marginal | All statements in this row are adhering to the truth membership function represented by (6) |
| Indeterminacy | IF there is a normal reduction in the ability to implement the primary function | ………………… | Then the risk severity on the power transformer is a rather-marginal | All statements in this row are adhering to the Indeterminacy membership function represented by (7) |

| | | | | |
|---|---|---|---|---|
| Falsity | IF there is an extremely reduction in the ability to implement the primary function | ………………… | Then the risk severity on the power transformer is a marginal | All statements in this row are adhering to the falsity membership function represented by (8) |

Table (4): Severity Rules distributed to its neutrosophic bias

Suppose the following variables represent the corresponding statements:

Table 5: Encoding the truth, indeterminacy, and falsity statements to its corresponding variable $\#_i$.

| **Neutrosophic Variables $\#_i$** | **Corresponding Statements** | **Neutrosophic bias** |
|---|---|---|
| $\#_1$ | the primary function can be done | Truth bias |
| $\#_2$ | un-urgent repair is required | Truth bias |
| $\#_3$ | the risk severity on the power transformer is minor-minor | Truth bias |
| $\#_4$ | urgent repair is required | Indet. bias |
| $\#_5$ | the risk severity on the power transformer is rather-minor | Indet. bias |
| $\#_6$ | very-urgent repair is required | Falsity bias |
| $\#_7$ | The risk severity on the power transformer is minor | Falsity bias |
| $\#_8$ | There is a few reduction in the ability to implement the primary function | Truth bias |
| $\#_9$ | The risk severity on the power transformer is a mini-marginal | Truth bias |
| $\#_{10}$ | There is a normal reduction in the ability to implement the primary function | Indet. bias |
| $\#_{11}$ | The risk severity on the power transformer is a rather-marginal | Indet. bias |
| $\#_{12}$ | There is an extremely reduction in the ability to implement the primary function | Falsity bias |
| $\#_{13}$ | The risk severity on the power transformer is a marginal | Falsity bias |

For all $\#_i$ , $i = 1,2,3,8,9$ the following truth membership function is recommended

$$\xi_A(\#_i) = \begin{cases} 0 & \#_i \le 0.4 \\ (\frac{\#_i - 0.4}{2})^2 & 0.4 < \#_i \le 2 \\ 1 & \#_i > 2 \end{cases} \quad (6)$$

For all $\#_i$ , $i = 4,5,10,11$ the following indeterminacy membership function should be taken

$$\varpi_A(\#_i, 0.4, 2) = \begin{cases} (\frac{\#_i - 0.4}{1.6})^2 & 0.4 \le \#_i < 1.6 \\ \frac{1}{2} - \left(\frac{\#_i - 1.6}{2}\right)^2 & 1.6 \le \#_i < 2 \\ 0 & 2 \le \#_i < 0.4 \end{cases} \quad (7)$$

Finally, those $\#_i$ , $i = 6,7,12,13$ have to be shapes according to the following falsity membership function:

$$\Upsilon_A(\#_i) = \begin{cases} 1 & \#_i \le 0.4 \\ 1 - (\frac{\#_i - 0.4}{2})^2 & 0.4 < \#_i \le 2 \\ 0 & \#_i > 2 \end{cases} \quad (8)$$

It should be noticed that the expert has free choice in adopting the truth membership function, but, once he/ she decides to adopt a specific one, the remaining indeterminacy and falsity functions must follow the behaviour of his choice as it refers to in the equation (2).

## 5. Build Neutrosophic Inference System Using Custom Functions

Due to the fuzzyLogicDesigner app and the MATLAB® command lines are not supplied with the ability to specify the neutrosophic statements to their appropriate truth, indeterminacy, and falsity membership functions, as well as, fuzzyLogicDesigner lacks to contain other operators except the traditional inference functions (AND, OR), which are inadequate for the representation of neutrosophic argues, therefore, this section has been dedicated to program new operators (i.e. custom functions) ANOR, NOT ANOR, in addition to program all neutrosophic inference system requirements.

### 5.1 Programming the Truth Membership Function

Now, the creation custom truth membership function is the aiming step, regarded as the preparation step to use it in the neutrosophic inference system. It is clear that the following MATLAB commands are plotting the truth membership function that mathematically represents the equation (6), and it lies between 0 and 1.

```
% plot the diagram of truth membership function
clc;
clear;
close;
syms x
f1=((x-0.4)/2)^2;
figure
obj=fplot(f1,[0.4 2.4])
hold on
fplot(0,[0 0.4])
fplot(1,[2.4 3.0])
hold off
title('figure1:truthmf')
xlabel('input values')
ylabel('truth membership function')
ylim([-0.06 1.06])
```

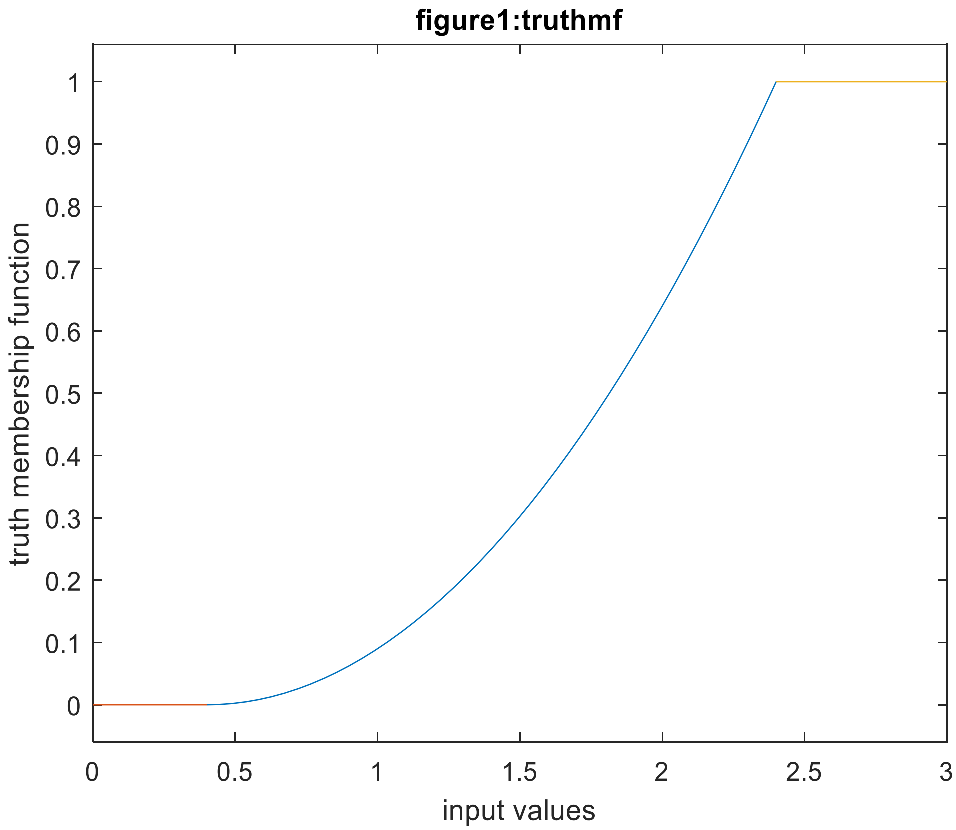



Figure 5.1: the graph of truth membership function $\xi_A(\#_i), i = 1,2,3,8,9$

## 5.2 Programming the Falsity Membership Function

Again using the similarly commands goes to plot the mathematical representation of falsity membership function that was presented in equation (8),

```
% plot the diagram of falsity membership function
clc;
clear;
close; syms x
f2=1-(((x-0.4)/2)^2);
figure
obj=fplot(f2,[0.4 2.4])
hold on
fplot(1,[0 0.4])
```

```
fplot(0,[2.4 3.0])
hold off
title('figure2:falsitymf')
xlabel('input values')
ylabel('falsity membership function')
ylim([-0.06 1.06])
```

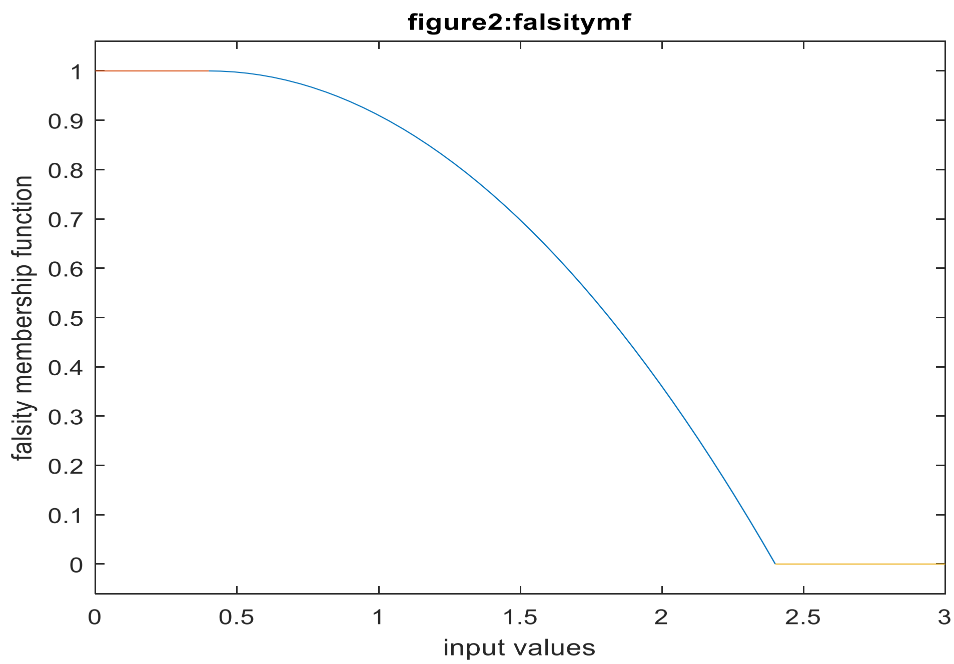



Figure 5.2: the graph of falsity membership function $\Upsilon_A(\#_i), i = 6,7,12,13$

## 5.3 Programming the Indeterminacy Membership Function

Again, the concept of the indeterminacy function is that function which swings between the truth and the falsity membership functions for the same neutrosophic object (variable, element, number… etc.), that is, the intersection of them represents the indeterminacy function. The following MATLAB syntax demonstrates the region

```
% plot the curve of the indeterminacy memebership function which is the
% intersection of both truth membership function and falsity membership % function
```

```
clc;
clear;
close;
syms x
f1=((x-0.4)/2)^2;
f2=0.5-(((x-0.4)/2)^2);
figure
obj=fplot(f2,[1.4 1.82])
hold on
obj=fplot(f1,[0.4 1.4])
fplot(0,[0 0.4])
fplot(0,[1.80091 3])
hold off
title('figure3:indeterminacymf')
xlabel('input values')
ylabel('indeterminacy membership function')
ylim([-0.06 1.06])
```

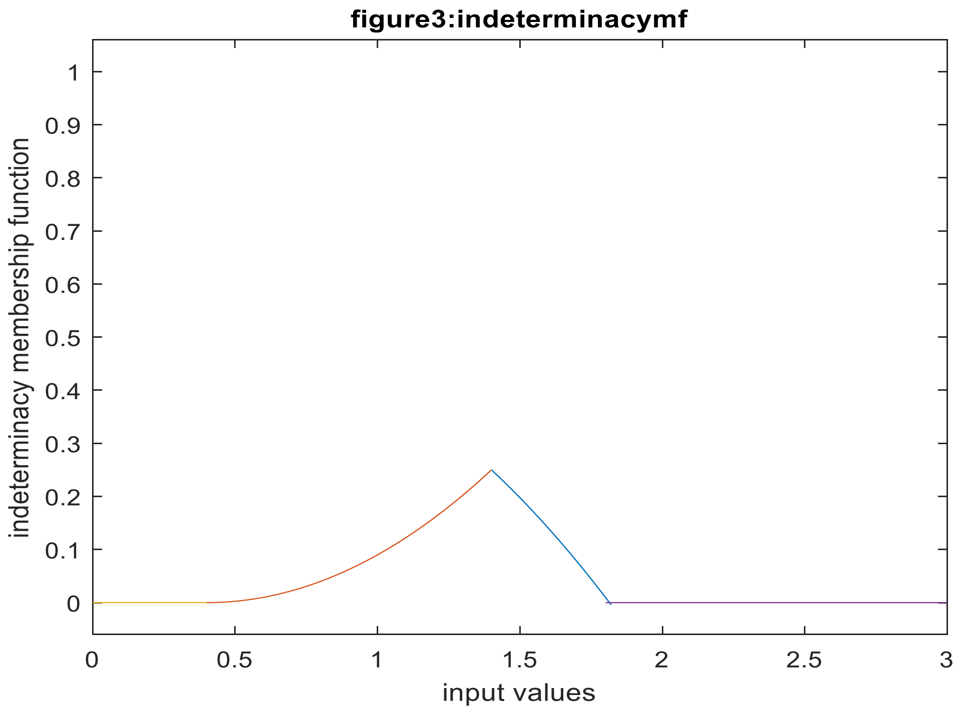



Figure 5.3: the graph of indeterminacy membership function $\varpi_A(\#_i, 0.4, 2), i = 4,5,10,11$

## 5.4 Create Custom Inference Functions

Since the MATLAB Toolbox of fuzzy inference systems (FIS) is void of features that support the operators ANOR, and NOT ANOR. Furthermore, the MATLAB Toolbox completely unsupported with Neutrosophic Inference System (NIS) which built-in AND, OR, implication, aggregation, and deneutrosophication, this forced us to the strategy of partial use of a FIS with some modification through custom-specific operators and adapted functions of (FIS) to be appropriate for neutrosophic Inference System (NIS), this will be done by the following clauses:

Note that: when the custom inference system has been created, we should save it in our current working folder or on the MATLAB path, this will enable us to design a NIS that uses the custom inference function at the command line or in *AddaptedFuzzyLogicDesigner* app.

In Neutrosophic Logic Toolbox™ software that should be built in Matlab we have:

- AND inference function performs an element by element matrix operation, similar to the command min, see the following example:

```
clc;
clear;
close;
x=[0.5 1;3 0.7];
y=[0.32 5;0.79 2];
ANDOperator=min(x,y)
```

```
ANDOperator =
  0.3200    1.0000
  0.7900    0.7000
```

- OR inference function performs an element by element matrix operator, similar to the command max, the following example illustrates the operator:

```
clc;
clear;
close;
x=[0.5 1;3 0.7];
y=[0.32 5;0.79 2];
OROperator=max(x,y)
```

```
OROperator =
  0.5000    5.0000
  3.0000    2.0000
```

- ANOR inference function performs an element by element matrix operator, similar to the command (max+min)/2, the following example illustrates the operator:

```
clc;
```

```
clear;
close;
x=[0.5 1;3 0.7;2.1 0.56];
y=[0.32 5;0.79 2;8.1 1.03];
ANOROperator=(max(x,y)+min(x,y))/2
```

```
ANOROperator =
 0.4100    3.0000
 1.8950    1.3500
 5.1000    0.7950
```

- NOT ANOR inference function performs an element by element matrix operator, similar to the command 1-(max+min)/2, the upcoming example is demonstrating the requirement:

```
x=[0.5 1;3 0.7;2.1 0.56];
y=[0.32 5;0.79 2;8.1 1.03];
ANOROperator=1-((max(x,y)+min(x,y))/2)
```

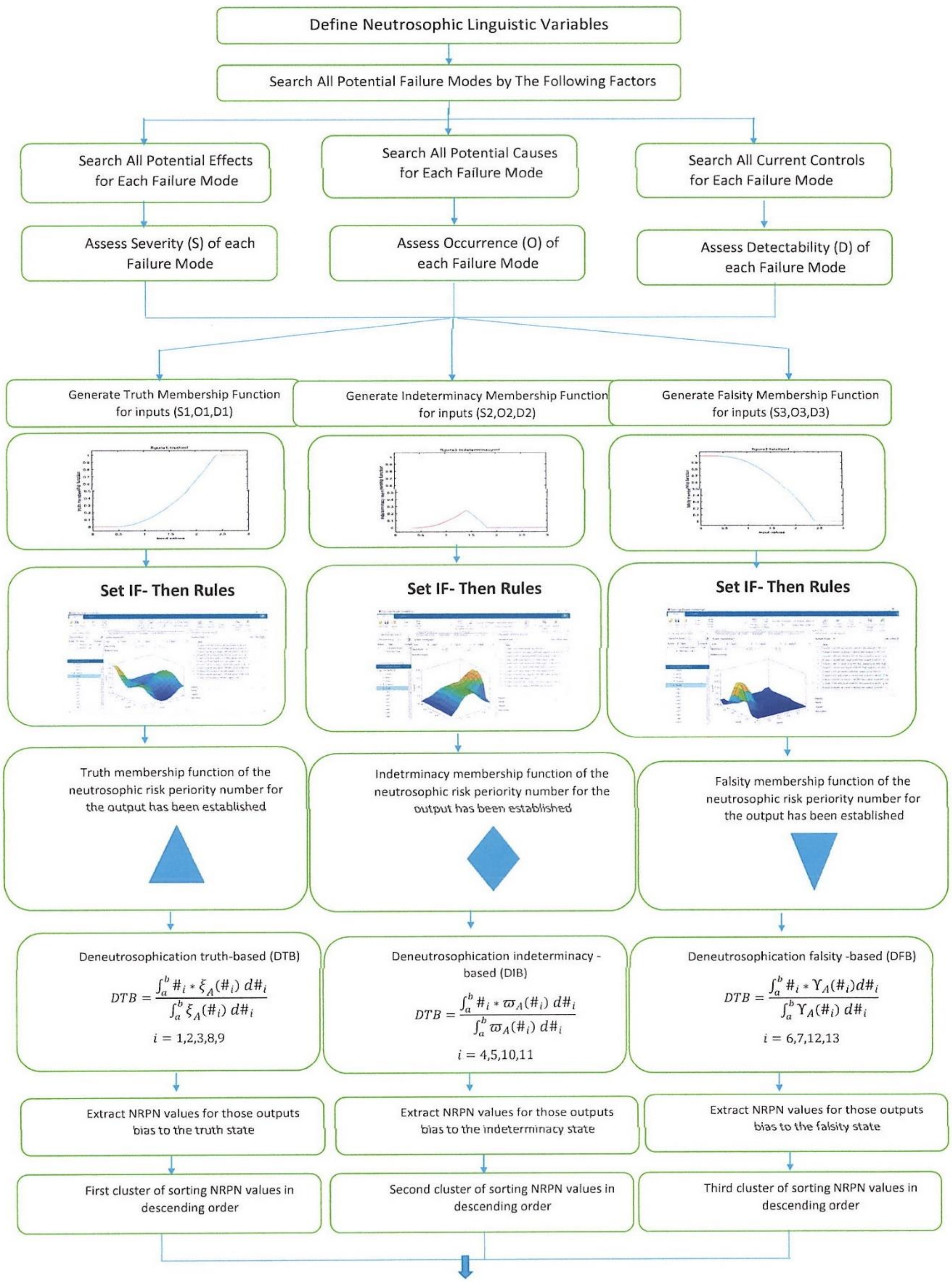
```
ANOROperator =
  0.5900   -2.0000
 -0.8950   -0.3500
 -4.1000    0.2050
```

## 6. Neutrosophic FMEA Flow Chart

The scaling of the three factors (i.e. Severity, Detection, and Occurrence) which were manifested in Tables (1,2 & 3), leads us to produce the flowchart of NFMEA which is presented in Figure 2 containing the algorithm of neutrosophic failure mode effect analysis to conduct the neutrosophic risk priority ranking.

We should state that the general defect analysis of the power transformers in this manuscript concerning its severity classification and its occurrence classification are scaling by using adapted MIL-STD-1629 standard, this adaptation is to meet our requirements that came from the neutrosophic theory were tabulated in tables (1&2). furthermore, our adaptation technique of the CIGRE working group on power transformer [11] is used to scale the detection factor for diagnostic tests tabulated in Table (3).

Without loss of generality, our algorithm for the neutrosophic inference system dependent on Mamdani type-1- method with customizing (truth, indeterminacy, and falsity) membership functions for inputs and outputs, and the neutrosophic risk priority number NRPN will be partitioned into three categories, NRPN truth-biased, NRPN indeterminacy-biased, finally NRPN falsity-biased. For each faulty cause, the controls that are currently in place in order to reduce or eliminate the risk linked with potential defective reason should be noted.


Define Neutrosophic Linguistic Variables
Search All Potential Failure Modes by The Following Factors
Search All Potential Effects for Each Failure Mode
Search All Potential Causes for Each Failure Mode
Search All Current Controls for Each Failure Mode
Assess Severity (S) of each Failure Mode
Assess Occurrence (O) of each Failure Mode
Assess Detectability (D) of each Failure Mode
Generate Truth Membership Function for inputs (S1,O1,D1)
Generate Indeterminacy Membership Function for inputs (S2,O2,D2)
Generate Falsity Membership Function for inputs (S3,O3,D3)
Set IF- Then Rules
Set IF- Then Rules
Set IF- Then Rules
Truth membership function of the neutrosophic risk periority number for the output has been established
Indetrminacy membership function of the neutrosophic risk periority number for the output has been established
Falsity membership function of the neutrosophic risk periority number for the output has been established
Deneutrosophication truth-based (DTB)
$DTB = \frac{\int_a^b \#_i * \xi_A(\#_i)\, d\#_i}{\int_a^b \xi_A(\#_i)\, d\#_i}$
$i = 1,2,3,8,9$
Deneutrosophication indeterminacy -based (DIB)
$DTB = \frac{\int_a^b \#_i * \varpi_A(\#_i)\, d\#_i}{\int_a^b \varpi_A(\#_i)\, d\#_i}$
$i = 4,5,10,11$
Deneutrosophication falsity -based (DFB)
$DTB = \frac{\int_a^b \#_i * \Upsilon_A(\#_i) d\#_i}{\int_a^b \Upsilon_A(\#_i)\, d\#_i}$
$i = 6,7,12,13$
Extract NRPN values for those outputs bias to the truth state
Extract NRPN values for those outputs bias to the indeterminacy state
Extract NRPN values for those outputs bias to the falsity state
First cluster of sorting NRPN values in descending order
Second cluster of sorting NRPN values in descending order
Third cluster of sorting NRPN values in descending order

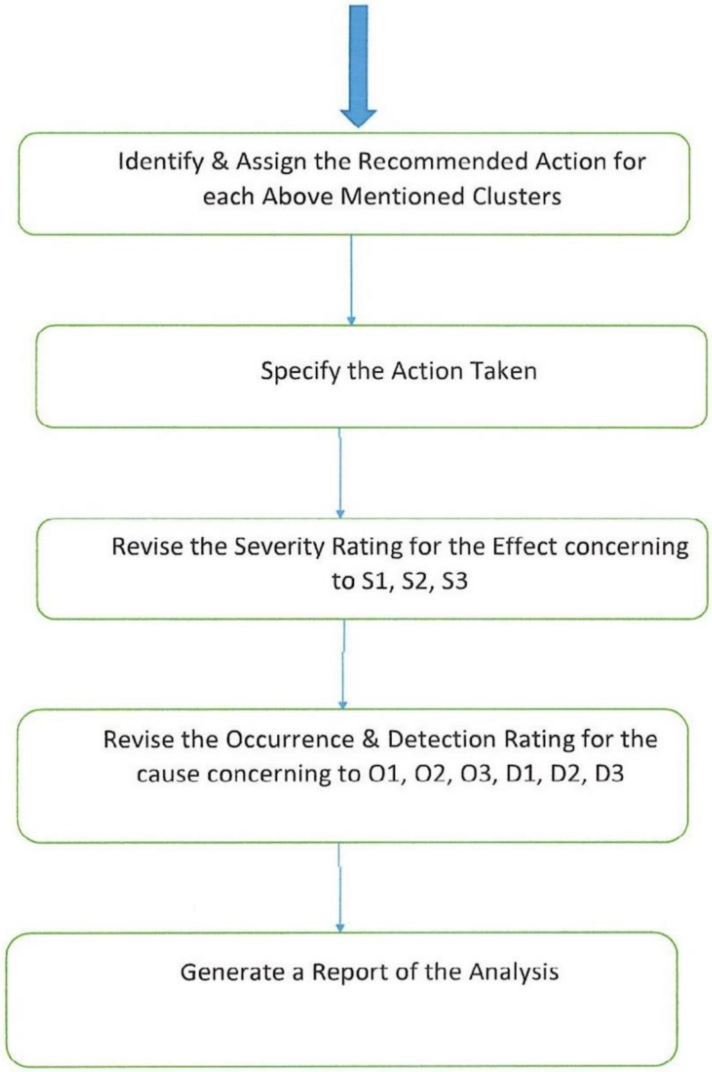



Figure 2: Algorithm of NFMEA creation

## 7. Components of the Power Transformer Dependent in this Work

In this work, we will depend on the same components of the oil-immersed power transformers that have been adopted by [2], which are:

1- The active part is composed of the Core and its function is to concentrate the magnetic flux. Windings have the function of carrying current.

2- The insulation system consists of two parts, a liquid part called transformer oil, and a solid part which is cellulose.

3- Accessories are firstly composed of bushings that insulate a high-voltage conductor passing through a metal enclosure. The second component is the tap changer which is the most complex component of the transformer and its function is to regular the voltage level by adding or subtracting turns from the transformer windings. The third component is the cooling System consists of cooling fans which is designed to remove the heat caused by copper and iron losses. The fourth component is the tank is primarily the container for the oil and physical protection for the active part. The fifth component is the mechanical structure which includes clamping, coil blocking, and lead support, their function is to support the active part of the transformer firmly in its place and withstand against mechanical stresses. The sixth and final part is winding connections which are between windings, tap leads, and bushings, their function is to provide the required electrical connection between these elements.

4- Protection is the primary objective of transformer protection which is used to detect internal faults in the transformer with a high degree of sensitivity and cause subsequent de-energization and, at the same time be immune to faults external to the transformer.

## 8. Power Transformer Neutrosophic Failure Mode Effect Analysis (NFMEA)

Similar to the severity rules distributed to the neutrosophic bias that has been done in table (4), we will create the same distributed rules for the occurrence and the severity of failure modes:

| Neutrosophic Bias | IF + Antecedent Statement | THEN + Consequence Statement | The membership function concerning to the neutrosophic bias |
|---|---|---|---|
| Truth | IF a single failure mode probability of occurrence is less than 0.001 | THEN the probability of the risk occurrence on the power transformer is extremely unlikely. | All statements in this row are adhering to the truth membership function represented by (6) |

| | | | |
|---|---|---|---|
| Truth | IF a single failure mode probability of occurrence is less than 0.01 | THEN the probability of the risk occurrence on the power transformer is unlikely. | All statements in this row are adhering to the truth membership function represented by (6) |
| Indeterminacy | IF a single failure mode probability of occurrence is less than 0.1 | THEN the probability of the risk occurrence on the power transformer is occasional. | All statements in this row are adhering to the Indeterminacy membership function represented by (7) |
| Indeterminacy | IF a single failure mode probability of occurrence is less than 0. | THEN the probability of the risk occurrence on the power transformer is reasonably probable. | All statements in this row are adhering to the Indeterminacy membership function represented by (7) |
| Falsity | IF a single failure mode probability of occurrence is greater than 0.2 | THEN the probability of the risk occurrence on the power transformer is sometimes frequent. | All statements in this row are adhering to the falsity membership function represented by (8) |
| Falsity | IF a single failure mode probability of occurrence is greater than 0.3 | THEN the probability of the risk occurrence on the power transformer is permanently frequent. | The last two statements in this row are adhering to the falsity membership function represented by (8) |

Table (6): Occurrence Rules Distributed to its Neutrosophic Bias

| Neutrosophic Bias | IF + Antecedent Statement | THEN + Consequence Statement | The membership function concerning to the neutrosophic bias |
|---|---|---|---|
| Truth | IF the problem has been well identified | THEN the problem will be completely fixed before the electricity services reach to the customer. | All statements in this row are adhering to the truth membership function represented by (6) |

| | | | |
|---|---|---|---|
| Truth | IF the problem has been fairly identified | THEN the problem will be fixed before the electricity services reach to the customer. | All statements in this row are adhering to the truth membership function represented by (6) |
| Indeterminacy | IF the problem has been well detected AND rough identification | THEN the problem will be nearly fixed before the electricity services reach to the customer. | All statements in this row are adhering to the Indeterminacy membership function represented by (7) |
| Indeterminacy | IF the problem has been fairly detected | THEN the problem will cause a delay in reaching the electricity services to the customer. | All statements in this row are adhering to the Indeterminacy membership function represented by (7) |
| Falsity | IF the problem has been roughly detected | THEN the problem will cause a temporary pause in the system. | All statements in this row are adhering to the falsity membership function represented by (8) |
| Falsity | IF the system needs to complementary test | THEN the problem will cause a pause in the system. | All statements in this row are adhering to the falsity membership function represented by (8) |

Table (7): Detection Rules Distributed to its Neutrosophic Bias

The following table (8) illustrates sixty-three of (IF-AND-THEN-ANOR) rules for seven components of the power transformer:

1- Solid Insulation: which has the function of insulation of windings, where its failure mode is physical and chemistry.
2- Oil Insulation: which has the function of isolating and cooling the active part of the transformer, where its failure mode is physical and chemistry.
3- Windings: which has the function of conducting current, where its failure mode is mechanical.

4- Tank: This has the function of enclosing oil and protecting the active part of the transformer, where its failure mode is chemical and physical.

5- Bushings: has the function of connecting windings with the net, and isolating between tank and windings. Also, its failure mode is physical and chemical.

6- Core: has the function of containing the magnetic field, and its failure mode is thermal.

7- Diverter switch: has the function of maintaining a coherent current, and its failure mode is electrical.

able (8): IF-THEN Rules for Some Power Transformer Components

| Components | Number of rules and Neutrosophic Bias | IF (Antecedent) or/and/anor (Antecedent) THEN (Consequence) or/and/anor (Consequence) |
|---|---|---|
| Oil Insulation | R1 Truth | IF the probability of particle contamination occurrence is less than 0.001 OR less than 0.01 THEN the reduction of the electrical strength AND the reduction of the breakdown voltage AND the increase in dielectric loss of oil is extremely unlikely OR unlikely.<br>**ANOR**<br>IF the probability of particle contamination occurrence is less than 0.001 or less than 0.01 THEN the overheating AND short circuit in the transformer is extremely unlikely OR unlikely.<br>**ANOR**<br>IF there is a pump bearing monitor AND there is a correct oil sampling procedure THEN the breakdown voltage will be completely fixed before the electricity services reach the customer. |
| | R2 Indeterminacy | IF the probability of particle contamination occurrence is less than 0.1 OR less than 0.2 THEN the reduction of the electrical strength AND the reduction of the |

| | | |
|---|---|---|
| | | breakdown voltage AND the increase in dielectric loss of oil is occasional OR reasonably probable<br>**ANOR**<br>IF the probability of particle contamination occurrence is less than 0.1 or less than 0.2 THEN the overheating AND short circuit in the transformer is occasional OR reasonably probable<br>**ANOR**<br>IF there is well detection for the pump bearing monitor AND there is rough identification for the correct oil sampling THEN there will be nearly correction of the breakdown voltage before the electricity services reach the customer |
| | R3 Falsity | IF the probability of particle contamination occurrence is greater than 0.2 OR greater than 0.3 THEN the reduction of the electrical strength AND the reduction of the breakdown voltage AND the increase in dielectric loss of oil is sometimes frequent OR permanently frequent.<br>**ANOR**<br>IF the probability of particle contamination occurrence is greater than 0.2 or greater than 0.3 THEN the overheating AND short circuit in the transformer is sometimes frequent OR permanently frequent.<br>**ANOR**<br>IF there is not a pump bearing monitor AND there is not a correct oil sampling procedure THEN the breakdown voltage will never fixed before the electricity services reach the customer. |

| | | |
|---|---|---|
| Solid Insulation | R4 Truth | IF the probability of the excessive moisture occurrence is less than 0.001 OR less than 0.01 THEN the reduction of the dielectric AND the reduction of the mechanical strength of paper is extremely unlikely OR unlikely<br>**ANOR**<br>IF the probability of the excessive moisture occurrence is less than 0.001 OR less than 0.01 THEN the Mechanical damage AND fault in insulation is extremely unlikely OR unlikely<br>**ANOR**<br>IF we prevent free transportation of the oil AND there is the prevention of direct entry of moisture from the air by the proper sealing THEN the oil moisture will be completely avoidable |
| | R5 Indeterminacy | IF the probability of the excessive moisture occurrence is less than 0.1 OR less than 0.2 THEN the reduction of the dielectric AND the reduction of the mechanical strength of paper is occasional OR reasonable probable.<br>**ANOR**<br>IF the probability of the excessive moisture occurrence is less than 0.1 OR less than 0.2 THEN the Mechanical damage AND fault in insulation is occasional OR reasonable probable.<br>**ANOR**<br>IF we often prevent the free transportation of the oil AND there is often prevention of direct entry of moisture from the air by the proper sealing THEN the oil moisture will be nearly avoidable. |

| | | |
|---|---|---|
| | R6 Falsity | IF the probability of the excessive moisture occurrence is greater than 0.2 OR greater than 0.3 THEN the reduction of the dielectric AND the reduction of the mechanical strength of paper is sometimes frequent OR permanently frequent.<br>**ANOR**<br>IF the probability of the excessive moisture occurrence is greater than 0.2 OR greater than 0.3 THEN the Mechanical damage AND fault in insulation is sometimes frequent OR permanently frequent.<br>**ANOR**<br>IF we do not prevent free transportation of the oil AND there is not prevention of direct entry of moisture from the air by the proper sealing THEN the oil moisture will not be avoidable |
| Windings | R7 Truth | IF the probability of the loose clamping occurrence is less than 0.001 OR less than 0.01 THEN the winding deformation is extremely unlikely OR unlikely.<br>**ANOR**<br>IF the probability of the loose clamping occurrence is less than 0.001 OR less than 0.01 THEN the high through current faults AND high inrush current AND protective relay tripping are extremely unlikely OR unlikely<br>**ANOR**<br>IF we use higher density insulation AND use of higher clamping pressures during manufacturing AND we use spring dashpot assemblies on the coil clamping structure THEN the loose clamping will be completely avoidable |

| | | |
|---|---|---|
| | R8 Indeterminacy | IF the probability of the loose clamping occurrence is less than 0.1 OR less than 0.2 THEN the winding deformation is occasional OR reasonable probable.<br>**ANOR**<br>IF the probability of the loose clamping occurrence is less than 0.1 OR less than 0.2 THEN the high through current faults AND high inrush current AND protective relay tripping are occasional OR reasonable probable.<br>**ANOR**<br>IF we often use higher density insulation AND often use of higher clamping pressures during manufacturing AND we often use spring dashpot assemblies on the coil clamping structure THEN the loose clamping will be nearly avoidable |
| | R9 Falsity | IF the probability of the loose clamping occurrence is greater than 0.2 OR greater than 0.3 THEN the winding deformation is sometimes frequent OR permanently frequent.<br>**ANOR**<br>IF the probability of the loose clamping occurrence is greater than 0.2 OR greater than 0.3 THEN the high through current faults AND high inrush currents AND protective relay tripping is sometimes frequent OR permanently frequent.<br>**ANOR**<br>IF we could not use higher density insulation AND could not use of higher clamping pressures during manufacturing AND we could not use spring dashpot |

| | | |
|---|---|---|
| | | assemblies on the coil clamping structure THEN the loose clamping will not be avoidable. |
| Tank | R10 Truth | IF the probability of the insufficient maintenance occurrence is less than 0.001 OR less than 0.01 THEN the corrosion is extremely unlikely OR unlikely. **ANOR** IF the probability of the insufficient maintenance occurrence is less than 0.001 OR less than 0.01 THEN the leakage is extremely unlikely OR unlikely. **ANOR** IF we use monitoring of the inhibitor content according to IEC 60666 AND there is external examination for oil leaks THEN any corrosion will be completely avoidable. |
| | R11 Indeterminacy | IF the probability of the insufficient maintenance occurrence is less than 0.1 OR less than 0.2 THEN the corrosion is occasional OR reasonably probable. **ANOR** IF the probability of the insufficient maintenance occurrence is less than 0.1 OR less than 0.2 THEN the leakage is occasional OR reasonably probable. **ANOR** IF we often use the monitoring of the inhibitor content according to IEC 60666 AND often there is an external examination for oil leaks THEN any corrosion will be nearly avoidable. |
| | R12 Falsity | IF the probability of the insufficient maintenance occurrence is greater than 0.2 OR greater than 0.3 THEN the corrosion is sometimes frequent OR permanently frequent. |

| | | |
|---|---|---|
| | | **ANOR**<br>IF the probability of the insufficient maintenance occurrence is greater than 0.2 OR greater than 0.3 THEN the leakage is sometimes frequent OR permanently frequent.<br>**ANOR**<br>IF we do not use monitoring of the inhibitor content according to IEC 60666 AND there is not external examination for oil leaks THEN any corrosion will not be avoidable. |
| Bushings | R13 Truth | IF the lack of maintenance is less than 0.001 OR less than 0.01 THEN the external contamination corrosion AND the discharge current on the external surface of the insulation is extremely unlikely OR unlikely.<br>**ANOR**<br>IF the lack of maintenance is less than 0.001 OR less than 0.01 THEN the short circuit AND the personal danger is extremely unlikely OR unlikely.<br>**ANOR**<br>IF there is periodic maintenance THEN the power factor will adhere the standard (IEC 137)/ tan delta. |
| | R14 Indeterminacy | IF the lack of maintenance is less than 0.1 OR less than 0.2 THEN the external contamination corrosion AND the discharge current on the external surface of the insulation is occasional OR reasonable probable.<br>**ANOR**<br>IF the probability of the lack of maintenance is less than 0.1 OR less than 0.2 THEN the short circuit AND the personal danger is occasional OR reasonable probable. |

| | | |
|---|---|---|
| | | **ANOR**<br>IF there is often periodic maintenance THEN the power factor will often adhere the standard (IEC 137)/ tan delta. |
| | R15 Falsity | IF the lack of maintenance is greater than 0.2 OR greater than 0.3 THEN the external contamination corrosion AND the discharge current on the external surface of the insulation is sometimes frequent OR permanently frequent.<br>**ANOR**<br>IF the lack of maintenance is greater than 0.2 OR greater than 0.3 THEN the short circuit AND the personal danger is sometimes frequent OR permanently frequent.<br>**ANOR**<br>IF there is not periodic maintenance THEN the power factor will not adhere the standard (IEC 137)/ tan delta. |
| Core | R16 Truth | IF the probability of the inexistence of the frame to earth circulating currents is less than 0.001 OR less than 0.01 THEN the increased core temperature is extremely unlikely OR unlikely.<br>**ANOR**<br>IF the probability of the inexistence of the frame to earth circulating currents is less than 0.001 OR less than 0.01 THEN the loss of efficiency is extremely unlikely OR unlikely.<br>**ANOR**<br>IF there exist the frame to earth circulating currents THEN there is not increased core temperature. |

| | | |
|---|---|---|
| | R17 Indeterminacy | IF the probability of the inexistence of the frame to earth circulating currents is less than 0.1 OR less than 0.2 THEN the increased core temperature is occasional OR reasonably probable.<br>**ANOR**<br>IF the probability of the inexistence of the frame to earth circulating currents is less than 0.1 OR less than 0.2 THEN the loss of efficiency is occasional OR reasonably probable.<br>**ANOR**<br>IF the often inexistence of the frame to earth circulating currents THEN the increased core temperature is often detected and often identification by Furfuraldehyde Analysis (FFA). |
| | R18 Falsity | IF the probability of the inexistence of the frame to earth circulating currents is greater than 0.2 OR greater than 0.3 THEN the increased core temperature is sometimes frequent OR permanently frequent.<br>**ANOR**<br>IF the probability of the inexistence of the frame to earth circulating currents is greater than 0.2 OR greater than 0.3 THEN the loss of efficiency is sometimes frequent OR permanently frequent.<br>**ANOR**<br>IF the inexistence of the frame to earth circulating currents THEN the increased core temperature is detected and identification by Furfuraldehyde Analysis (FFA). |
| | R19 Truth | IF the probability of the worry contact occurrence is less than 0.001 OR less than 0.01 THEN the existence of |

| | | |
|---|---|---|
| Diverter Switch | | a high carbon build-up is extremely unlikely OR unlikely.<br>**ANOR**<br>IF the probability of the worry contact occurrence is less than 0.001 OR less than 0.01 THEN the existence of possible flash over is extremely unlikely OR unlikely.<br>**ANOR**<br>IF there is not worry contact THEN we do not need to the contact replacement after the specified performance number according to the manufacturer suggestions. |
| | R20 Indeterminacy | IF the probability of the worry contact occurrence is less than 0.1 OR less than 0.2 THEN the existence of a high carbon build-up is occasional OR reasonable probable.<br>**ANOR**<br>IF the probability of the worry contact occurrence is less than 0.1 OR less than 0.2 THEN the existence of possible flash over is occasional OR reasonable probable.<br>**ANOR**<br>IF there is often a worry contact THEN we will often contact replacement after the specified performance number according to the manufacturer's suggestions. |
| | R21 Falsity | IF the probability of the worry contact occurrence is greater than 0.2 OR greater than 0.3 THEN the existence of a high carbon build-up is sometimes frequent OR permanently frequent.<br>**ANOR**<br>IF the probability of the worry contact occurrence is greater than 0.2 OR less than 0.3 THEN the existence of |

| | | |
|---|---|---|
| | | possible flash over is sometimes frequent OR permanently frequent.<br>**ANOR**<br>IF there is worry contact THEN the contact replacement after the specified performance number according to the manufacturer suggestions. |

## 9. Implement NFMEA using NeutrosophicLogicDesigner

As previously discussed, the MATLAB Toolbox suffers from uncontained for a neutrosophic logic based-inference system, therefore, the researcher was forced to use the fuzzyLogicDesigner after adapting it by using specific membership functions, specific operators...etc. However, the implementation will be on the type-1 Mamdani inference system.

To abbreviate the simulation, we tried to apply the Type-1 Mamdani inference method on just three rules of the above table (8), especially the following rules

1- IF there is a pump bearing monitor AND there is a correct oil sampling procedure THEN the breakdown voltage will be completely fixed before the electricity services reach the customer.

2- IF there is well detection for the pump bearing monitor AND there is rough identification for the correct oil sampling THEN there will be nearly correction of the breakdown voltage before the electricity services reach the customer.

3- IF there is not a pump bearing monitor AND there is not a correct oil sampling procedure THEN the breakdown voltage will never fixed before the electricity services reach the customer.

In the neutrosophic inference rules, we noticed that there are different composites from the traditional patterns, where we linked every two opposite statements in one input by two opposite membership functions using the command (Evenly Distributed MFs), Also we gave opposite weights of opposite rules. As well as we also interpreted the statements of indeterminate bias by those membership functions that are embedded between two opposite membership functions (i.e. truth and false)

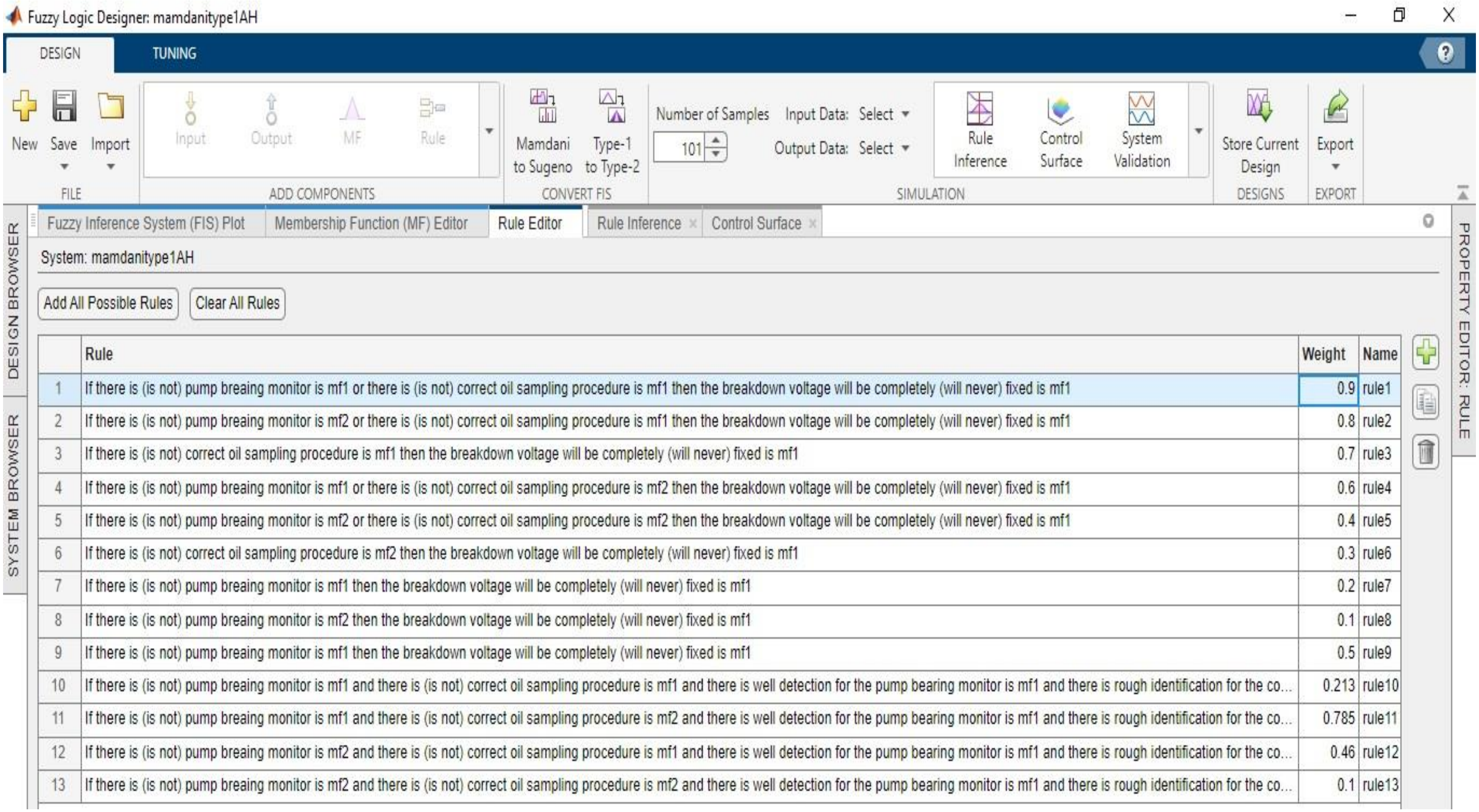



Figure 9.1: Some of the Defined Rules to Combine Neutrosophic (Severity, Occurrence and Detection) in MATLAB R2023a

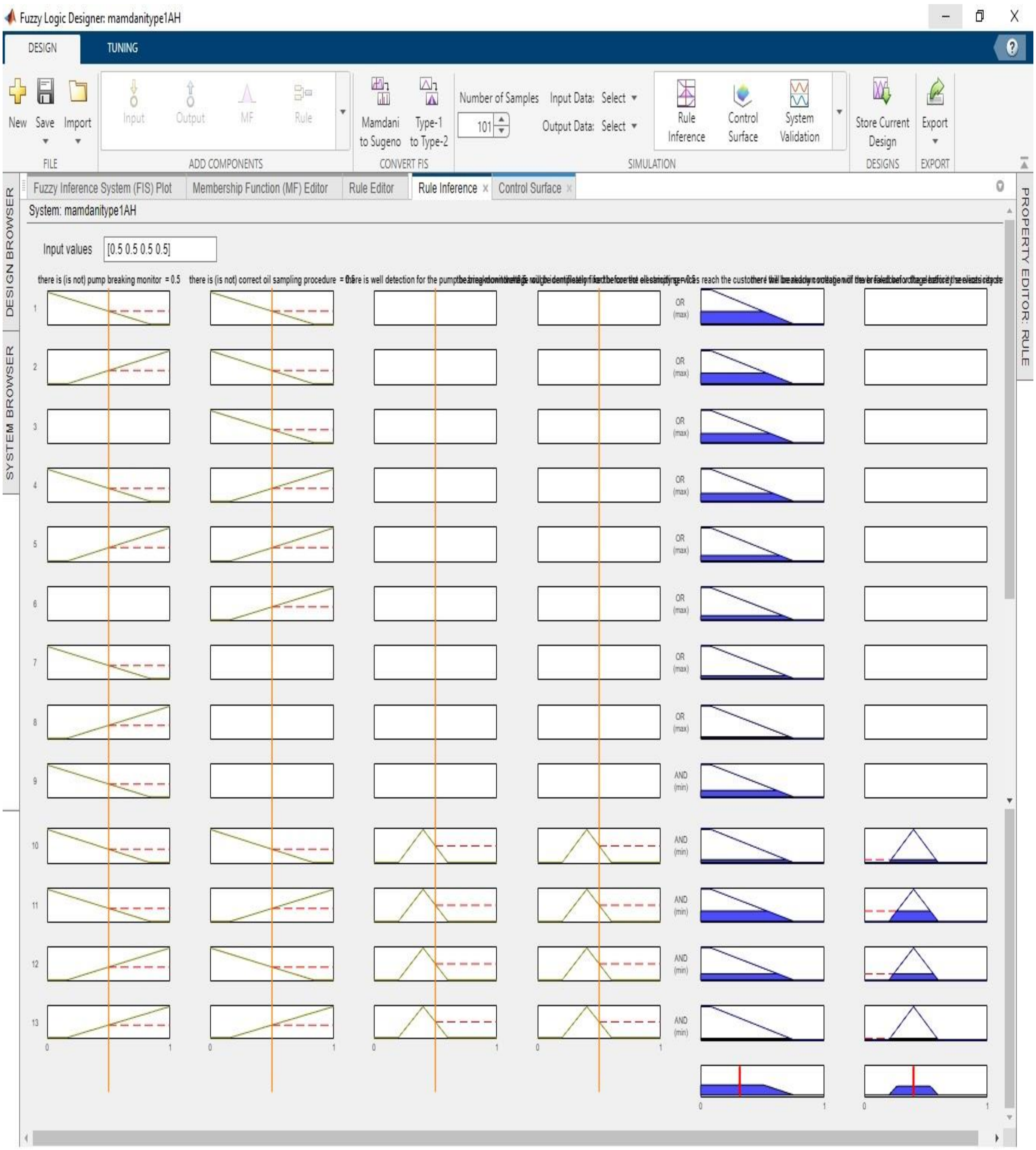


Figure 9.2: Implementation the Neutrosophic Inference Rules R1, R2, R3

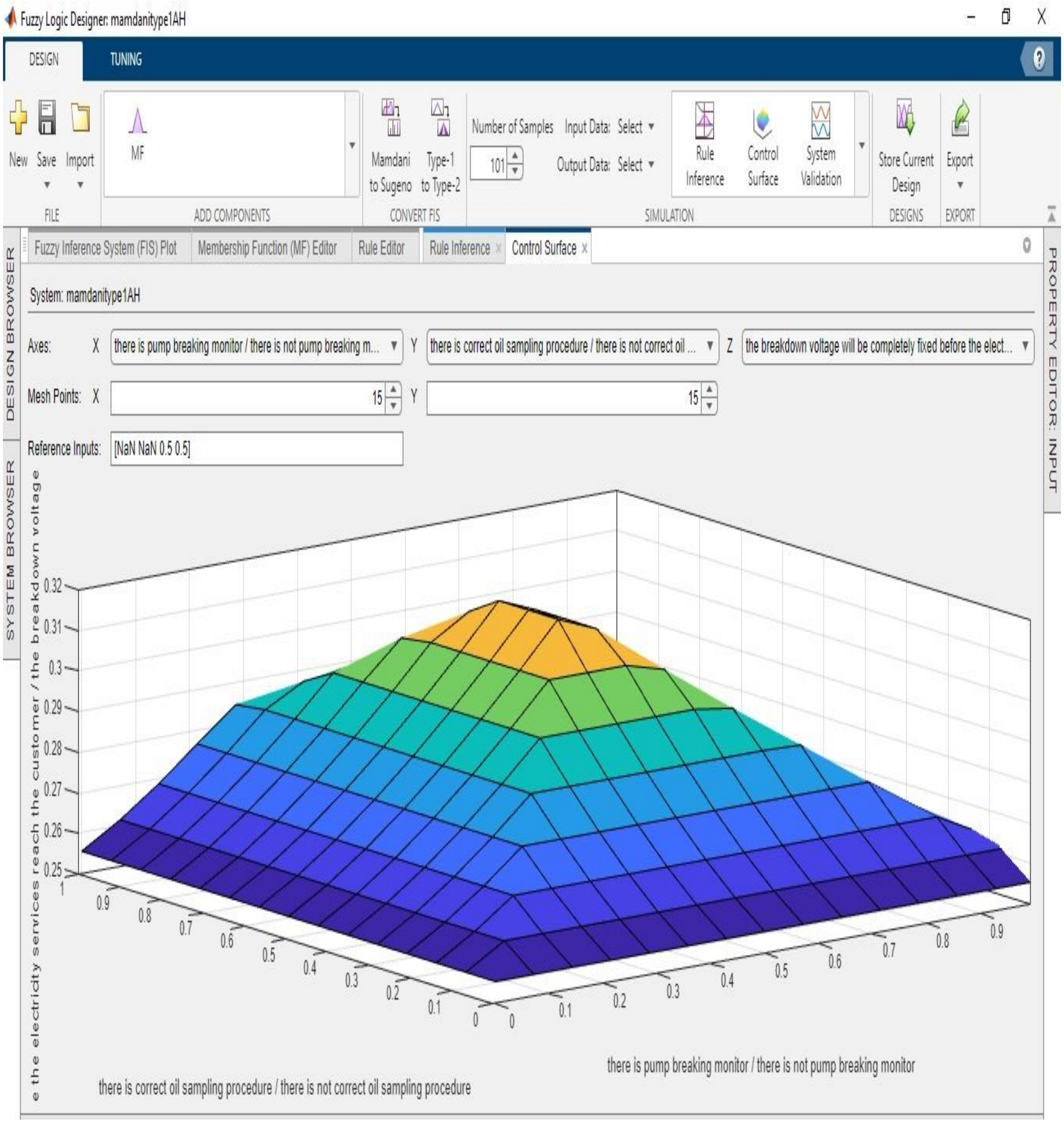



Figure 9.3: Surface of the Neutrosophic Inference System (NIS)

## 10. Neutrosophic Risk Priority Numbers

Analogs to the concept of the traditional FMEA, and fuzzy FMEA, the neutrosophic failure mode effect analysis assigns numerical values to every risk associated with causing failure, using severity, occurrence, and detection by calculating the neutrosophic risk priority number (NRPN) for each failure cause.

We should notice that, in the NFMEA, the definition of failure modes, failure causes and failure effects depend on the level of analysis and system failure criteria. As the analysis progress, the failure effects identified at the lower level may become failure modes at the higher level. The failure modes at the lower level may become the failure causes at the higher level, and so on [2].

## 10.1. Analysis results of the inferencing R1, R2, R3

The sake of this study is to put the principles of the neutrosophic inference system for the power transformers, and after putting all (IF-HEN) rules concerning the (severity, occurrence, not detection) of each failure mode belonging to seven components (Solid Insulation, Oil Insulation, Windings, Tank, Bushings, Core, and Diverter Switch), and since the MATLAB toolbox does not support the NIS which lead us to use the fuzzyLogicDesigner to build adaptive NeutrosophicLogicDesigner. Therefore, we tried to restrict the analysis to only three rules out of sixty-three rules stated in table (8). Also, the generalization concepts of the neutrosophic theory and its superiority to fuzzy logic, as well as, its superiority to classical logic, led to the creation of the operator (ANOR) which was never ever previously created neither in classical inference systems nor fuzzy inference systems. Again, the existence of the operator (ANOR) is regarded as a challenge preventing us from using all 63 rules stated in Table (8) for the same reason of unsupported fuzzyLogicDesigner to this kind of operator.

Now, if we track the trace of the thirteen rules resulting from R1, R2, and R3 (see figures 9.1 & 9.2) with different impacts in their weights for each rule depending upon the (severity, occurrence, not detection) if their bias to the truth state or to the indeterminate state or to the falsity state.

The ranking of each failure mode caused by its severity, occurrence, and not detection will be ranked in decreasingly order (the largest number of $NRPN = 6$ is the more truth situation having lowest risk impact that has lowest priority, while the smallest number of $NRPN = 1$ is the more falsity or more indeterminate situation having highest risk impact that has largest priority) of the neutrosophic risk priority number (NRPN) as demonstrated in the following table

Table (8): Neutrosophic Failure Mode Analysis in Power Transformers

| No. of Rules | Control (s) Factor | Neutrosophic bias | Weight | Ranking of NRPN |
|---|---|---|---|---|
| 1 | Pump bearing factor+ breakdown voltage+ correct oil sampling | Truth+Truth+Truth | 0.9 | 6 |
| 2 | Pump bearing factor+ breakdown voltage+ correct oil sampling | Truth+ Falsity+Truth | 0.8 | 5 |
| 3 | Pump bearing factor+ breakdown voltage+ correct oil sampling | Truth+Truth+Falsity | 0.7 | 4 |
| 4 | Pump bearing factor+ breakdown voltage+ correct oil sampling | Truth+Indet.+Truth | 0.6 | 4 |
| 5 | Pump bearing factor+ breakdown voltage+ correct oil sampling | Truth+Indet.+Falsity | 0.4 | 3 |
| 6 | Pump bearing factor+ breakdown voltage+ correct oil sampling | Truth+Indet.+Indet. | 0.3 | 3 |
| 7 | Pump bearing factor+ breakdown voltage+ correct oil sampling | Truth+Truth+Indet | 0.2 | 2 |
| 8 | Pump bearing factor+ breakdown voltage+ correct oil sampling | Falsity+Falsity+Falsity | 0.1 | 1 |
| 9 | Pump bearing factor+ breakdown voltage+ correct oil sampling | Indet.+indet+indet | 0.5 | 3 |
| 10 | Pump bearing factor+ breakdown voltage+ correct oil sampling | Indet.+Indet.+Falsity | 0.213 | 2 |
| 11 | Pump bearing factor+ breakdown voltage+ correct oil sampling | Indet.+Indet.+Truth | 0.785 | 4 |
| 12 | Pump bearing factor+ breakdown voltage+ correct oil sampling | Indet.+falsity+Truth | 0.46 | 3 |
| 13 | Pump bearing factor+ breakdown voltage+ correct oil sampling | Falsity+Indet.+Falsity | 0.1 | 1 |

## 11. Conclusion

In view of neutrosophic theory, many new concepts and procedures have been introduced in this manuscript, a modifying MIL-STD-1629A by re-presenting new categories of severity factors of failure mode that were stated and used since 1980. Also, the authors re-framed all failure modes (severity, occurrence, and not detection) of the electrical power transformer according to their neutrosophic bias. we create a mathematical tool for implementing the concept of neutrosophic inference systems, a consistent new algorithm of neutrosophic failure mode analysis has been presented, set up the neutrosophic risk priority numbers, and implement the new algorithm for inferencing and assessing the reliability and the stability of the system by using thirteen (IF-Then) rules derived from R1, R2, and R3.

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# The Graphical Method for Finding the Optimal Solution for Neutrosophic linear Models and Taking Advantage of Non-Negativity Constraints to Find the Optimal Solution for Some Neutrosophic linear Models in Which the Number of Unknowns is More than Three

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**Abstract:**
The linear programming method is one of the important methods of operations research that has been used to address many practical issues and provided optimal solutions for many institutions and companies, which helped decision makers make ideal decisions through which companies and institutions achieved maximum profit, but these solutions remain ideal and appropriate in If the conditions surrounding the work environment are stable, because any change in the data provided will affect the optimal solution and to avoid losses and achieve maximum profit, we have, in previous research, reformulated the linear models using the concepts of neutrosophic science, the science that takes into account the instability of conditions and fluctuations in the work environment and leaves nothing to chance. While taking data, neutrosophic values carry some indeterminacy, giving a margin of freedom to decision makers. In another research, we reformulated one of the most important methods used to solve linear models, which is the simplex method, using the concepts of this science, and as a continuation of what we did in the previous two researches, we will reformulate in this research. The graphical method for solving linear models using the concepts of neutrosophics. We will also shed light on a case that is rarely mentioned in most operations research references, which is that when the difference between the number of unknowns and the number of constraints is equal to one, two, or three, we can also find the optimal solution graphically for some linear models. This is done by taking advantage of the conditions of non-negativity that linear models have, and we will explain this through an example in which the difference is equal to two. Also, through examples, we will explain the difference between using classical values and neutrosophic values and the extent of this's impact on the optimal solution.

**Keywords:** linear programming; Neutrosophic science; Neutrosophic linear models; Graphical method for solving linear models; Graphical method for solving neutrosophic linear models.

## Introduction:

A continuation of what we have done in previous research, the purpose of which was to reformulate some operations research methods using the concepts of neutrosophic science. See [1-14] The science that made a great revolution in all fields of science, which grew and developed very quickly, as many topics were reformulated using the concepts of this Science, and we find neutrosophic groups, neutrosophic differentiation, neutrosophic integration, and neutrosophic statistics ... [14-16], and given the importance of the graphic method used to find the optimal solution for linear models, which is a graph of the model and is one of the easiest ways to solve linear programming problems, but it is not sufficient to address All linear programming problems because they often contain a large number of variables, and the use of the graphical method is limited to the following cases:

- The number of unknowns is n = 1 or n = 2 or n = 3.
- In linear models whose constraints are equal constraints, if the number of unknowns and the number of equations meet one of the following conditions: n – m = 1 or n – m = 2 or n – m = 3.Here we can transform the model into a function of one variable or two variables or three variables, respectively, by taking advantage of the non-negativity constraints that the variables of the linear model have. In this research, we will present a reformulation of the graphical method for solving linear models using the concepts of neutrosophic, as well as the graphical method for solving linear models that Its restrictions are equal restrictions, and the difference between the number of unknowns and the number of restrictions is equal to one, two, or three.

**Discussion:**

The graphical method is one of the important ways to find the optimal solution for the linear and nonlinear models, so in the research [3] we reformulated it for the neutrosophic nonlinear models, and in this research we will present the graphic method to find the optimal solution for the neutrosophic linear models that were presented in the research [1], we know that the model the script is written in the following abbreviated form:

$$Z = \sum_{j=1}^{n} c_j x_j \rightarrow (Max\ \ or\ \ Min\,)$$

**Restrictions:**

$$\sum_{j=1}^{n} a_{ij} x_j \begin{pmatrix} \geq \\ \leq \\ = \end{pmatrix} b_i \qquad ; i = 1,2,\dots,m$$

$$x_j \geq 0 \quad ; j = 1,2,\dots,n$$

If at least one of the values $c_j$ , $a_{ij}$ , $b_i$ , is a neutrosophic value then the linear model is a neutrosophic linear model.

**First: The graphical method for solving linear models: [17-20]**

Through the studies presented according to classical logic in many references, we know that to find the optimal solution for linear models in which the number of variables is one, two, or three graphically, we represent the area of common solutions for the constraints in one of the spaces $R \ or \ R^2 \ or \ R^3$This depends on the number of variables in this sentence, for example if the number of variables is two, i.e. the solution is in space $R^2$ (where work is done on models that contain one or three variables with the same steps)

**We find the optimal solution according to the following steps:**

1. We determine the half-planes defined by the inequalities of the constraints by drawing the straight lines resulting from converting the inequalities of the constraints into equals. The drawing is done by specifying two points that satisfy the constraint, and then we connect the two points to obtain the straight line corresponding to the constraint. This straight line divides the plane into two halves in order to determine the half-plane that satisfies the constraint. We choose A point at the top of the mapping from one of the two half-planes. We substitute the coordinates of this point into the inequality. If it is satisfied, then the region in which this point is located is the solution region. If it is not achieved, then the opposite region is the solution region.
2. We define the common solutions region, which is the region resulting from the intersection of the halves of the levels defined by constraint inequalities. This region must be non-empty so that we can proceed with the solution.
3. In order to represent the objective function, we note that its relationship contains three unknowns, $Z$ , $x_1$ ,$x_2$ . Therefore, we must know a value for $Z$, which is unknown to us. Here we assume a value, let it be $Z_1 = 0$, draw the equation of the objective function $Z_1$ give another value, let it be $Z_2$, and represent the equation. We get a line parallel to the first line, and by continuing, we obtain a set of parallel lines representing the target function.
4. We draw ray $\vec{C} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$where $c_1$is coefficient of $x_1$and $c_2$ is coefficient of $x_2$in the objective function statement, and the direction of its increasing function is the direction of ray$\vec{C} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, and the direction of its decreasing function is the opposite direction. This ray, that is, the drawing is done according to the type of objective function (maximization or minimization). In clearer terms, we find the optimal solution point by pulling the line representing $Z_1$parallel to itself towards the ray $\vec{C} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$to find the maximum value of the objective function, (And reverse this direction to find the smallest value), until it passes through the last point of the common solutions region and this point is the optimal solution point, which is located on the borders of the common solutions region and any other displacement, no matter how small, takes it out of it.

**The graphical method for finding the optimal solution for neutrosophic linear models:**
From the definition of neutrosophic linear models, we find that we can apply the same previous steps to obtain the optimal solution, which is a neutrosophic value suitable for all conditions. We explain the above through the following example:

**Example 1:**
A company produces two types of products $A_1$ , $A_2$ and uses three types of raw materials $B_1$ , $B_2$ , $B_3$ in the production process, if the available quantities of each of the raw materials are$B_i$ ; $i = 1,2,3$, and the quantity needed to produce one unit of each products $A_j$ ; $j = 1,2$, and the profit accruing from one unit of each of the products $A_1$ , $A_2$ is shown in the following table:

| products / raw materials | $A_1$ | $A_2$ | available quantities |
|---|---|---|---|
| $B_1$ | **6** | **4** | **36** |
| $B_2$ | **2** | **3** | **12** |
| $B_3$ | **5** | **0** | **10** |
| profit | $[6,8]$ | $[2,4]$ | |

**Table Issue data**

**Required:**
Determine the quantities that must be produced of each products $A_j$ ; $j = 1,2$ so that the company achieves maximum profit:
the solution :
Let $x_j$ be the quantity produced from product $j$ , where $j = 1,2$, then we can formulate the following neutrosophic linear mathematical model:

$$Z = [6,8]x_1 + [2,4]x_2 \to Max$$

**Restrictions:**

$$6x_1 + 4x_2 \leq 36 \quad (1)$$

$$2x_1 + 3x_2 \leq 12 \quad (2)$$

$$5x_1 \leq 15$$

$$x_1, x_2 \geq 0$$

**The previous model is a linear neutrosophic model because there is indeterminacy in variables coefficients in objective function. To find the optimal solution for the previous model, we will use the graphical method according to the following steps**:

The first constraint: We draw the straight line representing the first constraint:

$$6x_1 + 4x_2 = 36$$

We impose:

$$x_1 = 0 \Rightarrow 4x_2 = 36 \Rightarrow x_2 = 9$$

We get the first point: $A(0,9)$.

We impose:

$$x_2 = 0 \Rightarrow 6x_1 = 36 \Rightarrow x_1 = 6$$

We get the second point: $B(6,0)$

If we take a point at the top of the designation from one of the two halves of the resulting plane after drawing the straight through the two points $A(0,9)$ and $B(6,0)$, let it be the point $O(0,0)$and substitute it in the inequality of the first entry, we find that the inequality is fulfilled, that is, the half of the plane that the point $O(0,0)$belongs to it, which is half the solution plane of the first-constraint inequality.
We proceed in the same way for the second and third restrictions and obtain the following graphical representation: Figure No. (1)

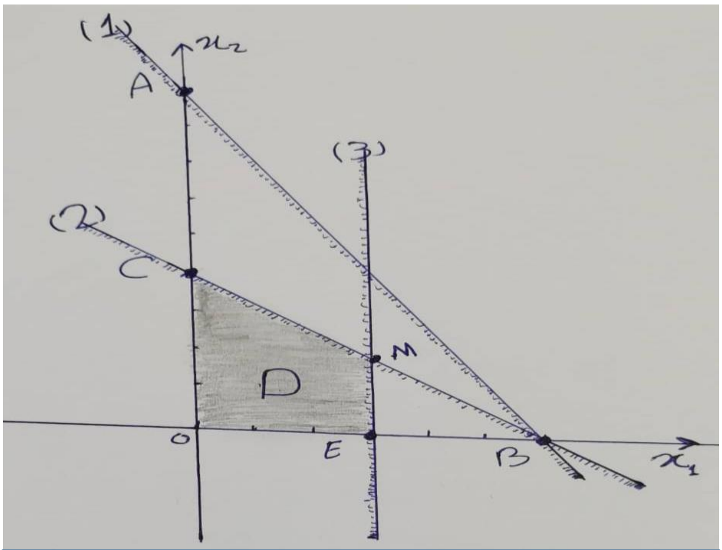


Figure No. (1) Graphic representation of the limitations of the linear model in Example 1

After representing the constraints, we notice that the common solution area is bounded by the polygon whose vertices are the points, $O(0,0)$ , $E(3,0)$ , $M$ and $C(0,4)$ .

The point $M$ is the point of intersection, the second and third constraints we get their coordinates by solving the following two equations:

$$2x_1 + 3x_2 = 12$$

$$5x_1 = 15$$

We fiend**:** $M(3,2)$

Substituting the coordinates of the vertex points into the objective function expression, we get:

$$Z_O = 0$$
$$Z_E \in [12,16]$$
$$Z_M \in [22,32]$$
$$Z_C \in [8,16]$$

That is, the greatest value of the function Z is achieved at point $M(3,2)$ , that is, the company must produce three units of the first product and two units of the second product, then it will achieve maximum profit.

$$Max\ Z = Z_M \in [22,32]$$

**Note:**

The process of substituting the objective function with the coordinates of the points of the vertices of the common solution area is possible when the number of points is small, as we can easily replace them in the objective function, and the point that gives the best value for the objective function represents the optimal solution, but when there are a large number of constraints, we get a large number From the vertical points located on the perimeter of the common solution region. In this case, the method of finding the coordinates of all these points and substituting them into the objective function becomes impractical, so we resort to representing the objective function and determining the optimal solution point as we mentioned previously.

**Second: How to take advantage of the conditions of non-negativity to find the optimal solution for some neutrosophic linear models using the graphical method:**

**Example 2:**

Find the optimal solution for the following linear neutrosophic model:

$$Z = x_1 - x_2 - 3x_3 + x_4 + [2,5]x_5 - x_6 + 2x_7 - [10,15] \to Max$$

**Restrictions:**

$$x_1 - x_2 + x_3 = 5 \qquad (1)$$

$$2x_1 - x_2 - x_3 - x_4 = -11 \qquad (2)$$

$$x_1 + x_2 - x_5 = -4 \qquad (3)$$

$$x_2 + x_6 = 6 \qquad (4)$$

$$2x_1 - 3x_2 - x_6 + 2x_7 = 8 \qquad (5)$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

**The solution:**

We note that the number of constraints $m = 5$ and the number of variables $n = 7$, meaning that $n - m = 2$. Therefore, we can, relying on the non-negativity constraints, find the optimal solution for the previous model using the graphical method according to the following steps:

1- We calculate five variables in terms of only two variables.
2- Since the variables of the linear model satisfy the non-negativity constraints, then we obtain from the variables that we calculated five inequalities of the type greater than or equal to.
3- Substituting the five variables into the objective function, we get an objective function with only two variables.
4- We write the new model, which is a linear model with two variables, so the optimal solution can be found graphically.

**We apply the previous steps to Example 2:**

We find:

$$x_3 = 5 - x_1 + x_2 \qquad (1)'$$

$$x_4 = 3x_1 - 2x_2 + 6 \qquad (2)'$$

$$x_5 = x_1 + x_2 + 4 \qquad (3)'$$

$$x_6 = 6 - x_2 \qquad (4)'$$

$$x_7 = 7 - x_1 + x_2 \qquad (5)'$$

Substituting in the objective function, we get:

$$Z = [1,4]x_1 + [3,6]x_2 + [8,25]$$

Since, $x_3, x_4, x_5, x_6, x_7 \geq 0$ from $(1)'$ ، $(2)'$ ،$(3)'$ ، $(4)'$ ، $(5)'$, we get the following set of constraints:

$$5 - x_1 + x_2 \geq 0$$
$$-3x_1 + 2x_2 - 3 \geq 0$$

$$x_1 + x_2 + 4 \geq 0$$
$$6 - x_2 \geq 0$$
$$7 - x_1 + x_2 \geq 0$$

**Then the neutrosophic linear mathematical model becomes:**

Find:

$$Z = [1,4]x_1 + [3,6]x_2 + [8,25] \rightarrow Max$$

Restrictions:

$$5 - x_1 + x_2 \geq 0$$
$$3x_1 - 2x_2 + 6 \geq 0$$
$$x_1 + x_2 + 4 \geq 0$$
$$6 - x_2 \geq 0$$
$$7 - x_1 + x_2 \geq 0$$
$$x_1, x_2 \geq 0$$

The model has two variables, so the optimal solution can be found graphically by following the same steps mentioned in Example (1).
We obtain the representation. Figure No. (2) Is the required graphic representation:

‘

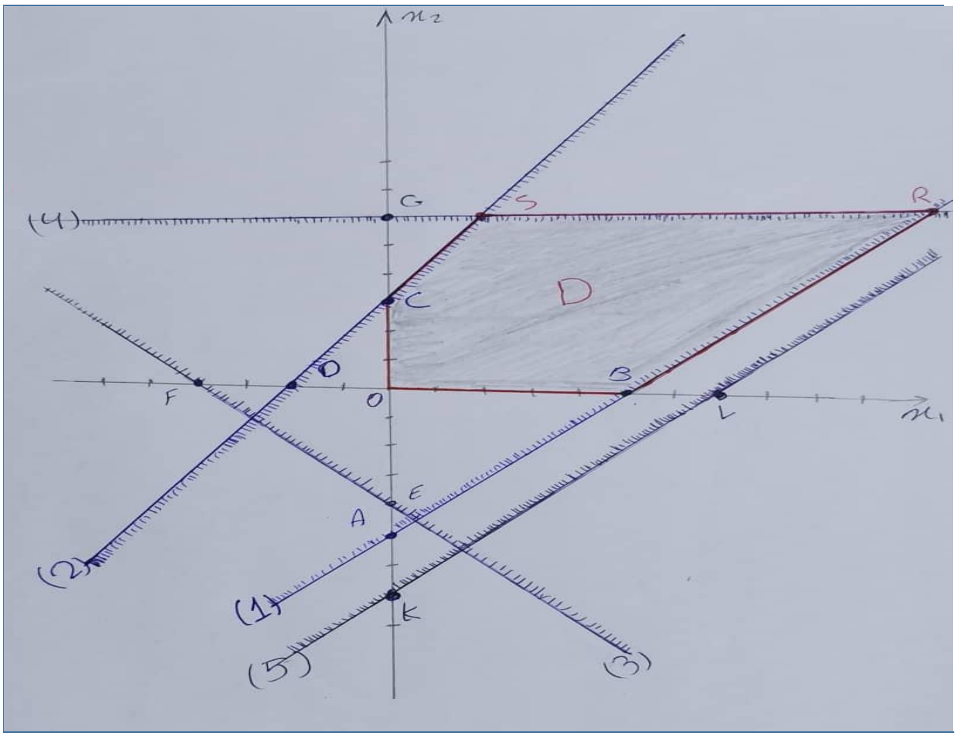


Figure No. (2): Graphical representation of the constraints of the linear model in Example 2

Region $D$ is the region of joint solutions and is defined by the polygon $OBRSC$, where, $O(0,0)$ , $B(5,0)$ , $C(0,3)$ , and for the two points $R$ , $S$ we find:
Point $R$ is the point of intersection of the first and fourth entries.
We obtain its coordinates by solving the set of equations:

$$5 - x_1 + x_2 = 0$$
$$6 - x_2 = 0$$

We get: $R(11,6)$

Point $S$ is the point of intersection of the second and fourth entries.
We obtain its coordinates by solving the set of equations:

$$3x_1 - 2x_2 + 6 = 0$$
$$6 - x_2 = 0$$

We get: $S(2,6)$

Since the optimal solution is located at one of the vertices of the common solution region, we substitute the coordinates of these points with the objective function:

At point, $O(0,0)$

$$Z_O = 0$$

At point, $B(5,0)$

$$Z_B = [13,45]$$

At point , $R(11,6)$

$$Z_R \in [37,105]$$

At point , $S(2,6)$

$$Z_S \in [28,69]$$

At point , $C(0,3)$

$$Z_C \in [17,43]$$

The greatest value of the objective function is at the point,$R(11,6)$ that is, $x_1 = 11$ and $x_2 = 6$.

We calculate the values of the remaining variables by $(1)'$ ، $(2)'$ ،$(3)'$ ، $(4)'$ ، $(5)'$ .

We find, $x_3 = 0$ , $x_4 = 27$ , $x_5 = 21$ , $x_6 = 0$ , $x_7 = 2$.

Substituting in the objective function of the original model we obtain the maximum value of the $Z$ function, which is.

$$MaxZ \in [68,126]$$

**Important Notes:**

1- The graphical solution applies to a vertical point in space $R^n$.
2- The number of components of the ideal solution is non-existent because the ideal solution applies to a vertical point, and the vertical point is the result of the intersection of a number of lines or planes, and the number of non-existent components is at least $n - m$ components.
3- The model may include some conditions that do not play a role in the solution process.
4- The ideal solution may be a single point, or it may be an infinite number of points, when one of the sides of the common solution area that passes through the point of the ideal solution is parallel to the straight line $Z = 0$. Therefore, when the straight line representing the objective function is drawn, this straight line will apply to the

parallel side, and all the points of that side, the number of which are infinite, will be they are perfect solutions.

5- If the region of acceptable solutions is open in terms of increasing the function $Z$ then we cannot stop at a specific ideal solution, and then we say that the objective function has an infinite number of acceptable solutions that give us greater and greater values of $Z$.

6- The state of not having an ideal solution (acceptable solution) when the conditions contradict each other and then the region of possibilities is an empty set (the problem is impossible to solve).

**Conclusion and results:**

In the previous study, we presented the graphical method for finding the optimal solution for neutrosophic linear models, and also a method that is rarely discussed in classical operations research references, which is how to take advantage of non-negativity constraints to find the graphically optimal solution for some neutrosophic linear models, but we must be aware that we may encounter neutrosophic linear models with two variables, but There may be difficulty in arriving at the common solution area, or there may be difficulty in determining the optimal solution after obtaining the common solution area. Therefore, it is preferable to use the Cymex neutrosophic method. As a result, the main goal is to obtain the optimal solution, so the researcher must determine the appropriate method for the model being solved.

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# Double neutrosophic integrals

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**Abstract:** Several previous papers dealt with neutrosophic integrals without introducing the idea of double neutrosophic integrals. In this article, double integrals were discussed by presenting several theories in double neutrosophic integrals over a rectangle and over a general region, the most important of which is the neutrosophic Fubini's theorem. In addition to studying the applications of double neutrosophic integrals in calculating areas.

**Keywords:** neutrosophic integrals, double neutrosophic integrals, neutrosophic Fubini's theorem, area.

## 1. Introduction

In contrast to the current logics, Smarandache suggested the Neutrosophic Logic to describe a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, and contradiction. Smarandache introduced the concept of neutrosophy as a new school of philosophy [4]. He presented the definition of the standard form of neutrosophic real number [3-5], studying the concept of the Neutrosophic probability [6], the Neutrosophic statistics [5-7], and professor Smarandache entered the concept of preliminary calculus of the differential and integral calculus, where he introduced for the first time the notions of neutrosophic mereo-limit, mereo-continuity, mereoderivative, and mereo-integral [1]. A number of studies in the area of integration and differentiation were given by Y. Alhasan [9-12-15], also he presented the definition of the concept of neutrosophic complex numbers and its properties, in addition, he studied the general exponential form of a neutrosophic complex number [2-10]. Madeleine Al- Taha presented results on single valued neutrosophic (weak) polygroups [13]. The AH isometry was used to study many structures such as conic sections, real analysis concepts, and geometrical surfaces [11-16].

The calculation of area, volume, and arc length is one of the most essential uses of integration in human life. In our reality, there are things that cannot be precisely defined and contain an element of indeterminacy.

There are four sections of paper. first section, which includes a study of neutrosophic science, serves as an introduction. The second portion deals with a neutrosophic integrals theories and rules. The double neutrosophic integral and its applications are discussed in the third part. The fourth section offers the paper's conclusion.

## 2. Preliminaries

### 2.1. Neutrosophic integration by substitution method [15]

**Definition 1**

Let $f: D_f \subseteq R \to R_f \cup \{I\}$, to evaluate $\int f(x)dx$

put: $x = g(u) \quad \Rightarrow \quad dx = \acute{g}(u)du$

by substitution, we get:

$$\int f(x)dx = \int f(u)\acute{g}(u)du$$

**Definition 3.2[16]**

Let $f$: $R(I) \to R(I)$; $f = f(X)$ and $X = x + yI \in R(I)$ then f is called a neutrosophic real function with one neutrosophic variable. a neutrosophic real function $f(X)$ written as follows:

$$f(X) = f(x + yI) = f(x) + I[f(x + y) - f(x)]$$

## 3. Double neutrosophic integrals over a rectangle

**Theorem 2** (neutrosophic Fubini's theorem)

Let $f(x, y, I)$ integrable over the rectangle

$R \cup I = \{(x, y, I)$: $\dot{a} + \dot{a}_0 I \le x \le \dot{b} + \dot{b}_0 I$ $and$ $\dot{c} + \dot{c}_0 I \le y \le \dot{d} + \dot{d}_0 I\}$ , where: $\dot{a}, \dot{a}_0, \dot{b}, \dot{b}_0, \dot{c}, \dot{c}_0, \dot{d}, \dot{d}_0$ are real numbers, while $I$ = indeterminacy.

Then we can write the double neutrosophic integrals over a rectangle $R \cup I$ by the following formula:

$$\iint_{R\cup I} f(x, y, I) \; dA = \int_{\dot{a}+\dot{a}_0 I}^{\dot{b}+\dot{b}_0 I} \int_{\dot{c}+\dot{c}_0 I}^{\dot{d}+\dot{d}_0 I} f(x, y, I) \; dydx = \int_{\dot{c}+\dot{c}_0 I}^{\dot{b}+\dot{b}_0 I} \int_{\dot{a}+\dot{a}_0 I}^{\dot{b}+\dot{b}_0 I} f(x, y, I) \; dxdy$$

Integration according to the horizontal slice:

$$\int_{\dot{c}+\dot{c}_0 I}^{\dot{d}+\dot{d}_0 I} \int_{\dot{a}+\dot{a}_0 I}^{\dot{b}+\dot{b}_0 I} f(x, y, I) \; dxdy$$

Integration according to the vertical slice:

$$\int_{\dot{a}+\dot{a}_0 I}^{\dot{b}+\dot{b}_0 I} \int_{\dot{c}+\dot{c}_0 I}^{\dot{d}+\dot{d}_0 I} f(x, y, I) \; dydx$$

**Example 1**

Let $R \cup I = \{(x, y, I)$: $0 \le x \le 2 + 2I$ $and$ $1 + I \le y \le 4 + 4I\}$, then let find:

$$\iint_{R\cup I} (x^2 + 2Ixy) \; dA$$

Solution:

Integration according to the vertical slice:

$$\int_{0}^{2+2I}\int_{1+I}^{4+4I}(x^2+2Ixy)dydx=\int_{0}^{2+2I}(x^2y+Ixy^2)|_{1+I}^{4+4I}dx$$

$$=\int_{0}^{2+2I}([(4+4I)x^2+Ix(4+4I)^2]-[(1+I)x^2+Ix(1+I)^2])dx$$

$$=\int_{0}^{2+2I}[(3+3I)x^2+60Ix]dx=\left((1+I)x^3+30Ix^2\right)\Big|_{0}^{2+2I}$$

$$=(1+I)(2+2I)^3+30I(2+2I)^2$$

$$=(1+I)(8+56I)+480I$$

$$=8+56I+8I+56I+480I=\boxed{8+600I}$$

Integration according to the horizontal slice:

$$\int_{1+I}^{4+4I}\int_{0}^{2+2I}(x^2+2Ixy)dxdy=\int_{1+I}^{4+4I}\left(\frac{x^3}{3}+Ix^2y\right)\Bigg|_{0}^{2+2I}dy$$

$$=\int_{1+I}^{4+4I}\left(\left[\left(\frac{(2+2I)^3}{3}+I(2+2I)^2y\right)\right]-[0]\right)dy$$

$$=\int_{1+I}^{4+4I}\left(\frac{(8+56I)}{3}+16Iy\right)dy$$

$$=\left(\frac{(8+56I)}{3}y+8Iy^2\right)\Bigg|_{1+I}^{4+4I}$$

$$=\left[\frac{(8+56I)}{3}(4+4I)+8I(4+4I)^2\right]-\left[\frac{(8+56I)}{3}(1+I)+8Iy(1+I)^2\right]$$

$$=(1+I)[(8+56I)+296I]$$

$$=(1+I)(8+296I)$$

$$=8+296I+296I+8I=\boxed{8+600I}$$

note that we got the same result.

### 3.1 Double neutrosophic integrals over a general region

**Theorem 3**

Let $f(x,y,I)$ is continuous on the region

$R \cup I = \{(x, y, I) \colon \dot{a} + \dot{a}_0 I \le x \le \dot{b} + \dot{b}_0 I \ \ and \ \ g_1(x, I) \le y \le g_2(x, I)\}$, where: $\dot{a}, \dot{a}_0, \dot{b}, \dot{b}_0$ are real numbers, while $I$ = indeterminacy, and $g_1(x, I)$, $g_2(x, I)$ continuous neutrosophic functions, where $g_1(x, I) \le \ g_2(x, I)$, for all $x \in [\dot{a} + \dot{a}_0 I \ , \dot{b} + \dot{b}_0 I]$.

Then we can write the double neutrosophic integrals a general region $R \cup I$ by the following formula:

$$\iint\limits_{R \cup I} f(x, y, I) \ \ dA = \int\limits_{\dot{a}+\dot{a}_0 I}^{\dot{b}+\dot{b}_0 I} \int\limits_{g_1(x,I)}^{g_2(x,I)} f(x, y, I) \ \ dydx$$

**Example 2**

Let $R \cup I = \{(x, y, I) \colon \ 0 \le x \le 4 + 4I \ \ and \ \ 0 \le y \le (1 + I)x\}$, then let find:

$$\iint\limits_{R \cup I} \left(2e^{x^2} - siny\right) \ dA$$

Solution:

$$\int\limits_0^{4+4I} \int\limits_0^{(1+I)x} \left(2e^{x^2} - \sin y\right) dydx = \int\limits_0^{4+4I} \left(2ye^{x^2} + \cos y\right)\Big|_0^{(1+I)x} dx$$

$$= \int\limits_0^{4+4I} \left(\left[2(1 + I)xe^{x^2} + cos(1 + I)x\right] - [0 + cos \ 0]\right)dx$$

$$= \int\limits_0^{4+4I} \left(\left[2(1 + I)xe^{x^2} + cos(1 + I)x\right] - [1]\right)dx$$

$$= \left((1 + I)e^{x^2} + \frac{1}{1 + I} sin(1 + I)x - x\right)\Bigg|_0^{4+4I}$$

$$= \left[(1 + I)e^{(4+4I)^2} + \frac{1}{1 + I} sin(1 + I)(4 + 4I) - 4 - 4I\right] - \left[(1 + I)e^0 + \frac{1}{1 + I} sin(0) - 0\right]$$

$$= (1 + I)e^{(4+4I)^2} + \frac{1}{1 + I} sin(1 + I)(4 + 4I) - 4 - 4I - 1 - I$$

$$= (1 + I)e^{16+48I} + \left(1 - \frac{1}{2} I\right) sin(4 + 12I) - 5 - 5I$$

$$= (1 + I)(e^{16} + I[e^{64} - e^{16}]) + \left(1 - \frac{1}{2} I\right) (sin(4) + I \ [sin(16) - sin(4)]) - 5 - 5I$$

**Example 2**

Let $R \cup I = \{(x, y, I) \colon \ 0 \le x \le 1 + I \ \ and \ \ 0 \le y \le x\}$, then let find:

$$\iint\limits_{R \cup I} \frac{sin\left(\frac{\pi}{3} + \frac{\pi}{4} I\right) x}{x} \ \ dA$$

Solution:

$$\iint\limits_{R\cup I} \frac{sin\left(\frac{\pi}{3}+\frac{\pi}{4}I\right)x}{x} \quad dA = \int_{0}^{1+I}\int_{0}^{x} \frac{sin\left(\frac{\pi}{3}+\frac{\pi}{4}I\right)x}{x} dydx$$

$$= \int_{0}^{1+I} \frac{sin\left(\frac{\pi}{3}+\frac{\pi}{4}I\right)x}{x} \quad y\Big|_{0}^{x} dx$$

$$= \int_{0}^{1+I} \left(\left[\frac{sin\left(\frac{\pi}{3}+\frac{\pi}{4}I\right)x}{x} \quad x\right] - \left[\frac{sin\left(\frac{\pi}{3}+\frac{\pi}{4}I\right)x}{x} \quad (0)\right]\right) dx$$

$$= \int_{0}^{1+I} sin\left(\frac{\pi}{3}+\frac{\pi}{4}I\right) x\, dx$$

$$= \frac{-1}{\frac{\pi}{3}+\frac{\pi}{4}} cos\left(\frac{\pi}{3}+\frac{\pi}{4}I\right) x\Big|_{0}^{1+I}$$

$$= \left[\frac{-1}{\frac{\pi}{3}+\frac{\pi}{4}} cos\left(\frac{\pi}{3}+\frac{\pi}{4}I\right)(1+I)\right] - \left[\frac{-1}{\frac{\pi}{3}+\frac{\pi}{4}} cos\left(\frac{\pi}{3}+\frac{\pi}{4}I\right)(0)\right]$$

$$= \left[\frac{-1}{\frac{\pi}{3}+\frac{\pi}{4}} cos\left(\frac{\pi}{3}+\frac{\pi}{4}I\right)(1+I)\right] - \left[\frac{-1}{\frac{\pi}{3}+\frac{\pi}{4}}\right]$$

$$= \left[\left(\frac{-3}{\pi}+\frac{9}{7\pi}I\right) cos\left(\frac{\pi}{3}+\frac{5\pi}{6}I\right)\right] - \left[\frac{-3}{\pi}+\frac{9}{7\pi}I\right]$$

$$= \left(\frac{-3}{\pi}+\frac{9}{7\pi}I\right)\left[cos\left(\frac{\pi}{3}\right) + I\left[cos\left(\frac{7\pi}{6}\right) - cos\left(\frac{\pi}{3}\right)\right]\right] - \left[\frac{-3}{\pi}+\frac{9}{7\pi}I\right]$$

$$= \left(\frac{-3}{\pi}+\frac{9}{7\pi}I\right)\left[\frac{1}{2} + I\left[\frac{-\sqrt{3}}{2} - \frac{1}{2}\right]\right] - \left[\frac{-3}{\pi}+\frac{9}{7\pi}I\right]$$

$$= \frac{1}{2}\left(\frac{-3}{\pi}+\frac{9}{7\pi}I\right) + I\left(\frac{-\sqrt{3}}{2}-\frac{1}{2}\right)\left(\frac{-3}{\pi}+\frac{9}{7\pi}I\right) - \left(\frac{-3}{\pi}+\frac{9}{7\pi}I\right)$$

$$= \frac{-1}{2}\left(\frac{-3}{\pi}+\frac{9}{7\pi}I\right) + I\left(\frac{-\sqrt{3}}{2}-\frac{1}{2}\right)\left(\frac{-3}{\pi}+\frac{9}{7\pi}I\right)$$

$$= \frac{3}{2\pi} - \frac{9}{14\pi}I + \frac{3\sqrt{3}}{2\pi}I - \frac{9\sqrt{3}}{14\pi}I + \frac{3}{2\pi}I - \frac{9}{14\pi}I$$

$$= \frac{3}{2\pi} + \left(\frac{3+12\sqrt{3}}{14\pi}\right)I$$

**Example 3**

$$\int_{0}^{1+\frac{1}{2}I}\int_{0}^{x} x\sin\left(\frac{\pi}{3}+\frac{\pi}{6}I\right)y\,dydx$$

$$=\int_{0}^{1+\frac{1}{2}I} \frac{-1}{\frac{\pi}{3}+\frac{\pi}{6}I}x\cos\left(\frac{\pi}{3}+\frac{\pi}{6}I\right)y\Bigg|_{0}^{x} dx$$

$$=\int_{0}^{1+\frac{1}{2}I}\left(\left[\frac{-1}{\frac{\pi}{3}+\frac{\pi}{6}I}x\cos\left(\frac{\pi}{3}+\frac{\pi}{6}I\right)y\right]-\left[\frac{-1}{\frac{\pi}{3}+\frac{\pi}{6}I}x\cos\left(\frac{\pi}{3}+\frac{\pi}{6}I\right)(0)\right]\right)dx$$

$$=\int_{0}^{1+\frac{1}{2}I}\left(\left[\left(\frac{-3}{\pi}-\frac{1}{\pi}I\right)x\cos\left(\frac{\pi}{3}+\frac{\pi}{6}I\right)x\right]-\left[\frac{-3}{\pi}-\frac{1}{\pi}I\right]x\right)dx$$

| derivation | integration |
|---|---|
| $(+)\left(\frac{-3}{\pi}-\frac{1}{\pi}I\right)\ x$ | $\cos\left(\frac{\pi}{3}+\frac{\pi}{6}I\right)x$ |
| $(-)\quad\left(\frac{-3}{\pi}-\frac{1}{\pi}I\right)$ | $\left(\frac{-3}{\pi}-\frac{1}{\pi}I\right)\sin\left(\frac{\pi}{3}+\frac{\pi}{6}I\right)x$ |
| $0$ | $\left(\frac{-3}{\pi}-\frac{1}{\pi}I\right)^{2}\cos\left(\frac{\pi}{3}+\frac{\pi}{6}I\right)x$ |

$$=\left(\left[\left(\frac{-3}{\pi}-\frac{1}{\pi}I\right)^{2}x\sin\left(\frac{\pi}{3}+\frac{\pi}{6}I\right)x-\left(\frac{-3}{\pi}-\frac{1}{\pi}I\right)^{3}\cos\left(\frac{\pi}{3}+\frac{\pi}{6}I\right)x\right]-\left[\frac{-3}{\pi}-\frac{1}{\pi}I\right]\frac{x^{2}}{2}\right)\Bigg|_{0}^{1+\frac{1}{2}I}$$

$$=\left(\left[\left(\frac{-3}{\pi}-\frac{1}{\pi}I\right)^{2}\left(1+\frac{1}{2}I\right)\sin\left(\frac{\pi}{3}+\frac{\pi}{6}I\right)\left(1+\frac{1}{2}I\right)-\left(\frac{-3}{\pi}-\frac{1}{\pi}I\right)^{3}\cos\left(\frac{\pi}{3}+\frac{\pi}{6}I\right)\left(1+\frac{1}{2}I\right)\right]\right.$$
$$\left.-\left[\frac{-3}{\pi}-\frac{1}{\pi}I\right]\frac{\left(1+\frac{1}{2}I\right)^{2}}{2}\right)-\left(\frac{-3}{\pi}-\frac{1}{\pi}I\right)^{3}$$

$$=\left(\left[\left(\frac{9}{\pi^{2}}-\frac{3}{\pi^{2}}I\right)\sin\left(\frac{\pi}{3}+\frac{7\pi}{12}I\right)-\left(\frac{27}{\pi^{3}}+\frac{29}{\pi^{3}}I\right)\cos\left(\frac{\pi}{3}+\frac{7\pi}{12}I\right)\right]-\left[\frac{-3}{2\pi}-\frac{3}{4\pi}I\right]\right)-\left(\frac{27}{\pi^{3}}+\frac{29}{\pi^{3}}I\right)$$

$$=\left(\frac{9}{\pi^{2}}-\frac{3}{\pi^{2}}I\right)\left(\sin\left(\frac{\pi}{3}\right)+I\left[\sin\left(\frac{11\pi}{12}\right)-\sin\left(\frac{\pi}{3}\right)\right]\right)-\left(\frac{27}{\pi^{3}}+\frac{29}{\pi^{3}}I\right)\left(\cos\left(\frac{\pi}{3}\right)+I\left[\cos\left(\frac{11\pi}{12}\right)-\cos\left(\frac{\pi}{3}\right)\right]\right)$$
$$+\frac{3}{2\pi}+\frac{3}{4\pi}I-\left(\frac{27}{\pi^{3}}+\frac{29}{\pi^{3}}I\right)$$

$$= \left(\frac{9}{\pi^2}-\frac{3}{\pi^2}I\right)\left(\frac{\sqrt{3}}{2}+I\left[\frac{\sqrt{6}-\sqrt{2}}{4}-\frac{\sqrt{3}}{2}\right]\right)-\left(\frac{27}{\pi^3}+\frac{29}{\pi^3}I\right)\left(\frac{1}{2}+I\left[\frac{-\sqrt{6}-\sqrt{2}}{4}-\frac{1}{2}\right]\right)+\frac{3}{2\pi}+\frac{3}{4\pi}I - \left(\frac{27}{\pi^3}+\frac{29}{\pi^3}I\right)$$

$$= \left(\frac{9}{\pi^2}-\frac{3}{\pi^2}I\right)\left(\frac{\sqrt{3}}{2}+\frac{\sqrt{6}-\sqrt{2}-2\sqrt{3}}{4}I\right)-\left(\frac{27}{\pi^3}+\frac{29}{\pi^3}I\right)\left(\frac{1}{2}+\frac{-\sqrt{6}-\sqrt{2}-2}{4}I\right)+\frac{3}{2\pi}+\frac{3}{4\pi}I - \left(\frac{27}{\pi^3}+\frac{29}{\pi^3}I\right)$$

$$= \left(\frac{9}{\pi^2}-\frac{3}{\pi^2}I\right)\left(\frac{\sqrt{3}}{2}+\frac{\sqrt{6}-\sqrt{2}-2\sqrt{3}}{4}I\right)-\left(\frac{27}{\pi^3}+\frac{29}{\pi^3}I\right)\left(\frac{1}{2}+\frac{-\sqrt{6}-\sqrt{2}-2}{4}I\right)+\frac{3}{2\pi}+\frac{3}{4\pi}I - \left(\frac{27}{\pi^3}+\frac{29}{\pi^3}I\right)$$

**Theorem 4**

Let $f(x,y,I)$ is continuous on the region

$R \cup I = \{(x,y,I)\colon\ \dot{c}+\dot{c}_0 I \le y \le \dot{d}+\dot{d}_0 I\ \ and\ \ h_1(y,I) \le y \le h_2(y,I)\}$, where: $\dot{c},\dot{c}_0,\dot{d},\dot{d}_0$ are real numbers, while $I$ = indeterminacy, and $h_1(y,I)$, $h_2(y,I)$ continuous neutrosophic functions, where $h_1(y,I) \le h_2(y,I)$, for all $y \in [\dot{c}+\dot{c}_0 I\ ,\dot{d}+\dot{d}_0 I]$.

Then we can write the double neutrosophic integrals a general region $R \cup I$ by the following formula:

$$\iint_{R\cup I} f(x,y,I)\ dA = \int_{\dot{c}+\dot{c}_0 I}^{\dot{d}+\dot{d}_0 I} \int_{h_1(y,I)}^{h_2(y,I)} f(x,y,I)\ dxdy$$

**Example 4**

Let $R \cup I = \{(x,y,I)\colon\ 1+I \le y \le 3+3I\ \ and\ \ (1+I)y \le x \le y^2\}$, then let find:

$$\iint_{R\cup I} 5\ dA$$

Solution:

$$\int_{1+I}^{3+3I}\int_{(1+I)y}^{y^2} 5\ dxdy = \int_{1+I}^{3+3I} 5x\Big|_{(1+I)y}^{y^2}\ dy$$

$$= 5\int_{1+I}^{3+3I} (y^2-(1+I)y)\ dy$$

$$= 5\left(\frac{y^3}{3}-(1+I)\frac{y^2}{2}\right)\Bigg|_{1+I}^{3+3I}$$

$$= 5\left(\left[\frac{(3+3I)^3}{3}-(1+I)\frac{(3+3I)^2}{2}\right]-\left[\frac{(1+I)^3}{3}-(1+I)\frac{(1+I)^2}{2}\right]\right)$$

$$= 5\left(\left[\frac{9(1+I)^3}{2}\right]-\left[\frac{(1+I)^3}{6}\right]\right)=\frac{65(1+I)^3}{3}$$

$$=\frac{65(1+7I)}{3}=\frac{65}{3}+\frac{455}{3}$$

**Theorem 5**

Let $f(x,y,I)$ and $g(x,y,I)$ be integrable over region $R\cup I \subset \mathbb{R}^2$, and let $\dot{c}+\dot{c}_0 I$ any constant, where $\dot{c},\dot{c}_0$ are real numbers, while $I$ = indeterminacy. Then:

$(i)$ $$\iint_{R\cup I}(\dot{c}+\dot{c}_0 I)f(x,y,I)\ dA=(\dot{c}+\dot{c}_0 I)\iint_{R\cup I} f(x,y,I)\ dA$$

$(ii)$ $$\iint_{R\cup I}[f(x,y,I)+g(x,y,I)]\ dA=\iint_{R\cup I} f(x,y,I)\ dA+\iint_{R\cup I} f(x,y,I)\ dA$$

$(iii)$ If $R\cup I=R_1\cup R_2\cup I$, where $R_1,R_2$ are nonoverlapping region, then:

$$\iint_{R\cup I} f(x,y,I)\ dA=\iint_{R\cup I} f(x,y,I)\ dA+\iint_{R\cup I} f(x,y,I)\ dA$$

### 3.2 Applications of double neutrosophic integrals

The area of region t can be calculated using a double neutrosophic integrals:

$$A=\iint_{R\cup I} dxdy=\iint_{R\cup I} dydx$$

**Example 5**

Using a double neutrosophic integrals to find the area of the plane region bounded by the curve of $y=x^2$ and $y=(1+I)x$.

Solution:

Let find the intersection points of the two equations:

$$x^2=(1+I)x$$

$$x^2-(1+I)x=0$$

$$x(x-1-I)=0 \qquad \Rightarrow \begin{cases} x=0 \\ x=1+I \end{cases}$$

$$\Rightarrow \begin{cases} y=0 \\ y=1+3I \end{cases}$$

so, the two equations intersect at the points: $(0\ ,0)$ and $(1+I\ ,1+3I)$

- Integration according to the horizontal slice:

$$\int_0^{1+I}\int_{x^2}^{(1+I)x} dydx=\int_0^{1+I} y\big|_{x^2}^{(1+I)x}dx$$

$$=\int_0^{1+I}\left((1+I)x-x^2\right)dx$$

$$=\left((1+I)\frac{x^2}{2}-\frac{x^3}{3}\right)\Bigg|_0^{1+I}$$

$$=(1+I)\frac{(1+I)^2}{2}-\frac{(1+I)^3}{3}$$

$$=\frac{(1+I)^3}{6}=\boxed{\frac{1}{6}+\frac{7}{6}I}$$

- Integration according to the vertical slice:

$$\int_0^{1+3I}\int_{(1-\frac{1}{2}I)y}^{\sqrt{y}} dxdy=\int_0^{1+3I} x|_{(1-\frac{1}{2}I)y}^{\sqrt{y}}dx$$

$$=\int_0^{1+3I}\left(\sqrt{y}-(1-\frac{1}{2}I)y\right)dx$$

$$=\left(\frac{2}{3}\sqrt{y^3}-(1-\frac{1}{2}I)\frac{y^2}{2}\right)\Bigg|_0^{1+3I}$$

$$=\left[\frac{2}{3}\sqrt{(1+3I)^3}-(1-\frac{1}{2}I)\frac{(1+3I)^2}{2}\right]-\left[\frac{2}{3}\sqrt{0^3}-(1-\frac{1}{2}I)\frac{0^2}{2}\right]$$

$$=\frac{2}{3}\sqrt{1+63I}-(1-\frac{1}{2}I)\frac{(1+15I)}{2}$$

$$=\frac{2}{3}(1+7I)-\left(\frac{1}{2}+\frac{15}{2}I-\frac{1}{4}I-\frac{15}{4}I\right)$$

$$=\frac{2}{3}(1+7I)-\left(\frac{1}{2}+\frac{7}{2}I\right)$$

$$=\frac{2}{3}+\frac{14I}{3}-\frac{1}{2}-\frac{7}{2}I=\boxed{\frac{1}{6}+\frac{7}{6}I}$$

note that we got the same result.

**Example 6**

Using a double neutrosophic integrals to find the area of the plane region bounded by the curve of $y=9+7I-x^2$ and $x-axis$.

Solution:

$$9+7I-x^2=0$$

$$x^2=9+7I$$

by root the both sides, we get on:

$$\Rightarrow \begin{cases} x = 3 + I \\ x = -3 - I \end{cases} \quad (1) \qquad or \qquad \begin{cases} x = 3 - 7I \\ x = -3 + 7I \end{cases} \quad (2)$$

where:

$$\sqrt{9 + 7I} = \gamma + \delta I$$

$$9 + 7I = \gamma^2 + 2\gamma\delta I + \delta^2 I$$

$$9 + 7I = \gamma^2 + (2\gamma\delta + \delta^2)I$$

then:

$$\begin{cases} \gamma^2 = 9 \\ 2\gamma\delta + \delta^2 = 7 \end{cases}$$

$$\begin{cases} \gamma = \pm 3 \\ \delta^2 + 2\gamma\delta - 7 = 0 \end{cases}$$

find the values of $\beta$:

- When $\gamma = 3 \quad \Rightarrow \quad \delta^2 + 6\delta - 7 = 0$

$$(\delta + 7)(\delta - 1) = 0 \qquad \Rightarrow \delta = -7 \ , \delta = 1$$

- When $\gamma = -3 \quad \Rightarrow \quad \delta^2 - 6\delta - 7 = 0$

$$(\delta - 7)(\delta + 1) = 0 \qquad \Rightarrow \delta = 7 \ , \delta = -1$$

$$\sqrt{9 + 7I} = 3 + I$$
$$or \qquad = -3 - I$$
$$or \qquad = 3 - 7I$$
$$or \qquad = -3 + 7I$$

case (1):

$$A = \int_{-3-I}^{3+I} \int_{0}^{9+7I-x^2} dydx = \int_{-3-I}^{3+I} y|_0^{9+7I-x^2} dx$$

$$= \int_{-3-I}^{3+I} (9 + 7I - x^2)dx$$

$$= \left( (9 + 7I)x - \frac{x^3}{3} \right) \Big|_{-3-I}^{3+I}$$

$$= \left[ (9 + 7I)(3 + I) - \frac{(3 + I)^3}{3} \right] - \left[ (9 + 7I)(-3 - I) - \frac{(-3 - I)^3}{3} \right]$$

$$= 2\left[ (9 + 7I)(3 + I) - \frac{(3 + I)^3}{3} \right]$$

$$= 2\left[27 + 37I - \left(\frac{27 + 27I + 9I + I}{3}\right)\right]$$

$$= 2\left[27 + 37I - 9 - \frac{37I}{3}\right] = 36 + \frac{148}{3}I$$

case (2):

$$A = \int_{-3+7I}^{3-7I} \int_{0}^{9+7I-x^2} dydx = \int_{-3+7I}^{3-7I} y|_0^{9+7I-x^2} dx$$

$$= \int_{-3+7I}^{3-7I} (9 + 7I - x^2)dx$$

$$= \left((9 + 7I)x - \frac{x^3}{3}\right)\Bigg|_{-3+7I}^{3-7I}$$

$$= \left[(9 + 7I)(3 - 7I) - \frac{(3 - 7I)^3}{3}\right] - \left[(9 + 7I)(-3 + 7I) - \frac{(-3 + 7I)^3}{3}\right]$$

$$= \left[(9 + 7I)(3 - 7I) - \frac{(3 - 7I)^3}{3}\right] + \left[(9 + 7I)(3 - 7I) - \frac{(3 - 7I)^3}{3}\right]$$

$$= 2\left[(9 + 7I)(3 - 7I) - \frac{(3 - 7I)^3}{3}\right]$$

$$= 2\left[27 - 91I - \left(\frac{27 - 189I + 441I - 343I}{3}\right)\right]$$

$$= 2\left[27 + 91I - 9 + \frac{91I}{3}\right] = 36 + \frac{364}{3}I$$

## 4. Conclusions

The significance of this paper stems from the fact that it explained the concept of double neutrosophic integrals. where double neutrosophic integrals over a rectangle and over a general region were presented. In addition, integrations were calculated according to the horizontal and vertical slice, and we got the same results in both cases. Also, we introduced the applications of double neutrosophic integrals.

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# BS-Neutrosophic Structures in BCK/BCI-Algebras

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**Abstract:** In this article we generalized a Smarandache's neutrosophic set as BS-neutrosophic set and it is applied to BCK/BCI-algebras. The concept of BS-neutrosophic subalgebra, BS-neutrosophic ideal and related properties are investigated.

**Keywords:** Neutrosophic set (NSS); BS-neutrosophic set (BS-NSS); BS-neutrosophic subalgebra (BS-NSSA); BS-neutrosophic ideal (BS-NSI).

## 1. Introduction

L.A.Zadeh [1], a professor of computer science at the University of California, introduced the concept of fuzzy set (FS) in 1965.Fuzzy sets analyzed the degree of membership of elements of set. In 1986 Atanassove [2] generalized a fuzzy set to an Intuitionistic Fuzzy Set (IFS) by including another function called a non-membership function. The neutrosophic Set (NS) concept was developed by Smarandache ([3],[4]) and is a more general framework that extends the concepts of Classical Set, fuzzy set, Intuitionistic fuzzy set, Interval valued fuzzy(Intuitionistic) set. Neutrosophic algebraic structures in BCK/BCI-algebras are described in articles [5],[6],[7],[8],[9],[10],[11],[12],[13],[14],[15] we know that Smarandache's NSS have many generalizations. The purpose of this paper is to consider a new generalization of the NS. A NS has true, false and indeterminate membership functions which are fuzzy sets. When we considering the generalization of a NS, an interval-valued fuzzy set is used as non-membership function, since the interval-valued fuzzy set is a generalization of the fuzzy set. We introduce the concept of BS-neutrosophic set, and we apply it to BCK/BCI-algebras. Also, the concept of BS-neutrosophic subalgebra, BS-neutrosophic ideal are introduced and the associated properties are investigated. We consider homomorphic inverse image of the BS-neutrosophic subalgebra and discuss the translation of the BS- neutrosophic Subalgebra. In a BCI-algebra, we provide conditions for a BS- neutrosophic ideal to be a BS-neutrosophic subalgebra.

## 2. Preliminaries

**Definition: 2.1**([16],[17],[18]) Let $\mathcal{K}$ be a non-empty set with a binary operation "$*$" and a constant "0" is called a BCI-algebra if it satisfies the following axioms for all $p_0, r_0, u_0 \in \mathcal{K}$

i. $((\wp_0 * r_0) * (\wp_0 * u_0)) * (u_0 * r_0) = 0$

ii. $(\wp_0 * (\wp_0 * r_0)) * r_0 = 0$

iii. $\wp_0 * \wp_0 = 0$

iv. $\wp_0 * r_0 = 0, r_0 * \wp_0 = 0 \Rightarrow \wp_0 = r_0$.

If a BCI-algebra $\mathcal{K}$ satisfies the following identity

v. $0 * \wp_0 = 0$ for all $\wp_0 \in \mathcal{K}$ then $\mathcal{K}$ is called a BCK-algebra

The following properties are hold in any BCK/BCI-algebra

i. $\wp_0 * 0 = 0$

ii. $\wp_0 \le r_0 \Rightarrow \wp_0 * u_0 \le r_0 * u_0, u_0 * r_0 \le u_0 * \wp_0$

iii. $(\wp_0 * r_0) * u_0 = (\wp_0 * u_0) * r_0$

iv. $(\wp_0 * u_0) * (r_0 * u_0) \le \wp_0 * r_0$ for all $\wp_0, r_0, u_0 \in \mathcal{K}$.

Where $\wp_0 \le r_0$ if and only if $\wp_0 * r_0 = 0$.

The following conditions are hold in any BCI-algebra $\mathcal{K}$ [16]

i. $\wp_0 * (\wp_0 * (\wp_0 * r_0)) = \wp_0 * r_0$

ii. $0 * (\wp_0 * r_0) = (0 * \wp_0) * (0 * r_0)$

**Definition: 2.2**[16] A BCI-algebra $\mathcal{K}$ is said to be p-semisimple if $0 * (0 * \wp_0) = \wp_0$ for all $\wp_0 \in \mathcal{K}$

In a p-semisimple BCI-algebra $\mathcal{K}$, the following holds for all $\wp_0, r_0 \in \mathcal{K}$

a. $0 * (\wp_0 * r_0) = r_0 * \wp_0$

b. $\wp_0 * (\wp_0 * r_0) = r_0$

**Definition: 2.3**[16] A BCI-algebra $\mathcal{K}$ is said to be associative if $(\wp_0 * r_0) * u_0 = (\wp_0 * u_0) * r_0$ for all $\wp_0, r_0, u_0 \in \mathcal{K}$

**Definition: 2.4** [18] An (s)-BCK-algebra, we mean a BCK-algebra $\mathcal{K}$ such that, for any $\wp_0, r_0 \in \mathcal{K}$ the set $\{u_0 \in \mathcal{K} / u_0 * \wp_0 \le r_0\}$ has the greatest element, written by $\wp_0 \circ r_0$.

**Definition: 2.5** A non-empty sub set $\mathcal{H}$ of a BCK/BCI-algebra $\mathcal{K}$ is called a sub algebra of $\mathcal{K}$ if $\wp_0 * r_0 \in \mathcal{H}$ for all $\wp_0, r_0 \in \mathcal{H}$.

**Definition: 2.6** A non-empty sub set $\mathcal{H}$ of a BCK/BCI-algebra $\mathcal{K}$ is called an ideal of $\mathcal{K}$ if $0 \in \mathcal{H}, and\ r_0, \wp_0 * r_0 \in \mathcal{H} \Rightarrow \wp_0 \in \mathcal{H}$ for all $\wp_0, r_0 \in \mathcal{K}$.

**Definition: 2.7** A non-empty sub set $\mathcal{H}$ of a BCI-algebra $\mathcal{K}$ is called a closed ideal of $\mathcal{K}$ if it is an ideal of $\mathcal{K}$ which satisfies $\wp_0 \in \mathcal{H} \Rightarrow 0 * \wp_0 \in \mathcal{H}$ for all $\wp_0 \in \mathcal{K}$

**Definition: 2.8**[1] Let $\mathcal{K}$ be non-empty set. A fuzzy set in $\mathcal{K}$ is a mapping $\mathcal{N}_T: \mathcal{K} \to [0,1]$

**Definition: 2.9**[1] The complement of fuzzy set $\mathcal{N}_T$ denoted by $(\mathcal{N}_T)^c$ is also a fuzzy set defined as $(\mathcal{N}_T)^c = 1 - \mathcal{N}_T$ for all $\wp_0 \in \mathcal{K}$. Also $((\mathcal{N}_T)^c)^c = \mathcal{N}_T$.

**Definition: 2.10** A fuzzy set $\mathcal{N}_T: \mathcal{K} \to [0,1]$ is called fuzzy sub-algebra of $\mathcal{K}$, if $\mathcal{N}_T(\wp_0 * r_0) \ge min\{\mathcal{N}_T(\wp_0), \mathcal{N}_T(r_0)\}$.

By an interval number we mean a closed subinterval $\widehat{m} = [m^-, m^+]$ $of$ $[I]$ where $0 \le m^- \le m^+ \le 1$. Denote by $[I]$ the set of all interval numbers. Let us define what is known as refined minimum (briefly, rmin) and refined maximum (briefly, rmax) of two elements in $[I]$. We also define the symbols " $\preccurlyeq$ " ," $\succcurlyeq$ "," $=$ " in case of two elements in $[I]$. Consider two interval numbers $\widehat{m}_1 = [m_1{}^-, m_1{}^+]$ and $\widehat{m}_2 = [m_2{}^-, m_2{}^+]$. Then

$rmin\{\widehat{m}_1, \widehat{m}_2\} = [min\{m_1{}^-, m_2{}^-\}, \min\{m_1{}^+, m_2{}^+\}]$,

$rmax\{\widehat{m}_1, \widehat{m}_2\} = [max\{m_1{}^-, m_2{}^-\}, \max\{m_1{}^+, m_2{}^+\}]$,

$\hat{m}_1 \succcurlyeq \hat{m}_2 \Leftrightarrow m_1^- \geq m_2^-, m_1^+ \geq m_2^+$, and similarly, we may have $\hat{m}_1 \preccurlyeq \hat{m}_2$ and $\hat{m}_1 = \hat{m}_2$. To say $\hat{m}_1 \succ \hat{m}_2$ (resp. $\hat{m}_1 \prec \hat{m}_2$) we mean $\hat{m}_1 \succcurlyeq \hat{m}_2$ and $\hat{m}_1 \neq \hat{m}_2$ (resp. $\hat{m}_1 \preccurlyeq \hat{m}_2$ *and* $\hat{m}_1 \neq \hat{m}_2$). Let $\hat{m}_i \in [I]$ where $i \in \sqcap$. We define

$\underset{i\in\sqcap}{rinf}\ \hat{m}_i = [\underset{i\in\sqcap}{inf} m_i^-, \underset{i\in\sqcap}{inf} m_i^+]$ *and* $\underset{i\in\sqcap}{rsup}\ \hat{m}_i = [\underset{i\in\sqcap}{sup} m_i^-, \underset{i\in\sqcap}{sup} m_i^+]$.

**Definition: 2.11**[19] Let $\mathcal{K}$ be a non-empty set. A function $\widehat{\mathcal{N}}: \mathcal{K} \to [I]$ is called an interval-valued fuzzy set (briefly, an IVF set) in $\mathcal{K}$. Let $[I]^{\mathcal{K}}$ stand for the set of all IVF sets in $\mathcal{K}$. For every $\widehat{\mathcal{N}} \in [I]^{\mathcal{K}}$ and $\wp_0 \in \mathcal{K}$, $\widehat{\mathcal{N}}(\wp_0) = [\mathcal{N}^-(\wp_0), \mathcal{N}^+(\wp_0)]$ is called the degree of membership of an element $\wp_0 \in \widehat{\mathcal{N}}$, where $\mathcal{N}^-: \mathcal{K} \to [I]$ and $\mathcal{N}^+: \mathcal{K} \to [I]$ are fuzzy sets in $\mathcal{K}$ which are called a lower fuzzy set and an upper fuzzy set in $\mathcal{K}$, respectively. For simplicity, we denote $\widehat{\mathcal{N}} = [\mathcal{N}^-, \mathcal{N}^+]$.

**Definition: 2.12**[4] Let $\mathcal{K}$ be a non-empty set. A neutrosophic set (NS) in $\mathcal{K}$ is a structure of the form $\mathcal{N} = \{\langle \wp_0; \mathcal{N}_T(\wp_0), \mathcal{N}_I(\wp_0), \mathcal{N}_F(\wp_0)\rangle : \wp_0 \in \mathcal{K}\}$ where $\mathcal{N}_T: \mathcal{K} \to [0,1]$ is a degree of membership, $\mathcal{N}_I: \mathcal{K} \to [0,1]$ is a degree of indeterminacy, and $\mathcal{N}_F: \mathcal{K} \to [0,1]$ degree of non-membership. For the sake of simplicity, we shall use the symbol $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ for the neutrosophic set $\mathcal{N} = \{\langle \wp_0; \mathcal{N}_T(\wp_0), \mathcal{N}_I(\wp_0), \mathcal{N}_F(\wp_0)\rangle : \wp_0 \in \mathcal{K}\}$.

## 3. BS-Neutrosophic Structures

**Definition: 3.1** Let $\mathcal{K}$ be a non-empty set. BS-neutrosophic set in $\mathcal{K}$ , is a structure of the form $\mathcal{N} = \{\langle \wp_0; \mathcal{N}_t(\wp_0), \mathcal{N}_i(\wp_0), \widehat{\mathcal{N}_f}(\wp_0)\rangle : \wp_0 \in \mathcal{K}\}$ where $\mathcal{N}_t$, $\mathcal{N}_i$ are fuzzy sets in $\mathcal{K}$, which are called a degree of indeterminacy and degree of non-membership, respectively, and $\widehat{\mathcal{N}_f}$ is an interval valued fuzzy set in $\mathcal{K}$ which is called an interval valued degree of non-membership

For the sake of simplicity, we shall use the symbol $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ for the BS-NSS $\mathcal{N} = \{\langle \wp_0; \mathcal{N}_t(\wp_0), \mathcal{N}_i(\wp_0), \widehat{\mathcal{N}_f}(\wp_0)\rangle : \wp_0 \in \mathcal{K}\}$.

In a BS-NSS $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ if we take $\widehat{\mathcal{N}_f}: \mathcal{K} \to [I]$, $\wp_0 \mapsto [\mathcal{N}_f^-(\wp_0), \mathcal{N}_f^+(\wp_0)]$ with $\mathcal{N}_f^-(\wp_0) = \mathcal{N}_f^+(\wp_0)$ then $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a neutrosophic set in $\mathcal{K}$.

**Definition: 3.2** Let $\mathcal{K}$ be a BCK/BCI algebra. A BS-NSS $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ in $\mathcal{K}$ is called a BS-neutrosophic subalgebra of $\mathcal{K}$ if it satisfies

(BS-NSSA 1) $\mathcal{N}_t(\wp_0 * r_0) \geq min\{\mathcal{N}_t(\wp_0), \mathcal{N}_t(r_0)\}$

(BS-NSSA 2) $\mathcal{N}_i(\wp_0 * r_0) \geq min\{\mathcal{N}_i(\wp_0), \mathcal{N}_i(r_0)\}$

(BS-NSSA 3) $\widehat{\mathcal{N}_f}(\wp_0 * r_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(\wp_0), \widehat{\mathcal{N}_f}(r_0)\}$ for all $\wp_0, r_0 \in \mathcal{K}$.

**Example: 3.3** Consider a set $\mathcal{K} = \{0, a, b, c\}$ with the binary operation $*$ which is given in table.1

| * | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | a | 0 | 0 | a |
| b | b | a | 0 | b |
| c | c | c | c | 0 |

Table.1 BCK-algebra

Then $(\mathcal{K}; *, 0)$ is a BCK-algebra. Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in $\mathcal{K}$ defined by table.2

| $\mathcal{K}$ | $\mathcal{N}_t(\wp_0)$ | $\mathcal{N}_i(\wp_0)$ | $\widehat{\mathcal{N}_f}(\wp_0)$ |
|---|---|---|---|
| 0 | 0.9 | 1 | [0.1,0.4] |
| a | 0.4 | 0.5 | [0.3,0.5] |

| | | | |
|---|---|---|---|
| b | 0.3 | 0.3 | [0.2,0.6] |
| c | 0 | 0.1 | [0.4,1] |

Table.2 BS-NSSA

It is routine to verify that $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$.

**Proposition: 3.4** If $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$ then $\mathcal{N}_t(0) \geq \mathcal{N}_t(p_0)$ $\mathcal{N}_i(0) \geq \mathcal{N}_i(p_0)$ and $\widehat{\mathcal{N}_f}(0) \preccurlyeq \widehat{\mathcal{N}_f}(p_0)$ for all $p_0 \in \mathcal{K}$

**Proof:** For any $p_0 \in \mathcal{K}$, we have

$\mathcal{N}_t(0) = \mathcal{N}_t(p_0 * p_0) \geq min\{\mathcal{N}_t(p_0), \mathcal{N}_t(p_0)\} = \mathcal{N}_t(p_0)$

$\mathcal{N}_i(0) = \mathcal{N}_i(p_0 * p_0) \geq min\{\mathcal{N}_i(p_0), \mathcal{N}_i(p_0)\} = \mathcal{N}_i(p_0)$

$\widehat{\mathcal{N}_f}(0) = \widehat{\mathcal{N}_f}(p_0 * p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(p_0)\} = rmax\{[\mathcal{N}_f^-(p_0), \mathcal{N}_f^+(p_0)], [\mathcal{N}_f^-(p_0), \mathcal{N}_f^+(p_0)]\}$

$= [\mathcal{N}_f^-(p_0), \mathcal{N}_f^+(p_0)] = \widehat{\mathcal{N}_f}(p_0)$

Hence the proof is completed.

**Proposition: 3.5** Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$ if there exists a sequence $\{p_{0_n}\}$ in $\mathcal{K}$ such that $\lim_{n\to\infty} \mathcal{N}_t(p_{0_n}) = 1$ , $\lim_{n\to\infty} \mathcal{N}_i(p_{0_n}) = 1$ and $\lim_{n\to\infty} \widehat{\mathcal{N}_f}(p_{0_n}) = [0,0]$, then $\mathcal{N}_t(0) = 1$, $\mathcal{N}_i(0) = 1$ and $\widehat{\mathcal{N}_f}(0) = [0,0]$.

**Proof:** Using the proposition 3.4, we know that $\mathcal{N}_t(0) \geq \mathcal{N}_t(p_{0_n})$ $\mathcal{N}_i(0) \geq \mathcal{N}_i(p_{0_n})$ and $\widehat{\mathcal{N}_f}(0) \preccurlyeq \widehat{\mathcal{N}_f}(p_{0_n})$ for every positive integer n. Note that

$1 \geq \mathcal{N}_t(0) \geq \lim_{n\to\infty} \mathcal{N}_t(p_{0_n}) = 1,$

$1 \geq \mathcal{N}_i(0) \geq \lim_{n\to\infty} \mathcal{N}_i(p_{0_n}) = 1,$

$[0,0] \preccurlyeq \widehat{\mathcal{N}_f}(0) \preccurlyeq \lim_{n\to\infty} \widehat{\mathcal{N}_f}(p_{0_n}) = [0,0].$

Therefore $\mathcal{N}_t(0) = 1$, $\mathcal{N}_i(0) = 1$ and $\widehat{\mathcal{N}_f}(0) = [0,0]$.

**Theorem: 3.6** Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSS in $\mathcal{K}$. Then $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA in $\mathcal{K}$ if and only if $\mathcal{N}_t, \mathcal{N}_i, (\mathcal{N}_f^-)^c$ *and* $(\mathcal{N}_f^+)^c$ are fuzzy subalgebras of $\mathcal{K}$.

**Proof:** Suppose that $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA in $\mathcal{K}$ then for all $p_0, r_0 \in \mathcal{K}$ we have

$\mathcal{N}_t(p_0 * r_0) \geq rmin\{\mathcal{N}_t(p_0), \mathcal{N}_t(r_0)\}$

$\mathcal{N}_i(p_0 * r_0) \geq min\{\mathcal{N}_i(p_0), \mathcal{N}_i(r_0)\}$

$\widehat{\mathcal{N}_f}(p_0 * r_0) \preccurlyeq max\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(r_0)\}$

$[\mathcal{N}_f^-(p_0 * r_0), \mathcal{N}_f^+(p_0 * r_0)] \preccurlyeq rmax\{[\mathcal{N}_f^-(p_0), \mathcal{N}_f^+(p_0)], [\mathcal{N}_f^-(r_0), \mathcal{N}_f^+(r_0)]\}$

$=[max\{\mathcal{N}_f^-(p_0), \mathcal{N}_f^-(r_0)\}, max\{\mathcal{N}_f^+(p_0), \mathcal{N}_f^+(r_0)\}]$

Therefore $\mathcal{N}_f^-(p_0 * r_0) \leq max\{\mathcal{N}_f^-(p_0), \mathcal{N}_f^-(r_0)\}$

$\Rightarrow 1 - \mathcal{N}_f^-(p_0 * r_0) \geq 1 - max\{\mathcal{N}_f^-(p_0), \mathcal{N}_f^-(r_0)\}$

$\Rightarrow 1 - \mathcal{N}_f^-(p_0 * r_0) \geq min\{1 - \mathcal{N}_f^-(p_0), 1 - \mathcal{N}_f^-(r_0)\}$

$\Rightarrow (\mathcal{N}_f^-)^c(p_0 * r_0) \geq min\{(\mathcal{N}_f^-)^c(p_0), (\mathcal{N}_f^-)^c(r_0)\}$ and

$\mathcal{N}_f^+(p_0 * r_0) \leq max\{\mathcal{N}_f^+(p_0), \mathcal{N}_f^+(r_0)\}$

$\Rightarrow 1 - \mathcal{N}_f^+(p_0 * r_0) \geq 1 - max\{\mathcal{N}_f^+(p_0), \mathcal{N}_f^+(r_0)\}$

$\Rightarrow 1 - \mathcal{N}_f^+(p_0 * r_0) \geq min\{1 - \mathcal{N}_f^+(p_0), 1 - \mathcal{N}_f^+(r_0)\}$

$\Rightarrow (\mathcal{N}_f^+)^c(p_0 * r_0) \geq min\{(\mathcal{N}_f^+)^c(p_0), (\mathcal{N}_f^+)^c(r_0)\}.$

Hence $\mathcal{N}_t, \mathcal{N}_i, (\mathcal{N}_f^{\,-})^c$ *and* $(\mathcal{N}_f^{\,+})^c$ are fuzzy subalgebras of $\mathcal{K}$.

Converse part is obvious.

**Definition: 3.7** Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSS in $\mathcal{K}$, we define the following sets

$\mathcal{U}(\mathcal{N}_t; l) = \{\wp_0 \in \mathcal{K}: \mathcal{N}_t(\wp_0) \geq l\}$

$\mathcal{U}(\mathcal{N}_i; m) = \{\wp_0 \in \mathcal{K}: \mathcal{N}_i(\wp_0) \geq m\}$

$\mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2]) = \{\wp_0 \in \mathcal{K}: \widehat{\mathcal{N}_f}(\wp_0) \preccurlyeq [n_1, n_2]\}$

Where $l, m \in [0,1]$ and $[n_1, n_2] \in [I]$

**Theorem: 3.8** A BS-NSS $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ in $\mathcal{K}$ is a BS-NSSA of $\mathcal{K}$ if and only if the non-empty sets $\mathcal{U}(\mathcal{N}_t; l)$, $\mathcal{U}(\mathcal{N}_i; m)$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ are subalgebras of $\mathcal{K}$ for all $l, m \in [0,1]$ and $[n_1, n_2] \in [I]$

**Proof:** Suppose that $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$.

Let $l, m \in [0,1]$ and $[n_1, n_2] \in [I]$ be such that $\mathcal{U}(\mathcal{N}_t; l)$, $\mathcal{U}(\mathcal{N}_i; m)$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ are non-empty. For any $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathcal{K}$ if $a_1, a_2 \in \mathcal{U}(\mathcal{N}_t; l)$ , $b_1, b_2 \in \mathcal{U}(\mathcal{N}_i; m)$ and $c_1, c_2 \in \mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ then

$\mathcal{N}_t(a_1 * a_2) \geq rmin\{\mathcal{N}_t(a_1), \mathcal{N}_t(a_2)\} \geq rmin\{l, l\} = l$

$\mathcal{N}_i(b_1 * b_2) \geq min\{\mathcal{N}_i(b_1), \mathcal{N}_i(b_2)\} \geq min\{m, m\} = m$

$\widehat{\mathcal{N}_f}(c_1 * c_2) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(c_1), \widehat{\mathcal{N}_f}(c_2)\} \preccurlyeq rmax\{[n_1, n_2], [n_1, n_2]\} = [n_1, n_2]$

Therefore $a_1 * a_2 \in \mathcal{U}(\mathcal{N}_t; l)$, $b_1 * b_2 \in \mathcal{U}(\mathcal{N}_i; m)$ and $c_1 * c_2 \in \mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$

Hence $\mathcal{U}(\mathcal{N}_t; l)$, $\mathcal{U}(\mathcal{N}_i; m)$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ are subalgebras of $\mathcal{K}$.

Conversely, assume that the non-empty sets $\mathcal{U}(\mathcal{N}_t; l)$ , $\mathcal{U}(\mathcal{N}_i; m)$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ are subalgebras of $\mathcal{K}$ for all $l, m \in [0,1]$ and $[n_1, n_2] \in [I]$

If $\mathcal{N}_t(a_0 * b_0) < min\{\mathcal{N}_t(a_0), \mathcal{N}_t(b_0)\}$ for some $a_0, b_0 \in \mathcal{K}$, then $a_0, b_0 \in \mathcal{U}(\mathcal{N}_t; l_0)$ but $a_0 * b_0 \notin \mathcal{U}(\mathcal{N}_t; l_0)$ for $l_0 = min\{\mathcal{N}_t(a_0), \mathcal{N}_t(b_0)\}$. This is a contradiction, and thus

$\mathcal{N}_t(\wp_0 * r_0) \geq min\{\mathcal{N}_t(\wp_0), \mathcal{N}_t(r_0)\}$ for all $\wp_0, r_0 \in \mathcal{K}$. Similarly, we can show that $\mathcal{N}_i(\wp_0 * r_0) \geq min\{\mathcal{N}_i(\wp_0), \mathcal{N}_i(r_0)\}$ for all $\wp_0, r_0 \in \mathcal{K}$.

Suppose that $\widehat{\mathcal{N}_f}(a_0 * b_0) \succ rmax\{\widehat{\mathcal{N}_f}(a_0), \widehat{\mathcal{N}_f}(b_0)\}$ for some $a_0, b_0 \in \mathcal{K}$.

Let $\widehat{\mathcal{N}_f}(a_0) = [\delta_1, \delta_2]$, $\widehat{\mathcal{N}_f}(b_0) = [\delta_3, \delta_4]$ and $\widehat{\mathcal{N}_f}(a_0 * b_0) = [n_1, n_2]$

Then $[n_1, n_2] \succ rmax\{[\delta_1, \delta_2], [\delta_3, \delta_4]\} = [max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}]$ and so $n_1 > max\{\delta_1, \delta_3\}$ and $n_2 > max\{\delta_2, \delta_4\}$

Taking $[\eta_1, \eta_2] = \frac{1}{2}\left[\widehat{\mathcal{N}_f}(a_0 * b_0) + rmax\{\widehat{\mathcal{N}_f}(a_0), \widehat{\mathcal{N}_f}(b_0)\}\right]$

$$= \frac{1}{2}\left[[n_1, n_2] + [max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}]\right]$$

$$= \left[\frac{1}{2}(n_1 + max\{\delta_1, \delta_3\}), \frac{1}{2}(n_2 + max\{\delta_2, \delta_4\})\right]$$

It follows that

$$n_1 > \eta_1 = \frac{1}{2}(n_1 + max\{\delta_1, \delta_3\}) > max\{\delta_1, \delta_3\} \text{ and } n_2 > \eta_2 = \frac{1}{2}(n_2 + max\{\delta_2, \delta_4\}) > max\{\delta_2, \delta_4\}$$

Hence $[max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}] \prec [\eta_1, \eta_2] \prec [n_1, n_2] = \widehat{\mathcal{N}_f}(a_0 * b_0)$

Therefore $a_0 * b_0 \notin \mathcal{U}(\widehat{\mathcal{N}_f}; [n_1, n_2])$. On the other hand

$\widehat{\mathcal{N}_f}(a_0) = [\delta_1, \delta_2] \preccurlyeq [max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}] \prec [\eta_1, \eta_2]$

$\widehat{\mathcal{N}_f}(b_0) = [\delta_3, \delta_4] \preccurlyeq [max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}] \prec [\eta_1, \eta_2]$ that is $a_0, b_0 \in \mathcal{U}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ .This is a contradiction and therefore $\widehat{\mathcal{N}_f}(\wp_0 * r_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(\wp_0), \widehat{\mathcal{N}_f}(r_0)\}$ for all $\wp_0, r_0 \in \mathcal{K}$.

Consequently $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$.

**Corollary: 3.9** If $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$,then the sets $\mathcal{K}_{\mathcal{N}_t} = \{p_0 \in \mathcal{K}: \mathcal{N}_t(p_0) = \mathcal{N}_t(0)\}$, $\mathcal{K}_{\mathcal{N}_i} = \{p_0 \in \mathcal{K}: \mathcal{N}_i(p_0) = \mathcal{N}_i(0)\}$ and $\mathcal{K}_{\widehat{\mathcal{N}_f}} = \{p_0 \in \mathcal{K}: \widehat{\mathcal{N}_f}(p_0) = \widehat{\mathcal{N}_f}(0)\}$ are subalgebras of $\mathcal{K}$.

We say that the subalgebras as $\mathcal{U}(\mathcal{N}_t; l)$, $\mathcal{U}(\mathcal{N}_i; m)$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ are BS-subalgebras of $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$

**Theorem: 3.10** Every subalgebra of $\mathcal{K}$ can be realized as BS-subalgebra of a BS-NSSA of $\mathcal{K}$.

**Proof:** Let $\mathcal{J}$ be a subalgebra of $\mathcal{K}$ and let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in $\mathcal{K}$ defined by

$$\mathcal{N}_t(p_0) = \begin{cases} l & if \ \ p_0 \in \mathcal{J}, \\ 0 & otherwise, \end{cases} \qquad \mathcal{N}_i(p_0) = \begin{cases} m & if \ \ p_0 \in \mathcal{J}, \\ 0 & otherwise, \end{cases} \quad \text{and} \quad \widehat{\mathcal{N}_f}(p_0) = \begin{cases} [\eta_1, \eta_2] & if \ \ p_0 \in \mathcal{J}, \\ [1,1] & otherwise, \end{cases}$$

Where $l, m \in (0,1]$ and $\eta_1, \eta_2 \in [0,1)$ with $\eta_1 < \eta_2$. It is clear that $\mathcal{U}(\mathcal{N}_t; l) = \mathcal{J}$, $\mathcal{U}(\mathcal{N}_i; m) = \mathcal{J}$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [\eta_1, \eta_2]) = \mathcal{J}$. Let $p_0, r_0 \in \mathcal{K}$. If $p_0, r_0 \in \mathcal{J}$ then $p_0 * r_0 \in \mathcal{J}$ and so

$\mathcal{N}_t(p_0 * r_0) = l = min\{l, l\} = min\{\mathcal{N}_t(p_0), \mathcal{N}_t(r_0)\}$

$\mathcal{N}_i(p_0 * r_0) = m = min\{m, m\} = min\{\mathcal{N}_i(p_0), \mathcal{N}_i(r_0)\}$

$\widehat{\mathcal{N}_f}(p_0 * r_0) = [\eta_1, \eta_2] = rmax\{[\eta_1, \eta_2], [\eta_1, \eta_2]\} = rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(r_0)\}$.

If any one of $p_0$ *and* $r_0$ is contained in $\mathcal{J}$, say $p_0 \in \mathcal{J}$, then $\mathcal{N}_t(p_0) = l, \mathcal{N}_i(p_0) = m, \widehat{\mathcal{N}_f}(p_0) = [\eta_1, \eta_2], \mathcal{N}_t(r_0) = 0, \mathcal{N}_i(r_0) = 0, and \ \ \widehat{\mathcal{N}_f}(r_0) = [1,1]$. Hence

$\mathcal{N}_t(p_0 * r_0) \geq 0 = min\{l, 0\} = min\{\mathcal{N}_t(p_0), \mathcal{N}_t(r_0)\}$

$\mathcal{N}_i(p_0 * r_0) \geq 0 = min\{m, 0\} = min\{\mathcal{N}_i(p_0), \mathcal{N}_i(r_0)\}$

$\widehat{\mathcal{N}_f}(p_0 * r_0) \preccurlyeq [1,1] = rmax\{[\eta_1, \eta_2], [1,1]\} = rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(r_0)\}$.

If $p_0, r_0 \notin \mathcal{J}$, then

$\mathcal{N}_t(p_0) = 0, \mathcal{N}_i(p_0) = 0, \widehat{\mathcal{N}_f}(p_0) = [1,1], \mathcal{N}_t(r_0) = 0, \mathcal{N}_i(r_0) = 0, and \ \ \widehat{\mathcal{N}_f}(r_0) = [1,1]$ it follows that

$\mathcal{N}_t(p_0 * r_0) \geq 0 = min\{0,0\} = min\{\mathcal{N}_t(p_0), \mathcal{N}_t(r_0)\}$

$\mathcal{N}_i(p_0 * r_0) \geq 0 = min\{0,0\} = min\{\mathcal{N}_i(p_0), \mathcal{N}_i(r_0)\}$

$\widehat{\mathcal{N}_f}(p_0 * r_0) \preccurlyeq [1,1] = rmax\{[1,1], [1,1]\} = rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(r_0)\}$.

Therefore $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$.

**Theorem:3.11** For any non-empty set $\mathcal{J}$ of $\mathcal{K}$, Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in $\mathcal{K}$ defined by

$$\mathcal{N}_t(p_0) = \begin{cases} l & if \ \ p_0 \in \mathcal{J}, \\ 0 & otherwise, \end{cases} \qquad \mathcal{N}_i(p_0) = \begin{cases} m & if \ \ p_0 \in \mathcal{J}, \\ 0 & otherwise, \end{cases} \quad \text{and} \quad \widehat{\mathcal{N}_f}(p_0) = \begin{cases} [\eta_1, \eta_2] & if \ \ p_0 \in \mathcal{J}, \\ [1,1] & otherwise, \end{cases}$$

Where $l, m \in (0,1]$ and $\eta_1, \eta_2 \in [0,1)$ with $\eta_1 < \eta_2$. If $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$,then $\mathcal{J}$ is subalgebra of $\mathcal{K}$.

**Proof:** let $p_0, r_0 \in \mathcal{J}$ then $\mathcal{N}_t(p_0) = l, \mathcal{N}_i(p_0) = m, \widehat{\mathcal{N}_f}(p_0) = [\eta_1, \eta_2]$, $\mathcal{N}_t(r_0) = l, \mathcal{N}_i(r_0) = m, and \ \ \widehat{\mathcal{N}_f}(r_0) = [\eta_1, \eta_2]$. Thus

$\mathcal{N}_t(p_0 * r_0) \geq min\{\mathcal{N}_t(p_0), \mathcal{N}_t(r_0)\} = l$

$\mathcal{N}_i(p_0 * r_0) \geq min\{\mathcal{N}_i(p_0), \mathcal{N}_i(r_0)\} = m$

$\widehat{\mathcal{N}_f}(p_0 * r_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(r_0)\} = [\eta_1, \eta_2]$ and therefore $p_0 * r_0 \in \mathcal{J}$.

Hence $\mathcal{J}$ is a subalgebra of $\mathcal{K}$.

**Theorem: 3.12** Given a BS-NSSA $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ of a BCI-algebra $\mathcal{K}$, Let $\mathcal{N}^* = ({\mathcal{N}_t}^*, {\mathcal{N}_i}^*, \widehat{\mathcal{N}_f}^*)$ is a BS-NSS defined by ${\mathcal{N}_t}^*(p_0) = \mathcal{N}_t(0 * p_0)$, ${\mathcal{N}_i}^*(p_0) = \mathcal{N}_i(0 * p_0)$ and $\widehat{\mathcal{N}_f}^*(p_0) = \widehat{\mathcal{N}_f}(0 * p_0)$ for all $p_0 \in \mathcal{K}$ then $\mathcal{N}^* = ({\mathcal{N}_t}^*, {\mathcal{N}_i}^*, \widehat{\mathcal{N}_f}^*)$ is a BS-NSSA of $\mathcal{K}$.

**Proof:** Note that $0 * (p_0 * r_0) = (0 * p_0) * (0 * r_0)$ for all $p_0, r_0 \in \mathcal{K}$. We have

${\mathcal{N}_t}^*(p_0 * r_0) = \mathcal{N}_t(0 * (p_0 * r_0)) = \mathcal{N}_t((0 * p_0) * (0 * r_0))) \geq min\{\mathcal{N}_t(0 * p_0), \mathcal{N}_t(0 * r_0)\}$
$= min\{{\mathcal{N}_t}^*(p_0), {\mathcal{N}_t}^*(r_0)\}$

${\mathcal{N}_i}^*(p_0 * r_0) = \mathcal{N}_i(0 * (p_0 * r_0)) = \mathcal{N}_i((0 * p_0) * (0 * r_0))) \geq min\{\mathcal{N}_i(0 * p_0), \mathcal{N}_i(0 * r_0)\}$
$= min\{{\mathcal{N}_i}^*(p_0), {\mathcal{N}_i}^*(r_0)\}$

$\widehat{\mathcal{N}_f}^*(p_0 * r_0) = \widehat{\mathcal{N}_f}(0 * (p_0 * r_0)) = \widehat{\mathcal{N}_f}((0 * p_0) * (0 * r_0))) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(0 * p_0), \widehat{\mathcal{N}_f}(0 * r_0)\} = rmax\{\widehat{\mathcal{N}_f}^*(p_0), \widehat{\mathcal{N}_f}^*(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$

Therefore $\mathcal{N}^* = ({\mathcal{N}_t}^*, {\mathcal{N}_i}^*, \widehat{\mathcal{N}_f}^*)$ is a BS-NSSA of $\mathcal{K}$.

**Theorem: 3.13** Let $\Psi: \mathcal{K} \to \mathcal{Y}$ be a homomorphism of BCK/BCI-algebras. If $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{Y}$, then $\Psi^{-1}(\mathcal{N}) = (\Psi^{-1}(\mathcal{N}_t), \Psi^{-1}(\mathcal{N}_i), \Psi^{-1}(\widehat{\mathcal{N}_f}))$ is a BS-NSSA of $\mathcal{K}$. Where $\Psi^{-1}(\mathcal{N}_t)(p_0) = \mathcal{N}_t(\Psi(p_0))$ , $\Psi^{-1}(\mathcal{N}_i)(p_0) = \mathcal{N}_i(\Psi(p_0))$ and $\Psi^{-1}(\widehat{\mathcal{N}_f})(p_0) = \widehat{\mathcal{N}_f}(\Psi(p_0))$ for all $p_0 \in \mathcal{K}$.

**Proof:** Let $p_0, r_0 \in \mathcal{K}$. Then

$\Psi^{-1}(\mathcal{N}_t)(p_0 * r_0) = \mathcal{N}_t(\Psi(p_0 * r_0)) = \mathcal{N}_t(\Psi(p_0) * \Psi(r_0)) \geq min\{\mathcal{N}_t(\Psi(p_0)), \mathcal{N}_t(\Psi(r_0))\} = min\{\Psi^{-1}(\mathcal{N}_t)(p_0), \Psi^{-1}(\mathcal{N}_t)(r_0)\}$,

$\Psi^{-1}(\mathcal{N}_i)(p_0 * r_0) = \mathcal{N}_i(\Psi(p_0 * r_0)) = \mathcal{N}_i(\Psi(p_0) * \Psi(r_0)) \geq min\{\mathcal{N}_i(\Psi(p_0)), \mathcal{N}_i(\Psi(r_0))\} = min\{\Psi^{-1}(\mathcal{N}_i)(p_0), \Psi^{-1}(\mathcal{N}_i)(r_0)\}$,

And

$\Psi^{-1}(\widehat{\mathcal{N}_f})(p_0 * r_0) = \widehat{\mathcal{N}_f}(\Psi(p_0 * r_0)) = \widehat{\mathcal{N}_f}(\Psi(p_0) * \Psi(r_0)) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(\Psi(p_0)), \widehat{\mathcal{N}_f}(\Psi(r_0))\} = rmax\{\Psi^{-1}(\widehat{\mathcal{N}_f})(p_0), \Psi^{-1}(\widehat{\mathcal{N}_f})(r_0)\}$.

Hence $\Psi^{-1}(\mathcal{N}) = (\Psi^{-1}(\mathcal{N}_t), \Psi^{-1}(\mathcal{N}_i), \Psi^{-1}(\widehat{\mathcal{N}_f}))$ is a BS-NSSA of $\mathcal{K}$

Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in $\mathcal{K}$. We denote

$\mathfrak{S} = 1 - sup\{\mathcal{N}_t(p_0): p_0 \in \mathcal{K}\}$

$\mathfrak{K} = 1 - sup\{\mathcal{N}_i(p_0): p_0 \in \mathcal{K}\}$

$\mathfrak{B} = r\inf\{\widehat{\mathcal{N}_f}(p_0): p_0 \in \mathcal{K}\}$.

For any $a \in [0, \mathfrak{S}]$ , $b \in [0, \mathfrak{K}]$ and $\widehat{c} \in [[0,0], \mathfrak{B}]$ we define ${\mathcal{N}_t}^a(p_0) = \mathcal{N}_t(p_0) + a$ , ${\mathcal{N}_i}^b(p_0) = \mathcal{N}_i(p_0) + b$ and $\widehat{\mathcal{N}_f}^{\widehat{c}} = \widehat{\mathcal{N}_f}(p_0) - \hat{c}$ then $\mathcal{N}^T = ({\mathcal{N}_t}^a, {\mathcal{N}_i}^b, \widehat{\mathcal{N}_f}^{\widehat{c}})$ is a BS-NSS in $\mathcal{K}$,which is called a $(a, b, \widehat{c}) -$ translative BS-NSS of $\mathcal{K}$.

**Theorem: 3.14** If $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ BS-NSSA of $\mathcal{K}$, then the $(a, b, \widehat{c}) -$ translative BS-NSS of $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is also a BS-NSSA of $\mathcal{K}$.

**Proof:** For any $p_0, r_0 \in \mathcal{K}$,we get

${\mathcal{N}_t}^a(p_0 * r_0) = \mathcal{N}_t(p_0 * r_0) + a \geq min\{\mathcal{N}_t(p_0), \mathcal{N}_t(r_0)\} + a = min\{\mathcal{N}_t(p_0) + a, \mathcal{N}_t(r_0) + a\} = min\{{\mathcal{N}_t}^a(p_0), {\mathcal{N}_t}^a(r_0)\}$,

${\mathcal{N}_i}^b(p_0 * r_0) = \mathcal{N}_i(p_0 * r_0) + b \geq min\{\mathcal{N}_i(p_0), \mathcal{N}_i(r_0)\} + b = min\{\mathcal{N}_i(p_0) + b, \mathcal{N}_i(r_0) + b\} = min\{{\mathcal{N}_i}^b(p_0), {\mathcal{N}_i}^b(r_0)\}$, and

$\widehat{\mathcal{N}_f}^{\widehat{c}}(p_0 * r_0) = \widehat{\mathcal{N}_f}(p_0 * r_0) - \widehat{c} \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(r_0)\} - \widehat{c} = rmax\{\widehat{\mathcal{N}_f}(p_0) - \widehat{c}, \widehat{\mathcal{N}_f}(r_0) - \widehat{c}\} = rmax\left\{\widehat{\mathcal{N}_f}^{\widehat{c}}(p_0), \widehat{\mathcal{N}_f}^{\widehat{c}}(r_0)\right\}$. Therefore $\mathcal{N}^T = (\mathcal{N}_t^{\hat{a}}, \mathcal{N}_i^{b}, \widehat{\mathcal{N}_f}^{c})$ is a BS-NSSA of $\mathcal{K}$.

**Theorem: 3.15** Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in $\mathcal{K}$ such that its $(a, b, \widehat{c})-$ translative BS-NSS is a BS-NSSA of $\mathcal{K}$ for $a \in [0, \mathfrak{S}], b \in [0, \mathfrak{K}]$ and $\widehat{c} \in \big[[0,0], \mathfrak{B}\big]$. Then $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$is a BS-NSSA of $\mathcal{K}$.

**Proof:** Assume that $\mathcal{N}^T = (\mathcal{N}_t^{a}, \mathcal{N}_i^{b}, \widehat{\mathcal{N}_f}^{\widehat{c}})$ is a BS-NSSA of $\mathcal{K}$ for $a \in [0, \mathfrak{S}], b \in [0, \mathfrak{K}]$ and $\widehat{c} \in \big[[0,0], \mathfrak{B}\big]$. Let $p_0, r_0 \in \mathcal{K}$. Then

$\mathcal{N}_t(p_0 * r_0) + a = \mathcal{N}_t^{a}(p_0 * r_0) \geq min\{\mathcal{N}_t^{a}(p_0), \mathcal{N}_t^{a}(r_0)\} = min\{\mathcal{N}_t(p_0) + a, \mathcal{N}_t(r_0) + a\} = rmin\{\mathcal{N}_t(p_0), \mathcal{N}_t(r_0)\} + a,$

$\mathcal{N}_i(p_0 * r_0) + b = \mathcal{N}_i^{b}(p_0 * r_0) \geq min\{\mathcal{N}_i^{b}(p_0), \mathcal{N}_i^{b}(r_0)\} = min\{\mathcal{N}_i(p_0) + b, \mathcal{N}_i(r_0) + b\} = min\{\mathcal{N}_i(p_0), \mathcal{N}_i(r_0)\} + b,$ and

$\widehat{\mathcal{N}_f}(p_0 * r_0) - \widehat{c} = \widehat{\mathcal{N}_f}^{\widehat{c}}(p_0 * r_0) \preccurlyeq rmax\left\{\widehat{\mathcal{N}_f}^{\widehat{c}}(p_0), \widehat{\mathcal{N}_f}^{\widehat{c}}(r_0)\right\} = rmax\{\widehat{\mathcal{N}_f}(p_0) - \widehat{c}, \widehat{\mathcal{N}_f}(r_0) - \widehat{c}\} = rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(r_0)\} - \widehat{c}$. It follows that

$\mathcal{N}_t(p_0 * r_0) \geq min\{\mathcal{N}_t(p_0), \mathcal{N}_t(r_0)\}$

$\mathcal{N}_i(p_0 * r_0) \geq min\{\mathcal{N}_i(p_0), \mathcal{N}_i(r_0)\}$

$\widehat{\mathcal{N}_f}(p_0 * r_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Hence $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$is a BS-NSSA of $\mathcal{K}$.

## 4. BS-Neutrosophic Ideal (BS-NSI)

**Definition:4.1** Let $\mathcal{K}$ be a BCK/BCI-algebra. A BS-NSS $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ in $\mathcal{K}$ is called a BS-NSI of $\mathcal{K}$ if it satisfies

(BS-NSI 1) $\mathcal{N}_t(0) \geq \mathcal{N}_t(p_0), \mathcal{N}_i(0) \geq \mathcal{N}_i(p_0)$ and $\widehat{\mathcal{N}_f}(0) \preccurlyeq \widehat{\mathcal{N}_f}(x)$ for all $p_0 \in \mathcal{K}$

(BS-NSI 2) $\mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(p_0 * r_0), \mathcal{N}_t(r_0)\}$

(BS-NSI 3) $\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(p_0 * r_0), \mathcal{N}_i(r_0)\}$

(BS-NSI 4) $\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

**Example:4.2** Consider a set $\mathcal{K} = \{0,1,2,a\}$ with the binary operation '$*$' which is given in the table:3 Then $(\mathcal{K}\ ;\ *, 0)$ is a BCI-algebra.

| * | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| a | a | 0 | 0 | 1 |
| b | b | b | 0 | 1 |
| 1 | 1 | 1 | 1 | 0 |

Table.3 BCI-algebra

Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in $\mathcal{K}$ defined in table:4

| $\mathcal{K}$ | $\mathcal{N}_t(p_0)$ | $\mathcal{N}_i(p_0)$ | $\widehat{\mathcal{N}_f}(p_0)$ |
|---|---|---|---|
| 0 | 0.9 | 0.8 | [0.2,0.5] |

| | | | |
|---|---|---|---|
| a | 0.7 | 0.6 | [0.4,0.7] |
| b | 0.4 | 0.3 | [0.7,0.9] |
| 1 | 0.2 | 0.1 | [0.9,1] |

Table.4 BS-NSI

It is routine to verify that $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI of $\mathcal{K}$.

**Proposition: 4.3** Let $\mathcal{K}$ be a BCK/BCI-algebra. Then every BS-NSI $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ of $\mathcal{K}$ satisfies the following assertion $p_0 * r_0 \leq u_0 \Rightarrow \mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(r_0), \mathcal{N}_t(u_0)\}$ , $\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(r_0), \mathcal{N}_i(u_0)\}$, $\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(r_0), \widehat{\mathcal{N}_f}(u_0)\}$ for all $p_0, r_0, u_0 \in \mathcal{K}$.

**Proof:** Let $p_0, r_0, u_0 \in \mathcal{K}$ be such that $p_0 * r_0 \leq u_0$. Then

$\mathcal{N}_t(p_0 * r_0) \geq min\{\mathcal{N}_t((p_0 * r_0) * u_0), \mathcal{N}_t(u_0)\} = min\{\mathcal{N}_t(0), \mathcal{N}_t(u_0)\} = \mathcal{N}_t(u_0)$,

$\mathcal{N}_i(p_0 * r_0) \geq min\{\mathcal{N}_i((p_0 * r_0) * u_0), \mathcal{N}_i(u_0)\} = min\{\mathcal{N}_i(0), \mathcal{N}_i(u_0)\} = \mathcal{N}_i(u_0)$, and

$\widehat{\mathcal{N}_f}(p_0 * r_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}((p_0 * r_0) * u_0), \widehat{\mathcal{N}_f}(u_0)\} = rmax\{\widehat{\mathcal{N}_f}(0), \widehat{\mathcal{N}_f}(u_0)\} = \widehat{\mathcal{N}_f}(u_0)$.

It follows that

$\mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(p_0 * r_0), \mathcal{N}_t(r_0)\} \geq min\{\mathcal{N}_t(u_0), \mathcal{N}_t(r_0)\}$,

$\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(p_0 * r_0), \mathcal{N}_i(r_0)\} \geq min\{\mathcal{N}_i(u_0), \mathcal{N}_i(r_0)\}$,

$\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\} \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(u_0), \widehat{\mathcal{N}_f}(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Hence the proof is completed.

**Theorem: 4.4** Every BS-NSS in a BCK/BCI-algebra $\mathcal{K}$ satisfying (BS-NSI 1) and assertion $p_0 * r_0 \leq u_0 \Rightarrow \mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(r_0), \mathcal{N}_t(u_0)\}$ , $\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(r_0), \mathcal{N}_i(u_0)\}$ , $\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(r_0), \widehat{\mathcal{N}_f}(u_0)\}$ for all $p_0, r_0, u_0 \in \mathcal{K}$ is a BS-NSI of $\mathcal{K}$.

**Proof:** Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in $\mathcal{K}$ satisfying (BS-NSI 1) and assertion $p_0 * r_0 \leq u_0 \Rightarrow \mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(r_0), \mathcal{N}_t(u_0)\}$ , $\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(r_0), \mathcal{N}_i(u_0)\}$ , $\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(r_0), \widehat{\mathcal{N}_f}(u_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Note that $p_0 * (p_0 * r_0) \leq r_0$ for all $p_0, r_0 \in \mathcal{K}$. So, we have

$\mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(p_0 * r_0), \mathcal{N}_t(r_0)\}$,

$\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(p_0 * r_0), \mathcal{N}_i(r_0)\}$,

$\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\}$. There fore $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI of $\mathcal{K}$.

**Theorem: 4.5** Given a BS-NSS $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ in a BCK/BCI-algebra $\mathcal{K}$. Then $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI if and only if $\mathcal{N}_t, \mathcal{N}_i, (\mathcal{N}_f^-)^c, and\ (\mathcal{N}_f^+)^c$ are fuzzy ideals of $\mathcal{K}$.

**Proof:** suppose that $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSI in $\mathcal{K}$. Then we have $\mathcal{N}_t(0) \geq \mathcal{N}_t(p_0), \mathcal{N}_i(0) \geq \mathcal{N}_i(p_0)$ and $\widehat{\mathcal{N}_f}(0) \preccurlyeq \widehat{\mathcal{N}_f}(x)$ for all $p_0 \in \mathcal{K}$

$\mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(p_0 * r_0), \mathcal{N}_t(r_0)\}$

$\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(p_0 * r_0), \mathcal{N}_i(r_0)\}$

$\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$

Now $\widehat{\mathcal{N}_f}(0) \preccurlyeq \widehat{\mathcal{N}_f}(p_0) \Rightarrow [\mathcal{N}_f^-(0), \mathcal{N}_f^+(0)] \preccurlyeq [\mathcal{N}_f^-(p_0) \ , \mathcal{N}_f^+(p_0)]$

$\Rightarrow \mathcal{N}_f^-(0) \leq \mathcal{N}_f^-(p_0)$ *and* $\mathcal{N}_f^+(0) \leq \mathcal{N}_f^+(p_0)$

$\Rightarrow (\mathcal{N}_f^-)^c(0) \geq (\mathcal{N}_f^-)^c(p_0)$ *and* $(\mathcal{N}_f^+)^c(0) \geq (\mathcal{N}_f^+)^c(p_0)$

$\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\}$

$$\Rightarrow [\mathcal{N}_f^-(p_0), \mathcal{N}_f^+(p_0)] \preccurlyeq rmax\{[\mathcal{N}_f^-(p_0 * r_0), \mathcal{N}_f^+(p_0 * r_0)], [\mathcal{N}_f^-(r_0), \mathcal{N}_f^+(r_0)]\}$$
$$= [max\{\mathcal{N}_f^-(p_0 * r_0), \mathcal{N}_f^-(r_0)\}, max\{\mathcal{N}_f^+(p_0 * r_0), \mathcal{N}_f^+(r_0)\}]$$

Therefore

$\mathcal{N}_f^{\,-}(p_0) \le max\{\mathcal{N}_f^{\,-}(p_0 * r_0), \mathcal{N}_f^{\,-}(r_0)\}$ and $\mathcal{N}_f^{\,+}(p_0) \le max\{\mathcal{N}_f^{\,+}(p_0 * r_0), \mathcal{N}_f^{\,+}(r_0)\}$
$\Rightarrow 1 - \mathcal{N}_f^{\,-}(p_0) \ge 1 - max\{\mathcal{N}_f^{\,-}(p_0 * r_0), \mathcal{N}_f^{\,-}(r_0)\}$
$\Rightarrow (\mathcal{N}_f^{\,-})^c(p_0) \ge min\{1 - \mathcal{N}_f^{\,-}(p_0 * r_0), 1 - \mathcal{N}_f^{\,-}(r_0)\}$
$\Rightarrow (\mathcal{N}_f^{\,-})^c(p_0) \ge min\{(\mathcal{N}_f^{\,-})^c(p_0 * r_0), (\mathcal{N}_f^{\,-})^c(r_0)\}$
Similarly
$(\mathcal{N}_f^{\,+})^c(p_0) \ge min\{(\mathcal{N}_f^{\,+})^c(p_0 * r_0), (\mathcal{N}_f^{\,+})^c(r_0)\}$
Therefore $\mathcal{N}_t, \mathcal{N}_i, (\mathcal{N}_f^{\,-})^c, and\ (\mathcal{N}_f^{\,+})^c$ are fuzzy ideals of $\mathcal{K}$.
Converse part is obvious.

**Theorem: 4.6** A BS-NSS $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ in $\mathcal{K}$ is a BS-NSI of $\mathcal{K}$ if and only if the non-empty sets $\mathcal{U}(\mathcal{N}_t; l)$, $\mathcal{U}(\mathcal{N}_i; m)$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ are ideals of $\mathcal{K}$ for all $l, m \in [0,1]$ and $[n_1, n_2] \in [I]$

**Proof:** Suppose that $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI of $\mathcal{K}$.
Let $l, m \in [0,1]$ and $[n_1, n_2] \in [I]$ be such that $\mathcal{U}(\mathcal{N}_t; l)$, $\mathcal{U}(\mathcal{N}_i; m)$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ are non-empty.
Obviously $0 \in \mathcal{U}(\mathcal{N}_t; l)$, $0 \in \mathcal{U}(\mathcal{N}_i; m)$ and $0 \in \mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$
For any $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathcal{K}$ if $a_1 * a_2, a_2 \in (\mathcal{N}_t; l)$ , $b_1 * b_2, b_2 \in \mathcal{U}(\mathcal{N}_i; m)$ and $c_1 * c_2, c_2 \in \mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ then
$\mathcal{N}_t(a_1) \ge min\{\mathcal{N}_t(a_1 * a_2), \mathcal{N}_t(a_2)\} \ge min\{l, l\} = l$
$\mathcal{N}_i(b_1) \ge min\{\mathcal{N}_i(b_1 * b_2), \mathcal{N}_i(b_2)\} \ge min\{m, m\} = m$
$\widehat{\mathcal{N}_f}(c_1) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(c_1 * c_2), \widehat{\mathcal{N}_f}(c_2)\} \preccurlyeq rmax\{[n_1, n_2], [n_1, n_2]\} = [n_1, n_2]$
Therefore $a_1 \in \mathcal{U}(\mathcal{N}_t; l)$, $b_1 \in \mathcal{U}(\mathcal{N}_i; m)$ and $c_1 \in \mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$
Hence $\mathcal{U}(\mathcal{N}_t; l)$, $\mathcal{U}(\mathcal{N}_i; m)$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ are ideals of $\mathcal{K}$.
Conversely, assume that the non-empty sets $\mathcal{U}(\mathcal{N}_t; l)$, $\mathcal{U}(\mathcal{N}_i; m)$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ are ideals of $\mathcal{K}$ for all $l, m \in [0,1]$ and $[n_1, n_2] \in [I]$.
Suppose that $\mathcal{N}_t(0) < \mathcal{N}_t(p_0)$, $\mathcal{N}_i(0) < \mathcal{N}_i(p_0)$ and $\widehat{\mathcal{N}_f}(0) \succ \widehat{\mathcal{N}_f}(p_0)$ for some $p_0 \in \mathcal{K}$.
Then $0 \notin \mathcal{U}(\mathcal{N}_t; \mathcal{N}_t(p_0)) \cap \mathcal{U}(\mathcal{N}_i; \mathcal{N}_i(p_0)) \cap \mathcal{L}(\widehat{\mathcal{N}_f}; \widehat{\mathcal{N}_f}(p_0))$.which is a contradiction.
Hence $\mathcal{N}_t(0) \ge \mathcal{N}_t(p_0), \mathcal{N}_i(0) \ge \mathcal{N}_i(p_0)$ and $\widehat{\mathcal{N}_f}(0) \preccurlyeq \widehat{\mathcal{N}_f}(x)$ for all $p_0 \in \mathcal{K}$
If $\mathcal{N}_t(a_0) < min\{\mathcal{N}_t(a_0 * b_0), \mathcal{N}_t(b_0)\}$ for some $a_0, b_0 \in \mathcal{K}$, then $a_0 * b_0, b_0 \in \mathcal{U}(\mathcal{N}_t; l_0)$ but $a_0 \notin \mathcal{U}(\mathcal{N}_t; l_0)$ for $l_0 = min\{\mathcal{N}_t(a_0 * b_0), \mathcal{N}_t(b_0)\}$. This is a contradiction, and thus
$\mathcal{N}_t(a) \ge min\{\mathcal{N}_t(a * b), \mathcal{N}_t(b)\}$ for all $a, b \in \mathcal{K}$.
Similarly, we can show that $\mathcal{N}_i(a) \ge min\{\mathcal{N}_i(a * b), \mathcal{N}_i(b)\}$ for all $a, b \in \mathcal{K}$.
Suppose that $\widehat{\mathcal{N}_f}(a_0) \succ rmax\{\widehat{\mathcal{N}_f}(a_0 * b_0), \widehat{\mathcal{N}_f}(b_0)\}$ for some $a_0, b_0 \in \mathcal{K}$.
Let $\widehat{\mathcal{N}_f}(a_0 * b_0) = [\delta_1, \delta_2]$, $\widehat{\mathcal{N}_f}(b_0) = [\delta_3, \delta_4]$ and $\widehat{\mathcal{N}_f}(a_0) = [n_1, n_2]$
Then $[n_1, n_2] \succ rmax\{[\delta_1, \delta_2], [\delta_3, \delta_4]\} = [max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}]$ and so
$n_1 > max\{\delta_1, \delta_3\}$ and $n_2 > max\{\delta_2, \delta_4\}$

Taking $[\eta_1, \eta_2] = \frac{1}{2}\left[\widehat{\mathcal{N}_f}(a_0) + rmax\{\widehat{\mathcal{N}_f}(a_0 * b_0), \widehat{\mathcal{N}_f}(b_0)\}\right]$

$$= \frac{1}{2}\left[[n_1, n_2] + (max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\})\right]$$

$$= \left[\frac{1}{2}(n_1 + max\{\delta_1, \delta_3\}), \frac{1}{2}(n_2 + max\{\delta_2, \delta_4\})\right]$$

It follows that

$n_1 > \eta_1 = \frac{1}{2}(n_1 + max\{\delta_1, \delta_3\}) > max\{\delta_1, \delta_3\}$ and $n_2 > \eta_2 = \frac{1}{2}(n_2 + max\{\delta_2, \delta_4\}) > max\{\delta_2, \delta_4\}$

Hence $[max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}] \prec [\eta_1, \eta_2] \prec [n_1, n_2] = \widehat{\mathcal{N}_f}(a_0)$
Therefore $a_0 \notin \mathcal{U}(\widehat{\mathcal{N}_f}; [n_1, n_2])$. On the other hand
$\widehat{\mathcal{N}_f}(a_0 * b_0) = [\delta_1, \delta_2] \preccurlyeq [max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}] \prec [\eta_1, \eta_2]$
$\widehat{\mathcal{N}_f}(b_0) = [\delta_3, \delta_4] \preccurlyeq [max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}] \prec [\eta_1, \eta_2]$ that is $a_0 * b_0, b_0 \in \mathcal{U}(\widehat{\mathcal{N}_f}; [n_1, n_2])$.This is a contradiction and therefore $\widehat{\mathcal{N}_f}(a_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(a_0 * b_0), \widehat{\mathcal{N}_f}(b_0)\}$ for all $a_0, b_0 \in \mathcal{K}$ .Consequently $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI of $\mathcal{K}$.

**Theorem: 4.7** Given an ideal $\mathcal{J}$ of a BCK/BCI-algebra $\mathcal{K}$, let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be an BS-NSS in $\mathcal{K}$ defined by $\mathcal{N}_t(\wp_0) = \begin{cases} l & if\ \wp_0 \in \mathcal{J}, \\ 0 & otherwise, \end{cases}$ $\mathcal{N}_i(\wp_0) = \begin{cases} m & if\ \wp_0 \in \mathcal{J}, \\ 0 & otherwise, \end{cases}$ and

$\widehat{\mathcal{N}_f}(\wp_0) = \begin{cases} [\eta_1, \eta_2] & if\ \wp_0 \in \mathcal{J}, \\ [1,1] & otherwise. \end{cases}$ Where $l, m \in (0,1]$ and $\eta_1, \eta_2 \in [0,1)$ with $\eta_1 < \eta_2$ Then $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI of $\mathcal{K}$ such that $\mathcal{U}(\mathcal{N}_t; l) = \mathcal{J}$, $\mathcal{U}(\mathcal{N}_i; m) = \mathcal{J}$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [\eta_1, \eta_2]) = \mathcal{J}$.

**Proof:** Let $\wp_0, r_0 \in \mathcal{K}$
If $\wp_0 * r_0 \in \mathcal{J}$ and $r_0 \in \mathcal{J}$ then $\wp_0 \in \mathcal{J}$ and so
$\mathcal{N}_t(\wp_0) = l = min\{l, l\} = min\{\mathcal{N}_t(\wp_0 * r_0), \mathcal{N}_t(r_0)\}$
$\mathcal{N}_i(\wp_0) = m = min\{\mathcal{N}_i(\wp_0 * r_0), \mathcal{N}_i(r_0)\}$
$\widehat{\mathcal{N}_f}(\wp_0) = [\eta_1, \eta_2] = rmax\{[\eta_1, \eta_2], [\eta_1, \eta_2]\} = rmax\{\widehat{\mathcal{N}_f}(\wp_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\}$.
If any one of $\wp_0 * r_0$ *and* $r_0$ is contained in $\mathcal{J}$, say $\wp_0 * r_0 \in \mathcal{J}$, then $\mathcal{N}_t(\wp_0 * r_0) = l, \mathcal{N}_i(\wp_0 * r_0) = m, \widehat{\mathcal{N}_f}(\wp_0 * r_0) = [\eta_1, \eta_2], \mathcal{N}_t(r_0) = 0, \mathcal{N}_i(r_0) = 0, and\ \widehat{\mathcal{N}_f}(r_0) = [1,1]$. Hence
$\mathcal{N}_t(\wp_0) \geq 0 = min\{l, 0\} = min\{\mathcal{N}_t(\wp_0 * r_0), \mathcal{N}_t(r_0)\}$
$\mathcal{N}_i(\wp_0) \geq 0 = min\{m, 0\} = min\{\mathcal{N}_i(\wp_0 * r_0), \mathcal{N}_i(r_0)\}$
$\widehat{\mathcal{N}_f}(\wp_0) \preccurlyeq [1,1] = rmax\{[\eta_1, \eta_2], [1,1]\} = rmax\{\widehat{\mathcal{N}_f}(\wp_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\}$.
If $\wp_0 * r_0 \notin \mathcal{J}$ *and* $r_0 \notin \mathcal{J}$,then
$\mathcal{N}_t(\wp_0 * r_0) = 0, \mathcal{N}_i(\wp_0 * r_0) = 0, \widehat{\mathcal{N}_f}(\wp_0 * r_0) = [1,1], \mathcal{N}_t(r_0) = 0, \mathcal{N}_i(r_0) = 0, and\ \widehat{\mathcal{N}_f}(r_0) = [1,1]$ it follows that
$\mathcal{N}_t(\wp_0) \geq 0 = min\{0,0\} = min\{\mathcal{N}_t(\wp_0 * r_0), \mathcal{N}_t(r_0)\}$,
$\mathcal{N}_i(\wp_0) \geq 0 = min\{0,0\} = min\{\mathcal{N}_i(\wp_0 * r_0), \mathcal{N}_i(r_0)\}$,
$\widehat{\mathcal{N}_f}(\wp_0) \preccurlyeq [1,1] = rmax\{[1,1], [1,1]\} = rmax\{\widehat{\mathcal{N}_f}(\wp_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\}$.
It is obvious that $\mathcal{N}_t(0) \geq \mathcal{N}_t(\wp_0), \mathcal{N}_i(0) \geq \mathcal{N}_i(\wp_0)$ and $\widehat{\mathcal{N}_f}(0) \preccurlyeq \widehat{\mathcal{N}_f}(x)$ for all $\wp_0 \in \mathcal{K}$
Therefore $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI of $\mathcal{K}$.
Obviously, we have $\mathcal{U}(\mathcal{N}_t; l) = \mathcal{J}$, $\mathcal{U}(\mathcal{N}_i; m) = \mathcal{J}$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [\eta_1, \eta_2]) = \mathcal{J}$.

**Theorem:4.8** For any non-empty set $\mathcal{J}$ of $\mathcal{K}$, Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in $\mathcal{K}$ defined by $\mathcal{N}_t(\wp_0) = \begin{cases} l & if\ \wp_0 \in \mathcal{J}, \\ 0 & otherwise, \end{cases}$ $\mathcal{N}_i(\wp_0) = \begin{cases} m & if\ \wp_0 \in \mathcal{J}, \\ 0 & otherwise, \end{cases}$ and $\widehat{\mathcal{N}_f}(\wp_0) = \begin{cases} [\eta_1, \eta_2] & if\ \wp_0 \in \mathcal{J}, \\ [1,1] & otherwise. \end{cases}$ Where $l, m \in (0,1]$ and $\eta_1, \eta_2 \in [0,1)$ with $\eta_1 < \eta_2$ .If $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI of $\mathcal{K}$,then $\mathcal{J}$ is ideal of $\mathcal{K}$.

**Proof:** Obviously, $0 \in \mathcal{J}$ . Let $\wp_0, r_0 \in \mathcal{K}$ be such that $\wp_0 * r_0, r_0 \in \mathcal{J}$ then $\mathcal{N}_t(\wp_0 * r_0) = l, \mathcal{N}_i(\wp_0 * r_0) = m, \widehat{\mathcal{N}_f}(\wp_0 * r_0) = [\eta_1, \eta_2], \mathcal{N}_t(r_0) = l, \mathcal{N}_i(r_0) = m, and\ \widehat{\mathcal{N}_f}(r_0) = [\eta_1, \eta_2]$. Thus

$\mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(p_0 * r_0), \mathcal{N}_t(r_0)\} = l$

$\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(p_0 * r_0), \mathcal{N}_i(r_0)\} = m$

$\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\} = [\eta_1, \eta_2]$ and therefore $p_0 \in \mathcal{J}$. Hence $\mathcal{J}$ is an ideal of $\mathcal{K}$.

**Theorem:4.9** In a BCK-algebra $\mathcal{K}$, every BS-NSI is a BS-NSSA of $\mathcal{K}$.

**Proof:** Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSI of a BCK-algebra $\mathcal{K}$.

Since $(p_0 * r_0) * p_0 \leq r_0$ for $p_0, r_0 \in \mathcal{K}$, it follows from proposition 4.3 that

$\mathcal{N}_t(p_0 * r_0) \geq min\{\mathcal{N}_t(p_0), \mathcal{N}_t(r_0)\}$, $\mathcal{N}_i(p_0 * r_0) \geq min\{\mathcal{N}_i(p_0), \mathcal{N}_i(r_0)\}$

$\widehat{\mathcal{N}_f}(p_0 * r_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Hence $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSSA of a BCK-algebra $\mathcal{K}$.

The converse of the above theorem may not be true as seen in the following example.

**Example: 4.10** Consider a BCK-algebra $\mathcal{K} = \{0, a, b, c\}$ with a binary operation '$*$' which is given in table.5 Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in $\mathcal{K}$ defined by table.6 then $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is BS-NSSA of $\mathcal{K}$, but it is not an BS-NSI of a BCK-algebra $\mathcal{K}$. Since $\mathcal{N}_t(a) \ngeq smin\{\mathcal{N}_t(a * b), \mathcal{N}_t(b)\}$

| * | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | a | 0 | 0 | a |
| b | b | a | 0 | b |
| c | c | c | c | 0 |

Table.5 BCK-algebra

| $\mathcal{K}$ | $\mathcal{N}_t(p_0)$ | $\mathcal{N}_i(p_0)$ | $\widehat{\mathcal{N}_f}(p_0)$ |
|---|---|---|---|
| 0 | 1 | 0.8 | [0.2,0.4] |
| a | 0.3 | 0.5 | [0.4,0.6] |
| b | 0.3 | 0.8 | [0.5,0.7] |
| c | 0.5 | 0.5 | [0.7,0.9] |

Table.6 BS-NSSA

We give a condition for a BS-NSSA to be a BS-NSI in a BCK-algebra

**Theorem:4.11** Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSSA of a BCK-algebra $\mathcal{K}$ satisfying the conditions $p_0 * r_0 \leq u_0 \Rightarrow \mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(r_0), \mathcal{N}_t(u_0)\}$ , $\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(r_0), \mathcal{N}_i(u_0)\}$ , $\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(r_0), \widehat{\mathcal{N}_f}(u_0)\}$ for all $p_0, r_0, u_0 \in \mathcal{K}$. Then $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI of $\mathcal{K}$.

**Proof:** For any $p_0 \in \mathcal{K}$, we get

$\mathcal{N}_t(0) = \mathcal{N}_t(p_0 * p_0) \geq min\{\mathcal{N}_t(p_0), \mathcal{N}_t(p_0)\} = \mathcal{N}_t(p_0)$

$\mathcal{N}_i(0) = \mathcal{N}_i(p_0 * p_0) \geq min\{\mathcal{N}_i(p_0), \mathcal{N}_i(p_0)\} = \mathcal{N}_i(p_0)$,

$\widehat{\mathcal{N}_f}(0) = \widehat{\mathcal{N}_f}(p_0 * p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(p_0)\} \preccurlyeq rmax\{[\mathcal{N}_f^-(p_0), \mathcal{N}_f^+(p_0)], [\mathcal{N}_f^-(p_0), \mathcal{N}_f^+(p_0)]\}$

$= [\mathcal{N}_f^-(p_0), \mathcal{N}_f^+(p_0)] = \widehat{\mathcal{N}_f}(p_0)$.

Since $p_0 * (p_0 * r_0) \leq r_0$ for all $p_0, r_0 \in \mathcal{K}$. It follows that $\mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(p_0 * r_0), \mathcal{N}_t(r_0)\}$ , $\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(p_0 * r_0), \mathcal{N}_i(r_0)\}$ , $\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$ . Therefore $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI in a BCK-algebra $\mathcal{K}$.

**Definition:4.12** A BS- neutrosophic ideal of $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ of a BCI-algebra $\mathcal{K}$ is said to be closed if $\mathcal{N}_t(0 * p_0) \geq \mathcal{N}_t(p_0), \mathcal{N}_i(0 * p_0) \geq \mathcal{N}_i(p_0)$ and $\widehat{\mathcal{N}_f}(0 * p_0) \preccurlyeq \widehat{\mathcal{N}_f}(x)$ for all $p_0 \in \mathcal{K}$.

**Theorem:4.13** In a BCI-algebra $\mathcal{K}$, every closed BS-NSI is a BS-NSSA

**Proof:** Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a closed BS-NSI of a BCI-algebra $\mathcal{K}$

We have for all $p_0, r_0 \in \mathcal{K}$

$$\begin{aligned}\mathcal{N}_t(p_0 * r_0) &\geq min\{\mathcal{N}_t((p_0 * r_0) * p_0), \mathcal{N}_t(p_0)\} \ (\because \ \mathcal{N} \ is \ a \ \text{BS}-\text{neutrosophic ideal}) \\ &= min\{\mathcal{N}_t((p_0 * p_0) * r_0), \mathcal{N}_t(p_0)\} \ (\because \ (p_0 * r_0) * u_0 = (p_0 * u_0) * r_0) \\ &= min\{\mathcal{N}_t(0 * r_0), \mathcal{N}_t(p_0)\} \ (\because \ p_0 * p_0 = 0) \\ &\geq min\{\mathcal{N}_t(r_0), \mathcal{N}_t(p_0)\} \ (\because \ \mathcal{N} \ is \ a \ closed \ \text{BS}-\text{neutrosophic ideal})\end{aligned}$$

$$\begin{aligned}\mathcal{N}_i(p_0 * r_0) &\geq min\{\mathcal{N}_i((p_0 * r_0) * p_0), \mathcal{N}_i(p_0)\} \ (\because \ \mathcal{N} \ is \ a \ \text{BS}-\text{neutrosophic ideal}) \\ &= min\{\mathcal{N}_i((p_0 * p_0) * r_0), \mathcal{N}_i(p_0)\} \ (\because \ (p_0 * r_0) * u_0 = (p_0 * u_0) * r_0) \\ &= min\{\mathcal{N}_i(0 * r_0), \mathcal{N}_i(p_0)\} \ (\because \ p_0 * p_0 = 0) \\ &\geq min\{\mathcal{N}_i(r_0), \mathcal{N}_i(p_0)\} \ (\because \ \mathcal{N} \ is \ a \ closed \ \text{BS}-\text{neutrosophic ideal})\end{aligned}$$

And

$$\begin{aligned}\widehat{\mathcal{N}_f}(p_0 * r_0) &\preccurlyeq rmax\{\widehat{\mathcal{N}_f}((p_0 * r_0) * p_0), \widehat{\mathcal{N}_f}(p_0)\} \ (\because \ \mathcal{N} \ is \ a \ \text{BS}-\text{neutrosophic ideal}) \\ &= rmax\{\widehat{\mathcal{N}_f}((p_0 * p_0) * r_0), \widehat{\mathcal{N}_f}(p_0)\} \ (\because \ (p_0 * r_0) * u_0 = (p_0 * u_0) * r_0) \\ &= rmax\{\widehat{\mathcal{N}_f}(0 * r_0), \widehat{\mathcal{N}_f}(p_0)\} \ (\because \ p_0 * p_0 = 0) \\ &\preccurlyeq rmax\{\widehat{\mathcal{N}_f}(r_0), \widehat{\mathcal{N}_f}(p_0)\} \ (\because \ \mathcal{N} \ is \ a \ closed \ \text{BS}-\text{neutrosophic ideal})\end{aligned}$$

Hence $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$.

**Theorem:4.14** In a weakly BCK-algebra $\mathcal{K}$, every BS-NSI is closed.

**Proof:** Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSI of a weakly BCK-algebra $\mathcal{K}$. For any $p_0 \in \mathcal{K}$, we obtain

$\mathcal{N}_t(0 * p_0) \geq min\{\mathcal{N}_t((0 * p_0) * p_0), \mathcal{N}_t(p_0)\} = min\{\mathcal{N}_t(0), \mathcal{N}_t(p_0)\} = \mathcal{N}_t(p_0)$,

$\mathcal{N}_i(0 * p_0) \geq min\{\mathcal{N}_i((0 * p_0) * p_0), \mathcal{N}_i(p_0)\} = min\{\mathcal{N}_i(0), \mathcal{N}_i(p_0)\} = \mathcal{N}_i(p_0)$,

$\widehat{\mathcal{N}_f}(0 * p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}((0 * p_0) * p_0), \widehat{\mathcal{N}_f}(p_0)\} = rmax\{\widehat{\mathcal{N}_f}(0), \widehat{\mathcal{N}_f}(p_0)\} = \widehat{\mathcal{N}_f}(p_0)$.

Therefore $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a closed BS-NSI of $\mathcal{K}$.

**Corollary: 4.15** In a weakly BCK-algebra, every BS-NSI is a BS-NSSA of $\mathcal{K}$.

In a following example we show that any BS-NSSA is not an BS-NSI in a BCI-algebra.

**Example: 4.16** Consider a BCI-algebra $\mathcal{K} = \{0,1,2,3,4,5\}$ with binary operation '$*$' in table.7

Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in $\mathcal{K}$ defined by table.8 It is routine to verify that $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$. But it is not a BS-NSI of $\mathcal{K}$. Since $\mathcal{N}_t(4) \prec min\{\mathcal{N}_t(4 * 3), \mathcal{N}_t(3)\}$.

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 3 | 2 | 3 | 3 |
| 1 | 1 | 0 | 3 | 2 | 3 | 3 |
| 2 | 2 | 2 | 0 | 3 | 0 | 0 |
| 3 | 3 | 3 | 2 | 0 | 2 | 2 |
| 4 | 4 | 2 | 1 | 3 | 0 | 1 |
| 5 | 5 | 2 | 1 | 3 | 1 | 0 |

Table.7 BCI-algebra

| $\mathcal{K}$ | $\mathcal{N}_t(p_0)$ | $\mathcal{N}_i(p_0)$ | $\widehat{\mathcal{N}_f}(p_0)$ |
|---|---|---|---|
| 0 | 0.9 | 0.8 | [0.2,0.6] |
| 1 | 0.3 | 0.4 | [0.5,0.9] |

| | | | |
|---|---|---|---|
| 2 | 0.9 | 0.8 | [0.2,0.6] |
| 3 | 0.9 | 0.8 | [0.2,0.6] |
| 4 | 0.3 | 0.4 | [0.5,0.9] |
| 5 | 0.3 | 0.4 | [0.5,0.9] |

Table.8 BS-NSSA

**Theorem: 4.17** In a p-semi simple BCI-algebra $\mathcal{K}$, the following are equivalent

(i). $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a closed BS-NSI of $\mathcal{K}$.

(ii). $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$.

**Proof:** $(i) \Longrightarrow (ii)$ see theorem 4.12

$(ii) \Longrightarrow (i)$ let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$. For any $p_0 \in \mathcal{K}$, we get

$\mathcal{N}_t(0) = \mathcal{N}_t(p_0 * p_0) \geq min\{\mathcal{N}_t(p_0), \mathcal{N}_t(p_0)\} = \mathcal{N}_t(p_0)$,

$\mathcal{N}_i(0) = \mathcal{N}_i(p_0 * p_0) \geq min\{\mathcal{N}_i(p_0), \mathcal{N}_i(p_0)\} = \mathcal{N}_i(p_0)$,

$\widehat{\mathcal{N}_f}(0) = \widehat{\mathcal{N}_f}(p_0 * p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(p_0)\} = \widehat{\mathcal{N}_f}(p_0)$.

Hence $\mathcal{N}_t(0 * p_0) \geq min\{\mathcal{N}_t(0), \mathcal{N}_t(p_0)\} = \mathcal{N}_t(p_0)$

$\mathcal{N}_i(0 * p_0) \geq min\{\mathcal{N}_i(0), \mathcal{N}_i(p_0)\} = \mathcal{N}_i(p_0)$,$\widehat{\mathcal{N}_f}(0 * p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(0), \widehat{\mathcal{N}_f}(p_0)\} = \widehat{\mathcal{N}_f}(p_0)$ for all $p_0 \in \mathcal{K}$.

Let $p_0, r_0 \in \mathcal{K}$ then

$\mathcal{N}_t(p_0) = \mathcal{N}_t(r_0 * (r_0 * p_0)) \geq min\{\mathcal{N}_t(r_0), \mathcal{N}_t(r_0 * p_0)\}$

$= min\{\mathcal{N}_t(r_0), \mathcal{N}_t(0 * (p_0 * r_0))\} \geq min\{\mathcal{N}_t(p_0 * r_0), \mathcal{N}_t(r_0)\}$,

$\mathcal{N}_i(p_0) = \mathcal{N}_i(r_0 * (r_0 * p_0)) \geq min\{\mathcal{N}_i(r_0), \mathcal{N}_i(r_0 * p_0)\}$

$= min\{\mathcal{N}_i(r_0), \mathcal{N}_i(0 * (p_0 * r_0))\} \geq min\{\mathcal{N}_i(p_0 * r_0), \mathcal{N}_i(r_0)\}$, and

$\widehat{\mathcal{N}_f}(p_0) = \widehat{\mathcal{N}_f}(r_0 * (r_0 * p_0)) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(r_0), \widehat{\mathcal{N}_f}(r_0 * p_0)\}$

$= rmax\{\widehat{\mathcal{N}_f}(r_0), \widehat{\mathcal{N}_f}(0 * (p_0 * r_0))\} \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\}$.

Therefore $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a closed BS-NSI of $\mathcal{K}$.

Since every associative BCI-algebra is a p-semisimple, we have the following are corollary

**Corollary:4.18** In a associative BCI-algebra $\mathcal{K}$, the following are equivalent

(i). $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a closed BS-NSI of $\mathcal{K}$.

(ii). $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$.

**Definition: 4.19** Let $\mathcal{K}$ be an (s)-BCK-algebra. A BS-NSS $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is called a BS-neutrosophic $\circ$-subalgebra of $\mathcal{K}$ if the following assertions are valid

$\mathcal{N}_t(p_0 \circ r_0) \geq min\{\mathcal{N}_t(p_0), \mathcal{N}_t(r_0)\}$

$\mathcal{N}_i(p_0 \circ r_0) \geq min\{\mathcal{N}_i(p_0), \mathcal{N}_i(r_0)\}$

$\widehat{\mathcal{N}_f}(p_0 \circ r_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

**Lemma:4.20** Every BS-NSI of BCK/BCI-algebra $\mathcal{K}$ satisfies the following assertion

$p_0 \leq r_0 \Rightarrow \mathcal{N}_t(p_0) \geq \mathcal{N}_t(r_0), \mathcal{N}_i(p_0) \geq \mathcal{N}_i(r_0)$ and $\widehat{\mathcal{N}_f}(p_0) \preccurlyeq \widehat{\mathcal{N}_f}(r_0)$ for all $p_0, r_0 \in \mathcal{K}$.

**Proof:** Assume that $p_0 \preccurlyeq r_0$ for all $p_0, r_0 \in \mathcal{K}$ then $p_0 * r_0 = 0$ and so

$\mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(p_0 * r_0), \mathcal{N}_t(r_0)\} = min\{\mathcal{N}_t(0), \mathcal{N}_t(r_0)\} = \mathcal{N}_t(r_0)$

$\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(p_0 * r_0), \mathcal{N}_i(r_0)\} = min\{\mathcal{N}_i(0), \mathcal{N}_i(r_0)\} = \mathcal{N}_i(r_0)$

$\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\} = rmax\{\widehat{\mathcal{N}_f}(0), \widehat{\mathcal{N}_f}(r_0)\} = \widehat{\mathcal{N}_f}(r_0)$ for all $p_0, r_0 \in \mathcal{K}$.

Hence the proof is completed.

**Theorem:4.21** In a (s)-BCK-algebra, every BS-NSI is a BS-neutrosophic $\circ$-subalgebra.

**Proof:** Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSI of a (s)-BCK-algebra $\mathcal{K}$. Note that $(p_0 \circ r_0) * p_0 \leq r_0$ for $p_0, r_0 \in \mathcal{K}$. We have

$\mathcal{N}_t(p_0 \circ r_0) \geq min\{\mathcal{N}_t((p_0 \circ r_0) * p_0), \mathcal{N}_t(p_0)\} \geq min\{\mathcal{N}_t(r_0), \mathcal{N}_t(p_0)\}$,

$\mathcal{N}_i(p_0 \circ r_0) \geq min\{\mathcal{N}_i((p_0 \circ r_0) * p_0), \mathcal{N}_i(p_0)\} \geq min\{\mathcal{N}_i(r_0), \mathcal{N}_i(p_0)\}$, and

$\widehat{\mathcal{N}_f}(p_0 \circ r_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}((p_0 \circ r_0) * p_0), \widehat{\mathcal{N}_f}(p_0)\} \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(r_0), \widehat{\mathcal{N}_f}(p_0)\}$.

Therefore $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-neutrosophic $\circ$-subalgebra of $\mathcal{K}$.

**Theorem:4.22** Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in a (s)-BCK-algebra $\mathcal{K}$. Then $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI of $\mathcal{K}$ if and only if the following assertions are valid $\mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(r_0), \mathcal{N}_t(u_0)\}$, $\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(r_0), \mathcal{N}_i(u_0)\}$, $\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(r_0), \widehat{\mathcal{N}_f}(u_0)\}$ for all $p_0, r_0, u_0 \in \mathcal{K}$ with $p_0 \leq r_0 \circ u_0$

**Proof:** Assume that $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI of $\mathcal{K}$ and let $p_0, r_0, u_0 \in \mathcal{K}$ be such that $p_0 \leq r_0 \circ u_0$

Then we have

$\mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(p_0 * (r_0 \circ u_0)), \mathcal{N}_t(r_0 \circ u_0)\}$

$= min\{\mathcal{N}_t(0), \mathcal{N}_t(r_0 \circ u_0)\} = \mathcal{N}_t(r_0 \circ u_0) \geq min\{\mathcal{N}_t(r_0), \mathcal{N}_t(u_0)\}$ (By theorem 4.20)

$\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(p_0 * (r_0 \circ u_0)), \mathcal{N}_i(r_0 \circ u_0)\}$

$= min\{\mathcal{N}_i(0), \mathcal{N}_i(r_0 \circ u_0)\} = \mathcal{N}_i(r_0 \circ u_0) \geq min\{\mathcal{N}_i(r_0), \mathcal{N}_i(u_0)\}$ (By theorem 4.20)

$\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0 * (r_0 \circ u_0)), \widehat{\mathcal{N}_f}(r_0 \circ u_0)\}$

$= rmax\{\widehat{\mathcal{N}_f}(0), \widehat{\mathcal{N}_f}(r_0 \circ u_0)\} = \widehat{\mathcal{N}_f}(r_0 \circ u_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(r_0), \widehat{\mathcal{N}_f}(u_0)\}$ (By theorem 4.20)

Conversely, let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in a (s)-BCK-algebra $\mathcal{K}$ satisfying the conditions $\mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(r_0), \mathcal{N}_t(u_0)\}$, $\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(r_0), \mathcal{N}_i(u_0)\}$, $\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(r_0), \widehat{\mathcal{N}_f}(u_0)\}$ for all $p_0, r_0, u_0 \in \mathcal{K}$ with $p_0 \leq r_0 \circ u_0$

Since $0 \leq p_0 \circ p_0$ for all $p_0 \in \mathcal{K}$, we have

$\mathcal{N}_t(0) \geq min\{\mathcal{N}_t(p_0), \mathcal{N}_t(p_0)\} = \mathcal{N}_t(p_0)$

$\mathcal{N}_i(0) \geq min\{\mathcal{N}_i(p_0), \mathcal{N}_i(p_0)\} = \mathcal{N}_i(p_0)$

$\widehat{\mathcal{N}_f}(0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(p_0)\} = \widehat{\mathcal{N}_f}(p_0)$.

Since $p_0 \leq (p_0 * r_0) \circ r_0$ for all $p_0, r_0 \in \mathcal{K}$, we have

$\mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(p_0 * r_0), \mathcal{N}_t(r_0)\}$

$\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(p_0 * r_0), \mathcal{N}_i(r_0)\}$

$\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$. Therefore $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI of $\mathcal{K}$.

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# Plithogenic Cubic Vague Sets

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**Abstract:** To measure the level of accuracy of crisp, fuzzy, intuitionistic fuzzy sets and neutrosophic set various mathematical tools are available. The generalization of these four sets is Plithogenic set. In this paper, for the first time we introduce the concept of plithogenic cubic vague set and its generalization, plithogenic fuzzy cubic vague set, plithogenic intuitionistic fuzzy cubic vague set, plithogenic neutrosophic cubic vague set. It is the combination of cubic vague set and plithogenic set. It aims to address the problems involving multiple attribute decision making. This concept is suitable and the accuracy of the result is precise as the set is described by more value of attributes. An attribute value v has a corresponding (fuzzy, intuitionistic fuzzy, neutrosophic) degree of appurtenance d(x,v) of the element x to the set P, with respect to some given criteria. Its corresponding internal and external sets are also discussed with examples. Further, P-union, P-intersection as well as R-union and R-intersection are introduced for plithogenic cubic vague sets which acts as a tool to study some of their properties. Some examples of the newly developed concepts in our everyday life is offered in this article.

**Keywords:** cubic vague set, plithogenic fuzzy cubic vague set, plithogenic intuitionistic fuzzy cubic vague set, plithogenic neutrosophic cubic vague set.

## 1. Introduction

In real life, there may be an uncertainty about any degree of membership in the variable assumption. Zadeh [24] introduced fuzzy set in which each element is assigned a membership degree in the form of a single crisp value in the interval [0,1]. Fuzzy sets is an extension of crisp set. He also gave the perception of an interval valued fuzzy set as a cause of uncertainty in the membership. Grabisch et.al[9] represent an aggregation operator exhibits a set of mathematical properties, which depends on imposed axiomatic assumptions. A new definition of cardinality of fuzzy sets on the basis of membership value is introduced by Mamonidhar[16]. The generalization of fuzzy set and fuzzy logic to intuitionistic fuzzy sets (IFS) by adding the falsehood (f), the degree of non-membership was introduced by Atanassov [5] to have a better accuracy level. The definition for some operations on intuitionistic fuzzy set and its properties was given by Atanassov[6]. It is based only on membership and non-membership function, but it does not exist in the indeterminacy. The concept of intuitionistic fuzzy topological spaces was put forward by Coker [7]. Norsyahida Zulkifli[18] proposed the interval-valued intuitionistic fuzzy vague sets (IVIFVS) where membership and non-membership of interval-valued intuitionistic fuzzy sets are combined with truth membership and false membership of vague sets. Then the next evaluation of IFS is neutrosophic set introduced by Smarandache[20]. It is a generalization of fuzzy sets and IFS. It deals with membership, indeterminacy and non-membership degree which is highly helpful for dealing with uncertain, inadequate and varying data exist in real life. But it is applicable only on three attribute values. Data needs to be handled with more attribute values so as to raise the accuracy level in the stage of advanced research.

The theory of vague sets was proposed by Gau and Buehrer[8]. It has more powerful ability than fuzzy sets to process fuzzy information to some degree. Jun, Y.B[12] introduced the concept of cubic set and it is characterized by interval valued fuzzy set and fuzzy set, which is a more general tool to capture uncertainty and vagueness. The ideas of internal and external cubic sets and their characteristics were also presented. Cubic interval-valued intuitionistic fuzzy sets was introduced by Jun et.al [13]. Khaleed et.al [14] introduced the novel concept of cubic vague set by incorporating both the ideas of cubic set and vague set. Some new operations on intuitionistic neutrosophic set with examples for the implementation of the operations problems is introduced by Monoranjan et.al.[15]

As an extension of the neutrosophic set Wang et.al [22] proposed the definition of the interval valued neutrosophic set (INS). Interval neutrosophic sets and their application in multi-criteria decision making problems is defined by Hong et.al[11]. Hazwani Hashimcet.et.al [10] introduced the idea Interval Neutrosophic Vague Sets. Shawkat Alkhazaleh [19] introduced the concept of neutrosophic vague set as a combination of neutrosophic set and vague set. Anitha et.al[4] introduced the NGSR closed sets in neutrosophic topological spaces. Wang et.al[23] presented an instance of neutrosophic set called single valued neutrosophic set.

To, increase the preciseness, Smarandache [21] introduced plithogenic. It is a powerful tool which is a generalization of crisp set, fuzzy set, intuitionistic fuzzy set and neutrosophic set is collectively called plithogenic. It is the base for all plithogenic functions such as plithogenic set, plithogenic probability, plithogenic logic and plithogenic statistics. These sets which are characterized by a single appurtenance is plithogenic set. A plithogenic set, in general may have elements characterized by attributes with four or more attributes. It is a set whose components are described by at least one trait and each attribute may have numerous elements. He developed the aggregation operations on plithogenic set and proved that plithogenic set is the most generalized structure that can be efficiently applied to a variety of real life problems.

A procedure to come up with a methodical system to assess the infirmary serving under a framework of plithogenic theory was suggested by Abdel-Basset et al.[1] as an approach to be constructed on the connotation of plithogenic theory. To increase the accuracy of the evaluation Abdel-Basset et al.[2] proposed a method which is a combination of quality function deployment with plithogenic aggregation operations. Nivetha et.al[17] developed a concept of combined plithogenic hypersoft set and its application in multi attribute decision making. Based on the technique in order of preference by similarity to ideal solution and criteria importance through inter-criteria correlation methods to estimation of sustainable supply chain risk management Abdel-Basset et al.[3] proposed a methodology as a combination of plithogenic multi-criteria decision making approach.

In this paper, we introduced the generalization of plithogenic cubic vague sets for fuzzy, intuitionistic fuzzy and neutrosophic sets using the principles of cubic vague set and plithogenic set and its union and intersection. Plithogenic cubic vague set is the combination of cubic vague set and plithogenic set. It aims to address the problems involving multiple attribute decision making. This concept is suitable and the accuracy of the result is precise as the set is described by more value of attributes. The organization of this paper is as follows. Introduction is presented in Section 1. Section 2 provides some preliminaries for the proposed concept is given. Section 3 covers the notion of plithogenic cubic vague set and it is divided into four subsections. In 3.1 definition and examples of plithogenic fuzzy cubic vague set, in 3.2 plithogenic intuitionistic fuzzy cubic vague set, in 3.3 plithogenic neutrosophic cubic vague set and in 3.4 internal and external plithogenic cubic vague sets were presented. In Section 4 we define the basic operations namely, union and intersection of the developed set. Finally, Section 5 conclude this paper and provides the direction for future studies.

## 2. Preliminaries

**Definition: 2.1. [8]** A vague set A (VS) in the universe of discourse U is a characterized by membership functions given by: truth membership function $t_A: U \to [0,1]$ and false membership function $f_A: U \to [0,1]$ where $t_A(u)$ is a lower bound of the grade of membership of u derived from the evidence of u and $f_A(u)$ is a lower bound of the negation of u derived from the evidence against u and $t_A(u) + f_A(u) \leq 1$. Thus the grade of membership of u in the vague set A is bounded by a sub interval $[t_A(u), 1 - f_A(u)]$ of [0,1]. This indicates that if the actual grade of membership is $\mu(u)$ then $t_A(u) \leq \mu_A(u) \leq 1 - f_A(u)$. The vague set A is written as A $=\{(u, [t_A(u), 1 - f_A(u)])|u \in U\}$, where the internal $[t_A(u), 1 - f_A(u)]$ is called vague value of u in A and denoted by $V_A(u)$.

**Definition: 2.2. [12]** Let X be a non-empty set. A structure $A = \{\langle x, A(x), \lambda(x)\rangle : x \in X\}$ be a cubic set in X in which A is an IVFS and $\lambda$ is a fuzzy set in X.

**Definition: 2.3. [12]** Let X be a universal Set. A cubic vague set $\mathbb{A}^V$ defined over the universal set X is an ordered pair which is defined as follows $\mathbb{A}^V = \{\langle x, A_V(x), \lambda_V(x)\rangle : x \in X\}$ where $A_V = \langle A_V^t, A_V^{1-f}\rangle = \{\langle x, [t_{A_V}^-(x), t_{A_V}^+(x)], [1 - f_{A_V}^-(x), 1 - f_{A_V}^+(x)]\rangle : x \in X\}$ represents IVVS defined on X while $\lambda_V = \{(x, t_{\lambda V}(x), 1 - f_{\lambda V}(x)) : x \in X\}$ represents VS such that $t_{A_V}^+(x) + f_{A_V}^+(x) \leq 1$ and $t_{\lambda V}(x) + f_{\lambda V}(x) \leq 1$. For clarity we denote the pairs as $\mathbb{A}^V = \langle A_V, \lambda_V\rangle$, where $A_V = \langle [t_{A_V}^-, t_{A_V}^+], [1 - f_{A_V}^-, 1 - f_{A_V}^+]\rangle$ and $\lambda_V = (t_{\lambda V}, 1 - f_{\lambda V})$. $C_V^X$ denotes the set of all cubic vague sets in X.

**Definition: 2.4. [12]** Let X be a universal set and V be a non-empty vague set. A cubic vague set $\mathbb{A}^V = \langle A_v, \lambda_v\rangle$ is called an internal cubic vague set (brief. ICVS) if $A_V^-(x) \leq \lambda_V(x) \leq A_V^+(x)$ for all $x \in X$.

**Definition: 2.5. [12]** Let X be a universal set and V be a non-empty vague set. A cubic vague set $\mathbb{A}^V = \langle A_v, \lambda_v\rangle$ is called an external cubic vague set (brief. ECVS) if $\lambda_V(x) \notin (A_V^-(x), A_V^+(x))$ for all $x \in X$.

**Definition: 2.6. [8]** An interval valued vague sets $\widetilde{A}^V$ over a universe of discourse X is defined as an object of the form $\widetilde{A}^V = \{(x_i, [T_{\widetilde{A}^V}(x_i), F_{\widetilde{A}^V}(x_i)])|x_i \in X\}$, where $T_{\widetilde{A}^V}: X \to D[0,1]$ and $F_{\widetilde{A}^V}: X \to D[0,1]$ are called truth membership function and false membership function respectively and where D[0,1] is the set of all intervals within [0,1].

**Definition: 2.7. [19]** A neutrosophic vague set $A_{NV}$ (NVS in short) on the universe of discourse X written as $A_{NV} = \{\langle x, \hat{T}_{A_{NV}}, \hat{I}_{A_{NV}}, \hat{F}_{A_{NV}}\rangle x \in X\}$ whose truth membership, indeterminacy membership and falsity membership functions are defined as $\hat{T}_{A_{NV}}(x) = [T^-, T^+]$, $\hat{I}_{A_{NV}}(x) = [I^-, I^+]$, $\hat{F}_{A_{NV}}(x) = [F^-, F^+]$, where $T^+ = 1 - F^-$, $F^+ = 1 - T^-$ and $0^- \leq T^- + I^- + F^- \leq 2^+$.

**Definition: 2.8. [10]** An interval valued neutrosophic vague set $A_{INV}$ also known as INVS in the universe of discourse E. An IVNVS is characterized by truth membership, indeterminacy membership and falsity membership functions is defined as: $A_{INV} = \{< e, [\hat{V}_A^L(e), \hat{V}_A^U(e)], [\widehat{W}_A^L(e), \widehat{W}_A^U(e)], [\hat{X}_A^L(e), \hat{X}_A^U(e)] > |e \in E\}$,
$\hat{V}_A^L(e) = [V^{L^-}, V^{L^+}]$, $\hat{V}_A^U(e) = [V^{U^-}, V^{U^+}]$, $\widehat{W}_A^L(e) = [W^{L^-}, W^{L^+}]$, $\widehat{W}_A^L(e) = [W^{U^-}, W^{U^+}]$, $\hat{X}_A^L(e) = [X^{L^-}, X^{L^+}]$, $\hat{X}_A^U(e) = [X^{U^-}, X^{U^+}]$ where $V^{L^+} = 1 - X^{L^-}$, $X^{L^+} = 1 - V^{L^-}$, $V^{U^+} = 1 - X^{U^-}$, $X^{U^+} = 1 - V^{U^-}$ and
$0^- \leq V^{L^-} + V^{U^-} + W^{L^-} + W^{U^-} + X^{L^-} + X^{U^-} \leq 4^+$, $0^- \leq V^{L^+} + V^{U^+} + W^{L^+} + W^{U^+} + X^{L^+} + X^{U^+} \leq 4^+$.

**Definition: 2.9 [12]** $\mathcal{A} = \langle A, \lambda\rangle$ and $\mathcal{B} = \langle B, \mu\rangle$ be cubic sets in X. Then we define

(a) (Equality) $\mathcal{A} = \mathcal{B} \Leftrightarrow A = B$ and $\lambda = \mu$

(b) (P-order) $\mathcal{A} = \mathcal{B} \Leftrightarrow A \subseteq B$ and $\lambda \leq \mu$

(c) (R-order) $\mathcal{A} = \mathcal{B} \Leftrightarrow A \subseteq B$ and $\lambda \geq \mu$

## 3. PLITHOGENIC CUBIC VAGUE SETS

**Definition: 3.1**
Interval valued plithogenic fuzzy vague set (IPFVS) is defined as $\forall x \in P, d_v: P \times Q_v \rightarrow P([0,1])$ and $\forall q \in Q, d(x,q)$ is an (open, semi-open, closed) interval included in [0,1].

**Definition: 3.2**
Interval valued plithogenic intuitionistic fuzzy vague set (IPIFVS) is defined as $\forall x \in P, d_v: P \times Q_v \rightarrow P([0,1]^2)$ and $\forall q \in Q, d(x,q)$ is an (open, semi-open, closed) interval included in [0,1].

**Definition: 3.3**
Interval valued plithogenic Neutrosophic vague set (IPIFVS) is defined as $\forall x \in P, d_v: P \times Q_v \rightarrow P([0,1]^3)$ and $\forall q \in Q, d(x,q)$ is an (open, semi-open, closed) interval included in [0,1].

**Definition: 3.4**
A fuzzy vague set $A_{FV}$ (FVS in short) on the universe of discourse X written as $A_{FV} = \{\langle x, \hat{T}_{A_{FV}} \rangle x \in X\}$ whose truth membership is defined as $\hat{T}_{A_{FV}}(x) = [T^-, T^+]$ where $0 \leq T^- \leq T^+ \leq 1$.

**Definition: 3.5**
A Intuitionistic fuzzy vague set $A_{IFV}$ (IFVS in short) on the universe of discourse X written as $A_{IFV} = \{\langle x, \hat{T}_{A_{IFV}}, \hat{F}_{A_{IFV}} \rangle x \in X\}$ whose truth membership and falsity membership functions are defined as $\hat{T}_{A_{NV}}(x) = [T^-, T^+]$, $\hat{F}_{A_{NV}}(x) = [F^-, F^+]$.

### 3.1. Plithogenic Fuzzy Cubic Vague Sets

**Definition: 3.1.1**
Let U be a universal set. The set $\mathbb{A}_p{}^v = \{\langle x, A_V(x), \lambda_V(x)\rangle : x \in X\}$ is called plithogenic fuzzy cubic vague set in which $A_V$ is an interval valued plithogenic fuzzy vague set in X and $\lambda_V$ is the fuzzy vague set in X.

**Example: 3.1.2**
Let the attribute be "size" and the attribute values are {small, medium, big, very big}. Let's consider the dominant value of attribute size be small.
The attribute value contradictory degrees are:
c(small, small) = 0, c(small, medium) = 0.50, c(small, big) = 0.75, c(small, very big) = 1.
The degree of appurtenance $A_V$: {small, medium, big, very big} → [0,1].
$A_V$(small) = ([0.3,0.4],[0.4,0.6]), $A_V$(medium) = ([0.2,0.3],[0.4,0.5]),
$A_V$(big) = ([0.1,0.3],[0.3,0.6]), $A_V$(very big) = ([0.3,0.4],[0.4,0.5]).

| Degrees of contradiction | 0 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|
| Attribute values | small | Medium | Big | very big |
| Degrees of appurtenance $A_V(x)$ | ([0.3,0.4], [0.4,0.6]) | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.3,0.6]) | ([0.3,0.4], [0.4,0.5]) |
| $\lambda_V(x)$ | [0.3,0.5] | [0.2,0.4] | [0.3,0.5] | [0.4,0.5] |

### 3.2 Plithogenic Intuitionistic Fuzzy Cubic Vague Sets

**Definition: 3.2.1**

Let U be a universal set. The set $\mathbb{A}_p{}^v = \{\langle x, A_V(x), \lambda_V(x)\rangle : x \in X\}$ is called plithogenic intuitionistic fuzzy cubic vague set in which $A_V$ is an interval valued plithogenic intuitionistic fuzzy vague set in X and $\lambda_V$ is the intuitionistic fuzzy vague set in X.

**Example: 3.2.2**
Let the attribute be "Car Brands" and the attribute values are {Ford, Audi, Benz, BMW}. Let's consider the dominant value of attribute car brands be Ford.

| Degrees of contradiction | 0 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|
| Attribute values | Ford | Audi | Benz | BMW |
| Degrees of appurtenance $A_V(x)$ | {([0.4,0.6],[0.5,0.5]),([0.4,0.6],[0.5.0.5])} | {([0.3,0.8],[0.5,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.4.0.7])} | {([0.1,0.7],[0.2,0.4]),([0.3,0.9],[0.6.0.8])} |
| $\lambda_V(x)$ | [0.3,0.6],[0.4,0.7] | [0.2,0.5],[0.5,0.8] | [0.4,0.7],[0.3,0.6] | [0.5,0.6],[0.4,0.5] |

**3.3 Plithogenic Neutrosophic Cubic Vague Sets**

**Definition: 3.3.1**
Let U be a universal set. The set $\mathbb{A}_p{}^v = \{\langle x, A_V(x), \lambda_V(x)\rangle : x \in X\}$ is called plithogenic neutrosophic cubic vague set in which $A_V$ is an interval valued plithogenic neutrosophic vague set in X and $\lambda_V$ is the neutrosophic vague set in X.

**Example: 3.3.2**

Let the attribute be "Colleges" and the attribute values are {Arts, Engineering, Medical, Agriculture}. Let's consider the dominant value of attribute colleges be Arts.

| Degrees of contradiction | 0 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|
| Attribute values | Arts | Engineering | Medical | Agriculture |
| Degrees of appurtenance $A_V(x)$ | {([0.4,0.6],[0.5,0.5]),([0.1,0.8],[0.2,0.6]),([0.4,0.6],[0.5.0.5])} | {([0.3,0.8],[0.5,0.6]),([0.3,0.5],[0.4,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.2,0.5]),([0.4,0.8],[0.4.0.7])} | {([0.1,0.7],[0.2,0.4]),([0.4,0.5],[0.3,0.6]),([0.3,0.9],[0.6.0.8])} |
| $\lambda_V(x)$ | [0.2,0.5],[0.3,0.4],[0.5,0.8] | [0.4,0.7],[0.2,0.3],[0.3,0.6] | [0.3,0.6],[0.4,0.5],[0.4,0.7] | [0.5,0.6],[0.5,0.6],[0.4,0.5] |

**3.4. Internal Plithogenic Cubic Vague Set and External Plithogenic Cubic Vague Set**

**Definition: 3.4.1**
Let X be a universal set and V be a non-empty vague set. The plithogenic fuzzy cubic vague set $\mathbb{A}_p{}^v = \langle A_v, \lambda_v\rangle$ in X is called an internal plithogenic fuzzy cubic vague set (IPFCVS) if $A_{v_{d_i}}{}^{-}(x) \leq \lambda_{v_i}(x) \leq A_{v_{d_i}}{}^{+}(x)$ for all $x \in X$ and $d_i$ denotes the contradictory degree and its attribute values.

**Example: 3.4.2**
Let the attribute be "Continents" and the attribute values are {Asia, Africa, Europe, Australia}. Let's consider the dominant value of attribute continents as Asia.

| Degrees of contradiction | 0 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|

| Attribute values | Asia | Africa | Europe | Australia |
|---|---|---|---|---|
| Degrees of appurtenance $A_V(x)$ | ([0.3,0.4], [0.4,0.6]) | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.3,0.6]) | ([0.3,0.4], [0.4,0.5]) |
| $\lambda_V(x)$ | [0.3,0.5] | [0.2,0.4] | [0.1,0.5] | [0.3,0.4] |

**Definition: 3.4.3**

Let X be a universal set and V be a non-empty vague set. The plithogenic intuitionistic fuzzy cubic vague set $\mathbb{A}_p{}^v$ = $\langle A_v, \lambda_v \rangle$ in X is called an internal plithogenic intuitionistic fuzzy cubic vague set (IPIFCVS) if $A_{v_{d_i}}{}^{-}(x) \leq \lambda_{v_i}(x) \leq A_{v_{d_i}}{}^{+}(x)$ for all $x \in X$ and $d_i$ denotes the contradictory degree and its attribute values.

**Example: 3.4.4**

Let the attribute be "Vehicles" and the attribute values are {Car, Bus, Bicycle, Scooter}. Let's consider the dominant value of the attribute Vehicles as Car.

| Degrees of contradiction | 0 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|
| Attribute values | Car | Bus | Bicycle | Scooter |
| Degrees of appurtenance $A_V(x)$ | {([0.3,0.8],[0.5,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.4,0.7],[0.3,0.5]),([0.3,0.6],[0.5.0.7])} | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.4.0.7])} | {([0.4,0.6],[0.5,0.5]),([0.4,0.6],[0.5.0.5])} |
| $\lambda_V(x)$ | [0.4,0.6],[0.3,0.4] | [0.5,0.5],[0.3,0.6] | [0.3,0.4],[0.5,0.5] | [0.2,0.5],[0.4,0.5] |

**Definition: 3.4.5**

Let X be a universal set and V be a non-empty vague set. The plithogenic neutrosophic cubic vague set $\mathbb{A}_p{}^v$ = $\langle A_v, \lambda_v \rangle$ in X is said to be internal plithogenic neutrosophic cubic vague set (IPNCVS) in X if it satisfies the following equations.

(i) truth-internal (briefly, T-internal) if $A_{v_{d_i}}{}^{-T}(x) \leq \lambda_{v_i}{}^{T}(x) \leq A_{v_{d_i}}{}^{+T}(x)$ for all $x \in x$ and $d_i$ denotes the contradictory degree and its attribute values. (Condition 1)

(ii) indeterminacy-internal (briefly, I-internal) if $A_{v_{d_i}}{}^{-I}(x) \leq \lambda_{v_i}{}^{I}(x) \leq A_{v_{d_i}}{}^{+I}(x)$ for all $x \in x$ and $d_i$ denotes the contradictory degree and its attribute values. (Condition 2)

(iii) falsity-internal (briefly, F-internal) if $A_{v_{d_i}}{}^{-F}(x) \leq \lambda_{v_i}{}^{F}(x) \leq A_{v_{d_i}}{}^{+F}(x)$ for all $x \in x$ and $d_i$ denotes the contradictory degree and its attribute values. (Condition 3)

**Example: 3.4.6**

Let the attribute be "Weather" and the attribute values are {Sunny, Cloudy, Rain, Snow}. Let's consider the dominant value of the attribute Weather as Sunny.

| Degrees of contradiction | 0 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|
| Attribute values | Sunny | Cloudy | Rain | Snow |
| Degrees of appurtenance $A_V(x)$ | {([0.1,0.7],[0.2,0.4]),([0.4,0.5],[0.3,0.6]),([0.3,0.9],[0.6.0.8])} | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.2,0.5]),([0.4,0.8],[0.4.0.7])} | {([0.3,0.8],[0.5,0.6]),([0.3,0.5],[0.4,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.4,0.6],[0.5,0.5]),([0.1,0.8],[0.2,0.6]),([0.4,0.6],[0.5.0.5])} |
| $\lambda_V(x)$ | [0.2,0.3],[0.4,0.4],[0.5,0.8] | [0.3,0.5],[0.5,0.5],[0.5,0.6] | [0.4,0.6],[0.3,0.5],[0.2,0.4] | [0.5,0.5],[0.2,0.6],[0.4,0.5] |

**Definition: 3.4.7**

Let X be a universal set and V be a non-empty vague set. The plithogenic fuzzy cubic vague set $\mathbb{A}_p{}^v = \langle A_v , \lambda_v \rangle$ in X is called an external plithogenic fuzzy cubic vague set (EPFCVS) if $\lambda_{v_i}(x) \notin (A_{v_{d_i}}{}^{-}(x), A_{v_{d_i}}{}^{+}(x))$ for all $x \in X$ and $d_i$ denotes the contradictory degree and its attribute values.

**Example: 3.4.8**

Let the attribute be "Languages" and the attribute values are {Tamil, French, Malayalam, English}. Let's consider the dominant value of the attribute Languages as Tamil.

| Degrees of contradiction | 0 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|
| Attribute values | Tamil | French | Malayalam | English |
| Degrees of appurtenance $A_V(x)$ | ([0.3,0.4], [0.4,0.6]) | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.3,0.5]) | ([0.3,0.5], [0.4,0.5]) |
| $\lambda_V(x)$ | [0.2,0.7] | [0.1,0.6] | [0.4,0.6] | [0.2,0.7] |

**Definition: 3.4.9**

Let X be a universal set and V be a non-empty vague set. The plithogenic intuitionistic fuzzy cubic vague set $\mathbb{A}_p{}^v = \langle A_v , \lambda_v \rangle$ in X is called an external plithogenic intuitionistic fuzzy cubic vague set (IPIFCVS) if $\lambda_{v_i}(x) \notin (A_{v_{d_i}}{}^{-}(x), A_{v_{d_i}}{}^{+}(x))$ for all $x \in X$ and $d_i$ denotes the contradictory degree and its attribute values.

**Example: 3.4.10**

Let the attribute be "Country" and the attribute values are {India, Japan, Malaysia, Korea}. Let's consider the dominant value of the attribute country as India.

| Degrees of contradiction | 0 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|
| Attribute values | India | Japan | Malaysia | Korea |
| Degrees of appurtenance $A_V(x)$ | {([0.3,0.8],[0.5,0.6]),([ 0.2,0.7],[0.4.0.5])} | {([0.4,0.7],[0.3,0.5]),([ 0.3,0.6],[0.5.0.7])} | {([0.2,0.6],[0.3,0.6]),([ 0.4,0.8],[0.4.0.7])} | {([0.4,0.6],[0.5,0.5]),([ 0.4,0.6],[0.5.0.5])} |
| $\lambda_V(x)$ | [0.2,0.8],[0.1,0.7] | [0.3,0.6],[0.2,0.8] | [0.1,0.7],[0.2,0.3] | [0.1,0.6],[0.2,0.6] |

**Definition: 3.4.11**

Let X be a universal set and V be a non-empty vague set. The plithogenic neutrosophic cubic vague set $\mathbb{A}_p{}^v = \langle A_v , \lambda_v \rangle$ in X is said to be external plithogenic neutrosophic cubic vague set (EPNCVS) in X if it satisfies the following equations.

(i) truth-external (briefly, T- external) if $\lambda_{v_i}{}^{T}(x) \notin (A_{v_{d_i}}{}^{-T}(x), A_{v_{d_i}}{}^{+T}(x))$ for all $x \in x$ and $d_i$ denotes the contradictory degree and its attribute values. (Condition 4)

(ii) indeterminacy- external (briefly, I- external) if $\lambda_{v_i}{}^{I}(x) \notin (A_{v_{d_i}}{}^{-I}(x), A_{v_{d_i}}{}^{+I}(x))$ for all $x \in x$ and $d_i$ denotes the contradictory degree and its attribute values. (Condition 5)

(iii) falsity- external (briefly, F- external) if $\lambda_{v_i}{}^{F}(x) \notin (A_{v_{d_i}}{}^{-F}(x), A_{v_{d_i}}{}^{+F}(x))$ for all $x \in x$ and $d_i$ denotes the contradictory degree and its attribute values. (Condition 6)

**Example: 3.4.12**

Let the attribute be "Month" and the attribute values are {July, May, April, March}. Let's consider the dominant value of the attribute Month as July.

| Degrees of contradiction | 0 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|
| Attribute values | July | May | April | March |

| | | | | |
|---|---|---|---|---|
| Degrees of appurtenance $A_V(x)$ | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.2,0.5]),([0.4,0.8],[0.4.0.7])} | {([0.4,0.6],[0.5,0.5]),([0.3,0.8],[0.2,0.6]),([0.4,0.6],[0.5.0.5])} | {([0.3,0.8],[0.5,0.6]),([0.3,0.5],[0.4,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.4,0.6],[0.1,0.2]),([0.4,0.5],[0.3,0.6]),([0.4,0.6],[0.8.0.9])} |
| $\lambda_V(x)$ | [0.1,0.7],[0.3,0.8],[0.2,0.8] | [0.3,0.7],[0.2,0.9],[0.3,0.6] | [0.1,0.9],[0.2,0.7],[0.1,0.6] | [0.1,0.5],[0.2,0.8],[0.2,0.5] |

**4. Union and Intersection**

In this section, we introduce the definitions of union and intersection of plithogenic cubic vague sets with examples.

**Definition:4.1** Let $\mathbb{A}_p{}^v = \{\langle x, A_V(x), \lambda_V(x)\rangle : x \in X, v \in V\}$ and $\mathbb{B}_p{}^v = \{\langle x, B_V(x), \mu_V(x)\rangle : x \in X, v \in V\}$ be two plithogenic cubic vague sets in X then we have

1. $\mathbb{A}_p{}^v = \mathbb{B}_p{}^v$ if and only if $A_V(x) = B_V(x)$ and $\lambda_V(x) = \mu_V(x)$.
2. $\mathbb{A}_p{}^v$ and $\mathbb{B}_p{}^v$ are two plithogenic cubic vague sets in X then we define and denote P-order as $\mathbb{A}_p{}^v \leq_P \mathbb{B}_p{}^v$ if and only if $A_V(x) \subseteq B_V(x)$ and $\lambda_V(x) \leq \mu_V(x)$ for all $x \in X$.
3. $\mathbb{A}_p{}^v$ and $\mathbb{B}_p{}^v$ are two plithogenic cubic vague sets in X then we define and denote R-order as $\mathbb{A}_p{}^v \leq_R \mathbb{B}_p{}^v$ if and only if $A_V(x) \subseteq B_V(x)$ and $\lambda_V(x) \geq \mu_V(x)$ for all $x \in X$.

**Definition:4.2** Let $\mathbb{A}_p{}^v = \{\langle x, A_V(x), \lambda_V(x)\rangle : x \in X, v \in V\}$ and $\mathbb{B}_p{}^v = \{\langle x, B_V(x), \mu_V(x)\rangle : x \in X, v \in V\}$ be two plithogenic cubic vague sets in X. Then we have

1. $\mathbb{A}_p{}^v \cup_P \mathbb{B}_p{}^v = \{\langle (x), \sup(A_V(x), B_V(x)), \sup(\lambda_V(x), \mu_V(x))\rangle \mid x \in X, v \in V\}$ (P-union)
2. $\mathbb{A}_p{}^v \cap_P \mathbb{B}_p{}^v = \{\langle (x), \inf(A_V(x), B_V(x)), \inf(\lambda_V(x), \mu_V(x))\rangle \mid x \in X, v \in V\}$ (P-intersection)
3. $\mathbb{A}_p{}^v \cup_R \mathbb{B}_p{}^v = \{\langle (x), \sup(A_V(x), B_V(x)), \inf(\lambda_V(x), \mu_V(x))\rangle \mid x \in X, v \in V\}$ (R-union)
4. $\mathbb{A}_p{}^v \cap_R \mathbb{B}_p{}^v = \{\langle (x), \inf(A_V(x), B_V(x)), \sup(\lambda_V(x), \mu_V(x))\rangle \mid x \in X, v \in V\}$ (R-intersection)

**4.1 Plithogenic Fuzzy Cubic Vague Union and Intersection**

**Example: 4.1.1 (P-Order)**

The expert values between "color" and "height" and their values are Color = {red, blue, green} and Height = {tall, medium} then the object elements are characterized by the Cartesian product

Color × Height $=\{(\text{red}, \text{tall}), (\text{red}, \text{medium}), (\text{blue}, \text{tall}), (\text{blue}, \text{medium}), (\text{green}, \text{tall}), (\text{green}, \text{medium})\}$.

Let's consider the dominant value of attribute "Color" be "red" and of attribute "Height" be "tall". The attribute value contradiction degrees are:

c(red, red) = 0, c(red, blue) = $\frac{1}{3}$, c(red, green) = $\frac{2}{3}$, c(tall, tall) = 0, c(tall, medium) = $\frac{1}{3}$.

We have two plithogenic cubic vague sets A and B and we consider the fuzzy, intuitionistic fuzzy and neutrosophic appurtenance degree of attribute values to the sets.

$\mathbb{A}_p{}^v$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| Degrees of appurtenance $A_V(x)$ | ([0.2,0.3], [0.4,0.6]) | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.3,0.6]) | | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.4,0.5]) |
| $\lambda_V(x)$ | [0.2,0.5] | [0.2,0.4] | [0.1,0.8] | | [0.3,0.5] | [0.1,0.4] |

$\mathbb{B}_p{}^v$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |

| | | | | | | |
|---|---|---|---|---|---|---|
| Degrees of appurtenance $B_V(x)$ | ([0.3,0.4], [0.4,0.6]) | ([0.3,0.4], [0.4,0.5]) | ([0.2,0.3], [0.4,0.6]) | | ([0.3,0.4], [0.4,0.5]) | ([0.3,0.4], [0.4,0.5]) |
| $\mu_V(x)$ | [0.5,0.5] | [0.3,0.4] | [0.2,0.8] | | [0.4,0.5] | [0.4,0.4] |

Then P-Union denoted by $\mathbb{A}_p{}^v \cup_P \mathbb{B}_p{}^v$ and P-Intersection denoted by $\mathbb{A}_p{}^v \cap_P \mathbb{B}_p{}^v$

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| $A_V \cup B_V$ | ([0.3,0.4], [0.4,0.6]) | ([0.3,0.4], [0.4,0.5]) | ([0.2,0.3], [0.4,0.6]) | | ([0.3,0.4], [0.4,0.5]) | ([0.3,0.4], [0.4,0.5]) |
| $\lambda_V \cup \mu_V$ | [0.5,0.5] | [0.3,0.4] | [0.2,0.8] | | [0.4,0.5] | [0.4,0.4] |
| $A_V \cap B_V$ | ([0.2,0.3], [0.4,0.6]) | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.3,0.6]) | | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.4,0.5]) |
| $\lambda_V \cap \mu_V$ | [0.2,0.5] | [0.2,0.4] | [0.1,0.8] | | [0.3,0.5] | [0.1,0.4] |

**Example: 4.1.2 (R-Order)**

$\mathbb{A}_p{}^v$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| Degrees of appurtenance $A_V(x)$ | ([0.2,0.3], [0.4,0.6]) | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.3,0.6]) | | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.4,0.5]) |
| $\lambda_V(x)$ | [0.4,0.5] | [0.3,0.4] | [0.2,0.8] | | [0.4,0.5] | [0.4,0.4] |

$\mathbb{B}_p{}^v$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| Degrees of appurtenance $B_V(x)$ | ([0.3,0.4], [0.4,0.6]) | ([0.3,0.4], [0.4,0.5]) | ([0.2,0.3], [0.4,0.6]) | | ([0.3,0.4], [0.4,0.5]) | ([0.3,0.4], [0.4,0.5]) |
| $\mu_V(x)$ | [0.3,0.5] | [0.2,0.4] | [0.1,0.8] | | [0.3,0.5] | [0.1,0.4] |

Then R-Union denoted by $\mathbb{A}_p{}^v \cup_R \mathbb{B}_p{}^v$ and R-Intersection denoted by $\mathbb{A}_p{}^v \cap_R \mathbb{B}_p{}^v$

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| $A_V \cup B_V$ | ([0.3,0.4], [0.4,0.6]) | ([0.3,0.4], [0.4,0.5]) | ([0.2,0.3], [0.4,0.6]) | | ([0.3,0.4], [0.4,0.5]) | ([0.3,0.4], [0.4,0.5]) |
| $\lambda_V \cup \mu_V$ | [0.3,0.5] | [0.2,0.4] | [0.1,0.8] | | [0.3,0.5] | [0.1,0.4] |
| $A_V \cap B_V$ | ([0.2,0.3], [0.4,0.6]) | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.3,0.6]) | | ([0.2,0.3], [0.4,0.5]) | ([0.1,0.3], [0.4,0.5]) |
| $\lambda_V \cap \mu_V$ | [0.4,0.5] | [0.3,0.4] | [0.2,0.8] | | [0.4,0.5] | [0.4,0.4] |

**4.2 Plithogenic Intuitionistic Fuzzy Cubic Vague Union and Intersection**

**Example: 4.2.1 (P-Order)**

$\mathbb{A}_p{}^v$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| Degrees of appurtenance $A_V(x)$ | {([0.2,0.6],[0.5,0.5]),([0.4,0.8],[0.5.0.5])} | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.4.0.7])} | {([0.4,0.5],[0.1,0.4]),([0.5,0.6],[0.6.0.9])} | | {([0.1,0.7],[0.2,0.4]),([0.3,0.9],[0.6.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.7.0.8])} |
| $\lambda_V(x)$ | [0.2,0.5],[0.5,0.8] | [0.4,0.5],[0.5,0.6] | [0.2,0.4],[0.6,0.8] | | [0.1,0.5],[0.5,0.9] | [0.2,0.4],[0.6,0.8] |

$\mathbb{B}_p{}^v$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| Degrees of appurtenance $B_V(x)$ | {([0.3,0.8],[0.5,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.3,0.8],[0.5,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.4,0.6],[0.5.0.5]),([0.4,0.6],[0.5.0.5])} | | {([0.6,0.9],[0.2,0.5]),([0.1,0.4],[0.5.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.7.0.8])} |
| $\mu_V(x)$ | [0.2,0.5],[0.5,0.8] | [0.4,0.5],[0.5,0.6] | [0.2,0.4],[0.6,0.8] | | [0.1,0.5],[0.5,0.9] | [0.2,0.4],[0.6,0.8] |

Then P-Union denoted by $\mathbb{A}_p{}^v \cup_P \mathbb{B}_p{}^v$ and P-Intersection denoted by $\mathbb{A}_p{}^v \cap_P \mathbb{B}_p{}^v$

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| $A_V \cup B_V$ | {([0.3,0.8],[0.5,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.3,0.8],[0.5,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.4,0.6],[0.5,0.5]),([0.4,0.6],[0.5.0.5])} | | {([0.6,0.9],[0.2,0.5]),([0.1,0.4],[0.5.0.8]) | {([0.2,0.6],[0.2,0.3]),([0.4,0.8],[0.7.0.8])} |
| $\lambda_V \cup \mu_V$ | [0.2,0.5], [0.5,0.8] | [0.4,0.5],[0.5,0.6] | [0.2,0.4],[0.6,0.8] | | [0.1,0.5],[0.5,0.9] | [0.2,0.4],[0.6,0.8] |
| $A_V \cap B_V$ | {([0.2,0.6],[0.5,0.5]),([0.4,0.8],[0.5.0.5])} | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.4.0.7])} | {([0.4,0.5],[0.1,0.4]),([0.5,0.6],[0.6.0.9])} | | {([0.1,0.7],[0.2,0.4]),([0.3,0.9],[0.6.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.7.0.8])} |
| $\lambda_V \cap \mu_V$ | [0.2,0.5],[0.5,0.8] | [0.4,0.5],[0.5,0.6] | [0.2,0.4],[0.6,0.8] | | [0.1,0.5],[0.5,0.9] | [0.2,0.4],[0.6,0.8] |

**Example: 4.2.2 (R-Order)**

$\mathbb{A}_p{}^v$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| Degrees of appurtenance $A_V(x)$ | {([0.2,0.6],[0.5,0.5]),([0.4,0.8],[0.5.0.5])} | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.4.0.7])} | {([0.4,0.5],[0.1,0.4]),([0.5,0.6],[0.6.0.9])} | | {([0.1,0.7],[0.2,0.4]),([0.3,0.9],[0.6.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.7.0.8])} |
| $\lambda_V(x)$ | [0.4,0.7],[0.3,0.6] | [0.1,0.3],[0.7,0.9] | [0.7,0.8],[0.2,0.3] | | [0.5,0.6],[0.4,0.5] | [0.3,0.5],[0.5,0.7] |

$\mathbb{B}_p{}^v$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| Degrees of appurtenance $B_V(x)$ | {([0.3,0.8],[0.5,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.3,0.8],[0.5,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.4,0.6],[0.5.0.5]),([0.4,0.6],[0.5.0.5])} | | {([0.6,0.9],[0.2,0.5]),([0.1,0.4],[0.5.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.7.0.8])} |
| $\mu_V(x)$ | [0.4,0.7],[0.3,0.6] | [0.1,0.3],[0.7,0.9] | [0.7,0.8],[0.2,0.3] | | [0.5,0.6],[0.4,0.5] | [0.3,0.5],[0.5,0.7] |

Then R-Union denoted by $\mathbb{A}_p{}^v \cup_R \mathbb{B}_p{}^v$ and R-Intersection denoted by $\mathbb{A}_p{}^v \cap_R \mathbb{B}_p{}^v$

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| $A_V \cup B_V$ | {([0.3,0.8],[0.5,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.3,0.8],[0.5,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.4,0.6],[0.5.0.5]),([0.4,0.6],[0.5.0.5])} | | {([0.6,0.9],[0.2,0.5]),([0.1,0.4],[0.5.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.7.0.8])} |
| $\lambda_V \cup \mu_V$ | [0.4,0.7],[0.3,0.6] | [0.1,0.3],[0.7,0.9] | [0.7,0.8],[0.2,0.3] | | [0.5,0.6],[0.4,0.5] | [0.3,0.5],[0.5,0.7] |
| $A_V \cap B_V$ | {([0.2,0.6],[0.5,0.5]),([0.4,0.8],[0.5.0.5])} | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.4.0.7])} | {([0.4,0.5],[0.1,0.4]),([0.5,0.6],[0.6.0.9])} | | {([0.1,0.7],[0.2,0.4]),([0.3,0.9],[0.6.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.7.0.8])} |
| $\lambda_V \cap \mu_V$ | [0.4,0.7],[0.3,0.6] | [0.1,0.3],[0.7,0.9] | [0.7,0.8],[0.2,0.3] | | [0.5,0.6],[0.4,0.5] | [0.3,0.5],[0.5,0.7] |

**4.3 Plithogenic Neutrosophic Cubic Vague Union and Intersection**

**Example: 4.3.1 (P-Order)**

$\mathbb{A}_p{}^v$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | Tall | Medium |
| Degrees of appurtenance $A_V(x)$ | {([0.2,0.6],[0.5,0.5]),([0.4,0.6],[0.2,0.6]),([0.4,0.8],[0.5.0.5])} | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.4,0.6]),([0.4,0.8],[0.4.0.7])} | {([0.4,0.5],[0.1,0.4]),([0.5,0.8],[0.2,0.6]),([0.5,0.6],[0.6.0.9])} | | {([0.1,0.7],[0.2,0.4]),([0.4,0.7],[0.3,0.6]),([0.3,0.9],[0.6.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.4,0.6]),([0.5,0.8],[0.7.0.8])} |

| | | | | | | |
|---|---|---|---|---|---|---|
| $\lambda_V(x)$ | [0.2,0.5],[0.2,0.3],[0.5,0.8] | [0.4,0.5],[0.5,0.5],[0.5,0.6] | [0.2,0.4],[0.2,0.4],[0.6,0.8] | | [0.1,0.5],[0.3,0.4],[0.5,0.9] | [0.2,0.4],[0.3,0.5],[0.6,0.8] |

$\mathbb{B}_p{}^v$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | Medium |
| Degrees of appurtenance $B_V(x)$ | {([0.3,0.8],[0.5,0.6]),([0.3,0.5],[0.1,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.3,0.8],[0.5,0.6]),([0.3,0.5],[0.2,0.5]),([0.2,0.7],[0.4.0.5])} | {([0.4,0.6],[0.5,0.5]),([0.1,0.5],[0.1,0.3]),([0.4,0.6],[0.5.0.5])} | | {([0.6,0.9],[0.2,0.5]),([0.3,0.5],[0.2,0.5]),([0.1,0.4],[0.5.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.4,0.6]),([0.5,0.8],[0.7.0.8])} |
| $\mu_V(x)$ | [0.2,0.5],[0.3,0.4],[0.5,0.8] | [0.4,0.5],[0.6,0.6],[0.5,0.6] | [0.2,0.4],[0.3,0.5],[0.6,0.8] | | [0.1,0.5],[0.5,0.6],[0.5,0.9] | [0.2,0.4],[0.7,0.8],[0.6,0.8] |

Then P-Union denoted by $\mathbb{A}_p{}^v \cup_P \mathbb{B}_p{}^v$ and P-Intersection denoted by $\mathbb{A}_p{}^v \cap_P \mathbb{B}_p{}^v$ are

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | Green | | tall | medium |
| $A_V \cup B_V$ | {([0.3,0.8],[0.5,0.6]),([0.3,0.5],[0.1,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.3,0.8],[0.5,0.6]),([0.3,0.5],[0.2,0.5]),([0.2,0.7],[0.4.0.5])} | {([0.4,0.6],[0.5,0.5]),([0.1,0.5],[0.1,0.3]),([0.4,0.6],[0.5.0.5])} | | {([0.6,0.9],[0.2,0.5]),([0.3,0.5],[0.2,0.5]),([0.1,0.4],[0.5.0.8]) | {([0.2,0.6],[0.2,0.3]),([0.4,0.7],[0.3,0.5]),([0.4,0.8],[0.7.0.8])} |
| $\lambda_V \cup \mu_V$ | [0.2,0.5],[0.2,0.3],[0.5,0.8] | [0.4,0.5],[0.5,0.5],[0.5,0.6] | [0.2,0.4],[0.2,0.4],[0.6,0.8] | | [0.1,0.5],[0.3,0.4],[0.5,0.9] | [0.2,0.4],[0.3,0.5],[0.6,0.8] |
| $A_V \cap B_V$ | {([0.2,0.6],[0.5,0.5]),([0.4,0.6],[0.2,0.6]),([0.4,0.8],[0.5.0.5])} | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.4,0.6]),([0.4,0.8],[0.4.0.7])} | {([0.4,0.5],[0.1,0.4]),([0.5,0.8],[0.2,0.6]),([0.5,0.6],[0.6.0.9])} | | {([0.1,0.7],[0.2,0.4]),([0.4,0.7],[0.3,0.6]),([0.3,0.9],[0.6.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.4,0.6]),([0.5,0.8],[0.7.0.8])} |
| $\lambda_V \cap \mu_V$ | [0.2,0.5],[0.3,0.4],[0.5,0.8] | [0.4,0.5],[0.6,0.6],[0.5,0.6] | [0.2,0.4],[0.3,0.5],[0.6,0.8] | | [0.1,0.5],[0.5,0.6],[0.5,0.9] | [0.2,0.4],[0.7,0.8],[0.6,0.8] |

**Example: 4.3.2 (R-Order)**

$\mathbb{A}_p{}^v$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | Medium |
| Degrees of appurtenance $A_V(x)$ | {([0.2,0.6],[0.5,0.5]),([0.4,0.6],[0. | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.4,0.6]),([0.4,0.8],[0.4.0.7])} | {([0.4,0.5],[0.1,0.4]),([0.5,0.8],[0.2, | | {([0.1,0.7],[0.2,0.4]),([0.4,0.7],[0.3, | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.4,0.6 |

| | | | | | | |
|---|---|---|---|---|---|---|
| | 2,0.6]),([0.4,0.68],[0.5.0.5])} | | 0.6]),([0.5,0.6],[0.6.0.9])} | | 0.6]),([0.3,0.9],[0.6.0.8])} | ]),([0.5,0.8],[0.7.0.8])} |
| $\lambda_V(x)$ | [0.4,0.7],[0.3,0.4],[0.3,0.6] | [0.1,0.3],[0.5,0.6],[0.7,0.9] | [0.7,0.8],[0.3,0.5],[0.2,0.3] | | [0.5,0.6],[0.5,0.6],[0.4,0.5] | [0.3,0.5],[0.6,0.6],[0.5,0.7] |

$\mathbb{B}_p{}^v$:

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | Medium |
| Degrees of appurtenance $B_V(x)$ | {([0.3,0.8],[0.5,0.6]),([0.3,0.5],[0.1,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.3,0.8],[0.5,0.6]),([0.3,0.5],[0.2,0.5]),([0.2,0.7],[0.4.0.5])} | {([0.4,0.6],[0.5,0.5]),([0.1,0.5],[0.1,0.3]),([0.4,0.6],[0.5.0.5])} | | {([0.6,0.9],[0.2,0.5]),([0.3,0.5],[0.2,0.5]),([0.1,0.4],[0.5.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.4,0.6]),([0.5,0.8],[0.7.0.8])} |
| $\mu_V(x)$ | [0.4,0.7],[0.2,0.3],[0.3,0.6] | [0.1,0.3],[0.3,0.4],[0.7,0.9] | [0.7,0.8],[0.1,0.3],[0.2,0.3] | | [0.5,0.6],[0.2,0.],[0.4,0.5] | [0.3,0.5],[0.4,0.5],[0.5,0.7] |

Then R-Union denoted by $\mathbb{A}_p{}^v \cup_R \mathbb{B}_p{}^v$ and R-Intersection denoted by $\mathbb{A}_p{}^v \cap_R \mathbb{B}_p{}^v$

| Degrees of contradiction | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | | 0 | $\frac{1}{3}$ |
|---|---|---|---|---|---|---|
| Attribute values | red | blue | green | | tall | medium |
| $A_V \cup B_V$ | {([0.3,0.8],[0.5,0.6]),([0.3,0.5],[0.1,0.6]),([0.2,0.7],[0.4.0.5])} | {([0.3,0.8],[0.5,0.6]),([0.3,0.5],[0.2,0.5]),([0.2,0.7],[0.4.0.5])} | {([0.4,0.6],[0.5,0.5]),([0.1,0.5],[0.1,0.3]),([0.4,0.6],[0.5.0.5])} | | {([0.6,0.9],[0.2,0.5]),([0.3,0.5],[0.2,0.5]),([0.1,0.4],[0.5.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.4,0.6]),([0.5,0.8],[0.7.0.8])} |
| $\lambda_V \cup \mu_V$ | [0.4,0.7],[0.2,0.3],[0.3,0.6] | [0.1,0.3],[0.3,0.4],[0.7,0.9] | [0.7,0.8],[0.1,0.3],[0.2,0.3] | | [0.5,0.6],[0.2,0.],[0.4,0.5] | [0.3,0.5],[0.4,0.5],[0.5,0.7] |
| $A_V \cap B_V$ | {([0.2,0.6],[0.5,0.5]),([0.4,0.6],[0.2,0.6]),([0.4,0.8],[0.5.0.5])} | {([0.2,0.6],[0.3,0.6]),([0.4,0.8],[0.4,0.6]),([0.4,0.8],[0.4.0.7])} | {([0.4,0.5],[0.1,0.4]),([0.5,0.8],[0.2,0.6]),([0.5,0.6],[0.6.0.9])} | | {([0.1,0.7],[0.2,0.4]),([0.4,0.7],[0.3,0.6]),([0.3,0.9],[0.6.0.8])} | {([0.2,0.5],[0.2,0.3]),([0.5,0.8],[0.4,0.6]),([0.5,0.8],[0.7.0.8])} |
| $\lambda_V \cap \mu_V$ | [0.4,0.7],[0.3,0.4],[0.3,0.6] | [0.1,0.3],[0.5,0.6],[0.7,0.9] | [0.7,0.8],[0.3,0.5],[0.2,0.3] | | [0.5,0.6],[0.5,0.6],[0.4,0.5] | [0.3,0.5],[0.6,0.6],[0.5,0.7] |

**Theorem: 4.3.2**

Let X be a non-empty set and $\mathbb{A}_p{}^v = \langle A_v, \lambda_v \rangle$ be a PNCVS in X which is not external. Then there exists $x \in X$ such that $\lambda_{v_i}{}^T(x) \in (A_{v_{d_i}}{}^{-T}(x), A_{v_{d_i}}{}^{+T}(x))$ , $\lambda_{v_i}{}^I(x) \in (A_{v_{d_i}}{}^{-I}(x), A_{v_{d_i}}{}^{+I}(x))$ , $\lambda_{v_i}{}^F(x) \in (A_{v_{d_i}}{}^{-F}(x), A_{v_{d_i}}{}^{+F}(x))$ where $d_i$ denotes the contradictory degree and its attribute values.

**Proof:** Straight forward

**Theorem: 4.3.3**

Let X be a non-empty set and $\mathbb{A}_p{}^v$ = $\langle A_v , \lambda_v \rangle$ be a PNCVS in X. If $\mathbb{A}_p{}^v$ is both T-internal and T-external, then, $(\forall\, x \in X)\left(\lambda_{v_i}{}^T(x) \in \left\{(A_{v_{d_i}}{}^{-T}(x) | x \in X\right\} \cup \left\{A_{v_{d_i}}{}^{+T}(x) | x \in \mathrm{X}\right\}\right)$ where $d_i$ denotes the contradictory degree and its attribute values.

**Proof:** Consider the conditions 1 and 4 which implies that $A_{v_{d_i}}{}^{-T}(x) \leq \lambda_{v_i}{}^T(x) \leq A_{v_{d_i}}{}^{+T}(x)$ and $\lambda_{v_i}{}^T(x) \notin (A_{v_{d_i}}{}^{-T}(x), A_{v_{d_i}}{}^{+T}(x))$ for all $x \in X$. Then it follows that $\lambda_{v_i}{}^T(x) = A_{v_{d_i}}{}^{-T}(x)$ or $\lambda_{v_i}{}^T(x) = A_{v_{d_i}}{}^{+T}(x)$, and hence $\left(\lambda_{v_i}{}^T(x) \in \left\{(A_{v_{d_i}}{}^{-T}(x) | x \in X\right\} \cup \left\{A_{v_{d_i}}{}^{+T}(x) | x \in \mathrm{X}\right\}\right)$ where $d_i$ denotes the contradictory degree and its attribute values. Hence proved.

**Remark:** Similarly the consequent theorems holds for indeterminacy and falsity values.

**Theorem: 4.3.4**

Let X be a non-empty set and $\mathbb{A}_p{}^v$ = $\langle A_v , \lambda_v \rangle$ be a PNCVS in X. If $\mathbb{A}_p{}^v$ is both I-internal and I-external, then, $(\forall\, x \in X)\left(\lambda_{v_i}{}^I(x) \in \left\{(A_{v_{d_i}}{}^{-I}(x) | x \in X\right\} \cup \left\{A_{v_{d_i}}{}^{+I}(x) | x \in \mathrm{X}\right\}\right)$ where $d_i$ denotes the contradictory degree and its attribute values.

**Theorem: 4.3.5**

Let X be a non-empty set and $\mathbb{A}_p{}^v$ = $\langle A_v , \lambda_v \rangle$ be a PNCVS in X. If $\mathbb{A}_p{}^v$ is both F-internal and F-external, then, $(\forall\, x \in X)\left(\lambda_{v_i}{}^F(x) \in \left\{(A_{v_{d_i}}{}^{-F}(x) | x \in X\right\} \cup \left\{A_{v_{d_i}}{}^{+F}(x) | x \in \mathrm{X}\right\}\right)$ where $d_i$ denotes the contradictory degree and its attribute values.

**Definition: 4.3.6**

Let X be a non-empty set. The complement of $\mathbb{A}_p{}^v$ = $\langle A_v , \lambda_v \rangle$ is said to be PNCVS, $(\mathbb{A}_p{}^v)^c$ = $\langle A^C{}_v, \lambda^C{}_v \rangle$

where $A_v{}^c = \{(A^C{}_{v_{d_i}}{}^T(x),\ A^C{}_{v_{d_i}}{}^I(x), A^C{}_{v_{d_i}}{}^F(x))\}$ is an interval valued PNCVS in X and $\lambda^C{}_v = \{(\lambda^C{}_v{}^T(x), \lambda^C{}_v{}^I(x), \lambda^C{}_v{}^F(x))\}$ is a neutrosophic set in Y.

**Theorem: 4.3.7**

Let X be a non-empty set and $\mathbb{A}_p{}^v$ = $\langle A_v , \lambda_v \rangle$ be a PNCVS in X. If $\mathbb{A}_p{}^v$ is both T-internal and T-external, then the complement $(\mathbb{A}_p{}^v)^c$ = $\langle A^C{}_v, \lambda^C{}_v \rangle$ of $\mathbb{A}_p{}^v$ = $\langle A_v , \lambda_v \rangle$ is a T-internal and T-external PNCVS in X.

**Proof:**

Let X be a non-empty set and if $\mathbb{A}_p{}^v$ = $\langle A_v , \lambda_v \rangle$ is an T-internal and T-external PNCVS in X, then $A_{v_{d_i}}{}^{-T}(x) \leq \lambda_{v_i}{}^T(x) \leq A_{v_{d_i}}{}^{+T}(x)$ and $\lambda_{v_i}{}^T(x) \notin (A_{v_{d_i}}{}^{-T}(x), A_{v_{d_i}}{}^{+T}(x))$ for all $x \in X$. It follows that $1 - A_{v_{d_i}}{}^{-T}(x) \leq 1 - \lambda_{v_i}{}^T(x) \leq 1 - A_{v_{d_i}}{}^{+T}(x)$ and $1 - \lambda_{v_i}{}^T(x) \notin (1 - A_{v_{d_i}}{}^{-T}(x), 1 - A_{v_{d_i}}{}^{+T}(x))$. Therefore $(\mathbb{A}_p{}^v)^c$ = $\langle A^C{}_v, \lambda^C{}_v \rangle$ of $\mathbb{A}_p{}^v$ = $\langle A_v , \lambda_v \rangle$ is a T-internal and T-external PNCVS in X.

**Remark:** Similarly the consequent theorems holds for indeterminacy and falsity values.

**Theorem: 4.3.8**

Let X be a non-empty set and $\mathbb{A}_p{}^v$ = $\langle A_v , \lambda_v \rangle$ be a PNCVS in X. If $\mathbb{A}_p{}^v$ is both I-internal and I-external, then the complement $(\mathbb{A}_p{}^v)^c$ = $\langle A^C{}_v, \lambda^C{}_v \rangle$ of $\mathbb{A}_p{}^v$ = $\langle A_v , \lambda_v \rangle$ is a I-internal and I-external PNCVS in X.

**Theorem: 4.3.9**

Let X be a non-empty set and $\mathbb{A}_p{}^v = \langle A_v, \lambda_v \rangle$ be a PNCVS in X. If $\mathbb{A}_p{}^v$ is both F-internal and F-external, then the complement $(\mathbb{A}_p{}^v)^c = \langle A^C{}_v, \lambda^C{}_v \rangle$ of $\mathbb{A}_p{}^v = \langle A_v, \lambda_v \rangle$ is a F-internal and F-external PNCVS in X.

**5 Conclusion and Future Work**

The objective of this paper is to define the concept of plithogenic cubic vague set and its generalization; plithogenic fuzzy cubic vague set, plithogenic intuitionistic fuzzy cubic vague set, plithogenic neutrosophic cubic vague set with examples. Its corresponding internal and external cubic vague sets are also defined. We studied the union and intersection of P and R order also some basic properties. The work presented in this paper delivers the theoretical framework for further study on plithogenic cubic vague set. In the future work, we will study AND and OR operations, similarity measures of plithogenic cubic vague set.

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# Enumeration of Neutrosophic Involutions over Finite Commutative Neutrosophic Rings

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## Abstract:

A finite commutative ring involution is the multiplicative inverse of the element attribute R is the element itself. This classical characteristic of a finite commutative ring makes Neutrosophic involutions possible, which are counted, listed and assessed in this work. Assume that the Neutrosophic ring R(I) is the finite commutative ring with unity 1 over the ring R under the indeterminate $\boldsymbol{I}$. We first establish some useful necessary and sufficient conditions for the Neutrosophic components of the type $\boldsymbol{a+bI}$ is involutory in order to understand how to count Neutrosophic involutions of R(I). The behavior of the Neutrosophic composition table for identifying Neutrosophic involutions and counting the number of 1s that appear on the primary diagonal of the composition table of R(I) is also investigated in this work.

**Keywords:** Involutions, Neutrosophic Involutions, Neutrosophic Units, Pure Neutrosophic Involution, Neutrosophic Ring.

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### 1. Introduction

An involution is a special element in any ring $R$ with unity and it is a self-multiplicative inverse element under multiplication defined over $R$. For a finite ring $R$, let $\mathcal{J}n(R)$ denote the set of involutions of$R$, and $|\mathcal{J}n(R)|$ represents the number of involutions of $R$. Because R's involutions are systematically arranged mathematical objects that don't require any additional resources to implement, they have received a great deal of attention for their potential applications in security systems, coding-decoding systems, combinatorial designs, the creation of self-intelligent systems, etc. [1–11].Due to the fact that involutions have been crucial to the development, interpretation, and design of electronic devices. Every commutative ring with unity is known to contain at least one involution. However, the theory of finite commutative rings has two intriguing subcategories. One is cyclic rings and theother is non-cyclic rings. A ring $R$ is called cyclic if the group $(R,+)$ is a cyclic group under addition defined by $R$. Otherwise, it is called a noncyclic ring. Every cyclic ring is commutative, and a finite cyclic ring with unity of order $n$ is isomorphic to the ring $Z_n$, integersunder addition and multiplication modulo $n$.On the other hand, the rings $Z_m \times Z_n$ and $Z_n[i]$ are all noncyclic rings for every integer$m, n > 1$.

In 1987, Smarandache and Vasantha Kandasamy introduced a basic setup of the theory of Neutrosophic structures through indeterminacy $I$, because simply they had a natural and necessary role of $I$ to play in the development of the Neutrosophic algebraic systems. Now a days, they rapidly become flourishing systems, because the structure $R$ and indeterminate$I$ are needful in modern mathematical systems and many intelligent systems like Neutrosophic decision systems, Neutrosophic error detection systems, algorithms for digital communication systems [12-15];all these typesof systems employ the Neutrosophic structure $R(I)$.

Because they had a natural and essential role for indeterminate $I$ to play in the growth of the Neutrosophic algebraic systems, Smarandache and Vasantha Kandasamy established the theory of Neutrosophic structures through indeterminacy $I$ in 1987. They are now developing rapidly because many smart systems and quality systems, such as product quality systems, Neutrosophic virtual reality systems, and uncertainty systems [16–18], all use the Neutrosophic structure and logic, which is necessary in modern mathematical systems.

The classic Neutrosophic Rings, written by Florentin Smarandache and Vasantha Kandasamy, and published in 2006, sparked the growth of two contemporary mathematics fields that are closely related to one another: The mathematical concept of "Neutrosophic ring" and Neutrosophic logic. The value of the symbols T(True), F(False), and I(Indeterminate) and their corresponding laws was illustrated in Chapter 2 of Florentin Smarandache and Vasantha Kandasamy's book [19].Neutrosophic logic is interested in how we think in order to draw conclusions about mathematics. That book will help us in studying this paper. Now starts a simple introduction about the structure$R(I)$. Mathematically, a Neutrosophic ring is a system with the following components: a ring R, an indeterminate I, two Neutrosophic binary operations on R, and a set of axioms that the elements of R satisfy via the indeterminate I. For any ring $R$, there exists a new structure $R(I)$, called a Neutrosophic ring, and is engendered by $R$ and $I$, which is represented by a Neutrosophic set "$R(I) = \langle R, I \rangle = \{a + bI : a, b \in R \ \ and \ \ I^2 = I\}$", where $I$ is the indeterminate of the system with algebraic properties: $0I = 0$, $1I = I$, $I^2 = I$, and $I^{-1}$ does not exist The Neutrosophic set $R(I) = \langle R, I \rangle$ of Neutrosophic elements of the form $a + bI$ forms a Neutrosophic ring under Neutrosophic addition "$(a + bI) + (c + dI) = (a + c) + (b + d)I$", and Neutrosophic multiplication "$(a + bI)(c + dI) = ac + (ad + bc + bd)I$", for every $a + bI$ and $c + dI$ in $R(I)$.`

In Neutrosophic algebra, the algebraic structures $R$ and $R(I)$ areplaying dominant roles, and they are also specific mathematical tools for developing and studying many Neutrosophic research fields like QuadrupleNeutrosophic rings, Neutrosophic zero rings, Neutrosophic number theory, Neutrosophic Boolean rings, Neutrosophic vector spaces, Refined Neutrosophic rings, and so on. For example, see [20].

The purpose of this paper is to prepare and enumerate the Neutrosophic involutions in the Neutrosophic group of units of a finite commutative Neutrosophic ring with unity and to examine and compare the properties of the classical involutions in a group of units. For this first, we shall define involutions in various fields of mathematics and their other related algebraic concepts. Generally, in modern mathematics and other related computational systems, involution is a map $f$ and it is equal to its inverse. This means that $f(f(x)) = x$ for all $x$ in the domain of a function $f$. So, the involution is a bijection. For this reason, many fields in modern mathematics contain the term involution such as Group theory, Ring theory, and Vector spaces. Moreover, in the Euclidean and the Projective geometry, the involution is a reflection through the origin, and an involution is a projectivity of period 2, respectively. In mathematical logic, the operation of complement in Boolean algebra is called Boolean involution, and in classical logic, the negation that satisfies the law of double negation is called involution. Lastly, in Computer science, the XOR bitwise operation with a given value for one constraint is also an involution, and RC4 cryptographic cipher is involution, as encryption and decryption operations use the same map.

In [21-23],there is a classical and simple problem between the composition table and the corresponding finite ring $R$ with unity1, that is, how many $1s$ appear in the principle diagonal of the composition table of$R$? This question produces the number of solutions of the equation$a^2 = 1$ in a finite ring $R$ with unity1. This paper extends this procedure to Neutrosophic rings $R(I)$ for enumerating Neutrosophic involutions, and we shall show that the Neutrosophic involutions to the Neutrosophic ring $R(I)$ over the finite commutative $R$ with unity 1come in a multiple of four through the relations $\mathcal{J}n(R(I)) = \mathcal{J}n(R) \cup (1-2I)\mathcal{J}n(R)$ and $|\mathcal{J}n(R(I))| = 2|\mathcal{J}n(R)|$.

## 2. Properties of Finite Neutrosophic Fields

This section introduces the concept of Neutrosophic involution and shows how to determine the number of such Neutrosophic involutions. Recall that the element $a$ in $R$ is involution if $a^2 = 1$, and the set of involutions of $R$ is $\mathcal{J}n(R)$ and notated as $\mathcal{J}n(R) = \{a \in R: a = a^{-1}\}$. For conveniently, it can be defined as$\mathcal{J}n(R) = \{a \in R: a^2 = 1\}$. For example, $\mathcal{J}n(Z_8) = \{1,3,5,7\}$, $\mathcal{J}n(Z_{10}) = \{1,9\}$, and $\mathcal{J}n(Z_{12}) = \{1,5,7,11\}$. Consequently, any undefined notions and results of classical involutions are standard as in [21].

Our next definition provides a considerably more efficient variant of this classical involution of a finite commutative ring $R$ with unity 1.

**Definition 2.1.**We say that a Neutrosophic element $a + bI$ of a Neutrosophic ring $R(I)$ is a Neutrosophic involution if$(a + bI)^2 = 1$, where $1 = 1 + 0I$ is the unity in$R(I)$.

The set of Neutrosophic involutions of $R(I)$ is denoted by $\mathcal{J}n(R(I))$ with the conditions

**(1)** $\mathcal{J}n(R) \subseteq \mathcal{J}n(R(I))$

**(2)** $\mathcal{J}n(R(I)) = \mathcal{J}n(U(R)) \cup \mathcal{J}n\left(U(R(I))\right)$,

where $U(R)$ and $U(R(I))$ are units and Neutrosophic units of $R$ and $R(I)$, respectively.

Now we begin our discussion with two simple examples.

**Example 2.2.**Neutrosophic involution, by definition, the involutions of the Neutrosophic ring $Z_3(I) = \{a + bI: a, b \in Z_3 \ \ and \ \ I^2 = I\}$ is the set $\mathcal{J}n(Z_3(I)) = \{1,2,1-2I, 2-I\}$, where $1^2 = 1$, $2^2 = 1$, $(1-2I)^2 = 1$, and $(2-I)^2 = 1$.

**Example 2.3.** Because $1^2 = 1$, $3^2 = 1$, $(1-2I)^2 = 1$, and $(3-2I)^2 = 1$, the involutions of the Neutrosophic ring $Z_4(I) = \{a + bI: a, b \in Z_4 \ \ and \ \ I^2 = I\}$ is the set $\mathcal{J}n(Z_3(I)) = \{1,3,1-2I, 3-2I\}$.

The above examplespresent the following two confluences of involutions and Neutrosophic involutions.

(1) $(a + bI)^2 = 1$ in $R(I)$ if and only if $a^2 = 1$ in $R$.

(2) $a + bI$ is a Neutrosophic involution implies $b + aI$ need not be a Neutrosophic involution, and vice versa.

These two confluences proposed the following necessary and sufficient conditions on $a$ and $b$ for $a + bI$ is a Neutrosophic involution.

**Theorem 2.4**. A necessary and sufficient condition for the Neutrosophic element $a + bI$ is a Neutrosophic involution in $R(I)$ is $(a - 2aI)^2 = a^2$.

**Proof.** Let$a + bI$ be a nonzero element in $R(I)$. Then there exists $(a + bI)^2$ in $R(I)$ such that

$a + bI$ be a Neutrosophic involution in $R(I) \Leftrightarrow (a + bI)^2 = 1$

$\Leftrightarrow a^2 + b^2 I + 2abI = 1$

$\Leftrightarrow a^2 = 1$, and $b^2 + 2ab = 0$

$\Leftrightarrow a^2 = 1$, and $b(b + 2a) = 0$ in $R$.

Let us starts two cases on the element $b$ in $R$.

**Case 1**. Suppose $b = 0$ in $R$. Then the Neutrosophic form reduces to classical form. This case concludes that

$a + bI$ be a Neutrosophic involution in $R(I) \Leftrightarrow a^2 = 1$, and $b = 0$ in $R$.

**Case 1**. Suppose $b \neq 0$ in $R$. Then

$a + bI$ be a Neutrosophic involution in $R(I) \Leftrightarrow a^2 = 1$, and $b + 2a = 0$ in $R$.

Therefore, $a^2 = 1$, and $b = -2a$ in $R$. These two conditions confirm that

$(a - 2aI)^2 = a^2$,

and clearly $a^2 = 1$ in $R$ if and only if $(-2a)^2 \neq 1$ in $R$. Hence, we end up with $a - 2aI$as a Neutrosophic involution in $R(I)$ whenever $a$ is an involution in $R$. This completes the proof. ■

**Corollary 2.5.** There is no Neutrosophic involution of the form $a + bI$, $b \neq 0$ in $R(I)$ if and only if $char(R)$ is 2. In other words, $\mathcal{J}n(R) = \mathcal{J}n(R(I))$ if and only if $char(R)$ is 2.

**Proof.**The widely recognized outcome makes it abundantly clear that

$char(R)$ is $2 \Leftrightarrow -2 \notin R$

$\Leftrightarrow -2aI \notin R(I)$

$\Leftrightarrow a - 2aI \notin R(I)$

$\Leftrightarrow (a - 2aI)^2 \neq a^2$ in $R(I)$.

By the Theorem [2.4 ],we clear that there is no Neutrosophic involution of the form $a + bI$, $b \neq 0$ in $R(I)$ if and only if $char(R)$ is 2. Hence, we conclude that

$$\mathcal{J}n(R) = \mathcal{J}n(R(I)) \text{ if and only if } char(R) \text{ is } 2,$$

because $\mathcal{J}n(R) \subseteq \mathcal{J}n(R(I))$ ■

The next example establishes the correctness of the above result.

**Example 2.6.** Since $\mathbb{F}_4 = \{0,1,\alpha,1+\alpha: \alpha^2+\alpha+1=0\}$ is a field of characteristic 2. So, there exists a Neutrosophic field $\mathbb{F}_4(I)$ same characteristic 2 such that

$$\mathbb{F}_4(I) = \{a+bI: a,b \in \mathbb{F}_4 \ \ and \ \ I^2 = I\}.$$

Obviously, $\mathcal{J}n(\mathbb{F}_4(I)) = \mathcal{J}n(\mathbb{F}_4)$ because $(a+bI)^2 = 1$ in $\mathbb{F}_4(I)$ if and only if $b = 0$ in $\mathbb{F}_4$.

In the classical ring theory, it is well known that $\mathcal{J}n(R) \subseteq U(R)$ and $U(R) \nsubseteq \mathcal{J}n(R)$ for any finite commutative ring that $R$ with unity, and similar manner, in the theory of Neutrosophic rings, these subset inclusions are both true, that is, $\mathcal{J}n(R(I)) \subseteq U(R(I))$ and $U(R(I)) \nsubseteq \mathcal{J}n(R(I))$, where $U(R(I))$ is the set of Neutrosophic units of $R(I)$. However, $\mathcal{J}n(R(I)) \subseteq U(R(I))$ and $U(R(I)) \subseteq \mathcal{J}n(R(I))$ are both true, that is, $U(R(I)) = \mathcal{J}n(R(I))$ if an only if $a+bI$ is in $U(R(I))$ with $b \neq 0$. For example, if $b \neq 0$ in $a+bI$, we have $U(Z_8(I))$ can be written as

$$U(Z_8(I)) = \mathcal{J}n(Z_8(I)) = \{1-2I, 3-6I, 5-2I, 7-6I\},$$

and which is equal to $\mathcal{J}n(Z_8(I))_{b\neq 0}$, because $1-2I$, , $3-6I$, $5-2I$, and $7-6I$ are all Neutrosophic involutions with $b \neq 0$, that is

$$\mathcal{J}n(Z_8(I))_{b\neq 0} = \{1-2I, 3-6I, 5-2I, 7-6I\}.$$

This illustration supports the following definition.

**Definition 2.7.** A Neutrosophic involution $a+bI$ in $R(I)$ is called *pure* Neutrosophic involution if $b \neq 0$.

The set of pure Neutrosophic involutions of $R(I)$ is denoted by of

$$\mathcal{J}n(R(I))_{b\neq 0} = \{a+bI \in R(I): \ (a+bI)^2 = a+bI \ \ and \ \ b \neq 0\}$$

The following theorem supports this observation.

**Theorem 2.8.** Let $R$ be a finite commutative ring with unity 1 and let $|R| > 2$. Then $\mathcal{J}n(R(I)) = U(R(I))$ if and only if $a+bI$ is pure in $R(I)$.

**Proof.** Because of $\mathcal{J}n(R(I)) \subseteq U(R(I))$, it is enough to prove that the other subset inclusion $U(R(I)) \subseteq \mathcal{J}n(R(I))$. For this, we shall show that every Neutrosophic unit is a Neutrosophic involution. Suppose $a+bI$ is pure in $R(I)$. Then there exists $c+dI$ in $U(R(I))$ such that $a+bI \neq c+dI$ and

$$(a+bI)(c+dI) = 1$$
$$\Leftrightarrow ac + (bc+bd+ad)I = 1$$
$$\Leftrightarrow ac = 1 \text{ and } bc+bd+ad = 0$$
$$\Leftrightarrow ac = 1 \text{ and } bc+bd+ad+ac = 1$$
$$\Leftrightarrow ac = 1 \text{ and } (a+b)(c+d) = 1$$
$$\Leftrightarrow a = 1,\ b = 0,\ c = 1 \text{ and } d = 0 \text{ in the ring } R.$$

Consequently, $R = \{0,1\}$, and $|R| = 2$, which is a contradiction to our hypothesis that $|R| > 2$. Thus $a+bI = c+dI$ is always true in $U(R(I))$ if and only if $b \neq 0$ in $R$. This implies that

$$(a+bI)(a+bI) = 1 \text{ for every } a+bI \text{ in } U(R(I)).$$
$$\Leftrightarrow (a+bI)^2 = 1 \text{ for every } a+bI \text{ in } U(R(I)).$$

$\Leftrightarrow a + bI \in \mathcal{I}n(R(I))$ for every $a + bI$ in $U(R(I))$.

Therefore, $U(R(I)) \subseteq \mathcal{I}n(R(I))$ is true in $U(R(I))$, and hence $\mathcal{I}n(R(I)) = U(R(I))$. ■

The subsequent description helps us to estimate the cardinality of $\mathcal{I}n(R(I))$.

First, we construct two tableswhich are employed Neutrosophic involutions along with the earlier results.

The first table describes Neutrosophic involutions arising in the cyclic Neutrosophic ring $Z_n(I)$ from $n = 1$ to $10$.

| $n$ | $\mathcal{I}n(Z_n(I))$ |
|---|---|
| 1 | $\emptyset$ |
| 2 | $\{1\}$ |
| 3 | $\{1,2\} \cup \{1 - 2I, 2 - I\}$ |
| 4 | $\{1,3\} \cup \{1 - 2I, 3 - 2I\}$ |
| 5 | $\{1,4\} \cup \{1 - 2I, 4 - 3I\}$ |
| 6 | $\{1,5\} \cup \{1 - 2I, 5 - 4I\}$ |
| 7 | $\{1,6\} \cup \{1 - 2I, 6 - 5I\}$ |
| 8 | $\{1,3,5,7\} \cup \{1 - 2I, 3 - 6I, 5 - 2I, 7 - 6I\}$ |
| 9 | $\{1,8\} \cup \{1 - 2I, 8 - 7I\}$ |
| 10 | $\{1,3,7,9\} \cup \{1 - 2I, 3 - 6I, 7 - 4I, 9 - 8I\}$ |

We now turn to noncyclic Neutrosophic rings over cyclic rings for determining Neutrosophic rings. For some positive integer $n$, there exists finite commutative ring $Z_n[i]$ such that

$$Z_n[i] = \{z = x + iy : x, y \in Z_n \quad and \quad i^2 = -1\}$$

And also for each $Z_n[i]$ there exists Neutrosophic ring $Z_n[i, I]$ such that

$$Z_n[i, I] = \{z + z'I : z, z' \in Z_n \ [i] \ \ and \ \ I^2 = I\}.$$

It is clear that $Z_n[i, I]$ is also non cyclic Neutrosophic ring, because

$$Z_n[i] = Z_n + iZ_n, \text{ and } Z_n[i, I] = Z_n + iZ_n + iIZ_n.$$

Here we notice that $|Z_n| = n$, $|Z_n[i]| = n^2$and $|Z_n[i, I]| = n^4$, and for more information about $Z_n[i]$ reader refer to [21].Next, the following second table illustrates the Neutrosophic Gaussian involutions from $n = 1$ to $5$.

| $n$ | $\mathcal{I}n(Z_n[i, I])$ |
|---|---|
| 1 | $\emptyset$ |
| 2 | $\{1, i\}$ |
| 3 | $\{1,2\} \cup \{1 - 2I, 2 - I\}$ |
| 4 | $\{1,3,1 + 2i, 3 + 2i\} \cup \{1 - 2I, 3 - 2I, (1 + 2i) - 2I, (3 + 2i) - 2I\}$ |
| 5 | $\{1,4,2i, 3i\} \cup \{1 - 2I, 4 - 3I, 2i - 4iI, 3i - iI\}$ |

By virtueofthe above tables, there are exact powers of 2 Neutrosophic involutionsthat exist in$R(I)$, these being related to the classical involutions in$R$. Also, the collection $\mathcal{I}n(R(I))$ contains a Neutrosophic element $1 - 2I$ as an element in $R(I)$ if and only if $|R| > 2$. So, one consequence of what has just been observed is that, in those finite commutative Neutrosophic ring $R(I)$ with unity cases in which a Neutrosophic involution exists, we can now state exactly how many there are.

**Theorem 2.9.** Let$char(R) \neq 2$. Then $\mathcal{J}n\big(R(I)\big) = (1-2I)\mathcal{J}n(R)$, where $\mathcal{J}n(R) = \{a \in R : a^2 = 1\}$.

**Proof.** By the theorem [2.4], it is well known that $b = 0$ in $\mathcal{J}n\big(R(I)\big)$ if and only if $a^2 = 1$ in $\mathcal{J}n(R)$ if and only if $\mathcal{J}n(R) \neq \mathcal{J}n\big(R(I)\big)$. Now suppose $b \neq 0$ in $\mathcal{J}n\big(R(I)\big)$. Then

$$\begin{aligned}\mathcal{J}n\big(R(I)\big) &= \{a + bI \in R(I) : (a+bI)^2 = 1, b \neq 0\ \}\\ &= \{a + bI \in R(I) : a \in \mathcal{J}n(R), and\ \ b + 2a = 0\ \}\\ &= \{a - 2aI \in R(I) : a \in \mathcal{J}n(R)\}\\ &= \{a(1-2I) \in R(I) : a \in \mathcal{J}n(R)\}\\ &= \mathcal{J}n(R) \cup (1-2I)\mathcal{J}n(R), \text{ since } \mathcal{J}n(R) \subseteq \mathcal{J}n\big(R(I)\big). \ \blacksquare\end{aligned}$$

The next theorem illustrates an extremely useful enumerating technique for enumerating Neutrosophic involutions, often used next results. First, we notice that $\mathcal{J}n(R) = \mathcal{J}n\big(R(I)\big)$ if and only if $char(R) = 2$.

**Theorem 2.10.** Let $char(R) \neq 2$. Then$\big|\mathcal{J}n\big(R(I)\big)\big| = 2|\mathcal{J}n(R)|$.

**Proof.** Let $\mathcal{J}n(R) \neq \mathcal{J}n\big(R(I)\big)$. Then $char(R) \neq 2$ and $charR(I) \neq 2$ but $char(R) = charR(I)$. So there exists an element $-2 \in R$ such that $-2I \in R(I)$. Therefore, $1 - 2I \in R(I)$, and we have

$$(1-2I)^2 = (1-2I)(1-2I) = 1 - 4I + 4I = 1 \text{ in } R(I).$$

This yields the order $|1-2I|$ of the Neutrosophic element $1-2I$ in $R(I)$ is 2. Using $\mathcal{J}n\big(R(I)\big) = \mathcal{J}n(R) \cup (1-2I)\mathcal{J}n(R)$ and also there is a one to one correspondence $f : \mathcal{J}n(R) \longrightarrow (1-2I)\mathcal{J}n(R)$ defined by the relation

$$f(a) = (1-2I)a$$

for every element $a$ in $\mathcal{J}n(R)$. So that $|\mathcal{J}n(R)| = |(1-2I)\mathcal{J}n(R)|$. Hence

$$\begin{aligned}\big|\mathcal{J}n\big(R(I)\big)\big| &= |\mathcal{J}n(R) \cup (1-2I)\mathcal{J}n(R)|\\ &= |\mathcal{J}n(R)| + |(1-2I)\mathcal{J}n(R)|, \text{ since } \mathcal{J}n(R) \cap (1-2I)\mathcal{J}n(R) = \emptyset.\\ &= |\mathcal{J}n(R)| + |\mathcal{J}n(R)| = 2|\mathcal{J}n(R)|. \ \blacksquare\end{aligned}$$

Let's apply the aforementioned to a concrete example now.

The cardinalities of $\mathcal{J}n(R)$ and $\mathcal{J}n\big(R(I)\big)$ are shown in the following brief table.

| **Involutions** ↓ | $n = 1$ → | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert\mathcal{J}n(Z_n)\rvert$ | 0 | 1 | 2 | 2 | 2 | 2 | 2 | 4 | 2 | 4 |
| $\big\lvert\mathcal{J}n\big(Z_n(I)\big)\big\rvert$ | 0 | 1 | 4 | 4 | 4 | 4 | 4 | 8 | 4 | 8 |
| $\lvert\mathcal{J}n(Z_n[i])\rvert$ | 0 | 2 | 2 | 4 | 4 | 4 | 2 | 8 | 2 | 8 |
| $\lvert\mathcal{J}n(Z_n[i,I])\rvert$ | 0 | 2 | 4 | 8 | 8 | 8 | 4 | 16 | 4 | 16 |

Let us attention to the fact that, in the above table, it is necessary to stipulate that $\big|\mathcal{J}n\big(R(I)\big)\big| \leq \big|\mathcal{J}n\big(S(I)\big)\big|$ whenever $R$ is a cyclic ring and $S$ is a noncyclic ring. Further attention depends on finite fields.Since only finite Neutrosophic fields $\mathbb{F}_{2^n}(I)$ of characteristic2 is of even order, and in this sense the Neutrosophic equation $(a+bI)^2 = 1$ has no Neutrosophic solution in $\mathbb{F}_{2^n}(I)$, because

$-2 \notin \mathbb{F}_{2^n}(I)$. In this case $\mathcal{J}n(\mathbb{F}_{2^n}) = \mathcal{J}n\big(\mathbb{F}_{2^n}(I)\big)$. Moreover, since the finite Neutrosophic field $\mathbb{F}_{p^n}(I)$ of characteristic $p$ has order $p^{2n}$ for some odd prime $p$and for some positive integer $n$. In this system $\mathbb{F}_{p^n}(I)$, the Neutrosophic equation $(a+bI)^2 = 1$ is solvable and it has Neutrosophic solutions, because $-2 \in \mathbb{F}_{p^n}(I)$. This theorem proves that there are an infinite number of solutions to the quadratic equation$(a+bI)^2 = 1$ over a finite Neutrosophic field of odd order.

**Theorem 2.11.** Over the finite Neutrosophic field $\mathbb{F}_{p^n}(I)$ corresponding to the odd prime $p$ and the integer $n \geq 1$, the Neutrosophic equation

$$(a+bI)^2 = 1$$

has exactly four solutions. In particular, $\left|\mathcal{J}n\left(\mathbb{F}_{p^n}(I)\right)\right| = 4$.

**Proof.** For any odd prime $p$, there exists field $\mathbb{F}_{p^n}$ and Neutrosophic field $\mathbb{F}_{p^n}(I)$ of odd orders $p^n$ and $p^{2n}$, respectively. Classically, you always the equation $a^2 = 1$ exists $\mathbb{F}_{p^n}$ and is also factorable, like

$$(a-1)(a+1) = 0$$

in $\mathbb{F}_{p^n}$. Since $\mathbb{F}_{p^n}$ is a field with no zero divisors, we must have $a = \pm 1$. Thus, $\mathcal{J}n\big(\mathbb{F}_{p^n}\big) = \{1, -1\}$. Further, since $-2 \in \mathbb{F}_{p^n}(I)$, there exists a Neutrosophic element $1-2I$ in $\mathbb{F}_{p^n}(I)$ such that

$$(1-2I)^2 = 1$$

in $\mathbb{F}_{p^n}(I)$. Using the Theorem [2.4],

$$\begin{aligned}\mathcal{J}n\left(\mathbb{F}_{p^n}(I)\right) &= \mathcal{J}n\big(\mathbb{F}_{p^n}\big) \cup (1-2I)\mathcal{J}n\big(\mathbb{F}_{p^n}\big)\\ &= \{1,-1\} \cup (1-2I)\{1,-1\}\\ &= \{1,-1,1\cdot(1-2I), -1\cdot(1-2I)\}\\ &= \{1,-1,1-2I,-1+2I\}.\end{aligned}$$

Hence, $\left|\mathcal{J}n\left(\mathbb{F}_{p^n}(I)\right)\right| = 4$. ∎

Further on the total number of Neutrosophic involutions to the $R(I)$ over $R$ we have the subsequent result.

**Theorem 2.12.**Neutrosophic involutions to the ring $R(I)$ over the finite commutative $R$ with unity $1$ come in multiple of four.

**Proof.**Let$R$ be any finite commutative with unity $1$ and let $cha(R) \neq 2$. Then there exists at least two involutions in $R$, namely unity $u$ and its additive inverse $-u$ whenever $u^2 = 1$.This means that the least number of involutions in a finite commutative ring $R$ with unity $1$ is two if and only if $cha(R) \neq 2$. Consequently,

$$\mathcal{J}n(R) = \langle u, -u : u^2 = 1 \ \ in \ \ R\rangle,$$

and similarly

$$(1-2I)\mathcal{J}n(R) = \langle (1-2I)u, -(1-2I)u : u^2 = 1 \ \ in \ \ R\rangle.$$

Therefore,

$|\mathfrak{J}n(R)| = |\langle u, -u: u^2 = 1\rangle| \geq 2$, and $|(1-2I)\mathfrak{J}n(R)| = |\langle (1-2I)u, -(1-2I)u: u^2 = 1\rangle| \geq 2$.

By the Theorem [2.4 ],the structure $\mathfrak{J}n(R(I))$ can be written as

$$\mathfrak{J}n(R(I)) = \mathfrak{J}n(R) \cup (1-2I)\mathfrak{J}n(R)$$
$$= \langle u, -u: u^2 = 1\rangle \cup \langle (1-2I)u, -(1-2I)u: u^2 = 1\rangle.$$

This shows that $|\mathfrak{J}n(R(I))| \geq (2)(2) = 4$, and also $|\mathfrak{J}n(R(I))| = 2|\mathfrak{J}n(R)|$. Hence, Neutrosophic involutions to the Neutrosophic ring $R(I)$ over the finite commutative $R$ with unity 1 comes in multiple of four. ■

**Corollary 2.13.** The least number of Neutrosophic involutions of$R(I)$is four if and only if $cha(R) \neq 2$.

**Proof.** This is easily understood based on a common observation.For any $R(I)$with$char(R(I)) \neq 2$, you always have the four Neutrosophic involutions$u, -u$, $(1-2I)u$ and $-(1-2I)u$ whenever $u^2 = 1$in $R$, and viceversa. ■

### 3. Neutrosophic Involutions of $R(I) \times S(I)$

In this section, we give some procedures of the determination of Neutrosophic involutions of $R(I) \times S(I)$ along with the involutions of$R \times S$. It is well known that if $R$ and $S$ are commutative rings with unity, then their Cartesian product $R \times S$ of $R$ and $S$ is also commutative rig with unity under the usual component-wise addition and component-wise multiplication. So for each system$R \times S$, there exists a Neutrosophic system $R(I) \times S(I)$ such that

$$R(I) \times S(I) = \{(a+bI, c+dI): a+bI \in R(I), c+dI \in S(I)\}$$

which is a commutative Neutrosophic ring with unity (1,1) under the component-wise Neutrosophic addition and Neutrosophic multiplication.

The following basic result associates the set of Neutrosophic involutions of $R(I) \times S(I)$ to Neutrosophic involutions of $R(I)$ and$S(I)$, and this association depends on component-wise Neutrosophic multiplication.

**Theorem 3.1.** Let $R$ and $S$ be commutative rigs with the same unity 1. Then

$$\mathfrak{J}n(R(I) \times S(I)) = \mathfrak{J}\text{n}(R(I)) \times \mathfrak{J}n(S(I)).$$

**Proof.** It is sufficient to prove that a Neutrosophic element of $(a+bI, c+dI)$ in of $R(I) \times S(I)$ is a Neutrosophic involution if and only if of $a+bI$ is a Neutrosophic involution in of $R(I)$, and of $c+dI$ is a Neutrosophic involution in of $S(I)$. Indeed,

$$(a+bI, c+dI)^2 = (1,1) \Leftrightarrow (a+bI, c+dI)(a+bI, c+dI) = (1,1)$$
$$\Leftrightarrow ((a+bI)^2, (c+dI)^2) = (1,1)$$
$$\Leftrightarrow (a+bI)^2 = 1 \text{ and } (,(c+dI)^2) = 1$$
$$\Leftrightarrow a+bI \in \mathfrak{J}\text{n}(R(I)) \text{ and } c+dI \in \mathfrak{J}\text{n}(S(I)).$$ ■

The next example presents one to one corresponding involution behavior between $\mathfrak{J}n(R(I) \times S(I))$ and $\mathfrak{J}\text{n}(R(I)) \times \mathfrak{J}n(S(I))$ for computing their corresponding cardinalities.

**Example 3.2.** Consider the Neutrosophic involution structures$\mathfrak{J}n(Z_4(I) \times Z_8(I))$and$\mathfrak{J}n(Z_4(I)) \times \mathfrak{J}n(Z_8(I))$, where

$\mathcal{J}n(Z_4(I) \times Z_8(I)) = \{(1,1),(1,7),(1,1-2I),(1,7-6I),(3,1),(3,7),(3,1-2I),(3,7-6I),(1-2I,1),(1-2I,7),(1-2I,1-2I),(1-2I,7-6I),(3-2I,1),(3-2I,7),(3-2I,1-2I),(3-2I,7-6I)\}$,and

$$\mathcal{J}n(Z_4(I)) \times \mathcal{J}n(Z_8(I)) = \{1,\ 3,\ 1-2I, 3-2I\} \times \{1,\ 7, 1-2I, 7-6I\}$$
$$= \{(1,1),(1,7),(1,1-2I),(1,7-6I),(3,1),(3,7),(3,1-2I),(3,7-6I), (1-2I,1),(1-2I,7),(1-2I,1-2I),(1-2I,7-6I),(3-2I,1), (3-2I,7),(3-2I,1-2I),(3-2I,7-6I)\}.$$

Let $m$ and $n$ be any two positive integers greater 1. Then $Z_m \times Z_n$ is a commutative ring with unity. So the following statement associates the set of involutions of $Z_m \times Z_n$to involutions of $Z_m$and$Z_n$. In the light of this basic argument, the following theorem is necessary and the proof is clear.

$$(a,b) \in \mathcal{J}n(Z_m \times Z_n) \Leftrightarrow a \in \mathcal{J}n(Z_m) \text{ and } b \in \mathcal{J}n(Z_n).$$

Now the following result of the immediate consequence of the above statement.

**Theorem 3.3.** Let $m$ and $n$ be any two positive integers greater 1. Then

$$|\mathcal{J}n(Z_m(I) \times Z_n(I))| = |\mathcal{J}n(Z_m(I))||\mathcal{J}n(Z_n(I))|.$$

**Proof.** Define a map $f$: $\mathcal{J}n(Z_m(I) \times Z_n(I)) \rightarrow \mathcal{J}n(Z_m(I)) \times \mathcal{J}n(Z_n(I))$ by the relation

$$f((a+bI, c+dI)) = \begin{cases} (a,c) & if \;\; b=0, d=0 \\ ((1-2I)a, (1-2I)c) & if \;\; b \neq 0, d \neq 0 \end{cases}$$

Let us suppose $b = 0$ and $d = 0$. Then there is nothing to prove because the map $f$: $\mathcal{J}n(Z_m(I) \times Z_n(I)) \rightarrow \mathcal{J}n(Z_m(I)) \times \mathcal{J}n(Z_n(I))$ is trivially a Neutrosophic ring isomorphism. Now we can prove that this for the case $b \neq 0$ and $d \neq 0$.

***f* is one to one**. Let $(a+bI, c+dI)$, $(a'+b'I, c'+d'I) \in \mathcal{J}n(Z_m(I) \times Z_n(I))$. Then

$$f((a+bI,c+dI)) = f((a'+b'I,c'+d'I)) \Rightarrow ((1-2I)a,(1-2I)c) = ((1-2I)a',(1-2I)c')$$
$$\Rightarrow (1-2I)a = (1-2I)a',\ (1-2I)c = (1-2I)c'$$
$$\Rightarrow a - 2aI = a' - 2a'I,\ c - 2cI = c' - 2c'I$$
$$\Rightarrow a + bI = a' + b'I,\ c + dI = c' + d'I,$$

where $b = -2a$, $b' = -2a'$, $d = -2c$, and $d' = -2c'$. So, the map $f$ is one to one.

***f* is onto.**The range of the function $f$: $\mathcal{J}n(Z_m(I) \times Z_n(I)) \rightarrow \mathcal{J}n(Z_m(I)) \times \mathcal{J}n(Z_n(I))$ is defined by

$$f\left(\mathcal{J}n(Z_m(I) \times Z_n(I))\right) = \{f((a+bI,c+dI)) : (a+bI,c+dI) \in \mathcal{J}n(Z_m(I) \times Z_n(I))\}$$
$$= \{f((a+bI,c+dI)) : (a+bI,c+dI) \in \mathcal{J}n(Z_m(I) \times Z_n(I))\}$$
$$= \{((1-2I)a,(1-2I)c) : a \in \mathcal{J}n(Z_m), c \in \mathcal{J}n(Z_n)\}$$
$$= \{(1-2I)a : a \in \mathcal{J}n(Z_m)\} \times \{(1-2I)c : c \in \mathcal{J}n(Z_n)\}$$
$$= \mathcal{J}n(Z_m) \cup (1-2I)\mathcal{J}n(Z_m) \times \mathcal{J}n(Z_n) \cup (1-2I)\mathcal{J}n(Z_n)$$
$$= \mathcal{J}n(Z_m(I)) \times \mathcal{J}n(Z_n(I)),$$

where $\mathcal{J}n(Z_m(I)) = \mathcal{J}n(Z_m) \cup (1-2I)\mathcal{J}n(Z_m)$ and $\mathcal{J}n(Z_n(I)) = \mathcal{J}n(Z_n) \cup (1-2I)\mathcal{J}n(Z_n)$ . Consequently the map $f$ is onto.

***f* is a Neutrosophic ring isomorphism.** For this let $\alpha = (a + bI, c + dI)$, $\beta = (a' + b'I, c' + d'I) \in \mathcal{I}n(Z_m(I) \times Z_n(I))$, then

$$f(\alpha + \beta) = f\big((a + bI, c + dI) + (a' + b'I, c' + d'I)\big)$$
$$= f\Big(\big((a + a') + (b + b')I, (c + c') + (d + d')I\big)\Big)$$
$$= \big((1 - 2I)(a + a'), (1 - 2I)(c + c')\big)$$
$$= \big((1 - 2I)a + (1 - 2I)a', (1 - 2I)c + (1 - 2I)c'\big)$$
$$= \big((1 - 2I)a, (1 - 2I)c\big) + \big((1 - 2I)a', (1 - 2I)c'\big)$$
$$= f\big((a + bI, c + dI)\big) + f(a' + b'I, c' + d'I) = f(\alpha) + f(\beta),$$

and similarly we can show that

$$f(\alpha\beta) = f(\alpha)f(\beta) \text{ for every } \alpha \text{ and } \beta \text{ in } \mathcal{I}n(Z_m(I) \times Z_n(I)).$$

Thus the map $f$ is a Neutrosophic ring isomorphism from $\mathcal{I}n(Z_m(I) \times Z_n(I))$ onto the $\mathcal{I}n(Z_m(I)) \times \mathcal{I}n(Z_n(I))$ with $f(1,1) = (1 - 2I, 1 - 2I)$, and hence

$$\mathcal{I}n(Z_m(I) \times Z_n(I)) \cong \mathcal{I}n(Z_m(I)) \times \mathcal{I}n(Z_n(I)).$$

This identity implies that

$$|\mathcal{I}n(Z_m(I) \times Z_n(I))| = |\mathcal{I}n(Z_m(I))||\mathcal{I}n(Z_n(I))|. \blacksquare$$

## 4.Diagonal Property of Neutrosophic Elements in $R(I)$

This section introduces the diagonal property of finite commutative Neutrosophicrings.First of all we have, for any finite commutative Neutrosophic ring $R(I)$ with unity, there exists a multiplicative composition table of all elements of $R(I)$, and this table is associated to one to one correspondence of the matrix network $R(I) \times R(I)$ with the size $|R(I)| \times |R(I)|$. Classically, it is well known that there is an element 1 at the position of the entry $(a, b)$ in the composition table of $R$. Then obviously $ab = 1 = ba$ in $R$. So automatically there a connection between $1s$ in $R$ and entries of the composition table of $R$, see [22,23 ].Here, we can establish same theory on to Neutrosophic rings.

**Definition 4.1.**A Neutrosophic ring $R(I)$ with unity 1 has diagonal property if all $1s$ appeared in the main diagonal of the composition table of $R(I)$.

In [ 23],the author Sunil proved the following necessary and sufficient condition for $1s$ appeared in the main diagonal of the composition table of $Z_n$ and divisors of 24.

**Theorem 4.2.**[23].The multiplication table for the cyclic ring $Z_n$ contains $1s$ only on the diagonal of the multiplicative composition table of $Z_n$ if and only if $n$ is a divisor of 24.

Consequently, this result also obviously true in Neutrosophic rings, that is it can be stated as follows.

**Theorem 4.3**. The multiplication table for the Neutrosophic cyclic ring $Z_n(I)$ contains $1s$ only on the diagonal of the multiplicative composition table if and only if $n$ is a divisor of 24.

Subsequently, if $n$ is not a divisor of 24, then the above results are not true. For example, $n$ is 5 which is not a divisor of 24, then there exists a Neutrosophic ring $Z_5(I) = \{a + bI: a, b \in Z_5;\ I^2 = I\}$, which does not satisfies diagonal property, because $(2 + 0I)(3 + 0I) = 6 + 0I = 1$ under modulo 5. This failure concept concludes that the condition$b = 0$exists in the form $a + bI$. However, all elements of Neutrosophic system $Z_5(I) = \{a + bI: a, b \in Z_5;\ I^2 = I\}$ is also satisfies diagonal property whenever $b \neq 0$ in the form$a + bI$, and this success illustrates the following Neutrosophic composition table for $Z_5(I)$with$b \neq 0$.

For$b \neq 0$, the pure Neutrosophic involutions of$Z_5(I)$ can be written as

$$\mathcal{I}n\big(Z_5(I)\big)_{b\neq 0} = \mathcal{I}n\Big(U\big(Z_5(I)\big)\Big) - \mathcal{I}n\big(U(Z_5)\big) = \{1-2I, 4-3I\}.$$

| $\odot_5$ | $1-2I$ | $4-3I$ |
|---|---|---|
| $1-2I$ | **1** | 4 |
| $4-3I$ | 4 | **1** |

Consequently the following result is more eminent for satisfying diagonal property of $any$Neutrosophic ring$R(I)$.

**Theorem 4.5**. The multiplication table for Neutrosophic units $U\big(R(I)\big) - U(R) = \{a + bI \in R(I): a, b \in R, and\ \ b \neq 0\}$ of any finite commutative Neutrosophic ring$R(I)$ contains $1s$ only on the diagonal of the multiplicative composition table if and only if $char(R) \neq 2$.

**Proof.** It is clear by theTheorem [2.8],we have

$\mathcal{I}n\big(R(I)\big) = U\big(R(I)\big)$ if and only if $a + bI$ is in $U\big(R(I)\big)$ with $b \neq 0$.

$\mathcal{I}n\big(R(I)\big) = U\big(R(I)\big)$ if and only if $a + bI$ is a pure Neutrosophic involutions of $R(I)$.

Hence, the number of 1s appear on the principal diagonal of the table of

$$U\big(R(I)\big) = \{a + bI \in R(I): a, b \in R, and\ \ b \neq 0\}$$

is equal to $2^k$ for some integer $k \geq 1$, because the cardinality of $U\big(R(I)\big) = \{a + bI \in R(I): a, b \in R, and\ \ b \neq 0\}$ is even and it is greater than or equal to the power of2. ∎

## 5. Conclusions

In this paper, we have analytically studied Neutrosophic involutory behavior of the Neutrosophic elements of the finite commutative Neutrosophic ring $R(I)$. A necessary and sufficient for the Neutrosophic element$a + bI$ being a Neutrosophic involution has been obtained. From this criterion we have developed a general procedure to enumerate Neutrosophic involutions of the form $a + bI$ over $R(I)$ from given classical involutions over the corresponding finite commutative ring $R$. The proposed technique can be used to determine more desired Neutrosophic involutions of $R(I)$.

## 6. Future Work

A Neutrosophic involution over a finite commutative Neutrosophic ring $R(I)$ is an element property whose multiplicative inverse is itself. Owing to this property, we will prepare and produce techniques for enumerating Neutrosophic involutions which are applied in Computational systems like the XOR bitwise operations with a given value for one parameter with indeterminate, and develop RC4 cryptographic ciphers, further we will use these Neutrosophic involutions for studying liminalities and minimalities of Reversible Rings.

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# Generating Random Variables that follow the Beta Distribution Using the Neutrosophic Acceptance-Rejection Method

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**Abstract**:

Analysis using simulation is a natural and logical extension of the analytical and mathematical models inherent in operations research. Simulation has become a modern tool that helps in studying many systems that we could not study or predict the results that we could obtain during the operation of these systems over time before the existence of Simulation, since the main interest in statistical analysis is to obtain a series of random variables that follow the probability distribution in which the system under study operates, through a series of random numbers that follow a uniform distribution over the domain [0, 1], using scientific methods provided by the efforts of researchers. In the field of modeling and simulation, such as the reverse transformation method, the rejection and acceptance method, and other methods that we have reformulated using the concepts of neutrosophic science in previous research. In summary of what we have done previously, we present in this research a study whose purpose is to generate random variables that follow the beta distribution, which is used in many applications. In administrative processes, especially in analyzing network diagrams using the neutrosophic rejection and acceptance method.

**Keywords:** modeling and simulation; neutrosophic science; rejection and acceptance method for generating neutrosophic random variables; neutrosophic uniform distribution; neutrosophic random numbers; Neutrosophic beta distribution.

## Introduction:

Operations research has provided many scientific methods that have contributed to the great scientific development witnessed in our contemporary world. The importance of these methods increases when they are reformulated using the concepts of neutrosophic science, the science that relies on neutrosophic data that leaves nothing to chance and takes into account all the circumstances and fluctuations facing decision makers. Therefore, researchers have presented many Among the researches through which some operations research methods were reformulated using the concepts and information presented by the founder of this science, we mention [1-18]. Neutrosophic statistical studies are an extension of traditional statistical studies, as they depend on neutrosophic data, which are groups where any $a$ value such as a in a group denoted by $a_N$ which means ($a$ neutrosophic), ($a$ imprecise), or ($a$ non-specific). $a_N$ may be a neighbor of $a$ or an interval containing $a$, and in general it can be considered any set close to $a$. See [19].

In any probability distribution, if there is a quantity that contains some indeterminacy, then this distribution is a neutrosophic probability distribution. Therefore, the random numbers and random variables that we obtain based on this probability distribution are neutrosophic random numbers and random variables. See [20].

**Discussion**:

Probability distributions are the mainstay of the simulation process. Therefore, the interest of researchers and scholars interested in the simulation method has focused on providing scientific studies that help in obtaining random variables that follow the probability distributions most used in practical applications. In classical studies, we find many algorithms that are concerned with transforming random numbers that follow a uniform distribution into The domain [1, 0] refers to the probability distribution with which the system to be simulated operates, and based on this importance of probability distributions and after the emergence of neuroscientific science, which paid great attention to probability distributions, many probability distributions were reformulated using the concepts of this science, and we presented in previous research how Generating neutrosophic random numbers and methods for converting these numbers into neutrosophic random variables that follow the exponential distribution and others that follow the uniform distribution using the inverse transformation method, which was reformulated using the concepts of neutrosophic science [21-23]. In this research, and given the importance of the beta distribution, we present a study whose goal is to use the method Neutrosophic rejection and acceptance, which was presented in the paper [24], to generate neutrosophic random variables that follow the beta distribution, which is defined using classical values as the following probability density function:

$$\boldsymbol{f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}}\quad ;\ \boldsymbol{0\le x\le 1}\qquad \boldsymbol{(1)}$$

Where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the medians used to define this distribution and $\boldsymbol{\alpha>0}$, $\boldsymbol{\beta>0}$

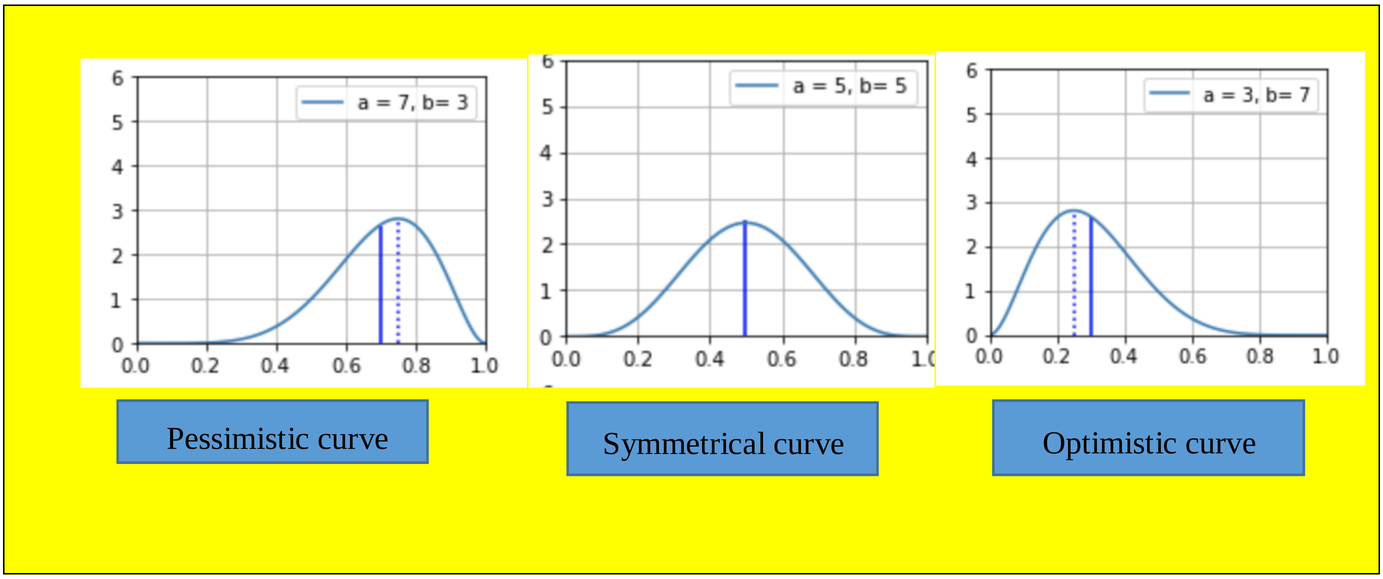
The symbol $\boldsymbol{\Gamma(c)}$ is the value of the integral (gamma), defined by the following relationship:

$$\boldsymbol{\Gamma(c)=\int_0^{\infty}x^{c-1}\,e^{-x}dx}$$

The beta distribution curve takes many shapes depending on the values of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and we can be classified into three types:

1- Pessimistic curve in this case is $\boldsymbol{\alpha>\beta}$
2- Symmetrical curve $\boldsymbol{\alpha=\beta}$
3- Optimistic curve $\boldsymbol{\alpha<\beta}$

As in the following figures:




**Neutrosophic function: [25]**

The neutrosophic function $f: A \to B$ is a function that has some indeterminacy, taking into account the definition of its domain, its corresponding domain, and the relationship between the elements of the domain and the elements of the corresponding domain.

If α , $\beta$ , or both of them carry some indeterminacy, that is, they take one or the other Where $\alpha_N$ and $\beta_N$ are neutrosophic values written as follows:

$$\beta_N = \beta + \delta \text{ و } \alpha_N = \alpha + \varepsilon$$

Where $\varepsilon$ is an indeterminacy and takes one form $\varepsilon \in \{\lambda_1, \lambda_2\}$ or $\varepsilon \in [\lambda_1, \lambda_2]$ or other than that.

Where $\delta$ is an indeterminacy and takes one form $\delta \in \{\mu_1, \mu_2\}$ or $\delta \in [\mu_1, \mu_2]$ or other than that.

Then we obtain the neutrosophic beta distribution, which has a probability density function given by the following formula:

$$f(x) = \frac{\Gamma(\alpha_N + \beta_N)}{\Gamma(\alpha_N)\Gamma(\beta_N)} x^{\alpha_N - 1}(1 - x)^{\beta_N - 1} \quad ; \quad 0 \le x \le 1 \qquad (2)$$

Based on what has been reported about this beta distribution in classical studies, we can present these neutrosophic types of beta distribution: [24]

The graphic curve of the beta neutrosophic distribution takes many shapes depending on the values of $\alpha_N$ and $\beta_N$ and can be classified into three types:

1- Pessimistic curve in this case is $\alpha_N > \beta_N$
2- Symmetrical curve $\alpha_N = \beta_N$
3- Optimistic curve $\alpha_N < \beta_N$

**Generating random variables that follow a beta distribution:**

We know that the process of neutrosophic simulation depends on generating neutrosophic random numbers that follow the regular neutrosophic distribution and then converting these random numbers into random variables that follow the probability distribution with which the system to be simulated operates. There are several methods that can be used for the conversion process, including we mentioned in previous research the inverse transformation method and the rejection method. And acceptance [21-23], where the appropriate method is used for the probability density function because in the inverse transformation method we need the inverse function of the cumulative distribution function. As we know, in many functions, the inverse function does not exist or obtaining it requires complex operations. Here we resort to the method Rejection and acceptance. As is clear from relationship (1), it is not easy to obtain the inverse function of the probability density function for the beta distribution. Therefore, we will use the inverse transformation method and apply it to the beta distribution according to the following algorithms:

**Rejection and acceptance algorithm: [23,24]**

We calculate $\boldsymbol{M_N}$, the maximum value taken by the probability density function over its defined domain, and we obtain it by calculating the derivative of this function and setting it equal to zero, i.e.:

$$M_N = \left.\frac{df_N(x)}{dx}\right|_{x=0}$$

And in simplified form, we define the derivative of the neutrosophic function as follows [24]:

$$f_N(x) = \lim_{h\to 0} \frac{[inf\ f(x+h) - inf\ f(x) sup f(x+h) - sup\ f(x)]}{h}$$

This is a generalization of the traditional derivative definition, where the function and variables are conventional. It can be written as:

$$[inf\ H\ , sup\ H] = h$$
$$inf\ (x+H) = sup(x+H) = f(x+h)$$
$$inf\ f(x) = sup\ f(x) = f(x)$$

Here, $\boldsymbol{H}$ is a closed, open, half-closed, or half-open interval.

By applying the above definition to the function defined by the relationship (2), we obtain:

$$f'(x) = c.(\alpha_N - 1)x^{\alpha_N - 2}(1-x)^{\beta_N - 1} - c(\beta_N - 1)x^{\alpha_N - 1}(1-x)^{\beta_N - 2}$$

To find the solution, we set the derivative equal to zero:

$$f'(x) = c.(\alpha_N - 1)x^{\alpha_N - 2}(1-x)^{\beta_N - 1} - c(\beta_N - 1)x^{\alpha_N - 1}(1-x)^{\beta_N - 2} = 0$$

In short, by finding the common limits, we get:

$$Nx = \frac{\alpha_N - 1}{\alpha_N + \beta_N - 2} = mod \qquad (3)$$

This value of x corresponds to the maximum value of the function, and therefore:

$$M_N = \frac{\alpha_N - 1}{\alpha_N + \beta_N - 2}$$

After obtaining the values of $\boldsymbol{\alpha_N}$ and $\boldsymbol{\beta_N}$ and substituting them into equations (2) and (3), we apply the following algorithm:

1. Generate two random numbers, $\boldsymbol{R_1}$and $\boldsymbol{R_2}$ following a uniform distribution in the range [0, 1]. We use the method of squaring the average to generate the random numbers $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ as follows [24,25]:

$$R_{i+1} = Mid[R_i^2] \quad ; i = 0,1,2,----\qquad (4)$$

where, Mid refers to the middle four digits of $\boldsymbol{R_1^2}$ and $\boldsymbol{R_0}$ is an initial random number consisting of four digits (called the seed) that doesn't contain zero in any of its four digits.

2. Convert the numbers $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ into neutrosophic function random numbers. To convert random numbers following a uniform distribution into neutrosophic random numbers, we follow the method introduced in [22], where three forms of neutrosophic random numbers are distinguished based on the uncertainty associated with the range [0, 1]:

a. **Form 1**: Uncertainty in the lower limit, i.e., $[\mathbf{0+\varepsilon}\,,\mathbf{1}]$. where, each classical random number is mapped to a neutrosophic function random number using the following relationship:

$$NR_0 = \frac{R_0-\varepsilon}{1-\varepsilon} = \frac{[R_0\,,\ R_0-n]}{[1\,,1-n]} \in [\,R_0, \frac{R_0-n}{1-n}]$$

b. **Form 2**: Uncertainty in the upper limit, i.e., $[\mathbf{0}\,,\mathbf{1}+\boldsymbol{\varepsilon}]$**.** where, each classical random number is mapped to a neutrosophic function random number using the following relationship:

$$NR_0 = \frac{R_0}{1+\varepsilon} = \frac{R_0}{[1\,,\ n+1]} \in [R_0\,, \frac{R_0}{n+1}]$$

c. **Form 3**: Uncertainty in both upper and lower limits, i.e. $[\mathbf{0}\,,\mathbf{1+\varepsilon}]$. where, each classical random number is mapped to a neutrosophic function random number using the following relationship:

$$NR_0 = R_0 - \varepsilon \in [R_0\,, R_0 - n]$$

In the three previous forms, we have $\boldsymbol{\varepsilon} \in [\mathbf{0}\,,\boldsymbol{n}]$ ***and*** $\mathbf{0} < \boldsymbol{n} < \mathbf{1}$.

In this study, we will use the third form: non-deterministic in the upper and lower bounds, i.e., $[\mathbf{0}+\boldsymbol{\varepsilon}\,,\mathbf{1}+\boldsymbol{\varepsilon}]$. Here, we associate each classical random number with a non-deterministic random number using the following relationship:

$$NR_0 = R_0 - \varepsilon \in [R_0\,, R_0 - n]$$

Where $\boldsymbol{\varepsilon} \in [\mathbf{0}\,,\boldsymbol{n}]$ ***and*** $\mathbf{0} < \boldsymbol{n} < \mathbf{1}$

3. We take one of the two numbers, let it be $\boldsymbol{NR_1}$ , and transform it appropriately for the uniform distribution. We know that when the boundaries of the domain are neutrosophic values, we apply the following [22] for the upper and lower limits of the domain:

$[\boldsymbol{b}+\boldsymbol{\varepsilon}\,,\boldsymbol{a}+\boldsymbol{\varepsilon}]$, where $\boldsymbol{a} = \boldsymbol{a_N} = \boldsymbol{a} + \varepsilon$ and $\boldsymbol{b_N} = \mathbf{b} + \varepsilon$, and $\boldsymbol{\varepsilon} \in [\mathbf{0}\,,\boldsymbol{n}]$ ***and*** $\boldsymbol{a} < \boldsymbol{n} < \boldsymbol{b}$.

We use the following relationship:

$$Nx_1 = (b-a)NR_1 + a$$

4. We test whether $\boldsymbol{NR_2}$satisfies the inequality:

$$NR_2 \le \frac{f(NR_1)}{M} \qquad (5)$$

That is,

$$NR_2 \leq \frac{f_N(NR_1)}{M_N} \;=\; \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}(NR_1)^{\alpha_N-1}\left[1-(NR_1)^{\beta_N-1}\right]$$

5. If the inequality (5) is satisfied, then we accept that $\boldsymbol{Nx_1 = NR_1}$follows the Beta distribution defined by equation (1).

6. If $\boldsymbol{NR_2}$ does not satisfy the inequality (5), then we reject the two numbers $\boldsymbol{NR_1}$ , $\boldsymbol{NR_2}$and return to the first step to generate new random numbers.

**We explain the above through the following example**:

**Example 1:**

We have a system that operates according to the beta function defined by the following probability density function:

$$\boldsymbol{f(x) = \frac{\Gamma(10)}{\Gamma(3)\Gamma(7)} x^2(1-x)^6 \quad ; \quad 0 \leq x \leq 1}$$

That is, $\boldsymbol{\alpha = 3}$ and $\boldsymbol{\beta = 7}$

What is required is to generate random variables that follow this distribution. In this example, we will take the beta function in the classical form and neutrosophic random numbers:

We apply the rejection and acceptance algorithm according to the previously mentioned steps:

1. We calculate the largest value of the density function over its defined field from the following relationship:

$$\boldsymbol{M = \frac{\alpha - 1}{\alpha + \beta - 2}}$$

We find:

$$\boldsymbol{M = \frac{3-1}{3+7-2} = \frac{2}{8} = \frac{1}{4}}$$

2. We use the method of squaring to generate two random numbers that follow the uniform distribution over the domain $[\boldsymbol{0}\,,\boldsymbol{1}]$**.** We take the seed $\boldsymbol{R_0 = 0.1273}$ and obtain the following random numbers:

$$\boldsymbol{R_1 = 0.6205 \quad , R_2 = 0.5020}$$

3. We convert the classical random numbers to random numbers that follow the non-deterministic uniform distribution over the domain $[0+\varepsilon\,,1+\varepsilon]$. We take the non-deterministic bounds $\boldsymbol{\varepsilon} \in [\boldsymbol{0}\;,\boldsymbol{0.02}]$and obtain the following neutrosophic non-deterministic random numbers:

$$NR_1 = R_1 - \varepsilon \in [R_1\,, R_1 - 0.02]$$
$$NR_1 = R_1 - \varepsilon = 0.6205 - [0\;,0.02] = [0.6205\,,0.6005] \Rightarrow NR_1 \in [0.6205\,,0.6005]$$
$$NR_2 = R_2 - \varepsilon \in [R_2\,, R_2 - 0.02]$$
$$NR_2 = R_2 - \varepsilon = 0.5020 - [0\;,0.02] = [0.5020\,,0.482] \Rightarrow NR_2 \in [0.5020\,,0.482]$$

4. We take one of the numbers and form an appropriate transformation for the uniform

distribution over the domain $[\mathbf{0}\,,\mathbf{1}]$ **.** Here, since the distribution is defined over the domain $[\mathbf{0}\,,\mathbf{1}]$**,** we take one of the non-deterministic random numbers, let's say $\boldsymbol{NR_1} \in [\mathbf{0.6205}\,,\mathbf{0.6005}]$, and calculate the value of the probability density function at that point:

$$\boldsymbol{f(NR_1)} = \frac{\boldsymbol{\Gamma(10)}}{\boldsymbol{\Gamma(3)\Gamma(7)}}([\mathbf{0.6205}\,,\mathbf{0.6005}])^{\mathbf{2}}(\mathbf{1}-([\mathbf{0.6205}\,,\mathbf{0.6005}]))^{\mathbf{6}}$$

We know that if $\boldsymbol{n}$ is a positive integer, then $(\boldsymbol{n}+\mathbf{1}) = \boldsymbol{n}!$ **,** therefore:

$$\frac{\mathbf{\Gamma(10)}}{\mathbf{\Gamma(3)\Gamma(7)}} = \frac{\mathbf{9}!}{\mathbf{2}!\,\mathbf{6}!} = \mathbf{252}$$

$$\boldsymbol{f(NR_1)} \in (\mathbf{252}\times[\mathbf{0.385},\mathbf{0.3606}]\times[\mathbf{0.003},\mathbf{0.0041}]) = [\mathbf{0.2911},\mathbf{0.3726}]$$

1. We test the inequality (5) and for this, we calculate:

$$\frac{f(NR_1)}{M} \in \frac{[0.2911{,}0.3726]}{0.25} = [1.1644{,}1.4904]$$

We have $NR_2 \in [0.5020\,,0.482]$. We note that:

$$[0.5020\,,0.482] \leq [1.1644{,}1.4904]$$

Therefore, inequality (5) is satisfied, i.e.:

$$NR_2 \leq \frac{f(NR_1)}{M}$$

Here we accept that $N(R_1) \in [1.1644{,}1.4904]$follows the beta distribution given in the example.

**Generating random numbers following the neutrosophic beta distribution from classical random numbers**:

**Example 2**: We have a system that operates according to the beta function defined by the neutrosophic probability density function as in the following relationship:

$$f(x) = \frac{\Gamma(\alpha_N+\beta_N)}{\Gamma(\alpha_N)\Gamma(\beta_N)}x^{\alpha_N-1}(1-x)^{\beta_N-1} \quad ; \; 0 \leq x \leq 1$$

$\alpha_N$and $\beta_N$ are neutrosophic values of the form $\alpha_N = \alpha+\varepsilon$ , $\beta_N = \beta+\delta$, which are adjacent to the real values $\alpha = 3$ and $\beta = 7$.

Where $\varepsilon$ and $\delta$ are the indeterminacy in these values. We take them in this example as follows: $\varepsilon \in [\lambda_1,\lambda_2\,] = [0\,,0.2\,]$and $\delta \in [\mu_1,\mu_2] = [0\,,0.1\,]$ we get $\alpha_N \in [3\,,3.2\,]$ and $\beta_N \in [7\,,7.1\,]$ then the probability density function is written as follows:

$$f_N(x) = \frac{\Gamma([10\,,10\,.3\,])}{\Gamma([3\,,3.2\,])\Gamma([7\,,7.1\,])}x^{[2\,,2.2\,]}(1-x)^{[6\,,6.2\,]} \;\; ; \;\; 0 \leq x \leq 1$$

What is required is to generate random numbers that follow the previous distribution, based on classical random numbers. We apply the rejection and acceptance algorithm according to the steps mentioned previously, and as we did in Example 1, we obtain what is required.

**Example 3**: We have a system that operates according to the beta function defined by the neutrosophic probability density function as in the following relationship:

$$f(x) = \frac{\Gamma(\alpha_N+\beta_N)}{\Gamma(\alpha_N)\Gamma(\beta_N)}x^{\alpha_N-1}(1-x)^{\beta_N-1} \quad ; \; 0 \leq x \leq 1$$

$\alpha_N$and $\beta_N$are neutrosophic values of the form $\alpha_N = \alpha+\varepsilon$ , $\beta_N = \beta+\delta$, which are adjacent to the real values $\alpha = 3$ , $\beta = 7$.

And $\varepsilon$ and $\delta$ are the indeterminacy in these values. We take them in this example as follows: $\varepsilon \in [\lambda_1,\lambda_2\,] = [0\,,0.2\,]$and $\delta \in [\mu_1,\mu_2] = [0\,,0.1\,]$ we get $\alpha_N \in [3\,,3.2\,]$ and $\beta_N \in [7\,,7.1\,]$ then the probability density function is written as follows:

$$f_N(x) = \frac{\Gamma([10\,,10\,.3\,])}{\Gamma([3\,,3.2\,])\Gamma([7\,,7.1\,])}x^{[2\,,2.2\,]}(1-x)^{[6\,,6.2\,]} \;\; ; \;\; 0 \leq x \leq 1$$

What is required is to generate random numbers that follow the previous distribution, based on neutrosophic random numbers. We apply the rejection and acceptance algorithm according to the steps mentioned previously, and as we did in Example 1, we obtain what is required.

**The difference between the second and third examples:**

**In the second example**, we generate random numbers that follow a uniform distribution over the domain $[0\ ,1]$using the mean square method. For example, we take the seed $R_0 = 0.1273$ from the relationship (3), which is:

$$R_1 = 0.6205 \quad , R_2 = 0.5020$$

Then we use the two numbers to implement the rejection and acceptance algorithm, as we did in the first example.

**In the third example**, we generate two random numbers that follow a uniform distribution over the domain $[0\ ,1]$using the mean square method. For example, we take the seed $R_0 = 0.1273$ from the relationship (3), which is:

$$R_1 = 0.6205 \quad , R_2 = 0.5020$$

Then we transform them into two random numbers that follow the uniform neutrosophic distribution over the domain$[0 + \varepsilon, 1 + \varepsilon]$. We take the indeterminacy $\varepsilon \in [0\ \ ,0.02]$. We get the following two-neutrosophic random numbers:

$$NR_1 \in [0.6205\ ,0.6005]\ \ ,\ \ NR_2 \in [0.5020\ ,0.482]$$

Then we use two-neutrosophic random numbers to implement the rejection and acceptance algorithm, as we did in the first example.

**Conclusion and results:**

In order to obtain more accurate results and enjoy a margin of freedom when simulating systems that operate according to the beta distribution, which is one of the important distributions that has many uses in many fields, we presented in this research a study through which we are able to obtain neutrosophic random variables that follow this distribution, using neutrosophic rejection and acceptance method. Thanks to the indeterminacy of neutrosophic values, we are able to provide simulation results suitable for all circumstances and achieve the desired goal for decision makers.

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# Normality Testing under Neutrosophic Statistics

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**Abstract**

The normal distribution has been extensively been used in the statistical inference and decision making problems. One of the main assumptions of the existing tests in classical statistics is that the data should be from normally distributed population.Shapiro- Wilk test is widely used to test the normality of data when the observations are precise or exact in nature. But in many areas including agriculture, engineering and reliability, the data may be in interval form, indeterminate form or uncertain. In such cases, the existed Classical Shapiro- Wilk test fails to test the normality of data. In this paper, we proposed the Shapiro- Wilk test under neutrosophic environment to check whether the uncertain data is from neutrosophic normal distribution or not. The hypothesis testing process is executed on the observations based on lifetime of batteries. The comparative analysis has been done with the existed Shapiro- Wilk test.The comparison shows that the proposed test is efficacious, appropriate and well-suited to be applied in scenarios involving indeterminacy.

**Keywords:** Neutrosophic Statistics, Shapiro- Wilk Test, Neutrosophic Normal Distribution, Classical Statistics, Hypothesis Testing.

## 1. Introduction

Now a day, inferential statistics have been used commonly in all fields of the research to test the hypotheses and making predictions on the bases of data. The statistical tests in inferential statistics have common assumption about data that it should be taken from a population following a specific distribution. It is a crucial assumption for the selection of relevant test. The distribution from which the sample has been taken is always unknown in advance. In classical statistics, there are two ways to check the distribution of the data. First is graphical approach in which the graphs are formed on the bases of given data. Another strategy that yields more trustworthy and superior results is “goodness of fit”. These tests used the “cumulative distribution function” of the fitted or underlying distribution. For the testing of the assumption that the data follows the generalized Pareto distribution or not, Anderson- Darling test proposed by Arshad et al. [1]. Various tests, most notably "goodness-of-fit," are used to determine whether the sample has been taken from a specific distribution. The most commonly used “goodness-of-fit” tests are; Jarque–Bera [2-3], Kolmogorov–Smirnov [4-5], Lilliefors [6], “Pearson's chi-square” [7], “Cramèr–von Mises” [8-9], D'Agostino–Pearson [10] and Anderson–Darling [11-12]. Aslam [13] proposed a new test of “goodness of fit” under the presence of neutrosophic parameters. Ahsan-ul-Haq [14] discussed the Cramèr–von Mises test under uncertainty. Smarandache [15] proposed a generalisation of fuzzy logic known as neutrosophic logic. The advantages of neutrosophic logic over fuzzy logic and interval-based analysis were shown by Smarandache and Khalid [16].The more literature, related articles and books on the neutrosophic statistics can be view in [17-18]. Shapiro [19] proposed the Shapiro–Wilk test for checking the assumption of normality. Nornadiah and Yap [20] proved that the Shapiro–Wilk test is most powerful normality test as comparison to the Kolmogorov–Smirnov test, Lilliefors test and Anderson–Darling test.Jeyaraman et al. [21] defined the neutrosophic norms and made some finding regarding the respective categories. Uma and Nandhitha[22] determined the Quick Switching System using the Neutrosophic Poisson distribution and compared with the Fuzzy Poisson distribution by means of Operating Characteristic (OC) curves. Dey and Ray[23] investigated some properties of the redefined neutrosophic composite relations. Utilizing the principles of neutrosophic science, Jdid and Smarandache [24]reformulated the Lagrangian multiplier technique originally designed for nonlinear models constrained by equality.

Under uncertainty, the existed Classical Shapiro- Wilk test is failed to test the normality assumption. Here, we proposed the Shapiro-Wilk test under neutrosophic environment to check whether the uncertain data is from neutrosophic normal distribution or not. We discussed the hypothesis testing procedure on lifetime of batteries. The comparative analysis has been done with the existed Shapiro-Wilk test.

**2. Preliminaries**

Let us consider $A_N = A_l + A_u D_N$ are the neutrosophic numbers, such that $D_N \in [D_l, D_u]$ is an indeterminacy interval, follows that neutrosophic normal distribution (NND) [17-18]with the neutrosophic mean $\mu_N = \mu_l + \mu_u D_N$ ; $D_N \in [D_l, D_u]$ and neutrosophic variance $\sigma_N^2 = \sigma_l^2 + \sigma_u^2 D_N$ ;$D_N \in [D_l, D_u]$. The "probability density function" of the NND is given by

$$f_N(A_N) = \frac{1}{\sigma_N \sqrt{2\pi}} exp\left\{-\frac{(A_N - \mu_N)^2}{2\sigma_N^2}\right\} ; \mu_N \in [\mu_l, \mu_u] , \sigma_N^2 \in [\sigma_l^2, \sigma_u^2], D_N \in [D_l, D_u] \quad (1)$$

The NND given by equation (1) is the generalized version of existed normal distribution. NND will reduce to the classical normal distribution if the $D_l = 0$.

**3. Shapiro- Wilk Test Under Neutrosophic Statistics (NSW)**

The Shapiro- Wilk (SW) test is a frequently used test in classical statistics to access the normality of the data. Here, we extend the SW test under the neutrosophic environment. We'll go over the algorithm for determining whether the given data follows the NND or not. The neutrosophic parameters of NND are assumed to be unknown and estimated from the given neutrosophic data in order to develop the proposed test. When compared to the traditional SW test, the proposed test will produce the results in terms of interval of indeterminacy. The assumptions of the NSW test are that the data should consist neutrosophic observations and observations should be independent of each other. The null and alternative hypothesis for the NSW test is:

$H_{0N}$: The data has been taken from the NND.

$H_{aN}$: The data has not been from the NND.

The intended application procedure for the neutrosophic Shapiro-Wilks test is outlined as follows:

**Step 1:** Find the mean of neutrosophic observations $\boldsymbol{A_{(i)N}}$ where $\boldsymbol{i}$ =1, 2,…, n i.e.

$$\overline{A_{(\iota)N}} = \frac{1}{n}\sum_{\boldsymbol{i=1}}^{n} \boldsymbol{A_{(i)N}} \quad (2)$$

**Step 2:** Find the Neutrosophic Sum of Squares (NSS) by subtracting the neutrosophic mean value from neutrosophic observations then squaring and summing the obtained neutrosophic values as given below:

$$NSS = \sum_{i=1}^{n}\left(A_{(i)N} - \overline{A_{\iota N}}\right)^2 \quad (3)$$

**Step 3:** Now calculated$g_N$as given below

$$g_N = \left(\sum_{i=1}^{n} w_i\left(A_{(n+1-i)N} - A_{(i)N}\right)\right)^2 \quad (4)$$

where $w_i$ weights taken from Shapiro-Wilk Table [19].

$A_{(i)N}$ is the ith smallest neutrosophic number or ith neutrosophic order statistic. The median value is not included if n is odd.

**Step 4:** Compute the value of test statistic for NSW is given by

$$W_N = \frac{\left(\sum_{i=1}^{n} w_i (A_{(n+1-i)N} - A_{(i)N})\right)^2}{\sum_{i=1}^{n} \left(A_{(i)N} - \overline{A_{iN}}\right)^2} \; ; \; W_N \in [W_l, W_u] \; and D_N \in [D_l, D_u] \quad (5)$$

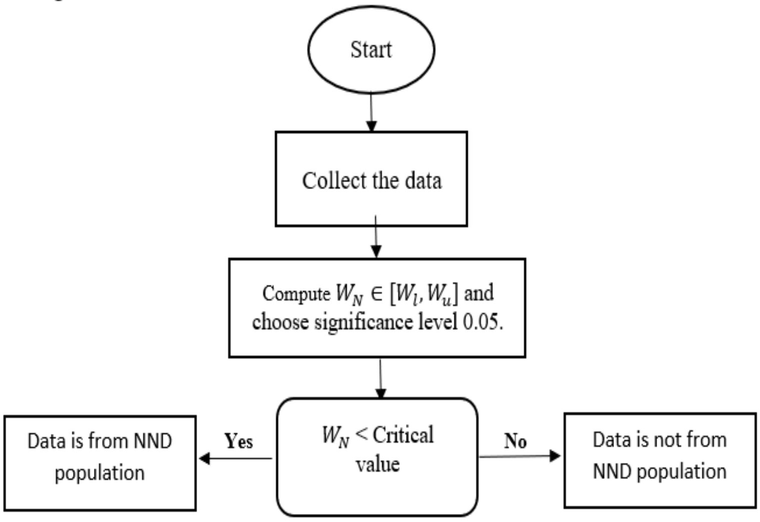


**Figure 1**. Testing procedure of NSW test

From the Shapiro-Wilks table, we choose the critical value corresponding to significance level $\alpha$. The null hypothesis, which assumes that the data conforms to theNND, is considered valid if the calculated Shapiro-Wilk test statistic ($W_N$) lies within the range of critical values. Otherwise, we can conclude that the data do not conforms the NND

## 4. Application

In this section, we will delve into the implementation of the suggested test by utilizing data of life-span of batteries in which the life time is observed for twenty-three batteries. This data set was utilized by [13]. In practice, the failure time of the batteries cannot be measured precisely becauseit's tough to know exactly when they'll stop working.But it can be measured in neutrosophic form. The lifetime in 100h of twenty-three batteries are given below:

**Table 1.** The failure time of 23 batteries (in 100h)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2.9,3.99 | 5.24,7.2 | 6.56,9.02 | 7.14,9.82 | 11.6,15.96 | 12.14,16.69 | 12.65,17.4 | 13.24,18.21 |
| 13.67,18.79 | 13.88,19.09 | 15.64,21.51 | 17.05,23.45 | 17.4,23.93 | 17.8,24.48 | 19.01,26.14 | 19.34,26.59 |
| 23.13,31.81 | 23.34,32.09 | 26.07,35.84 | 30.29,41.65 | 43.97,60.46 | 48.09,66.13 | 73.48,98.04 | |

The engineers make some claim that the average failure time of the batteries is somewhere 3000 hours. For this they need to access whether the data has been taken from normally distributed population or not. From table 1, the data is in interval form that is indeterministic in nature. Hence, it is not possible to use the SW Test under classical statistics. Therefore, we use here proposed NSW to check the normality of data.

$H_{0N}$: The sample of batteries came from the neutrosophic Normal distribution.

$H_{aN}$: The sample of batteries do not come from the neutrosophic Normal distribution.

**Step 1:**the mean of neutrosophic observations given in table 1 is

$$\overline{A_{(i)N}} = \frac{1}{n}\sum_{i=1}^{n} A_{(i)N} = \frac{1}{23}(\,473.63,647.66)$$

**Step 2:**By equation (3), theNeutrosophic Sum of Squares (NSS)is given by

$$\sum_{i=1}^{n}\left(A_{(i)N} - \overline{A_{\iota N}}\right)^2 = (6912.268,11460.13)$$

**Step 3:**The value of sample size (n) = 23 which is odd. From Shapiro-Wilk Table [19], the weights$w_i$ values corresponding to n =23 are 0.4542, 0.3126, 0.2563, 0.2139, 0.1787, 0.1480, 0.1201, 0.0941, 0.0696, 0.0459, 0.0228, 0.0000.

$$\left(\sum_{i=1}^{n} w_i (A_{(n+1-i)N} - A_{(i)N})\right)^2 = (3736.699,9183.037)$$

**Step 4:**The value of test statistic for NSW is given by

$$W_N = \frac{(3736.699,9183.037)}{(6912.268,11460.13)}$$

$$\mathrm{W_N} = (0.540589,0.801303) \text{ where } W_l = 0.540589, W_u = 0.801303$$

We found that the critical value corresponding to n = 23 and significance level 0.01 is (0.881,0.881). Since $W_N < (0.881,0.881)$, for the lifetime of batteries data. Hence it can be concluded that the lifetime of batteries follows the NND.

## 5. Comparative Study & Discussion

In this analysis, we compared the effectiveness of the provided NSW test with the traditional SW test. When working with data characterized by imprecise, uncertain, or ambiguous observations, the suggested test demonstrates greater efficiency by delivering outcomes in the form of indeterminacy.The obtained value of NSWtest statistic is $W_N \in (0.5406, 0.8013)$. It can be written as $W_N = 0.5406 + 0.8013D_N$ ; $D_N \in [0, 0.3254]$. The critical value of neutrosophic Shapiro-Wilk Test is (0.881,0.881) at 0.01significance level. Here the value 0.5406 represents the value of Classical Shapiro-Wilk Test when the $D_N = 0$ .For the significance level 0.01, "the probability of the rejecting the null hypothesis when it is true is 0.01 and probability of accepting the null hypothesis when it is true will be 0.99 and measure of indeterminacy is 0.3254". Hence it can be said that the proposed Shapiro-Wilk Test of normality under neutrosophic statistics gives the test statistic value with the measure indeterminacy $(D_N)$ while the existed Shapiro-Wilk Test of normality under classical statistics fails to provide any information about the measure of indeterminacy. Therefore, the proposed test of normality under uncertainty or neutrosophic statistic is more effective than the existed Shapiro-Wilk Test and Classical Shapiro-Wilk Test becomes special case of the suggested Shapiro-Wilk Test under neutrosophic environment when the indeterminacy $(D_N) = 0$.

## 6. Conclusion

In this paper, we suggested the Shapiro-Wilk Test under the presence of neutrosophic data. We obtained the value of test statistic of the proposed test and decision rule. We take the data of the lifetime of batteries in neutrosophic form and used the proposed model to check whether the data is taken from neutrosophic normally distribution population or not. From the obtained value of test statistic, we conclude that the lifetime of batteries follows the neutrosophic normal distribution and data can be used for further analysis under neutrosophic statistical inferences where it is necessary that the obtained data should be from neutrosophic normally distributed population. From the comparative study, the proposed test of normality under uncertainty or neutrosophic statistic is more effective than the existed Shapiro-Wilk Test and Classical Shapiro-Wilk Test becomes special case of the proposed Shapiro-Wilk Test under neutrosophic statistics when the indeterminacy $(D_N) = 0$. In future, the proposed test can be applied on some other data to check the neutrosophic normality.

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# Novel Single Valued Neutrosophic Prioritized Aczel Alsina Aggregation Operators and Their Applications in Multi-Attribute Decision Making

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**Abstract:** The Multi-attribute decision-making (MADM) approaches are utilized to aggregate ambiguous and imprecise information based on different aggregation operators (AOs). The aim of this article is to explore the notion of single-valued neutrosophic (SVN) set (SVNS), wich is the modified structure of an intuitionistic fuzzy sets and picture fuzzy sets. Some appropriate operations of Aczel Alsina tools under the system of SVN information are also presented. By using the theory of prioritization aggregation model, we developed a class of new approaches including SVN Aczel Alsina prioritized average (SVNAAPA) and SVN Aczel Alsina prioritized geometric (SVNAAPG) operators. We also presented a series of new methodologies in the light of SVN information such as SVN Aczel Alsina prioritized weighted average (SVNAAPWA), and SVN Aczel Alsina prioritized weighted geometric (SVNAAPWG) operators. To verify discussed aggregation approaches, we also presented some notabe characteristics. We established a MADM technique to solve complexities and difficulties during decision-making in our real-life problems. By utilizing a practical numerical example to select an appropriate research scientist for the vacant post of a public university. To find the validity and flexibility of our invented approaches, sensitive analysis, and comparative study by comparing the results of existing approaches with currently proposed aggregation techniques.

**Keywords:** Neutrosophic values, Single valued neutrosophic values, Aczel Alsina Aggregation operators, and Multi-attribute decision-making approach.

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## 1. Introduction

In order to choose the optimal option based on a set of criteria, decision-making is a common and daily activity in human existence. The last several years have seen extensive research and useful decision-making applications to management, economics, and other fields because of its outstanding ability to express information uncertainty. Fuzzy set theory has become more common in recent years as a way to resolve decision-making issues due to the uncertainty of decision data. Zadeh [1] anticipated the fuzzy set (FS) concepts, which have gained popularity among intellectuals. In order to deal with uncertain conditions, numerous theoretical advancements in FS have been made to date. However, in many circumstances, the notion of FS is effective. For instance, the FS theory is unable to deal with the knowledge supplied to a person in the form of positive membership value (PMV) and negative membership value (NMV). To address these issues, Atanassov [2] created the intuitionistic FS (IFS) theory by incorporating the concept of PMV into the drawbacks of FS. The limitation that the addition of the PMV and NMV is between $[0, 1]$ makes IFS significantly more useful than the current FS. When dealing with complex and unreliable data in decision-making scenarios, IFS is a thorough and powerful strategy. Many researchers have used IFS theory in a variety of fields [3], [4]. However, the IFS cannot handle such values if the sum of them exceeds the unit interval $[0, 1]$. To address such difficulties, Yager [5] investigated Pythagorean FS (PyFS). The PyFS is more effective for tackling complex, unreliable information in real-life situations. Yager [6] also modified and explored the theory of the PyFS in the framework of q-rung orthopair FS (q-ROFS), with the additional limitation that the sum of the qth power of PMV and NMV cannot be more than the unit interval $[0,1]$. The concepts of q-ROFS have received much use and have received more interest from researchers because of their structure. Many authors have applied the q-ROFS theory in ways that have been detrimental to a number of areas. Cuong [7], [8] extended the concepts of IFS, PyFS and q-ROFS using the characteristics PMV, abstinence membership value (AMV), and NMV such that the sum of PMV, AMV, and NMV restricted on interval $[0, 1]$. In order to convey ambiguous and conflicting data, Smarandache [9] anticipated the neutrosophic set (NS). A NS having PMV, AMV, and NMV are separately represented and lies in real standard or nonstandard subsets of $]^{-}0, 1^{+}[$. We may have faced many difficulties when we explored the results in nonstandard close intervals. In order to overcome this complexity, Wang et al. [10] gave the concepts of SVNS and provided the idea of interval NS [11]. Ye [12] explored the work of IFS, PFS, and NS using the system of simplified NSs to deal effectively with uncertain and inaccurate data during the decision-making process. Many research scientists explored the concepts of NS, and SVNS in the different fuzzy environments [13]–[15].

The AOs are reliable and convenient mathematical tools to easily handle inaccurate and uncertain information during aggregation. Due to the significance of AOs, several research scientists worked on different fuzzy environments. Xu [16] explored the idea of arithmetic and geometric tools using the framework of weighted averaging and geometric operators depending on IFS. Rahman et al. [17] gave some AOs of PyFSs by using the concepts of algebraic sum and algebraic product to handle imprecision information. Jan et al. [18] explored the notions of PyFS by applying the interval-valued PyFS (IVPyFS) structure to cope with ambiguous and uncertain information. Liu and Wang [19] presented AOs of q-ROFSs to solve real-life problems under a MADM approach. Garg [20] expanded the theory of IFS using the way of PFSs and anticipated some innovative AOs to handle the complexities of the fuzziness. Riaz and Farid [21] explored the theory of PFSs to handle unpredictable and imprecision information during the decision-making process by developing certain approaches. Jdid et al. [22] proposed a strong mechanism for checking the qualities of final products and developed some new mathematical approaches for the inspection of goods under their cost and benefits. A novel approach for the improvement of the sustainability and resilience of supply chain enterprises based on the theory of industry 5.0 was presented by Gamal et al. [23]. This theory has a great capability to provide strong decision under considering the decision-making process. Riaz and Hashmi [24] extended the ideology of FSs regarding Linear Diophantine FS to introduce some valuable AOs on the basis of the fundamental operations of PFSs. Liu and Jiang [25] explored the conception of distance measures in the form of interval-valued IFS (IVIFS) and used a number of AOs to deal with real-life problems under the MADM process. Mahnaz et al. [26] present detailed certain approaches to T-SFSs by utilizing the concepts of frank operators to cope with inaccurate and impression information. Ahmad et al. [27] provided some specific approaches of SFSs to deal with real-life issues under MADM techniques. Ali et al. [28] explored the theory of T-SFSs and explored the basic operation of T-SFSs. Ye [29] presented some AOs by using the correlation coefficients tools of SVNSs and interval-valued SVNSs. Wei and Zhang [30] present a few certain methodologies of Bonferroni power operators applying SVN information. Chen and Ye [31] anticipated certain approaches based on Dombi operations under the system of SVNSs. Mahmood and Ali [32] explored the theory of SVNS by utilizing complex SVNS (CSVNS) to develop certain approaches by using mathematical tools like prioritized Muirhead Mean operators. Fan et al. [33] invent a series of certain approaches to SVNS by utilizing innovative linguistic variables for the solution of complicatedness in different fuzziness. Hussain et al. [34] anticipated a series of complex IFS by using the theory of Hamy mean tools and established an application in the tourism industry. Garg [35] explored the theory of SVNS to overcome the loss of information during the aggregation process with the help of certain mathematical tools like Frank operators. Liu et al. [36] elaborated the theories of NS to develop a series of certain approaches to cope with vague and impression information in the fuzzy environment. Akram et al. [37] elaborated the structure of energy cell under the system of interval valued T-SFSs to develop certain models of Bonferroni mean operators. Ali Khan et al. [38] gave a series of certain approaches of PyFSs based on prioritized mathematical tools to express the ambiguous and vague information.

Aczel and Alsina [39] explored the theories of t-norm (TNM) and t-conorm (TCNM) to develop an innovative idea for Aczel Alsina tools in 1982. Farahbod and Eftekhari [40] compared other TNMs and TCNMs to evaluate and categorize more reliable TNMs and TCNMs after investigation. Recently several research scientists worked on different fuzzy environments to cope with uncertain and imprecise information. Senapati et al. [41] explored the idea of IFSs to establish a list of certain approaches by using the basic operations of Aczel Alsina tools to deal with real-life problems under a MADM approach. Senapati et al. [42] also utilized the basic operations of Aczel Alsina tools to develop a few certain approaches based on IVIFSs. Hussain et al. [43] generalized the concept of Aczel Alsina tools in framework of PyFSs and gave a series of certain approaches to aggregate ambiguous and uncertain information. Khan et al. [44] generalized the structure of q-ROFS and anticipated a series of certain approaches by using the basic operations of Aczel Alsina tools. Naeem et al. [45] expanded the concept of PFSs and anticipated detailed certain approaches by using basic operations of Aczel Alsina tools. Mahmood et al. [46] explored the meanings of IFS in terms of complex IFS to introduce a list of certain approaches using the basic operations of Aczel Alsina tools. Several research scientists also conceptualized the ideas of Aczel Alsina tools in different fuzzy environments seen in the references [47]–[49].

In order to handle vague information, we studied several aggregation models under considering different fuzzy circumstances. Sometimes decision-makers cannot approach an appropriate optimal option due to insufficient information on human opinions. To serve this purpose, the SVNS is a well-known aggregation model that provides the decision maker freedom in their decisions. Aczel Alsina aggregation tools have an attractive aggregation tool and play an essential role in the decision-making process. By inspiring the theory of prioritization and Aczel Alsian aggregation tools, we explored the theory of SVNSs. The main contribution of our research work is established as follows:

a) To expose the theory of SVNS with some specific properties.
b) To explore operations of Aczel Alsina aggregation tools under considering the system of SVN information.
c) By utilizing the degree of preferences of the attributes, we developed a class of new approaches based on Aczel Alsina aggregation models such as SVNAAPA and SVNAAPG operators.
d) We also proposed a series of new methodologies under the system of SVNS, including SVNAAPWA and SVNAAPWG operators.
e) To illustrate the applicability and effectiveness of our invented approaches, some notable characteristics are also demonstrated.
f) We established an algorithm of the MADM technique to resolve several real-life applications.
g) We gave a practical numerical example to find a suitable candidate for the vacant post of general manager for a multinational company. To find validity and flexibility of our invented approaches, we discussed sensitive analysis and comparative study by contrasting the results of existing approaches.
h) Additionally, some remarkable points related to our research work are expressed in the conclusion.

The structure of this manuscript is given as follows: In section 2, we studied the notion of SVNSs and its primary operations. In section 3, we revised the concepts of prioritized AOs based on SVNVs and some existing AOs based on SVNVs. In section 4, we improved the fundamental OLs of SVNVs based on Aczel Alsina operations. Section 5 listed certain approaches of SVNAAPA and SVNAAPG operators based on Aczel Alsina operations. In section 6, we anticipated the AOs of SVNAAPWA and SVNAAPWG operators with the help of weight vectors based on Aczel Alsina operations. In section 7, we evaluate a MADM technique to select a suitable research scientist by utilizing the SVNAAPWA and SVNAAPWG operators for a public university and observe the effects on the results of alternatives for different parametric values. In section 8, find the validity and reliability of our discussed approaches by contrasting the outcomes of current AOs with the result of our invented approaches. In section 9, the entire article was condensed into one paragraph and discussed the advantages of our research work.

## 2. Preliminaries

This section will study the basic definition of the neutrosophic set (NS) and single-valued neutrosophic set (SVNS). We also study some fundamental OLs of SVN value (SVNV) for further development of this article. We also provide a list of all abbreviations in Table 1.

**Definition 1:** [9] Let $X$ be a be a non-empty set and a NS $A$ in $X$ is characterized by positive membership value (PMV), abstinence membership value (AMV) and negative membership value (NMV).Then, all the terms of membership are restricted in such intervals, $\varphi_A(r) \in ]0^-, 1^+[, \delta_A(r) \in ]0^-, 1^+[$ and $\sigma_A(r) \in ]0^-, 1^+[$.

Where a PMV is denoted by $\varphi_A(r)$, AMV is denoted by $\delta_A(r)$ and a NMV is denoted by $\sigma_A(r)$.

Table 1 shows abbreviations and their meanings.

| Abbreviations | Meanings | Abbreviations | Meanings |
|---|---|---|---|
| MADM | Multi-attribute decision making | NMV | negative membership value |
| OLs | Operational laws | AMV | abstinence membership value |
| NS | neutrosophic set | FS | Fuzzy set |
| SVNS | Single-valued neutrosophic set | IFS | Intuitionistic fuzzy set |
| SVNV | Single-valued neutrosophic value | PyFS | Pythagorean fuzzy set |
| AOs | Aggregation operators | q-ROFS | q-rung orthopair fuzzy set |
| NMV | positive membership value | PFS | Picture fuzzy set |
| TNM | t-norm | TCNM | t-conorm |
| SVNAAPA | Single valued neutrosophic Aczel Alsina prioritized average. | | |
| SVNAAPG | Single-valued neutrosophic Aczel Alsina prioritized geometric. | | |
| SVNAAPWA | Single valued neutrosophic Aczel Alsina prioritized weighted average. | | |
| SVNAAPWG | Single-valued neutrosophic Aczel Alsina prioritized weighted geometric. | | |

**Definition 2:** [10] A SVNS $A$ is defined as:

$$A = \{(r, \varphi_A(r), \delta_A(r), \sigma_A(r)) | r \in X\}$$

Where $\varphi_A(r): X \to [0, 1]$, $\delta_A(r): X \to [0, 1]$ and $\sigma_A(r): X \to [0, 1]$ represent the PMV, AMV, and NMV, respectively. A SVNS satisfies such condition:

$$0 \leq \varphi_A(r) + \delta_A(r) + \sigma_A(r) \leq 3$$

A SVNV is denoted by the $\alpha = (\varphi_\alpha, \delta_\alpha, \sigma_\alpha)$.

**Definition 3:** [10] Let $\alpha = (\varphi_\alpha, \delta_\alpha, \sigma_\alpha)$ be a SVNV. Then, a score function $\mathcal{G}(\alpha)$ can be particularized as:

$$\mathcal{G}(\alpha) = \frac{2 + (\varphi_\alpha - \delta_\alpha - \sigma_\alpha)}{3} \quad (1)$$

Here, $\mathcal{G}(\alpha) \in [0, 1]$.

**Definition 4:** [10] Let $\alpha = (\varphi_\alpha, \delta_\alpha, \sigma_\alpha)$ be a SVNV. Then, an accuracy function $\mathfrak{Q}(\alpha)$ can be particularized as:

$$\mathfrak{Q}(\alpha) = \varphi_\alpha + \sigma_\alpha \tag{2}$$

Here, $\mathfrak{Q}(\alpha) \in [0,1]$.

**Definition 5:** [10] Let $\alpha = (\varphi_\alpha, \delta_\alpha, \sigma_\alpha)$ and $\beta = (\varphi_\beta, \delta_\beta, \sigma_\beta)$ be two SVNVs and $\mathcal{G}(\alpha) = \frac{2+(\varphi_\alpha - \delta_\alpha - \sigma_\alpha)}{3}$ and $\mathcal{G}(\beta) = \frac{2+(\varphi_\beta - \delta_\beta - \sigma_\beta)}{3}$ be the score values of $\alpha$ and $\beta$ respectively. Let $\mathfrak{Q}(\alpha) = \varphi_\alpha + \sigma_\alpha$ and $\mathfrak{Q}(\beta) = \varphi_\beta + \sigma_\beta$ be the accuracy values of $\alpha$ and $\beta$ respectively. Then,

i. If $\mathcal{G}(\alpha) < \mathcal{G}(\beta)$, then $\alpha < \beta$

ii. If $\mathcal{G}(\alpha) = \mathcal{G}(\beta)$ then,

a. If $\mathfrak{Q}(\alpha) < \mathfrak{Q}(\beta)$, then $\alpha < \beta$

b. If $\mathfrak{Q}(\alpha) = \mathfrak{Q}(\beta)$, then $\alpha = \beta$

**Definition 6:** [32] Let $\alpha = (\varphi_\alpha, \delta_\alpha, \sigma_\alpha)$ and $\beta = (\varphi_\beta, \delta_\beta, \sigma_\beta)$ be two SVNVs. Then, some basic operations of SVNVs are given as:

i. $\alpha \oplus \beta = (\varphi_\alpha + \varphi_\beta - \varphi_\alpha\varphi_\beta, \delta_\alpha\delta_\beta, \sigma_\alpha\sigma_\beta)$

ii. $\alpha \otimes \beta = (\varphi_\alpha\varphi_\beta, \delta_\alpha + \delta_\beta - \delta_\alpha\delta_\beta, \sigma_\alpha + \sigma_\beta - \sigma_\alpha\sigma_\beta)$

iii. $\mu\alpha = (1-(1-\varphi_\alpha)^\mu, (\delta_\alpha)^\mu, (\sigma_\alpha)^\mu)\ ,\ \mu > 0$

iv. $(\alpha)^\mu = ((\varphi_\alpha)^\mu, 1-(1-\delta_\alpha)^\mu, 1-(1-\sigma_\alpha)^\mu),\ \mu > 0$

Definition 7: [50] Let $ß = (ß_1, ß_2, \ldots, ß_n)$ be the collection of characteristics and there is a prioritization between the attributes which is represented by linear ordering $ß_1 > ß_2 > \cdots > ß_n$ shows that attribute $ß_\rho$ has a maximum priority than $ß_k$, if $\rho < k$. The values $ß_\rho(r)$ shows the performance of any alternative $r$ under the attribute $ß_\rho$, and satisfies $ß_\rho(r) \in [0,1]$. The prioritized average operator (PA) is defined as if it satisfies such axiom:

$$PA\left(\tau_\rho(r)\right) = \sum_{\rho=1}^{n} \omega_\rho \tau_\rho(r) \tag{3}$$

Where $\boldsymbol{\omega}_\rho = \frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}, \boldsymbol{\varepsilon}_\rho = \prod_{k=1}^{\rho-1}\mathcal{G}(\boldsymbol{\tau}_k), \rho = \mathbf{2}, \mathbf{3}, \ldots, \boldsymbol{n}$. The initial value $\boldsymbol{\varepsilon}_1 = \mathbf{1}$ and $\mathcal{G}(\boldsymbol{\tau}_k)$ represents score values of $\boldsymbol{k}^{th}$ SVNVs. Then, PA is called the prioritized averaging (PA) operator.

Definition 8: [50] Let $\tau_\rho = (\varphi_\rho, \delta_\rho, \sigma_\rho), \rho = 1, 2, 3, \ldots, n$ be the collection of SVNVs, with PA $\omega_\rho = \frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}$ operator for each $\tau_\rho$. Then, SVN prioritized averaging (SVNPA) operator is particularized as:

$$SVNPA(\tau_1, \tau_2, \ldots, \tau_n) = \varepsilon_1\tau_1 \oplus \varepsilon_2\tau_2 \oplus \ldots \oplus \varepsilon_n\tau_n$$

Where $\varepsilon_\rho = \prod_{k=1}^{\rho-1} \mathcal{G}(\tau_\rho), \rho = 2, 3, \ldots, n$. The initial value of $\varepsilon_1 = 1$ and $\mathcal{G}(\tau_k)$ represents the score value of $k^{th}$ SVNVs.

Definition 9: [50] Let $\tau_\rho = (\varphi_\rho, \delta_\rho, \sigma_\rho), \rho = 1, 2, 3, \dots, n$ be the collection of SVNVs with PA $\omega_\rho = \frac{\varepsilon_\rho}{\sum_{\rho=1}^{n} \varepsilon_\rho}$ operator for each $\tau_\rho$. Then, the SVN prioritized geometric (SVNPG) operator is particularized as:

$$SVNPG(\tau_1, \tau_2, \dots, \tau_n) = \tau_1^{\varepsilon_1} \otimes \tau_2^{\varepsilon_2} \otimes \dots \otimes \tau_n^{\varepsilon_n}$$

Where $\boldsymbol{\varepsilon}_\rho = \prod_{k=1}^{\rho-1} \mathcal{G}(\boldsymbol{\tau}_\rho), \rho = \mathbf{2}, \mathbf{3}, \dots, \boldsymbol{n}$. The initial value of $\boldsymbol{\varepsilon_1} = \mathbf{1}$ and $\mathcal{G}(\boldsymbol{\tau_k})$ represents the score values of $\boldsymbol{k^{th}}$ SVNVs.

## 3. Basic Operations of Aczel Alsina tools Based on Single-Valued Neutrosophic Information

In this section, we will demonstrate Aczel Alsina operations in the form of sum, product, scalar multiplication, and power rule using SVNV data.

**Definition 10:** Let $\tau = (\varphi, \delta, \sigma), \tau_1 = (\varphi_1, \delta_1, \sigma_1)$ and $\tau_2 = (\varphi_2, \delta_2, \sigma_2)$ be the three SVNVs, $\not{C} \geq 1$ and $\mu > 0$. Then, we illustrate some basic operations of Aczel Alsina tools in the following form:

i. $$\tau_1 \oplus \tau_2 = \begin{pmatrix} 1 - e^{-\left(\left(-ln(1-\varphi_1)\right)^{\not{C}} + \left(-ln(1-\varphi_2)\right)^{\not{C}}\right)^{\frac{1}{\not{C}}}}, \\ e^{-\left(\left(-ln(\delta_1)\right)^{\not{C}} + \left(-ln(\delta_2)\right)^{\not{C}}\right)^{\frac{1}{\not{C}}}}, \\ e^{-\left(\left(-ln(\sigma_1)\right)^{\not{C}} + \left(-ln(\sigma_2)\right)^{\not{C}}\right)^{\frac{1}{\not{C}}}} \end{pmatrix}$$

ii. $$\tau_1 \otimes \tau_2 = \begin{pmatrix} e^{-\left(\left(-ln(\varphi_1)\right)^{\not{C}} + \left(-ln(\varphi_2)\right)^{\not{C}}\right)^{\frac{1}{\not{C}}}}, \\ 1 - e^{-\left(\left(-ln(1-\delta_1)\right)^{\not{C}} + \left(-ln(1-\delta_2)\right)^{\not{C}}\right)^{\frac{1}{\not{C}}}}, \\ 1 - e^{-\left(\left(-ln(1-\sigma_1)\right)^{\not{C}} + \left(-ln(1-\sigma_2)\right)^{\not{C}}\right)^{\frac{1}{\not{C}}}} \end{pmatrix}$$

iii. $$\mu\tau = \begin{pmatrix} 1 - e^{-\left(\mu\left(-ln(1-\varphi_r)\right)^{\not{C}}\right)^{\frac{1}{\not{C}}}}, \\ e^{-\left(\mu\left(-ln(\delta_r)\right)^{\not{C}}\right)^{\frac{1}{\not{C}}}}, \\ e^{-\left(\mu\left(-ln(\sigma_r)\right)^{\not{C}}\right)^{\frac{1}{\not{C}}}} \end{pmatrix}$$

iv. $$\tau^\mu = \begin{pmatrix} e^{-\left(\mu\left(-ln(\varphi_r)\right)^{\not{C}}\right)^{\frac{1}{\not{C}}}}, \\ 1 - e^{-\left(\mu\left(-ln(1-\delta_r)\right)^{\not{C}}\right)^{\frac{1}{\not{C}}}}, \\ 1 - e^{-\left(\mu\left(-ln(1-\sigma_r)\right)^{\not{C}}\right)^{\frac{1}{\not{C}}}} \end{pmatrix}$$

**Theorem 1:** Let $\tau = (\varphi, \delta, \sigma), \tau_1 = (\varphi_1, \delta_1, \sigma_1)$ and $\tau_2 = (\varphi_2, \delta_2, \sigma_2)$ be the three SVNVs with $\not{C} \geq 1$ and $\mu > 0$. Then, a few fundamental OLs are defined as follows:

i. $\tau_1 \oplus \tau_2 = \tau_2 \oplus \tau_1$

ii. $\tau_1 \otimes \tau_2 = \tau_2 \otimes \tau_1$

iii. $\mu(\tau_1 \oplus \tau_2) = \mu\tau_1 \oplus \mu\tau_2, \mu > 0$

iv. $(\mu_1 + \mu_2)\tau = \mu_1\tau + \mu_2\tau, \mu_1, \mu_2 > 0$

v. $(\tau_1 \otimes \tau_2)^{\mu} = \tau_1^{\mu} \otimes \tau_2^{\mu}, \mu > 0$

vi. $\tau^{\mu_1} \otimes \tau^{\mu_2} = \tau^{(\mu_1+\mu_2)} \;, \mu_1, \mu_2 > 0$

## 4. Single Valued Neutrosophic Aczel Alsina Prioritized Aggregation Operators

In this section, we will narrate some certain approaches of SVNVs based on Aczel Alsina operations and elaborate on some characteristics of our aimed work. We also extend our work to the weighted averaging and weighted geometric operators.

**Definition 11:** Let $\tau_{\rho} = (\varphi_{\rho}, \delta_{\rho}, \sigma_{\rho}), \rho = 1, 2, 3, \dots, n$ be the collection of SVNVs with PA $\omega_{\rho} = \frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n} \varepsilon_{\rho}}$ operator for each $\tau_{\rho}$. Then SVNAAPA operator is particularized as:

$$SVNAAPA(\tau_1, \tau_2, \dots, \tau_n) = \bigoplus_{\rho=1}^{n} \left( \frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n} \varepsilon_{\rho}} \tau_{\rho} \right)$$
$$SVNAAPA(\tau_1, \tau_2, \dots, \tau_n) = \left( \frac{\varepsilon_1}{\sum_{\rho=1}^{n} \varepsilon_{\rho}} \right) \tau_1 \oplus \left( \frac{\varepsilon_2}{\sum_{\rho=1}^{n} \varepsilon_{\rho}} \right) \tau_2 \oplus \dots \oplus \left( \frac{\varepsilon_n}{\sum_{\rho=1}^{n} \varepsilon_{\rho}} \right) \tau_n \quad (4)$$

Where $\varepsilon_{\rho} = \prod_{k=1}^{\rho-1} \mathcal{G}(\tau_k), \rho = 2, 3, \dots, n.$

**Theorem 2**: Let $\tau_{\rho} = (\varphi_{\rho}, \delta_{\rho}, \sigma_{\rho}), \rho = 1,2,3, \dots, n$ be the collection of SVNVs with PA $\omega_{\rho} = \frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n} \varepsilon_{\rho}}$ operator for each $\tau_{\rho}$. Then, the SVNAAPA operator is particularized as:

$$SVNAAPA = \sum_{\rho=1}^{n} \left( \frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n} \varepsilon_{\rho}} \tau_{\rho} \right) = \begin{pmatrix} 1 - e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n} \varepsilon_{\rho}}\right)\left(-ln(1-\varphi_{\rho})\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}}, \\ e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n} \varepsilon_{\rho}}\right)\left(-ln(\delta_{\rho})\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}}, \\ e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n} \varepsilon_{\rho}}\right)\left(-ln(\sigma_{\rho})\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}} \end{pmatrix} \quad (5)$$

**Proof:** We will proof this theorem with the help of a mathematical induction technique in the following way:

**i.** Take the value of $\rho = 2$ depends on Aczel Alsina operations of SVNVs, we get,

$$\left( \frac{\varepsilon_1}{\sum_{\rho=1}^{n} \varepsilon_{\rho}} \tau_1 \right) = \begin{pmatrix} 1 - e^{-\left(\left(\frac{\varepsilon_1}{\sum_{\rho=1}^{n} \varepsilon_{\rho}}\right)\left(-ln(1-\varphi_1)\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}}, \\ e^{-\left(\left(\frac{\varepsilon_1}{\sum_{\rho=1}^{n} \varepsilon_{\rho}}\right)\left(-ln(\delta_1)\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}}, \\ e^{-\left(\left(\frac{\varepsilon_1}{\sum_{\rho=1}^{n} \varepsilon_{\rho}}\right)\left(-ln(\sigma_1)\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}} \end{pmatrix}$$

$$\left(\frac{\varepsilon_2}{\sum_{\rho=1}^{n}\varepsilon_\rho}\tau_2\right)=\begin{pmatrix}1-e^{-\left(\left(\frac{\varepsilon_2}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(1-\varphi_2)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\left(\frac{\varepsilon_2}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\delta_2)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\left(\frac{\varepsilon_2}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\sigma_2)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}}\end{pmatrix}$$

By using the above Definition 11, we have:

$$SVNAAPA(\tau_1,\tau_2)=\left(\begin{pmatrix}1-e^{-\left(\left(\frac{\varepsilon_1}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(1-\varphi_1)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\left(\frac{\varepsilon_1}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\delta_1)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\left(\frac{\varepsilon_1}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\sigma_1)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}}\end{pmatrix}\oplus\begin{pmatrix}1-e^{-\left(\left(\frac{\varepsilon_2}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(1-\varphi_2)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\left(\frac{\varepsilon_2}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\delta_2)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\left(\frac{\varepsilon_2}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\sigma_2)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}}\end{pmatrix}\right)$$

$$=\begin{pmatrix}1-e^{-\left(\left(\left(\frac{\varepsilon_1}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(1-\varphi_1)\right)^{\mathbb{C}}\right)\oplus\left(\left(\frac{\varepsilon_2}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(1-\varphi_2)\right)^{\mathbb{C}}\right)\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\left(\left(\frac{\varepsilon_1}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\delta_1)\right)^{\mathbb{C}}\right)\oplus\left(\left(\frac{\varepsilon_2}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\delta_2)\right)^{\mathbb{C}}\right)\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\left(\left(\frac{\varepsilon_1}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\sigma_1)\right)^{\mathbb{C}}\right)\oplus\left(\left(\frac{\varepsilon_2}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\sigma_1)\right)^{\mathbb{C}}\right)\right)^{\frac{1}{\mathbb{C}}}}\end{pmatrix}$$

$$=\begin{pmatrix}1-e^{-\left(\sum_{\rho=1}^{2}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln\left(1-\varphi_\rho\right)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\sum_{\rho=1}^{2}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln\left(\delta_\rho\right)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\sum_{\rho=1}^{2}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln\left(\sigma_\rho\right)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}}\end{pmatrix}$$

Hence, this is true for $\rho = 2$.

**ii.** Now, suppose that this is true for $\rho = k$. Then, we have:

$$SVNAAPA(\tau_1,\tau_2,\dots,\tau_n\ ) = \sum_{\rho=1}^{k}\left(\frac{\varepsilon_k}{\sum_{\rho=1}^{n}\varepsilon_\rho}\tau_k\right) = \begin{pmatrix} 1-e^{-\left(\sum_{\rho=1}^{k}\left(\frac{\varepsilon_k}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-\ ln(1-\varphi_k)\right)^{\cancel{C}}\right)^{\frac{1}{\cancel{C}}}}, \\ e^{-\left(\sum_{\rho=1}^{k}\left(\frac{\varepsilon_k}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\delta_k)\right)^{\cancel{C}}\right)^{\frac{1}{\cancel{C}}}}, \\ e^{-\left(\sum_{\rho=1}^{k}\left(\frac{\varepsilon_k}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\sigma_k)\right)^{\cancel{C}}\right)^{\frac{1}{\cancel{C}}}} \end{pmatrix}$$

Now, for $\rho = k+1$. We get,

$$SVNAAPA(\tau_1,\tau_2,\dots,\tau_k,\tau_{k+1}) = \bigoplus_{\rho=1}^{k}\left(\left(\frac{\varepsilon_k}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\tau_k \oplus \left(\frac{\varepsilon_{k+1}}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\tau_{k+1}\right)$$

$$= \begin{pmatrix} \begin{pmatrix} 1-e^{-\left(\sum_{\rho=1}^{k}\left(\frac{\varepsilon_k}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-\ ln(1-\varphi_k)\right)^{\cancel{C}}\right)^{\frac{1}{\cancel{C}}}}, \\ e^{-\left(\sum_{\rho=1}^{k}\left(\frac{\varepsilon_k}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\delta_k)\right)^{\cancel{C}}\right)^{\frac{1}{\cancel{C}}}}, \\ e^{-\left(\sum_{\rho=1}^{k}\left(\frac{\varepsilon_k}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\sigma_k)\right)^{\cancel{C}}\right)^{\frac{1}{\cancel{C}}}} \end{pmatrix} \oplus \\ \begin{pmatrix} 1-e^{-\left(\left(\frac{\varepsilon_{k+1}}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(1-\varphi_{k+1})\right)^{\cancel{C}}\right)^{\frac{1}{\cancel{C}}}}, \\ e^{-\left(\left(\frac{\varepsilon_{k+1}}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\delta_{k+1})\right)^{\cancel{C}}\right)^{\frac{1}{\cancel{C}}}}, \\ e^{-\left(\left(\frac{\varepsilon_{k+1}}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\sigma_{k+1})\right)^{\cancel{C}}\right)^{\frac{1}{\cancel{C}}}} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1-e^{-\left(\sum_{\rho=1}^{k+1}\left(\frac{\varepsilon_{k+1}}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-\ ln(1-\varphi_{k+1})\right)^{\cancel{C}}\right)^{\frac{1}{\cancel{C}}}}, \\ e^{-\left(\sum_{\rho=1}^{k+1}\left(\frac{\varepsilon_{k+1}}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\delta_{k+1})\right)^{\cancel{C}}\right)^{\frac{1}{\cancel{C}}}}, \\ e^{-\left(\sum_{\rho=1}^{k+1}\left(\frac{\varepsilon_{k+1}}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\sigma_{k+1})\right)^{\cancel{C}}\right)^{\frac{1}{\cancel{C}}}} \end{pmatrix}$$

Hence proved.

**Example 1:** Let $(0.54, 0.98, 0.27), (0.87, 0.55, 0.61)$, $(0.49, 0.33, 0.72)$ and $(0.11, 0.39, 0.27)$ be the four SVNVs with $\cancel{C} = 3$. Then, SVNAAPA can be calculated as:

$$SVNAAPA(\tau_1,\tau_2,\tau_3,\tau_4)=\begin{pmatrix}1-e^{-\left(\begin{array}{c}\left(\frac{\varepsilon_1}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)(-ln(1-\varphi_1))^{\mathbb{C}}+\left(\frac{\varepsilon_2}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)(-ln(1-\varphi_2))^{\mathbb{C}}+\\ \left(\frac{\varepsilon_3}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)(-ln(1-\varphi_3))^{\mathbb{C}}+\left(\frac{\varepsilon_4}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)(-ln(1-\varphi_4))^{\mathbb{C}}\end{array}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\begin{array}{c}\left(\frac{\varepsilon_1}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)(-ln(\delta_1))^{\mathbb{C}}+\left(\frac{\varepsilon_2}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)(-ln(\delta_2))^{\mathbb{C}}+\\ \left(\frac{\varepsilon_3}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)(-ln(\delta_3))^{\mathbb{C}}+\left(\frac{\varepsilon_4}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)(-ln(\delta_4))^{\mathbb{C}}\end{array}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\begin{array}{c}\left(\frac{\varepsilon_1}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)(-ln(\sigma_1))^{\mathbb{C}}+\left(\frac{\varepsilon_2}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)(-ln(\sigma_2))^{\mathbb{C}}+\\ \left(\frac{\varepsilon_3}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)(-ln(\sigma_3))^{\mathbb{C}}+\left(\frac{\varepsilon_4}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)(-ln(\sigma_4))^{\mathbb{C}}\end{array}\right)^{\frac{1}{\mathbb{C}}}}\end{pmatrix}$$

$$=\begin{pmatrix}1-e^{-\left(\begin{array}{c}\left(\frac{1}{1.7927}\right)(-ln(1-0.54))^3+\left(\frac{0.4300}{1.7927}\right)(-ln(1-0.87))^3+\\ \left(\frac{0.2451}{1.7927}\right)(-ln(1-0.49))^3+\left(\frac{0.1176}{1.7927}\right)(-ln(1-0.11))^3\end{array}\right)^{\frac{1}{3}}},\\ e^{-\left(\begin{array}{c}\left(\frac{1}{1.7927}\right)\left((-ln(0.98))^3\right)+\left(\frac{0.4300}{1.7927}\right)\left((-ln(0.55))^3\right)+\\ \left(\frac{0.2451}{1.7927}\right)\left((-ln(0.33))^3\right)+\left(\frac{0.1176}{1.7927}\right)\left((-ln(0.39))^3\right)\end{array}\right)^{\frac{1}{3}}},\\ e^{-\left(\begin{array}{c}\left(\frac{1}{1.7927}\right)\left((-ln(0.27))^3\right)+\left(\frac{0.4300}{1.7927}\right)\left((-ln(0.61))^3\right)+\\ \left(\frac{0.2451}{1.7927}\right)\left((-ln(0.72))^3\right)+\left(\frac{0.1176}{1.7927}\right)\left((-ln(0.27))^3\right)\end{array}\right)^{\frac{1}{3}}}\end{pmatrix}$$

$$SVNAAPA(\tau_1,\tau_2,\tau_3,\tau_4)=(0.7349, 0.5149, 0.3239)$$

**Theorem 3:** If all $\tau_\rho=(\varphi_\rho,\delta_\rho,\sigma_\rho), \rho=1,2,3,\dots,n$ are equal, that is, $\tau_\rho=\tau$ for all $\tau$. Then, we have:

$$SVNAAPA(\tau_1,\tau_2,\tau_3,\dots,\tau_n)=\tau$$

**Proof:** Since $\tau_\rho=(\varphi_\rho,\delta_\rho,\sigma_\rho), \rho=1,2,3,\dots,n$. Then,

$$SVNAAPA(\tau_1,\tau_2,\dots,\tau_n)=\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\tau_\rho\right)$$

$$=\begin{pmatrix}1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(1-\varphi_\rho)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\delta_\rho)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\sigma_\rho)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}}\end{pmatrix}$$

$$=\left(1-e^{-\left(-ln\left(1-(\varphi_\rho)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}}, e^{-\left(\left(-ln(\delta_\rho)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}}, e^{-\left(\left(-ln(\sigma_\rho)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}}\right)=\tau$$

Thus, it is obvious that $SVNAAPA(\tau_1,\tau_2,\dots,\tau_n)=\tau$ holds.

**Theorem 4:** Let $\tau_\rho=(\varphi_\rho,\delta_\rho,\sigma_\rho), \rho=1,2,3,\dots,n$ be the collection of SVNVs, with PA $\omega_\rho=\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}$ operator for each $\tau_\rho$. Let $\tau^-=min(\tau_1,\tau_2,\dots,\tau_n)$ and $\tau^+=max(\tau_1,\tau_2,\dots,\tau_n)$. So,

$$\tau^-\le SVNAAPA(\tau_1,\tau_2,\dots,\tau_n)\le\tau^+$$

**Proof:** Let $\tau_{\rho} = (\varphi_{\rho}, \delta_{\rho}, \sigma_{\rho}), \rho = 1,2,3, \dots, n$ be the collection of SVNVs, with PA $\omega_{\rho} = \frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n} \varepsilon_{\rho}}$ operator for each $\tau_{\rho}$. Let $\tau^{-} = min_{\rho}(\tau_1, \tau_2, \dots, \tau_n) = (\varphi^{-}, \delta^{-}, \sigma^{-})$, and $\tau^{+} = max_{\rho}(\tau_1, \tau_2, \dots, \tau_n) = (\varphi^{+}, \delta^{+}, \sigma^{+})$. We have, $\varphi^{-} = min_{\rho}\{\varphi_{\rho}\}$, $\delta^{-} = max_{\rho}\{\delta_{\rho}\}$ and $\sigma^{-} = max_{\rho}\{\sigma_{\rho}\}$ and $\varphi^{+} = max_{\rho}\{\varphi_{\rho}\}$, $\delta^{+} = min_{\rho}\{\delta_{\rho}\}$, and $\sigma^{+} = min_{\rho}\{\sigma_{\rho}\}$. Hence, there is the following result for the inequalities:

$$1 - e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(1-\varphi^{-}))^{\not{c}}\right)^{\frac{1}{\not{c}}}} \leq 1 - e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(1-\varphi_{\rho}))^{\not{c}}\right)^{\frac{1}{\not{c}}}} \leq 1 - e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(1-\varphi^{+}))^{\not{c}}\right)^{\frac{1}{\not{c}}}}$$

$$e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(\delta^{-}))^{\not{c}}\right)^{\frac{1}{\not{c}}}} \leq e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(\delta_{\rho}))^{\not{c}}\right)^{\frac{1}{\not{c}}}} \leq e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(\delta^{+}))^{\not{c}}\right)^{\frac{1}{\not{c}}}}$$

Similarly,

$$e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(\sigma^{-}))^{\not{c}}\right)^{\frac{1}{\not{c}}}} \leq e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(\sigma_{\rho}))^{\not{c}}\right)^{\frac{1}{\not{c}}}} \leq e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(\sigma^{+}))^{\not{c}}\right)^{\frac{1}{\not{c}}}}$$

So,

$$\tau^{-} \leq SVNAAPA(\tau_1, \tau_2, \dots, \tau_n) \leq \tau^{+}$$

**Theorem 5**: Let $\tau_{\rho}$ and $\tau'_{\rho}, \rho = 1, 2, 3, \dots, n$ be two sets of SVNVs, if $\tau_{\rho} \leq \tau'_{\rho}$ For all $\tau$. So, $SVNAAPA(\tau_1, \tau_2, \dots, \tau_n) \leq SVNAAPA(\tau'_1, \tau'_2, \dots, \tau'_n)$

**Proof:** Let $\tau_{\rho}$ and $\tau'_{\rho}, \rho = 1, 2, 3, \dots, n$ be two sets of SVNVs, we can write in the following form:

$$1 - e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(1-\varphi_{\rho}))^{\not{c}}\right)^{\frac{1}{\not{c}}}} \leq 1 - e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(1-\varphi'_{\rho}))^{\not{c}}\right)^{\frac{1}{\not{c}}}}$$

$$e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(\delta_{\rho}))^{\not{c}}\right)^{\frac{1}{\not{c}}}} \geq e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(\delta'_{\rho}))^{\not{c}}\right)^{\frac{1}{\not{c}}}}$$

$$e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(\sigma_{\rho}))^{\not{c}}\right)^{\frac{1}{\not{c}}}} \geq e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(\sigma'_{\rho}))^{\not{c}}\right)^{\frac{1}{\not{c}}}}$$

Hence, it is proved that $SVNAAPA(\tau_1, \tau_2, \dots, \tau_n) \leq SVNAAPA(\tau'_1, \tau'_2, \dots, \tau'_n)$.

**Definition 12:** Let $\tau_{\rho} = (\varphi_{\rho}, \delta_{\rho}, \sigma_{\rho}), \rho = 1, 2, 3, \dots, n$ be the collection of SVNVs, with PA $\omega_{\rho} = \frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n} \varepsilon_{\rho}}$ operator for each $\tau_{\rho}$. Then, the SVNAAPG operator is particularized as:

$$SVNAAPG(\tau_1, \tau_2, \dots, \tau_n) = \bigotimes_{\rho=1}^{n}\left(\tau_n^{\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)}\right) \tag{6}$$

$$SVNAAPG(\tau_1, \tau_2, \dots, \tau_n) = \tau_1^{\left(\frac{\varepsilon_1}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)} \otimes \tau_2^{\left(\frac{\varepsilon_2}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)} \otimes \dots \otimes \tau_{\not{c}}^{\left(\frac{\varepsilon_n}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)}$$

Where $\varepsilon_{\rho} = \prod_{k=1}^{\rho-1} \mathcal{G}(\tau_k), \rho = 2, 3, \dots, n$. The initial value of $\varepsilon_1 = 1$ and $\mathcal{G}(\tau_k)$ represents the score values of $k^{th}$ SVNVs.

**Theorem 6**: Let $\tau_{\rho} = (\varphi_{\rho}, \delta_{\rho}, \sigma_{\rho}), \rho = 1, 2, 3, \ldots, n$ be the collection of SVNVs, with PA $\omega_{\rho} = \frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n} \varepsilon_{\rho}}$ operator for each $\tau_{\rho}$. Then, the SVNAAPG operator is particularized as:

$$SVNAAPG = \prod_{\rho=1}^{n}\left(\tau_{\rho}^{\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)}\right) = \begin{pmatrix} e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)\left(-ln(\varphi_{\rho})\right)^{\cent}\right)^{\frac{1}{\cent}}}, \\ 1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)\left(-ln(1-\delta_{\rho})\right)^{\cent}\right)^{\frac{1}{\cent}}}, \\ 1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)\left(-ln(1-\sigma_{\rho})\right)^{\cent}\right)^{\frac{1}{\cent}}} \end{pmatrix} \quad (7)$$

**Proof:** Proof is similar to the theorem 2.

**Example 2:** Let $(0.59, 0.73, 0.34), (0.45, 0.56, 0.67), (0.78, 0.89, 0.9)$ and $(0.51, 0.82, 0.65)$ be the four SVNVs with $\varepsilon_1 = 1, \varepsilon_2 = 0.5067, \varepsilon_3 = 0.2060$ and $\varepsilon_4 = 0.0680$ and $\cent = 3$. Then, SVNAAPG can be calculated as:

$$SVNAAPG(\tau_1,\tau_2,\tau_3,\tau_4) == \begin{pmatrix} e^{-\left(\begin{matrix}\left(\frac{\varepsilon_1}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(\varphi_1))^{\cent}+\left(\frac{\varepsilon_2}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(\varphi_2))^{\cent}+ \\ \left(\frac{\varepsilon_3}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(\varphi_3))^{\cent}+\left(\frac{\varepsilon_4}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(\varphi_4))^{\cent}\end{matrix}\right)^{\frac{1}{\cent}}}, \\ 1-e^{-\left(\begin{matrix}\left(\frac{\varepsilon_1}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(1-\delta_1))^{\cent}+\left(\frac{\varepsilon_2}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(1-\delta_2))^{\cent}+ \\ \left(\frac{\varepsilon_3}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(1-\delta_3))^{\cent}+\left(\frac{\varepsilon_4}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(1-\delta_4))^{\cent}\end{matrix}\right)^{\frac{1}{\cent}}}, \\ 1-e^{-\left(\begin{matrix}\left(\frac{\varepsilon_1}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(1-\sigma_1))^{\cent}+\left(\frac{\varepsilon_2}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(1-\sigma_2))^{\cent}+ \\ \left(\frac{\varepsilon_3}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(1-\sigma_3))^{\cent}+\left(\frac{\varepsilon_4}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}\right)(-ln(1-\sigma_4))^{\cent}\end{matrix}\right)^{\frac{1}{\cent}}} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-\left(\begin{matrix}\left(\frac{1}{1.7807}\right)(-ln(0.59))^{3}+\left(\frac{0.5067}{1.7807}\right)(-ln(0.45))^{3}+ \\ \left(\frac{0.2060}{1.7807}\right)(-ln(0.78))^{3}+\left(\frac{0.0680}{1.7807}\right)(-ln(0.51))^{3}\end{matrix}\right)^{\frac{1}{3}}}, \\ 1-e^{-\left(\begin{matrix}\left(\frac{1}{1.7807}\right)(-ln(1-0.73))^{3}+\left(\frac{0.5067}{1.7807}\right)(-ln(1-0.56))^{3}+ \\ \left(\frac{0.2060}{1.7807}\right)(-ln(1-0.89))^{3}+\left(\frac{0.0680}{1.7807}\right)(-ln(1-0.82))^{3}\end{matrix}\right)^{\frac{1}{3}}}, \\ 1-e^{-\left(\begin{matrix}\left(\frac{1}{1.7807}\right)(-ln(1-0.34))^{3}+\left(\frac{0.5067}{1.7807}\right)(-ln(1-0.67))^{3}+ \\ \left(\frac{0.2060}{1.7807}\right)(-ln(1-0.9))^{3}+\left(\frac{0.0680}{1.7807}\right)(-ln(1-0.65))^{3}\end{matrix}\right)^{\frac{1}{3}}} \end{pmatrix}$$

$$SVNAAPG(\tau_1,\tau_2,\tau_3,\tau_4) = (0.5368, 0.7579, 0.7092)$$

**Theorem 7:** If all $\tau_{\rho} = (\varphi_{\rho}, \delta_{\rho}, \sigma_{\rho}), \rho = 1, 2, 3, \ldots, n$ are equal, that is, $\tau_n = \tau$ for all $\tau$. Then,

$$SVNAAPG(\tau_1, \tau_2, \ldots, \tau_n) = \tau.$$

**Proof:** Since $\tau_{\rho} = (\varphi_{\rho}, \delta_{\rho}, \sigma_{\rho}), \rho = 1,2,3,\ldots,n$. Then,

$$SVNAAPG(\tau_1, \tau_2, \ldots, \tau_n) = \prod_{\rho=1}^{n}\left(\tau_{\rho}^{\frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\varepsilon_{\rho}}}\right)$$

$$SVNAAPG(\tau_1,\tau_2,\dots,\tau_n) = \begin{pmatrix} e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\varphi_\rho)\right)^{\not{c}}\right)^{\frac{1}{\not{c}}}}, \\ 1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(1-\delta_\rho)\right)^{\not{c}}\right)^{\frac{1}{\not{c}}}}, \\ 1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(1-\sigma_\rho)\right)^{\not{c}}\right)^{\frac{1}{\not{c}}}} \end{pmatrix}$$

$$= \left( e^{-\left(\left(-ln(\varphi_{\not{c}})\right)^{\not{c}}\right)^{\frac{1}{\not{c}}}}, 1-e^{-\left(\left(-ln(1-\delta_{\not{c}})\right)^{\not{c}}\right)^{\frac{1}{\not{c}}}}, 1-e^{-\left(\left(-ln(1-\sigma_{\not{c}})\right)^{\not{c}}\right)^{\frac{1}{\not{c}}}} \right) = \tau$$

Thus, it is obvious that $SVNAAPG(\tau_1,\tau_2,\dots,\tau_n) = \tau$ holds.

**Theorem 8:** Let $\tau_\rho = (\varphi_\rho, \delta_\rho, \sigma_\rho), \rho = 1,2,3,\dots,n$ be the collection of SVNVs, with PA $\omega_\rho = \frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}$ operator for each $\tau_\rho$. Let $\tau^- = min(\tau_1,\tau_2,\dots,\tau_n)$ and $\tau^+ = max(\tau_1,\tau_2,\dots,\tau_n)$. Then,

$$\tau^- \le SVNAAPG(\tau_1,\tau_2,\dots,\tau_n) \le \tau^+.$$

**Proof:** Let $\tau_\rho = (\varphi_\rho, \delta_\rho, \sigma_\rho), \rho = 1,2,3,\dots,n$ be the collection of SVNVs with PA $\omega_\rho = \frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}$ operator for each $\tau_\rho$. Let $\tau^- = min_\rho(\tau_1,\tau_2,\dots,\tau_n) = (\varphi^-,\delta^-,\sigma^-)$, and $\tau^+ = max_\rho(\tau_1,\tau_2,\dots,\tau_n) = (\varphi^+,\delta^+,\sigma^+)$. We have, $\varphi^- = min_\rho\{\varphi_\rho\}$, $\delta^- = max_\rho\{\delta_\rho\}$ and $\sigma^- = max_\rho\{\sigma_\rho\}$ and $\varphi^+ = max_\rho\{\varphi_\rho\}$, $\delta^+ = min_\rho\{\delta_\rho\}$, and $\sigma^+ = min_\rho\{\sigma_\rho\}$. Hence, there is the following result for the inequalities:

$$e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\varphi^-)\right)^{\not{c}}\right)^{\frac{1}{\not{c}}}} \le e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\varphi_\rho)\right)^{\not{c}}\right)^{\frac{1}{\not{c}}}} \le e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\varphi^+)\right)^{\not{c}}\right)^{\frac{1}{\not{c}}}}$$

$$1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(1-\delta^-)\right)^{\not{c}}\right)^{\frac{1}{\not{c}}}} \le 1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(1-\delta_\rho)\right)^{\not{c}}\right)^{\frac{1}{\not{c}}}} \le 1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(1-\delta^+)\right)^{\not{c}}\right)^{\frac{1}{\not{c}}}}$$

$$1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(1-\sigma^-)\right)^{\not{c}}\right)^{\frac{1}{\not{c}}}} \le 1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(1-\sigma_\rho)\right)^{\not{c}}\right)^{\frac{1}{\not{c}}}} \le 1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(1-\sigma^+)\right)^{\not{c}}\right)^{\frac{1}{\not{c}}}}$$

So, $\tau^- \le SVNAAPG(\tau_1,\tau_2,\dots,\tau_n) \le \tau^+$ holds.

Theorem 9: Let $\tau_\rho$ and $\tau'_\rho, \rho = 1,2,3,\dots,n$ be two sets of SVNVs, if $\tau_\rho \le \tau'_\rho$ for all $\tau$. Then, $SVNAAPG(\tau_1,\tau_2,\dots,\tau_n) \le SVNAAPG(\tau'_1,\tau'_2,\dots,\tau'_n)$.

Proof: Let $\tau_\rho$ and $\tau'_\rho, \rho = 1,2,3,\dots,n$ be two sets of SVNVs. Then, we can write in the following way:

$$e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\varphi_\rho)\right)^{\not{c}}\right)^{\frac{1}{\not{c}}}} \le e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln(\varphi'_\rho)\right)^{\not{c}}\right)^{\frac{1}{\not{c}}}}$$

And,

$$1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln\left(1-\delta_\rho\right)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}} \geq 1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln\left(1-\delta'_\rho\right)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}}$$

Similarly, we get:

$$1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln\left(1-\sigma_\rho\right)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}} \geq 1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}\right)\left(-ln\left(1-\sigma'_\rho\right)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}}$$

From this, we can conclude that $\boldsymbol{SVNAAPG(\tau_1,\tau_2,\dots,\tau_n)} \leq \boldsymbol{SVNAAPG(\tau'_1,\tau'_2,\dots,\tau'_n)}$ holds.

## 5. Single Valued Neutrosophic Aczel Alsina Prioritized Weighted Aggregation Operators

In this section, we demonstrate many AOs of SVNAAPWA and SVNAAPWG based on Aczel Alsina operations with some specific characteristics by using our methodology.

**Definition 13:** Let $\tau_\rho=\left(\varphi_\rho,\delta_\rho,\sigma_\rho\right), \rho=1,2,3,\dots,n$ be the collection of SVNVs with associated weight vectors (WVs) $\Psi=(\Psi_1,\Psi_2,\dots,\Psi_n)^T$ of $\tau_\rho$, $\rho=1,2,3,\dots,n$ such that $\Psi_\rho\in[0,1]$ and $\sum_{\rho=1}^{n}\Psi_\rho=1$ with PA $\omega_\rho=\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}$ operator for each $\tau_\rho$. Then, the SVNAAPWA operator is particularized as:

$$SVNAAPWA(\tau_1,\tau_2,\dots,\tau_n)=\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\tau_\rho\right) \tag{8}$$

$$SVNAAPWA(\tau_1,\tau_2,\dots,\tau_n)=\left(\frac{\Psi_1\varepsilon_1}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\tau_1\right)\oplus\left(\frac{\Psi_2\varepsilon_2}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\tau_2\right)\oplus\dots\oplus\left(\frac{\Psi_n\varepsilon_n}{\sum_{\rho=1}^{n}\Psi_n\varepsilon_n}\tau_n\right)$$

Where $\varepsilon_\rho=\prod_{k=1}^{\rho-1}\mathcal{G}(\tau_k), \rho=2,3,\dots,n$. The initial value is $\varepsilon_1=1$ and $\mathcal{G}(\tau_k)$ be the score value of $k^{th}$ SVNVs.

**Theorem 9:** Let $\tau_\rho=\left(\varphi_\rho,\delta_\rho,\sigma_\rho\right), \rho=1,2,3,\dots,n$ be the collection of SVNVs with WVs $\Psi=(\Psi_1,\Psi_2,\dots,\Psi_n)^T$ and $\Psi_{\mathbb{C}}\in[0,1]$, $\sum_{\rho=1}^{n}\Psi_{\mathbb{C}}=1$ associated with PA $\omega_\rho=\frac{\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}$ operator. Then SVNAAPWA operator is particularized as:

$$SVNAAPWA=\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\tau_\rho\right)=\begin{pmatrix} 1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln\left(1-\varphi_\rho\right)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}}, \\ e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln\left(\delta_\rho\right)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}}, \\ e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln\left(\sigma_\rho\right)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}} \end{pmatrix} \tag{9}$$

**Proof:** We will proof this theorem with the help of a mathematical induction technique in the following way:

**i.** Take the value of $\rho=2$ depends on Aczel Alsina operations of SVNVs, we get,

$$\left(\frac{\Psi_1\varepsilon_1}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\tau_1\right)=\begin{pmatrix}1-e^{-\left(\left(\frac{\Psi_1\varepsilon_1}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(1-\varphi_1)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\left(\frac{\Psi_1\varepsilon_1}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\delta_1)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\left(\frac{\Psi_1\varepsilon_1}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\sigma_1)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}}\end{pmatrix}$$

$$\left(\frac{\Psi_2\varepsilon_2}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\tau_2\right)=\begin{pmatrix}1-e^{-\left(\left(\frac{\Psi_2\varepsilon_2}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(1-\varphi_2)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\left(\frac{\Psi_2\varepsilon_2}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\delta_2)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\left(\frac{\Psi_2\varepsilon_2}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\sigma_2)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}}\end{pmatrix}$$

By using the above Definition 13, we have:

$$SVNAAPWA(\tau_1,\tau_2)=\begin{pmatrix}\begin{pmatrix}1-e^{-\left(\left(\frac{\Psi_1\varepsilon_1}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(1-\varphi_1)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\left(\frac{\Psi_1\varepsilon_1}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\delta_1)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\left(\frac{\Psi_1\varepsilon_1}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\sigma_1)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}}\end{pmatrix}\oplus\\ \begin{pmatrix}1-e^{-\left(\left(\frac{\Psi_2\varepsilon_2}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(1-\varphi_2)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\left(\frac{\Psi_2\varepsilon_2}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\delta_2)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\left(\frac{\Psi_2\varepsilon_2}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\sigma_2)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}}\end{pmatrix}\end{pmatrix}$$

$$=\begin{pmatrix}1-e^{-\left(\sum_{\rho=1}^{2}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(1-\varphi_\rho)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\sum_{\rho=1}^{2}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\delta_\rho)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}},\\ e^{-\left(\sum_{\rho=1}^{2}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\sigma_\rho)\right)^{\mathbb{C}}\right)^{\frac{1}{\mathbb{C}}}}\end{pmatrix}$$

Hence, this is true for $\rho = 2$.

**ii.** Now suppose that this is true for $\rho = k$. Then, we have:

$$SVNAAPWA(\tau_1,\tau_2,\dots,\tau_n)=\sum_{\rho=1}^{n}\left(\frac{\Psi_k\varepsilon_k}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\tau_k\right)$$

$$=\begin{pmatrix} 1-e^{-\left(\sum_{\rho=1}^{k}\left(\frac{\Psi_k\varepsilon_k}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-\ ln(1-\varphi_k)\right)^{\cancel{c}}\right)^{\frac{1}{\cancel{c}}}}, \\ e^{-\left(\sum_{\rho=1}^{k}\left(\frac{\Psi_k\varepsilon_k}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\delta_k)\right)^{\cancel{c}}\right)^{\frac{1}{\cancel{c}}}}, \\ e^{-\left(\sum_{\rho=1}^{k}\left(\frac{\Psi_k\varepsilon_k}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\sigma_k)\right)^{\cancel{c}}\right)^{\frac{1}{\cancel{c}}}} \end{pmatrix}$$

Now, for $\rho = k+1$. We get,

$$SVNAAPWA(\tau_1,\tau_2,\tau_k,\ldots,\tau_{k+1}) = \sum_{\rho=1}^{k+1}\left(\left(\frac{\Psi_k\varepsilon_k}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\tau_k\right)\oplus\left(\frac{\Psi_{k+1}\varepsilon_{k+1}}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\tau_{k+1}\right)\right)$$

$$=\begin{pmatrix} \begin{pmatrix} 1-e^{-\left(\sum_{\rho=1}^{k}\left(\frac{\Psi_k\varepsilon_k}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-\ ln(1-\varphi_k)\right)^{\cancel{c}}\right)^{\frac{1}{\cancel{c}}}}, \\ e^{-\left(\sum_{\rho=1}^{k}\left(\frac{\Psi_k\varepsilon_k}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\delta_k)\right)^{\cancel{c}}\right)^{\frac{1}{\cancel{c}}}}, \\ e^{-\left(\sum_{\rho=1}^{k}\left(\frac{\Psi_k\varepsilon_k}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\sigma_k)\right)^{\cancel{c}}\right)^{\frac{1}{\cancel{c}}}} \end{pmatrix} \oplus \\ \begin{pmatrix} 1-e^{-\left(\left(\frac{\Psi_{k+1}\varepsilon_{k+1}}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(1-\varphi_{k+1})\right)^{\cancel{c}}\right)^{\frac{1}{\cancel{c}}}}, \\ e^{-\left(\left(\frac{\Psi_{k+1}\varepsilon_{k+1}}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\delta_{k+1})\right)^{\cancel{c}}\right)^{\frac{1}{\cancel{c}}}}, \\ e^{-\left(\left(\frac{\Psi_{k+1}\varepsilon_{k+1}}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\sigma_{k+1})\right)^{\cancel{c}}\right)^{\frac{1}{\cancel{c}}}} \end{pmatrix} \end{pmatrix}$$

$$=\begin{pmatrix} 1-e^{-\left(\sum_{\rho=1}^{k+1}\left(\frac{\Psi_{k+1}\varepsilon_{k+1}}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-\ ln(1-\varphi_{k+1})\right)^{\cancel{c}}\right)^{\frac{1}{\cancel{c}}}}, \\ e^{-\left(\sum_{\rho=1}^{k+1}\left(\frac{\Psi_{k+1}\varepsilon_{k+1}}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\delta_{k+1})\right)^{\cancel{c}}\right)^{\frac{1}{\cancel{c}}}}, \\ e^{-\left(\sum_{\rho=1}^{k+1}\left(\frac{\Psi_{k+1}\varepsilon_{k+1}}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\sigma_{k+1})\right)^{\cancel{c}}\right)^{\frac{1}{\cancel{c}}}} \end{pmatrix}$$

**Example 3:** Let $\tau_1 = (0.98, 0.45, 0.32), (0.56, 0.76, 0.3), (0.11, 0.23, 0.66)$ and $(0.45, 0.6, 0.29)$ be the four SVNVs with WVs $(0.3783, 0.4180, 0.1045, 0.0992)$ and $\cancel{c} = 3$. Then, SVNAAPWA can be calculated as:

$$SVNAAPWA = \sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\tau_\rho\right) = \begin{pmatrix} 1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(1-\varphi_\rho)\right)^{\cancel{c}}\right)^{\frac{1}{\cancel{c}}}}, \\ e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\delta_\rho)\right)^{\cancel{c}}\right)^{\frac{1}{\cancel{c}}}}, \\ e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\sigma_\rho)\right)^{\cancel{c}}\right)^{\frac{1}{\cancel{c}}}} \end{pmatrix}$$

$$SVNAAPWA(\tau_1,\tau_2,\tau_3,\tau_4)=\begin{pmatrix}1-e^{-\left(\begin{array}{c}\left(\frac{\Psi_1\varepsilon_1}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(1-\varphi_1)\right)^{\text{Ȼ}}+\left(\frac{\Psi_2\varepsilon_2}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(1-\varphi_2)\right)^{\text{Ȼ}}+\\ \left(\frac{\Psi_3\varepsilon_3}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(1-\varphi_3)\right)^{\text{Ȼ}}+\left(\frac{\Psi_4\varepsilon_4}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(1-\varphi_4)\right)^{\text{Ȼ}}\end{array}\right)^{\frac{1}{\text{Ȼ}}}},\\ e^{-\left(\begin{array}{c}\left(\frac{\Psi_1\varepsilon_1}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\delta_1)\right)^{\text{Ȼ}}+\left(\frac{\Psi_2\varepsilon_2}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\delta_2)\right)^{\text{Ȼ}}+\\ \left(\frac{\Psi_3\varepsilon_3}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\delta_3)\right)^{\text{Ȼ}}+\left(\frac{\Psi_4\varepsilon_4}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\delta_4)\right)^{\text{Ȼ}}\end{array}\right)^{\frac{1}{\text{Ȼ}}}},\\ e^{-\left(\begin{array}{c}\left(\frac{\Psi_1\varepsilon_1}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\sigma_1)\right)^{\text{Ȼ}}+\left(\frac{\Psi_2\varepsilon_2}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\sigma_2)\right)^{\text{Ȼ}}+\\ \left(\frac{\Psi_3\varepsilon_3}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\sigma_3)\right)^{\text{Ȼ}}+\left(\frac{\Psi_4\varepsilon_4}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\sigma_4)\right)^{\text{Ȼ}}\end{array}\right)^{\frac{1}{\text{Ȼ}}}}\end{pmatrix}$$

$$=\begin{pmatrix}1-e^{-\left(\begin{array}{c}\left(\frac{0.2000}{0.5287}\right)\left(-ln(1-0.98)\right)^{3}+\left(\frac{0.2210}{0.5287}\right)\left(-ln(1-0.56)\right)^{3}+\\ \left(\frac{0.0553}{0.5287}\right)\left(-ln(1-0.11)\right)^{3}+\left(\frac{0.0524}{0.5287}\right)\left(-ln(1-0.45)\right)^{3}\end{array}\right)^{\frac{1}{3}}},\\ e^{-\left(\begin{array}{c}\left(\frac{0.2000}{0.5287}\right)\left(-ln(0.45)\right)^{3}+\left(\frac{0.2210}{0.5287}\right)\left(-ln(0.76)\right)^{3}+\\ \left(\frac{0.0553}{0.5287}\right)\left(-ln(0.23)\right)^{3}+\left(\frac{0.0524}{0.5287}\right)\left(-ln(0.6)\right)^{3}\end{array}\right)^{\frac{1}{3}}},\\ e^{-\left(\begin{array}{c}\left(\frac{0.2000}{0.5287}\right)\left(-ln(0.32)\right)^{3}+\left(\frac{0.2210}{0.5287}\right)\left(-ln(0.3)\right)^{3}+\\ \left(\frac{0.0553}{0.5287}\right)\left(-ln(0.66)\right)^{3}+\left(\frac{0.0524}{0.5287}\right)\left(-ln(0.29)\right)^{3}\end{array}\right)^{\frac{1}{3}}}\end{pmatrix}$$

$$SVNAAPWA(\tau_1,\tau_2,\tau_3,\tau_4)=(0.9416,0.4416,0.3196)$$

**Theorem 10:** If all $\tau_\rho=(\varphi_\rho,\delta_\rho,\sigma_\rho),\rho=1,2,3,\dots,n$ are equal, that is, $\tau_n=\tau$ for all $\tau$. Then, we have:

$$SVNAAPWA(\tau_1,\tau_2,\dots,\tau_n)=\tau$$

**Proof:** Since $\tau_\rho=(\varphi_\rho,\delta_\rho,\sigma_\rho),\rho=1,2,3,\dots,n$. Then,

$$SVNAAPWA(\tau_1,\tau_2,\dots,\tau_n)=\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\tau_\rho$$

$$=\begin{pmatrix}1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(1-\varphi_\rho)\right)^{\text{Ȼ}}\right)^{\frac{1}{\text{Ȼ}}}},\\ e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\delta_\rho)\right)^{\text{Ȼ}}\right)^{\frac{1}{\text{Ȼ}}}},\\ e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\sigma_\rho)\right)^{\text{Ȼ}}\right)^{\frac{1}{\text{Ȼ}}}}\end{pmatrix}$$

$$=\left(1-e^{-\left(-\ ln\left(1-(\varphi_\rho)\right)^{\text{Ȼ}}\right)^{\frac{1}{\text{Ȼ}}}},e^{-\left(\left(-ln(\delta_\rho)\right)^{\text{Ȼ}}\right)^{\frac{1}{\text{Ȼ}}}},e^{-\left(\left(-ln(\sigma_\rho)\right)^{\text{Ȼ}}\right)^{\frac{1}{\text{Ȼ}}}}\right)=\tau$$

Thus, it is obvious that $SVNAAPWA(\tau_1,\tau_2,\dots,\tau_n)=\tau$ holds.

**Theorem 11:** Let $\tau_\rho=(\varphi_\rho,\delta_\rho,\sigma_\rho),\rho=1,2,3,\dots,n$ be the collection of SVNVs with associated WVs $\Psi=(\Psi_1,\Psi_2,\dots,\Psi_n)^T$ of $\tau_\rho$, $\rho=1,2,3,\dots,n$ such that $\Psi_\rho\in[0,1]$ and $\sum_{\rho=1}^{n}\Psi_\rho=1$ with PA $\omega_\rho=\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\varepsilon_\rho}$ operator for each $\tau_\rho$. Let $\tau^-=min(\tau_1,\tau_2,\dots,\tau_n)$, and $\tau^+=max(\tau_1,\tau_2,\dots,\tau_n)$, then $\tau^-\le SVNAAPWA(\tau_1,\tau_2,\dots,\tau_n)\le\tau^+$.

**Proof:** Let $\tau_{\rho} = (\varphi_{\rho}, \delta_{\rho}, \sigma_{\rho}), \rho = 1,2,3, \dots, n$ be the collection of SVNVs with associated WVs $\Psi = (\Psi_1, \Psi_2, \dots, \Psi_n)^T$ of $\tau_{\rho}$, $\rho = 1, 2, 3, \dots, n$ such that $\Psi_{\rho} \in [0,1]$ and $\sum_{\rho=1}^{n} \Psi_{\rho} = 1$ with PA $\omega_{\rho} = \frac{\Psi_{\rho}\varepsilon_{\rho}}{\sum_{\rho=1}^{n} \varepsilon_{\rho}}$ operator for each $\tau_{\rho}$. Let $\tau^- = min(\tau_1, \tau_2, \dots, \tau_n) = (\varphi^-, \delta^-, \sigma^-)$, and $\tau^+ = max(\tau_1, \tau_2, \dots, \tau_n) = (\varphi^+, \delta^+, \sigma^+)$ . We have, $\varphi^- = min_{\rho}\{\varphi_{\rho}\}, \delta^- = max_{\rho}\{\delta_{\rho}\}$ and $\sigma^- = max_{\rho}\{\sigma_{\rho}\}$ and $\varphi^+ = max_{\rho}\{\varphi_{\rho}\}, \delta^+ = min_{\rho}\{\delta_{\rho}\}$, and $\sigma^+ = min_{\rho}\{\sigma_{\rho}\}$ . Hence, there is the following result for the inequalities:

$$1 - e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_{\rho}\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\Psi_{\rho}\varepsilon_{\rho}}\right)\left(-ln\left(1-\varphi^-\right)\right)^{\cent}\right)^{\frac{1}{\cent}}} \leq 1 - e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_{\rho}\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\Psi_{\rho}\varepsilon_{\rho}}\right)\left(-ln\left(1-\varphi_{\rho}\right)\right)^{\cent}\right)^{\frac{1}{\cent}}}$$
$$\leq 1 - e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_{\rho}\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\Psi_{\rho}\varepsilon_{\rho}}\right)\left(-ln\left(1-\varphi^+\right)\right)^{\cent}\right)^{\frac{1}{\cent}}}$$

And,

$$e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_{\rho}\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\Psi_{\rho}\varepsilon_{\rho}}\right)\left(-ln\left(\delta^-\right)\right)^{\cent}\right)^{\frac{1}{\cent}}} \leq e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_{\rho}\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\Psi_{\rho}\varepsilon_{\rho}}\right)\left(-ln\left(\delta_{\rho}\right)\right)^{\cent}\right)^{\frac{1}{\cent}}} \leq e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_{\rho}\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\Psi_{\rho}\varepsilon_{\rho}}\right)\left(-ln\left(\delta^+\right)\right)^{\cent}\right)^{\frac{1}{\cent}}}$$

Similarly,

$$e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_{\rho}\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\Psi_{\rho}\varepsilon_{\rho}}\right)\left(-ln\left(\sigma^-\right)\right)^{\cent}\right)^{\frac{1}{\cent}}} \leq e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_{\rho}\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\Psi_{\rho}\varepsilon_{\rho}}\right)\left(-ln\left(\sigma_{\rho}\right)\right)^{\cent}\right)^{\frac{1}{\cent}}} \leq e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_{\rho}\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\Psi_{\rho}\varepsilon_{\rho}}\right)\left(-ln\left(\sigma^+\right)\right)^{\cent}\right)^{\frac{1}{\cent}}}$$

So, $\tau^- \leq SVNAAPWA(\tau_1, \tau_2, \dots, \tau_n) \leq \tau^+$ holds.

**Theorem 12**: Let $\tau_{\rho}$ and $\tau'_{\rho}, \rho = 1, 2, 3, \dots, n$ be two sets of SVNVs, if $\tau_{\rho} \leq \tau'_{\rho}$ for all $\tau$. Then, $SVNAAPWA(\tau_1, \tau_2, \dots, \tau_n) \leq SVNAAPWA(\tau'_1, \tau'_2, \dots, \tau'_n)$.

**Proof:** Let $\tau_{\rho}$ and $\tau'_{\rho}, \rho = 1, 2, 3, \dots, n$ be two sets of SVNVs, we can say that:

$$1 - e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_{\rho}\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\Psi_{\rho}\varepsilon_{\rho}}\right)\left(-ln\left(1-\varphi_{\rho}\right)\right)^{\cent}\right)^{\frac{1}{\cent}}} \leq 1 - e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_{\rho}\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\Psi_{\rho}\varepsilon_{\rho}}\right)\left(-ln\left(1-\varphi'_{\rho}\right)\right)^{\cent}\right)^{\frac{1}{\cent}}}$$
$$e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_{\rho}\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\Psi_{\rho}\varepsilon_{\rho}}\right)\left(-ln\left(\delta_{\rho}\right)\right)^{\cent}\right)^{\frac{1}{\cent}}} \geq e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_{\rho}\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\Psi_{\rho}\varepsilon_{\rho}}\right)\left(-ln\left(\delta'_{\rho}\right)\right)^{\cent}\right)^{\frac{1}{\cent}}}$$

Similarly,

$$e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_{\rho}\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\Psi_{\rho}\varepsilon_{\rho}}\right)\left(-ln\left(\sigma_{\rho}\right)\right)^{\cent}\right)^{\frac{1}{\cent}}} \geq e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_{\rho}\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\Psi_{\rho}\varepsilon_{\rho}}\right)\left(-ln\left(\sigma'_{\rho}\right)\right)^{\cent}\right)^{\frac{1}{\cent}}}$$

**Definition 14:** Let $\tau_{\rho} = (\varphi_{\rho}, \delta_{\rho}, \sigma_{\rho}), \rho = 1, 2, 3, \dots, n$ be the collection of SVNVs with associated WVs $\Psi = (\Psi_1, \Psi_2, \dots, \Psi_n)^T$ of $\tau_{\rho}$, $\rho = 1, 2, 3, \dots, n$ such that $\Psi_{\rho} \in [0,1]$ and $\sum_{\rho=1}^{n} \Psi_{\rho} = 1$ with PA $\omega_{\rho} = \frac{\varepsilon_{\rho}}{\sum_{\rho=1}^{n} \varepsilon_{\rho}}$ operator. Then, the SVNAAPWG operator is particularized as:

$$SVNAAPWG(\tau_1, \tau_2, \dots, \tau_n) = \prod_{\rho=1}^{n}\left(\tau_{\rho}^{\frac{\Psi_{\rho}\varepsilon_{\rho}}{\sum_{\rho=1}^{n}\Psi_{\rho}\varepsilon_{\rho}}}\right) \tag{10}$$

$$SVNAAPWG(\tau_1, \tau_2, \dots, \tau_n) = \tau_1^{\left(\frac{\Psi_1 \varepsilon_1}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)} \otimes \tau_2^{\left(\frac{\Psi_2 \varepsilon_2}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)} \otimes \dots \otimes \tau_n^{\left(\frac{\Psi_n \varepsilon_n}{\sum_{\rho=1}^{n} \Psi_n \varepsilon_n}\right)}$$

Where $\varepsilon_\rho = \prod_{k=1}^{\rho-1} \mathcal{G}(\tau_k), \rho = 2, 3, \dots, n$. The initial value is $\varepsilon_1 = 1$ and $\mathcal{G}(\tau_k)$ be the score value of $k^{th}$ SVNVs.

**Theorem 10:** Let $\tau_\rho = (\varphi_\rho, \delta_\rho, \sigma_\rho), \rho = 1, 2, 3, \dots, n$ be the collection of SVNVs with WVs $\Psi = (\Psi_1, \Psi_2, \dots, \Psi_n)^T$ and $\Psi_\rho \in [0,1]$, $\sum_{\rho=1}^{n} \Psi_\rho = 1$ associated with PA $\omega_\rho = \frac{\varepsilon_\rho}{\sum_{\rho=1}^{n} \varepsilon_\rho}$ operator. Then, the SVNAAPWG operator is particularized as:

$$SVNAAPWG = \prod_{\rho=1}^{n} \left( \tau_\rho^{\frac{\Psi_\rho \varepsilon_\rho}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}} \right) = \begin{pmatrix} e^{-\left(\sum_{\rho=1}^{n} \left(\frac{\Psi_\rho \varepsilon_\rho}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)\left(-ln(\varphi_\rho)\right)^{₡}\right)^{\frac{1}{₡}}}, \\ 1 - e^{-\left(\sum_{\rho=1}^{n} \left(\frac{\Psi_\rho \varepsilon_\rho}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)\left(-ln(1-\delta_\rho)\right)^{₡}\right)^{\frac{1}{₡}}}, \\ 1 - e^{-\left(\sum_{\rho=1}^{n} \left(\frac{\Psi_\rho \varepsilon_\rho}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)\left(-ln(1-\sigma_\rho)\right)^{₡}\right)^{\frac{1}{₡}}} \end{pmatrix} \quad (11)$$

**Proof:** Proof is similar to Theorem 9.

**Example 3:** Let $\tau_1 = (0.39, 0.69, 0.41), (0.19, 0.43, 0.71), (0.8, 0.37, 0.99)$ and $(0.55, 0.51, 0.25)$ be the four SVNVs with WVs $(0.5307, 0.3423, 0.0599, 0.0671)$ and $₡ = 3$. Then, SVNAAPWG can be calculated as:

$$SVNAAPWG = \prod_{\rho=1}^{n} \left( \tau_\rho^{\frac{\Psi_\rho \varepsilon_\rho}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}} \right) = \begin{pmatrix} e^{-\left(\sum_{\rho=1}^{n} \left(\left(\frac{\Psi_\rho \varepsilon_\rho}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)\left(-ln(\varphi_\rho)\right)^{₡}\right)\right)^{\frac{1}{₡}}}, \\ 1 - e^{-\left(\sum_{\rho=1}^{n} \left(\left(\frac{\Psi_\rho \varepsilon_\rho}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)\left(-ln(1-\delta_\rho)\right)^{₡}\right)\right)^{\frac{1}{₡}}}, \\ 1 - e^{-\left(\sum_{\rho=1}^{n} \left(\left(\frac{\Psi_\rho \varepsilon_\rho}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)\left(-ln(1-\sigma_\rho)\right)^{₡}\right)\right)^{\frac{1}{₡}}} \end{pmatrix}$$

$$SVNAAPWG(\tau_1, \tau_2, \tau_3, \tau_4) = \begin{pmatrix} e^{-\begin{pmatrix} \left(\frac{\Psi_1 \varepsilon_1}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)\left(-ln(\varphi_1)\right)^{₡} + \left(\frac{\Psi_2 \varepsilon_2}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)\left(-ln(\varphi_2)\right)^{₡} \\ + \left(\frac{\Psi_3 \varepsilon_3}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)\left(-ln(\varphi_3)\right)^{₡} + \left(\frac{\Psi_4 \varepsilon_4}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)\left(-ln(\varphi_4)\right)^{₡} \end{pmatrix}^{\frac{1}{₡}}}, \\ 1 - e^{-\begin{pmatrix} \left(\frac{\Psi_1 \varepsilon_1}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)\left(-ln(1-\delta_1)\right)^{₡} + \left(\frac{\Psi_2 \varepsilon_2}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)\left(-ln(1-\delta_2)\right)^{₡} + \\ \left(\frac{\Psi_3 \varepsilon_3}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)\left(-ln(1-\delta_3)\right)^{₡} + \left(\frac{\Psi_4 \varepsilon_4}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)\left(-ln(1-\delta_4)\right)^{₡} \end{pmatrix}^{\frac{1}{₡}}}, \\ 1 - e^{-\begin{pmatrix} \left(\frac{\Psi_1 \varepsilon_1}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)\left(-ln(1-\sigma_1)\right)^{₡} + \left(\frac{\Psi_2 \varepsilon_2}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)\left(-ln(1-\sigma_2)\right)^{₡} + \\ \left(\frac{\Psi_3 \varepsilon_3}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)\left(-ln(1-\sigma_3)\right)^{₡} + \left(\frac{\Psi_4 \varepsilon_4}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)\left(-ln(1-\sigma_4)\right)^{₡} \end{pmatrix}^{\frac{1}{₡}}} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-\left( \begin{array}{c} \left(\frac{0.2000}{0.3769}\right)(-ln(0.39))^3 + \left(\frac{0.1290}{0.3769}\right)(-ln(0.19))^3 + \\ \left(\frac{0.0226}{0.3769}\right)(-ln(0.8))^3 + \left(\frac{0.0253}{0.3769}\right)(-ln(0.55))^3 \end{array} \right)^{\frac{1}{3}}}, \\ 1 - e^{-\left( \begin{array}{c} \left(\frac{0.2000}{0.3769}\right)(-ln(1-0.69))^3 + \left(\frac{0.1290}{0.3769}\right)(-ln(1-0.43))^3 + \\ \left(\frac{0.0226}{0.3769}\right)(-ln(1-0.37))^3 + \left(\frac{0.0253}{0.3769}\right)(-ln(1-0.51))^3 \end{array} \right)^{\frac{1}{3}}}, \\ 1 - e^{-\left( \begin{array}{c} \left(\frac{0.2000}{0.3769}\right)(-ln(1-0.41))^3 + \left(\frac{0.1290}{0.3769}\right)(-ln(1-0.71))^3 + \\ \left(\frac{0.0226}{0.3769}\right)(-ln(1-0.99))^3 + \left(\frac{0.0253}{0.3769}\right)(-ln(1-0.25))^3 \end{array} \right)^{\frac{1}{3}}} \end{pmatrix}$$

$$SVNAAPWG(\tau_1, \tau_2, \tau_3, \tau_4) = (0.2830, 0.6250, 0.8465)$$

**Theorem 13:** If all $\tau_\rho = (\varphi_\rho, \delta_\rho, \sigma_\rho), \rho = 1, 2, 3, \dots, n$ are equal, that is, $\tau_n = \tau$ for all $\tau$. Then, $SVNAAPWG(\tau_1, \tau_2, \dots, \tau_n) = \tau$.

**Proof:** Since $\tau_\rho = (\varphi_\rho, \delta_\rho, \sigma_\rho), \rho = 1,2,3, \dots, n$. Then,

$$SVNAAPWG(\tau_1, \tau_2, \dots, \tau_n) = \prod_{\rho=1}^{n} \left( \tau_\rho^{\frac{\Psi_\rho \varepsilon_\rho}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}} \right)$$

$$SVNAAPWG(\tau_1, \tau_2, \dots, \tau_n) = \begin{pmatrix} e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho \varepsilon_\rho}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)(-ln(\varphi_\rho))^{₡}\right)^{\frac{1}{₡}}}, \\ 1 - e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho \varepsilon_\rho}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)(-ln(1-\delta_\rho))^{₡}\right)^{\frac{1}{₡}}}, \\ 1 - e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho \varepsilon_\rho}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)(-ln(1-\sigma_\rho))^{₡}\right)^{\frac{1}{₡}}} \end{pmatrix}$$

$$= \left( e^{-\left((-ln(\varphi_{₡}))^{₡}\right)^{\frac{1}{₡}}}, 1 - e^{-\left((-ln(1-\delta_{₡}))^{₡}\right)^{\frac{1}{₡}}}, 1 - e^{-\left((-ln(1-\sigma_{₡}))^{₡}\right)^{\frac{1}{₡}}} \right) = \tau$$

Thus, it is obvious that $SVNAAPWG(\tau_1, \tau_2, \dots, \tau_n) = \tau$ holds.

**Theorem 14:** Let $\tau_\rho = (\varphi_\rho, \delta_\rho, \sigma_\rho), \rho = 1,2,3, \dots, n$ be the collection of SVNVs with associated WVs $\Psi = (\Psi_1, \Psi_2, \dots, \Psi_n)^T$ of $\tau_\rho$, $\rho = 1, 2, 3, \dots, n$ such that $\Psi_\rho \in [0,1]$ and $\sum_{\rho=1}^{n} \Psi_\rho = 1$ with PA $\omega_\rho = \frac{\Psi_\rho \varepsilon_\rho}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}$ operator for each $\tau_\rho$. Let $\tau^- = min(\tau_1, \tau_2, \dots, \tau_n)$ and $\tau^+ = max(\tau_1, \tau_2, \dots, \tau_n)$. Then, $\tau^- \leq SVNAAPWG(\tau_1, \tau_2, \dots, \tau_n) \leq \tau^+$.

**Proof:** Let $\tau_\rho = (\varphi_\rho, \delta_\rho, \sigma_\rho), \rho = 1,2,3, \dots, n$ be the collection of SVNVs with associated WVs $\Psi = (\Psi_1, \Psi_2, \Psi_3, \dots, \Psi_n)^T$ of $\tau_\rho$, $\rho = 1, 2, 3, \dots, n$ such that $\Psi_\rho \in [0,1]$ and $\sum_{\rho=1}^{n} \Psi_\rho = 1$ with PA $\omega_\rho = \frac{\Psi_\rho \varepsilon_\rho}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}$ operator for each $\tau_\rho$. Let $\tau^- = min(\tau_1, \tau_2, \dots, \tau_n) = (\varphi^-, \delta^-, \sigma^-)$, and $\tau^+ = max(\tau_1, \tau_2, \dots, \tau_n) = (\varphi^+, \delta^+, \sigma^+)$. We have, $\varphi^- = min_\rho\{\varphi_\rho\}, \delta^- = max_\rho\{\delta_\rho\}$ and $\sigma^- = max_\rho\{\sigma_\rho\}$ and $\varphi^+ = max_\rho\{\varphi_\rho\}, \delta^+ = min_\rho\{\delta_\rho\}$, and $\sigma^+ = min_\rho\{\sigma_\rho\}$. Hence, there is the following result for the inequalities:

$$e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho \varepsilon_\rho}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)(-ln(\varphi^-))^{₡}\right)^{\frac{1}{₡}}} \leq e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho \varepsilon_\rho}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)(-ln(\varphi_\rho))^{₡}\right)^{\frac{1}{₡}}} \leq e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho \varepsilon_\rho}{\sum_{\rho=1}^{n} \Psi_\rho \varepsilon_\rho}\right)(-ln(\varphi^+))^{₡}\right)^{\frac{1}{₡}}}$$

$$1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln\left(1-\delta^{-}\right)\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}} \leq 1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln\left(1-\delta_\rho\right)\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}}$$
$$\leq 1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln\left(1-\delta^{+}\right)\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}}$$

$$1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln\left(1-\sigma^{-}\right)\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}} \leq 1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln\left(1-\sigma_\rho\right)\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}}$$
$$\leq 1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln\left(1-\sigma^{+}\right)\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}}$$

So, $\tau^{-} \leq SVNAAPWG(\tau_1, \tau_2, \dots, \tau_n) \leq \tau^{+}$ holds.

**Theorem 15**: Let $\tau_\rho$ and $\tau'_\rho, \rho = 1,2,3, \dots, n$ be two sets of SVNVs, if $\tau_\rho \leq \tau'_\rho$ for all $\tau$. Then, $SVNAAPWG(\tau_1, \tau_2, \dots, \tau_n) \leq SVNAAPWG(\tau'_1, \tau'_2, \dots, \tau'_n)$.

**Proof:** Let $\tau_\rho$ and $\tau'_\rho, \rho = 1,2,3, \dots, n$ be two sets of SVNVs. Then, we use the following way to prove it:

$$e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln\left(\varphi_\rho\right)\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}} \leq e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln\left(\varphi'_\rho\right)\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}}$$

And,

$$1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln\left(1-\delta_\rho\right)\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}} \geq 1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln\left(1-\delta'_\rho\right)\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}}$$

In the same way,

$$1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln\left(1-\sigma_\rho\right)\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}} \geq 1-e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln\left(1-\sigma'_\rho\right)\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}}$$

From the above, we can conclude that:

$$SVNAAPWG(\tau_1, \tau_2, \dots, \tau_n) \leq SVNAAPWG(\tau'_1, \tau'_2, \dots, \tau'_n)$$

## 6. MADM Techniques of SVNAAPWA and SVNAAPWG Operations

In this section, we shall use the SVNAAPWA and SVNAAPWG operators to solve the MADM technique by using the information of SVNVs. Suppose that $ꞥ = (ꞥ_1, ꞥ_2, \dots, ꞥ_n)$ be the set of alternatives and $ß = (ß_1, ß_2, ß_3, \dots, ß_n)$ be the set of attributes with the degree of weights $\Psi = (\Psi_1, \Psi_2, \dots, \Psi_n)^T, \rho = (1, 2, 3, \dots, n)$ such that $\Psi_\rho \in [0,1]$ and $\sum_{\rho=1}^{n}\Psi_\rho = 1$. The decision maker also explores the theory of prioritization between attributes which is represented as linear ordering $\Sigma_1 > \Sigma_2 > \dots > \Sigma_n$. The following decision matrix $Ꝁ = \left(Y_{\eta\rho}\right)_{\varkappa\times ῆ}$ contained information in the form of SVNVs.

$$Ꝁ = \left(Y_{\eta\rho}\right)_{\varkappa\times ῆ} = \begin{pmatrix} \left(\varphi_{\tau_{11}}, \delta_{\tau_{11}}, \sigma_{\tau_{11}}\right) & \left(\varphi_{\tau_{12}}, \delta_{\tau_{12}}, \sigma_{\tau_{12}}\right) & \dots & \left(\varphi_{\tau_{1n}}, \delta_{\tau_{1n}}, \sigma_{\tau_{1ῆ}}\right) \\ \left(\varphi_{\tau_{21}}, \delta_{\tau_{21}}, \sigma_{\tau_{21}}\right) & \left(\varphi_{\tau_{22}}, \delta_{\tau_{22}}, \sigma_{\tau_{22}}\right) & \cdots & \left(\varphi_{\tau_{2n}}, \delta_{\tau_{2n}}, \sigma_{\tau_{2ῆ}}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\varphi_{\tau_{\varkappa1}}, \delta_{\tau_{\varkappa1}}, \sigma_{\tau_{\varkappa1}}\right) & \left(\varphi_{\tau_{\varkappa2}}, \delta_{\tau_{\varkappa2}}, \sigma_{\tau_{\varkappa2}}\right) & \vdots & \left(\varphi_{\tau_{\varkappa ῆ}}, \delta_{\tau_{\varkappa ῆ}}, \sigma_{\tau_{\varkappa ῆ}}\right) \end{pmatrix}$$

In this decision matrix $\left(\varphi_{\tau_{\varkappa ῆ}}, \delta_{\tau_{\varkappa ῆ}}, \sigma_{\tau_{\varkappa ῆ}}\right)$ represents the value of SVNV and $\varphi_{\tau_{\varkappa ῆ}} \in [0,1]$, $\delta_{\tau_{\varkappa ῆ}} \in [0,1]$ and $\sigma_{\tau_{\varkappa ῆ}} \in [0,1]$ such that $0 \leq \varphi_{\tau_{\varkappa ῆ}} + \delta_{\tau_{\varkappa ῆ}} + \sigma_{\tau_{\varkappa ῆ}} \leq 3$. There are two kinds of attributes: cost factor and beneficial factor. If the cost factor is involved in the decision matrix, then the decision matrix transforms to normalize matrix:

$$Ꞧ = \left(Y_{\eta \rho}\right)_{\varkappa \times ῆ} = \begin{cases} \left(\varphi_{\tau_{\varkappa ῆ}}, \delta_{\tau_{\varkappa ῆ}}, \sigma_{\tau_{\varkappa ῆ}}\right) & if \;\; benefit \;\; factor \\ \left(\varphi_{\tau_{\varkappa ῆ}}, \delta_{\tau_{\varkappa ῆ}}, \sigma_{\tau_{\varkappa ῆ}}\right) & if \;\; cost \;\; factor \end{cases}$$

Now, we will describe the following steps of the algorithm for solving given a MADM technique by the decision maker.

*6.1 Algorithm*

**Step 1:** In the first step, the decision maker collects information and arranges a decision matrix under the system of SVNVs.

**Step 2:** We must convert the decision matrix into a normalizer matrix if the cost factor involves in the set of attributes; otherwise, there is no need.

**Step 3:** We utilized our proposed methodologies to solve a MADM technique by using the SVNAAWA and SVNAAWG operators.

Step 4: Shows the results of SVNAAWA and SVNAAWG operators in a table.

Step 5: Calculate score values by using the consequences of SVNAAWA and SVNAAWG operators. We evaluate suitable alternatives after ranking and ordering of the score values.

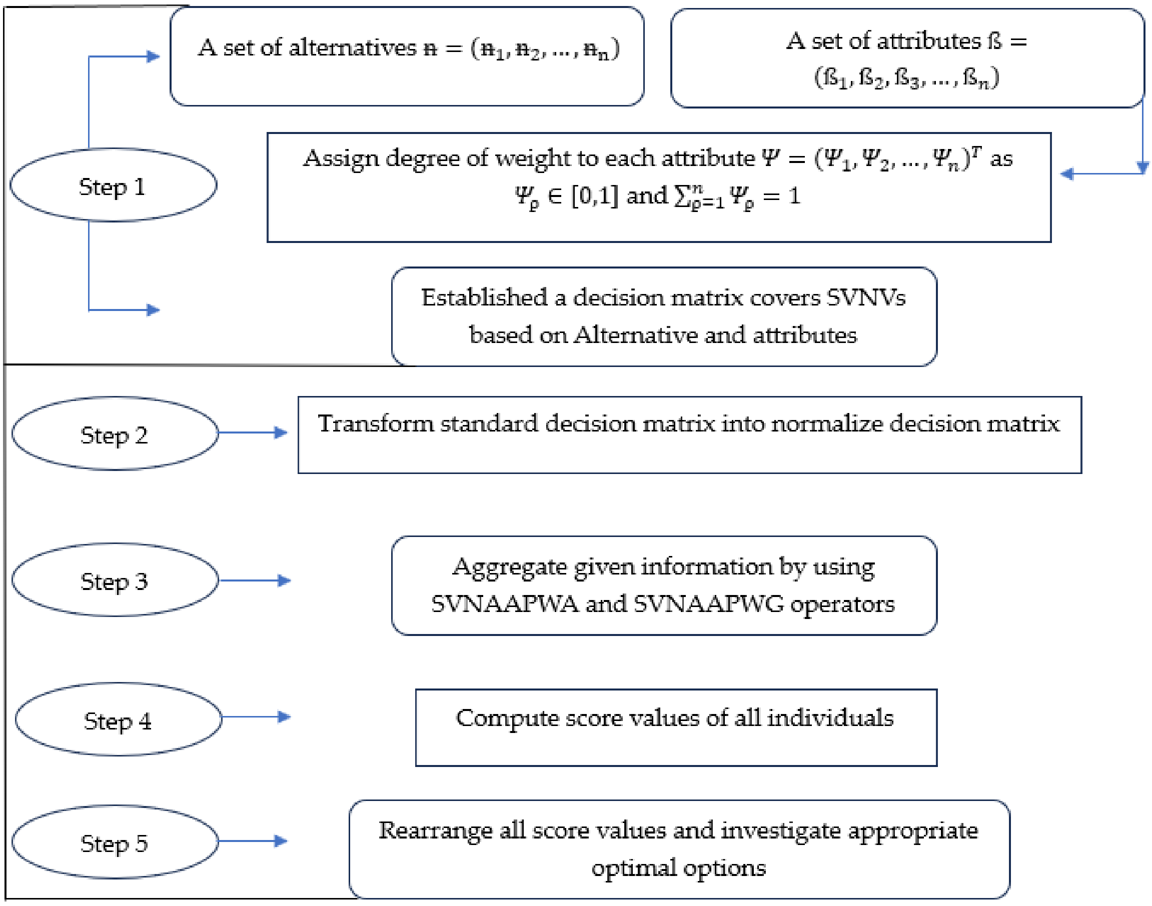


Figure 1 Flow chart of an algorithm.

*6.2. Application*

Research scientists are present in various alternative domains, such as mathematics, chemistry, biology, software engineering, environmental science, medicine, nano technology, human science, history, political science and so on. They develop a conceptual model for collecting information, and findings respond to inquiries about individuals and the universe. Research scientists are employed by various institutions, including universities and colleges, government agencies, organizations, and businesses engaged in production and innovation. Research scientists generally hold master's or doctoral degrees in their respective professions. Most research scientists hold postgraduate degrees in their specialized disciplines. While master's degrees are frequently sufficient for employment in the general financial industry, PhDs are typically necessary for research scientist careers at colleges and universities. Research scientists are generally interested. Their task involves analytical skills and sensitive, caring attention in order to put up a repeatable approach and recommend the right results. For their discoveries to be communicated in publications and oral presentations, research scientists must be effective communicators and editors.

*6.3. Numerical Example*

Consider a public university wanted to fill its vacant post with a research scientist, and the selection committee selects from five different applicants $ꞩ_i = (ꞩ_1, ꞩ_2, \dots, ꞩ_n)$ based on the following four characteristics. $ß_1$ : represents the qualification/ academic history, $ß_2$: represents the publication and its citations, $ß_3$: experience in teaching related to the research field, $ß_4$: Personality/ digital skills/ communication skills/ moral value.

The decision maker selects a suitable candidate under above-discussed characteristics. Consider WVs $\Psi = (0.20, 0.30, 0.15, 0.35)$ associated with the collected information in the form of SVNVs. We aggregated information given by the decision maker in Table 2 by following the steps of the algorithm.

**Table 2** shows the information in the form of SNNVSs given by the decision maker.

| | $ß_1$ | $ß_2$ | $ß_3$ | $ß_4$ | $ß_5$ |
|---|---|---|---|---|---|
| $ħ_1$ | $(0.23, 0.45, 0.56)$ | $(0.66, 0.65, 0.78)$ | $(0.12, 0.97, 0.32)$ | $(0.21, 0.78, 0.78)$ | $(0.65, 0.54, 0.76)$ |
| $ħ_2$ | $(0.67, 0.78, 0.98)$ | $(0.89, 0.12, 0.32)$ | $(0.99, 0.88, 0.76)$ | $(0.23, 0.32, 0.71)$ | $(0.65, 0.55, 0.61)$ |
| $ħ_3$ | $(0.8, 0.39, 0.19)$ | $(0.12, 0.34, 0.54)$ | $(0.33, 0.9, 0.1)$ | $(0.62, 0.56, 0.69)$ | $(0.78, 0.61, 0.32)$ |
| $ħ_4$ | $(0.7, 0.39, 0.88)$ | $(0.78, 0.1, 0.2)$ | $(0.2, 0.4, 0.5)$ | $(0.11, 0.77, 0.19)$ | $(0.77, 0.22, 0.11)$ |

**Step 1:** The information gathered by the decision-maker using the system of SVNVs is represented in Table 2.

**Step 2:** As no cost factor is included in the attributes set data, we have not transformed the decision matrix into the normalized matrix.

**Step 3:** Applied the techniques of SVNAAPWA and SVNAAPWG operators to aggregate information given by the decision maker, which is depicted in Table 2.

$$SVNAAPWA = \sum_{\rho=1}^{n}\left(\frac{\Psi_\rho \varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\tau_\rho\right) = \begin{pmatrix} 1 - e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(1-\varphi_\rho)\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}}, \\ e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\delta_\rho)\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}}, \\ e^{-\left(\sum_{\rho=1}^{n}\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\sigma_\rho)\right)^{\text{₡}}\right)^{\frac{1}{\text{₡}}}} \end{pmatrix}$$

And,

$$SVNAAPWG = \prod_{\rho=1}^{n}\left(\tau_\rho^{\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}}\right) = \begin{pmatrix} e^{-\left(\sum_{\rho=1}^{n}\left(\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(\varphi_\rho)\right)^{\text{₡}}\right)\right)^{\frac{1}{\text{₡}}}}, \\ 1 - e^{-\left(\sum_{\rho=1}^{n}\left(\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(1-\delta_\rho)\right)^{\text{₡}}\right)\right)^{\frac{1}{\text{₡}}}}, \\ 1 - e^{-\left(\sum_{\rho=1}^{n}\left(\left(\frac{\Psi_\rho\varepsilon_\rho}{\sum_{\rho=1}^{n}\Psi_\rho\varepsilon_\rho}\right)\left(-ln(1-\sigma_\rho)\right)^{\text{₡}}\right)\right)^{\frac{1}{\text{₡}}}} \end{pmatrix}$$

**Table 3** shows the results of our proposed work.

| SVNAAPWA | SVNAAPWG |
|---|---|
| $(\mathbf{0.6067, 0.4812, 0.5002})$ | $(0.2768, 0.6612, 0.9359)$ |
| $(\mathbf{0.7990, 0.1950, 0.3747})$ | $(0.3459, 0.5610, 0.6983)$ |
| $(\mathbf{0.9468, 0.6972, 0.3062})$ | $(0.1510, 0.9553, 0.6059)$ |
| $(\mathbf{0.3138, 0.4944, 0.5376})$ | $(0.2250, 0.7457, 0.7587)$ |

| | |
|---|---|
| $(\mathbf{0.6877, 0.4317, 0.3256})$ | $(0.5387, 0.5375, 0.6947)$ |

**Step 4**: After the computation of information using our proposed methodologies, we displayed all outcomes in Table 3.

**Step 5**: Investigate the results of the score values obtained by SVNAAPWA and SVNAAPWG operators shown in Table 3. Invested results of all individual by the SVNAAPWA and SVNAAPWG operators listed in Table 4.

**Table 4** shows the score values of AOs of SVNAAPWA and SVNAAPWG

| Operators | $\mathcal{G}(\text{ꞥ}_1)$ | $\mathcal{G}(\text{ꞥ}_2)$ | $\mathcal{G}(\text{ꞥ}_3)$ | $\mathcal{G}(\text{ꞥ}_4)$ | $\mathcal{G}(\text{ꞥ}_5)$ | Ranking and ordering |
|---|---|---|---|---|---|---|
| SVNAAPWA | 0.5418 | 0.7431 | 0.6478 | 0.4273 | 0.6435 | $\text{ꞥ}_2 > \text{ꞥ}_3 > \text{ꞥ}_5 > \text{ꞥ}_1 > \text{ꞥ}_4$ |
| SVNAAPWG | 0.2266 | 0.3622 | 0.1966 | 0.2402 | 0.4355 | $\text{ꞥ}_5 > \text{ꞥ}_2 > \text{ꞥ}_4 > \text{ꞥ}_1 > \text{ꞥ}_3$ |

We noticed the ranking and ordering of score values $\text{ꞥ}_2 > \text{ꞥ}_5 > \text{ꞥ}_3 > \text{ꞥ}_1 > \text{ꞥ}_4$ and $\text{ꞥ}_2 > \text{ꞥ}_5 > \text{ꞥ}_1 > \text{ꞥ}_3 > \text{ꞥ}_4$ for SVNAAPWA and SVNAAPWG, respectively. $\text{ꞥ}_2$ is a suitable applicant for the vecant post. Similarly, $\text{ꞥ}_5$ is the best applicants for a research scientist of a public university. We also show the outcomes of the score values acquired from the SVNAAWA and SVNAAWG operators as a graphical representation of the following Figure 2.

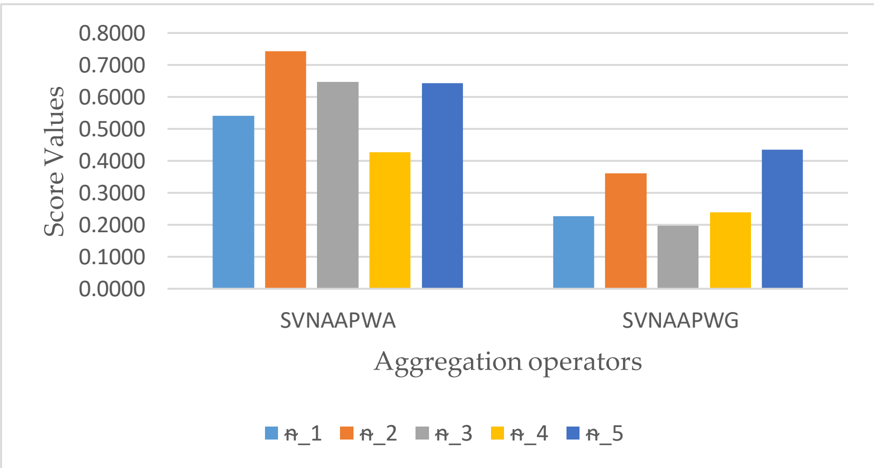


**Figure 2** Covers the geometrical representation of all score values, which are listed in Table 4.

*6.4. Behavior of Different Parameters of ₵ on our Purposed Methodologies*

We modified several values of ₵ in step 4 of the recommended MADM approach to explore the impact of different parameter values ₵ on the ranking of all alternatives. The derived outcomes are displayed in Tables 5-6. From Table 5, we noticed when the valued of ₵ increases, score values gained through the SVNAAPWA and SVNAAPWG operators also increase. Moreover, we noticed that the ranking and ordering sequence of the score values remain the same when we change the parametric values of ₵ for our invented approaches SVNAAPWA and SVNAAPWG operators. To see this increasing sequence of the parameter value ₵ and outcomes obtained from our discussed approaches is shown the isotonicity property.

**Table 5** shows the results of SVNAAPWA operators for the variation of ₵.

| | $\mathcal{G}(ꞥ_1)$ | $\mathcal{G}(ꞥ_2)$ | $\mathcal{G}(ꞥ_3)$ | $\mathcal{G}(ꞥ_4)$ | $\mathcal{G}(ꞥ_5)$ | Ranking and Ordering |
|---|---|---|---|---|---|---|
| **₵ = 1** | 0.4352 | 0.6502 | 0.4841 | 0.2984 | 0.5522 | $ꞥ_2 > ꞥ_5 > ꞥ_3 > ꞥ_1 > ꞥ_4$ |
| **₵ = 3** | 0.5418 | 0.7431 | 0.6478 | 0.4273 | 0.6435 | $ꞥ_2 > ꞥ_3 > ꞥ_5 > ꞥ_1 > ꞥ_4$ |
| **₵ = 25** | 0.7041 | 0.8436 | 0.8061 | 0.6675 | 0.7905 | $ꞥ_2 > ꞥ_3 > ꞥ_5 > ꞥ_1 > ꞥ_4$ |
| **₵ = 75** | 0.7281 | 0.8566 | 0.8222 | 0.6915 | 0.8075 | $ꞥ_2 > ꞥ_3 > ꞥ_5 > ꞥ_1 > ꞥ_4$ |
| **₵ = 105** | 0.7315 | 0.8585 | 0.8244 | 0.6949 | 0.8101 | $ꞥ_2 > ꞥ_3 > ꞥ_5 > ꞥ_1 > ꞥ_4$ |
| **₵ = 155** | 0.7343 | 0.8601 | 0.8262 | 0.6976 | 0.8122 | $ꞥ_2 > ꞥ_3 > ꞥ_5 > ꞥ_1 > ꞥ_4$ |
| **₵ = 201** | 0.7356 | 0.8608 | 0.8271 | 0.6989 | 0.8132 | $ꞥ_2 > ꞥ_3 > ꞥ_5 > ꞥ_1 > ꞥ_4$ |
| **₵ = 255** | 0.7365 | 0.8614 | 0.8277 | 0.6999 | 0.8139 | $ꞥ_2 > ꞥ_3 > ꞥ_5 > ꞥ_1 > ꞥ_4$ |
| **₵ = 275** | 0.7368 | 0.8615 | 0.8279 | 0.7001 | 0.8141 | $ꞥ_2 > ꞥ_3 > ꞥ_5 > ꞥ_1 > ꞥ_4$ |
| **₵ = 321** | 0.7372 | 0.8618 | 0.8282 | 0.7006 | 0.8145 | $ꞥ_2 > ꞥ_3 > ꞥ_5 > ꞥ_1 > ꞥ_4$ |
| **₵ = 375** | 0.7376 | 0.8620 | 0.8284 | 0.7010 | 0.8148 | $ꞥ_2 > ꞥ_3 > ꞥ_5 > ꞥ_1 > ꞥ_4$ |
| **₵ = 421** | 0.7379 | 0.8621 | 0.8286 | 0.7012 | 0.8150 | $ꞥ_2 > ꞥ_3 > ꞥ_5 > ꞥ_1 > ꞥ_4$ |
| **₵ = 463** | 0.7381 | 0.8623 | 0.8287 | 0.7014 | 0.8152 | $ꞥ_2 > ꞥ_3 > ꞥ_5 > ꞥ_1 > ꞥ_4$ |

**Table 6** shows the results of SVNAAPWG operators for the variation of ₵.

| | $\mathcal{G}(ꞥ_1)$ | $\mathcal{G}(ꞥ_2)$ | $\mathcal{G}(ꞥ_3)$ | $\mathcal{G}(ꞥ_4)$ | $\mathcal{G}(ꞥ_5)$ | Ranking and Ordering |
|---|---|---|---|---|---|---|
| **₵ = 1** | 0.3007 | 0.5041 | 0.2636 | 0.2610 | 0.4807 | $ꞥ_2 > ꞥ_5 > ꞥ_1 > ꞥ_3 > ꞥ_4$ |
| **₵ = 3** | 0.2266 | 0.3622 | 0.1966 | 0.2402 | 0.4355 | $ꞥ_5 > ꞥ_2 > ꞥ_4 > ꞥ_1 > ꞥ_3$ |
| **₵ = 25** | 0.1591 | 0.2445 | 0.1366 | 0.2212 | 0.3423 | $ꞥ_5 > ꞥ_2 > ꞥ_4 > ꞥ_1 > ꞥ_3$ |
| **₵ = 75** | 0.1489 | 0.2348 | 0.1285 | 0.2183 | 0.3254 | $ꞥ_5 > ꞥ_2 > ꞥ_4 > ꞥ_1 > ꞥ_3$ |
| **₵ = 105** | 0.1473 | 0.2334 | 0.1271 | 0.2179 | 0.3229 | $ꞥ_5 > ꞥ_2 > ꞥ_4 > ꞥ_1 > ꞥ_3$ |
| **₵ = 155** | 0.1460 | 0.2323 | 0.1258 | 0.2175 | 0.3209 | $ꞥ_5 > ꞥ_2 > ꞥ_4 > ꞥ_1 > ꞥ_3$ |
| **₵ = 201** | 0.1454 | 0.2318 | 0.1253 | 0.2173 | 0.3199 | $ꞥ_5 > ꞥ_2 > ꞥ_4 > ꞥ_1 > ꞥ_3$ |
| **₵ = 255** | 0.1450 | 0.2314 | 0.1249 | 0.2172 | 0.3192 | $ꞥ_5 > ꞥ_2 > ꞥ_4 > ꞥ_1 > ꞥ_3$ |
| **₵ = 275** | 0.1448 | 0.2313 | 0.1247 | 0.2171 | 0.3190 | $ꞥ_5 > ꞥ_2 > ꞥ_4 > ꞥ_1 > ꞥ_3$ |
| **₵ = 321** | 0.1446 | 0.2311 | 0.1245 | 0.2171 | 0.3187 | $ꞥ_5 > ꞥ_2 > ꞥ_4 > ꞥ_1 > ꞥ_3$ |
| **₵ = 375** | 0.1444 | 0.2309 | 0.1244 | 0.2170 | 0.3184 | $ꞥ_5 > ꞥ_2 > ꞥ_4 > ꞥ_1 > ꞥ_3$ |
| **₵ = 421** | 0.1443 | 0.2308 | 0.1243 | 0.2170 | 0.3182 | $ꞥ_5 > ꞥ_2 > ꞥ_4 > ꞥ_1 > ꞥ_3$ |
| **₵ = 463** | 0.1442 | 0.2308 | 0.1242 | 0.2169 | 0.3181 | $ꞥ_5 > ꞥ_2 > ꞥ_4 > ꞥ_1 > ꞥ_3$ |

Further, we explored all the results obtained by the SVNAAPWA and SVNAAPWG operators in the graphical representation of Figure 3 and Figure 4.

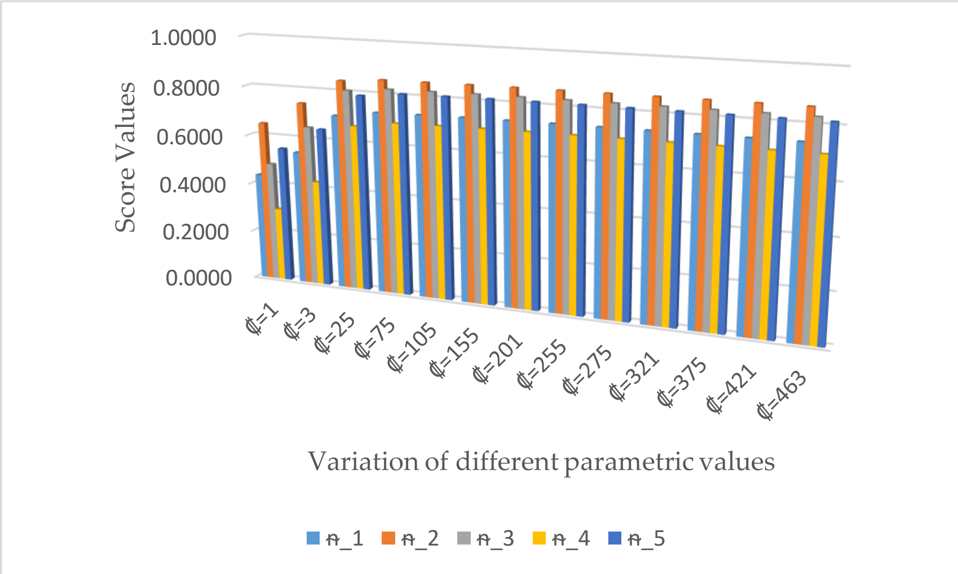


**Figure 3** Graphical representation of score values depicted in Table 5.

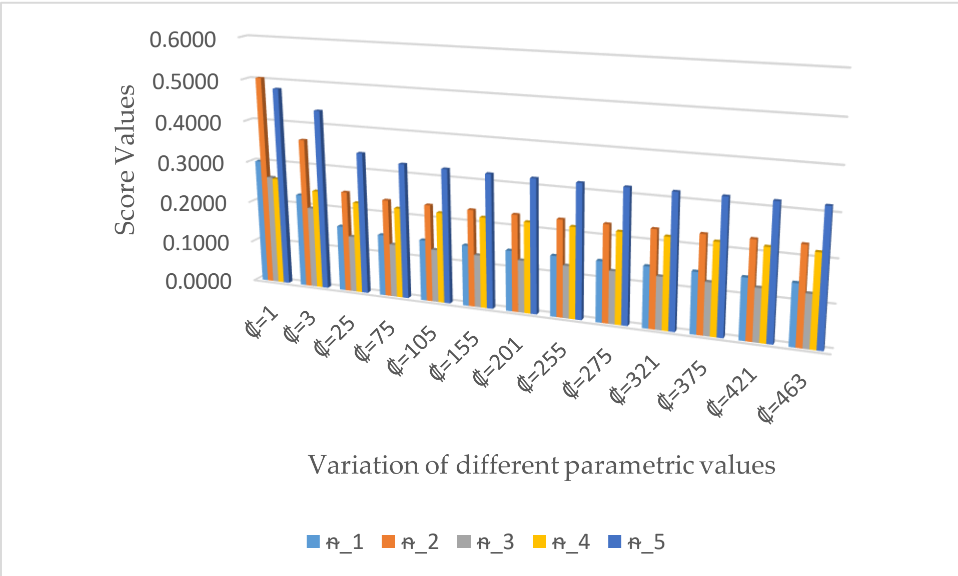


**Figure 4** Graphical representation of score values depicted in Table 6.

## 7. Comparative Study

To show the effectiveness and applacability of our discussed approaches, we make a comparison of the outcomes of the current discussed approaches with the consequences of the existing approaches. For this purpose, we utilized a few numbers of used AOs on the data of SVNVs presented by the decision maker and shown in Table 2. AOs of SVN Dombi weighted average and SVN Dombi weighted geometric operators anticipated by Chen and Ye [31], AOs of SVN weighted average and SVN weighted geometric operators presented by Peng et al. [51], AOs of SVN Einstein weighted average and SVN Einstein weighted geometric operators anticipated by the Ye et al. [52], and AOs of complex SVNVs (CSVNVs) based on Prioritized Muirhead Mean tools given by the Mahmood and

Ali [32]. All the results obtained by the existing AOs [31], [32], [51], [52] are shown in the following Table 7.

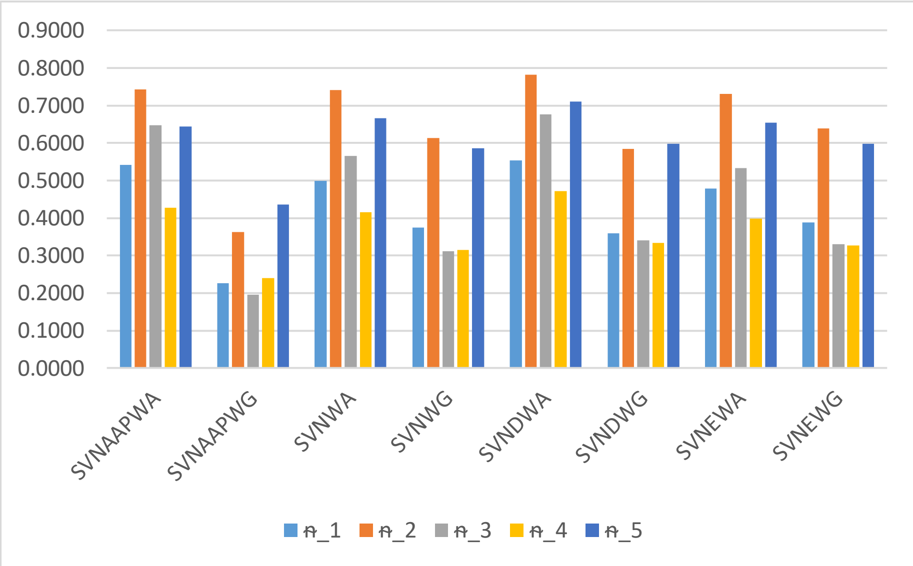


Figure 5. Shows the results of the comparative study in a graphical representation.

**Table 7.** Shows the results of a comparative study.

| AOs | Score values | Ranking and ordering |
|---|---|---|
| Current work | $\mathcal{G}(\hbar_1) = 0.5418, \mathcal{G}(\hbar_2) = 0.7431, \mathcal{G}(\hbar_3) = 0.6478, \mathcal{G}(\hbar_4) = 0.4273, \mathcal{G}(\hbar_5) = 0.6435$ | $\hbar_2 > \hbar_3 > \hbar_5 > \hbar_1 > \hbar_4$ |
| Current work | $\mathcal{G}(\hbar_1) = 0.2266, \mathcal{G}(\hbar_2) = 0.3622, \mathcal{G}(\hbar_3) = 0.1966, \mathcal{G}(\hbar_4) = 0.2402, \mathcal{G}(\hbar_5) = 0.4355$ | $\hbar_5 > \hbar_2 > \hbar_4 > \hbar_1 > \hbar_3$ |
| SVNWA **[51]** | $\mathcal{G}(\hbar_1) = 0.4985, \mathcal{G}(\hbar_2) = 0.7415, \mathcal{G}(\hbar_3) = 0.5654, \mathcal{G}(\hbar_4) = 0.4160, \mathcal{G}(\hbar_5) = 0.6655$ | $\hbar_2 > \hbar_5 > \hbar_3 > \hbar_1 > \hbar_4$ |
| SVNWG **[51]** | $\mathcal{G}(\hbar_1) = 0.3751, \mathcal{G}(\hbar_2) = 0.6136, \mathcal{G}(\hbar_3) = 0.3120, \mathcal{G}(\hbar_4) = 0.3158, \mathcal{G}(\hbar_5) = 0.5862$ | $\hbar_2 > \hbar_5 > \hbar_1 > \hbar_3 > \hbar_4$ |
| SVNDWA **[31]** | $\mathcal{G}(\hbar_1) = 0.5539, \mathcal{G}(\hbar_2) = 0.7815, \mathcal{G}(\hbar_3) = 0.6759, \mathcal{G}(\hbar_4) = 0.4714, \mathcal{G}(\hbar_5) = 0.7115$ | $\hbar_2 > \hbar_5 > \hbar_3 > \hbar_1 > \hbar_4$ |
| SVNDWG **[31]** | $\mathcal{G}(\hbar_1) = 0.3597, \mathcal{G}(\hbar_2) = 0.5842, \mathcal{G}(\hbar_3) = 0.3416, \mathcal{G}(\hbar_4) = 0.3346, \mathcal{G}(\hbar_5) = 0.5980$ | $\hbar_2 > \hbar_5 > \hbar_1 > \hbar_3 > \hbar_4$ |
| SVNEWA **[52]** | $\mathcal{G}(\hbar_1) = 0.4797, \mathcal{G}(\hbar_2) = 0.7309, \mathcal{G}(\hbar_3) = 0.5333, \mathcal{G}(\hbar_4) = 0.3979, \mathcal{G}(\hbar_5) = 0.6549$ | $\hbar_2 > \hbar_5 > \hbar_3 > \hbar_1 > \hbar_4$ |
| SVNEWG **[52]** | $\mathcal{G}(\hbar_1) = 0.3882, \mathcal{G}(\hbar_2) = 0.6397, \mathcal{G}(\hbar_3) = 0.3313, \mathcal{G}(\hbar_4) = 0.3271, \mathcal{G}(\hbar_5) = 0.5977$ | $\hbar_2 > \hbar_5 > \hbar_1 > \hbar_4 > \hbar_3$ |
| Mahmood and Ali **[32]** | CSVNVs | Failed |

From Table 7, we examined the results of existing approaches and concluded that invented methodologies are superior to other ones. Due to the parametric value of Aczel Alsina aggregation tools, Decision makers can acquire results of score values according to their preferences by setting

different parametric values of Aczel Alsina aggregation tools. We also observed the consistency and effectiveness of our invented approaches in Tables 5-6.
Following graphical representation shows the results of existing approaches obtained by the decision matrix of Table 2 and shown in Figure 5.

## 8. Conclusion

The decision information is more appropriately described in terms of SVNVs during the decision-making process due to the increasing uncertainties and complexity of practical situations. In this article, we exposed the notion of SVNSs to cope with ambiguous and vague information about human opinions. The SVNS is the modified version of an IFSs and PFSs, which provides freedom to decision-makers in the decision-making process and contains more extensive information than other frameworks of fuzzy systems. Aczel Alsina aggregation tools are superior to other aggregation tools. By using the theory of Aczel Alsina aggregation tools, we proposed a class of new approaches based on SVN information, including SVNAAPA and SVNAAPG operators. We also generalized the theory of SVNSs with properties of Aczel Alsina aggregation tools and presented a series of new approaches like SVNAAPWA and SVNAAPWG operators. To reveal the intensity and effectiveness of our invented methodologies, some notable characteristics are also explored. We established an algorithm for the MADM problem under the system of SVN information. We discussed a numerical example to find the most appropriate candidate for the vacant post of a general manger for the multinational company. To find the validity and flexibility of our methods, we evaluated the effects of the results on the alternatives for several parametric values. The advantages of our presented methodologies are also presented by comparing the findings of existing approaches with currently proposed AOs.

Sometimes decision-makers cannot find an appropriate optimal option due to insufficient information about weight vectors. We can use the concepts of power operators and entropy measures to handle this situation. We also apply our invented approaches to resolve different applications such as artificial intelligence, game theory, waste management, and social selection. Furthermore, we will explore our invented approaches in the framework of the bipolar soft set [53], [54], picture fuzzy sets [55], spherical fuzzy sets, and complex spherical fuzzy sets [56]. Next, we will apply our invented approaches to improve the healthcare system's reliability and establish a strong model for the waste materials under the system of NS [57].

Furthermore, we also attached a list of variables used throughout this article.

| **Symbols** | **Meanings** | **Symbols** | **Meanings** |
|---|---|---|---|
| $X$ | Non-empty set | $\mathfrak{Q}$ | Accuracy function |
| $\varphi$ | PMV | $\boldsymbol{\Psi}$ | Weight vector |
| $\delta$ | AMV | ß | Attribute |
| $\sigma$ | NMV | $\mathfrak{n}$ | Alternative |

| $\mathscr{r}$ | Element from non-empty set | $\not\subset$ | Parametric values |
|---|---|---|---|
| $\alpha$ | SVNV | $\mathcal{G}$ | Score function |

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# Determining the Best Plastic Recycling Technology Using the MABAC Method in a Single-Valued Neutrosophic Fuzzy Approach

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**Abstract**. In recent years, waste management approaches have shifted to recycling and recovery, and waste is now viewed as a potentially new resource. Several research projects have developed extensive plans to observe the planning in these waste management systems. These plastic recycling methods contribute to the creation of environmentally friendly products from waste. In this study, we present a method of multi-attribute decision making (MADM) to provide an efficient way to choose the best plastic recycling method from the selected four recycling alternatives. The multi-attribute border approximation area comparison (MABAC) method is used to evaluate the alternatives under the single-valued neutrosophic fuzzy set (SVNFS). Additionally, we used SWARA in computing the weights of the attributes. Finally, a numerical illustration is given for this problem.

**Keywords:** Multi-attribute decision making; single-valued neutrosophic fuzzy set; plastic recycling; MABAC.

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## 1. **Introduction**

Plastics, once a rare commodity, are now our most serious threat. Plastic is widely used, durable, and cheap, so its use has become a part of all sectors. The use of plastic products in our daily lives has inevitably increased. Plastics are mainly used in daily life items, medical and industrial equipment, and electrical appliances. It plays an important role in the products that people mostly use in all fields. The reason for using this plastic is that it is very easy to carry and the price is cheap, so people are using it more and more.

A reference to plastic materials in the Earth's environment that affect organisms living on the Earth's surface is called "plastic pollution". When plastics are burned, they pollute land, water, the oceans, and the air. Many environmental problems are caused by pollution due to its use and improper management. Balancing the production of plastics and their recycling and reuse after use is a major challenge in today's environment. We can't stop using plastic right now and make it better, but we can certainly find and use alternatives to reduce the use and production of plastic in the world. This research paper explains how to control the effects of plastic waste and how to recycle it. Reusing plastic can help reduce overproduction. The process of converting waste materials into new products is called recycling. This study discusses recycling methods for plastic and thermoplastic polymers. Also, we discussed the plastic recycling methods (PRM) and their processes.

Sabino Armenise et al.[1] fully examined and reviewed in the area of plastic recycling with pyrolysis methods. Martinez [2] studied plastic pyrolysis methods in American countries. Anuar et al. [3] reviewed some literature about pyrolysis methodologies in plastic wastes. Harish Jeswani et al. [4] disguised about pyrolysis of mixed plastic waste. Sofie Huysman et al. [5] studied the pertinence of the recyclability benefit index concepts. Pakiya Pradeep and Gowthaman [6] explained waste plastic as potential alternative sources for fossil fuel. Goodship [7] provided a compendious of the quantities and the main effects of recycling on the plastic material. Adeleka et al. [8] explained the sustainable utilization of energy from waste in South Africa. Chen et al. [9] discussed the various recycling energy recovery technologies. Wilson et al. [10] summarized pyrolysis technology in plastic waste management; the experimental results on the pyrolysis of thermoplastic polymers are discussed on single and mixed waste plastics. Based on a real-world case study, Gu et al. [11] assessed mechanical plastic recycling practices. Pacheco et al. [12] overview and investigated of plastic recycling difficulties were in the Metropolitan area of Rio de Janeiro. Shanker et al. [13] proposed recycling technological options for India and reprocessing infrastructure for PWR in India. Plastics recycling worldwide overviewed by dAmbrires [14]. Challenges, and opportunities of recycling plastics in Western Australia reported by Cceres Ruiz and Zaman [15]. Many researchers examined various recycling technologies. All of these studies examined and aimed to identify the most viable plastic recycling method. As a result, the study's goal was to develop a general framework for selecting the most appropriate PRM based on environmental and social factors.

Real-life decision-making problems are made more difficult by ambiguity and fuzzy logic. Zadeh [16, 17] created the concept of fuzzy set theory, which describes and converts data that is imprecise rather than accurate. The theory of fuzzy logic demonstrates an empirical basis for gathering information about the risks and unpredictability associated with human cognitive abilities such as reasoning and comprehension [18]. Due to the intricate nature of data and the vagueness of the way humans think, the ability to identify members of the set of fuzzy numbers is not always adequate for determining the features of issues. To overcome this restriction, Atanassov [19] converted the fuzzy set into an intuitionistic fuzzy set (IFS) by introducing the not-being-a-member and unwillingness functions. An IFS may indicate situations in three ways: superiority, complex inferiority, and skepticism, with intuitionistic fuzzy numbers (IFNs) typically representing these [20]. Smarandache [21] recommended the neutrosophic set and neutrosophic possibility in 1998, in addition to the reasoning behind them, which includes three distinct sense ideas such as truthfulness, indeterminacy, and untruthfulness. This idea additionally encompasses the idea of trepidation, which contributes to the research having a significant impact in specific research areas. In a neutrosophic fuzzy set (NSS), truthfulness is expressed by $T$, indeterminacy by $I$, and falsehood by $F$. All of these are separate, adding up to $0 \leq T + I + F \leq 3$. Although the level of membership as well as non-membership determines the ambiguity of IFS, the indeterminacy associated with NFS is not dependent on truth and untruth values. NFNs can be used to define the ambiguity, falsity, and unwillingness of information in an everyday issue. Karaaslan and Hunu [22] used the TOPSIS approach to determine and explain single-valued neutrosophic sets (SVNS) and their applications in multiple attribute group decision making (MCGDM).

Balwada et al. [23] identify a better waste collection system using AHP for packaging plastic waste. Geetha et al. [24] proposed a suitable recycling method for plastics under hesitant pythagorean fuzzy ELECTRE III. Vinodh et al. [25] examined the best recycling method using integrated MCDM methods. Soni et al. [26] proposed a triangular fuzzy weighted bonferroni mean operator AHP-TOPSIS model for selecting an appropriate composition for developing floor tiles from recycled waste plastics. Chakraborty and Saha [27] presented a new GDM process that combines the AHP model and WASPAS under LR fuzzy numbers to convey and model expert linguistic judgments. Afzal and Aslam [28] introduced a novel methodology to establish the relationship between capacitance and resistance when dealing with imprecise data obtained from LCR meters. Using a neutrosophic set, Abdelhafeez et al. [29] proposed a mean weighting methodology for analysing and selecting the best criteria in smart farming. The AHP is combined with the SVNS to deal with uncertain data in the assessment process of underwater vehicles studied by Mohamed et al. [30]. Gamal and Mohamed [31] examined the integrated MCDM methods for the industrial robot selection problem. Abdel-Basset et al. [32] suggested a hybrid MCDM method for choosing the components of a sustainable RES in unpredictable circumstances, employing various triangular neutrosophic numbers for dealing with ambiguous data. Abdl-Basset et al. [33] described a novel hybrid MCDM framework for classifying and selecting third-party reverse logistics provider identification. Rani et al. [34] investigated a novel single-valued neutrosophic mixed compromise solution approach for selecting renewable energy resources. Ali Salamai [35] explored a neutrosophic SWARA and VIKOR integrated technique for ranking strategies in energy problems related to decision-making. Based on the SVNFS, Stanujkic et al. [36] suggested a multiple-criteria evaluation model. The MABAC model was used by Sahin and Altun [37] in a probabilistic single-valued neutrosophic hesitant scenario. Wang et al. [38] elevated the MABAC procedure for MCGDM in a fuzzy Q-rung environment.

Many studies have focused on the application of plastic recycling methods in the existing literature. The goal of this study is to develop a new MCDM model for the plastic recycling problem. There has been no research using the MABAC method with a single-valued neutrosophic fuzzy set. As a result, it is critical to address the lack of research in plastic recycling treatments. The MABAC model successfully adapts to the relevance needs for plastic recycling techniques, which motivated us to research and develop our proposed model for PRM, which may substantially minimise plastic waste while also protecting the environment and society. This is urgently needed. The contribution of the study is to use the proposed method to choose the best plastic recycling method in terms of minimal operating expenses, a small amount of contamination, additional social benefits, and fewer harms to the environment. In this research, we combined the MABAC model with the SWARA weight finding method and performed an analysis of comparison to validate the suggested method's suitability for PRM problems against existing methods such as EDAS and WASPAS. Furthermore, sustainability was examined and provided as a sensitivity analysis.

In this study, we use the SVNFS to present an improved and trustworthy solution to the plastic recycling challenge under MABAC and SWARA methods. Moreover, numerous studies investigated plastic recycling methods using a variety of fuzzy sets with different MCDM approaches. To fill this research gap for this problem, we use the proposed method under SVNFN.

This paper is organized as follows: Section 2 - preliminaries; Section 3 - mathematical methods; Section 4 - application; Section 5 - numerical example of the application; Section 6 - comparative and sensitivity analysis of the obtained solutions; Section 7-conclusions.

## 2. Preliminaries

**Definition 2.1.** [34, 36] Let $U$ be a universal set. A fuzzy set $F$ on $U$ is a form

$$F = \{(a, \mu_F(a))|a \in U, 0 \leq \mu_F(a) \leq 1\}$$

Where $\mu_F(a)$ denoted the membership degree of $a \in U$ to $F$.

**Definition 2.2.** [34, 36] A neutrosophic fuzzy set $N$ on $U$ is a form:

$$N = \{(a, T_N(a), I_N(a), F_N(a)) : a \in U\}$$

where $T_N(a), I_N(a), F_N(a) \in [0,1], 0 \leq T_N(a) + I_N(a) + F_N(u) \leq 3$ for all $a \in U$, $T_N(a)$ is membership, $I_N(a)$ is indeterminacy and $F_N(a)$ is non-membership degree. Here, $T_N(a)$ and $F_N(a)$ are dependent and $I_N(a)$ is an independent components.

Table 1. Linguistic scale

| Linguistic term | Single-valued neutrosophic fuzzy number |
|---|---|
| Extremely High Preferred (EHP) | (0.85, 0.20, 0.15) |
| Very High Preferred (VHP) | (0.80, 0.25, 0.20) |
| High Preferred (HP) | (0.75, 0.25, 0.25) |
| Moderate (M) | (0.70, 0.30, 0.30) |
| Moderate Preferred (MP) | (0.65, 0.30, 0.35) |
| Low Moderate Preferred (LMP) | (0.60, 0.35, 0.40) |
| Extremely Moderate Preferred (EMP) | (0.55, 0.40, 0.45) |

**Definition 2.3.** [36] A SVNF set $S$ in $U$ is a form:

$$S = < a, T_S(a), I_S(a), F_S(a) > | a \in U$$

Where $T_S : U \to [0,1]$ is called the truth-membership grade of $a \in U$ to $S$, $I_S : U \to [0,1]$ is indeterminacy-membership, $F_S : U \to [0,1]$ is called the falsity-membership grade. They satisfy $0 \leq T_S(a) + I_S(a) + F_S(a) \leq 3$ for $a \in U$.

**Definition 2.4.** [36, 37] Let $h =< T, I, F >$ be a SVNFN. The score function $A_h$ of $h$ is a follows:

$$A_h = (1 + T - 2I - F)/2$$

Where $A_h \in [-1, 1]$.

**Definition 2.5** (31,39)**.** Let $m = (k_1, f_1)$ and $n = (k_2, f_2)$ be two SVNNs, then the single-valued neutrosophic fuzzy normalized hamming distance (SVNFNHD) is

$$D_{Ham}(m, n) = \frac{1}{3n} \sum_{i=1}^{n} (|T_m(a_i) - T_n(b_i)| + |I_m(a_i) - I_n(b_i)| + |F_m(a_i) - F_n(b_i)|)$$

**Definition 2.6.** Linguistic variables deal with many more complex and uncertain real-world decision-making problems [40]. Table 1 shows the linguistic variables with SVNFNs used to evaluate the PRM based on selected attributes and the linguistic scale.

## 3. **Mathematical Methods**

Pamucar and Cirovi [33] established the MABAC, which is an innovative distance-based approach. This strategy's policy is based on calculating criterion function parameters for alternatives and expressing the criterion function's distance from the border approximation area. As a result, all alternatives can be included into the approximation area's border ($G$), upper ($G+$), or lower ($G-$).

The procedure of the fuzzy MABAC method is discussed below.

## 4. The fuzzy MABAC method

Here, $m$ alternatives $X_1, X_2, ..., X_m$, $n$ attributes $Y_1, Y_2, ..., Y_n$ are given with weight $W_j$, and the decision-making procedures of the traditional MABAC method are explained below.

**Step 1:** Create the initial decision matrix $F = [X_{ij}], i = 1, 2, ..., m, j = 1, 2, ..., n$ as:

$$F = [X_{ij}] = \begin{bmatrix} x_{11} & x_{12} & ... & x_{1n} \\ x_{21} & x_{22} & ... & x_{2n} \\ \vdots & \vdots & ... & \vdots \\ x_{m1} & x_{m2} & ... & x_{mn} \end{bmatrix} \tag{1}$$

where $x_{ij}$ represents the evaluation information of alternative $X_i$ based on attributes $Y_j$ by decision maker $E$.

**Step 2:** Normalize the initial decision matrix (NDM) $N = [X_{ij}]$ based on beneficial and non-beneficial attributes which are given below:
For beneficial attributes:

$$N_{ij} = X_{ij}, i = 1, 2, ..., m, j = 1, 2, ..., n \tag{2}$$

For non-beneficial attributes:

$$N_{ij} = 1 - X_{ij}, i = 1, 2, ..., m, j = 1, 2, ..., n \tag{3}$$

**Step 3:** In accordance with the NDM $N_{ij}$ and attribute's weight values $w_j$; the weighted normalized matrix (WNDM) $V_{ij} = w_j N_{ij}$ can be calculated as:

$$V_{ij} = w_j N_{ij} \tag{4}$$

**Step 4:** Calculate the border approximation area (BAA) values and the BAA matrix $B = [b_j]_{1\times n}$ can be obtained as below:

$$b_j = (\prod_{i=1}^{m} V_{ij}) \tag{5}$$

**Step 5:** Obtain the distance $D = [d_{ij}]_{m\times n}$ between each alternative and the BAA is given below:

$$d_{ij} = \begin{cases} d(V_{ij}, b_j) & if \quad V_{ij} > b_j \\ Otherwise & if \quad V_{ij} = b_j \\ -d(V_{ij}, b_j) & if \quad V_{ij} < b_j \end{cases} \tag{6}$$

where $d(V_{ij}, b_j)$ denotes the distance from $V_{ij}$ to $b_j$. Based on the values of $d_{ij}$,

- if $d_{ij} > 0$, the alternatives are in the upper approximation area $G^+(UAA)$;
- if $d_{ij} = 0$, the alternatives are in the border approximation area $G(BAA)$;
- if $d_{ij} < 0$, the alternatives are in the lower approximation area $G^-(LAA)$;

Clearly, the best alternatives are in $G^{+}(UAA)$ and the worst alternatives are in $G^{-}(LAA)$.
**Step 6:** Sum the values of each alternatives $d_{ij}$ is given below:

$$R_i = \sum_{j=1}^{n} d_{ij} \tag{7}$$

Here, the highest value of the alternative $R_i$ is the best choice from the evaluation results.

### 4.1. *The Step-wise Weight Assessment Ratio Analysis weighting method*

The SWARA model developed by Kresuliene et al. [34] helps experts find criteria weights. The SWARA model procedure is as follows:

**Step 1:** Sort the criteria by priority.

**Step 2:** Obtain its relative importance $\delta_j$.

**Step 3:** Compute $\gamma_j$, where $\gamma_j = \delta_j + 1$.

**Step 4:** Determine the starting weights $\eta_j, \eta_j = \frac{\delta_j}{\delta}$.

**Step 5:** Finally, in order to determine the final ranking of the criteria $W_j$, where $W_j = \frac{\eta_j}{\sum \eta_j}$.

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## 5. **Application**

Plastics are a type of chemical-based or partly synthetic substance that consists mainly of polymers. Because of their flexibility, plastics can be shaped, ejected, or transformed into solid things of various shapes. This versatility, along with additional features such as thinness, longevity, adaptability, and low manufacturing expenses, has contributed to its broad adoption. Plastics are usually manufactured using human industrial machinery. Most present-day plastics contain chemicals produced from fossil fuels such as natural gas or gasoline.

Plastic waste (PW) is a major source of solid waste pollution all over the world. PWs slow degradation rate kills billions of living organisms. People all over the world have tried different types of methods to degrade or convert PW into usable materials in order to dispose of it. Incineration involves the combustion of PW, which produces toxic gases. Recycling is another method for converting PW into new plastic products. Recycling is required because almost all plastic is non-biodegradable and thus accumulates in the environment, causing harm.

Modern landfill technology is already vastly superior to older landfills and open-air dumps. New landfills in the various countries are better designed and built in safer locations to reduce or prevent seepage of noxious water or gases into the environment. Especially by recycling plastic materials, the biggest impact can be avoided. The use of plastic waste to build roads can lead to quality roads. Learning basic knowledge about recycling processes will definitely help us fight plastic pollution and choose the most suitable recycling method. Modern policies are now in place to recycle plastics.

Mechanical recycling is the process of making secondary materials without significantly changing the chemical structure of the plastic. Thermoplastic is recycled in this manner [1, 4]. Pyrolysis is a chemical process that breaks down plastic by recycling it. When plastic waste is separated and extracted from pyrolysis, its raw materials reveal an excess of crude oil. Stabilization of plastic waste at different temperatures $(300 - 900^{o}C)$ in anoxic or low oxygen conditions is called "pyrolysis". In this case, their hydrocarbon composition is cracked instead of being heated. Pyrolysis is the process by which discarded plastic is converted into a valuable resource in the form of fuel and monomers. This recycling offers many advantages over conventional plastic waste management; another recycling method reduces the plasticity compared with pyrolysis. As the plastic is repeatedly recycled, its strength and flexibility decrease [1].

Cold plasma pyrolysis is the most advanced method of pyrolysis. In common, the method of pyrolysis is thermal decomposition with limited oxygen at temperatures between $400 and 650C$. From this process, one can generate electricity and fuels; in particular, when cold plasma is added to the pyrolysis process, waste plastics give off hydrogen, methane, and ethylene. Green energy can be generated from plastic waste. Hydrogen and methane are able to be employed as environmentally friendly energy sources due to their low emissions of $CO_2$, while ethylene is the basic component of most plastics. Rather than wasting plastics, cold plasma pyrolysis may preserve valuable substances that can be used to manufacture other kinds of plastic [35].

PRM is critical as a waste management method and as an essential part of the new circular economy and no-waste systems, all of which are designed to decrease waste and improve ecological sustainability. Only a small percentage of plastic waste is recycled. There are several reasons for this, and while our plastic waste is increasing, technological advancements and changes in how we recycle are assisting in making it more successful and efficient. In this paper, we propose the best plastic recycling method based on a single-valued neutrosophic fuzzy approach, using the fuzzy MABAC and SWARA methods.

## 6. **Numerical Illustration**

In this section, we discuss the PR treatment problem under the single-valued neutrosophic fuzzy set using the MABAC and SWARA model. Here, the decision maker evaluate this problem based on the four attributes which are $Y_1$ - Environment, $Y_2$ - Technology, $Y_3$ - Economic, and $Y_4$ - Social aspects. The PR technologies are $X_1$ - Cold Plasma Recycling, $X_2$ - Mechanical recycling, $X_3$ - Pyrolysis Recycling and $X_4$ - New trends in Landfill.

Table 2. Weight values of the criteria

| Criteria | $\delta_j$ | $\gamma_j = \delta_j + 1$ | $\eta_j = \frac{\delta_{j-1}}{\delta}$ | $W_j = \frac{\eta_j}{\sum \eta_j}$ |
|---|---|---|---|---|
| $Y_1$ | 0 | 1 | 1 | 0.4186 |
| $Y_4$ | 0.32 | 1.32 | 0.7575 | 0.3171 |
| $Y_2$ | 0.48 | 1.8 | 0.4208 | 0.1761 |
| $Y_3$ | 0.20 | 2 | 0.2104 | 0.0880 |

To address this issue, experts use the proposed method to evaluate PRM. A decision matrix is constructed using the linguistic scale.

### 6.1. *SWARA method:*

Using the SWARA model procedure, we get the weight values of the attributes, which are shown in Table 2.

### 6.2. *The fuzzy MABAC method:*

The plastic recycling techniques and attributes are given below:

$X_1$ – Cold Plasma Recycling

$X_2$ – Mechanical recycling

$X_3$ – Pyrolysis Recycling

$X_4$ – New trends in Landfill

Attributes are as follows:

$Y_1$ – Environment

$Y_2$ – Technology

$Y_3$ – Economic (8)

$Y_4$ – Social aspects

**Step 1:** The evaluation attribute chosen by the decision maker is used to evaluate the plastic recycling techniques. Table 3 shows the initial decision matrix along with the assessments in the format of SVNFNs obtained from transforming the linguistic factors from Table 1.

**Step 2:** Table 4 shows the results of obtaining the NDM by using equations (2) and (3).

**Step 3:** The weighted NDM can be calculated using equation (4), which is given in Table 5.

**Step 4:** Using equation (5), calculate the BAA values shown in Table 6.

**Step 5:** Obtained the distance $d_{ij}$ by applying the equation (6) which is shown in Table 7.

Table 3. Initial decision matrix

| Alternatives / Attribute | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ |
|---|---|---|---|---|
| $X_1$ | (0.70, 0.35, 0.20) | (0.70, 0.30, 0.30) | (0.60, 0.30, 0.35) | (0.70, 0.25, 0.45) |
| $X_2$ | (0.55, 0.30, 0.40) | (0.65, 0.35, 0.20) | (0.85, 0.25, 0.30) | (0.60, 0.25, 0.30) |
| $X_3$ | (0.80, 0.25, 0.45) | (0.75, 0.30, 0.25) | (0.55, 0.25, 0.35) | (0.80, 0.35, 0.30) |
| $X_4$ | (0.65, 0.40, 0.30) | (0.85, 0.35, 0.20) | (0.80, 0.25, 0.40) | (0.85, 0.30, 0.35) |

Table 4. Normalized decision matrix

| Alternatives / Attribute | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ |
|---|---|---|---|---|
| $X_1$ | (0.70, 0.35, 0.20) | (0.70, 0.30, 0.30) | (0.40, 0.70, 0.65) | (0.70, 0.25, 0.45) |
| $X_2$ | (0.55, 0.30, 0.40) | (0.65, 0.35, 0.20) | (0.15, 0.75, 0.70) | (0.60, 0.25, 0.30) |
| $X_3$ | (0.80, 0.25, 0.45) | (0.75, 0.30, 0.25) | (0.45, 0.75, 0.65) | (0.80, 0.35, 0.30) |
| $X_4$ | (0.65, 0.40, 0.30) | (0.85, 0.35, 0.20) | (0.20, 0.75, 0.60) | (0.85, 0.30, 0.35) |

Table 5. Weighted normalized decision matrix

| Alternatives / Attribute | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ |
|---|---|---|---|---|
| $X_1$ | (0.2930, 0.1465, 0.0837) | (0.0792, 0.2219, 0.2219) | (0.0704, 0.1232, 0.1144) | (0.0264, 0.066, 0.0484) |
| $X_2$ | (0.2302, 0.1255, 0.1674) | (0.1109, 0.2061, 0.2536) | (0.0264, 0.1320, 0.1232) | (0.0352, 0.066, 0.0616) |
| $X_3$ | (0.3348, 0.1046, 0.1883) | (0.0792, 0.2219, 0.2378) | (0.0792, 0.1320, 0.1144) | (0.0176, 0.0572, 0.0616) |
| $X_4$ | (0.2720, 0.1674, 0.1255) | (0.0475, 0.2061, 0.2536) | (0.0352, 0.1320, 0.1056) | (0.0132, 0.0616, 0.0572) |

Table 6. BAA values

| $b_j$ | values |
|---|---|
| $b_1$ | (0.2120, 0.1014, 0.1021) |
| $b_2$ | (0.0657, 0.0285, 0.0205) |
| $b_3$ | (0.1134, 0.3085, 0.2716) |
| $b_4$ | (0.1286, 0.0500, 0.0607) |

**Step 6:** Sum the values of each alternatives $R_i$ is calculated by using equation (7). The final ranking results are shown in Table 8.

From this Table 8, $X_1-$Cold plasma recycling technology is the most suitable and environment friendly method for plastic recycling method.

TABLE 7. Distance $d_{ij}$ values

| | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ |
|---|---|---|---|---|
| $X_1$ | 0.0917 | 0.0905 | 0.0586 | 0.0790 |
| $X_2$ | -0.1052 | 0.0824 | -0.0566 | 0.0812 |
| $X_3$ | 0.0970 | 0.0869 | 0.0685 | -0.0902 |
| $X_4$ | -0.0972 | -0.0886 | -0.0560 | -0.0871 |

TABLE 8. The final ranking results

| Alternatives | $R_i$ values | Ranking result |
|---|---|---|
| $X_1$ | 0.3198 | 1 |
| $X_2$ | 0.0018 | 3 |
| $X_3$ | 0.1622 | 2 |
| $X_4$ | -0.3289 | 4 |

## 7. Comparison and sensitivity analysis

In this section, we analyze the proficiency of this suggested method through the comparison of existing methods such as EDAS and WASPAS in the case of a SVNFN. For this study, sensitivity analysis was also established.

### 7.1. *Comparison Analysis*

Although contrasted to the EDAS and WASPAS shown in Figure 1, the MABAC is more readily compatible with our application. The analysis of comparison in this study generates more realistic and consistent outcomes if compared with different approaches. In short, instead of the EDAS and WASPAS techniques, the MABAC requires an alternative comparison. Furthermore, the selection results generated by the suggested approach provide more data in the form of a reliability index of outranking relationships among alternatives, which is more beneficial for the suggested approach. Table 9 indicates the order of importance provided for the two methods as well as the order of ranking results, and the graphic depictions are shown in Figure 1. We focused on just four criteria to determine alternatives in this paper; however, future research can employ the proposed approach to consider additional aspects such as expenses for operations and societal benefits. The proposed ranking produces results that more differ from the existing EDAS and WASPAS methods. As a result, the proposed approach produces more reliable results when compared to other MCDM model.

Table 9. Comparison analysis results

| Alternatives | EDAS | Rank | VIKOR | Rank | Proposed method | Rank |
|---|---|---|---|---|---|---|
| $X_1$ | 0.2375 | 2 | 0.4216 | 1 | 0.3198 | 1 |
| $X_2$ | 0.1818 | 3 | 0.4169 | 2 | 0.0018 | 3 |
| $X_3$ | 0.3906 | 1 | 0.4033 | 4 | 0.1622 | 2 |
| $X_4$ | -0.1901 | 4 | 0.4137 | 3 | -0.3289 | 4 |

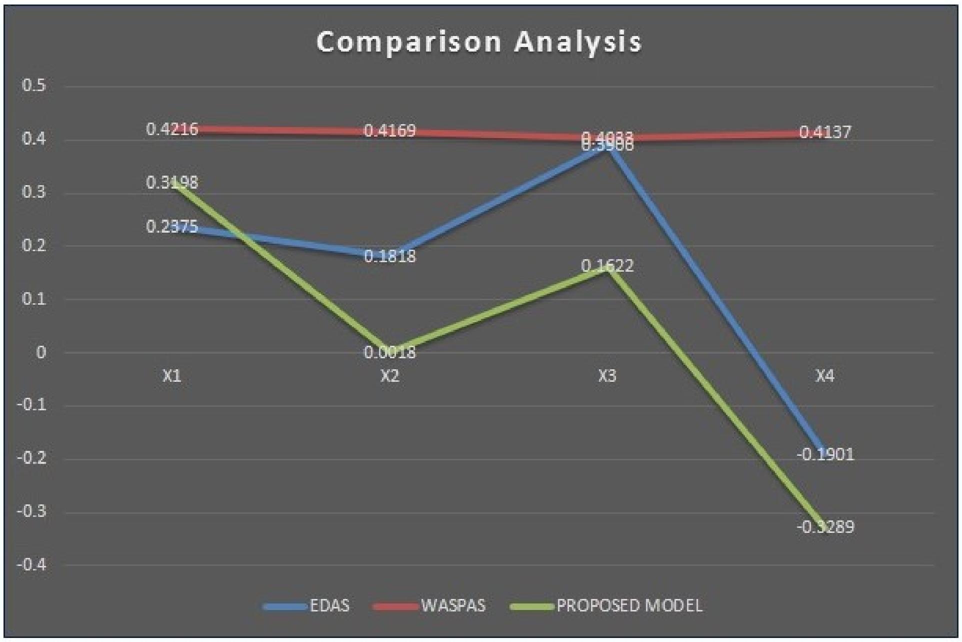


Figure 1. Graphical representation for comparison analysis

7.2. *Sensitivity analysis*

The sensitivity evaluation of this framework is contrasted with the outcomes of three cases, as shown in Table 10. These cases are discovered by varying the weights of the criteria. Case 1 is the study's outcome, and Cases 2 and 3 are each of the results obtained by applying various attribute weights. Sensitivity analysis reveals that changing the attribute weights has an impact on the overall order, as shown in Table 11, and Figure 2 depicts their graphical representation.

7.3. *Results and discussion*

There are numerous advantages to the cold plasma recycling of plastics. This results in a decrease in the utilization of novel goods and energy, lowering carbon dioxide emissions. As a

Table 10. Weights in sensitivity analysis

| Attribute | Case 1 | Case 2 | Case 3 |
|---|---|---|---|
| $Y_1$ | 0.4186 | 0.3171 | 0.1761 |
| $Y_2$ | 0.3171 | 0.0880 | 0.4186 |
| $Y_3$ | 0.1761 | 0.4186 | 0.0880 |
| $Y_4$ | 0.0880 | 0.1761 | 0.3171 |

Table 11. Sensitivity analysis results

| Alternatives | Case 1 | Rank | Case 2 | Rank | Case 3 | Rank |
|---|---|---|---|---|---|---|
| $M_1$ | 0.3198 | 1 | 0.2185 | 2 | 0.1496 | 2 |
| $M_2$ | 0.0018 | 3 | -0.3793 | 4 | 0.0803 | 3 |
| $M_3$ | 0.1622 | 2 | 0.3927 | 1 | 0.2427 | 1 |
| $M_4$ | -0.3289 | 4 | -0.0592 | 3 | 0.0649 | 4 |

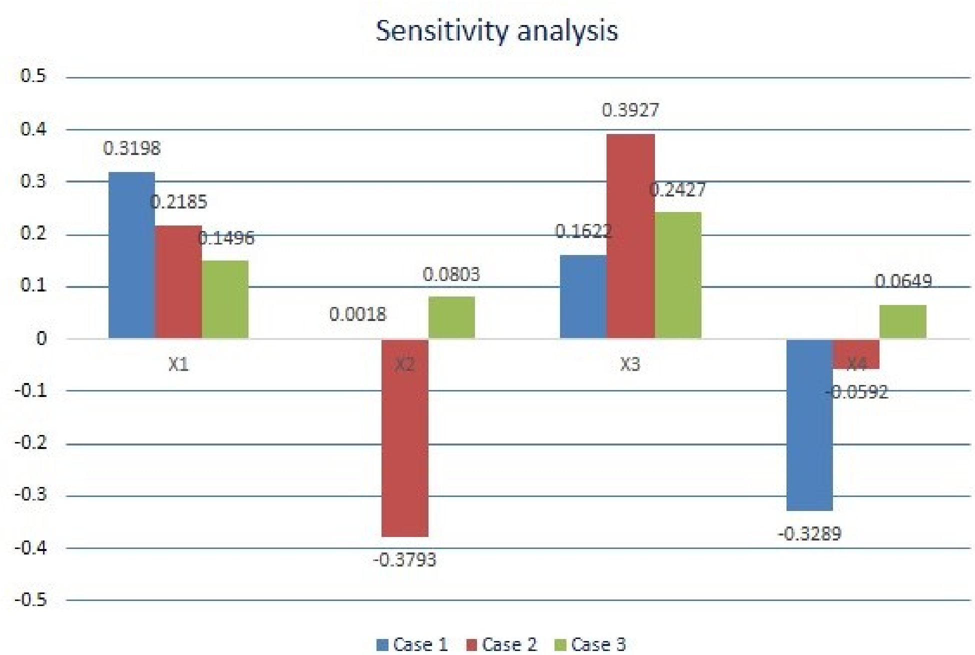


Figure 2. Graphical representation for sensitivity analysis

result, the cold plasma recycling method is a good plastic recycling method that contributes to lowering plastic waste [41]. Cold plasma recycling alternative $X_1$ is the best PRM technique. Furthermore, the overall ranking of the choices for this problem proposed a one-of-a-kind method. Because experts struggled to evaluate choices among the various levels of contentment and frustration in order to estimate unknowns, this research study used the SVNN to handle the data. Based on the end ranking results (Fig. 2), an enhanced MABAC model looks to be more precise and suitable to address a various of other issues in MADM. The contrast between suggested model and other ranking strategies reveals that proposed model provides a more realistic and suitable solution that is easier to implement. The obtained weight values of criteria show that this approach was combined with the advanced, rapid, and precisely calculated SWARA weight finding method. In this paper, we propose a cohesive MCDM model for the plastic recycling problem. The MABAC method is used to determine the order of the alternatives, and SWARA is used to calculate the weights of the criteria. These processes are particularly effective when contrasted with some MCDM methods. The MABAC model computes the distance between each alternative and the bore estimation area. This model has numerous benefits over other MCDM methods, including shorter processing times, greater ease and stability, and fewer numerical computations [42]. As a result, decision makers will be able to use fuzzy MABAC to select the best alternative with regard to processing time, reliability, and expenses. Compared to other criteria weight determination methods (such as AHP), the SWARA method has reduced computational challenges and greater consistency. As a result of these benefits, the SWARA method has been used to solve real-world issues in a variety of scenarios [43]. SWARA gives more plausible weight values than other weighting methods due to more consistent computations.

## 8. **Conclusion**

In this regard, the mathematical model is an important tool for the evaluation of plastic waste management systems and illustrates an efficient implementation of plastic recycling approach to the plastic recycling methods. The SVNFs decision techniques have distinguished similar rankings among the administrative choices when target weights are allocated to the criteria. An alternate ranking is obtained just with the weight set which vigorously needs technical/operation indicators. Consequently, attribute weighting is an important process in decision making. In this paper, we conclude cold plasma recycling method is the best alternative solution for the plastic recycling planning. This gives energy producing technique also and environmental friendly.

Plastic recycling methods, among others, have been recognised as a promising solution to the issue of intricately made and growing waste from plastics in advanced nations like the European Union and the US. The government and policymakers, on the other hand, continue to face major obstacles in determining appropriate plastic recycling techniques for establishing effective waste disposal systems. As a result, this research was conducted in order to suggest an overall organised structure that can assist policymakers in determining the most suitable technology. In this paper, MABAC and SWARA methods under single-valued neutrosophic fuzzy environment were presented. The characteristics of every option are represented by SVNFNs. According to the suggested strategy for determining a suitable solution for this problem, cold plasma recycling has been selected as the most secure and best-performing PRM technique in the present scenario. This method reduces waste while generating energy, which will aid in addressing future energy challenges. In the future, we are interested in extending the proposed method to other issues, such as microplastic disposal. Furthermore, applying the suggested methodology to the Pythagorean neutrosophic fuzzy approach is an intriguing avenue for future research.

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# A Neutrosophic Approach to Edge-Based Anomaly Detection in Smart Farming Systems

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**Abstract**: Neutrosophic sets have emerged as a powerful tool for addressing uncertainty and imprecision in diverse domains, and their potential in anomaly detection within smart farming systems is the central focus of this paper. We present a cutting-edge Neutrosophic Approach to Edge-Based Anomaly Detection, specifically designed to cater to the intricacies of smart farming data. By harnessing the unique attributes of single-valued neutrosophic sets, in conjunction with single-valued neutrosophic decision matrices, our methodology adeptly handles the challenges posed by uncertain, dynamic, and multi-dimensional farm data. Through a comprehensive analysis of sample data, we illustrate the precision and adaptability of our approach, allowing for the quantification of intricate attribute relationships and the precise identification of anomalies. By employing neutrosophic statistics and a weighted correlation coefficient, our approach provides profound insights into the complex interactions within smart farming systems. This research stands as a pivotal contribution within the scope of neutrosophic-based anomaly detection, promising to advance the state of the art in the realm of precision agriculture.

**Keywords**: Neutrosophic Logic; Edge Computing; Anomaly Detection; Smart Farming; Sensor Networks; Agricultural IoT; MCDM; Neutrosophic Sets.

## 1. Introduction

The utilization of neutrosophic sets has witnessed a significant surge in recent years, offering a versatile framework to tackle complex problems characterized by ambiguity, uncertainty, and imprecision. Neutrosophic set theory, an extension of classical fuzzy set theory, introduces a third component—indeterminacy—alongside membership and non-membership degrees, enabling a more comprehensive representation of uncertain information [1]. This novel framework has found application in diverse fields such as medicine, image processing, decision-making, and pattern recognition [2].

The domain of smart farming, characterized by its amalgamation of advanced technologies, has ushered in a new era of data-driven agriculture. Within this landscape, edge-based anomaly detection plays a pivotal role in ensuring the seamless operation and optimization of farming systems. However, the very attributes that make smart farming systems so powerful—the vast and dynamic streams of data generated by sensors, devices, and machinery—also introduce inherent challenges. Uncertainty abounds in every facet of smart farming data. Environmental conditions fluctuate, sensor readings exhibit variability, and unforeseen events can disrupt the expected patterns. These uncertainties are further compounded by the intricate interplay of multiple variables and attributes within the farming ecosystem [3].

This pressing need for precise, real-time anomaly detection in the face of pervasive uncertainty provides the impetus for our exploration of neutrosophic sets. Neutrosophic sets, with their ability to handle not only membership and non-membership degrees but also indeterminacy, offer a nuanced understanding of uncertain data [4]. Their adaptive nature aligns perfectly with the volatile environment of smart farming systems, where anomalies can be subtle and evolving. By embracing the philosophy of neutrosophic sets, we embark on a journey to harness the power of uncertainty, transforming it from a challenge into an opportunity [5].

In line with the increasing relevance of neutrosophic sets, this paper is centered on their application in the realm of anomaly detection within smart farming systems. Smart farming, characterized by its integration of cutting-edge technologies such as IoT devices, sensors, and data analytics, has ushered in a new era of precision agriculture [4]. However, the vast and dynamic nature of the data generated by these systems presents intricate challenges for anomaly detection. Traditional methods often fall short in handling the nuanced uncertainties inherent in smart farming data. This is where neutrosophic sets, particularly single-valued neutrosophic sets, take center stage, offering a comprehensive approach to address the intricacies of anomaly detection in this context [5-6]. In light of the challenges posed by smart agriculture and the potential of neutrosophic logic and edge intelligence, this paper sets out specific research objectives [8]. Our primary goal is to develop and evaluate a novel approach to anomaly detection in smart agriculture systems. We aim to harness the power of edge intelligence to process and analyze agricultural data in real time and utilize neutrosophic logic to model and detect anomalies effectively. Through rigorous experimentation and validation, we seek to demonstrate the feasibility and advantages of our proposed approach [7-11].

The paper is organized into five sections to comprehensively address the development of a neutrosophic approach for anomaly detection in smart agriculture systems using edge intelligence. In Section 2, we review existing literature and research efforts related to smart agriculture, anomaly detection, edge intelligence, and neutrosophic logic. This section offers a contextual foundation for our work by highlighting gaps in the current body of knowledge. Section 3 presents the intricacies of our proposed approach, elucidating the integration of edge intelligence and neutrosophic logic for anomaly detection. In Section 4, we provide a detailed account of our experimental findings and data analysis, showcasing the effectiveness of our methodology in real-world smart agriculture scenarios. Section 5 offers a concise summary of our key findings, contributions, and future directions.

**2. Background**

This section provides a comprehensive review of the existing literature related to anomaly detection in smart farming systems, focusing on both traditional and emerging approaches. We explore how various techniques have addressed the challenges of anomaly detection and highlight the rationale for introducing a neutrosophic approach as a novel and promising avenue for enhancing the precision and robustness of anomaly detection systems in smart farming. Dhole et al. [12] presented a review of brain tumor detection from MRI images using hybrid approaches. Although their work focused on medical imaging, it underscored the importance of hybrid techniques in image analysis. This review highlighted the relevance of combining multiple methods, which potentially inspired hybrid approaches in the context of anomaly detection in smart farming systems. The study by Garcia-Lamont et al. [15] provided insights into image segmentation using color features. While their focus was on a different application domain, the segmentation techniques they discussed had relevance in preprocessing and feature extraction for smart farming system anomaly detection. Zakaria et al. [16] discussed the use of graph cuts for image segmentation in the context of COVID-19 X-ray image analysis. Their work showcased the effectiveness of segmentation methods in medical image analysis. In the context of smart farming, similar segmentation techniques might have been employed for the preprocessing of agricultural images. Qi et al. [18] provided a comprehensive review of computer vision-based hand gesture recognition for human-robot interaction. Although their focus was on different applications, their discussion on computer vision methods was relevant, as computer vision played a pivotal role in many smart farming applications, including anomaly detection. The work by Ashfaq [19] touched on retrospective image registration for medical image analysis. While the domain differed, image registration techniques could have been adapted to align images in the context of smart farming, potentially aiding in anomaly detection. Beebe [22] provided a complete bibliography of publications in computer networks, which might have contained relevant references for advanced data communication and networking techniques that were important for edge-based anomaly detection systems in smart farming.

**3. Proposed Method**

In this section, we elucidate the methodology employed in our research to develop a robust and innovative approach for anomaly detection in smart agriculture systems using the amalgamation of edge intelligence and neutrosophic logic. The methodology presented herein outlines the systematic framework and techniques we employed to address the intricate challenges associated with real-time anomaly detection in agricultural environments.

***The neutrosophic theory*** is a mathematical framework that deals with indeterminacy, uncertainty, and incomplete information. It extends classical set theory to handle situations where information is imprecise, vague, or contradictory. In the context of anomaly detection in smart farming, neutrosophic sets can represent the uncertainty associated with sensor readings and anomalies. A neutrosophic set is defined as a triple $(T, I, F)$, Suppose $\alpha_{\tilde{a}}$ , $\theta_{\tilde{a}}$ , $\beta_{\tilde{a}} \in [0,1]$and $a_1$ , $a_2$ , $a_3$ , $a_4 \in$ R, where $a_1 \le a_2 \le a_3 \le a_4$. Then, a single-valued trapezoidal neutrosophic number $\tilde{a}$=⟨($a_1$ , $a_2$ , $a_3$, $a_4$); $\alpha_{\tilde{a}}$ , $\theta_{\tilde{a}}$ , $\beta_{\tilde{a}}$⟩ is a special neutrosophic set on the real line set R, whose truth-membership, indeterminacy-membership and falsity-membership functions are defined as

$T$ represents the truth-membership function.

$$T_{\tilde{a}}(x) = \begin{cases} \alpha_{\tilde{a}}\left(\frac{x-a_1}{a_2-a_1}\right) & (a_1 \le x \le a_2) \\ \alpha_{\tilde{a}} & (a_2 \le x \le a_3) \\ \alpha_{\tilde{a}}\left(\frac{a_4-x}{a_4-a_3}\right) & (a_3 \le x \le a_4) \\ 0 & otherwise \end{cases} \tag{1}$$

$I$ represents the indeterminacy-membership function.

$$I_{\tilde{a}}(x) = \begin{cases} \frac{(a_2-x+\theta_{\tilde{a}}(x-a_1))}{(a_2-a_1)} & (a_1 \le x \le a_2) \\ \alpha_{\tilde{a}} & (a_2 \le x \le a_3) \\ \frac{(x-a_3+\theta_{\tilde{a}}(a4-x))}{(a_4-a_3)} & (a_3 \le x \le a_4) \\ 1 & otherwise \end{cases}, \tag{2}$$

$F$ represents the falsity-membership function.

$$F_{\tilde{a}}(x) = \begin{cases} \frac{(a_2-x+\beta_{\tilde{a}}(x-a_1))}{(a_2-a_1)} & (a_1 \le x \le a_2) \\ \alpha_{\tilde{a}} & (a_2 \le x \le a_3) \\ \frac{(x-a_3+\beta_{\tilde{a}}(a4-x))}{(a_4-a_3)} & (a_3 \le x \le a_4) \\ 1 & otherwise, \end{cases} \tag{3}$$

*Neutrosophic logic operators*, analogous to classical logic operators, are used to perform operations on neutrosophic sets. The neutrosophic logical operators include:

***Neutrosophic AND (M) Operator:***

For two neutrosophic sets $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)_1$ the neutrosophic AND operator is defined as follows:

$$AA_AB = (T_A - T_B, I_A + I_B - I_A \cdot I_B, F_A + F_B - F_A - F_B) \tag{4}$$

***Neutrosophic OR(V) Operator:***

For $A$ and $B$ as defined above, the neutrosophic $OR$ operator is defined as:

$$A \vee_n B = (T_A + T_B - T_A \cdot T_B, I_A - I_B, F_A \cdot F_B) \quad (5)$$

***Neutrosophic NOT ($\sim_n$) Operator.***

For a neutrosophic set $A = (T_A, I_A, F_A)$, the neutrosophic NOT operator is defined as:

$$\sim \text{n}\, A = (1 - T_A, I_A, F_A) \quad (6)$$

In the context of anomaly detection, sensor data is represented as neutrosophic sets. The degree of anomaly for a particular data point can be calculated using the neutrosophic anomaly score (N.S.S) Let $X = (T_X, I_X, F_X)$ be the neutrosophic representation of a data point. The neutrosophic anomaly score can be calculated as follows:

$$NAS = 1 - (T_X + I_X + F_X) \quad (7)$$

A NAS value closer to 1 indicates a higher degree of anomaly.

In decision-making processes, neutrosophic sets can be used to represent different criteria. Aggregation operators, such as the neutrosophic weighted average, can be employed to combine these criteria into an overall decision. For a set of neutrosophic criteria $C_1, C_2, \dots, C_n$, with corresponding weights $w_1, w_2, \dots, w_n$, the neutrosophic weighted average ($NWA$) is calculated as:

$$NWA = \frac{\sum_{i=1}^{V_i} (w_i \cdot T_i, w_i \cdot I_i, w_i \cdot F_i)}{\sum_{i=1}^{n_i} w_i} \quad (8)$$

where $T_{i_1} I_{i_2}$ and $F_1$ represent the truth, indeterminacy, and falsity membership functions of criterion $C_i$ respectively.

In the context of anomaly detection in smart farming systems, effectively managing uncertain information is of paramount importance. The complexity and dynamism of the data, coupled with the intricate interplay between factors that may lead to anomalies, present formidable challenges. Handling uncertain information is a critical objective in anomaly detection. In real-world agricultural scenarios, numerous variables contribute to the data collected from smart farming sensors. These variables are often influenced by various factors, such as weather conditions, soil composition, and the presence of pests or diseases. The resulting data can be inherently uncertain, imprecise, and subject to fluctuations.

To address these challenges, this paper leverages the power of neutrosophic sets. Neutrosophic sets are a powerful tool for handling uncertainty, indeterminacy, and contradiction in data and decision-making. In our proposed approach for edge-based anomaly detection in smart farming, we use neutrosophic sets to model these three aspects. Uncertainty represents the degree to which we lack precise knowledge about a data point. In the context of smart farming, uncertainty may arise from various sources such as sensor noise, environmental variability, or measurement errors. Neutrosophic sets can model uncertainty through their membership functions.

Uncertainty Membership Function ($U_X$): For a neutrosophic set $X = (T_x, I_x, F_x)$, the uncertainty membership function $U_X$ is defined as:

$$U_X = F_X \tag{9}$$

This formula quantifies the indeterminacy component of the neutrosophic set, which represents uncertainty.

Indeterminacy represents the degree to which a data point is ambiguous or lacks a definite attribute value. In our context, indeterminacy can occur when sensor readings are imprecise or when the interpretation of data is not clear. Contradiction arises when there are conflicting pieces of information within a data point. In smart farming, contradiction may occur when different sensors provide conflicting readings or when historical data contradicts current observations.

*Contradiction Membership Function* ($C_X$) : For a neutrosophic set $X = (T_X, I_X, F_X)$, the contradiction membership function $C_X$ is defined as:

$$C_X = F_X \tag{10}$$

This formula quantifies the falsity component of the neutrosophic set, which represents a contradiction. For example, let's consider an example using a neutrosophic set $X$ representing the temperature reading from a smart farming sensor:

$$X = (0.7, 0.2, 0.1) \tag{11}$$

In this case $T_X = 0.7$ represents the truth component, indicating a high likelihood that the temperature reading is accurate. $I_X = 0.2$ represents the indeterminacy component, signifying some uncertainty or imprecision in the reading. $F_X = 0.1$ represents the falsity component, suggesting a minor degree of contradiction or inconsistency.

This paper proposes the utilization of a single-valued neutrosophic set, a specialized form of neutrosophic set. In the context of anomaly detection in smart farming, the single-valued neutrosophic set is generated using triangular neutrosophic numbers. This approach allows for a more structured representation of uncertainty, where each data point is associated with a single-valued neutrosophic set that encapsulates its truth, indeterminacy, and falsity components. To quantify the relationships between data points and detect anomalies effectively, this paper introduces an improved weighted correlation coefficient formula. This formula takes into account the specific characteristics of single-valued neutrosophic sets and their triangular neutrosophic number representations. The detailed description of our anomaly detection approach involves the following essential steps. These steps collectively form a comprehensive framework for identifying anomalies within the smart farming system:

Step 1: In this phase of our anomaly detection approach, we handle the generation of neutrosophic numbers for farm anomaly data ($A$) and test sample data ($C$). This step is crucial in preparing the data for subsequent analysis. For anomaly template set $A = \{A_1, A_2, \dots, A_m\}$ and test sample set $C = \{C_1, C_2, \dots, C_n\}$, we generate neutrosophic numbers to represent the inherent uncertainty in the data. Specifically, we create three neutrosophic numbers for each data point: the lower membership function ($L$), the upper membership function ($U$), and the midpoint ($M$).

*Lower Membership Function* ($L$): The lower membership function represents the lower bound of the data's uncertainty. It quantifies the minimum possible value that the data point can take. This is computed as:

$$L(x) = \frac{x - \min(A_i)}{\max(A_i) - \min(A_i)} \tag{12}$$

where $x$ is the value of the data point and $\min(A_i)$ and $\max(A_i)$ are the minimum and maximum values in the anomaly template set $A_i$ for $i = 1,2, \dots, m$.

Upper Membership Function ( $U$ ): The upper membership function represents the upper bound of the data's uncertainty. It quantifies the maximum possible value that the data point can take. This is computed as:

$$U(x) = \frac{\max(A_i) - x}{\max(A_i) - \min(A_i)} \tag{13}$$

where $x$ is the value of the data point, and $\min(A_i)$ and $\max(A_i)$ are the minimum and maximum values in the anomaly template set $A_i$ for $i = 1,2, \dots, m$.

*Midpoint* ($M$): The midpoint represents the central value within the data's uncertainty range. It is calculated as the average of the lower and upper bounds:

$$M(x) = \frac{L(x) + U(x)}{2} \tag{14}$$

In our anomaly detection approach, we utilize triangular neutrosophic numbers to represent data points. A triangular neutrosophic number is characterized by specific properties that allow us to quantify the inherent uncertainty in the data. These properties are as follows:

*Largest Value*: In a triangular neutrosophic number, the largest value corresponds to the right end value of the triangle. This value represents the upper bound of uncertainty and signifies the maximum possible value that the data point can take. It quantifies the most optimistic scenario.

*Minimum Value*: Conversely, the minimum value in a triangular neutrosophic number is located at the left endpoint of the triangle. This value represents the lower bound of uncertainty and signifies the minimum possible value that the data point can take. It quantifies the most pessimistic scenario.

*Average Value*: The average value of a triangular neutrosophic number is situated at the upper-end value of the triangle. This value represents the central tendency within the data's uncertainty range. It is calculated as the midpoint between the minimum and maximum values.

*Height of the Triangle*: A critical characteristic of a triangular neutrosophic number is that the height of the triangle is equal to 1. This height signifies the degree of uncertainty or fuzziness associated with the data point. A taller triangle indicates a higher degree of uncertainty, while a shorter triangle suggests greater confidence in the data's precision.

The graphical representation of a triangular neutrosophic number is depicted in Figure 1, where the triangle illustrates the range of uncertainty, and the location of its apex, base, and height aligns with the specific properties mentioned above.

Step 2: In the second step of our anomaly detection approach, we perform a critical comparison between the neutrosophic numbers representing the attributes of test samples $(C_j)$ and those of the farm anomaly templates $(A_i)$. This comparison allows us to quantify the degrees of determinacy membership $(T_{A_i}(C_j))$, non-membership $(F_{A_i}(C_j))$, and indeterminacy-membership $(I_{A_i}(C_j))$. These degrees provide insights into the level of conformity or deviation between test samples and farm anomaly templates, facilitating effective anomaly detection. For each attribute $(A_i)$ of a test sample $(C_j)$, the degrees of membership, non-membership, and indeterminacy-membership are calculated using the following formulas:

*Degree of Determinacy-Membership* $(T_{A_i}(C_j))$ : This degree represents the extent to which the attribute $A_i$ of the test sample $C_j$ belongs to the farm anomaly template $A_i$. Calculated as the minimum of the upper-bound values of the neutrosophic numbers:

$$T_{A_i}(C_j) - \min\{U_{A_i}(C_j), U_{A_i}(A_i)\} \tag{15}$$

*Degree of Non-Membership* $(F_{A_i}(C_j))$*:* This degree quantifies the extent to which the attribute $A_i$ of the test sample $C_j$ does not belong to the farm anomaly template $A_i$. Calculated as the maximum of the lower-bound values of the neutrosophic numbers:

$$F_{A_i}(C_j) - \max\{L_{A_i}(C_j), L_{A_i}(A_i)\} \tag{16}$$

*Degree of Indeterminacy-Membership* $(I_{A_i}(C_j))$: This degree captures the degree of ambiguity or uncertainty associated with the attribute $A_i$ of the test sample $C_j$ concerning the farm anomaly template $A_i$. Calculated as the complement of the sum of the degrees of determinacy-membership and non-membership:

$$I_{A_i}(C_j) - 1 - T_{A_i}(C_j) - F_{A_i}(C_j) \tag{17}$$

These degrees $\left(T_{A_i}(C_j), F_{A_i}(C_j), I_{A_i}(C_j)\right)$ provide a comprehensive understanding of the similarity or dissimilarity between test samples and farm anomaly templates for each attribute.

Step 3: In the next step of our anomaly detection approach, we transition from the parameters $T_{A_i}(C_j), F_{A_i}(C_j)$, and $I_{A_i}(C_j)$ to single-valued neutrosophic sets $(a_{ij})$. These single-valued neutrosophic sets encapsulate the degrees of determinacy-membership, nonmembership, and indeterminacy-membership for each attribute of a test sample $C_j$ concerning the corresponding attribute in the anomaly template $A_i$.

$$D = (a_{ij})_{m\times n} = \begin{bmatrix} < t_{11}, f_{11}, i_{11} > & < t_{12}, f_{12}, i_{12} > & \cdots & < t_{in}, f_{ij}, i_{ij} > \\ < t_{21}, f_{21}, i_{21} > & < t_{22}, f_{22}, i_{22} > & \cdots & < t_{2n}, f_{2n}, i_{2n} > \\ \vdots & \vdots & & \vdots \\ < t_{m1}, f_{m1}, i_{m1} > & < t_{m2}, f_{m2}, i_{m2} > & \cdots & < t_{mn}, f_{mn}, i_{mn} > \end{bmatrix} \tag{18}$$

Each single-valued neutrosophic set $a_{ij}$ is represented as a triple $\langle t_{ij}, f_{ij}, i_{ij}\rangle$, where $t_{ij}$ denotes the degree of determinacy-membership. $f_{ij}$ represents the degree of non-membership. $i_{ij}$ signifies the degree of indeterminacy-membership. By expressing the parameters $T_{A_i}(C_j), F_{A_i}(C_j)$, and $I_{A_i}(C_j)$ in the form of single-valued neutrosophic sets $a_{ij}$, we encapsulate the uncertainty and relationships between attributes within a structured neutrosophic framework.

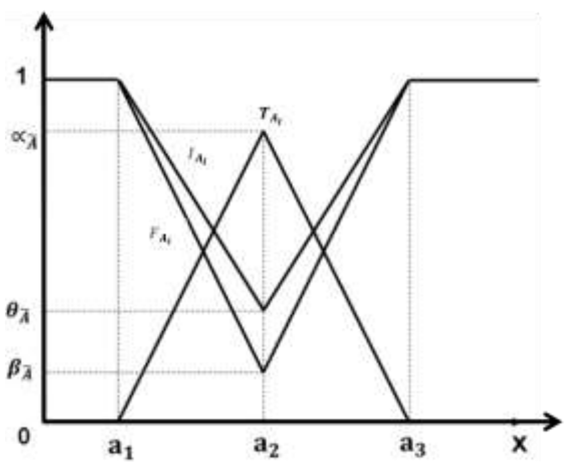


**Figure** 1. Visualizing a Triangular neutrosophic Number Geometrically.

Step 4: In the subsequent step of our anomaly detection approach, we derive ideal single-valued neutrosophic numbers for each attribute $j$ $(where\ j = 1,2, \dots, n)$. These ideal single-valued neutrosophic numbers serve as reference points for assessing the degree of similarity or dissimilarity between test samples and anomaly templates. The generation process is performed column-wise based on the single-valued neutrosophic set decision matrix $D$.

$$a^*{}_j =< t^*{}_j, f^*{}_j, i^*{}_j >=< \max_i(t_{ij}), \min_i(f_{ij}), \min_i(i_{ij}) > \tag{19}$$

Step 5: In this step of our anomaly detection approach, we generate a weighted correlation coefficient to quantitatively assess the degree of similarity between the single-valued neutrosophic sets in the decision matrix $D$ and the ideal single-valued neutrosophic number. This coefficient serves as a crucial indicator for identifying anomalies based on the deviation from the ideal values. The calculation formula for the weighted correlation coefficient $WCC$ is as follows:

$$W(A_i, B) = \frac{2 \cdot \sum_{j=1}^{n} w_j [t_{ij} \cdot t^*{}_j + f_{ij} \cdot f^*{}_j + i_{ij} \cdot i^*{}_j]}{\sum_{j=1}^{n} w_j [t_{ij}{}^2 + f_{ij}{}^2 + i_{ij}{}^2] + \sum_{j=1}^{n} w_j [t^*{}_j{}^2 + f^*{}_j{}^2 + i^*{}_j{}^2]} \tag{20}$$

**4. Results and Analysis**

The outcomes of our research and the in-depth analysis of the experimental data, offering a comprehensive assessment of the effectiveness of our proposed methodology for anomaly detection in smart agriculture systems are presented. In our study, we leveraged real-world farm anomaly data to conduct a comprehensive analysis of anomaly detection within smart farming systems. A pivotal aspect of this analysis involved the acquisition of triangular neutrosophic numbers for diverse attributes, a critical step in the neutrosophic framework we employed for anomaly detection. Table 1 presents the resulting triangular neutrosophic numbers obtained under various attributes. These neutrosophic numbers encapsulate the inherent uncertainty and variability within the data, capturing the complex and dynamic nature of smart farming environments. Our study meticulously considered attributes such as temperature, humidity, soil moisture, and spectral data, among others, to provide a holistic view of anomaly detection in agriculture. The acquisition of these neutrosophic numbers represents a foundational element of our research, facilitating the subsequent neutrosophic analysis that drives the accurate identification of anomalies. In the following sections, we delve into the outcomes of our anomaly detection approach, offering insights into its effectiveness and real-world applicability within the realm of smart farming.

**Table 1.** Triangular neutrosophic Numbers for Attributes in Farm Anomaly Data

| | Minimum | Mean | Maximum | Zone |
|---|---|---|---|---|
| **A12-A18** | 0.0551 | 0.2232 | 0.2928 | 0.2516 |
| **A22-A28** | 0.1180 | 0.1528 | 0.4736 | 0.1129 |
| **A32-A38** | 0.0278 | 0.2001 | 0.2381 | 0.0199 |
| **A42-A48** | 0.3117 | 4.3342 | 4.6856 | 2.1545 |
| **B12-B18** | 0.1253 | 0.2009 | 0.2296 | 0.0236 |
| **B22-B28** | 0.3031 | 0.4025 | 0.4551 | 0.0220 |
| **B32-B38** | 0.1339 | 0.2658 | 0.3574 | 0.0677 |

| | | | | |
|---|---|---|---|---|
| **B42-B48** | 4.0418 | 4.7958 | 9.0371 | 2.4010 |
| **C12-C18** | 0.2749 | 0.3803 | 0.3990 | 0.0235 |
| **C22-C28** | 0.2278 | 0.3486 | 0.4082 | 0.0423 |
| **C32-C38** | 0.0959 | 0.1688 | 0.2776 | 0.0357 |
| **C42-C48** | 9.3726 | 9.8447 | 10.1851 | 0.3635 |

In our analysis of the sample data, a fundamental component of our study involved the extraction of triangular neutrosophic numbers for diverse attributes, a key element within the neutrosophic framework we employed for anomaly detection. As demonstrated in Table 2, we obtained these triangular neutrosophic numbers under various attributes, each representing the inherent uncertainty and variability observed within the sample dataset. Attributes such as temperature, humidity, soil moisture, and spectral data were meticulously considered, providing a comprehensive view of anomaly detection in the context of our study. These triangular neutrosophic numbers lay the foundation for the subsequent neutrosophic analysis, enabling us to quantitatively assess the degree of similarity between the sample data attributes and ideal values.

**Table 2.** Triangular Neutrosophic Numbers for Attributes in Analyzed Sample Data

| | Minimum | Mean | Maximum | Zone |
|---|---|---|---|---|
| **A1** | 0.10898 | 0.20047 | 0.31950 | 0.02117 |
| **A2** | 0.03912 | 0.20983 | 0.33639 | 0.18393 |
| **A3** | 0.09619 | 0.23943 | 0.31954 | 0.04330 |
| **A4** | 0.11814 | 0.15975 | 0.24484 | 0.00999 |
| **A5** | 4.03510 | 4.10292 | 4.19177 | 0.12442 |

In our analysis, we conducted an in-depth examination of the analyzed sample data, focusing on the matching of sample attributes (represented as $X_k$, where $k = 1,2,3,4$ denotes specific attributes) with various anomaly categories ($G_k^{1-5}$, where $G = X, Y, Z$ represents three distinct types of anomalies - A, B, C). This matching process allowed us to systematically evaluate the degree of similarity and dissimilarity between the attributes of the sample data and the three anomaly categories. Subsequently, we calculated neutrosophic statistics encompassing the determined-membership degree ($T$), nonmembership degree ($F$), and indeterminacy-membership degree ($I$). These statistics, presented in Table 3, offer valuable insights into the dynamic interplay between the sample data attributes and the anomaly categories. The statistics shed light on the varying degrees of membership, non-membership, and indeterminacy, providing a comprehensive understanding of the anomaly detection process within our study. In the following sections, we delve deeper into the implications and significance of these findings, elucidating their relevance in real-world applications.

**Table 3**. Neutrosophic Statistics for Sample Data Attributes Matched with Anomaly Categories

| Anomaly Template | Neutrosophic Number |
|---|---|
| **A12-A18** | (0.8143,0.0523,0.8483) |
| **A22-A28** | (0,1,0.5431) |
| **A32-A38** | (0,1,0.6321) |
| **A42-A48** | (0.7063,0.2633,0.7002) |
| **B12-B18** | (0,1,0.6024) |
| **B22-B28** | (0.0103,0.9874,0.6722) |
| **B32-B38** | (0.0127,0.9787,0.6809) |
| **B42-B48** | (0,1,1) |
| **C12-C18** | (0.9121,0.0164,0.6952) |
| **C22-C28** | (0.9887,0.0072,0.9801) |
| **C32-C38** | (0.0806,0.9293,0.9088) |
| **C42-C48** | (0,1,0.616257) |

In our examination of the farming anomaly data template and the neutrosophic sample attributes represented by $X$, we embarked on a comprehensive analysis aimed at quantifying the relationships between these attributes. Our objective was to construct a structured representation that captures the nuanced interplay between attributes, essential for effective anomaly detection. As a result of this analysis, we have generated a single-valued neutrosophic decision matrix, which is presented in Table 4. This decision matrix encapsulates the degree of similarity and dissimilarity between the various attributes of the farming anomaly data template and the corresponding attributes within the neutrosophic sample. Each entry in this matrix reflects the outcome of our neutrosophic analysis, quantifying the degree to which attributes align or deviate from one another. The decision matrix serves as a cornerstone for our anomaly detection approach, offering a comprehensive view of attribute relationships and guiding us in the identification of anomalies within smart farming systems. In the subsequent sections, we delve into the practical implications and insights derived from our anomaly detection methodology, highlighting its effectiveness in real-world applications.

**Table 4.** Single-Valued Neutrosophic Decision Matrix for Attribute Relationships

| Anomaly | A12-A18 | B12-B18 | C12-C18 |
|---|---|---|---|
| **A1** | (0.0, 0.7384,0.5728) | (0.1905,0.0,1) | (0.1367, 0.6227, 0.3160) |
| **A2** | (0.1903,1,0.2589) | (0.6327,1.0,0.1212) | (0.5734, 0.4212, 0.6390) |
| **A3** | (0.4062, 0.5660, 0.3213) | (0.4010,0.0,0.5598) | (0.2363, 0.7829, 0.0) |
| **A4** | (0.3593, 0, 0) | (0.3718,0.1022, 0.3867) | (0.3953, 0.3674, 1) |

## 5. Conclusions

This study has introduced and demonstrated the efficacy of a neutrosophic approach to edge-based anomaly detection within smart farming systems. By harnessing the power of neutrosophic sets and single-valued neutrosophic decision matrices, we have successfully addressed the challenges posed by uncertain and dynamic farm data. Our methodology has proven capable of quantifying attribute relationships, facilitating the identification of anomalies with precision and sensitivity. Through the analysis of sample data and the generation of neutrosophic statistics, we have gained valuable insights into the complexities of anomaly detection in agriculture. These findings underscore the adaptability and real-world applicability of our approach, offering the potential to enhance the resilience and efficiency of smart farming systems. As we move forward, further research and refinement of our methodology promise to contribute significantly to the advancement of anomaly detection and decision-making processes in the realm of precision agriculture.

Data Availability: All data generated or analyzed during this study are included in this article.

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# A New Approach to Neutrosophic Hypersoft Rough Sets.

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**Abstract**. The aim of this work is to introduce the new notion of Neutrosophic hypersoft rough set and study its properties. Neutrosophic hypersoft rough set is a generalization of Neutrosophic soft rough set. The notion of Neutrosophic hypersoft rough set has not been reported in the literature. The concept of Neutrosophic hypersoft approximation space is presented with illustrative examples and some of its properties are established. The notions of equality, reduct and core among Neutrosophic hypersoft rough sets are studied with suitable examples. Some directions for applications and future research in this area are also indicated.

**Keywords:** Hypersoft sets, Neutrosophic sets, Neutrosophic soft sets, Neutrosophic hypersoft sets, Neutrosophic hypersoft rough sets, Equality, Reduct and Core.

## 1. Introduction

Extension of soft set to hypersoft set was discussed by Smarandache.F [13,14]. Al-Quran.A et al. [1] presented a novel approach to Neutrosophic soft rough set in 2019. Some basic operations on hypersoft sets was studied by Mujahid.A et al. [10]. Jafar.M.N et al. [8] proposed trigonometry similarity measures of Neutrosophic hypersoft set and investigated the basic operations on them. They have also presented an application to revolvable energy source selection. Jafar.M.N et al. [7] have proposed different similarity measures with the help of distances and max-min operators defined on Neutrosophic hypersoft sets. They have proved some properties of this similarity measures and presented an application in site selection for solid waste management system. Jafar.M.N and Saeed.M [6] have also presented an algorithm based on a score function to solve a multi attribute multi criteria decision making problem. Aggregate operators on Neutrosophic hypersoft sets was studied by Saqlain.M et al. [12]. Huseyin.K [5] investigated on hybrid structure of hypersoft sets and rough sets. Das.M et al. [4] expanded the scope of rough set, soft set, and Neutrosophic set by combining Neutrosophic soft set with rough set theory. Broumi.S et al. [1,2] developed a hybrid structure called rough neutrosophic set and discussed its properties. Maji.P.K [9] defined some operations on Neutrosophic soft set and established some properties. Ozturk.T.Y et al. [11] have redefined some operations on Neutrosophic soft sets with illustrative examples. Yolcu.A et al. [15] have broadened the scope of rough, soft and Neutrosophic sets by developing the notion of Neutrosophic soft rough set. They have also presented examples and established some properties of the new structure.

From the above literature study it can be seen that hybrid structures combing Neutrosophic sets and soft sets, Neutrosophic sets and rough sets as well as Neutrosophic hyper sets and soft sets have been considered by different authors for different applications. No work in the literature exists combining Neutrosophic hypersoft sets and rough sets.

In this paper we propose to introduce the hybrid structure Neutrosophic hypersoft rough set ($\mathcal{NHSRS}$) and discuss some of its basic properties like union, intersection and complementation with illustrative examples. The notions of equality between $\mathcal{NHSRS}s$, the reduct and core of a $\mathcal{NHSRS}$ are studied.

The rest of the paper is organized as follows. Section 2, deals with the necessary preliminaries. In section 3, we present the definition of $\mathcal{NHSRS}$ and give an example. Some properties of Neutrosophic hypersoft approximation space are also established. In section 4, equality between $\mathcal{NHSRS}s$ is defined and some interesting results are also established. Section 5, deals with reduct and core of a $\mathcal{NHSRS}$. Some theoretical results connecting core and reduct are proved with necessary examples.

## 2. Preliminaries

The necessary fundamental definitions such as neutrosophic set, hypersoft set, rough set, some properties of neutrosophic hypersoft set and neutrosophic hypersoft rough set can be found in [11, 15].

## 3. Neutrosophic Hypersoft Rough Sets ($\mathcal{NHSRS}s$).

In this section we introduce the notion of $\mathcal{NHSRS}s$.

**Definition 3.1.** Let $U$ be a non-empty universe set and $P_N(U)$ be the set of all neutrosophic sets over $U$. Let $E$ denote the set of parameters. We assume that $E = \{\Delta_1, \Delta_2, ..., \Delta_n\}$, where $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j$. Let $\delta_j \subseteq \Delta_j$ $j \in \{1, 2, ..., n\}$. Then $\Pi_{j=1}^n \delta_j \subseteq \Pi_{j=1}^n \Delta_j$. The pair $\left(\mathbb{N},\ \Pi_{j=1}^n \delta_j\right) = P_{NH}(U)$, where $\mathbb{N}$ is mapping defined by $\mathbb{N} : \Pi_{j=1}^n \delta_j \to P_N(U)$ is called a Neutrosophic hypersoft set ($\mathcal{NHSS}$). The triplet $\left(U, \mathbb{N}, \Pi_{j=1}^n \delta_j\right)$ is called a neutrosophic hypersoft approximation space. The $\underline{lower}$ and $\overline{upper}$ neutrosophic hypersoft approximation spaces of $K \in P_{NH}(U)$ with respect to $\left(U, \mathbb{N}, \Pi_{j=1}^{n}\delta_j\right)$ are denoted by $\underline{apr}_{\mathcal{NHSS}}(K)$ and $\overline{apr}_{\mathcal{NHSS}}(K)$ respectively, defined by

$$\underline{apr}_{\mathcal{NHSS}}(K) = \left\{ \left( \Pi_{j=1}^{n}\delta_j, < \frac{\varkappa}{\underline{\mu}_{\Pi_{j=1}^{n}\delta_j}(\varkappa), \underline{\eta}_{\Pi_{j=1}^{n}\delta_j}(\varkappa), \underline{\nu}_{\Pi_{j=1}^{n}\delta_j}(\varkappa)} > \right), \forall\ \varkappa \in U \right\}.$$

$$\overline{apr}_{\mathcal{NHSS}}(K) = \left\{ \left( \Pi_{j=1}^{n}\delta_j, < \frac{\varkappa}{\overline{\mu}_{\Pi_{j=1}^{n}\delta_j}(\varkappa), \overline{\eta}_{\Pi_{j=1}^{n}\delta_j}(\varkappa), \overline{\nu}_{\Pi_{j=1}^{n}\delta_j}(\varkappa)} > \right), \ \forall\ \varkappa \in U \right\}.$$

where,

$$\underline{\mu}_{\Pi_{j=1}^{n}\delta_j}(\varkappa) = \left\{ \bigwedge \mu_{\Pi_{j=1}^{n}\delta_j}(\varkappa) : \mu_{\Pi_{j=1}^{n}\delta_j}(\varkappa) \in K \cap (\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j); (\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j) \subseteq K, \forall \varkappa \in U \right\}$$
$$\underline{\eta}_{\Pi_{j=1}^{n}\delta_j}(\varkappa) = \left\{ \bigwedge \eta_{\Pi_{j=1}^{n}\delta_j}(\varkappa) : \eta_{\Pi_{j=1}^{n}\delta_j}(\varkappa) \in K \cap (\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j); (\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j) \subseteq K, \forall \varkappa \in U \right\}$$
$$\underline{\nu}_{\Pi_{j=1}^{n}\delta_j}(\varkappa) = \left\{ \bigvee \nu_{\Pi_{j=1}^{n}\delta_j}(\varkappa) : \nu_{\Pi_{j=1}^{n}\delta_j}(\varkappa) \in K \cap (\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j); (\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j) \subseteq K, \forall \varkappa \in U \right\}$$
$$\overline{\mu}_{\Pi_{j=1}^{n}\delta_j}(\varkappa) = \left\{ \bigvee \mu_{\Pi_{j=1}^{n}\delta_j}(\varkappa) : \mu_{\Pi_{j=1}^{n}\delta_j}(\varkappa) \in K \cup (\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j); (\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j) \subseteq K, \forall \varkappa \in U \right\}$$
$$\overline{\eta}_{\Pi_{j=1}^{n}\delta_j}(\varkappa) = \left\{ \bigvee \eta_{\Pi_{j=1}^{n}\delta_j}(\varkappa) : \eta_{\Pi_{j=1}^{n}\delta_j}(\varkappa) \in K \cup (\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j); (\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j) \subseteq K, \forall \varkappa \in U \right\}$$
$$\overline{\nu}_{\Pi_{j=1}^{n}\delta_j}(\varkappa) = \left\{ \bigwedge \nu_{\Pi_{j=1}^{n}\delta_j}(\varkappa) : \nu_{\Pi_{j=1}^{n}\delta_j}(\varkappa) \in K \cup (\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j); (\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j) \subseteq K, \forall \varkappa \in U \right\}$$

where "min" and "max" operators are denoted by "$\bigwedge$" and "$\bigvee$", respectively. It is easy to see that $\underline{apr}_{NHSS}(K)$ and $\overline{apr}_{NHSS}(K)$ are two $\mathcal{NHSS}$s over $P_{NH}(U)$. It is said that $K$ is a neutrosophic hypersoft definable set if $\underline{apr}_{\mathcal{NHSS}}(K) = \overline{apr}_{\mathcal{NHSS}}(K)$, otherwise it is referred to as a Neutrosophic hypersoft rough sets ($\mathcal{NHSRS}s$).

**Example 3.2.** Let $U = \{\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4\}$. Define the attributes sets by,
$\Delta_1 = \{e_{11}, e_{12}\}, \Delta_2 = \{e_{21}, e_{22}\}, \Delta_3 = \{e_{31}, e_{32}\}$.
Let $\delta_1 = \{e_{11}, e_{12}\}, \delta_2 = \{e_{21}, e_{22}\}, \delta_3 = \{e_{31}\}$ that is $\Pi_{j=1}^{n}\delta_j \subseteq \Pi_{j=1}^{n}\Delta_j$, $j = 1, 2, 3$. Let the $\mathcal{NHSS}$,

$$\begin{aligned}\left(\mathbb{N}_1, \Pi_{j=1}^{3}\delta_i\right) = \Big\{ & ((e_{11}, e_{21}, e_{31}), \{\frac{\varkappa_1}{< .5, .2, .3 >}, \frac{\varkappa_2}{< .7, .3, .2 >}\}), \\ & ((e_{13}, e_{23}, e_{33}), \{\frac{x_2}{< .8, .4, .2 >}, \frac{x_3}{< .7, .8, .9 >}, \frac{x_4}{< .6, .1, .4 >}\}), \\ & ((e_{11}, e_{22}, e_{31}), \{\frac{\varkappa_2}{< .3, .2, .5 >}\}), \\ & ((e_{12}, e_{21}, e_{31}), \{\frac{\varkappa_3}{< .8, .4, .1 >}, \frac{\varkappa_4}{< .1, .5, .5 >}\}), \\ & ((e_{12}, e_{22}, e_{31}), \{\frac{\varkappa_1}{< .5, .2, .3 >}, \frac{\varkappa_4}{< .4, .3, .2 >}\}), \\ & ((e_{13}, e_{21}, e_{31}), \{\frac{x_2}{< .8, .9, .2 >}, \frac{x_3}{< .4, .2, .7 >}\})\Big\}\end{aligned}$$

$\alpha_1 = \{e_{11}\}, \alpha_2 = \{e_{21}, e_{22}\}, \alpha_3 = \{e_{31}, e_{32}\}$ that is $\Pi_{j=1}^{n}\alpha_j \subseteq \Pi_{j=1}^{n}\Delta_j$ $j = 1, 2, 3$. Let the $\mathcal{NHSS}$,

$$\begin{aligned}\big(\mathbb{N}_2, \Pi_{i=1}^{3}\alpha_i\big) =& \Big\{((e_{11}, e_{21}, e_{31}), \{\frac{\varkappa_2}{<.2, .5, .6>}, \frac{\varkappa_3}{<.8, .6, .1>}\}),\\ &((e_{13}, e_{23}, e_{33}), \{\frac{x_2}{<.6, .3, .8>}, \frac{x_3}{<.2, .7, .3>}\}),\\ &((e_{11}, e_{22}, e_{31}), \{\frac{\varkappa_2}{<.6, .2, .3>}\}),\\ &((e_{11}, e_{21}, e_{32}), \{\frac{\varkappa_1}{<.4, .3, .5>}, \frac{\varkappa_4}{<.7, .3, .2>}\}),\\ &((e_{11}, e_{22}, e_{32}), \{\frac{\varkappa_3}{<.4, .4, .2>}, \frac{\varkappa_4}{<.1, .3, .8>}\}),\\ &((e_{13}, e_{21}, e_{31}), \{\frac{x_2}{<.9, .2, .4>}, \frac{x_3}{<.8, .1, .9>}\})\}\end{aligned}$$

$\beta_1 = \{e_{11}, e_{12}\}, \beta_2 = \{e_{21}\}, \beta_3 = \{e_{31}, e_{32}\}$ that is $\Pi_{j=1}^{n}\beta_j \subseteq \Pi_{j=1}^{n}\Delta_j$ $j = 1, 2, 3$. Let the $\mathcal{NHSS}$,

$$\begin{aligned}\big(\mathbb{N}_3, \Pi_{j=1}^{3}\beta_j\big) =& \Big\{((e_{11}, e_{21}, e_{31}), \{\frac{\varkappa_2}{<.7, .4, .5>}, \frac{\varkappa_3}{<.9, .6, .7>}, \frac{\varkappa_4}{<.4, .6, .8>}\}),\\ &((e_{13}, e_{23}, e_{33}), \{\frac{x_2}{<.4, .6, .8>}, \frac{x_3}{<.8, .4, .9>}\}),\\ &((e_{12}, e_{21}, e_{31}), \{\frac{\varkappa_2}{<.5, .6, .3>}, \frac{\varkappa_4}{<.6, .3, .4>}\}),\\ &((e_{11}, e_{21}, e_{32}), \{\frac{\varkappa_1}{<.7, .9, .3>}, \frac{\varkappa_3}{<.6, .7, .8>}, \frac{\varkappa_4}{<.3, .8, .2>}\}),\\ &((e_{12}, e_{21}, e_{32}), \{\frac{\varkappa_1}{<.6, .3, .5>}, \frac{\varkappa_3}{<.4, .5, .9>}\}),\\ &((e_{13}, e_{21}, e_{31}), \{\frac{x_3}{<.8, .4, .1>}\})\}\end{aligned}$$

Let $K$ be a $\mathcal{NHSS}$ defined as

$$\begin{aligned}K =& \Big\{((e_{11}, e_{21}, e_{31}), \{\frac{\varkappa_1}{<.6, .4, .9>}, \frac{\varkappa_2}{<.5, .6, .9>}, \frac{\varkappa_3}{<.2, .5, .6>}, \frac{\varkappa_4}{<.8, .6, .6>}\}),\\ &((e_{11}, e_{22}, e_{31}), \{\frac{\varkappa_2}{<.8, .5, .2>}\}),\\ &((e_{12}, e_{21}, e_{31}), \{\frac{\varkappa_2}{<.7, .4, .9>}, \frac{\varkappa_3}{<.5, .7, .8>}, \frac{\varkappa_4}{<.6, .4, .8>}\}),\\ &((e_{12}, e_{22}, e_{31}), \{\frac{\varkappa_1}{<.7, .8, .2>}, \frac{\varkappa_4}{<.5, .6, .3>}\}),\\ &((e_{11}, e_{21}, e_{32}), \{\frac{\varkappa_1}{<.6, .3, .8>}, \frac{\varkappa_3}{<.6, .6, .2>}, \frac{\varkappa_4}{<.7, .5, .8>}\}),\\ &((e_{11}, e_{22}, e_{32}), \{\frac{\varkappa_3}{<.3, .7, .9>}, \frac{\varkappa_4}{<.8, .9, .2>}\}),\\ &((e_{12}, e_{21}, e_{32}), \{\frac{\varkappa_1}{<.6, .3, .8>}, \frac{\varkappa_3}{<.4, .6, .9>}\})\Big\}\end{aligned}$$

Then the $\underline{lower}$ and $\overline{upper}$ $\mathcal{NHSS}$ approximation of $K$ are calculated as

$$\underline{apr}_{\mathcal{NHSS}}(K) = \Big\{((e_{11}, e_{21}, e_{31}), \{\frac{\varkappa_2}{<.2, .3, .9>}\})\Big\}$$

$$\begin{aligned}\overline{apr}_{\mathcal{NHSS}}(K) = \Big\{ & ((e_{11}, e_{21}, e_{31}), \{\frac{\varkappa_1}{< .6, .4, .3 >}, \frac{\varkappa_2}{< .7, .6, .2 >}, \frac{\varkappa_3}{< .9, .6, .1 >}, \frac{\varkappa_4}{< .8, .6, .6 >}\}),\\ & ((e_{11}, e_{22}, e_{31}), \{\frac{\varkappa_2}{< .8, .5, .2 >}\}),\\ & ((e_{12}, e_{21}, e_{31}), \{\frac{\varkappa_2}{< .7, .6, .2 >}, \frac{\varkappa_3}{< .8, .7, .1 >}, \frac{\varkappa_4}{< .6, .5, .4 >}\}),\\ & ((e_{12}, e_{22}, e_{31}), \{\frac{\varkappa_1}{< .7, .8, .2 >}, \frac{\varkappa_4}{< .5, .6, .2 >}\}),\\ & ((e_{11}, e_{21}, e_{32}), \{\frac{\varkappa_1}{< .7, .9, .3 >}, \frac{\varkappa_3}{< .6, .7, .2 >}, \frac{\varkappa_4}{< .7, .8, .2 >}\}),\\ & ((e_{11}, e_{22}, e_{32}), \{\frac{\varkappa_3}{< .4, .7, .2 >}, \frac{\varkappa_4}{< .8, .9, .2 >}\}),\\ & ((e_{12}, e_{21}, e_{32}), \{\frac{\varkappa_1}{< .6, .3, .5 >}, \frac{\varkappa_3}{< .4, .6, .9 >}\})\Big\}.\end{aligned}$$

**Theorem 3.3.** *Let* $\left(U,\ \mathbb{N},\ \Pi_{j=1}^{n}\delta_j\right)$ *be a neutrosophic hypersoft approximation space, and* $K, L \in P_{NH}(U)$*, then the following properties hold.*

i. $\underline{apr}_{\mathcal{NHSS}}(K) \subseteq K \subseteq \overline{apr}_{\mathcal{NHSS}}(K)$
ii. $\underline{apr}_{\mathcal{NHSS}}(0_{(U,\ \Pi_{j=1}^{n}\delta_j)}) = 0_{(U,\ \Pi_{j=1}^{n}\delta_j)}$, $\overline{apr}_{\mathcal{NHSS}}(1_{(U,\ \Pi_{j=1}^{n}\delta_j)}) = 1_{(U,\ \Pi_{j=1}^{n}\delta_j)}$
iii. *If* $K \subseteq L$*, then* $\underline{apr}_{\mathcal{NHSS}}(K) \subseteq \underline{apr}_{\mathcal{NHSS}}(L)$
iv. *If* $K \subseteq L$*, then* $\overline{apr}_{\mathcal{NHSS}}(K) \subseteq \overline{apr}_{\mathcal{NHSS}}(L)$
v. $\underline{apr}_{\mathcal{NHSS}}(K \cap L) \subseteq \underline{apr}_{\mathcal{NHSS}}(K) \cap \underline{apr}_{\mathcal{NHSS}}(L)$
vi. $\underline{apr}_{\mathcal{NHSS}}(K \cup L) \subseteq \underline{apr}_{\mathcal{NHSS}}(K) \cup \underline{apr}_{\mathcal{NHSS}}(L)$
vii. $\overline{apr}_{\mathcal{NHSS}}(K \cap L) \subseteq \overline{apr}_{\mathcal{NHSS}}(K) \cap \overline{apr}_{\mathcal{NHSS}}(L)$
viii. $\overline{apr}_{\mathcal{NHSS}}(K \cup L) \subseteq \overline{apr}_{\mathcal{NHSS}}(K) \cup \overline{apr}_{\mathcal{NHSS}}(L)$

*Proof.* (i) From the Definition 3.1, we can conclude that $\underline{apr}_{\mathcal{NHSS}}(K) \subseteq K$.
In addition, from the definition of neutrosophic hypersoft upper approximation,
$\forall \left(\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j\right) \cap K \neq 0_{(U,\ \Pi_{j=1}^{n}\delta_j)}$, $\overline{\mu}, \overline{\nu}, \overline{\eta} \in K \cup \left(\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j\right)$.
Hence , $K \subseteq \overline{apr}_{\mathcal{NHSS}}(K)$.
Thus, $\underline{apr}_{\mathcal{NHSS}}(K) \subseteq K \subseteq \overline{apr}_{\mathcal{NHSS}}(K)$.
(ii) From Definition 3.1, the proof of (ii) naturally follows.
(iii) Let $K \subseteq L$ and $\left(\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j\right) \subseteq K, j = (1, 2, ..., n)$.
Then $\underline{apr}_{\mathcal{NHSS}}(K) = K \cap \left(\bigcap_{j=1}^{n} \left(\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j\right)\right)$.
Also, we have $(\left(\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j\right) \subseteq K$ then $\left(\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j\right) \subset L$.
Hence $\underline{apr}_{\mathcal{NHSS}}(L) = L \cap \left(\bigcap_{j=1}^{n} \left(\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j\right)\right)$.
This implies $\underline{apr}_{\mathcal{NHSS}}(K) \subseteq \underline{apr}_{\mathcal{NHSS}}(L)$.
(iv) Let $K \subseteq L$ and $\left(\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j\right) \cap K \neq \emptyset, j = 1, 2, ..., n$.
Then $\overline{apr}_{\mathcal{NHSS}}(K) = K \cup \left(\bigcup_{j=1}^{n} \left(\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j\right)\right)$. For $K \subseteq L$,
then $\left(\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j\right) \cap L \neq \emptyset$ and $\overline{apr}_{\mathcal{NHSS}}(L) = L \cup \left(\bigcup_{j=1}^{n} \left(\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j\right)\right)$.

This implies $\overline{apr}_{\mathcal{NHSS}}(K) \subseteq \overline{apr}_{\mathcal{NHSS}}(L)$.

(v) Let $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \underline{apr}_{\mathcal{NHSS}}(K \cap L)$.

There exist $\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right)$ such that $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right) \subseteq \underline{apr}_{\mathcal{NHSS}}(K \cap L)$, $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right) \subseteq K$ and $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right) \subseteq L$.

Therefore $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \underline{apr}_{\mathcal{NHSS}}(K)$ and

$\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \underline{apr}_{\mathcal{NHSS}}(L)$, implying $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \underline{apr}_{\mathcal{NHSS}}(K) \cap \underline{apr}_{\mathcal{NHSS}}(L)$.

Thus $\underline{apr}_{\mathcal{NHSS}}(K \cap L) \subseteq \underline{apr}_{\mathcal{NHSS}}(K) \cap \underline{apr}_{\mathcal{NHSS}}(L)$.

(vi) Let $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \notin \underline{apr}_{\mathcal{NHSS}}(K \cup L)$.

There exist $\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right)$ such that $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right) \nsubseteq \underline{apr}_{\mathcal{NHSS}}(K \cap L)$, hence it follows that $\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right) \nsubseteq K$ and $\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right) \nsubseteq L$.

Therefore $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \notin \underline{apr}_{\mathcal{NHSS}}(K)$ and $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \notin \underline{apr}_{\mathcal{NHSS}}(L)$,

implying $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \notin \underline{apr}_{\mathcal{NHSS}}(K) \cup \underline{apr}_{\mathcal{NHSS}}(L)$.

Thus $\underline{apr}_{\mathcal{NHSS}}(K \cup L) \subseteq \underline{apr}_{\mathcal{NHSS}}(K) \cup \underline{apr}_{\mathcal{NHSS}}(L)$.

(vii) Let $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \overline{apr}_{\mathcal{NHSS}}(K \cap L)$.

There exist $\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right)$ such that $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right) \cap (K \cap L) \neq 0_{(U,\ \Pi_{j=1}^{n}\delta_j)}$, $\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right) \cap (K) \neq 0_{(U,\ \Pi_{j=1}^{n}\delta_j)}$ and $\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right) \cap (L) \neq 0_{(U,\ \Pi_{j=1}^{n}\delta_j)}$.

Therefore $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \overline{apr}_{\mathcal{NHSS}}(K)$ and

$\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \overline{apr}_{\mathcal{NHSS}}(L)$, implying $\varkappa_{\mu,\nu,\eta}^{\Pi_{i=1}^{n}A_i} \in \overline{apr}_{\mathcal{NHSS}}(K) \cap \overline{apr}_{\mathcal{NHSS}}(L)$.

(viii) Let $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \overline{apr}_{\mathcal{NHSS}}(K \cup L)$.

There exist $\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right)$ such that $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right) \cap (K \cup L) \neq 0_{(U,\ \Pi_{j=1}^{n}\delta_j)}$, it follows that

$\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right) \cap (K) \neq 0_{(U,\ \Pi_{j=1}^{n}\delta_j)}$ or $\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right) \cap (L) \neq 0_{(U,\ \Pi_{j=1}^{n}\delta_j)}$.

Therefore $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \overline{apr}_{\mathcal{NHSS}}(K)$ or $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \overline{apr}_{\mathcal{NHSS}}(L)$.

Hence $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \overline{apr}_{\mathcal{NHSS}}(K) \cup \overline{apr}_{\mathcal{NHSS}}(L)$.

Thus $\overline{apr}_{\mathcal{NHSS}}(K \cup L) \subseteq \overline{apr}_{\mathcal{NHSS}}(K) \cup \overline{apr}_{\mathcal{NHSS}}(L)$ □

*The converse of properties (i), (iii), (iv), (v), (vi), (vii) and (viii) in Theorem 3.3 does not hold.*

**Theorem 3.4.** *Let $\left(U,\ \mathbb{N},\ \Pi_{j=1}^{n}\delta_j\right)$ be a neutrosophic hypersoft approximation space, and $K, L \in P_{NH}(U)$, then the following properties hold.*

i. $\underline{apr}_{\mathcal{NHSS}}\left(\underline{apr}_{\mathcal{NHSS}}(K)\right) = \underline{apr}_{\mathcal{NHSS}}(K)$
ii. $\overline{apr}_{\mathcal{NHSS}}\left(\overline{apr}_{\mathcal{NHSS}}(K)\right) \supseteq \overline{apr}_{\mathcal{NHSS}}(K)$
iii. $\overline{apr}_{\mathcal{NHSS}}(K) \subseteq \overline{apr}_{\mathcal{NHSS}}\left(\underline{apr}_{\mathcal{NHSS}}(K)\right)$.
iv. $\overline{apr}_{\mathcal{NHSS}}\left(\underline{apr}_{\mathcal{NHSS}}(K)\right) \supseteq \underline{apr}_{\mathcal{NHSS}}(K)$.

*Proof.* (i) Let $\varkappa_{(\mu,\eta,\nu)}^{\Pi_{j=1}^{n}\delta_j} \in \underline{apr}_{\mathcal{NHSS}}(K)$.
Then, we have $\varkappa_{(\mu,\eta,\nu)}^{\Pi_{j=1}^{n}\delta_j} \in \left(\mathbb{N}_j, \Pi_{j=1}^{n}\delta_j\right) \subseteq \underline{apr}_{\mathcal{NHSS}}(K)$.
So $\varkappa_{(\mu,\eta,\nu)}^{\Pi_{j=1}^{n}\delta_j} \in \underline{apr}_{\mathcal{NHSS}}\left(\underline{apr}_{\mathcal{NHSS}}(K)\right)$.
Therefore $\underline{apr}_{\mathcal{NHSS}}(K) \subseteq \underline{apr}_{\mathcal{NHSS}}\left(\underline{apr}_{\mathcal{NHSS}}(K)\right)$.
From the Theorem 3.3 $\underline{apr}_{\mathcal{NHSS}}(K) \subseteq K$ using (iii) of Theorem 3.3 we obtain $\underline{apr}_{\mathcal{NHSS}}\left(\underline{apr}_{\mathcal{NHSS}}(K)\right) \subseteq \underline{apr}_{\mathcal{NHSS}}(K)$.
Hence $\underline{apr}_{\mathcal{NHSS}}\left(\underline{apr}_{\mathcal{NHSS}}(K)\right) = \underline{apr}_{\mathcal{NHSS}}(K)$.
(ii) Let $P = \overline{apr}_{\mathcal{NHSS}}(K)$. Using property (i) of Theorem 3.3,
we get $P \subseteq \overline{apr}_{\mathcal{NHSS}}(K)$.
Hence $\overline{apr}_{\mathcal{NHSS}}\left(\overline{apr}_{\mathcal{NHSS}}(K)\right) \supseteq \overline{apr}_{\mathcal{NHSS}}(K)$.
(iii) Let $P = \overline{apr}_{\mathcal{NHSS}}(K)$.
Using property (i) of Theorem 3.3, we got $\underline{apr}_{\mathcal{NHSS}}(K) \subseteq P$.
Hence $\underline{apr}_{\mathcal{NHSS}}\left(\overline{apr}_{\mathcal{NHSS}}(K)\right) \subseteq \overline{apr}_{\mathcal{NHSS}}(K)$.
(iv) Let $Q = \underline{apr}_{\mathcal{NHSS}}(K)$.
Using property (i) of Theorem 3.3, we got $\overline{apr}_{\mathcal{NHSS}}(K) \supseteq Q$.
Hence $\underline{apr}_{\mathcal{NHSS}} \subseteq \overline{apr}_{\mathcal{NHSS}}(K)\left(\underline{apr}_{\mathcal{NHSS}}(K)\right)$. □

The converse of properties (ii), (iii) and (iv) in Theorem 3.4 does not hold.

**Remark 3.5.** Let $\left(U, \mathbb{N}, \Pi_{j=1}^{n}\delta_j\right)$ be a neutrosophic hypersoft approximation space, and $K, L \in P_{NH}(U)$, then the following properties hold.

i. $\underline{apr}_{\mathcal{NHSS}}(K^c) \neq [\overline{apr}_{\mathcal{NHSS}}(K)]^c$
ii. $\overline{apr}_{\mathcal{NHSS}}(K^c) \neq [\underline{apr}_{\mathcal{NHSS}}(K)]^c$

**Definition 3.6.** Let $\left(\mathbb{N}_1, \Pi_{j=1}^{n}\delta j\right)$ and $\left(\mathbb{N}_2, \Pi_{j=1}^{n}\delta j\right)$ be two $\mathcal{NHSS}$s over the same universe set $U$. Then "$\left(\mathbb{N}_1, \Pi_{j=1}^{n}\delta j\right)$ difference $\left(\mathbb{N}_2, \Pi_{j=1}^{n}\delta j\right)$" operation on them is denoted by $\left((\mathbb{N}_1 \setminus \mathbb{N}_2), \Pi_{j=1}^{n}\delta j\right)$ and is defined by
$\left((\mathbb{N}_1 \setminus \mathbb{N}_2), \Pi_{j=1}^{n}\delta j\right) = \left(\mathbb{N}_1, \Pi_{j=1}^{n}\delta j\right) \cap \left(\mathbb{N}_2, \Pi_{j=1}^{n}\delta j\right)^c$
$= \{\Pi_{j=1}^{n}\delta j, < \varkappa, (1-\mu)_{\mathbb{N}_j,\Pi_{j=1}^{n}\delta j}(\varkappa), (1-\eta)_{\mathbb{N}_j,\Pi_{j=1}^{n}\delta j}(\varkappa), (1-\nu)_{\mathbb{N}_j,\Pi_{j=1}^{n}\delta j}(\varkappa) >: \varkappa \in U : \Pi_{j=1}^{n}\delta j \subseteq \Pi_{j=1}^{n}\Delta_j\}$.
where
$\mu_{(\mathbb{N}_1\setminus\mathbb{N}_2),\Pi_{j=1}^{n}\delta j}(\varkappa) = min\{\mu_{\mathbb{N}_1,\Pi_{j=1}^{n}\delta j}(\varkappa), \mu_{\mathbb{N}_2,\Pi_{j=1}^{n}\delta j}(\varkappa)\}$
$\eta_{(\mathbb{N}_1\setminus\mathbb{N}_2),\Pi_{j=1}^{n}\delta j}(\varkappa) = min\{\eta_{\mathbb{N}_1,\Pi_{j=1}^{n}\delta j}(\varkappa), \eta_{\mathbb{N}_2,\Pi_{j=1}^{n}\delta j}(\varkappa)\}$
$\nu_{(\mathbb{N}_1\setminus\mathbb{N}_2),\Pi_{j=1}^{n}\delta j}(\varkappa) = max\{\nu_{\mathbb{N}_1,\Pi_{j=1}^{n}\delta j}(\varkappa), \nu_{\mathbb{N}_2,\Pi_{j=1}^{n}\delta j}(\varkappa)\}$.

**Definition 3.7.** Let $\underline{apr}_{\mathcal{NHSS}}(K)$ and $\overline{apr}_{\mathcal{NHSS}}(K)$ be neutrosophic hypersoft $\underline{lower}$ and $\overline{upper}$ approximations of $K \in P_{NH}(U)$ with respect to the neutrosophic hypersoft approximation space $K$, respectively. Then

$pos_{\mathcal{NHSS}}(K) = \underline{apr}_{\mathcal{NHSS}}(K)$

$neg_{\mathcal{NHSS}}(K) = (\overline{apr}_{\mathcal{NHSS}}(K))^c$

$bnd_{\mathcal{NHSS}}(K) = \overline{apr}_{\mathcal{NHSS}}(K) \setminus \underline{apr}_{\mathcal{NHSS}}(K)$

are called the neutrosophic hypersoft positive region, neutrosophic hypersoft negative region and neutrosophic hypersoft boundary region of $K$, respectively.

**Example 3.8.** Consider Example 3.2. The neutrosophic hypersoft rough positive, negative, and boundary region can then be computed as follows:

$$\begin{aligned} pos_{\mathcal{NHSS}}(K) =& \underline{apr}_{\mathcal{NHSS}}(K) \\ =& \Big\{((e_{11}, e_{21}, e_{31}), \{\frac{\varkappa_2}{< .2, .3, .9 >}\})\Big\} \end{aligned}$$

$$\begin{aligned} neg_{\mathcal{NHSS}}(K) =& (\overline{apr}_{\mathcal{NHSS}}(K))^c \\ =& \Big\{((e_{11}, e_{21}, e_{31}), \{\frac{\varkappa_1}{< .4, .6, .7 >}, \frac{\varkappa_2}{< .3, .4, .8 >}, \frac{\varkappa_3}{< .1, .4, .9 >}, \frac{\varkappa_4}{< .2, .4, .4 >}\}), \\ & ((e_{11}, e_{22}, e_{31}), \{\frac{\varkappa_2}{< .2, .5, .8 >}\}), \\ & ((e_{12}, e_{21}, e_{31}), \{\frac{\varkappa_2}{< .3, .4, .8 >}, \frac{\varkappa_3}{< .2, .3, .9 >}, \frac{\varkappa_4}{< .4, .5, .6 >}\}), \\ & ((e_{12}, e_{22}, e_{31}), \{\frac{\varkappa_1}{< .3, .2, .8 >}, \frac{\varkappa_4}{< .5, .4, .8 >}\}), \\ & ((e_{11}, e_{21}, e_{32}), \{\frac{\varkappa_1}{< .3, .1, .7 >}, \frac{\varkappa_3}{< .4, .3, .8 >}, \frac{\varkappa_4}{< .3, .2, .8 >}\}), \\ & ((e_{11}, e_{22}, e_{32}), \{\frac{\varkappa_3}{< .6, .3, .8 >}, \frac{\varkappa_4}{< .2, .1, .8 >}\}), \\ & ((e_{12}, e_{21}, e_{32}), \{\frac{\varkappa_1}{< .4, .7, .5 >}, \frac{\varkappa_3}{< .6, .4, .1 >}\})\Big\} \end{aligned}$$

$$(\underline{apr}_{\mathcal{NHSS}}(K))^c = \Big\{((e_{11}, e_{21}, e_{31}), \{\frac{\varkappa_2}{< .8, .7, .1 >}\})\Big\}$$

$$\begin{aligned} Bnd_{\mathcal{NHSS}}(K) =& \overline{apr}_{\mathcal{NHSS}}(K) \setminus \underline{apr}_{\mathcal{NHSS}}(K) \\ =& \overline{apr}_{\mathcal{NHSS}}(K) \cap (\underline{apr}_{\mathcal{NHSS}}(K))^c \\ =& \Big\{((e_{11}, e_{21}, e_{31}), \{\frac{\varkappa_2}{< .7, .6, .2 >}\})\Big\} \end{aligned}$$

Obviously, $\underline{apr}_{\mathcal{NHSS}}(K) \neq \overline{apr}_{\mathcal{NHSS}}(K)$. So $K$ is $\mathcal{NHSS}$ in the approximation space $\Big(U, \mathbb{N}, \Pi_{j=1}^{n} \delta_j\Big)$.

**Theorem 3.9.** *Let* $\Big(U,\ \mathbb{N},\ \Pi_{j=1}^{n} \delta_j\Big)$ *be a neutrosophic hypersoft approximation space, and* $K, L \in P_{NH}(U)$*, then* $\underline{apr}_{\mathcal{NHSS}}(K \setminus L) \subseteq \underline{apr}_{\mathcal{NHSS}}(K) \setminus \underline{apr}_{\mathcal{NHSS}}(L)$.

*Proof.* Let $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \underline{apr}_{\mathcal{NHSS}}(K \setminus L)$.
There exist $\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right)$ such that $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right) \subseteq \underline{apr}_{\mathcal{NHSS}}(K \setminus L)$, $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right) \subseteq K$ and $\varkappa_{1-\mu,1-\nu,1-\eta}^{\Pi_{j=1}^{n}\delta_j} \in \left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right) \subseteq L^c$.
Thus $\underline{apr}_{\mathcal{NHSS}}(K \setminus L) \subseteq \underline{apr}_{\mathcal{NHSS}}(K) \setminus \underline{apr}_{\mathcal{NHSS}}(L)$.
Therefore $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \underline{apr}_{\mathcal{NHSS}}(K)$ and
$\varkappa_{1-\mu,1-\nu,1-\eta}^{\Pi_{j=1}^{n}\delta_j} \in \underline{apr}_{\mathcal{NHSS}}(L^c)$, implying $\varkappa_{\mu,\nu,\eta}^{\Pi_{j=1}^{n}\delta_j} \in \underline{apr}_{\mathcal{NHSS}}(K) \setminus \underline{apr}_{\mathcal{NHSS}}(L)$.
Thus $\underline{apr}_{\mathcal{NHSS}}(K \cap L) \subseteq \underline{apr}_{\mathcal{NHSS}}(K \setminus \underline{apr}_{\mathcal{NHSS}}(L)$. □

The converse of Theorem 3.9 does not hold.

**Remark 3.10.** Let $\left(U,\ \mathbb{N},\ \Pi_{j=1}^{n}\delta_j\right)$ be a neutrosophic hypersoft approximation space, and $K, L \in P_{NH}(U)$, then

$$\overline{apr}_{\mathcal{NHSS}}(K \setminus L) \neq \overline{apr}_{\mathcal{NHSS}}(K) \setminus \overline{apr}_{\mathcal{NHSS}}(L).$$

**Example 3.11.** Let $\left(U,\ \mathbb{N},\ \Pi_{j=1}^{n}\delta_j\right)$ be a neutrosophic hypersoft approximation space, and $K, L \in P_{NH}(U)$, based on Example 3.2 defined as,

$$\begin{aligned} K = \Big\{ & \big((e_{11}, e_{21}, e_{31}), \{\frac{\varkappa_1}{<.6,.4,.9>}, \frac{\varkappa_2}{<.5,.6,.9>}, \frac{x_3}{<.2,.5,.6>}, \frac{\varkappa_4}{<.8,.6,.6>}\}\big), \\ & \big((e_{13}, e_{23}, e_{33}), \{\frac{\varkappa_2}{<.8,.6,.9>}, \frac{\varkappa_3}{<.4,.6,.8>}, \frac{\varkappa_4}{<.8,.4,.9>}\}\big), \\ & \big((e_{11}, e_{22}, e_{31}), \{\frac{\varkappa_2}{<.8,.5,.2>}\}\big), \\ & \big((e_{12}, e_{21}, e_{31}), \{\frac{\varkappa_2}{<.7,.4,.9>}, \frac{\varkappa_3}{<.5,.7,.8>}, \frac{\varkappa_4}{<.6,.4,.8>}\}\big), \\ & \big((e_{12}, e_{22}, e_{31}), \{\frac{\varkappa_1}{<.7,.8,.2>}, \frac{\varkappa_4}{<.5,.6,.3>}\}\big), \\ & \big((e_{11}, e_{21}, e_{32}), \{\frac{\varkappa_1}{<.6,.3,.8>}, \frac{\varkappa_3}{<.6,.6,.2>}, \frac{\varkappa_4}{<.7,.5,.8>}\}\big), \\ & \big((e_{11}, e_{22}, e_{32}), \{\frac{\varkappa_3}{<.3,.7,.9>}, \frac{\varkappa_4}{<.8,.9,.2>}\}\big), \\ & \big((e_{12}, e_{21}, e_{32}), \{\frac{\varkappa_1}{<.6,.3,.8>}, \frac{\varkappa_3}{<.4,.6,.9>}\}\big)\Big\} \end{aligned}$$

Then the $\underline{lower}$ and $\overline{upper}$ $\mathcal{NHSS}$ approximation of $K$ are calculated as

$$\begin{aligned} \underline{apr}_{\mathcal{NHSS}}(K) = \Big\{ & \big((e_{11}, e_{21}, e_{31}), \{\frac{\varkappa_2}{<.2,.3,.9>}\}\big), \\ & \big((e_{13}, e_{23}, e_{33}), \{\frac{\varkappa_2}{<.4,.3,.9>}, \frac{\varkappa_3}{<.2,.4,.9>}\}\big)\Big\} \end{aligned}$$

$$\begin{aligned}\overline{apr}_{\mathcal{NHSS}}(K) =\Big\{&((e_{11},e_{21},e_{31}),\{\frac{\varkappa_1}{<.6,.4,.3>},\frac{\varkappa_2}{<.7,.6,.2>},\frac{\varkappa_3}{<.9,.6,.1>},\frac{\varkappa_4}{<.8,.6,.6>}\}),\\&((e_{13},e_{23},e_{33}),\{\frac{\varkappa_2}{<.8,.6,.2>},\frac{\varkappa_3}{<.8,.8,.3>},\frac{\varkappa_4}{<.8,.4,.4>}\}),\\&((e_{11},e_{22},e_{31}),\{\frac{\varkappa_2}{<.8,.5,.2>}\}),\\&((e_{12},e_{21},e_{31}),\{\frac{\varkappa_2}{<.7,.6,.2>},\frac{\varkappa_3}{<.8,.7,.1>},\frac{\varkappa_4}{<.6,.5,.4>}\}),\\&((e_{12},e_{22},e_{31}),\{\frac{\varkappa_1}{<.7,.8,.2>},\frac{\varkappa_4}{<.5,.6,.2>}\}),\\&((e_{11},e_{21},e_{32}),\{\frac{\varkappa_1}{<.7,.9,.3>},\frac{\varkappa_3}{<.6,.7,.2>},\frac{\varkappa_4}{<.7,.8,.2>}\}),\\&((e_{11},e_{22},e_{32}),\{\frac{\varkappa_3}{<.4,.7,.2>},\frac{\varkappa_4}{<.8,.9,.2>}\}),\\&((e_{12},e_{21},e_{32}),\{\frac{\varkappa_1}{<.6,.3,.5>},\frac{\varkappa_3}{<.4,.6,.9>}\})\Big\}\end{aligned}$$

$$\begin{aligned}Let\ L =\Big\{&((e_{11},e_{21},e_{31}),\{\frac{\varkappa_1}{<.7,.5,.4>},\frac{\varkappa_2}{<.6,.7,.8>},\frac{\varkappa_3}{<.3,.6,.2>},\frac{\varkappa_4}{<.9,.6,.3>}\}),\\&((e_{11},e_{22},e_{31}),\{\frac{\varkappa_2}{<.9,.5,.1>}\}),\\&((e_{12},e_{21},e_{31}),\{\frac{\varkappa_2}{<.9,.8,.1>},\frac{\varkappa_3}{<.6,.8,.5>},\frac{\varkappa_4}{<.7,.5,.4>}\}),\\&((e_{11},e_{21},e_{32}),\{\frac{\varkappa_1}{<.8,.5,.6>},\frac{\varkappa_3}{<.8,.6,.1>},\frac{\varkappa_4}{<.8,.6,.3>}\}),\\&((e_{12},e_{21},e_{32}),\{\frac{\varkappa_1}{<.8,.5,.3>},\frac{\varkappa_3}{<.6,.8,.7>}\}),\\&((e_{13},e_{21},e_{31}),\{\frac{\varkappa_2}{<.7,.1,.3>},\frac{\varkappa_3}{<.9,.3,.8>}\})\Big\}\end{aligned}$$

Then the $\underline{lower}$ and $\overline{upper}$ $\mathcal{NHSS}$ approximation of $L$ are calculated as

$$\begin{aligned}\underline{apr}_{\mathcal{NHSS}}(L) =\Big\{&((e_{11},e_{21},e_{31}),\{\frac{\varkappa_2}{<.2,.3,.8>}\}),\\&((e_{13},e_{21},e_{31}),\frac{\varkappa_3}{<.4,.1,.9>}\})\Big\}\end{aligned}$$

$$\begin{aligned}\overline{apr}_{\mathcal{NHSS}}(L) =\Big\{&((e_{11},e_{21},e_{31}),\{\frac{\varkappa_1}{<.7,.5,.3>},\frac{\varkappa_2}{<.7,.7,.2>},\frac{\varkappa_3}{<.9,.6,.1>},\frac{\varkappa_4}{<.9,.6,.3>}\}),\\&((e_{11},e_{22},e_{31}),\{\frac{\varkappa_2}{<.9,.5,.1>}\}),\\&((e_{12},e_{21},e_{31}),\{\frac{\varkappa_2}{<.9,.8,.1>},\frac{\varkappa_3}{<.8,.8,.1>},\frac{\varkappa_4}{<.7,.5,.4>}\}),\\&((e_{11},e_{21},e_{32}),\{\frac{\varkappa_1}{<.8,.9,.1>},\frac{\varkappa_3}{<.6,.7,.8>},\frac{\varkappa_4}{<.8,.8,.2>}\}),\\&((e_{12},e_{21},e_{32}),\{\frac{\varkappa_1}{<.8,.5,.3>},\frac{\varkappa_3}{<.6,.8,.7>}\}),\\&((e_{13},e_{21},e_{31}),\{\frac{\varkappa_2}{<.9,.9,.2>},\frac{\varkappa_3}{<.9,.4,.1>}\})\Big\}\end{aligned}$$

$$\underline{apr}_{\mathcal{NHSS}}(K)/\underline{apr}_{\mathcal{NHSS}}(L) =\Big\{((e_{11},e_{21},e_{31}),\{\frac{\varkappa_2}{<.2,.3,.9>}\})\Big\}$$

$$\overline{apr}_{\mathcal{NHSS}}(K)/\overline{apr}_{\mathcal{NHSS}}(L) = \Big\{((e_{11}, e_{21}, e_{31}), \{\frac{\varkappa_1}{<.3, .4, .7>}, \frac{\varkappa_2}{<.3, .3, .8>}, \frac{\varkappa_3}{<.1, .4, .9>}, \frac{\varkappa_4}{<.1, .4, .7>}\}),$$
$$((e_{11}, e_{22}, e_{31}), \{\frac{\varkappa_2}{<.1, .5, .9>}\}),$$
$$((e_{12}, e_{21}, e_{31}), \{\frac{\varkappa_2}{<.1, .2, .9>}, \frac{\varkappa_3}{<.2, .2, .9>}, \frac{\varkappa_4}{<.3, .5, .6>}\}),$$
$$((e_{11}, e_{21}, e_{32}), \{\frac{\varkappa_1}{<.2, .1, .9>}, \frac{\varkappa_3}{<.4, .3, .2>}, \frac{\varkappa_4}{<.2, .2, .8>}\}),$$
$$((e_{12}, e_{21}, e_{32}), \{\frac{\varkappa_1}{<.2, .5, .7>}, \frac{\varkappa_3}{<.4, .2, .3>}\})\Big\}$$

$$L^c = \Big\{((e_{11}, e_{21}, e_{31}), \{\frac{\varkappa_1}{<.3, .5, .6>}, \frac{\varkappa_2}{<.4, .3, .2>}, \frac{\varkappa_3}{<.7, .4, .8>}, \frac{\varkappa_4}{<.1, .4, .7>}\}),$$
$$((e_{11}, e_{22}, e_{31}), \{\frac{\varkappa_2}{<.1, .5, .9>}\}),$$
$$((e_{12}, e_{21}, e_{31}), \{\frac{\varkappa_2}{<.1, .2, .9>}, \frac{\varkappa_3}{<.4, .2, .5>}, \frac{\varkappa_4}{<.2, .4, .7>}\}),$$
$$((e_{11}, e_{21}, e_{32}), \{\frac{\varkappa_1}{<.6, .3, .8>}, \frac{\varkappa_3}{<.2, .5, .7>}, \frac{\varkappa_4}{<.4, .2, .3>}\}),$$
$$((e_{12}, e_{21}, e_{32}), \{\frac{\varkappa_1}{<.3, .9, .7>}, \frac{\varkappa_3}{<.1, .7, .2>}\})\Big\}$$

$$K \setminus L = \Big\{((e_{11}, e_{21}, e_{31}), \{\frac{\varkappa_1}{<.3, .4, .9>}, \frac{\varkappa_2}{<.4, .3, .9>}, \frac{\varkappa_3}{<.2, .5, .8>}, \frac{\varkappa_4}{<.1, .4, .7>}\}),$$
$$((e_{11}, e_{22}, e_{31}), \{\frac{\varkappa_2}{<.1, .5, .9>}\}),$$
$$((e_{12}, e_{21}, e_{31}), \{\frac{\varkappa_2}{<.1, .2, .9>}, \frac{\varkappa_3}{<.4, .2, .8>}, \frac{\varkappa_4}{<.2, .4, .8>}\}),$$
$$((e_{11}, e_{21}, e_{32}), \{\frac{\varkappa_1}{<.6, .3, .8>}, \frac{\varkappa_3}{<.2, .5, .7>}, \frac{\varkappa_4}{<.4, .2, .8>}\}),$$
$$((e_{12}, e_{21}, e_{32}), \{\frac{\varkappa_1}{<.3, .3, .8>}, \frac{\varkappa_3}{<.1, .6, .9>}\})\Big\}$$

$$\underline{apr}_{\mathcal{NHSS}}(K \setminus L) = \Big\{((e_{11}, e_{21}, e_{31}), \{\frac{\varkappa_2}{<.2, .3, .9>}\})\Big\}$$

$$\overline{apr}_{\mathcal{NHSS}}(K \setminus L) = \Big\{((e_{11}, e_{21}, e_{31}), \{\frac{\varkappa_1}{<.5, .4, .3>}, \frac{\varkappa_2}{<.7, .5, .2>}, \frac{\varkappa_3}{<.9, .6, .1>}, \frac{\varkappa_4}{<.4, .6, .8>}\}),$$
$$((e_{11}, e_{22}, e_{31}), \{\frac{\varkappa_2}{<.6, .5, .3>}\}),$$
$$((e_{12}, e_{21}, e_{31}), \{\frac{\varkappa_2}{<.5, .6, .3>}, \frac{\varkappa_3}{<.8, .4, .1>}, \frac{\varkappa_4}{<.6, .5, .4>}\}),$$
$$((e_{11}, e_{21}, e_{32}), \{\frac{\varkappa_1}{<.7, .9, .3>}, \frac{\varkappa_3}{<.6, .7, .7>}, \frac{\varkappa_4}{<.7, .8, .2>}\}),$$
$$((e_{12}, e_{21}, e_{32}), \{\frac{\varkappa_1}{<.6, .3, .5>}, \frac{\varkappa_3}{<.4, .6, .9>}\})\Big\}$$

Hence,

$$\underline{apr}_{\mathcal{NHSS}}(K \setminus L) \subseteq \underline{apr}_{\mathcal{NHSS}}(K) \setminus \underline{apr}_{\mathcal{NHSS}}(L)$$
$$\overline{apr}_{\mathcal{NHSS}}(K \setminus L) \neq \overline{apr}_{\mathcal{NHSS}}(K) \setminus \overline{apr}_{\mathcal{NHSS}}(L).$$

## 4. Equality Properties on Neutrosophic Hypersoft Rough Sets

In this section, we define equity between neutrosophic hypersoft rough sets.

**Definition 4.1.** Let $\left(U,\ \mathbb{N},\ \Pi_{j=1}^{n}\delta_j\right)$ be a neutrosophic hypersoft approximation space, then $\forall\ K, L \in P_{NH}(U)$, we define the following binary relations:
(i). Sets $K$ and $L$ are in lower $\mathcal{NHSS}$ equal related iff
$K \preccurlyeq_{\mathcal{NHSS}}\ L \Longleftrightarrow \underline{apr}_{\mathcal{NHSS}}(K) = \underline{apr}_{\mathcal{NHSS}}(L)$.
(ii). Sets $K$ and $L$ are in upper $\mathcal{NHSS}$ equal related iff
$K \preccurlyeq_{\mathcal{NHSS}}\ L \Longleftrightarrow \overline{apr}_{\mathcal{NHSS}}(K) = \overline{apr}_{\mathcal{NHSS}}(L)$.
(iii). Sets $K$ and $L$ are $\mathcal{NHSS}$ equal related iff
$K \approx_{\mathcal{NHSS}} L \Longleftrightarrow \underline{apr}_{\mathcal{NHSS}}(K) = \underline{apr}_{\mathcal{NHSS}}(L)\ \&\ \overline{apr}_{\mathcal{NHSS}}(K) = \overline{apr}_{\mathcal{NHSS}}(L)$ .

**Theorem 4.2.** *Let $\left(U,\ \mathbb{N},\ \Pi_{j=1}^{n}\delta_j\right)$ be a neutrosophic hypersoft approximation space, then $\forall$ $K, L, M, N \in P_{NH}(U)$. Then the following hold:*
*(i). If $K \subseteq L$ and $L \preccurlyeq_{\mathcal{NHSS}} 0_{(U,\Pi_{j=1}^{n}\delta_j)}$, then $K \preccurlyeq_{\mathcal{NHSS}} 0_{(U,\Pi_{j=1}^{n}\delta_j)}$.*
*(ii). If $K \subseteq L$ and $K \preccurlyeq_{\mathcal{NHSS}} 1_{(U,\Pi_{j=1}^{n}\delta_j)}$, then $L \preccurlyeq_{\mathcal{NHSS}} 1_{(U,\Pi_{j=1}^{n}\delta_j)}$.*
*(iii) $K \preccurlyeq_{\mathcal{NHSS}} L \Longleftrightarrow K \preccurlyeq_{\mathcal{NHSS}} (K \cup L) \preccurlyeq_{\mathcal{NHSS}} L$.*
*(iv) $K \preccurlyeq_{\mathcal{NHSS}} L, M \preccurlyeq_{\mathcal{NHSS}} N \Longrightarrow (K \cup M) \preccurlyeq_{\mathcal{NHSS}} (L \cup N)$.*

*Proof.* (i). Given $K \subseteq L$ and $L \preccurlyeq_{\mathcal{NHSS}} 0_{(U,\Pi_{j=1}^{n}\delta_j)}$,
so that $\overline{apr}_{\mathcal{NHSS}}(K) \subseteq \overline{apr}_{\mathcal{NHSS}}(L)$ and $\overline{apr}_{\mathcal{NHSS}}(L) = 0_{(U,\Pi_{j=1}^{n}\delta_j)}$.
Hence, $\overline{apr}_{\mathcal{NHSS}}(K) = 0_{(U,\Pi_{j=1}^{n}\delta_j)} = \overline{apr}_{\mathcal{NHSS}}(L)$.
(ii). Given, $K \preccurlyeq_{\mathcal{NHSS}} 1_{(U,\Pi_{j=1}^{n}\delta_j)}$ and $K \subseteq L$ ,
then $\overline{apr}_{\mathcal{NHSS}}(K) = \overline{apr}_{\mathcal{NHSS}}(1_{(U,\Pi_{j=1}^{n}\delta_j)})$ and $\overline{apr}_{\mathcal{NHSS}}(K) \subseteq \overline{apr}_{\mathcal{NHSS}}(L)$.
But we know that $\overline{apr}_{\mathcal{NHSS}}(K) \subseteq \overline{apr}_{\mathcal{NHSS}}(1_{(U,\Pi_{j=1}^{n}\delta_j)})$,
hence $\overline{apr}_{\mathcal{NHSS}}(K) = \overline{apr}_{\mathcal{NHSS}}(1_{(U,\Pi_{j=1}^{n}\delta_j)})$.

We note here that $K \preccurlyeq_{\mathcal{NHSS}} L$ iff $K \cap L \preccurlyeq_{\mathcal{NHSS}} K$ and $K \cap L \preccurlyeq_{\mathcal{NHSS}} L$ is not true in general.

(iii) Assume that $K \preccurlyeq_{\mathcal{NHSS}} L$. By definition 4.1(ii), we have
$\overline{apr}_{\mathcal{NHSS}}(K) = \overline{apr}_{\mathcal{NHSS}}(L)$.
From Theorem 3.3, it is known that $\overline{apr}_{\mathcal{NHSS}}(K \cup L) = \overline{apr}_{\mathcal{NHSS}}(K) \cup \overline{apr}_{\mathcal{NHSS}}(L)$. There by we obtain $\overline{apr}_{\mathcal{NHSS}}(K \cup L) = \overline{apr}_{\mathcal{NHSS}}(K) = \overline{apr}_{\mathcal{NHSS}}(L)$.
Consequently, $K \preccurlyeq_{\mathcal{NHSS}} L \Rightarrow K \preccurlyeq_{\mathcal{NHSS}} (K \cup L) \preccurlyeq_{\mathcal{NHSS}} L$.
Conversely, if $K \preccurlyeq_{\mathcal{NHSS}} (K \cup L) \preccurlyeq_{\mathcal{NHSS}} L$ then it is obvious that $K \preccurlyeq_{\mathcal{NHSS}} L$ from the transitivity of $\preccurlyeq_{\mathcal{NHSS}}$.
(iv) Suppose that $K \preccurlyeq_{\mathcal{NHSS}} L$ and $M \preccurlyeq_{\mathcal{NHSS}} N$. By Definition 4.1(ii),
we can write $\overline{apr}_{\mathcal{NHSS}}(K) = \overline{apr}_{\mathcal{NHSS}}(L)$ and $\overline{apr}_{\mathcal{NHSS}}(M) = \overline{apr}_{\mathcal{NHSS}}(N)$. By considering Theorem 3.3, we have $\overline{apr}_{\mathcal{NHSS}}(K \cup L) = \overline{apr}_{\mathcal{NHSS}}(K) \cup \overline{apr}_{\mathcal{NHSS}}(L)$ and $\overline{apr}_{\mathcal{NHSS}}(M \cup N) =$

$\overline{apr}_{\mathcal{NHSS}}(M) \cup \overline{apr}_{\mathcal{NHSS}}(N)$.
Thereby, we conclude that $\overline{apr}_{\mathcal{NHSS}}(K \cup M) = \overline{apr}_{\mathcal{NHSS}}(L \cup N)$ and so $\overline{apr}_{\mathcal{NHSS}}(K \cup M) \preccurlyeq_{\mathcal{NHSS}} \overline{apr}_{\mathcal{NHSS}}(L \cup N)$. □

The converse of property (iv) in Theorem 4.2 does not hold.

**Example 4.3.** Let $\left(U,\ \mathbb{N},\ \Pi_{j=1}^{n}\delta_j\right)$ be a neutrosophic hypersoft approximation space, then $\forall\ K, L \in P_{NH}(U)$ based on Example 3.2, $K \preccurlyeq_{\mathcal{NHSS}}\ L \Longleftrightarrow \overline{apr}_{\mathcal{NHSS}}(K) = \overline{apr}_{\mathcal{NHSS}}(L)$,

$$\begin{aligned} K = \{&((e_{11}, e_{21}, e_{31}), \{\frac{x_1}{<.6,.4,.9>}, \frac{x_2}{<.5,.6,.9>}, \frac{x_3}{<.2,.5,.6>}, \frac{x_4}{<.8,.6,.6>}\}),\\ &((e_{11}, e_{22}, e_{31}), \{\frac{x_2}{<.8,.5,.2>}\}),\\ &((e_{12}, e_{21}, e_{31}), \{\frac{x_2}{<.7,.4,.9>}, \frac{x_3}{<.5,.7,.8>}, \frac{x_4}{<.6,.4,.8>}\}),\\ &((e_{11}, e_{21}, e_{32}), \{\frac{x_1}{<.6,.3,.8>}, \frac{x_3}{<.6,.6,.2>}, \frac{x_4}{<.7,.5,.8>}\}),\\ &((e_{12}, e_{21}, e_{32}), \{\frac{x_1}{<.6,.3,.8>}, \frac{x_3}{<.4,.6,.9>}\})\} \end{aligned}$$

The lower and upper $\mathcal{NHSS}$ approximation of $K$ are calculated as

$$\underline{apr}_{NHSS}(K) = \{((e_{11}, e_{21}, e_{31}), \{\frac{x_2}{<.2,.3,.9>}\})\}$$

$$\begin{aligned} \overline{apr}_{NHSS}(K) = \{&((e_{11}, e_{21}, e_{31}), \{\frac{x_1}{<.6,.4,.3>}, \frac{x_2}{<.7,.6,.2>}, \frac{x_3}{<.9,.6,.1>}, \frac{x_4}{<.8,.6,.6>}\}),\\ &((e_{11}, e_{22}, e_{31}), \{\frac{x_2}{<.8,.5,.2>}\}),\\ &((e_{12}, e_{21}, e_{31}), \{\frac{x_2}{<.7,.6,.2>}, \frac{x_3}{<.8,.7,.1>}, \frac{x_4}{<.6,.5,.4>}\}),\\ &((e_{11}, e_{21}, e_{32}), \{\frac{x_1}{<.7,.9,.3>}, \frac{x_3}{<.6,.7,.2>}, \frac{x_4}{<.7,.8,.2>}\}),\\ &((e_{12}, e_{21}, e_{32}), \{\frac{x_1}{<.6,.3,.5>}, \frac{x_3}{<.4,.6,.9>}\})\} \end{aligned}$$

$$\begin{aligned} L = \{&((e_{11}, e_{21}, e_{31}), \{\frac{x_1}{<.6,.4,.5>}, \frac{x_2}{<.6,.6,.8>}, \frac{x_3}{<.8,.6,.4>}, \frac{x_4}{<.8,.6,.6>}\}),\\ &((e_{11}, e_{22}, e_{31}), \{\frac{x_2}{<.8,.5,.2>}\}),\\ &((e_{12}, e_{21}, e_{31}), \{\frac{x_2}{<.7,.5,.6>}, \frac{x_3}{<.6,.7,.4>}, \frac{x_4}{<.6,.4,.8>}\}),\\ &((e_{11}, e_{21}, e_{32}), \{\frac{x_1}{<.7,.4,.6>}, \frac{x_3}{<.6,.7,.2>}, \frac{x_4}{<.7,.6,.5>}\}),\\ &((e_{12}, e_{21}, e_{32}), \{\frac{x_1}{<.6,.3,.7>}, \frac{x_3}{<.4,.6,.9>}\})\} \end{aligned}$$

The lower and upper $\mathcal{NHSS}$ approximation of $L$ are calculated as

$$\underline{apr}_{NHSS}(L) = \{((e_{11}, e_{21}, e_{31}), \{\frac{x_2}{<.2,.3,.8>}\})\}$$

$$\overline{apr}_{NHSS}(L) = \{((e_{11}, e_{21}, e_{31}), \{\frac{x_1}{<.6,.4,.3>}, \frac{x_2}{<.7,.6,.2>}, \frac{x_3}{<.9,.6,.1>}, \frac{x_4}{<.8,.6,.6>}\}),$$
$$((e_{11}, e_{22}, e_{31}), \{\frac{x_2}{<.8,.5,.2>}\}),$$
$$((e_{12}, e_{21}, e_{31}), \{\frac{x_2}{<.7,.6,.2>}, \frac{x_3}{<.8,.7,.1>}, \frac{x_4}{<.6,.5,.4>}\}),$$
$$((e_{11}, e_{21}, e_{32}), \{\frac{x_1}{<.7,.9,.3>}, \frac{x_3}{<.6,.7,.2>}, \frac{x_4}{<.7,.8,.2>}\}),$$
$$((e_{12}, e_{21}, e_{32}), \{\frac{x_1}{<.6,.3,.5>}, \frac{x_3}{<.4,.6,.9>}\})\}$$

**Theorem 4.4.** *Let* $\left(U,\ \mathbb{N},\ \Pi_{j=1}^{n}\delta_j\right)$ *be a neutrosophic hypersoft approximation space, then* $\forall$ $K, L, M, N \in P_{NH}(U)$. *Then the following hold:*

*(i). If* $K \subseteq L$ *and* $L \preccurlyeq_{\mathcal{NHSS}} 0_{(U,\Pi_{j=1}^{n}\delta_j)}$, *then* $K \preccurlyeq_{\mathcal{NHSS}} 0_{(U,\Pi_{j=1}^{n}\delta_j)}$.

*(ii). If* $K \subseteq L$ *and* $L \preccurlyeq_{\mathcal{NHSS}} 1_{(U,\Pi_{j=1}^{n}\delta_j)}$, *then* $K \preccurlyeq_{\mathcal{NHSS}} 1_{(U,\Pi_{j=1}^{n}\delta_j)}$.

*(iii)* $K \preccurlyeq_{\mathcal{NHSS}} L \Longleftrightarrow K \preccurlyeq_{\mathcal{NHSS}} (K \cup L) \preccurlyeq_{NHSS} L$.

*(iv)* $K \preccurlyeq_{\mathcal{NHSS}} L, M \preccurlyeq_{\mathcal{NHSS}} N \Longrightarrow (K \cup M) \preccurlyeq_{\mathcal{NHSS}} (L \cup N)$.

*Proof.* By considering Definition 4.1(i), and Theorem 3.3, it can be proved similar to the proof of Theorem 4.2. □

The converse of property (iv) in Theorem 4.4. does not hold.

**Example 4.5.** Let $\left(U,\ \mathbb{N},\ \Pi_{j=1}^{n}\delta_j\right)$ be a neutrosophic hypersoft approximation space, then $\forall\ K, L \in P_{NH}(U)$ based on Example 3.2, $K \preccurlyeq_{\mathcal{NHSS}}\ L \Longleftrightarrow \underline{apr}_{\mathcal{NHSS}}(K) = \underline{apr}_{\mathcal{NHSS}}(L)$,

$$K = \{((e_{11}, e_{21}, e_{31}), \{\frac{x_1}{<.6,.4,.9>}, \frac{x_2}{<.5,.6,.9>}, \frac{x_3}{<.2,.5,.6>}, \frac{x_4}{<.8,.6,.6>}\}),$$
$$((e_{11}, e_{22}, e_{31}), \{\frac{x_2}{<.8,.5,.2>}\}),$$
$$((e_{12}, e_{21}, e_{31}), \{\frac{x_2}{<.7,.4,.9>}, \frac{x_3}{<.5,.7,.8>}, \frac{x_4}{<.6,.4,.8>}\}),$$
$$((e_{11}, e_{21}, e_{32}), \{\frac{x_1}{<.6,.3,.8>}, \frac{x_3}{<.6,.6,.2>}, \frac{x_4}{<.7,.5,.8>}\}),$$
$$((e_{12}, e_{21}, e_{32}), \{\frac{x_1}{<.6,.3,.8>}, \frac{x_3}{<.4,.6,.9>}\})\}$$

The lower and upper $\mathcal{NHSS}$ approximation of $K$ are calculated as

$$\underline{apr}_{NHSS}(K) = \{((e_{11}, e_{21}, e_{31}), \{\frac{x_2}{<.2,.3,.9>}\})\}$$

$$\overline{apr}_{NHSS}(K) = \{((e_{11}, e_{21}, e_{31}), \{\frac{x_1}{<.6,.4,.3>}, \frac{x_2}{<.7,.6,.2>}, \frac{x_3}{<.9,.6,.1>}, \frac{x_4}{<.8,.6,.6>}\}),$$
$$((e_{11}, e_{22}, e_{31}), \{\frac{x_2}{<.8,.5,.2>}\}),$$
$$((e_{12}, e_{21}, e_{31}), \{\frac{x_2}{<.7,.6,.2>}, \frac{x_3}{<.8,.7,.1>}, \frac{x_4}{<.6,.5,.4>}\}),$$
$$((e_{11}, e_{21}, e_{32}), \{\frac{x_1}{<.7,.9,.3>}, \frac{x_3}{<.6,.7,.2>}, \frac{x_4}{<.7,.8,.2>}\}),$$
$$((e_{12}, e_{21}, e_{32}), \{\frac{x_1}{<.6,.3,.5>}, \frac{x_3}{<.4,.6,.9>}\})\}$$

$$L = \{((e_{11}, e_{21}, e_{31}), \{\frac{x_1}{<.7,.5,.4>}, \frac{x_2}{<.6,.7,.9>}, \frac{x_3}{<.3,.6,.2>}, \frac{x_4}{<.9,.6,.3>}\}),$$
$$((e_{11}, e_{22}, e_{31}), \{\frac{x_2}{<.9,.5,.1>}\}),$$
$$((e_{12}, e_{21}, e_{31}), \{\frac{x_2}{<.9,.8,.1>}, \frac{x_3}{<.6,.8,.5>}, \frac{x_4}{<.7,.5,.4>}\}),$$
$$((e_{11}, e_{21}, e_{32}), \{\frac{x_1}{<.8,.5,.6>}, \frac{x_3}{<.8,.6,.1>}, \frac{x_4}{<.8,.6,.3>}\}),$$
$$((e_{12}, e_{21}, e_{32}), \{\frac{x_1}{<.8,.5,.3>}, \frac{x_3}{<.6,.8,.7>}\})\}$$

The lower and upper $\mathcal{NHSS}$ approximation of $L$ are calculated as

$$\underline{apr}_{NHSS}(L) = \{((e_{11}, e_{21}, e_{31}), \{\frac{x_2}{<.2,.3,.9>}\})\}$$

$$\overline{apr}_{NHSS}(L) = \{((e_{11}, e_{21}, e_{31}), \{\frac{x_1}{<.7,.5,.3>}, \frac{x_2}{<.7,.7,.2>}, \frac{x_3}{<.9,.6,.1>}, \frac{x_4}{<.9,.6,.3>}\}),$$
$$((e_{11}, e_{22}, e_{31}), \{\frac{x_2}{<.9,.5,.1>}\}),$$
$$((e_{12}, e_{21}, e_{31}), \{\frac{x_2}{<.9,.8,.1>}, \frac{x_3}{<.8,.8,.1>}, \frac{x_4}{<.7,.5,.4>}\}),$$
$$((e_{11}, e_{21}, e_{32}), \{\frac{x_1}{<.8,.9,.1>}, \frac{x_3}{<.6,.7,.8>}, \frac{x_4}{<.8,.8,.2>}\}),$$
$$((e_{12}, e_{21}, e_{32}), \{\frac{x_1}{<.8,.5,.3>}, \frac{x_3}{<.6,.8,.7>}\})\}$$

## 5. **Reduct and Core of Neutrosophic Hypersoft Rough Sets**

In this section, we discuss reduct, core, dispensable, and indispensable neutrosophic hypersoft rough sets.

**Definition 5.1.** Let $(\acute{\acute{R}}_N, \Pi_{j=1}^n \acute{X}_j), (\acute{\acute{S}}_N, \Pi_{j=1}^n \acute{Y}_j)$ & $\acute{\acute{T}}_N, \Pi_{j=1}^n \acute{Z}_j)$ be $\mathcal{NHSS}$s on U, where $(\Pi_{j=1}^n \acute{X}_j, \Pi_{j=1}^n \acute{Y}_j, \Pi_{j=1}^n \acute{Z}_j) \subseteq \left(\mathbb{N}, \Pi_{j=1}^n \delta_j\right)$, and $\acute{\acute{R}}_N : \Pi_{j=1}^n \acute{X}_j \to P_N(U)$, $\acute{\acute{S}}_N : \Pi_{j=1}^n \acute{Y}_j \to P_N(U)$, $\acute{\acute{T}}_N : \Pi_{i=1}^n \acute{Z}_i \to P_N(U)$ are mappings.
Let $\left(\mathbb{N}, \Pi_{j=1}^n \delta_j\right) = \{(\acute{\acute{R}}_N : \Pi_{j=1}^n \acute{X}_j), (\acute{\acute{S}}_N : \Pi_{j=1}^n \acute{Y}_j) \& (\acute{\acute{T}}_N : \Pi_{j=1}^n \acute{Z}_j)\}$.
We define approximate neutrosophic hypersoft set, which is denoted by $APP$.
$APP\left(\mathbb{N}, \Pi_{j=1}^n \delta j\right)$
$=APP\Big\{(\acute{\acute{R}}_N, \Pi_{j=1}^n \acute{X}_i), (\acute{\acute{S}}_N, \Pi_{j=1}^n \acute{Y}_j) \& (\acute{\acute{T}}_N, \Pi_{j=1}^n \acute{Z}_j)\Big\}$

$= \Big\{ (r_{\Pi_{j=1}^n \alpha_j}, s_{\Pi_{i=1}^n \beta_j}, t_{\Pi_{j=1}^n \gamma_j}), \bigcap \{ \acute{\acute{R}}_N, (r_{\Pi_{j=1}^n \alpha_j}), \acute{\acute{S}}_N, (s_{\Pi_{j=1}^n \beta_j}), \acute{\acute{T}}_N, (t_{\Pi_{j=1}^n \gamma_j}) \}$
$/1 \leq \alpha_j, \beta_j, \gamma_j \leq n \Big\}$

$= \Big\{ (\Pi_{j=1}^n e_j,\ \mathbb{N}, \Pi_{j=1}^n (e_j)) \setminus \Pi_{j=1}^n e_j \in \Big(\mathbb{N}, \Pi_{j=1}^n \delta_j\Big) \Big\},$

where $r_{\Pi_{j=1}^n \alpha_j} \in \acute{\acute{R}}_N,\ s_{\Pi_{j=1}^n \beta_j} \in \acute{\acute{S}}_N,\ t_{\Pi_{j=1}^n \gamma_j} \in \acute{\acute{T}}_N$ and

$\Pi_{j=1}^n e_j \in \Big(\mathbb{N}, \Pi_{j=1}^n \delta_i\Big) \subseteq \acute{\acute{R}}_N \times \acute{\acute{S}}_N \times \acute{\acute{T}}_N,$

$(\mathbb{N}, \Pi_{j=1}^n (e_j)) = \bigcap \Big\{ \acute{\acute{R}}_N, (r_{\Pi_{j=1}^n \alpha_j}), \acute{\acute{S}}_N, (s_{\Pi_{j=1}^n \beta_j}), \acute{\acute{T}}_N, (t_{\Pi_{j=1}^n \gamma_j}) \Big\}.$

Also we write the difference in approximate neutrosophic hypersoft sets as

$APP\Big(\Big(\mathbb{N}, \Pi_{j=1}^n \delta_j\Big) - (\acute{\acute{R}}_N, \Pi_{j=1}^n \acute{X}_j)\Big) = APP\Big((\acute{\acute{S}}_N, \Pi_{j=1}^n \acute{Y}_j), (\acute{\acute{T}}_N, \Pi_{j=1}^n \acute{Z}_j)\Big).$

**Definition 5.2.** Two approximate neutrosophic hypersoft sets $APP(\mathbb{G}, \Pi_{j=1}^n \acute{\acute{B}}_j)$ and $APP(\mathbb{H}, \Pi_{k=1}^n \acute{\acute{C}}_k)$ are said to be equal, that is $APP(\mathbb{G}, \Pi_{j=1}^n \acute{\acute{B}}_j) = APP(\mathbb{H}, \Pi_{k=1}^n \acute{\acute{C}}_k)$ if for every $\Pi_{j=1}^n b_j \in \Pi_{j=1}^n \acute{\acute{B}}_j$ there exists one $\Pi_{k=1}^n c_k \in \Pi_{k=1}^n \acute{\acute{C}}_k$ such that $\mathbb{G}, (\Pi_{j=1}^n b_j) = \mathbb{H}, (\Pi_{k=1}^n c_k)$ for some $1 \leq j, k \leq n$ and for every $\Pi_{k=1}^n c_k \in \Pi_{k=1}^n \acute{\acute{C}}_k$ there exists one $\Pi_{j=1}^n b_j \in \Pi_{j=1}^n \acute{\acute{B}}_j$ such that $\mathbb{H}, (\Pi_{k=1}^n c_k) = \mathbb{H}, (\Pi_{k=1}^n c_k)$ for some $1 \leq j, k \leq n$, where $\Pi_{j=1}^n \acute{\acute{B}}_j, \Pi_{k=1}^n \acute{\acute{C}}_k \subseteq \Pi_{j,k=1}^n \Delta_{jk}$ and $G : \Pi_{j=1}^n \acute{\acute{B}}_j \to P_N(U), H : \Pi_{k=1}^n \acute{\acute{C}}_k \to P_N(U)$.

**Definition 5.3.** The neutrosophic hypersoft set $(\acute{\acute{R}}_N, \Pi_{j=1}^n \acute{X}_j))$ is dispensable in

$\Big\{ (\acute{\acute{R}}_N, \Pi_{j=1}^n \acute{X}_i), (\acute{\acute{S}}_N, \Pi_{j=1}^n \acute{Y}_j), (\acute{\acute{T}}_N, \Pi_{j=1}^n \acute{Z}_j)) \Big\}$

if $APP\Big(\Big(\mathbb{N}, \Pi_{j=1}^n \delta_j\Big)\Big) = APP\Big(\Big(\mathbb{N}, \Pi_{j=1}^n \delta_j\Big) - (\acute{\acute{R}}_N, \Pi_{j=1}^n \acute{X}_j)\Big).$

Suppose if $APP\Big(\Big(\mathbb{N}, \Pi_{j=1}^n \delta_j\Big) \neq APP\Big(\Big(\mathbb{N}, \Pi_{j=1}^n \delta_j\Big) - (\acute{\acute{R}}_N, \Pi_{j=1}^n \acute{X}_j)\Big),$

then $(\acute{\acute{R}}_N, \Pi_{j=1}^n \acute{X}_j)$ is indispensable in $\Big(\mathbb{N}, \Pi_{j=1}^n \delta_j\Big)$.

**Definition 5.4.** The neutrosophic hypersoft set $APP(\Big(\mathbb{N}, \Pi_{j=1}^n \delta_j\Big))$ is independent if each $(\acute{\acute{R}}_N, \Pi_{j=1}^n \acute{X}_j) \subseteq (\mathbb{N}, \Pi_{j=1}^n \delta_j)$ is indispensable in $(\mathbb{N}, \Pi_{j=1}^n \delta_j)$. Otherwise neutrosophic hypersoft set $(\mathbb{N}, \Pi_{j=1}^n \delta_j)$ is dependent.

**Definition 5.5.** The set of all indispensable neutrosophic hypersoft sets in $APP\left(\mathbb{N}, \Pi_{j=1}^n \delta_j\right)$ is called the core of $APP\left(\mathbb{N}, \Pi_{j=1}^n \delta_j\right)$ and is denoted $CORE\Big(APP\left(\mathbb{N}, \Pi_{j=1}^n \delta_j\right)\Big)$.

**Theorem 5.6.** $CORE\Big(APP\left(\mathbb{N}, \Pi_{j=1}^n \delta_j\right)\Big) = \bigcap RED\Big(APP\left(\mathbb{N}, \Pi_{j=1}^n \delta_j\right)\Big),$
*where* $RED\Big(APP\left(\mathbb{N}, \Pi_{j=1}^n \delta_j\right)\Big)$ *is the family of all reducts of* $APP\left(\mathbb{N}, \Pi_{j=1}^n \delta_j\right)$.

*Proof.* If $(\acute{\acute{R}}_N, \Pi_{j=1}^n \acute{X}_j)$ is a reduct of $\Big(\mathbb{N}, \Pi_{j=1}^n \delta_j\Big)$ and

$(\acute{\acute{S}}_N, \Pi_{j=1}^n \acute{Y}_j) \in \Big(\mathbb{N}, \Pi_{j=1}^n \delta Y_j\Big) - (\acute{\acute{R}}_N, \Pi_{j=1}^n \acute{X}_j),$

then $APP\left(\mathbb{N}, \Pi_{j=1}^n \delta_j\right) = APP(\acute{\acute{R}}_N, \Pi_{j=1}^n \acute{X}_j),$

$(\acute{\acute{R}}_N,\ \acute{\acute{R}}_N, \Pi_{j=1}^n \acute{X}_j) \subseteq \Big(\mathbb{N}, \Pi_{j=1}^n \delta_j\Big) - \Big\{ (\acute{\acute{R}}_N, \Pi_{j=1}^n \acute{Y}_j) \Big\} \subseteq \Big(\mathbb{N}, \Pi_{j=1}^n \delta_j\Big).$

Note that, if $\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right), (\acute{\acute{R}}_N, \Pi_{j=1}^{n}\acute{X}_j), (\acute{\acute{S}}_N, \Pi_{j=1}^{n}\acute{Y}_j)$,
then $APP(\acute{\acute{R}}_N, \Pi_{j=1}^{n}\acute{X}_j) = APP(\acute{\acute{S}}_N, \Pi_{j=1}^{n}\acute{Y}_j)$.
Assuming that $(\acute{\acute{S}}_N, \Pi_{j=1}^{n}\acute{Y}_j) = \left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right) - \left\{(\acute{\acute{S}}_N, \Pi_{j=1}^{n}\delta Y_j)\right\}$ we conclude that $(\acute{\acute{S}}_N, \Pi_{j=1}^{n}\acute{Y}_j)$ is superfluous, i.e. $(\acute{\acute{S}}_N, \Pi_{j=1}^{n}\delta Y_j) \notin CORE\left(\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right)\right)$ and
$CORE\left(\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right)\right) \subseteq \bigcap\left\{(\acute{\acute{R}}_N, \Pi_{j=1}^{n}\acute{X}_j) : (\acute{\acute{R}}_N, \Pi_{j=1}^{n}\acute{X}_j) \in RED\left(\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right)\right)\right\}$.
Suppose $\acute{\acute{S}}_N, \Pi_{j=1}^{n}\acute{Y}_j) \notin CORE\left(\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right)\right)$, i.e. $(\acute{\acute{S}}_N, \Pi_{j=1}^{n}\acute{Y}_j)$ is superfluous in $\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right)$. That means $APP(\left(\mathbb{N},\ prod_{j=1}^{n}\delta_j\right)) = APP\left(\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right) - \left\{(\acute{\acute{S}}_N, \Pi_{j=1}^{n}\acute{Y}_j)\right\}\right)$, which implies that there exists an independent subset $\acute{\acute{T}}_N, \Pi_{j=1}^{n}\acute{Z}_j \subseteq \left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right) - \{\acute{\acute{S}}_N, \Pi_{j=1}^{n}\acute{Y}_j)\}$, such that $APP(\acute{\acute{T}}_N, \Pi_{j=1}^{n}\acute{z}_j) = APP(\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right))$. Obviously $\acute{\acute{T}}_N$ is reduct of $\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right)$ and $\acute{\acute{S}}_N, \Pi_{j=1}^{n}\acute{Y}_j) \in \acute{\acute{T}}_N, \Pi_{j=1}^{n}\acute{Z}_j$. This shows that
$CORE\left(\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right)\right) \supseteq \bigcap\left\{(\acute{\acute{R}}_N, \Pi_{j=1}^{n}\acute{X}_j) : (\acute{\acute{R}}_N, \Pi_{j=1}^{n}\acute{X}_j) \in RED\left(\left(\mathbb{N}, \Pi_{j=1}^{n}\delta_j\right)\right)\right\}$ □

**Example 5.7.** Now we consider an apartment evaluation problem. Suppose $U = \{\varkappa_1, \varkappa_2, ..., \varkappa_9\}$ be a set of nine apartments, $E = \{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$ be the set of attributes, where $\{\Delta_1 - rate; \Delta_2 - condition; \Delta_3 - infra - structure; \Delta_4 - environs\}$. The values of rate are $\{e_{11} - high; e_{12} - normal; e_{13} - low\}$, the values of condition are $\{e_{21} - worth; e_{22} - not\ worth\}$, the values of infra-structure are $\{e_{31} - super; e_{32} - ok; e_{33} - worst\}$, and the values of environs are $\{e_{41} - quiet; e_{42} - a\ little\ noisy; e_{43} - noisy; e_{44} - quite\ noisy\}$. The evaluation results are listed below. The reduct and core are evaluated as follows.
Here, $\Pi_{j=1}^{4}\delta_j = \left\{\Pi_{j=1}^{3}\acute{W}_j, \Pi_{j=1}^{2}\acute{X}_j, \Pi_{j=1}^{3}\acute{Y}_j, \Pi_{j=1}^{4}\acute{Z}_j\right\}$
For attribute rate
$\acute{W}_j$ : $e_{11}$-high $= \{\varkappa_1, \varkappa_4, \varkappa_5, \varkappa_7\}$, $e_{12}$-normal $= \{\varkappa_2, \varkappa_8\}$, $e_{13}$-low $= \{\varkappa_3, \varkappa_6, \varkappa_9\}$;
For attribute condition:
$\acute{X}_j$ : $e_{21}$-worth $= \{\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_6\}$, $e_{22}$-not worth $= \{\varkappa_4, \varkappa_5, \varkappa_7, \varkappa_8, \varkappa_9\}$;
For attribute infrastructure:
$\acute{Y}_j$ : $e_{31}$-super $= \{\varkappa_1, \varkappa_2, \varkappa_3\}$, $e_{32}$-ok $= \{\varkappa_4, \varkappa_5, \varkappa_6, \varkappa_7, \varkappa_8\}$, $e_{33}$-worst $= \{x9\}$;
For attribute environs:
$\acute{Z}_j$ : $e_{41}$-quiet $= \{\varkappa_1, \varkappa_2\}$, $e_{42}$-a little noisy $= \{\varkappa_3, \varkappa_6\}$, $e_{43}$-noisy $= \{\varkappa_4, \varkappa_5, \varkappa_7\}$, $e_{44}$-quite noisy $= \{\varkappa_8, \varkappa_9\}$;
Let $\acute{\acute{Q}}_N, \Pi_{j=1}^{n}\acute{W}_j = \{\Delta_1 - rate\}$ $\acute{\acute{R}}_N, \Pi_{j=1}^{n}\acute{X}_j = \{\Delta_2 - condition\}$, $\acute{\acute{S}}_N, \Pi_{j=1}^{n}\acute{Y}_j = \{\Delta_3 - infrastructure\}$, $\acute{\acute{T}}_N, \Pi_{j=1}^{n}\acute{Z}_j = \{\Delta_4 - environs\} \subseteq \Pi_{j=1}^{4}\Delta_j$.

$$(\acute{\acute{Q}}_N, \Pi_{j=1}^{n}\acute{W}_j) = \Big\{((e_{11}, e_{12}, e_{13}), \acute{\acute{Q}}_N(e_{11}, e_{12}, e_{13}))\Big\}$$

$$= \Big\{((e_{11}, e_{12}, e_{13}), \{\frac{\varkappa_1}{< .3, .4, .9 >}, \frac{\varkappa_4}{< .5, .6, .6 >}, \frac{\varkappa_5}{< .2, .8, .6 >}, \frac{\varkappa_7}{< .8, .9, .6 >}\},$$

$$\{\frac{\varkappa_2}{< .8, .2, .7 >}, \frac{\varkappa_8}{< .7, .9, .4 >}\},$$

$$\{\frac{\varkappa_3}{< .6, .4, .8 >}, \frac{\varkappa_6}{< .6, .3, .2 >}, \frac{\varkappa_9}{< .7, .8, .8 >}\})\Big\}$$

$$(\acute{\acute{R}}_N, \Pi_{j=1}^{n}\acute{X}_j) = \Big\{((e_{21}, e_{22}), \acute{\acute{R}}_N(e_{21}, e_{22}))\Big\}$$

$$= \Big\{((e_{21}, e_{22}), \{\frac{\varkappa_1}{< .8, .5, .3 >}, \frac{\varkappa_2}{< .5, .9, .4 >}, \frac{\varkappa_3}{< .9, .3, .4 >}, \frac{\varkappa_6}{< .4, .8, .6 >}\},$$

$$\{\frac{\varkappa_4}{< .8, .3, .2 >}, \frac{\varkappa_5}{< .6, .6, .2 >}, \frac{\varkappa_7}{< .6, .4, .9 >}\frac{\varkappa_8}{< .5, .3, .1 >}, \frac{\varkappa_9}{< .9, .9, .2 >}\})\Big\}$$

$$(\acute{\acute{S}}_N, \Pi_{j=1}^{n}\acute{Y}_j) = \Big\{((e_{31}, e_{32}, e_{33}), \acute{\acute{S}}_N(e_{31}, e_{22}, e_{33}))\Big\}$$

$$= \Big\{((e_{31}, e_{32}, e_{33}), \{\frac{\varkappa_1}{< .6, .8, .7 >}, \frac{\varkappa_2}{< .2, .9, .2 >}, \frac{\varkappa_3}{< .3, .6, .1 >}\},$$

$$\{\frac{\varkappa_4}{< .7, .5, .3 >}, \frac{\varkappa_5}{< .9, .4, .6 >}, \frac{\varkappa_6}{< .2, .3, .8 >}, \frac{\varkappa_7}{< .6, .3, .9 >}, \frac{\varkappa_8}{< .6, .8, .3 >}\},$$

$$\{\frac{\varkappa_9}{< .6, .3, .8 >}\})\Big\}$$

$$(\acute{\acute{T}}_N, \Pi_{i=1}^{n}\acute{Z}_j) = \Big\{((e_{41}, e_{42}, e_{43}, e_{44}), \acute{\acute{T}}_N(e_{41}, e_{42}, e_{43}, e_{44}))\Big\}$$

$$= \Big\{((e_{41}, e_{42}, e_{43}, e_{44}), \{\frac{\varkappa_1}{< .6, .4, .9 >}, \frac{\varkappa_2}{< .5, .6, .9 >}\},$$

$$\{\frac{\varkappa_3}{< .8, .5, .2 >}, \frac{\varkappa_6}{< .7, .3, .4 >}\},$$

$$\{\frac{\varkappa_4}{< .7, .1, .9 >}, \frac{\varkappa_5}{< .8, .4, .1 >}, \frac{\varkappa_7}{< .5, .3, .8 >}\},$$

$$\{\frac{\varkappa_8}{< .7, .8, .2 >}, \frac{\varkappa_9}{< .6, .3, .8 >}\})\Big\}$$

$$APP\Big((\mathbb{N}, \Pi_{j=1}^{n}\delta_j)\Big) = APP\Big((\acute{\acute{Q}}_N, \Pi_{j=1}^{n}\acute{W}_j), (\acute{\acute{R}}_N, \Pi_{j=1}^{n}\acute{X}_j), (\acute{\acute{S}}_N, \Pi_{j=1}^{n}\acute{Y}_j), (\acute{\acute{T}}_N, \Pi_{j=1}^{n}\acute{Z}_j)\Big)$$

$$= \Big\{((e_{11}, e_{12}, e_{13}), (e_{21}, e_{22}), (e_{31}, e_{32}, e_{33}), (e_{41}, e_{42}, e_{43}, e_{44}), \{\frac{\varkappa_1}{< .3, .4, .9 >}\},$$

$$\{\frac{\varkappa_4}{< .5, .1, .9 >}, \frac{\varkappa_5}{< .2, .4, .6 >}, \frac{\varkappa_7}{< .5, .3, .9 >}\},$$

$$\{\frac{\varkappa_2}{< .2, .2, .9 >}\},$$

$$\{\frac{\varkappa_8}{< .5, .3, .4 >}\},$$

$$\{\frac{\varkappa_3}{< .3, .3, .8 >}\},$$

$$\{\frac{\varkappa_6}{< .2, .3, .8 >}\},$$

$$\{\frac{\varkappa_9}{< .6, .3, .8 >}\})\Big\}$$

$$APP\Big((\mathbb{N},\Pi_{j=1}^{n}\delta_j)-(\acute{\acute{Q}}_N,\Pi_{j=1}^{n}\acute{W}_i)\Big)=APP\Big((\acute{\acute{R}}_N,\Pi_{j=1}^{n}\acute{X}_i),(\acute{\acute{S}}_N,\Pi_{j=1}^{n}\acute{Y}_i),(\acute{\acute{T}}_N,\Pi_{j=1}^{n}\acute{Z}_j)\Big)$$
$$=\Big\{((e_{21},e_{22}),(e_{31},e_{32},e_{33}),(e_{41},e_{42},e_{43},e_{44}),\{\frac{\varkappa_1}{<.6,.4,.9>},\frac{\varkappa_2}{<.2,.6,.9>}\},$$
$$\{\frac{\varkappa_4}{<.7,.1,.9>},\frac{\varkappa_5}{<.6,.4,.6>},\frac{\varkappa_7}{<.5,.3,.9>}\},$$
$$\{\frac{\varkappa_3}{<.3,.3,.4>}\},$$
$$\{\frac{\varkappa_6}{<.2,.3,.8>}\},$$
$$\{\frac{\varkappa_8}{<.5,.3,.3>}\},$$
$$\{\frac{\varkappa_9}{<.6,.3,.8>}\})\Big\}$$
$$\neq APP\Big((\mathbb{N},\Pi_{j=1}^{n}\delta_j)\Big)$$

Hence, $(\acute{\acute{Q}}_N,\Pi_{j=1}^{n}\acute{W}_j)$ is indispensable in$(\mathbb{N},\Pi_{j=1}^{n}\delta_j)$.

$$APP\Big((\mathbb{N},\Pi_{j=1}^{n}\delta_j)-(\acute{\acute{R}}_N,\Pi_{j=1}^{n}\acute{X}_j)\Big)=APP\Big((\acute{\acute{Q}}_N,\Pi_{j=1}^{n}\acute{W}_j,\acute{\acute{S}}_N,\Pi_{j=1}^{n}\acute{Y}_j,(\acute{\acute{T}}_N,\Pi_{j=1}^{n}\acute{Z}_j)\Big)$$

$$=\Big\{((e_{11},e_{12},e_{13}),(e_{31},e_{32},e_{33}),(e_{41},e_{42},e_{43},e_{44}),\{\frac{\varkappa_1}{<.3,.4,.9>}\},$$
$$\{\frac{\varkappa_4}{<.5,.1,.9>},\frac{\varkappa_5}{<.2,.4,.6>},\frac{\varkappa_7}{<.5,.3,.9>}\},$$
$$\{\frac{\varkappa_2}{<.2,.2,.9>}\},$$
$$\{\frac{\varkappa_8}{<.6,.8,.4>}\},$$
$$\{\frac{\varkappa_3}{<.3,.4,.8>}\},$$
$$\{\frac{\varkappa_6}{<.2,.3,.8>}\},$$
$$\{\frac{\varkappa_9}{<.6,.3,.8>}\})\Big\}$$
$$=APP\Big((\mathbb{N},\Pi_{i=1}^{n}\delta_j)\Big)$$

Hence, $(\acute{\acute{R}}_N, \Pi_{j=1}^{n}\acute{X}_j)$ is dispensable in$(\mathbb{N}, \Pi_{j=1}^{n}\delta_j)$.

$$APP\Big((\mathbb{N}, \Pi_{j=1}^{n}\delta_j) - (\acute{\acute{S}}_N, \Pi_{j=1}^{n}\acute{Y}_j)\Big) = APP\Big((\acute{\acute{Q}}_N, \Pi_{j=1}^{n}\acute{W}_j, \acute{\acute{R}}_N, \Pi_{j=1}^{n}\acute{X}_j, (\acute{\acute{T}}_N, \Pi_{j=1}^{n}\acute{Z}_j)\Big)$$

$$=\Big\{((e_{11}, e_{12}, e_{13}), (e_{21}, e_{22}), (e_{41}, e_{42}, e_{43}, e_{44}), \{\frac{\varkappa_1}{<.3, .4, .9>}\},$$
$$\{\frac{\varkappa_4}{<.5, .1, .9>}, \frac{\varkappa_5}{<.2, .4, .6>}, \frac{\varkappa_7}{<.5, .3, .9>}\},$$
$$\{\frac{\varkappa_2}{<.5, .2, .9>}\},$$
$$\{\frac{\varkappa_8}{<.5, .3, .4>}\},$$
$$\{\frac{\varkappa_3}{<.6, .3, .8>}, \frac{\varkappa_6}{<.4, .3, .6>}\},$$
$$\{\frac{\varkappa_9}{<.6, .3, .8>}\})\Big\}$$
$$\neq APP\Big((\mathbb{N}, \Pi_{j=1}^{n}\delta_j)\Big)$$

Hence, $(\acute{\acute{S}}_N, \Pi_{j=1}^{n}\acute{Y}_j)$ is indispensable in$(\mathbb{N}, \Pi_{j=1}^{n}\delta_j)$.

$$APP\Big((\mathbb{N}, \Pi_{j=1}^{n}\delta_j) - (\acute{\acute{T}}_N, \Pi_{j=1}^{n}\acute{Z}_j)\Big) = APP\Big((\acute{\acute{Q}}_N, \Pi_{j=1}^{n}\acute{W}_j, \acute{\acute{R}}_N, \Pi_{j=1}^{n}\acute{X}_j, (\acute{\acute{S}}_N, \Pi_{j=1}^{n}\acute{Y}_j)\Big)$$

$$=\Big\{((e_{11}, e_{12}, e_{13}), (e_{21}, e_{22}), (e_{31}, e_{32}, e_{33}), \{\frac{\varkappa_1}{<.3, .4, .9>}\},$$
$$\{\frac{\varkappa_4}{<.5, .3, .6>}, \frac{\varkappa_5}{<.2, .4, .6>}, \frac{\varkappa_7}{<.6, .3, .9>}\},$$
$$\{\frac{\varkappa_2}{<.2, .2, .7>}\},$$
$$\{\frac{\varkappa_8}{<.5, .3, .4>}\},$$
$$\{\frac{\varkappa_3}{<.3, .3, .8>}\},$$
$$\{\frac{\varkappa_6}{<.2, .3, .8>}\},$$
$$\{\frac{\varkappa_9}{<.6, .3, .8>}\})\Big\}$$
$$= APP\Big((\mathbb{N}, \Pi_{j=1}^{n}\delta_j)\Big)$$

Hence, $(\acute{\acute{T}}_N, \Pi_{j=1}^{n}\acute{Z}_j)$ is dispensable in$(\mathbb{N}, \Pi_{j=1}^{n}\delta_j)$.

The set of four approximate neutrosophic hypersoft sets $APP\Big((\acute{\acute{Q}}_N, \Pi_{j=1}^{n}\acute{W}_j), (\acute{\acute{R}}_N, \Pi_{j=1}^{n}\acute{X}_j), (\acute{\acute{S}}_N, \Pi_{j=1}^{n}\acute{Y}_j), (\acute{\acute{T}}_N, \Pi_{j=1}^{n}\acute{Z}_j)\Big)$ is the same as the $APP\Big((\acute{\acute{Q}}_N, \Pi_{j=1}^{n}\acute{W}_j), (\acute{\acute{S}}_N, \Pi_{j=1}^{n}\acute{Y}_j), (\acute{\acute{T}}_N, \Pi_{j=1}^{n}\acute{Z}_j)\Big)$ and $APP\Big((\acute{\acute{Q}}_N, \Pi_{j=1}^{n}\acute{W}_j), (\acute{\acute{R}}_N, \Pi_{j=1}^{n}\acute{X}_j), (\acute{\acute{S}}_N, \Pi_{j=1}^{n}\acute{Y}_j)\Big)$. In order to find reducts of the approximate neutrosophic hypersoft sets $APP\Big((\mathbb{N}, \Pi_{j=1}^{n}\delta_j)\Big) = APP\Big((\acute{\acute{Q}}_N, \Pi_{j=1}^{n}\acute{W}_j), (\acute{\acute{R}}_N, \Pi_{j=1}^{n}\acute{X}_j), (\acute{\acute{S}}_N, \Pi_{j=1}^{n}\acute{Y}_j)\Big)$ we have to check

whether $APP\Big((\acute{\acute{Q}}_N, \Pi_{j=1}^n \acute{W}_j), (\acute{\acute{R}}_N, \Pi_{j=1}^n \acute{X}_j), (\acute{\acute{S}}_N, \Pi_{j=1}^n \acute{Y}_j)\Big)$ are independent or not.
Because $APP\Big((\acute{\acute{Q}}_N, \Pi_{j=1}^n \acute{W}_j), (\acute{\acute{S}}_N, \Pi_{j=1}^n \acute{Y}_j), (\acute{\acute{T}}_N, \Pi_{j=1}^n \acute{Z}_j)\Big) \neq APP((\acute{\acute{Q}}_N, \Pi_{j=1}^n \acute{W}_j))$
$APP\Big((\acute{\acute{Q}}_N, \Pi_{j=1}^n \acute{W}_j), (\acute{\acute{S}}_N, \Pi_{j=1}^n \acute{Y}_j), (\acute{\acute{T}}_N, \Pi_{j=1}^n \acute{Z}_j)\Big) \neq APP((\acute{\acute{S}}_N, \Pi_{j=1}^n \acute{Y}_j))$
$APP\Big((\acute{\acute{Q}}_N, \Pi_{j=1}^n \acute{W}_j), (\acute{\acute{S}}_N, \Pi_{j=1}^n \acute{Y}_j), (\acute{\acute{T}}_N, \Pi_{j=1}^n \acute{Z}_j)\Big) \neq APP((\acute{\acute{T}}_N, \Pi_{j=1}^n \acute{Z}_j))$, hence the approximate neutrosophic hypersoft sets are independent and consequently
$APP\Big((\acute{\acute{Q}}_N, \Pi_{j=1}^n \acute{W}_j), (\acute{\acute{S}}_N, \Pi_{j=1}^n \acute{Y}_j), (\acute{\acute{T}}_N, \Pi_{j=1}^n \acute{Z}_j)\Big)$ is reduct of $APP\Big((\mathbb{N}, \Pi_{j=1}^n \delta_j)\Big)$. Proceeding in the same way we find that $APP\Big((\acute{\acute{Q}}_N, \Pi_{j=1}^n \acute{W}_j), (\acute{\acute{R}}_N, \Pi_{j=1}^n \acute{X}_j), (\acute{\acute{S}}_N, \Pi_{j=1}^n \acute{Y}_j)\Big)$ is also a reduct of $APP\Big((\mathbb{N}, \Pi_{j=1}^n \delta_j)\Big)$.

Thus there are two reducts of the approximate neutrosophic hypersoft sets $APP\Big((\mathbb{N}, \Pi_{j=1}^n \delta_j)\Big)$,
$APP\Big((\acute{\acute{Q}}_N, \Pi_{j=1}^n \acute{W}_j), (\acute{\acute{S}}_N, \Pi_{j=1}^n \acute{Y}_j), (\acute{\acute{T}}_N, \Pi_{j=1}^n \acute{Z}_j)\Big)$
and $APP\Big((\acute{\acute{Q}}_N, \Pi_{j=1}^n \acute{W}_j), (\acute{\acute{R}}_N, \Pi_{j=1}^n \acute{X}_j), (\acute{\acute{S}}_N, \Pi_{j=1}^n \acute{Y}_j)\Big)$
From Theorem 5.6,

$$
\begin{aligned}
CORE\Big(APP\left(\mathbb{N}, \Pi_{j=1}^n \delta_j\right)\Big) &= \bigcap RED\Big(APP\left(\mathbb{N}, \Pi_{j=1}^n \delta_j\right)\Big) \\
&= \{APP\Big((\acute{\acute{Q}}_N, \Pi_{j=1}^n \acute{W}_j), (\acute{\acute{S}}_N, \Pi_{j=1}^n \acute{Y}_j), (\acute{\acute{T}}_N, \Pi_{j=1}^n \acute{Z}_j)\Big)\} \\
&\bigcap \\
&\{APP\Big((\acute{\acute{Q}}_N, \Pi_{j=1}^n \acute{W}_j), (\acute{\acute{R}}_N, \Pi_{j=1}^n \acute{X}_j), (\acute{\acute{S}}_N, \Pi_{j=1}^n \acute{Y}_j)\Big)\} \\
&= APP\Big((\acute{\acute{Q}}_N, \Pi_{j=1}^n \acute{W}_j), (\acute{\acute{S}}_N, \Pi_{j=1}^n \acute{Y}_j)\Big)
\end{aligned}
$$

is the core of $APP\Big((\mathbb{N}, \Pi_{j=1}^n \delta_i)\Big)$ and also $APP\Big((\acute{\acute{Q}}_N, \Pi_{j=1}^n \acute{W}_j), ((\acute{\acute{S}}_N, \Pi_{j=1}^n \acute{Y}_j)\Big)$ is indispensable in $APP\Big((\mathbb{N}, \Pi_{j=1}^n \delta_j)\Big)$. If these nine houses are the training samples, then we have two different kinds of evaluation references for other input samples $\{rate; infrastructure; environs\}$, $\{rate; condition; infrastructure\}$ and it is clear that the attributes $\{rate; infrastructure\}$ is the key attributes for the evaluation of apartments.

## 6. Conclusion

This article defines the neutrosophic hypersoft rough set by combining the notions three sets: neutrosophic set, hypersoft set, and rough set. The study of fundamental properties such as union, intersection, and complement are illustrated using examples. The lower and upper rough neutrosophic hypersoft approximations are then specified and validated. The relationship between the core and reduct on the neutrosophic hypersoft rough set is illustrated with examples. The ideas of reduct and core can be used in data reduction and identification of vital set of attributes in decision making problems. We propose to work on multi-attribute, multi-criteria decision making problems using the theoretical properties of reduct, core and equity defined in this work as our future research direction.

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**Conflicts of Interest:** The authors state that they do not have any conflicts of interest.

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# Neutrosophic Generalized Rayleigh Distribution with Application

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**Abstract**. In this paper, we consider the neutrosophic generalized Rayleigh distribution (NGRD). Various neutrosophic properties of NGRD are developed and discussed. The developed distribution is specifically more useful to model indeterminate data which are skewed lifetime data. Also, the neutrosophic parameters are estimated using the well-known method of maximum likelihood (ML) estimation based on a neutrosophic environment. A simulation study is carried out to establish the achievement of the estimated neutrosophic parameters. As a final point, the proposed NGRD applications in the real world have been discussed with the help of real data. A comparative studies also carried out with some recent proposed neutrosophic distributions.

**Keywords:** Neutrosophic Mean and variance; generating functions; parametric estimation; simulation study; neutrosophic generalized Rayleigh distribution; survival function.

## 1. Introduction

To model the real data, [9] invented twelve forms of new cumulative distribution functions. More researchers pay attention to two distributions among the twelve new distributions: Burr-Type X and Burr-Type XII distributions. The two-parameter Burr Type X distribution is developed by [35] under the name of *generalized Rayleigh distribution (GRD)*. Due to the increasing and decreasing hazard function nature of GRD, it is more applicable in survival analysis. The various applications of GRD in statistical inference, reliability, statistical quality control, sampling plans were studied by [22], [1], [25], [7], [19], [12], [17], [16], [18], [10], [24].

This paper aims to develop a new neutrosophic generalized Rayleigh distribution (NGRD) in lifetime data application. Nevertheless, it is more sensible to consider that the GRD is the best-depicted distribution for the lifetime data based on an interval set of quantities for vague parameters. In this situation, neutrosophic statistics is the better environment to address lifetimes based on interval data. The concept of the neutrosophic theory is invented and studied extensively by [29] for indeterminacies in the data. The new school of thought on neutrosophic theory is an expansion of fuzzy logics or fuzzy sets; for more details, see [6], [26], [30–32,34], [36].

Furthermore, neutrosophic statistics is pioneered by [33] and is an expansion of classical statistics, which addresses uncertain or vague data and corresponding statistical probability distributions. The generalization of interval statistics is neutrosophic and also studies fuzzy interval sets. Neutrosophic statistics becomes classical statistics when data is known or deterministic. Whereas in real-world applications, most of the data sets are vague, nondeterministic or unclear, partially unknown or incomplete than determinate data; in these situations, neutrosophic statistical procedures are desirable; for more details, refer [4], [15], [23].

In recent years, few researchers have been attracted to work on neutrosophic probability distributions. [5] developed neutrosophic Weibull distribution. Neutrosophic exponential distribution applications for complex data analysis studied by [11]. [28] presented neutrosophic beta distribution with properties and applications. [2] explored neutrosophic Kumaraswamy distribution with engineering application. [27] discussed the neutrosophic extension of the Maxwell model. [14] attempted on statistical development of neutrosophic gamma distribution with applications to complex data analysis.

Developing a new model is to initiate neutrosophic adaptation of the GRD. This type of conservatory can handle real-world practical issues dealing with undetermined data in either univarite or multivariate situations, mainly when the data reported interval statistics. The cumulative distribution function (cdf) and probability density function (pdf) of GRD are respectively given below:

$$F(x;\upsilon,\sigma)=\left[1-\exp\left\{-\left(\frac{x}{\sigma}\right)^2\right\}\right]^{\upsilon};\qquad x>0,\ \upsilon>0,\ \sigma>0. \tag{1}$$

and

$$f(x;\upsilon,\sigma)=\frac{2\upsilon}{\sigma^2}x\exp\{-\left(\frac{x}{\sigma}\right)^2\}\left[1-\exp\left\{-\left(\frac{x}{\sigma}\right)^2\right\}\right]^{\upsilon-1};\quad x>0,\ \upsilon>0,\ \sigma>0. \tag{2}$$

Where $\upsilon$ and $\sigma$ are shape and scale parameters, respectively.

The remaining paper is reported under a description of NGRD in Section 2. The various statistical properties of NGRD are presented in Section 3. In Section 4, the estimation of neutrosophic parameters is explained. The extensive simulation study is carried out in Section 5. An industrial application of the developed NGRD using the real-life data is given in Section 6, and Section 7 presents the concluding remarks and future study.

## 2. Neutrosophic Generalized Rayleigh Distribution

Neutrosophic statistics is the generalization of classical statistics. We administer with specific or crumple values in classical statistics, but in neutrosophic statistics, the sample values are chosen from a population with uncertainty environment. In neutrosophic statistics, the information can be vague, imprecise, ambiguous, uncertain, incomplete, or even unknown. Neutrosophic numbers have a standard form based on classical statistics, which is given below.

$$X_N = E + I$$

Data is broken down into two parts, $E$ and $I$, where $E$ is the exact or determined data, and $I$ is the uncertain, inexact, or indeterminate part of the data. It is equivalent to $X_N \in [X_L, X_U]$. A subscript $_N$ is used to distinguish the neutrosophic random variable, for example, $X_N$. Let us assume that $X_{N_i} \in [X_L, X_U]$, $i = 1, 2, \ldots, n_N$ is neutrosophic random variable following the neutrosophic generalized Rayleigh distribution (NGRD) with neutrosophic shape parameter $\upsilon_N \in [\upsilon_L, \upsilon_U]$ and neutrosophic scale parameter $\sigma_N \in [\sigma_L, \sigma_U]$. The neutrosophic cumulative distribution fucntion (ncdf) and probability density function (npdf) of NGRD are respectively given as follows:

$$F(x_N; \upsilon_N, \sigma_N) = \left[1 - \exp\left\{-\left(\frac{x_N}{\sigma_N}\right)^2\right\}\right]^{\upsilon_N}; \quad x_N > 0,\ \upsilon_N > 0,\ \sigma_N > 0. \tag{3}$$

and

$$f(x_N; \upsilon_N, \sigma_N) = \frac{2\upsilon_N}{\sigma_N^2} x_N \exp\left\{-\left(\frac{x_N}{\sigma_N}\right)^2\right\}\left[1 - \exp\left\{-\left(\frac{x_N}{\sigma_N}\right)^2\right\}\right]^{\upsilon_N - 1}; \quad x_N > 0,\ \upsilon_N > 0,\ \sigma_N > 0. \tag{4}$$

Where $\upsilon_N$ and $\sigma_N$ are neutrosophic shape and scale parameters, respectively.

The survival function and hazard function of NGRD are respectively given below:

$$s(x_N; \upsilon_N, \sigma_N) = 1 - \left[1 - \exp\left\{-\left(\frac{x_N}{\sigma_N}\right)^2\right\}\right]^{\upsilon_N} \tag{5}$$

and

$$h(x_N) = \frac{\frac{2\upsilon_N}{\sigma_N^2} x_N \exp\left\{-\left(\frac{x_N}{\sigma_N}\right)^2\right\}\left[1 - \exp\left\{-\left(\frac{x_N}{\sigma_N}\right)^2\right\}\right]^{\upsilon_N - 1}}{1 - \left(1 - \exp\{-\left(\frac{x_N}{\sigma_N}\right)^2\}\right)^{\upsilon_N}} \tag{6}$$

In Figure 1 presented various shapes of the NGRD for various scale and shape parameters. From Figure 1 it is noticed that the nature of NGRD is right-skewed, left-skewed and symmetrical shapes for the given shape parameters. In Figure 2 the various forms of CDF curves are displayed for various scale and shape parameters. The various natures of survival function and hazard function are plotted in Figures 3 and 4. From Figure 4 it is noticed that when shape parameter $\upsilon_N$ less than or equal to [0.75,0.75] the nature of hazard function is approximately bathtub type and when shape parameter $\upsilon_N$ greater than [0.75,0.75] the nature of hazard function as increasing. Hence the proposed model is more applicable in industrial data where the failure rate is in increasing tendency.

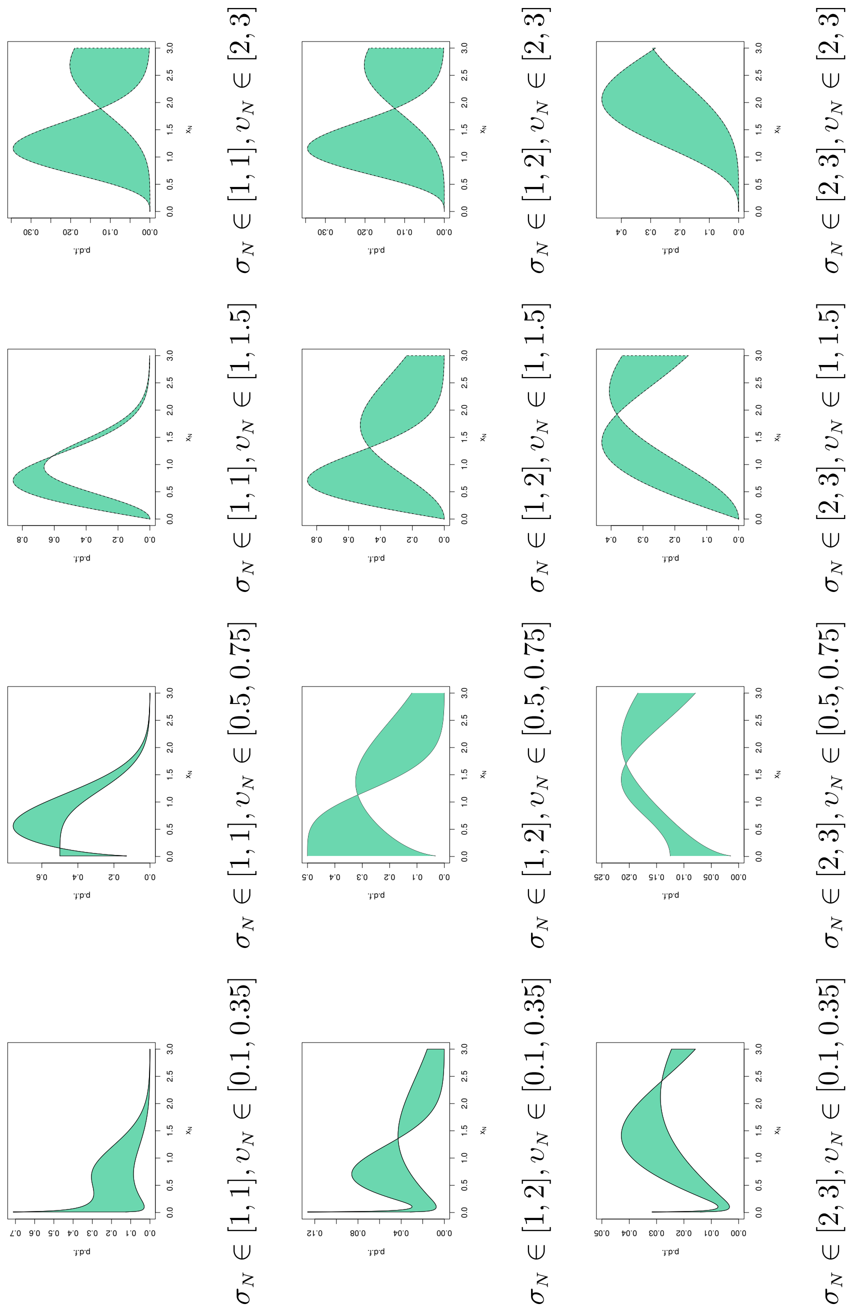


FIGURE 1. The p.d.f. plots of NGE distribution

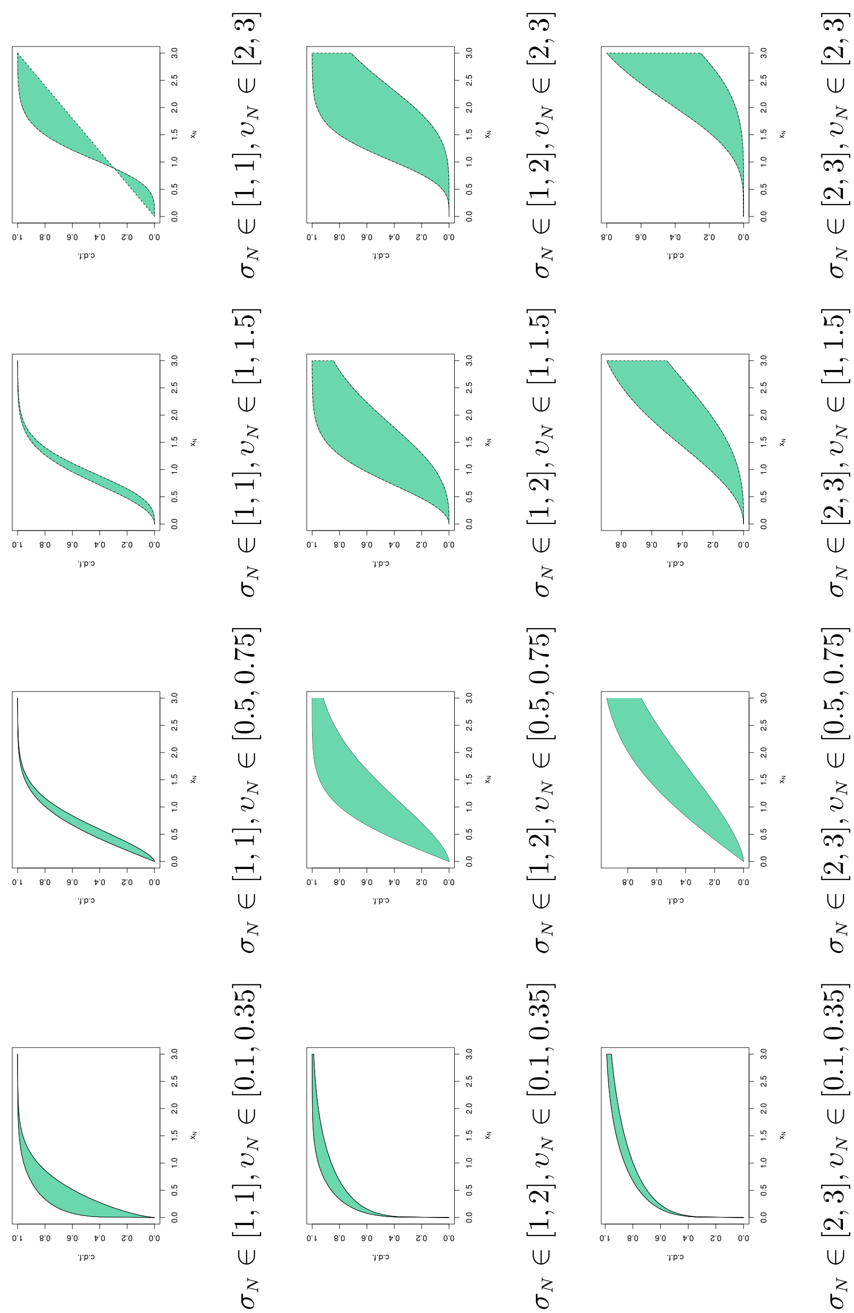


FIGURE 2. The c.d.f. plots of NGE distribution

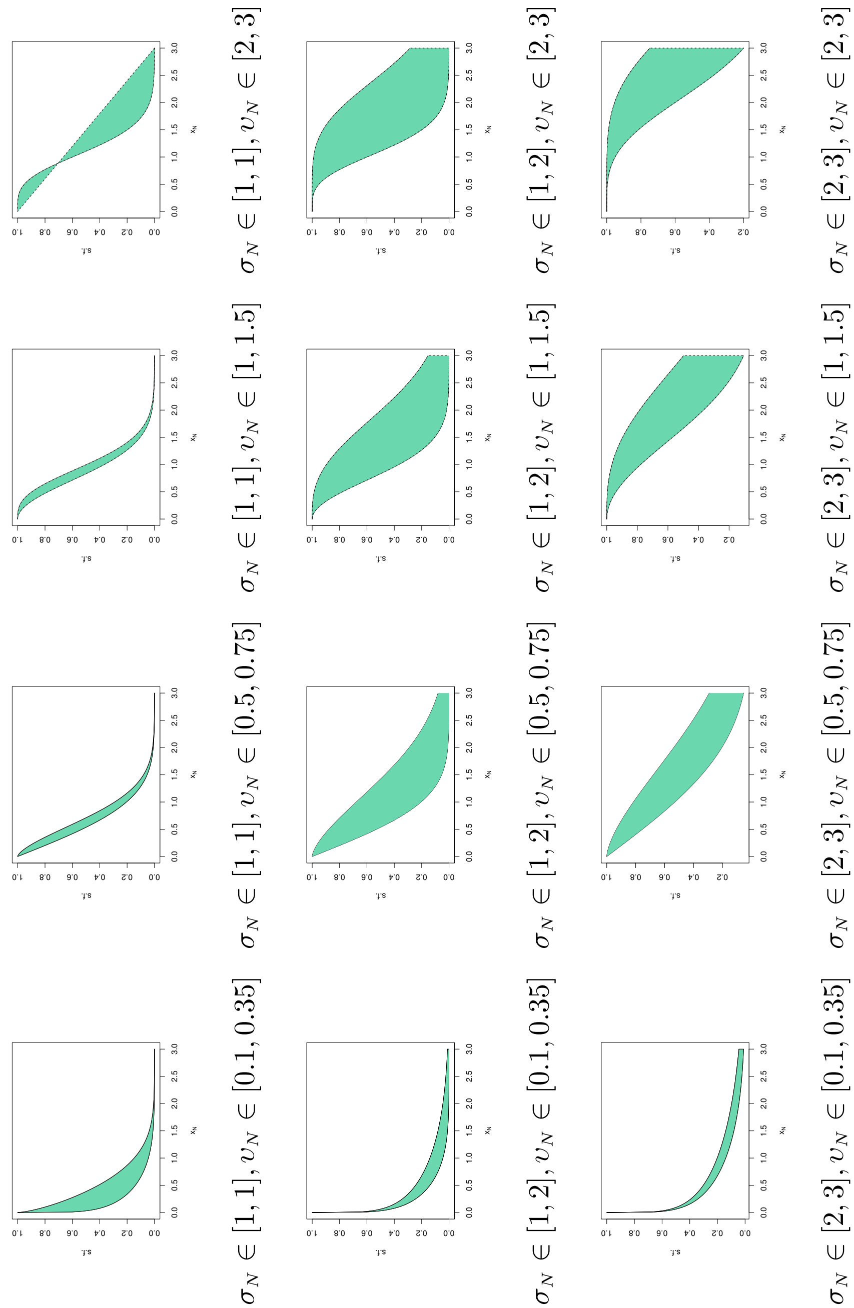


FIGURE 3. The survival function plots of NGE distribution

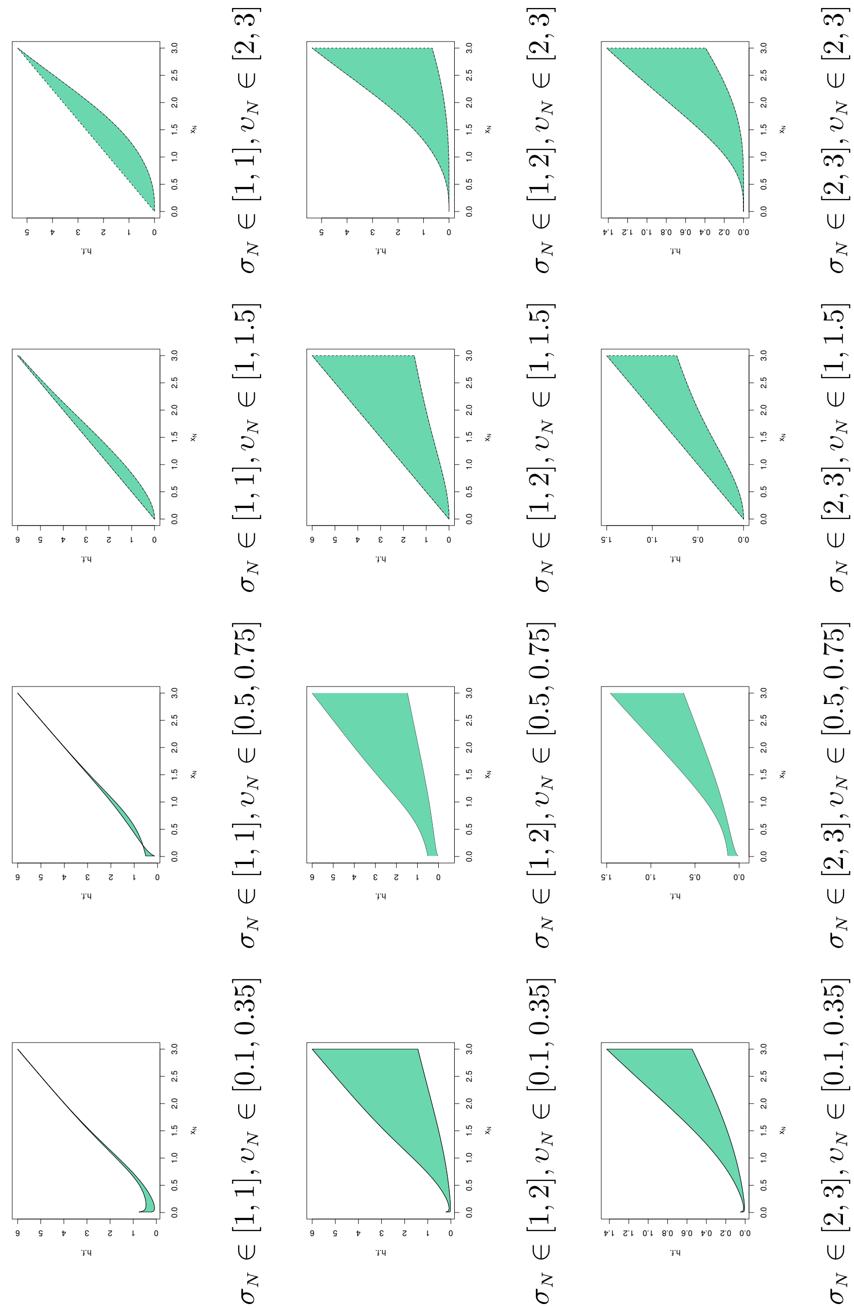


FIGURE 4. The hazard function plots of NGE distribution

## 3. Properties of NGRD

In this section, we discuss the some statistical properties of the NGRD and the result are brought out as under:

**Theorem 3.1.** *The $k^{th}$ moment about origin of NGRD is*

$$\mu_k' = \upsilon_N \sigma_N^k \Gamma\left(\upsilon_N\right) \Gamma\left(\frac{k}{2}+1\right) \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma\left(\upsilon_N - j\right) j! (j+1)^{\frac{k}{2}+1}} \tag{7}$$

**Proof.** By definition, the $k^{th}$ raw moment is given as

$$\begin{aligned}
\mu_k' &= \mathrm{E}\left[X_N^k\right] = \int_0^\infty x_N^k f\left(x_N; \upsilon_N, \sigma_N\right) dx_N \\
&= \int_0^\infty x_N^k \frac{2\upsilon_N}{\sigma_N^2} x_N \exp\left\{-\frac{x_N^2}{\sigma_N^2}\right\} \left[1 - \exp\left\{-\frac{x_N^2}{\sigma_N^2}\right\}\right]^{\upsilon_N - 1} dx_N \\
&= \int_0^\infty x_N^k \frac{2\upsilon_N}{\sigma_N^2} x_N \exp\left\{-\frac{x_N^2}{\sigma_N^2}\right\} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma\left(\upsilon_N\right) \exp\left\{-j\frac{x_N^2}{\sigma_N^2}\right\}}{\Gamma\left(\upsilon_N - j\right) j!} dx_N
\end{aligned}$$

where

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b) z^j}{\Gamma(b-j) j!}$$

Thus, we get

$$\begin{aligned}
\mu_k' &= \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma\left(\upsilon_N\right)}{\Gamma\left(\upsilon_N - j\right) j!} \int_0^\infty x_N^k \frac{2\upsilon_N}{\sigma_N^2} x_N \exp\left\{-\left(\frac{x_N}{\sigma_N}\right)^2\right\} \exp\left\{-j\left(\frac{x_N}{\sigma_N}\right)^2\right\} dx_N \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma\left(\upsilon_N\right)}{\Gamma\left(\upsilon_N - j\right) j!} \int_0^\infty \upsilon_N \sigma_N^k y_N^{\frac{k}{2}} \exp\left\{-(j+1) y_N\right\} dy_N
\end{aligned}$$

where $y = x_N^2/\sigma_N^2$. Therefore, we get

$$\mu_k' = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma\left(\upsilon_N\right) \upsilon_N \sigma_N^k}{\Gamma\left(\upsilon_N - j\right) j!} \frac{\Gamma\left(\frac{k}{2}+1\right)}{(j+1)^{\frac{k}{2}+1}},$$

where $\upsilon_N \in [\upsilon_{\mathrm{L}}, \upsilon_{\mathrm{U}}]$ and $\sigma_N \in [\sigma_{\mathrm{L}}, \sigma_{\mathrm{U}}]$. □

For $k = 1$, in Eq. (7) we get the first raw moment (mean) of the NGRD is given by

$$\text{Mean} = \mu_1' = \upsilon_N \sigma_N \Gamma\left(\upsilon_N\right) \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma\left(\upsilon_N - j\right) j! (j+1)^{\frac{3}{2}}}. \tag{8}$$

For $k = 2$, in Eq. 7 we get the second raw moment of the NGRD is given by

$$\mu_2' = \upsilon_N \sigma_N^2 \Gamma\left(\upsilon_N\right) \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma\left(\upsilon_N - j\right) j! (j+1)^2}.$$

Therefore, Neutrosophic variance (Nvar) is given by

$$\text{Nvar}\left(X_N\right) = \mu_2' - \left(\mu_1'\right)^2$$

Similarly, we can obtain other raw and central moments, using first four central moments one can obtain skewness and kurtosis to study the nature of the NGRD.

### 3.1. *Quantile function*

A useful and the important statistical property of NGRD is a quantile function and it also useful to generate random sample from NGRD for simulation work. The quantile function of NGRD is given by

$$Q_N(q) = F_N^{-1}(q) = \sigma_N \left[ -\ln\left(1 - q^{\frac{1}{\upsilon_N}}\right)\right]^{\frac{1}{2}},$$

where $\upsilon_N \in [\upsilon_L, \upsilon_U]$ and $\sigma_N \in [\sigma_L, \sigma_U]$. Thus,

$$\text{Median} = Q_N(0.5) = \sigma_N \left[ -\ln\left(1 - 0.5^{\frac{1}{\upsilon_N}}\right)\right]^{\frac{1}{2}}.$$

#### 3.1.1. *Measures of Skewness and Kurtosis based on Quantile Function*

The quantitative measure of skewness and Kurtosis based on quantile function is defined by [13] and [20], respectively. The following formula is used to determine the neutrosophic skewness and kurtosis of NGRD under neutrosophic environment.

$$\text{Skewness} = \frac{Q_N\left(\frac{6}{8}\right) - 2Q_N\left(\frac{4}{8}\right) + Q_N\left(\frac{2}{8}\right)}{Q_N\left(\frac{6}{8}\right) - Q_N\left(\frac{2}{8}\right)}$$

and

$$\text{Kurtosis} = \frac{Q_N\left(\frac{7}{8}\right) - Q_N\left(\frac{5}{8}\right) + Q_N\left(\frac{3}{8}\right) - Q_N\left(\frac{1}{8}\right)}{Q_N\left(\frac{6}{8}\right) - Q_N\left(\frac{2}{8}\right)}$$

The neutrosophic mean, neutrosophic variance, neutrosophic median, neutrosophic skewness, and neutrosophic kurtosis for various neutrosophic scale and shape parameters are displayed in Table 1. The results from Table 1 shows that for various statistic values are increases as neutrosophic share parametric values increases for fixed neutrosophic scale parameter.

## 4. **Neutrosophic Parametric Estimation**

To study the effectiveness of parametric estimation, a brief discussion is given in this section about neutrosophic maximum likelihood estimator(NMLE) of the parameters of NGRD. The asymptotic properties of NMLEs of $\upsilon_N$ and $\sigma_N$ also discussed. A simulation study is carried out to study the performance of classical MLEs of the parameters as well as other methods of parametric estimations have been considered widely in [16].

Let $X_{N_1}, \ldots, X_{N_n}$ be a random sample from NGRD, then the log-likelihood function can be expressed as follows:

$$l(\upsilon_N, \sigma_N) \cong n\ln(2) + n\ln(\upsilon_N) + 2n\ln(\sigma_N) + \sum_{i=1}^{n} \ln(x_{N_i})$$

$$-\sum_{i=1}^{n}\left(\frac{x_{N_i}}{\sigma_N}\right)^2+(\upsilon_N-1)\sum_{i=1}^{n}\ln\left(1-\exp\{-\left(\frac{x_N}{\sigma_N}\right)^2\}\right) \quad (9)$$

The NMLEs of $\upsilon_N$ and $\sigma_N$ can be obtained on maximize the Eq. (9) with respect to $\upsilon_N$ and $\sigma_N$. Thus NMLEs $\upsilon_N$ and $\sigma_N$ would be the solution of the following two non-linear equations:

$$\frac{\partial l}{\partial \upsilon_N}=\frac{n}{\upsilon_N}+\sum_{i=1}^{n}\ln\left(1-\exp\{-\left(\frac{x_{N_i}}{\sigma_N}\right)^2\right\}=0 \quad (10)$$

and

$$\frac{\partial l}{\partial \sigma_N}=\frac{-2n}{\sigma_N}+\frac{2}{\sigma_N}\sum_{i=1}^{n}\left(\frac{x_{N_i}}{\sigma_N}\right)^2-\frac{2(\upsilon_N-1)}{\sigma_N}\sum_{i=1}^{n}\frac{\left(\frac{x_{N_i}}{\sigma_N}\right)^2\exp\left\{-\left(\frac{x_{N_i}}{\sigma_N}\right)^2\right\}}{\left[1-\exp\{-\left(\frac{x_{N_i}}{\sigma_N}\right)^2\}\right]}=0 \quad (11)$$

The NLME of $\upsilon_N$ and $\sigma_N$, denoted by $\hat{\upsilon_N}$ and $\hat{\sigma_N}$ respectively, can be obtained by solving two non-linear Eqs. (10) and (11) simultaneously.

## 5. Simulation Study

To study the performance of the proposed NGRD distribution model, a simulation study is carried out. The accomplishment of NGRD estimated parameters and their performance are expressed as neutrosophic average estimates (AEs), neutrosophic average biased (Avg. Biases) and neutrosophic measure square error (MSEs) using simulation investigation. The simulation results of average Bias and MSE are summarized in Tables 2-4. It is noticed from tables that the average Bias and MSE are decrease when size of the sample increases, as

TABLE 1. The mean, variance, median, skewness and kurtosis for different neutrosophic parametric values

| $\sigma_N$ | $\upsilon_N$ | Mean | Variance | Median | Skewness | Kurtosis |
|---|---|---|---|---|---|---|
| $[1,1]$ | $[0.1,0.35]$ | $[0.188,0.502]$ | $[0.117,0.211]$ | $[0.031,0.385]$ | $[0.207,0.747]$ | $[1.158,1.936]$ |
| $[1,1]$ | $[0.5,0.75]$ | $[0.626,0.777]$ | $[0.221,0.222]$ | $[0.536,0.711]$ | $[0.094,0.138]$ | $[1.167,1.190]$ |
| $[1,1]$ | $[1,1.5]$ | $[0.886,1.039]$ | $[0.200,0.215]$ | $[0.833,0.997]$ | $[0.063,0.076]$ | $[1.204,1.219]$ |
| $[1,1]$ | $[2,3]$ | $[0.779,1.146]$ | $[0.187,0.894]$ | $[1.108,1.256]$ | $[0.056,0.058]$ | $[1.227,1.234]$ |
| $[1,2]$ | $[0.1,0.35]$ | $[0.188,1.003]$ | $[0.118,0.845]$ | $[0.031,0.771]$ | $[0.207,0.747]$ | $[1.158,1.936]$ |
| $[1,2]$ | $[0.5,0.75]$ | $[0.626,1.554]$ | $[0.222,0.884]$ | $[0.536,1.422]$ | $[0.094,0.138]$ | $[1.167,1.190]$ |
| $[1,2]$ | $[1,3]$ | $[0.886,2.659]$ | $[0.215,1.069]$ | $[0.833,1.994]$ | $[0.063,0.076]$ | $[1.204,1.219]$ |
| $[1,2]$ | $[2,3]$ | $[1.146,1.557]$ | $[0.187,3.574]$ | $[1.108,2.513]$ | $[0.056,0.058]$ | $[1.227,1.234]$ |
| $[2,3]$ | $[0.1,0.35]$ | $[0.376,1.505]$ | $[0.470,1.901]$ | $[0.063,1.156]$ | $[0.207,0.747]$ | $[1.158,1.936]$ |
| $[2,3]$ | $[0.5,0.75]$ | $[1.252,2.331]$ | $[0.886,1.989]$ | $[1.073,2.133]$ | $[0.094,0.138]$ | $[1.167,1.190]$ |
| $[2,3]$ | $[1,2]$ | $[1.7723.988]$ | $[0.858,2.404]$ | $[1.665,2.991]$ | $[0.063,0.076]$ | $[1.204,1.219]$ |
| $[2,3]$ | $[2,3]$ | $[2.292,2.336]$ | $[0.749,8.043]$ | $[2.216,3.769]$ | $[0.056,0.058]$ | $[1.227,1.234]$ |

expected. According to Tables 2-4, Bias of shape parameters is negative and scale parameter is positive at different values of shape parametric and scale parametric values.

TABLE 2. $\upsilon_N = [1, 1]$, $\sigma_N = [0.5, 0.75]$

| | AEs | | Avg. Biases | | MSEs | |
|---|---|---|---|---|---|---|
| | $\hat{\upsilon}_N$ | $\hat{\sigma}_N$ | $\hat{\upsilon}_N$ | $\hat{\sigma}_N$ | $\hat{\upsilon}_N$ | $\hat{\sigma}_N$ |
| 30 | 1.1005 | [0.4902,0.7377] | 0.1005 | [-0.0098,-0.0123] | 0.3204 | [0.0604,0.0894] |
| 50 | 1.0562 | [0.4942,0.7425] | 0.0562 | [-0.0058,-0.0075] | 0.2154 | [0.0465,0.0686] |
| 100 | 1.0274 | [0.4974,0.7452] | 0.0274 | [-0.0026,-0.0048] | 0.1415 | [0.0326,0.0493] |
| 200 | 1.0132 | [0.4985,0.7474] | 0.0132 | [-0.0015,-0.0026] | 0.0948 | [0.0231,0.0347] |
| 500 | 1.0051 | [0.4994,0.7494] | 0.0051 | [-0.0006,-0.0005] | 0.0598 | [0.0146,0.0221] |
| 1000 | 1.0025 | [0.4998,0.7497] | 0.0025 | [-0.0002,-0.0003] | 0.0419 | [0.0103,0.0155] |

TABLE 3. $\upsilon_N = [0.5, 0.75]$, $\sigma_N = [1, 1]$

| | AEs | | Avg. Biases | | MSEs | |
|---|---|---|---|---|---|---|
| | $\hat{\alpha}_N$ | $\hat{\sigma}_N$ | $\hat{\upsilon}_N$ | $\hat{\sigma}_N$ | $\hat{\upsilon}_N$ | $\hat{\sigma}_N$ |
| 30 | [0.5388,0.8168] | 0.9802 | [0.0388,0.0668] | -0.0198 | [0.1364,0.2132] | 0.1383 |
| 50 | [0.5225,0.7888] | 0.9873 | [0.0225,0.0388] | -0.0127 | [0.0942,0.1524] | 0.1070 |
| 100 | [0.5108,0.7692] | 0.9933 | [0.0108,0.0192] | -0.0067 | [0.0625,0.1008] | 0.0760 |
| 200 | [0.5052,0.7591] | 0.9963 | [0.0052,0.0091] | -0.0037 | [0.0424,0.0676] | 0.0536 |
| 500 | [0.5018,0.7538] | 0.9988 | [0.0018,0.0038] | -0.0012 | [0.0264,0.0434] | 0.0341 |
| 1000 | [0.5015,0.7525] | 0.9992 | [0.0015,0.0025] | -0.0008 | [0.0226,0.0363] | 0.029 |

TABLE 4. $\upsilon_N = [0.5, 0.75]$, $\sigma_N = [1, 3]$

| | AEs | | Avg. Biases | | MSEs | |
|---|---|---|---|---|---|---|
| | $\hat{\alpha}_N$ | $\hat{\sigma}_N$ | $\hat{\upsilon}_N$ | $\hat{\sigma}_N$ | $\hat{\upsilon}_N$ | $\hat{\sigma}_N$ |
| 30 | [0.5397,0.8185] | [0.9757,2.9472] | [0.0397,0.0685] | [-0.0243,-0.0528] | [0.1363,0.2205] | [0.1484,0.3877] |
| 50 | [0.5225,0.7888] | [0.9854,2.9676] | [0.0225,0.0388] | [-0.0146,-0.0324] | [0.0942,0.1524] | [0.1149,0.2977] |
| 100 | [0.5108,0.7692] | [0.9936,2.9791] | [0.0108,0.0192] | [-0.0064,-0.0209] | [0.0625,0.1008] | [0.0807,0.2138] |
| 200 | [0.5052,0.7591] | [0.9963,2.9889] | [0.0052,0.0091] | [-0.0037,-0.0111] | [0.0424,0.0676] | [0.057,0.1508] |
| 500 | [0.5018,0.7538] | [0.9985,2.9974] | [0.0018,0.0038] | [-0.0015,-0.0026] | [0.0264,0.0434] | [0.0361,0.0962] |
| 1000 | [0.5015,0.7525] | [0.9988,2.9989] | [0.0015,0.0025] | [-0.0012,-0.0011] | [0.0226,0.0363] | [0.0307,0.0818] |

## 6. **Real Data Applications**

A realistic attempt of NGE distribution model is studied with help a real data in this section. The Parameter estimates along with the values of AIC (Akaike's Information criteria), BIC (Bayesian Information criteria) and KS (Kolmogorov–Smirnov) statistic are provided for comparison neutrosophic exponential distribution (NED), neutrosophic generalized exponential distribution (NGED), neutrosophic Weibull distribution (NWD), neutrosophic Rayleigh distribution (NRD) and neutrosophic generalized Rayleigh distribution (NGRD).

To demonstrate a real example here we considered an rough population compactness of few villages in rural USA. This data is taken from [3] and they studied for neutrosophic W/S test based on the data follows to neutrosophic normal distribution. This data consists of the population of 17 villages in USA and their neutrosophic data, which is reproduced in Table 5 for ready reference. The results in Table 6 also shows that NGED is more suitable to fit the data than the NED.

Table 5. Neutrosophic population density of some villages in the USA

| Villages | Population density | Villages | Population density |
|---|---|---|---|
| Aranza | [4.13,4.14] | Charapan | [5.10,5.12] |
| Corupo | [4.53,4.55] | Comachuen | [5.25,5.27] |
| San Lorenzo | [4.69,4.70] | Pichataro | [5.36,5.38] |
| Cheranatzicurin | [4.76,4.78] | Quinceo | [5.94,5.96] |
| Nahuatzen | [4.77,4.79] | Nurio | [6.06,6.08] |
| Pomacuaran | [4.96,4.98] | Turicuaro | [6.19,6.21] |
| Servina | [4.97,4.99] | Urapicho | [6.30,6.32] |
| Arantepacua | [5.00,5.06] | Capacuaro | [7.73,7.98] |
| Cocucho | [5.04,5.06] | | |

Table 6. Estimates and Goodness-of-fit statistics for village data set

| Model | Parameter | Estimates | LogLikelihood | AIC | BIC | KS |
|---|---|---|---|---|---|---|
| NED | $\upsilon$ | [0.1861,0.1873] | [-45.58152,-45.4788] | [94.9576,95.16304] | [102.2905,102.4959] | [0.5385,0.5372] |
| NGED | shape | [41.6889,44.5078] | [-20.86095,-20.2097] | [44.4194,45.72189] | [51.7523,53.0547] | [0.2707,0.3270] |
| | rate | [7.7599,8.3348] | | | | |
| NWD | shape | [5.5773,5.8981] | [-23.94168,-23.04169] | [103.9359,107.1836] | [111.2687,114.5164] | [0.6552,0.6579] |
| | scale | [5.7621,5.7143] | | | | |
| NRD | $\upsilon$ | [3.8224,3.8495] | [-34.4533,-34.3024] | [72.6048,72.9067] | [79.9377,80.2395] | [0.4438,0.4457] |
| | $\sigma$ | - | - | - | - | - |
| NGRD | $\upsilon$ | [47.075,0.6187] | [-19.8662,-19.30757] | [42.6151,43.7323] | [49.9480,51.06525] | [0.1695,0.1728] |
| | $\sigma$ | [2.5501,2.5915] | | | | |

## 7. Conclusions

In this article, a generalization Rayleigh distribution is developed under neutrosophic statistics environment. Very few researchers are studied probability distributions based on neutrosophic statistics. The mathematical properties of the developed neutrosophic generalization

Rayleigh distribution are studied. The nature of the distribution is studies through various neutrosophic parametric combinations. Using the maximum likelihood method the parameters are estimated. A simulation study is carried out under neutrosophic environment. The average Bias and MSE are decreases as sample size increases, as expected. Finally, the application of the proposed neutrosophic generalized Rayleigh distribution is presented through a real data set. A comparative study with other distribution is also done based real data set. Based on real data example, we conclude that the proposed distribution furnishes better performance over existing distributions.

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# A Neutrosophic Approach for Multi-Factor Analysis of Uncertainty and Sustainability of Supply Chain Performance

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**Abstract:** The pursuit of Sustainable Supply Chain Management (SSCM) has become increasingly vital in the face of escalating environmental, social, and economic challenges. This paper presents a novel approach that harnesses the power of m-generalized q-neutrosophic numbers (mGqNN) within the MULTIMOORA-mGqNN method to evaluate and select SSCM performance and theory. By integrating mGqNN, we offer a versatile framework that adeptly navigates the uncertainties and vagueness inherent to SSCM decision-making. Through systematic linguistic assessments by multiple decision makers, our approach ranks SSCM alternatives, facilitating the identification of optimal strategies that enhance sustainability performance. This paper contributes to the evolving discourse on SSCM by introducing a robust methodological framework that addresses the multifaceted complexities of sustainability in supply chains. In an era where sustainability is paramount, the MULTIMOORA-mGqNN method offers researchers and practitioners a valuable tool to make informed decisions and guide supply chain strategies towards a more sustainable and responsible future. This innovative approach has the potential to reshape the landscape of SSCM, empowering organizations to forge a path towards enduring environmental stewardship, social responsibility, and economic resilience.

**Keywords**: Neutrosophic Sets, Sustainable Supply Chain Management (SSCM), m-Generalized q-Neutrosophic Numbers (mGqNN), MULTIMOORA-mGqNN Method, Decision-making, Uncertainty handling

## I. Introduction

Sustainable Supply Chain Management (SSCM) is a multidisciplinary approach that encompasses the integration of environmental, social, and economic considerations into all stages of a supply chain, from the sourcing of raw materials to the final delivery of products or services. It revolves around the responsible and ethical practices that organizations adopt to minimize their environmental footprint, support social well-being, and ensure economic viability while meeting the demands of the present without compromising the needs of future generations. SSCM extends beyond traditional supply chain optimization by recognizing that sustainability goes hand in hand with profitability and resilience. In essence, SSCM strives for a harmonious coexistence between ecological conservation, social equity, and economic growth within the context of supply chain operations [1-3].

The past few decades have witnessed a paradigm shift in the global business landscape, where sustainability has emerged as a central concern in supply chain management. This shift is driven by a growing awareness of the finite nature of natural resources, the social and ethical obligations of businesses, and the increasing scrutiny from consumers, regulators, and stakeholders [4]. The urgency to address climate change, reduce carbon emissions, and combat social inequalities has propelled sustainability to the forefront of corporate strategies. Businesses are recognizing that integrating sustainability into supply chains is not just a moral imperative but also a strategic advantage. It enhances brand reputation, mitigates risks associated with environmental and social disruptions, fosters innovation, and ultimately improves long-term financial performance. Consequently, the importance of sustainability in supply chains has never been more pronounced, making SSCM a critical field of study and practice [5].

SSCM is uniquely positioned to tackle a range of pressing challenges facing our planet today. Firstly, SSCM plays a pivotal role in addressing environmental challenges by minimizing the environmental footprint of supply chain activities. This involves reducing waste, conserving resources, optimizing transportation, and adopting eco-friendly technologies and practices. Secondly, SSCM is integral to addressing social challenges, such as labor rights, fair wages, and safe working conditions in global supply chains. It promotes the well-being of workers and the communities in which supply chain operations are embedded. Finally, SSCM is closely linked to economic challenges by fostering supply chain resilience and adaptability [6].

The contemporary landscape of SSCM is characterized by an intricate web of interdependencies, uncertainties, and dynamic challenges. In this complex environment, the need for systematic evaluation and assessment becomes paramount. Traditional supply chain management approaches often fall short in adequately addressing the multifaceted nature of sustainability, which includes environmental, social, and economic dimensions. Moreover, the pervasive presence of uncertainties, ambiguities, and indeterminacies in SSCM decision-making processes makes it imperative to adopt innovative methodologies capable of capturing and handling such complexities [7]. Robust evaluation and assessment techniques are the cornerstone of informed decision-making, allowing organizations to gauge the effectiveness of their sustainability initiatives, identify areas for improvement, and align their strategies with evolving global sustainability goals. SSCM has witnessed the application of various existing approaches and methodologies aimed at incorporating sustainability principles into supply chain operations. These traditional methods, although valuable, often confront inherent limitations when applied to SSCM [8]. Conventional supply chain management techniques tend to focus primarily on cost efficiency and optimization, overlooking the broader environmental and social impacts of supply chain activities. Moreover, they struggle to address the inherent uncertainties and complexities related to sustainability, which frequently involve vague or incomplete information, making it challenging to arrive at precise decisions. The limitations of traditional approaches in the context of SSCM underscore the need for innovative methodologies that can better accommodate the nuances of sustainability, acknowledge the intricacies of decision-making in the presence of ambiguity, and provide comprehensive insights into the multifaceted dimensions of sustainability [9-12]. In this regard, the adoption of neutrosophic sets emerges as a promising avenue to overcome these limitations, allowing for a more nuanced and inclusive evaluation of sustainability factors in supply chain management, which will be explored in detail in this paper.

Neutrosophic sets, a mathematical framework introduced to address the complexities of uncertainty and ambiguity, offer a powerful and versatile tool for modeling and analyzing phenomena in SSCM. Neutrosophic sets extend the traditional binary logic of true or false to a trilemma of true, false, and indeterminate, allowing for a more nuanced representation of information. In the context of SSCM, where decision-making often involves incomplete or imprecise data and where the assessment of sustainability factors inherently carries elements of ambiguity, neutrosophic sets offer a means to capture and manage this inherent uncertainty. By embracing the indeterminate aspect of neutrosophic sets, supply chain professionals can better grapple with the complexities of sustainability, integrating it into decision-making processes, and fostering a more holistic and adaptive approach to SSCM.

The primary objective of this research is to provide a comprehensive and inclusive examination of the application of neutrosophic set theory in the context of sustainable supply chain management. This study seeks to achieve the following key goals:

- Factor Assessment: Analyze and assess the multifaceted factors that influence the performance and sustainability of supply chains, accounting for their inherent uncertainty and ambiguity using neutrosophic sets.
- Theoretical Advancements: Explore the theoretical foundations of neutrosophic sets and their applicability in modeling and optimizing sustainable supply chain operations.
- Decision Support: Discuss how the integration of neutrosophic sets can enhance decision support systems for sustainable supply chain management, aiding organizations in making more robust and adaptive choices.
- Case Studies: Examine real-world case studies and practical applications of neutrosophic sets in sustainable supply chain management to illustrate the potential benefits and challenges of adopting this approach.

The organization of the remaining of this paper is structured as follows: Section II reviews related work, Section III presents the methodology employed, Section IV provides results and analysis, Section V concludes the paper. This systematic arrangement ensures a comprehensive exploration of the application of neutrosophic sets in the context of SSCM.

## II. Related Works

Herin, we delve into the existing body of research and literature related to both SSCM and the utilization of neutrosophic sets in decision-making. This review serves as the foundation for our study, offering insights into the current state of knowledge and highlighting gaps in the literature that our research aims to address. Shen et al. [6] proposed multi-attribute decision-making methods based on normal random variables for supply chain risk management, highlighting the importance of addressing uncertainties in supply chain operations. Yang and Guo [7] conducted research on the evaluation of public emergency management intelligence capability in probabilistic language environments, emphasizing the relevance of probabilistic approaches in assessing complex scenarios. Liu et al. [8] performed an uncertainty analysis for offshore wind power investment decisions, showcasing the applicability of real options approaches in managing uncertain investment environments. Biswas et al. [9] presented a multi-criteria-based stock selection framework for emerging markets, emphasizing the importance of multi-criteria decision-making in investment decisions. Vinogradova-Zinkevič et al. [10] conducted a comparative assessment of the stability of AHP and FAHP methods, shedding light on the comparative advantages of different decision-making techniques. Ecer et al. [11] evaluated cryptocurrencies for investment decisions using a multi-criteria methodology, underscoring the relevance of advanced decision-making techniques in the era of Industry 4.0. Bhattacharjee et al. [12] applied Failure Mode and Effects Analysis (FMEA) using interval number-based BWM-MCDM approaches, highlighting the role of decision-making techniques in risk management. Yang et al. [13] explored a Bayesian-based approach for NIMBY crisis transformation in municipal solid waste incineration, showcasing the utility of Bayesian methods in addressing public concerns. Ozturk [14] investigated structures on neutrosophic topological spaces, contributing to the theoretical foundation of neutrosophic set theory. Zakeri et al. [15] employed a grey approach for computing interactions between two groups of irrelevant variables in decision matrices, demonstrating the value of grey systems theory in decision analysis. Yin et al. [16] conducted research on module partition for remanufacturing parts to be assembled, highlighting the importance of efficient part management in sustainable practices. Zhang et al. [17] developed a decision framework for the location and selection of container multimodal hubs, emphasizing the role of decision support systems in the context of infrastructure development under initiatives like the Belt and Road Initiative.

## III. Methodology

Suggest an introductory paragraph to Methodology section that investigate different methods for

### 3.1. m-Generalized q-Neutrosophic Numbers (mGqNN)

mGqNN are a mathematical representation used to handle uncertainty, indeterminacy, and vagueness in a decision-making process, particularly in the context of SSCM. An mGqNN is characterized by three components: the membership degree (T), the indeterminacy degree (I), and the non-membership degree (F). These components quantify the degree of truth, indeterminacy, and falsity associated with a particular value or parameter. Mathematically, an mGqNN can be represented as:

$$A = (T_A, I_A, F_A), \quad (1)$$

Where $T_A$ represents the membership degree (degree of truth) of an element $A$ in a specific set, indicating the extent to which $A$ belongs to the set. $I_A$ represents the indeterminacy degree, reflecting the degree of uncertainty or ambiguity in assessing the membership of $A$ in the set. $F_A$ represents the non-membership degree (degree of falsity), signifying the extent to which AA does not belong to the set. The values of $T_A, I_A$,and $F_A$ lie within the interval $[0, 1]$, and they satisfy the following constraint:

$$T_A + I_A + F_A = 1 \quad (2)$$

The mGqNN framework provides a flexible and comprehensive way to represent and manipulate uncertainty in SSCM evaluation, allowing for a more nuanced understanding of the factors impacting sustainable supply chain performance and theory.

m-Generalized q-neutrosophic sets (mGqNS) are a mathematical framework used to represent and handle uncertainty, indeterminacy, and vagueness in the context of decision-making, particularly in SSCM. An mGqNS is defined over a universal set UU and consists of three components: the membership function ($\mu$), the indeterminacy function ($\lambda$), and the non-membership function ($\nu$). These functions quantify the degrees to which elements of $U$ belong to, are indeterminate with respect to, or do not belong to a specific set $A$. Mathematically, an mGqNS can be represented as:

$$A = \{(\mu_A(x), \lambda_A(x), \nu_A(x))\}, x \in U. \quad (3)$$

where $\mu_A(x)$ represents the membership function, indicating the degree to which element $x$ belongs to set $A$. $\lambda_A(x)$ represents the indeterminacy function, reflecting the degree of uncertainty or ambiguity in assessing the membership of element $x$ in set $A$. $\nu_A(x)$ represents the non-membership function, signifying the degree to which element $x$ does not belong to set $A$. These functions are defined for each element $x$ in the universal set $U$, and they satisfy the following constraints for every $x \in U$:

$$\mu_A(x) + \lambda_A(x) + \nu_A(x) = 1 \quad (4)$$

This implies that the sum of the membership, indeterminacy, and non-membership degrees for each element xx is equal to 1.

$$0 \leq \mu_A(x) + \lambda_A(x) + \nu_A(x) \leq 1 \quad (5)$$

This implies that the degrees are bounded within the interval [0, 1]. The mGqNS framework provides a versatile and comprehensive way to represent and manage uncertainty in SSCM decision-making, allowing for a more nuanced and context-aware assessment of factors impacting sustainable supply chain performance and theory within the universal set $U$.

The operations between mGqNNs are defined as a fundamental aspect of this mathematical framework, crucial for processing and manipulating uncertainty within the context of SSCM. When performing operations on mGqNNs, denoted as $\psi_1$ and $\psi_2$ a positive real number $\lambda$ plays a central role in scaling the degree of indeterminacy, allowing for the adjustment of ambiguity levels in the mathematical operations. This scaling factor provides the flexibility to control the influence of uncertainty during calculations, ensuring that mGqNNs can effectively capture and manage various degrees of vagueness and indistinctness in SSCM decision-making processes. These defined operations empower decision-makers to perform comprehensive and context-sensitive analyses, making mGqNNs a versatile tool for addressing the complexities of sustainability within supply chains.

$$\psi_1 \oplus \psi_2 = (1-(1-\zeta_1^q)(1-\zeta_2^q))^{\frac{1}{q}}, \vartheta_1\vartheta_2, \eta_1\eta_2, \quad (6)$$
$$\psi_1 \otimes \psi_2 = \zeta_1\zeta_2, (1-(1-\vartheta_1^q)(1-\vartheta_2^q))^{\frac{1}{q}}, (1-(1-\eta_1^q)(1-\eta_2^q))^{\frac{1}{q}}, \quad (7)$$
$$\lambda * \psi_1 = (1-(1-\zeta_1^q)^\lambda)^{\frac{1}{q}}, \vartheta_1^\lambda, \eta_1^\lambda, \quad (8)$$
$$\lambda \odot \psi_1 = \zeta_1^\lambda, (1-(1-\vartheta_1^q)^\lambda)^{\frac{1}{q}}, (1-(1-\eta_1^q)^\lambda)^{\frac{1}{q}}, \quad (9)$$
$$\psi_1^c = \eta_1, 1-\vartheta_1, \zeta_1. \quad (10)$$

The calculation of the mGqNN score function is determined by:

$$S(\psi) = \frac{3 + 3\zeta^q - 2\vartheta^q - \eta^q}{6}. \quad (11)$$

In the realm of mGqNNs, ranking plays a pivotal role in decision-making processes involving multiple elements or alternatives. The ranking procedure is established in descending order, primarily relying on score function values. In cases where two or more mGqNNs yield identical score function values, they are assigned the same rank, ensuring fairness and consistency in the ranking process. This ranking methodology serves as a crucial step in discerning the most preferred alternatives or elements within the context of SSCM. To facilitate aggregation and decision-making among these ranked mGqNNs, the mGqNWAA offers a formalized approach to combine their information, enabling SSCM practitioners to make informed and comprehensive choices while considering the intricacies of uncertainty and vagueness inherent in supply chain sustainability assessments. The calculation of the mGqNN score function is determined as follows:

$$\mathrm{mGqNWAA}(\psi_1, \dots, \psi_p) = \left(\frac{3}{m} - \prod_{k=1}^{p} \left(\frac{3}{m} - \zeta_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}, \prod_{k=1}^{p} \vartheta_k^{w_k}, \prod_{k=1}^{p} \eta_k^{w_k}. \tag{12}$$

The m-Generalized g-Neutrosophic Weighted Geometric Aggregation (mGqNWGA) operator is computed as follows:

$$\mathrm{mGqNWGA}(\psi_1, \dots, \psi_p) = \prod_{k=1}^{p} \zeta_k^{w_k}, \left(\frac{3}{m} - \prod_{k=1}^{p} \left(\frac{3}{m} - \vartheta_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}, \left(\frac{3}{m} - \prod_{k=1}^{p} \left(\frac{3}{m} - \eta_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}. \tag{13}$$

Expert opinions can be effectively integrated into the criteria weighting process to enhance the comprehensibility and applicability of the methodology. In this study, the mGqNN framework employs a subjective weighting approach to accommodate expert preferences. These experts, despite their extensive knowledge in the research field, may not be familiar with intricate methods, which can potentially diminish the validity of their assessments. This is particularly relevant when dealing with complex weighting or prioritization techniques that involve iterative or sequential comparisons. The adopted approach in this study, however, prioritizes the comfort and confidence of experts by providing them with a valid and secure platform to express their preferences. The process for obtaining criteria weight values follows specific steps, ensuring a methodological approach that aligns with both the complexity of the task and the expertise of the participants.

In the first step of the criteria weighting process, obtaining linguistic assessments from experts is a crucial endeavor. Experts, $(j = 1, \dots, p)$ $\varsigma_j^{(k)} = \zeta_j^{(k)}, \vartheta_j^{(k)}, \eta_j^{(k)}$, are tasked with evaluating the relative importance levels of various criteria $(j = 1, \dots, n)$, and they do so by employing a range of linguistic expressions provided in Table 1. These linguistic expressions serve as a structured and standardized framework that allows experts to convey their judgments in a clear and comprehensible manner. By using this linguistic scale, experts can express their subjective perceptions of the significance of each criterion, facilitating the subsequent steps of the weighting process. This step not only ensures that expert opinions are effectively captured but also contributes to the overall transparency and consistency of the criteria weighting procedure, aligning it with the experts' preferences and levels of familiarity with the methodology.

In the second step of the criteria weighting process, the determination of weights for expert evaluations is a pivotal task. This step involves translating the linguistic expressions from Table 1, which experts used to assess the importance levels of criteria, into numerical weight values. By mapping this linguistic expression to precise weight values, the subjective evaluations provided by the experts are transformed into quantifiable metrics. This conversion process is essential for creating a quantitative foundation upon which further calculations and analyses can be based. It not only ensures the rigor and consistency of the criteria weighting methodology but also respects the preferences, $\xi_k = (\zeta_k, \vartheta_k, \eta_k)$ , of the experts who, as previously discussed, prioritize simplicity and clarity in the assessment process. This step lays the groundwork for subsequent stages in the decision-making process, where the weighted criteria play a significant role in evaluating and ranking alternatives or elements within the SSCM context.

$$\acute{\upsilon}_k = \frac{\frac{3+3\zeta_k^q-2\vartheta_k^q-\eta_k^q}{6}}{\sum_{k=1}^{p}\frac{3+3\zeta_k^q-2\vartheta_k^q-\eta_k^q}{6}}\ k=(1,2,..,p), \tag{14}$$

In the third step of the criteria weighting process, the integration of experts' assessments is a pivotal stage in establishing the overall importance levels for criteria within the mGqNN framework. This integration is achieved through the application of Eq. (14), which serves as an aggregation mechanism for combining the mGqNNs associated with each linguistic expression provided by the experts. By aggregating these mGqNNs, a holistic and comprehensive representation of the criteria's importance levels is attained. This step embodies the essence of collective expert judgment, where the diverse opinions and assessments provided by experts are harmonized into a coherent and unified perspective on the relative importance of criteria in the context of SSCM. The resulting integrated importance levels serve as a foundational element for subsequent decision-making processes, allowing for well-informed and balanced evaluations of SSCM alternatives or elements.

$$\omega_j = \left(\frac{3}{m}-\prod_{k=1}^{p}\left(\frac{3}{m}-\zeta_j^{(k)\frac{qm}{3}}\right)^{\acute{\upsilon}_k}\right)^{\frac{3}{qm}}, \prod_{k=1}^{p}\vartheta_j^{(k)\acute{\upsilon}_k}, \prod_{k=1}^{p}\eta_j^{(k)\acute{\upsilon}_k} \tag{15}$$

In the fourth and final step of the criteria weighting process, the calculation of criteria weights is executed, providing a quantitative representation of the relative importance of each criterion. This calculation is carried out using the prescribed formula, which encapsulates the integrated importance levels obtained in the previous step.

$$w_j = \frac{\frac{3+3\zeta_j^q-2\vartheta_j^q-\eta_j^q}{6}}{\sum_{j=1}^{n}\frac{3+3\zeta_j^q-2\vartheta_j^q-\eta_j^q}{6}}\, j=(1,2,..,n), \tag{16}$$

Table 1: Linguistic Expressions for Expert Assessments of Criteria Importance Levels

| **Linguistic Expression for Evaluating the Significance of Criteria** | **Rating Linguistic Expression** | **Neutrosophic value** |
|---|---|---|
| Extremely High Significance (EHS) | Exceptionally Excellent (EE) | (0.95, 0.03, 0.02) |
| **Very Very High Significance (VVHS)** | Remarkably Good (RG) | (0.85, 0.10, 0.05) |
| **Very High Significance (VHS)** | Very Good (VG) | (0.75, 0.15, 0.10) |
| **High Significance (HS)** | Good (G) | (0.65, 0.20, 0.15) |
| **Above Average Significance (AAS)** | Moderately Good (MG) | (0.55, 0.25, 0.20) |
| **Average Significance (AS)** | Fair (F) | (0.50, 0.30, 0.20) |
| **Below Average Significance (BAS)** | Moderately Low (ML) | (0.45, 0.35, 0.20) |
| **Low Significance (LS)** | Low (L) | (0.35, 0.45, 0.20) |
| **Very Low Significance (VLS)** | Very Low (VL) | (0.25, 0.55, 0.20) |
| **Very Very Low Significance (VVLS)** | Remarkably Low (RL) | (0.15, 0.65, 0.20) |
| **Extremely Low Significance (ELS)** | Exceptionally Low (EL) | (0.05, 0.85, 0.10) |

The MULTIMOORA-mGqNN methodology represents a significant advancement in the field of decision-making, particularly in the complex domain of SSCM. Combining the MULTIMOORA (Multi-Objective Optimization by Ratio Analysis plus the Full Multiplicative Form) approach with the power of mGqNN, this hybrid framework offers a comprehensive solution to the multifaceted challenges SSCM practitioners face. MULTIMOORA, renowned for its effectiveness in multi-criteria decision analysis, provides a robust foundation for evaluating alternatives across various criteria. The integration of mGqNN enhances this approach by addressing the inherent uncertainties and vagueness associated with SSCM assessments. By leveraging mGqNN's capability to capture and manage degrees of truth, indeterminacy, and falsity, MULTIMOORA-mGqNN empowers decision-makers to make well-informed, context-aware choices while considering the intricate interplay of economic, social, and environmental factors in supply chain sustainability. This amalgamation of methodologies stands as a testament to the evolving landscape of decision science, ushering in a new era of informed, nuanced, and sustainable decision-making within supply chain management.

In the MULTIMOORA-mGqNN methodology, the decision-making process unfolds through a series of well-defined steps, each contributing to the comprehensive evaluation of alternatives within the context of Sustainable Supply Chain Management (SSCM).

**Step 1: Constructing the Decision Matrix** The initial step involves the construction of the decision matrix, a foundational component of decision analysis. In this stage, experts actively engage in assessing the various alternatives using the linguistic expressions provided in Table 1. These expressions facilitate a structured and consistent approach to evaluating the alternatives, capturing the experts' nuanced judgments regarding the criteria under consideration. This step forms the basis for subsequent analyses, ensuring that the subjective assessments of the alternatives align with the linguistic terms chosen to express their performance across the criteria.

**Step 2: Determining Weights of Experts' Evaluation** Following the assessment of alternatives, Step 2 focuses on determining the weights of experts' evaluations. This critical task aims to transform the linguistic assessments into numerical weight values, allowing for the quantification of the experts' subjective judgments. The importance levels assigned by experts to their evaluations are computed in accordance with the procedure specified in Step 2 of the mGqNN subjective weighting approach. This conversion of linguistic expressions into numerical weights establishes a robust foundation for subsequent calculations, facilitating a quantitative representation of the experts' assessments and their relative significance in the decision-making process.

**Step 3: Constructing the Integrated mGqNN Decision Matrix** In Step 3, the integrated mGqNN decision matrix takes shape as the evaluations provided by experts are harmonized into a unified representation. This integration is achieved through the application of Eq. (17), a mathematical mechanism designed to aggregate the individual mGqNN assessments. By combining the mGqNNs, this step creates a holistic view of the alternatives, incorporating the diverse perspectives of experts into a single, comprehensive matrix. The result is a powerful representation of the alternatives' performance that accounts for the degrees of truth, indeterminacy, and falsity present in the experts' judgments, setting the stage for a nuanced analysis of SSCM alternatives.

$$x_{ij} = \left(\left(\frac{3}{m} - \prod_{k=1}^{p}\left(\frac{3}{m} - t_{ij}^{(k)\frac{qm}{3}}\right)^{\acute{\upsilon}_k}\right)^{\frac{3}{qm}}, \prod_{k=1}^{p} b_{ij}^{(k)\acute{\upsilon}_k}, \prod_{k=1}^{p} f_{ij}^{(k)\acute{\upsilon}_k} \right) \; (i = 1,2,\dots,m;\; j = 1,2,..,n), \quad (17)$$

with $x_{ij} = (t_{ij}, b_{ij}, f_{ij})$

**Step 4: Application of the Ratio System** The final step, Step 4, introduces the application of the ratio system, a fundamental aspect of the MULTIMOORA-mGqNN methodology. This system, executed through Eq. (18), plays a pivotal role in evaluating and ranking the alternatives based on the integrated mGqNN decision matrix. By leveraging the ratio system, decision-makers can make informed choices, taking into account the weighted criteria, the integrated mGqNN assessments, and the intricacies of SSCM. This step transforms the extensive groundwork laid in previous stages into actionable insights, facilitating the selection of sustainable alternatives that align with the specific goals and priorities of the decision-makers in the SSCM domain.

$$Q_i = \sum_{j \in J_b} w_j x_{ij} + \left(\sum_{j \in J_c} w_j x_{ij}\right)^c \; (i = 1,2,\dots,m), \quad (18)$$

***Step 5,*** this step the MULTIMOORA-mGqNN methodology, the focus shifts to the application of the reference point, a critical stage that facilitates a more refined evaluation and ranking of alternatives within the context of SSCM. During this step, two crucial calculations are performed: the deviation from the reference point and the Min-Max metric of the Tchebycheff norm.

$$\min_i \left( \max_j \left| D\left(r_j - w_j x_{ij}\right) \right| \right) (i = 1,2,\dots,m; j = 1,2,..,n) \quad (19)$$

The reference point, in essence, serves as a benchmark or ideal state against which the performance of alternatives is assessed. It represents the desired values or attributes that SSCM practitioners aim to achieve or maintain within their supply chain processes. By establishing this reference point, decision-makers can gauge how well each alternative aligns with their sustainability goals and objectives.

$$r_j = \max_j \left(w_j x_{ij}\right) (i = 1,2,\dots,m; j = 1,2,..,n) \quad (20)$$

The deviation from the reference point is a quantitative measure of how far each alternative deviates from the ideal state represented by the reference point. This deviation calculation considers the mGqNNs associated with each alternative's performance across the criteria. It provides decision-makers with valuable insights into the extent to which each alternative fulfills or falls short of their sustainability targets.

$$D(\psi_1, \psi_2) = \sqrt{\frac{1}{3}\left(\left(\zeta_1^q - \zeta_2^q\right)^2 + \left(\vartheta_1^q - \vartheta_2^q\right)^2 + \left(\eta_1^q - \eta_2^q\right)^2\right)}. \quad (21)$$

The Min-Max metric of the Tchebycheff norm, on the other hand, offers a systematic and objective approach to evaluating alternatives based on their deviations from the reference point. This metric accounts for the degree of importance assigned to each criterion, as determined in earlier steps of the methodology. It enables decision-makers to identify the alternative that minimizes the maximum deviation across all criteria, reflecting a balanced and optimal solution within the constraints of SSCM.

Step 5, therefore, represents a critical phase where quantitative assessments are made, and the Min-Max metric allows for the identification of the most suitable alternative that best aligns with the reference point's sustainability objectives. This step empowers decision-makers to make data-driven choices, taking into consideration both the ideal state they strive to achieve and the real-world complexities of supply chain sustainability, ultimately enhancing their ability to select alternatives that contribute positively to SSCM goals.

Step 6 marks a pivotal stage in the MULTIMOORA-mGqNN methodology, where the focus is on implementing the full multiplicities form to further refine the evaluation of alternatives within the domain of Sustainable Supply Chain Management (SSCM). This step involves the minimization of a purely multiplicative utility function, a sophisticated approach that takes into account the complexities and nuances associated with SSCM assessments. By leveraging this utility function, decision-makers can effectively balance the impact of criteria and the performance of alternatives, resulting in a comprehensive evaluation that captures both the positive and negative aspects of each alternative. Step 6, therefore, enhances the robustness of the decision-making process by offering a more nuanced and balanced perspective on the sustainability of SSCM alternatives.

$$U_i = \frac{s(A_i)}{s(B_i)} (i = 1,2, \dots, m), \tag{22}$$

$$A_i = \prod_{j \in J_b} w_j x_{ij} \; (i = 1,2, \dots, m), \tag{23}$$

$$B_i = \prod_{j \in J_c} w_j x_{ij} \; (i = 1,2, \dots, m). \tag{24}$$

**Step 7: Ranking Alternatives** In the final step, Step 7, the culmination of the MULTIMOORA-mGqNN methodology occurs through the ranking of alternatives. After an exhaustive evaluation process that includes the integration of mGqNNs, the application of a reference point, and the utilization of the full multiplicities form, decision-makers are equipped with a wealth of information about the performance of alternatives across diverse criteria. This step compiles this information into a clear and concise ranking of alternatives based on their alignment with SSCM goals and objectives. By applying a systematic and rigorous ranking methodology, decision-makers can readily identify the most suitable alternatives that best address the specific sustainability challenges and priorities of their supply chain. Step 7 represents the culmination of the decision-making process, providing decision-makers with actionable insights and a ranked order of alternatives that can guide their choices and investments in SSCM.

## IV. Results and Analysis

The section serves as the heart of our study, where we delve into the substantive findings and comprehensive assessments derived from the application of the MULTIMOORA-mGqNN methodology to the domain of SSCM. In this section, we present a wealth of quantitative and qualitative data, shedding light on the performance of SSCM alternatives, the prioritization of criteria, and the intricate dynamics of decision-making within the realm of sustainability. Our analysis seeks to unravel the complexities, uncertainties, and nuances inherent in SSCM, offering a structured and data-driven approach to evaluating alternatives and informing strategic choices. In Table 2, we present a comprehensive breakdown of the criteria essential for the evaluation of Sustainable Supply Chain Management (SSCM). This table not only provides a list of these criteria but also offers detailed explanations for each, ensuring clarity and understanding of their relevance in the context of SSCM. The criteria outlined in Table 2 serve as the foundation for the subsequent evaluations and assessments, enabling a structured and systematic analysis of the sustainability aspects within the supply chain.

Table 2: Criteria and Explanations for SSCM

| Group Name | Criteria | Explanation |
|---|---|---|
| **Environmental Sustainability** | Environmental Impact (E1) | Assessing the ecological footprint of supply chain activities, including emissions and resource use. |
| | Energy Efficiency (E2) | Evaluating energy consumption and efficiency in supply chain activities, including transportation and operations. |
| | Waste Management (E3) | Examining waste reduction, recycling, and sustainable disposal practices within the supply chain. |
| **Social Responsibility** | Social Responsibility (S1) | Evaluating the supply chain's commitment to ethical labor practices, diversity, and community engagement. |
| | Supplier Relations (S2) | Assessing relationships with suppliers, including communication, collaboration, and ethical sourcing practices. |
| | Customer Satisfaction (S3) | Assessing the satisfaction levels of end customers in terms of product quality, delivery, and service. |
| **Economic Performance** | Economic Efficiency (C1) | Analyzing cost-effectiveness, resource utilization, and financial sustainability within the supply chain. |
| | Innovation and Technology Integration (C2) | Measuring the adoption of innovative technologies and practices to enhance supply chain efficiency and sustainability. |
| | Regulatory Compliance (C3) | Ensuring adherence to relevant laws, regulations, and industry standards across the supply chain. |
| **Product Quality** | Product Quality (P1) | Measuring the consistency and quality of products or services delivered throughout the supply chain. |
| | Supply Chain Transparency (P2) | Assessing the degree to which supply chain operations and processes are open and transparent to stakeholders. |
| | Resilience to Disruptions (P3) | Evaluating the supply chain's ability to adapt and recover from disruptions such as natural disasters or pandemics. |

Moving forward, Table 3 illustrates the outcomes of the experts' assessments regarding the importance levels assigned to the identified criteria. These assessments are a critical component of the decision-making process, as they reflect the expert perspectives on the relative significance of each criterion in achieving SSCM goals. Table 3 provides a transparent representation of these importance levels, establishing a quantitative basis for the subsequent weighting and integration processes.

Table 3: Experts' Assessments of Criteria Importance Levels

| Decision Maker (DM) | E1 | E2 | E3 | S1 | S2 | S3 | C1 | C2 | C3 | P1 | P2 | P3 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **DM1** | HS | HS | HS | HS | VVHS | VVHS | HS | AAS | AAS | HS | HS | HS |
| **DM2** | AAS | AAS | HS | AAS | AAS | HS | HS | VVHS | HS | AAS | AAS | AAS |
| **DM3** | HS | HS | HS | HS | AAS | AAS | AAS | AAS | HS | LS | LS | LS |
| **DM4** | HS | LS | AAS | LS | HS | HS | BAS | BAS | LS | LS | LS | AAS |
| **DM5** | VVHS | BAS | AAS | BAS | VVHS | HS | HS | HS | HS | HS | AAS | AAS |
| **DM6** | HS | HS | HS | HS | AAS | AAS | HS | HS | HS | LS | HS | AAS |
| **DM7** | HS | HS | HS | HS | VVHS | AAS | HS | HS | HS | HS | AAS | AAS |

Table 4 represents the outcomes of the integration of linguistic assessments from Table 4, resulting in numerical values that quantify the experts' evaluations. This integration is a pivotal step that transforms subjective linguistic expressions into quantitative data, facilitating rigorous analyses within the mGqNN framework. The values in Table 4 lay the groundwork for further calculations and assessments, offering a comprehensive picture of the criteria's relative importance.

Table 4: Integrated Importance Levels for SSCM Criteria

| | E1 | E2 | E3 | S1 | S2 | S3 |
|---|---|---|---|---|---|---|
| **Integrated** | (0.59, 0.33, 0.29) | (0.63, 0.31, 0.39) | (0.57, 0.22, 0.31) | (0.58, 0.38, 0.51) | (0.49, 0.51, 0.62) | (0.50, 0.36, 0.46) |
| $\omega_j$ | 0.9235 | 0.8343 | 0.8187 | 0.5874 | 0.5934 | 0.5988 |
| $\omega_j$ | 0.1452 | 0.0804 | 0.1017 | 0.2271 | 0.3669 | 0.0742 |
| | **C1** | **C2** | **C3** | **P1** | **P2** | **P3** |
| **Integrated** | (0.57, 0.37, 0.36) | (0.63, 0.32, 0.34) | (0.61, 0.37, 0.33) | (0.51, 0.37, 0.32) | (0.66, 0.25, 0.55) | (0.66, 0.33, 0.29) |
| $\omega_j$ | 0.9106 | 0.8429 | 0.7343 | 0.7876 | 0.9503 | 0.7377 |
| $\omega_j$ | 0.3015 | 0.2415 | 0.0646 | 0.2584 | 0.1731 | 0.1909 |

In Table 5, we present the linguistic evaluations assigned to the alternatives considered in the SSCM problem solution. These evaluations provide insights into how each alternative performs across the criteria, capturing the nuances and variations in their sustainability attributes. Table 5 serves as a crucial reference point for the subsequent steps in the decision-making process, enabling a holistic evaluation of SSCM alternatives.

Table 5: Linguistic Evaluations for SSCM Alternatives

| Decision Maker | Alternative | E1 | E2 | E3 | S1 | S2 | S3 | C1 | C2 | C3 | P1 | P2 | P3 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **DM1** | Alt1 | G | VG | F | EE | G | G | VG | VG | G | VG | EE | G |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **DM1** | Alt2 | VG | VG | G | VG | G | G | VG | EE | G | G | G | G |
| **DM1** | Alt3 | G | G | G | G | VG | VG | EE | G | G | L | L | L |
| **DM2** | Alt1 | G | VG | F | EE | G | G | VG | VG | G | VG | EE | G |
| **DM2** | Alt2 | VG | VG | G | VG | G | G | VG | EE | G | G | G | G |
| **DM2** | Alt3 | G | G | G | G | VG | VG | EE | G | G | L | L | L |
| **DM3** | Alt1 | G | VG | F | EE | G | G | VG | VG | G | VG | EE | G |
| **DM3** | Alt2 | VG | VG | G | VG | G | G | VG | EE | G | G | G | G |
| **DM3** | Alt3 | G | G | G | G | VG | VG | EE | G | G | L | L | L |
| **DM4** | Alt1 | G | VG | F | EE | G | G | VG | VG | G | VG | EE | G |
| **DM4** | Alt2 | VG | VG | G | VG | G | G | VG | EE | G | G | G | G |
| **DM4** | Alt3 | G | G | G | G | VG | VG | EE | G | G | L | L | L |
| **DM5** | Alt1 | G | VG | F | EE | G | G | VG | VG | G | VG | EE | G |
| **DM5** | Alt2 | VG | VG | G | VG | G | G | VG | EE | G | G | G | G |
| **DM5** | Alt3 | G | G | G | G | VG | VG | EE | G | G | L | L | L |
| **DM6** | Alt1 | G | VG | F | EE | G | G | VG | VG | G | VG | EE | G |
| **DM6** | Alt2 | VG | VG | G | VG | G | G | VG | EE | G | G | G | G |
| **DM6** | Alt3 | G | G | G | G | VG | VG | EE | G | G | L | L | L |
| **DM7** | Alt1 | G | VG | F | EE | G | G | VG | VG | G | VG | EE | G |
| **DM7** | Alt2 | VG | VG | G | VG | G | G | VG | EE | G | G | G | G |
| **DM7** | Alt3 | G | G | G | G | VG | VG | EE | G | G | L | L | L |

Moving on to Table 6, we present the integrated decision matrix, a key outcome of the MULTIMOORA-mGqNN methodology. This matrix consolidates the assessments of alternatives by integrating mGqNNs, providing a comprehensive view of their performance across the identified criteria. Table 6 encapsulates the complexities of SSCM evaluations, offering decision-makers a structured and data-driven foundation for their choices.

Table 6: Integrated Decision Matrix for SSCM Alternatives

| | **E1** | **E2** | **E3** | **S1** | **S2** | **S3** |
|---|---|---|---|---|---|---|
| **Alt1** | 0.316, 0.777, 0.765 | 0.777, 0.765, 0.411 | 0.765, 0.411, 0.136 | 0.411, 0.136, 0.897 | 0.136, 0.897, 0.068 | 0.897, 0.068, 0.42 |
| **Alt2** | 0.053, 0.033, 0.077 | 0.033, 0.077, 0.764 | 0.077, 0.764, 0.207 | 0.764, 0.207, 0.847 | 0.207, 0.847, 0.929 | 0.847, 0.929, 0.005 |
| **Alt3** | 0.274, 0.173, 0.769 | 0.173, 0.769, 0.121 | 0.769, 0.121, 0.608 | 0.121, 0.608, 0.271 | 0.608, 0.271, 0.78 | 0.271, 0.78, 0.419 |
| | **C1** | **C2** | **C3** | **P1** | **P2** | **P3** |
| **Alt1** | 0.199, 0.351, 0.29 | 0.351, 0.29, 0.219 | 0.29, 0.219, 0.506 | 0.219, 0.506, 0.749 | 0.506, 0.749, 0.27 | 0.749, 0.27, 0.085 |
| **Alt2** | 0.296, 0.523, 0.047 | 0.523, 0.047, 0.345 | 0.047, 0.345, 0.115 | 0.345, 0.115, 0.121 | 0.115, 0.121, 0.27 | 0.121, 0.27, 0.896 |
| **Alt3** | 0.565, 0.246, 0.176 | 0.246, 0.176, 0.226 | 0.176, 0.226, 0.022 | 0.226, 0.022, 0.874 | 0.022, 0.874, 0.912 | 0.874, 0.912, 0.112 |

## V. Conclusions

This paper has introduced an innovative approach for assessing and selecting Sustainable Supply Chain Management (SSCM) strategies and theories using the MULTIMOORA-mGqNN method. By integrating m-generalized q-neutrosophic numbers (mGqNN) into the decision-making framework, we have demonstrated the versatility and robustness of this approach in handling uncertainty and ambiguity inherent to SSCM. Through a systematic evaluation process involving linguistic assessments by multiple decision makers, we have effectively ranked SSCM alternatives and identified optimal strategies to enhance sustainability performance. Moreover, the incorporation of mGqNN in the MULTIMOORA-mGqNN method allows for comprehensive analysis and decision-making that considers various dimensions of SSCM, making it a valuable tool for researchers and practitioners striving to navigate the complex landscape of sustainability in supply chains. In the era of growing environmental, social, and economic challenges, this paper contributes to the ongoing discourse on sustainable supply chain management by providing a methodological framework that can enhance decision-making processes. As sustainability continues to be a focal point in supply chain strategies, the MULTIMOORA-mGqNN method offers a promising avenue for addressing the multifaceted complexities of SSCM, fostering informed choices, and ultimately facilitating the transition towards more sustainable and responsible supply chains.

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# Neutrosophic $\varkappa$-structures in an $AG$-groupoid

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**Abstract**. An $AG$-groupoid is the midway between commutative semigroup and groupoid. The core structure of Flock theory is an $AG$-groupoid, which focuses on motion replication and distance optimization and has numerous applications in physics and biology. Unfortunately, in many cases, modelling real-world problems in domains like computer science, operations research, artificial intelligence, control engineering, and robotics can be risky. Different theories, such as fuzzy sets, intuitionistic fuzzy sets, probability, soft sets, neutrosophic sets, and others, have been created to deal with similar situations. In this paper, We define the notions of neutrosophic $\varkappa$-ideal structures in an $AG$-groupoid and investigate their properties. We also obtain equivalent assertion of neutrosophic $\varkappa$-ideals and product of neutrosophic $\varkappa$-structures in $AG$-groupoid.

**Keywords:** AG-groupoid;neutrosophic $\varkappa$-structures; ideals; neutrosophic $\varkappa$-ideals; neutrosophic $\varkappa$-interior ideals.

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## 1. **Introduction**

In [1], Zadeh pioneered the fuzzy set theory to model imprecise ideas in the globe. Atanassov expanded fuzzy set theory principles and termed it Intuitionistic fuzzy set in [2]. In his opinion, there are two types of degrees of freedom in an universe: non-membership in a specific subset and membership in a vague subset. In [3], Rosenfeld proposed the notion of fuzziness in groups and produced a number of results. Recently, several authors studied their research in this field, and similar notions are used in variety of algebraic structures, including semigroups, semiring, ordered semigroups, rings (refer, [4] - [14], [18]- [21]).

To deal with the uncertainty that exists everywhere, Smarandache suggested the notions of neutrosophic sets in [15]. It's a combination of fuzzy sets and intuitionistic fuzzy sets that's been generalized. Neutrosophic sets are defined using these three properties, which include membership functions for truth ($T$), indeterminacy ($I$), and falsity ($F$). These sets can be used in a variety of fields to deal with the difficulties that result from ambiguous data. The relative and absolute membership functions can be distinguished by a neutrosophic set. Smarandache employed neutrosophic sets in non-standard analysis, such as control theory, decision making theory, sports decision (winning/losing/tie), and so on.

In BCK-algebra, Muhiuddin et al. discovered an association between $(\epsilon, \epsilon)$-neutrosophic subalgebra and $(\epsilon, \epsilon)$-neutrosophic ideal in [16], and Muhiuddin et al. created and investigated neutrosophic implicative $\varkappa$-ideal in [17]. Additionally, the connection between several neutrosophic implicative $\varkappa$-ideals were examined.

In semigroup, neutrosophic $\varkappa$-subsemigroup and the $\epsilon$-neutrosophic $\varkappa$-subsemigroup were defined and their different features were covered in [18] by Khan et al. We examined the properties of various neutrosophic $\varkappa$-structure notions, namely neutrosophic $\varkappa$-ideal structures in a semigroup, as inspiration from [18]. A neutrosophic $\varkappa$-ideals in a semigroup were suggested by B. Elavarasan et al. in [19] and different features were achieved. The comparable claims for the typical neutrosophic $\varkappa$-structure were also given.

Porselvi et al. studied a number of characteristics of the neutrosophic $\varkappa$-bi-ideal in a semigroup in [20], and neutrosophic $\varkappa$-interior ideal in [21]. We have established equivalent claims for regular semigroup. In [22], Elavarasan et al. presented and studied neutrosophic $\varkappa$-filters in semigroups. In [23], Muhiuddin and others proposed the concepts of neutrosophic $\varkappa$-structures in ordered semigroup, and examined their properties. Smarandache proposed neutrosophic topologies in [26], Runu Dhar studied compactness and neutrosophic topological space in [27], Sudeep Dey et al. presented neutrosophic composite relation in [28].

We present the ideas of neutrosophic $\varkappa$-ideal structures in an $AG$-groupoid in this paper. We prove that the product of two neutrosophic $\varkappa$-right-ideal is a neutrosophic $\varkappa$-bi-ideal, and neutrosophic $\varkappa$-right-ideal is equivalent to neutrosophic $\varkappa$-interior-ideal, under certain condition.

## 2. **Preliminaries**

Unless otherwise specified, $\mathscr{M}$ denotes an $AG$-groupoid throughout this paper. Here is a glossary of the definitions we have already used for your perusal.

For $M_1, M_2 \subseteq \mathscr{M}$, we denote $(M_1] = \{k \in \mathscr{M} \ : \ k \leq m$ for some $m \in M_1\}$ and $M_1M_2 = \{k_1k_2$ : for all $k_1 \in M_1$ and $k_2 \in M_2\}$. Following [24] and [25], an $AG$-groupoid, $\mathscr{M}$, is a groupoid whose elements hold the left invertive law: $(m_1m_2)k_3 = (k_3m_2)m_1$ for all $m_1, m_2, k_3 \in \mathscr{M}$. An $AG$-groupoid structure lies between a commutative semigroup and a groupoid. In $\mathscr{M}$, the medial law $(m_1m_2)(k_3k_4) = (m_1k_3)(m_2k_4)$ for all $m_1, m_2, k_3, k_4 \in \mathscr{M}$ holds. If there is an element $e \in \mathscr{M} \ni em = m \ \forall \ m \in \mathscr{M}$, then $e$ is the left identity. If $\mathscr{M}$ has a right identity, then $\mathscr{M}$ is said to be commutative monoid. If $\mathscr{M}$ is having a left identity, then $(m_1m_2)(k_3k_4) = (k_4k_3)(m_2m_1)$ holds for all $m_1, m_2, k_3, k_4 \in \mathscr{M}$. An element $m \in \mathscr{M}$ is said to be idempotent if $m^2 = m$.

Let $\mathscr{M}$ be an $AG$-groupoid and $\phi \neq M \subseteq \mathscr{M}$. Then $M$ is called a $AG$-subgroupoid of $\mathscr{M}$ (see [24]) if $M^2 \subseteq M$. A subset $M \neq \phi$ in $\mathscr{M}$ is called a left(respectively, right) ideal if $\mathscr{M}M \subseteq M$ (respectively, $M\mathscr{M} \subseteq M$), and $M$ is said to be an ideal if it is both a right and a left ideal of $\mathscr{M}$. A subset $M \neq \phi$ in $\mathscr{M}$ is said to be an interior ideal if $(\mathscr{M}M)\mathscr{M} \subseteq M$. A subset $M \neq \phi$ in $\mathscr{M}$ is known as bi-ideal if $(M\mathscr{M})M \subseteq M$. A subset $M \neq \phi$ in $\mathscr{M}$ is said to be a idempotent if $MM = M$.

Let $\mathscr{M}$ be an $AG$-groupoid. Then a function $\nu : \mathscr{M} \to [-1, 0]$ is the $\varkappa$-function on $\mathscr{M}$, and the set of all the $\varkappa$-functions is given by $F(\mathscr{M}, [-1, 0])$. A $\varkappa$-structure is an ordered pair $(\mathscr{M}, h)$ of $\mathscr{M}$ and an $\varkappa$-function $\nu$ on $\mathscr{M}$.

**Definition 2.1.** Let $\mathscr{M}$ be an $AG$-groupoid. A neutrosophic $\varkappa$-structure in $\mathscr{M}$ is given in the form:

$$\mathscr{M}_\zeta := \frac{\mathscr{M}}{(T_\zeta, I_\zeta, F_\zeta)} = \left\{ \frac{k}{(T_\zeta(k), I_\zeta(k), F_\zeta(k))} \mid k \in \mathscr{M} \right\},$$

where $T_\zeta, F_\zeta$ and $I_\zeta$ are the negative truth, negative falsity and negative indeterminacy membership functions respectively in $\mathscr{M}$ ($\varkappa$-functions). Clearly, $-3 \leq T_\zeta(m) + I_\zeta(m) + F_\zeta(m) \leq 0$ $\forall \ m \in \mathscr{M}$.

Throughout this section, we assume that $\mathscr{M}_\zeta$ and $\mathscr{M}_\xi$ are neutrosophic $\varkappa$-structures in $\mathscr{M}$, unless otherwise stated.

**Notation 1.** *We denote the set of*

*(i) neutrosophic $\varkappa$-left ideal by $\mathscr{M}_l$,*

*(ii) neutrosophic $\varkappa$-right ideal by $\mathscr{M}_r$,*

*(iii) neutrosophic $\varkappa$-ideal by $\mathscr{M}_i$,*

*(iv) neutrosophic $\varkappa$-bi-ideal by $\mathscr{M}_b$,*

*(v) neutrosophic $\varkappa$-interior ideal by $\mathscr{M}_n$,*

*(vi) neutrosophic $\varkappa$-AG-subgroupoid by $\mathscr{M}_s$.*

*(vii) neutrosophic $\varkappa$-idempotent by $\mathscr{M}_d$.*

**Definition 2.2.** Let $\mathscr{M}_\zeta \in \mathscr{M}$. Then $\mathscr{M}_\zeta \in \mathscr{M}_s$ provided the below condition is valid:

$$(\forall m_1, m_2 \in \mathscr{M}) \begin{pmatrix} T_\zeta(m_1m_2) \leq T_\zeta(m_1) \vee T_\zeta(m_2) \\ I_\zeta(m_1m_2) \geq I_\zeta(m_1) \wedge I_\zeta(m_2) \\ F_\zeta(m_1m_2) \leq F_\zeta(m_1) \vee F_\zeta(m_2) \end{pmatrix}.$$

Let $\nu, \gamma, \omega \in [-1, 0]$. Consider the sets:

$$T_\zeta^\nu = \{m_1 \in \mathscr{M} \mid T_\zeta(m_1) \leq \nu\},$$
$$I_\zeta^\gamma = \{m_1 \in \mathscr{M} \mid I_\zeta(m_1) \geq \gamma\},$$
$$F_\zeta^\omega = \{m_1 \in \mathscr{M} \mid F_\zeta(m_1) \leq \omega\}.$$

The set $\mathscr{M}_\zeta(\nu, \gamma, \omega) := \{m_1 \in \mathscr{M} \mid T_\zeta(m_1) \leq \nu,\ I_\zeta(m_1) \geq \gamma,\ F_\zeta(m_1) \leq \omega\}$ is known as $(\nu, \gamma, \omega)$-level set on $\mathscr{M}_\zeta$. Obviously, $\mathscr{M}_\zeta(\nu, \gamma, \omega) = T_\zeta^\nu \cap I_\zeta^\gamma \cap F_\zeta^\omega$.

**Definition 2.3.** Let $\mathscr{M}_\zeta \in \mathscr{M}$. Then $\mathscr{M}_\zeta \in \mathscr{M}_i$ provided the below conditions are valid:

$$(i)\ (\forall m_1, m_2 \in \mathscr{M}) \begin{pmatrix} T_\zeta(m_1m_2) \leq T_\zeta(m_2) \\ I_\zeta(m_1m_2) \geq I_\zeta(m_2) \\ F_\zeta(m_1m_2) \leq F_\zeta(m_2) \end{pmatrix}.$$

$$(ii)\ (\forall m_1, m_2 \in \mathscr{M}) \begin{pmatrix} T_\zeta(m_1m_2) \leq T_\zeta(m_1) \\ I_\zeta(m_1m_2) \geq I_\zeta(m_1) \\ F_\zeta(m_1m_2) \leq F_\zeta(m_1) \end{pmatrix}.$$

If condition $(i)$ hold, then $\mathscr{M}_\zeta \in \mathscr{M}_l$. If condition $(ii)$ hold, then $\mathscr{M}_\zeta \in \mathscr{M}_r$.

**Definition 2.4.** Let $\mathscr{M}_\zeta \in \mathscr{M}_s$.Then $\mathscr{M}_\zeta \in \mathscr{M}_b$ if the below assertion is valid:

$$(\forall a, k_1, k_2 \in \mathscr{M}) \begin{pmatrix} T_\zeta(k_1ak_2) \leq T_\zeta(k_1) \vee T_\zeta(k_2) \\ I_\zeta(k_1ak_2) \geq I_\zeta(k_1) \wedge I_\zeta(k_2) \\ F_\zeta(k_1ak_2) \leq F_\zeta(k_1) \vee F_\zeta(k_2) \end{pmatrix}.$$

It is obvious that for any $\mathscr{M}_\zeta \in \mathscr{M}_i$, we have $\mathscr{M}_\zeta \in \mathscr{M}_b$. The converse need not be true, as shown by an example.

**Example 2.5.** Suppose $\mathscr{M} := \{x_1, x_2, x_3, x_4, x_5\}$. Then $(\mathscr{M}, .)$ is an $AG$-groupoid as given below:

| $.$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|
| $x_1$ | $x_1$ | $x_4$ | $x_1$ | $x_4$ | $x_4$ |
| $x_2$ | $x_1$ | $x_2$ | $x_1$ | $x_4$ | $x_4$ |
| $x_3$ | $x_1$ | $x_4$ | $x_3$ | $x_4$ | $x_5$ |
| $x_4$ | $x_1$ | $x_4$ | $x_1$ | $x_4$ | $x_4$ |
| $x_5$ | $x_1$ | $x_4$ | $x_3$ | $x_4$ | $x_5$ |

Let

$$\mathscr{M}_\zeta = \left\{ \frac{x_1}{(-0.8,-0.2,-0.6)}, \frac{x_2}{(-0.5,-0.9,-0.1)}, \frac{x_3}{(-0.3,-0.4,-0.5)}, \frac{x_4}{(-0.8,-0.2,-0.6)}, \frac{x_5}{(-0.2,-0.5,-0.1)} \right\}.$$

Then $\mathscr{M}_\zeta \in \mathscr{M}_b$, and $\mathscr{M}_\zeta \notin \mathscr{M}_i$ as $T_{\mathscr{M}}(x_3x_5) = -0.2 > T_{\mathscr{M}}(x_3)$, $I_{\mathscr{M}}(x_3x_5) = -0.5 < I_{\mathscr{M}}(x_3)$ and $F_{\mathscr{M}}(x_3x_5) = -0.1 > F_{\mathscr{M}}(x_3)$.

**Definition 2.6.** Let $\mathscr{M}_\zeta \in \mathscr{M}_s$.Then $\mathscr{M}_\zeta \in \mathscr{M}_n$ provided the below assertion is valid:

$$(\forall a, m_1, m_2 \in \mathscr{M}) \left( \begin{array}{c} T_\zeta(m_1 a m_2) \leq T_\zeta(a) \\ I_\zeta(m_1 a m_2) \geq I_\zeta(a) \\ F_\zeta(m_1 a m_2) \leq F_\zeta(a) \end{array} \right).$$

It is obvious that for any $\mathscr{M}_\zeta \in \mathscr{M}_i$, we have $\mathscr{M}_\zeta \in \mathscr{M}_n$. The converse is not true, as shown by an example.

**Example 2.7.** Let $\mathscr{M}$ be the collection of all positive integers with 0 except 1. Then under usual multiplication, $\mathscr{M}$ is an $AG$-groupoid.

Let

$$\mathscr{M}_\zeta = \left\{ \frac{0}{(-0.8,-0.2,-0.8)}, \frac{2}{(-0.3,-0.4,-0.5)}, \frac{5}{(-0.5,-0.6,-0.6)}, \frac{10}{(-0.2,-0.5,-0.3)}, \frac{otherwise}{(-0.8,-0.2,-0.4)} \right\}.$$

Then $\mathscr{M}_\zeta \in \mathscr{M}_n$, and $\mathscr{M}_\zeta \notin \mathscr{M}_i$, as $T_{\mathscr{M}}(2.5) = -0.2 > T_{\mathscr{M}}(2)$ and $T_{\mathscr{M}}(2.5) = -0.2 > T_{\mathscr{M}}(5)$.

**Definition 2.8.** For any $\mathscr{L} \subseteq \mathscr{M}$, the characteristic neutrosophic $\chi$-structure in $\mathscr{M}$ is refered as

$$\chi_{\mathscr{L}}(\mathscr{M}_\zeta) = \frac{\mathscr{M}}{(\chi_{\mathscr{L}}(T)_\zeta, \chi_{\mathscr{L}}(I)_\zeta, \chi_{\mathscr{L}}(F)_\zeta)}$$

where

$$\chi_{\mathscr{L}}(T)_\zeta : \mathscr{M} \to [-1, 0],\ m_1 \mapsto \begin{cases} -1 & if\ m_1 \in \mathscr{L} \\ 0 & otherwise, \end{cases}$$

$$\chi_{\mathscr{L}}(I)_\zeta : \mathscr{M} \to [-1, 0],\ m_1 \mapsto \begin{cases} 0 & if\ m_1 \in \mathscr{L} \\ -1 & otherwise, \end{cases}$$

$$\chi_{\mathscr{L}}(F)_\zeta : \mathscr{M} \to [-1, 0],\ m_1 \mapsto \begin{cases} -1 & if\ m_1 \in \mathscr{L} \\ 0 & otherwise. \end{cases}$$

**Definition 2.9.** Let $\mathscr{M}_\xi := \frac{\mathscr{M}}{(T_\xi, I_\xi, F_\xi)} \in \mathscr{M}$ and $\mathscr{M}_\zeta := \frac{\mathscr{M}}{(T_\zeta, I_\zeta, F_\zeta)} \in \mathscr{M}$. Then

(i) $\mathscr{M}_\xi$ is said to be a neutrosophic $\varkappa$-substructure in $\mathscr{M}_\zeta$, denote by $\mathscr{M}_\zeta \subseteq \mathscr{M}_\xi$, if $T_\zeta(m_1) \geq T_\xi(m_1), I_\zeta(m_1) \leq I_\xi(m_1), F_\zeta(m_1) \geq F_\xi(m_1)$ for all $m_1 \in \mathscr{M}$.

If $\mathscr{M}_\xi \subseteq \mathscr{M}_\zeta$ and $\mathscr{M}_\zeta \subseteq \mathscr{M}_\xi$, then we write $\mathscr{M}_\xi = \mathscr{M}_\zeta$.

(ii) The union of $\mathscr{M}_\xi$ and $\mathscr{M}_\zeta$ over $\mathscr{M}$ is described as

$$\mathscr{M}_\xi \cup \mathscr{M}_\zeta = \mathscr{M}_{\xi\cup\zeta} = (\mathscr{M}; T_{\xi\cup\zeta}, I_{\xi\cup\zeta}, F_{\xi\cup\zeta}),$$

where $\forall\ m_1 \in \mathscr{M}$,

$$(T_\xi \cup T_\zeta)(m_1) = T_{\xi\cup\zeta}(m_1) = T_\xi(m_1) \wedge T_\zeta(m_1),$$
$$(I_\xi \cup I_\zeta)(m_1) = I_{\xi\cap\zeta}(m_1) = I_\xi(m_1) \vee I_\zeta(m_1),$$
$$(F_\xi \cup F_\zeta)(m_1) = F_{\xi\cup\zeta}(m_1) = F_\xi(m_1) \wedge F_\zeta(m_1).$$

(iii) The intersection of $\mathscr{M}_\xi$ and $\mathscr{M}_\zeta$ over $\mathscr{M}$ is described as

$$\mathscr{M}_\xi \cap \mathscr{M}_\zeta = \mathscr{M}_{\xi\cap\zeta} = (\mathscr{M}; T_{\xi\cap\zeta}, I_{\xi\cap\zeta}, F_{\xi\cap\zeta}),$$

$$\text{where } \forall\ m_1 \in \mathscr{M}, \quad \begin{array}{l} (T_\xi \cap T_\zeta)(m_1) = T_{\xi\cap\zeta}(m_1) = T_\xi(m_1) \vee T_\zeta(m_1), \\ (I_\xi \cap I_\zeta)(m_1) = I_{\xi\cap\zeta}(m_1) = I_\xi(m_1) \wedge I_\zeta(m_1), \\ (F_\xi \cap F_\zeta)(m_1) = F_{\xi\cap\zeta}(m_1) = F_\xi(m_1) \vee F_\zeta(m_1). \end{array} .$$

## 3. Main Results

We present some characteristics of neutrosophic $\varkappa$-ideal structures in an $AG$-groupoid $\mathscr{M}$. In $\mathscr{M}$, neutrosophic $\varkappa$-ideals are clearly neutrosophic $\varkappa$-interior ideals, but the converse is true under certain conditions.

**Theorem 3.1.** *For any $\mathscr{M}$, $(\mathscr{M}_\xi, \odot)$ is an AG-groupoid.*

*Proof.* It is clear that $(\mathscr{M}_\xi, \odot)$ is closed. Let $\mathscr{M}_\xi, \mathscr{M}_\zeta, \mathscr{M}_\mathscr{R} \in \mathscr{M}$. Then for any $t \in \mathscr{M}$,

$$\begin{aligned} ((T_\xi \circ T_\zeta) \circ T_\mathscr{R})(t) &= \{(T_\xi \circ T_\zeta)(y) \vee T_\mathscr{R}(z)\} \\ &= \bigwedge_{t=yz} \{ \bigwedge_{y=rs} \{T_\xi(r) \vee T_\zeta(s)\} \vee T_\mathscr{R}(z)\} \\ &= \bigwedge_{t=(rs)z} \{T_\xi(r) \vee T_\zeta(s) \vee T_\mathscr{R}(z)\} \\ &= \bigwedge_{t=(zs)r} \{T_\mathscr{R}(z) \vee T_\zeta(s) \vee T_\xi(r)\} \\ &= \bigwedge_{t=ur} \{(T_\mathscr{R} \circ T_\zeta)(u) \vee T_\xi(r)\} \\ &= ((T_\mathscr{R} \circ T_\zeta) \circ T_\xi)(t), \\ ((I_\xi \circ I_\zeta) \circ I_\mathscr{R})(t) &= \bigvee_{t=yz} \{(I_\xi \circ I_\zeta)(y) \wedge I_\mathscr{R}(z)\} \\ &= \bigvee_{t=yz} \{ \bigvee_{y=rs} \{I_\xi(r) \wedge I_\zeta(s)\} \wedge I_\mathscr{R}(z)\} \\ &= \bigvee_{t=(rs)z} \{I_\xi(r) \wedge I_\zeta(s) \wedge I_\mathscr{R}(z)\} \\ &= \bigvee_{t=(zs)r} \{I_\mathscr{R}(z) \wedge I_\zeta(s) \wedge I_\xi(r)\} \\ &= \bigwedge_{t=ur} \{(I_\mathscr{R} \circ I_\zeta)(u) \wedge I_\xi(r)\} \\ &= ((I_\mathscr{R} \circ I_\zeta) \circ I_\xi)(t), \end{aligned}$$

$$\begin{aligned}((F_\xi \circ F_\zeta) \circ F_{\mathscr{R}})(t) &= \bigwedge_{t=yz} \{(F_\xi \circ F_\zeta)(y) \vee F_{\mathscr{R}}(z)\} \\ &= \bigwedge_{t=yz} \{ \bigwedge_{y=rs} \{F_\xi(r) \vee F_\zeta(s)\} \vee F_{\mathscr{R}}(z)\} \\ &= \bigwedge_{t=(rs)z} \{F_\xi(r) \vee F_\zeta(s) \vee F_{\mathscr{R}}(z)\} \\ &= \bigwedge_{t=(zs)r} \{F_{\mathscr{R}}(z) \vee F_\zeta(s) \vee F_\xi(r)\} \\ &= \bigwedge_{t=ur} \{(F_{\mathscr{R}} \circ F_\zeta)(u) \vee F_\xi(r)\} \\ &= ((F_{\mathscr{R}} \circ F_\zeta) \circ F_\xi)(t).\end{aligned}$$

Therefore $(\mathscr{M}_\xi, \odot)$ is an $AG$-groupoid. □

**Corollary 3.2.** *For any $\mathscr{M}_\xi, \mathscr{M}_\zeta, \mathscr{M}_{\mathscr{R}}, \mathscr{M}_{\mathscr{Q}} \in \mathscr{M}$, $(\mathscr{M}_\xi \odot \mathscr{M}_\zeta) \odot (\mathscr{M}_{\mathscr{R}} \odot \mathscr{M}_{\mathscr{Q}}) = (\mathscr{M}_\xi \odot \mathscr{M}_{\mathscr{R}}) \odot (\mathscr{M}_\zeta \odot \mathscr{M}_{\mathscr{Q}})$.*

*Proof.* Let $\mathscr{M}_\xi, \mathscr{M}_\zeta, \mathscr{M}_{\mathscr{R}}, \mathscr{M}_{\mathscr{Q}} \in \mathscr{M}$. Then

$(T_\xi \circ T_\zeta) \circ (T_{\mathscr{R}} \circ T_{\mathscr{Q}}) = ((T_{\mathscr{R}} \circ T_{\mathscr{Q}}) \circ T_\zeta) \circ T_\xi) = ((T_\zeta \circ T_{\mathscr{Q}}) \circ T_{\mathscr{R}}) \circ T_\xi) = (T_\xi \circ T_{\mathscr{R}}) \circ (T_\zeta \circ T_{\mathscr{Q}})$,

$(F_\xi \circ F_\zeta) \circ (F_{\mathscr{R}} \circ F_{\mathscr{Q}}) = ((F_{\mathscr{R}} \circ F_{\mathscr{Q}}) \circ F_\zeta) \circ F_\xi) = ((F_\zeta \circ F_{\mathscr{Q}}) \circ F_{\mathscr{R}}) \circ F_\xi) = (F_\xi \circ F_{\mathscr{R}}) \circ (F_\zeta \circ F_{\mathscr{Q}})$

and

$(I_\xi \circ I_\zeta) \circ (I_{\mathscr{R}} \circ I_{\mathscr{Q}}) = ((I_{\mathscr{R}} \circ I_{\mathscr{Q}}) \circ I_\zeta) \circ I_\xi) = ((I_\zeta \circ I_{\mathscr{Q}}) \circ I_{\mathscr{R}}) \circ I_\xi) = (I_\xi \circ I_{\mathscr{R}}) \circ (I_\zeta \circ I_{\mathscr{Q}})$.

Hence $(\mathscr{M}_\xi \odot \mathscr{M}_\zeta) \odot (\mathscr{M}_{\mathscr{R}} \odot \mathscr{M}_{\mathscr{Q}}) = (\mathscr{M}_\xi \odot \mathscr{M}_{\mathscr{R}}) \odot (\mathscr{M}_\zeta \odot \mathscr{M}_{\mathscr{Q}})$. □

**Theorem 3.3.** *If $\mathscr{M}$ has left identity, then for any $\mathscr{M}_\xi, \mathscr{M}_\zeta, \mathscr{M}_{\mathscr{R}}, \mathscr{M}_{\mathscr{Q}} \in \mathscr{M}$, we have the following:*

*(i) $\mathscr{M}_\xi \odot (\mathscr{M}_\zeta \odot \mathscr{M}_{\mathscr{R}}) = \mathscr{M}_\zeta \odot (\mathscr{M}_\xi \odot \mathscr{M}_{\mathscr{R}})$,*

*(ii) $(\mathscr{M}_\xi \odot \mathscr{M}_\zeta) \odot (\mathscr{M}_{\mathscr{R}} \odot \mathscr{M}_{\mathscr{Q}}) = (\mathscr{M}_{\mathscr{Q}} \odot \mathscr{M}_{\mathscr{R}}) \odot (\mathscr{M}_\zeta \odot \mathscr{M}_\xi)$.*

*Proof.* (i) Let $m \in \mathscr{M}$. If $m \neq xy$ for any $x, y \in \mathscr{M}$, then

$$(T_\xi \circ (T_\zeta \circ T_{\mathscr{R}}))(m) = 0 = (T_\zeta \circ (T_\xi \circ T_{\mathscr{R}}))(m),$$

$$(I_\xi \circ (I_\zeta \circ I_{\mathscr{R}}))(m) = -1 = (I_\zeta \circ (I_\xi \circ I_{\mathscr{R}}))(m),$$

$$(F_\xi \circ (F_\zeta \circ F_{\mathscr{R}}))(m) = 0 = (F_\zeta \circ (F_\xi \circ F_{\mathscr{R}}))(m).$$

Suppose $m = yz$ for $y, z \in \mathscr{M}$. Then

$$
\begin{aligned}
(T_\xi \circ (T_\zeta \circ T_{\mathscr{R}}))(m) &= \bigwedge_{m=yz} \{T_\xi(y) \vee (T_\zeta \circ T_{\mathscr{R}})(z)\} \\
&= \bigwedge_{m=yz} \{T_\xi(y) \vee \bigwedge_{z=rs} \{T_\zeta(r) \vee T_{\mathscr{R}}(s)\}\} \\
&= \bigwedge_{m=y(rs)} \{T_\xi(y) \vee T_\zeta(r) \vee T_{\mathscr{R}}(s)\} \\
&= \bigwedge_{m=r(ys)} \{T_\zeta(r) \vee T_\xi(y) \vee T_{\mathscr{R}}(s)\} \\
&= \bigwedge_{m=rp} \{T_\zeta(r) \vee \bigwedge_{p=ys} \{T_\xi(y) \vee T_{\mathscr{R}}(s)\}\} \\
&= \bigwedge_{m=rp} \{T_\zeta(r) \vee (T_\xi \circ T_{\mathscr{R}})(p)\} \\
&= (T_\zeta \circ (T_\xi \circ T_{\mathscr{R}}))(m), \\
(I_\xi \circ (I_\zeta \circ I_{\mathscr{R}}))(m) &= \bigvee_{m=yz} \{I_\xi(y) \wedge (I_\zeta \circ I_{\mathscr{R}})(z)\} \\
&= \bigvee_{m=yz} \{I_\xi(y) \wedge \bigvee_{z=rs} \{I_\zeta(r) \wedge I_{\mathscr{R}}(s)\}\} \\
&= \bigvee_{m=y(rs)} \{I_\xi(y) \wedge I_\zeta(r) \wedge I_{\mathscr{R}}(s)\} \\
&= \bigvee_{m=r(ys)} \{I_\zeta(r) \wedge I_\xi(y) \wedge I_{\mathscr{R}}(s)\} \\
&= \bigvee_{m=rp} \{I_\zeta(r) \wedge \bigvee_{p=ys} \{I_\xi(y) \wedge I_{\mathscr{R}}(s)\}\} \\
&= \bigvee_{m=rp} \{I_\zeta(r) \wedge (I_\xi \circ I_{\mathscr{R}})(p)\} \\
&= (I_\zeta \circ (I_\xi \circ I_{\mathscr{R}}))(m), \\
(F_\xi \circ (F_\zeta \circ F_{\mathscr{R}}))(m) &= \bigwedge_{m=yz} \{F_\xi(y) \vee (F_\zeta \circ F_{\mathscr{R}})(z)\} \\
&= \bigwedge_{m=yz} \{F_\xi(y) \vee \bigwedge_{z=rs} \{F_\zeta(r) \vee F_{\mathscr{R}}(s)\}\} \\
&= \bigwedge_{m=y(rs)} \{F_\xi(y) \vee F_\zeta(r) \vee F_{\mathscr{R}}(s)\} \\
&= \bigwedge_{m=r(ys)} \{F_\zeta(r) \vee F_\xi(y) \vee F_{\mathscr{R}}(s)\} \\
&= \bigwedge_{m=rp} \{F_\zeta(r) \vee \bigwedge_{p=ys} \{F_\xi(y) \vee F_{\mathscr{R}}(s)\}\} \\
&= \bigwedge_{m=rp} \{F_\zeta(r) \vee (F_\xi \circ F_{\mathscr{R}})(p)\} \\
&= (F_\zeta \circ (F_\xi \circ F_{\mathscr{R}}))(m).
\end{aligned}
$$

Therefore $\mathscr{M}_\xi \odot (\mathscr{M}_\zeta \odot \mathscr{M}_{\mathscr{R}}) = \mathscr{M}_\zeta \odot (\mathscr{M}_\xi \odot \mathscr{M}_{\mathscr{R}})$.

(ii) Let $m \in \mathscr{M}$. If $m \neq xy$ for any $x, y \in \mathscr{M}$, then

$$((T_\xi \circ T_\zeta) \circ (T_\mathscr{R} \circ T_\mathscr{Q}))(m) = 1 = ((T_\mathscr{Q} \circ T_\mathscr{R}) \circ (T_\zeta \circ T_\xi))(m).$$

Suppose $m = yz$ for any $y, z \in \mathscr{M}$. Then

$$\begin{aligned}
((T_\xi \circ T_\zeta) \circ (T_\mathscr{R} \circ T_\mathscr{Q}))(m) &= \bigwedge_{m=yz} \{(T_\xi \circ T_\zeta)(y) \vee (T_\mathscr{R} \circ T_\mathscr{Q})(z)\} \\
&= \bigwedge_{m=yz} \{ \bigwedge_{y=pq} \{T_\xi(p) \vee T_\zeta(q)\} \vee \bigwedge_{z=rs} \{T_\mathscr{R}(r) \vee T_\mathscr{Q}(s)\}\} \\
&= \bigwedge_{m=(pq)(rs)} \{T_\xi(p) \vee T_\zeta(q) \vee T_\mathscr{R}(r) \vee T_\mathscr{Q}(s)\} \\
&= \bigwedge_{m=(sr)(qp)} \{T_\mathscr{Q}(s) \vee T_\mathscr{R}(r) \vee T_\zeta(q) \vee T_\xi(p)\} \\
&= \bigwedge_{m=vw} \{ \bigwedge_{v=sr} \{T_\mathscr{Q}(s) \vee T_\mathscr{R}(r)\} \vee \bigwedge_{w=qp} \{T_\zeta(q) \vee T_\xi(p)\}\} \\
&= \bigwedge_{m=vw} \{(T_\mathscr{Q} \circ T_\mathscr{R})(v) \vee (T_\zeta \circ T_\xi)(w)\} \\
&= ((T_\mathscr{Q} \circ T_\mathscr{R}) \circ (T_\zeta \circ T_\xi))(m), \\
((I_\xi \circ I_\zeta) \circ (I_\mathscr{R} \circ I_\mathscr{Q}))(m) &= \bigvee_{m=yz} \{(I_\xi \circ I_\zeta)(y) \wedge (I_\mathscr{R} \circ I_\mathscr{Q})(z)\} \\
&= \bigvee_{m=yz} \{ \bigvee_{y=pq} \{I_\xi(p) \wedge I_\zeta(q)\} \wedge \bigvee_{z=rs} \{I_\mathscr{R}(r) \wedge I_\mathscr{Q}(s)\}\} \\
&= \bigvee_{m=(pq)(rs)} \{I_\xi(p) \wedge I_\zeta(q) \wedge I_\mathscr{R}(r) \wedge I_\mathscr{Q}(s)\} \\
&= \bigvee_{m=(sr)(qp)} \{I_\mathscr{Q}(s) \wedge I_\mathscr{R}(r) \wedge I_\zeta(q) \wedge I_\xi(p)\} \\
&= \bigvee_{m=vw} \{ \bigvee_{v=sr} \{I_\mathscr{Q}(s) \wedge I_\mathscr{R}(r)\} \wedge \bigvee_{w=qp} \{I_\zeta(q) \wedge I_\xi(p)\}\} \\
&= \bigvee_{m=vw} \{(I_\mathscr{Q} \circ I_\mathscr{R})(v) \wedge (I_\zeta \circ I_\xi)(w)\} \\
&= ((I_\mathscr{Q} \circ I_\mathscr{R}) \circ (I_\zeta \circ I_\xi))(m), \\
((F_\xi \circ F_\zeta) \circ (F_\mathscr{R} \circ F_\mathscr{Q}))(m) &= \bigwedge_{m=yz} \{(F_\xi \circ F_\zeta)(y) \vee (F_\mathscr{R} \circ F_\mathscr{Q})(z)\} \\
&= \bigwedge_{m=yz} \{ \bigwedge_{y=pq} \{F_\xi(p) \vee F_\zeta(q)\} \vee \bigwedge_{z=rs} \{F_\mathscr{R}(r) \vee F_\mathscr{Q}(s)\}\} \\
&= \bigwedge_{m=(pq)(rs)} \{F_\xi(p) \vee F_\zeta(q) \vee F_\mathscr{R}(r) \vee F_\mathscr{Q}(s)\} \\
&= \bigwedge_{m=(sr)(qp)} \{F_\mathscr{Q}(s) \vee F_\mathscr{R}(r) \vee F_\zeta(q) \vee F_\xi(p)\} \\
&= \bigwedge_{m=vw} \{ \bigwedge_{v=sr} \{F_\mathscr{Q}(s) \vee F_\mathscr{R}(r)\} \vee \bigwedge_{w=qp} \{F_\zeta(q) \vee F_\xi(p)\}\}
\end{aligned}$$

$$= \bigwedge_{m=vw} \{(F_{\mathscr{Q}} \circ F_{\mathscr{R}})(v) \vee (F_\zeta \circ F_\xi)(w)\}$$
$$= ((F_{\mathscr{Q}} \circ F_{\mathscr{R}}) \circ (F_\zeta \circ F_\xi))(m).$$

Therefore $(\mathscr{M}_\xi \odot \mathscr{M}_\zeta) \odot (\mathscr{M}_{\mathscr{R}} \odot \mathscr{M}_{\mathscr{Q}}) = (\mathscr{M}_{\mathscr{Q}} \odot \mathscr{M}_{\mathscr{R}}) \odot (\mathscr{M}_\zeta \odot \mathscr{M}_\xi)$. □

**Theorem 3.4.** *Let $\mathscr{M}_\xi \in \mathscr{M}$. Then the listed conditions hold:*

*(i) $\mathscr{M}_\xi \in \mathscr{M}_s \Leftrightarrow \mathscr{M}_\xi \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$.*

*(ii) $\mathscr{M}_\xi \in \mathscr{M}_l \Leftrightarrow \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$ for any $\mathscr{M}_\zeta \in \mathscr{M}$.*

*(iii) $\mathscr{M}_\xi \in \mathscr{M}_r \Leftrightarrow \mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \subseteq \mathscr{M}_\xi$ for any $\mathscr{M}_\zeta \in \mathscr{M}$.*

*(iv) $\mathscr{M}_\xi \in \mathscr{M}_i \Leftrightarrow \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$ and $\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \subseteq \mathscr{M}_\xi$ for any $\mathscr{M}_\zeta \in \mathscr{M}$.*

*Proof.* (i) Assume $\mathscr{M}_\xi \in \mathscr{M}_s$. Now, for any $k \in \mathscr{M}$,

$$(T_\xi \circ T_\xi)(k) = \bigwedge_{k=k_1k_2} \{T_\xi(k_1) \vee T_\xi(k_2)\} \geq \bigwedge_{k=k_1k_2} T_\xi(k_1k_2) = T_\xi(k),$$
$$(I_\xi \circ I_\xi)(k) = \bigvee_{k=k_1k_2} \{I_\xi(k_1) \wedge I_\xi(k_2)\} \leq \bigvee_{k=k_1k_2} I_\xi(k_1k_2) = I_\xi(k),$$
$$(F_\xi \circ F_\xi)(k) = \bigwedge_{k=k_1k_2} \{F_\xi(k_1) \vee F_\xi(k_2)\} \geq \bigwedge_{k=k_1k_2} F_\xi(k_1k_2) = F_\xi(k).$$

So $\mathscr{M}_\xi \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$.

Conversely, assume $\mathscr{M}_\xi \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$. Now, for any $k_1, k_2 \in \mathscr{M}$,

$$T_\xi(k_1k_2) \leq (T_\xi \circ T_\xi)(k_1k_2) = \bigwedge_{k_1k_2} \{T_\xi(k_1) \vee T_\xi(k_2)\} \leq T_\xi(k_1) \vee T_\xi(k_2),$$
$$I_\xi(k_1k_2) \geq (I_\xi \circ I_\xi)(k_1k_2) = \bigvee_{k_1k_2} \{I_\xi(k_1) \wedge I_\xi(k_2)\} \geq I_\xi(k_1) \wedge I_\xi(k_2),$$
$$F_\xi(k_1k_2) \leq (F_\xi \circ F_\xi)(k_1k_2) = \bigwedge_{k_1k_2} \{F_\xi(k_1) \vee F_\xi(k_2)\} \leq F_\xi(k_1) \vee F_\xi(k_2).$$

So $\mathscr{M}_\xi \in \mathscr{M}_s$.

(ii) Assuming $\mathscr{M}_\xi \in \mathscr{M}_l$. Now for any $\mathscr{M}_\zeta \in \mathscr{M}$ and $k \in \mathscr{M}$,

$$(\chi_{\mathscr{M}}(T)_\zeta \circ T_\xi)(k) = \bigwedge_{k=k_1k_2} \{\chi_{\mathscr{M}}(T)_\zeta(k_1) \vee T_\xi(k_2)\}$$
$$= \bigwedge_{k=k_1k_2} T_\xi(k_2)$$
$$\geq T_\xi(k_1k_2)$$
$$= T_\xi(k),$$

$$\begin{aligned}(\chi_{\mathscr{M}}(I)_\zeta \circ I_\xi)(k) &= \bigvee_{k=k_1k_2} \{\chi_{\mathscr{M}}(I)_\zeta(k_1) \wedge I_\xi(k_2)\} \\ &= \bigvee_{k=k_1k_2} I_\xi(k_2) \\ &\leq I_\xi(k_1k_2) \\ &= I_\xi(k),\end{aligned}$$

$$\begin{aligned}(\chi_{\mathscr{M}}(F)_\zeta \circ F_\xi)(k) &= \bigwedge_{k=k_1k_2} \{\chi_{\mathscr{M}}(F)_\zeta(k_1) \vee F_\xi(k_2)\} \\ &= \bigwedge_{k=k_1k_2} F_\xi(k_2) \\ &\geq F_\xi(k_1k_2) \\ &= F_\xi(k).\end{aligned}$$

Therefore $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$.

Conversely, suppose $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$ for any $\mathscr{M}_\zeta \in \mathscr{M}$. Now for any $k_1, k_2 \in \mathscr{M}$,

$$\begin{aligned}T_\xi(k_1k_2) &\leq (\chi_{\mathscr{M}}(T)_\zeta \circ T_\xi)(k_1k_2) \\ &= \bigwedge_{k=k_1k_2} \{\chi_{\mathscr{M}}(T)_\zeta(k_1) \vee T_\xi(k_2)\} \\ &\leq \chi_{\mathscr{M}}(T)_\zeta(k_1) \vee T_\xi(k_2) \\ &= T_\xi(k_2), \\ I_\xi(k_1k_2) &\geq (\chi_{\mathscr{M}}(I)_\zeta \circ I_\xi)(k_1k_2) \\ &= \bigvee_{k=k_1k_2} \{\chi_{\mathscr{M}}(T)_\zeta(k_1) \wedge I_\xi(k_2)\} \\ &\geq \chi_{\mathscr{M}}(I)_\zeta(k_1) \wedge I_\xi(k_2) \\ &= I_\xi(k_2), \\ F_\xi(k_1k_2) &\leq (\chi_{\mathscr{M}}(F)_\zeta \circ F_\xi)(k_1k_2) \\ &= \bigwedge_{k=k_1k_2} \{\chi_{\mathscr{M}}(F)_\zeta(k_1) \vee F_\xi(k_2)\} \\ &\leq \chi_{\mathscr{M}}(F)_\zeta(k_1) \vee F_\xi(k_2) \\ &= F_\xi(k_2).\end{aligned}$$

Hence $\mathscr{M}_\xi \in \mathscr{M}_l$.

The proof of (iii) and (iv) is left to the reader. □

**Lemma 3.5.** *(i) If $\mathscr{M}_\xi, \mathscr{M}_\zeta \in \mathscr{M}_s$, then $\mathscr{M}_\xi \cap \mathscr{M}_\zeta \in \mathscr{M}_s$.*

*(ii) If $\mathscr{M}_\xi, \mathscr{M}_\zeta \in \mathscr{M}_l$, then $\mathscr{M}_\xi \cap \mathscr{M}_\zeta \in \mathscr{M}_l$.*

*(iii) If* $\mathscr{M}_\xi, \mathscr{M}_\zeta \in \mathscr{M}_r$*, then* $\mathscr{M}_\xi \cap \mathscr{M}_\zeta \in \mathscr{M}_r$.

*(iv) If* $\mathscr{M}_\xi, \mathscr{M}_\zeta \in \mathscr{M}_i$*, then* $\mathscr{M}_\xi \cap \mathscr{M}_\zeta \in \mathscr{M}_i$.

*Proof.* (i) Let $\mathscr{M}_\xi$ and $\mathscr{M}_\zeta$ be two neutrosophic $\varkappa$-$AG$-subgroupoids in $\mathscr{M}$. Now for $k_1, k_2 \in \mathscr{M}$,

$$\begin{aligned}(T_\xi \cap T_\zeta)(k_1k_2) &= T_\xi(k_1k_2) \vee T_\zeta(k_1k_2)\\ &\leq (T_\xi(k_1) \vee T_\xi(k_2)) \vee (T_\zeta(k_1) \vee T_\zeta(k_2))\\ &= (T_\xi(k_1) \vee T_\zeta(k_1)) \vee (T_\xi(k_2) \vee T_\zeta(k_2))\\ &= (T_\xi \cap T_\zeta)(k_1) \vee (T_\xi \cap T_\zeta)(k_2),\\ (I_\xi \cap I_\zeta)(k_1k_2) &= I_\xi(k_1k_2) \wedge I_\zeta(k_1k_2)\\ &\geq (I_\xi(k_1) \wedge I_\xi(k_2)) \wedge (I_\zeta(k_1) \wedge I_\zeta(k_2))\\ &= (I_\xi(k_1) \wedge I_\zeta(k_1)) \wedge (I_\xi(k_2) \wedge I_\zeta(k_2))\\ &= (I_\xi \cap I_\zeta)(k_1) \wedge (I_\xi \cap I_\zeta)(k_2),\\ (F_\xi \cap F_\zeta)(k_1k_2) &= F_\xi(k_1k_2) \vee F_\zeta(k_1k_2)\\ &\leq (F_\xi(k_1) \vee F_\xi(k_2)) \vee (F_\zeta(k_1) \vee F_\zeta(k_2))\\ &= (F_\xi(k_1) \vee F_\zeta(k_1)) \vee (F_\xi(k_2) \vee F_\zeta(k_2))\\ &= (F_\xi \cap F_\zeta)(k_1) \vee (F_\xi \cap F_\zeta)(k_2).\end{aligned}$$

So $\mathscr{M}_\xi \cap \mathscr{M}_\zeta \in \mathscr{M}_s$.

(ii) Let $\mathscr{M}_\xi,\ \mathscr{M}_\zeta \in \mathscr{M}_l$. Now for any $k_1, k_2 \in \mathscr{M}$,

$$(T_\xi \cap T_\zeta)(k_1k_2) = T_\xi(k_1k_2) \vee T_\zeta(k_1k_2) \leq T_\xi(k_2) \vee T_\zeta(k_2) = (T_\xi \cap T_\zeta)(k_2),$$
$$(I_\xi \cap I_\zeta)(k_1k_2) = I_\xi(k_1k_2) \wedge I_\zeta(k_1k_2) \geq I_\xi(k_2) \wedge I_\zeta(k_2) = (I_\xi \cap I_\zeta)(k_2),$$
$$(F_\xi \cap F_\zeta)(k_1k_2) = F_\xi(k_1k_2) \vee F_\zeta(k_1k_2) \leq F_\xi(k_2) \vee F_\zeta(k_2) = (F_\xi \cap F_\zeta)(k_2).$$

So $\mathscr{M}_\xi \cap \mathscr{M}_\zeta \in \mathscr{M}_l$.

The proof of (iii) and (iv) is left to the reader. □

**Lemma 3.6.** *If* $\mathscr{M}$ *is having left identity* $e$*, then* $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) = \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)$ *for any* $\mathscr{M}_\zeta \in \mathscr{M}$.

*Proof.* Let $k_1 \in \mathscr{M}$. Then $k_1 = ek_1$. Now,

$$(\chi_{\mathscr{M}}(T)_\zeta \circ \chi_{\mathscr{M}}(T)_\zeta)(k_1) = \bigwedge_{k_1=x_1x_2} \{\chi_{\mathscr{M}}(T)_\zeta(x_1) \vee \chi_{\mathscr{M}}(T)_\zeta(x_2)\} \leq \chi_{\mathscr{M}}(T)_\zeta(e) \vee \chi_{\mathscr{M}}(T)_\zeta(k_1) = 0$$

which implies $(\chi_{\mathscr{M}}(T)_\zeta \circ \chi_{\mathscr{M}}(T)_\zeta)(k_1) = 0 = \chi_{\mathscr{M}}(T)_\zeta(k_1)$.

$$(\chi_{\mathscr{M}}(I)_\zeta \circ \chi_{\mathscr{M}}(I)_\zeta)(k_1) = \bigvee_{k_1=x_1x_2} \{\chi_{\mathscr{M}}(I)_\zeta(x_1) \wedge \chi_{\mathscr{M}}(I)_\zeta(x_2)\} \geq \chi_{\mathscr{M}}(I)_\zeta(e) \wedge \chi_{\mathscr{M}}(I)_\zeta(k_1) = -1$$

which implies $(\chi_{\mathscr{M}}(I)_\zeta \circ \chi_{\mathscr{M}}(I)_\zeta)(k_1) = -1 = \chi_{\mathscr{M}}(I)_\zeta(k_1)$.

$$(\chi_{\mathscr{M}}(F)_\zeta \circ \chi_{\mathscr{M}}(F)_\zeta)(k_1) = \bigwedge_{k_1=x_1x_2} \{\chi_{\mathscr{M}}(F)_\zeta(x_1) \vee \chi_{\mathscr{M}}(F)_\zeta(x_2)\} \leq \chi_{\mathscr{M}}(F)_\zeta(e) \vee \chi_{\mathscr{M}}(F)_\zeta(k_1) = 0$$

which implies $(\chi_{\mathscr{M}}(F)_\zeta \circ \chi_{\mathscr{M}}(F)_\zeta)(k_1) = 0 = \chi_{\mathscr{M}}(F)_\zeta(k_1)$.

Therefore $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) = \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)$. □

**Lemma 3.7.** *If $\mathscr{M}$ has left identity $e$, then for any $\mathscr{M}_\zeta \in \mathscr{M}$, we have $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi = \mathscr{M}_\xi$ for every $\mathscr{M}_\xi \in \mathscr{M}_l$.*

*Proof.* Let $k_1 \in \mathscr{M}$. Then $k_1 = ek_1$. Now,

$$(\chi_{\mathscr{M}}(T)_\zeta \circ (T)_\xi)(k_1) = \bigwedge_{k_1=x_1x_2} \{(\chi_{\mathscr{M}}(T)_\zeta)(x_1) \vee (T)_\xi(x_2)\} \leq (\chi_{\mathscr{M}}(T)_\zeta)(e) \vee (T)_\xi(k_1) = (T)_\xi(k_1),$$

$$(\chi_{\mathscr{M}}(I)_\zeta \circ (I)_\xi)(k_1) = \bigvee_{k_1=x_1x_2} \{(\chi_{\mathscr{M}}(I)_\zeta)(x_1) \wedge (I)_\xi(x_2)\} \geq (\chi_{\mathscr{M}}(I)_\zeta)(e) \wedge (I)_\xi(k_1) = (I)_\xi(k_1),$$

$$(\chi_{\mathscr{M}}(F)_\zeta \circ (F)_\xi)(k_1) = \bigwedge_{k_1=x_1x_2} \{(\chi_{\mathscr{M}}(F)_\zeta)(x_1) \vee (F)_\xi(x_2)\} \leq (\chi_{\mathscr{M}}(F)_\zeta)(e) \vee (F)_\xi(k_1) = (F)_\xi(k_1).$$

So $\mathscr{M}_\xi \subseteq \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi$. By Theorem 3.4, $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$ and hence $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi = \mathscr{M}_\xi$. □

**Proposition 3.8.** *Suppose $\mathscr{M}$ is having left identity. If $\mathscr{M}_\xi$, $\mathscr{M}_\zeta \in \mathscr{M}_l$, then for any $\mathscr{M}_\mathscr{R}, \mathscr{M}_\mathscr{Q} \in \mathscr{M}$, $\mathscr{M}_\xi \odot \mathscr{M}_\mathscr{R} = \mathscr{M}_\zeta \odot \mathscr{M}_\mathscr{Q}$ implies $\mathscr{M}_\mathscr{R} \odot \mathscr{M}_\xi = \mathscr{M}_\mathscr{Q} \odot \mathscr{M}_\zeta$.*

*Proof.* Since $\mathscr{M}_\xi, \mathscr{M}_\zeta \in \mathscr{M}_l$, we have by Lemma 3.7, $\chi_{\mathscr{M}}(\mathscr{M}_\mathscr{S}) \odot \mathscr{M}_\xi = \mathscr{M}_\xi$ and $\chi_{\mathscr{M}}(\mathscr{M}_\mathscr{U}) \odot \mathscr{M}_\zeta = \mathscr{M}_\zeta$ for $\mathscr{M}_\mathscr{S}, \mathscr{M}_\mathscr{U} \in \mathscr{M}$. Now, for any $\mathscr{M}_\mathscr{M} \in \mathscr{M}$, $\mathscr{M}_\mathscr{R} \odot \mathscr{M}_\xi = (\chi_{\mathscr{M}}(\mathscr{M}_\mathscr{M}) \odot \mathscr{M}_\mathscr{R}) \odot \mathscr{M}_\xi = (\mathscr{M}_\xi \odot \mathscr{M}_\mathscr{R}) \odot \chi_{\mathscr{M}}(\mathscr{M}_\mathscr{M}) = (\mathscr{M}_\zeta \odot \mathscr{M}_\mathscr{Q}) \odot \chi_{\mathscr{M}}(\mathscr{M}_\mathscr{M}) = (\chi_{\mathscr{M}}(\mathscr{M}_\mathscr{M}) \odot \mathscr{M}_\mathscr{Q}) \odot \mathscr{M}_\zeta = \mathscr{M}_\mathscr{Q} \odot \mathscr{M}_\zeta$.

□

**Corollary 3.9.** *For any $\mathscr{M}_\xi, \mathscr{M}_\zeta, \mathscr{M}_\mathscr{R} \in \mathscr{M}$, the listed claims are equivalent:*

*(i)* $(\mathscr{M}_\xi \odot \mathscr{M}_\zeta) \odot \mathscr{M}_\mathscr{R} = \mathscr{M}_\zeta \odot (\mathscr{M}_\xi \odot \mathscr{M}_\mathscr{R})$,

*(ii)* $(\mathscr{M}_\xi \odot \mathscr{M}_\zeta) \odot \mathscr{M}_\mathscr{R} = \mathscr{M}_\zeta \odot (\mathscr{M}_\mathscr{R} \odot \mathscr{M}_\xi)$.

**Proposition 3.10.** *Let $\mathscr{M}_\xi \in \mathscr{M}_l$. If $\mathscr{M}_\xi \in \mathscr{M}_d$, then $\mathscr{M}_\xi \in \mathscr{M}_i$.*

*Proof.* Let $\mathscr{M}_\xi \in \mathscr{M}_l$ and $\mathscr{M}_\xi \in \mathscr{M}_d$. Then for any $\mathscr{M}_\zeta \in \mathscr{M}$, $\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) = (\mathscr{M}_\xi \odot \mathscr{M}_\xi) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) = (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi \odot \mathscr{M}_\xi = \mathscr{M}_\xi$, so $\mathscr{M}_\xi \in \mathscr{M}_i$. □

**Remark 3.11.** If $\mathscr{M}$ has left identity, then $\mathscr{M}_l = \mathscr{M}_r$.

**Theorem 3.12.** *Suppose $\mathscr{M}$ has left identity and $\mathscr{M}_\xi \in \mathscr{M}_d$. Then the listed claims holds:*

*(i)* $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi \in \mathscr{M}_d$ *for* $\mathscr{M}_\zeta \in \mathscr{M}$,

*(ii) Every* $\mathscr{M}_\zeta \in \mathscr{M}_l$ *commutes with* $\mathscr{M}_\xi$.

*Proof.* (i) It is clear from Corollary 3.2 and Lemma 3.6.

(ii) Let $\mathscr{M}_\xi, \mathscr{M}_\zeta \in \mathscr{M}$. Now, $\mathscr{M}_\xi \odot \mathscr{M}_\zeta = (\mathscr{M}_\xi \odot \mathscr{M}_\xi) \odot \mathscr{M}_\zeta = (\mathscr{M}_\zeta \odot \mathscr{M}_\xi) \odot \mathscr{M}_\xi \subseteq (\mathscr{M}_\zeta \odot \chi_{\mathscr{M}}(\mathscr{M}_\xi)) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\zeta \odot \mathscr{M}_\xi$. Also, $\mathscr{M}_\zeta \odot \mathscr{M}_\xi = \mathscr{M}_\zeta \odot (\mathscr{M}_\xi \odot \mathscr{M}_\xi) = \mathscr{M}_\xi \odot (\mathscr{M}_\zeta \odot \mathscr{M}_\xi) \subseteq \mathscr{M}_\xi \odot (\mathscr{M}_\zeta \odot \chi_{\mathscr{M}}(\mathscr{M}_\xi)) \subseteq \mathscr{M}_\xi \odot \mathscr{M}_\zeta$. □

**Lemma 3.13.** *If $\mathscr{M}$ has left identity and $\mathscr{M}_\xi \in \mathscr{M}_r$, then $\mathscr{M}_\xi \in \mathscr{M}_i$.*

*Proof.* Let $\mathscr{M}_\xi \in \mathscr{M}_r$. Then for $\mathscr{M}_\zeta \in \mathscr{M}$, $\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \subseteq \mathscr{M}_\xi$. By Lemma 3.6, $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi = (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot \mathscr{M}_\xi = (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \subseteq \mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \subseteq \mathscr{M}_\xi$. So $\mathscr{M}_\xi \in \mathscr{M}_l$ and hence $\mathscr{M}_\xi \in \mathscr{M}_i$. □

**Remark 3.14.** Suppose $\mathscr{M}$ has left identity. If $\mathscr{M}_\xi \in \mathscr{M}_r$, then $\mathscr{M}_\xi \cup (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi)$ and $\mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \mathscr{M}_\xi)$ are neutrosophic $\varkappa$- ideals for $\mathscr{M}_\zeta \in \mathscr{M}$.

**Theorem 3.15.** *Suppose $\mathscr{M}_\xi \in \mathscr{M}_l$ with left identity. Then $\mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta))$ and $\mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \mathscr{M}_\xi)$ are neutrosophic $\varkappa$-ideals for $\mathscr{M}_\zeta \in \mathscr{M}$.*

*Proof.* Now,

$$\begin{aligned}
(\mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta))) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) &= (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \cup ((\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \\
&= (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \cup ((\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot \mathscr{M}_\xi) \\
&= (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \cup (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \\
&= (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \cup \mathscr{M}_\xi \\
&= \mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)).
\end{aligned}$$

Thus $\mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \in \mathscr{M}_r$ and hence $\mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \in \mathscr{M}_i$ by Lemma 3.13.

Now, for any $\mathscr{M}_\zeta \in \mathscr{M}$,

$$\begin{aligned}
(\mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \mathscr{M}_\xi)) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) &= (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \cup ((\mathscr{M}_\xi \odot \mathscr{M}_\xi) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \\
&= (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \cup ((\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \odot \mathscr{M}_\xi) \\
&\subseteq (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \cup (\mathscr{M}_\xi \odot \mathscr{M}_\xi) \\
&= (\mathscr{M}_\xi \odot \mathscr{M}_\xi) \cup (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \\
&\subseteq (\mathscr{M}_\xi \odot \mathscr{M}_\xi) \cup \mathscr{M}_\xi \\
&= \mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \mathscr{M}_\xi).
\end{aligned}$$

Thus $\mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \mathscr{M}_\xi) \in \mathscr{M}_r$ and so $\mathscr{M}_\xi \cup (\mathscr{M}_\xi \odot \mathscr{M}_\xi) \in \mathscr{M}_i$ by Lemma 3.13. □

**Theorem 3.16.** *Suppose $\phi \neq U \subseteq \mathscr{M}$. Then the below claims are equivalent:*

*(i) $U$ is bi-ideal,*

*(ii) For any $\mathscr{M}_\xi \in \mathscr{M}$, $\chi_U(\mathscr{M}_\xi) \in \mathscr{M}_b$.*

*Proof.* This is similar to Theorem 3.1 in [20]. □

**Lemma 3.17.** *Let $\mathscr{M}_\xi \in \mathscr{M}_s$. Then the listed claims are equivalent:*

*(i) $\mathscr{M}_\xi \in \mathscr{M}_b$,*

*(ii) $(\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$ for any $\mathscr{M}_\zeta \in \mathscr{M}$.*

*Proof.* Assume $\mathscr{M}_\xi \in \mathscr{M}_b$ and let $k_1 \in \mathscr{M}$. Suppose $\exists\ x_1, x_2 \in \mathscr{M} \ni k_1 = x_1x_2$. Then

$$
\begin{aligned}
(((T)_\xi \circ \chi_{\mathscr{M}}(T)_\zeta) \circ (T)_\xi)(k_1) &= \bigwedge_{k_1=x_1x_2} \{((T)_\xi \circ \chi_{\mathscr{M}}(T)_\zeta)(x_1) \vee (T)_\xi(x_2)\} \\
&= \bigwedge_{k_1=x_1x_2} \{ \bigwedge_{x_1=x_3x_4} \{(T)_\xi(x_3) \vee \chi_{\mathscr{M}}(T)_\zeta(x_4)\} \vee (T)_\xi(x_2)\} \\
&= \bigwedge_{k_1=x_3x_4x_2} \{((T)_\xi(x_3) \vee (-1)) \vee (T)_\xi(x_2)\} \\
&= \bigwedge_{k_1=x_3x_4x_2} \{(T)_\xi(x_3) \vee (T)_\xi(x_2)\} \\
&\geq \bigwedge_{k_1=x_3x_4x_2} (T)_\xi(x_3x_4x_2) \\
&= (T)_\xi(k_1),
\end{aligned}
$$

$$\begin{aligned}(((I)_\xi \circ \chi_{\mathscr{M}}(I)_\zeta) \circ (I)_\xi)(k_1) &= \bigvee_{k_1=x_1x_2} \{((I)_\xi \circ \chi_{\mathscr{M}}(I)_\zeta)(x_1) \wedge (I)_\xi(x_2)\} \\ &= \bigvee_{k_1=x_1x_2} \{ \bigvee_{x_1=x_3x_4} \{(I)_\xi(x_3) \wedge \chi_{\mathscr{M}}(I)_\zeta(x_4)\} \wedge (I)_\xi(x_2)\} \\ &= \bigvee_{k_1=x_3x_4x_2} \{((I)_\xi(x_3) \wedge 0) \wedge (I)_\xi(x_2)\} \\ &= \bigvee_{k_1=x_3x_4x_2} \{(I)_\xi(x_3) \wedge (I)_\xi(x_2)\} \\ &\leq \bigvee_{k_1=x_3x_4x_2} (I)_\xi(x_3x_4x_2) \\ &= (I)_\xi(k_1), \\ (((F)_\xi \circ \chi_{\mathscr{M}}(F)_\zeta) \circ (F)_\xi)(k_1) &= \bigwedge_{k_1=x_1x_2} \{((F)_\xi \circ \chi_{\mathscr{M}}(F)_\zeta)(x_1) \vee (F)_\xi(x_2)\} \\ &= \bigwedge_{k_1=x_1x_2} \{ \bigwedge_{x_1=x_3x_4} \{(F)_\xi(x_3) \vee \chi_{\mathscr{M}}(F)_\zeta(x_4)\} \vee (F)_\xi(x_2)\} \\ &= \bigwedge_{k_1=x_3x_4x_2} \{((F)_\xi(x_3) \vee (-1)) \vee (F)_\xi(x_2)\} \\ &= \bigwedge_{k_1=x_3x_4x_2} \{(F)_\xi(x_3) \vee (F)_\xi(x_2)\} \\ &\geq \bigwedge_{k_1=x_3x_4x_2} (F)_\xi(x_3x_4x_2) \\ &= (F)_\xi(k_1).\end{aligned}$$

Suppose there is no $x_1, x_2 \in \mathscr{M} \ni k_1 = x_1x_2$. Then

$$(((T)_\xi \circ \chi_{\mathscr{M}}(T)_\zeta) \circ (T)_\xi)(k_1) = 0 \geq (T)_\xi(k_1),$$

$$(((I)_\xi \circ \chi_{\mathscr{M}}(I)_\zeta) \circ (I)_\xi)(k_1) = -1 \leq (I)_\xi(k_1),$$

$$(((F)_\xi \circ \chi_{\mathscr{M}}(F)_\zeta) \circ (F)_\xi)(k_1) = 0 \geq (F)_\xi(k_1).$$

Therefore $(\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$ for any $\mathscr{M}_\zeta \in \mathscr{M}$.

Conversely, assume $(\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$ for any $\mathscr{M}_\zeta \in \mathscr{M}$. Let $x_1, x_2 \in \mathscr{M}$. Then

$$(T)_\xi(x_1x_2) \leq ((T)_\xi \circ (T)_\xi)(x_1x_2) \leq (T)_\xi(x_1) \vee (T)_\xi(x_2),$$

$$(I)_\xi(x_1x_2) \geq ((I)_\xi \circ (I)_\xi)(x_1x_2) \geq (I)_\xi(x_1) \wedge (I)_\xi(x_2),$$

$$(F)_\xi(x_1x_2) \leq ((F)_\xi \circ (F)_\xi)(x_1x_2) \leq (F)_\xi(x_1) \vee (F)_\xi(x_2).$$

So $\mathscr{M}_\xi \in \mathscr{M}_s$.

Let $x_1, x_2, x_3 \in \mathscr{M}$. Then

$(T)_\xi(x_1x_2x_3) \leq (((T)_\xi \circ \chi_{\mathscr{M}}(T)_\zeta) \circ (T)_\xi)(x_1x_2x_3) \leq ((T)_\xi \circ \chi_{\mathscr{M}}(T)_\zeta)(x_1x_2) \vee (T)_\xi(x_3) \leq \{(T)_\xi(x_1) \vee \chi_{\mathscr{M}}(T)_\zeta(x_2)\} \vee (T)_\xi(x_3) = (T)_\xi(x_1) \vee (T)_\xi(x_3)$,
$(I)_\xi(x_1x_2x_3) \geq (((I)_\xi \circ \chi_{\mathscr{M}}(I)_\zeta) \circ (I)_\xi)(x_1x_2x_3) \geq ((I)_\xi \circ \chi_{\mathscr{M}}(I)_\zeta)(x_1x_2) \wedge (I)_\xi(x_3) \geq \{(I)_\xi(x_1) \wedge$

$\chi_{\mathscr{M}}(I)_\zeta(x_2)\} \wedge (I)_\xi(x_3) = (I)_\xi(x_1) \wedge (I)_\xi(x_3)$,
$(F)_\xi(x_1x_2x_3) \leq (((F)_\xi \circ \chi_{\mathscr{M}}(F)_\zeta) \circ (F)_\xi)(x_1x_2x_3) \leq ((F)_\xi \circ \chi_{\mathscr{M}}(F)_\zeta)(x_1x_2) \vee (F)_\xi(x_3) \leq \{(F)_\xi(x_1) \vee \chi_{\mathscr{M}}(F)_\zeta(x_2)\} \vee (F)_\xi(x_3) = (F)_\xi(x_1) \vee (F)_\xi(x_3)$. Therefore $\mathscr{M}_\xi \in \mathscr{M}_b$. □

**Lemma 3.18.** *Suppose $\mathscr{M}_\xi, \mathscr{M}_\zeta \in \mathscr{M}_r$ having left identity. Then $\mathscr{M}_\xi \odot \mathscr{M}_\zeta \in \mathscr{M}_b$ and $\mathscr{M}_\zeta \odot \mathscr{M}_\xi \in \mathscr{M}_b$.*

*Proof.* By Corollary 3.2, $(\mathscr{M}_\xi \odot \mathscr{M}_\zeta) \odot (\mathscr{M}_\xi \odot \mathscr{M}_\zeta) = (\mathscr{M}_\xi \odot \mathscr{M}_\xi) \odot (\mathscr{M}_\zeta \odot \mathscr{M}_\zeta) \subseteq \mathscr{M}_\xi \odot \mathscr{M}_\zeta$. Hence $\mathscr{M}_\xi \odot \mathscr{M}_\zeta \in \mathscr{M}_s$. Now, by Corollary 3.2 and Lemma 3.6, for any $\mathscr{M}_\mathscr{R} \in \mathscr{M}$,

$$\begin{aligned}((\mathscr{M}_\xi \odot \mathscr{M}_\zeta) \odot \chi_{\mathscr{M}}(\mathscr{M}_\mathscr{R})) \odot (\mathscr{M}_\xi \odot \mathscr{M}_\zeta) &= ((\mathscr{M}_\xi \odot \mathscr{M}_\zeta) \odot (\chi_{\mathscr{M}}(\mathscr{M}_\mathscr{R}) \odot \chi_{\mathscr{M}}(\mathscr{M}_\mathscr{R}))) \odot (\mathscr{M}_\xi \odot \mathscr{M}_\zeta)\\ &= ((\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\mathscr{R})) \odot (\mathscr{M}_\zeta \odot \chi_{\mathscr{M}}(\mathscr{M}_\mathscr{R}))) \odot (\mathscr{M}_\xi \odot \mathscr{M}_\zeta)\\ &\subseteq (\mathscr{M}_\xi \odot \mathscr{M}_\zeta) \odot (\mathscr{M}_\xi \odot \mathscr{M}_\zeta)\\ &\subseteq \mathscr{M}_\xi \odot \mathscr{M}_\zeta.\end{aligned}$$

By Lemma 3.17, $\mathscr{M}_\xi \odot \mathscr{M}_\zeta \in \mathscr{M}_b$. Similarly, $\mathscr{M}_\zeta \odot \mathscr{M}_\xi \in \mathscr{M}_b$. □

**Lemma 3.19.** *Let $\mathscr{M}_\xi, \mathscr{M}_\zeta \in \mathscr{M}_b$. Then $\mathscr{M}_\xi \cap \mathscr{M}_\zeta \in \mathscr{M}_b$.*

*Proof.* Let $\mathscr{M}_\xi, \mathscr{M}_\zeta \in \mathscr{M}_b$ and $k_1, k_2, a \in \mathscr{M}$. Then

$$\begin{aligned}(T_\xi \cap T_\zeta)(k_1ak_2) &= T_\xi(k_1ak_2) \vee T_\zeta(k_1ak_2)\\ &\leq (T_\xi(k_1) \vee T_\xi(k_2)) \vee (T_\zeta(k_1) \vee T_\zeta(k_2))\\ &= (T_\xi \cap T_\zeta)(k_1) \vee (T_\xi \cap T_\zeta)(k_2),\\ (I_\xi \cap I_\zeta)(k_1ak_2) &= I_\xi(k_1ak_2) \wedge I_\zeta(k_1ak_2)\\ &\geq (I_\xi(k_1) \wedge I_\xi(k_2)) \wedge (I_\zeta(k_1) \wedge I_\zeta(k_2))\\ &= (I_\xi \cap I_\zeta)(k_1) \wedge (I_\xi \cap I_\zeta)(k_2),\\ (F_\xi \cap F_\zeta)(k_1ak_2) &= F_\xi(k_1ak_2) \vee F_\zeta(k_1ak_2)\\ &\leq (F_\xi(k_1) \vee F_\xi(k_2)) \vee (F_\zeta(k_1) \vee F_\zeta(k_2))\\ &= (F_\xi \cap F_\zeta)(k_1) \vee (F_\xi \cap F_\zeta)(k_2).\end{aligned}$$

Hence $\mathscr{M}_\xi \cap \mathscr{M}_\zeta \in \mathscr{M}_b$. □

**Theorem 3.20.** *Let $\mathscr{M}_\xi \in \mathscr{M}_s$. Then $\mathscr{M}_\xi \in \mathscr{M}_n$ if and only if $(\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \subseteq \mathscr{M}_\xi$ for any $\mathscr{M}_\zeta \in \mathscr{M}$.*

*Proof.* This is similar to Theorem 3.18 in [21]. □

**Proposition 3.21.** *If every neutrosophic $\varkappa$-left ideal is neutrosophic $\varkappa$-idempotent in $\mathscr{M}$, then the following statements hold:*

*(i)* $\mathscr{M}_\xi \in \mathscr{M}_b$,

*(ii)* $\mathscr{M}_\xi \in \mathscr{M}_n$.

*Proof.* (i) Assume $\mathscr{M}_\xi \in \mathscr{M}_l$. Then $\mathscr{M}_\xi \odot \mathscr{M}_\xi = \mathscr{M}_\xi$. By Corollary 3.2, for any $\mathscr{M}_\zeta \in \mathscr{M}$, $(\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot \mathscr{M}_\xi = (\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot (\mathscr{M}_\xi \odot \mathscr{M}_\xi) = (\mathscr{M}_\xi \odot \mathscr{M}_\xi) \odot (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \subseteq \mathscr{M}_\xi \odot \mathscr{M}_\xi = \mathscr{M}_\xi$. Hence $\mathscr{M}_\xi \in \mathscr{M}_b$.

(ii) For any $\mathscr{M}_\zeta \in \mathscr{M}$, $(\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \subseteq \mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) = (\mathscr{M}_\xi \odot \mathscr{M}_\xi) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) = (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi \odot \mathscr{M}_\xi = \mathscr{M}_\xi$. Hence $\mathscr{M}_\xi \in \mathscr{M}_n$. □

**Lemma 3.22.** *Suppose $\mathscr{M}$ is having left identity $e$. Then the listed claims are equivalent:*

*(i)* $\mathscr{M}_\xi \in \mathscr{M}_r$,

*(ii)* $\mathscr{M}_\xi \in \mathscr{M}_n$.

*Proof.* Let $r \in \mathscr{M}$. Then $er = r$.

$(i) \Rightarrow (ii)$ Assume $\mathscr{M}_\xi \in \mathscr{M}_r$ and $x_1, x_2, x_3 \in \mathscr{M}$. Then

$$(T)_\xi((x_1x_2)x_3) \leq (T)_\xi(x_1x_2) = (T)_\xi((ex_1)x_2) = (T)_\xi((x_2x_1)e) \leq (T)_\xi(x_2x_1) \leq (T)_\xi(x_2),$$

$$(I)_\xi((x_1x_2)x_3) \geq (I)_\xi(x_1x_2) = (I)_\xi((ex_1)x_2) = (I)_\xi((x_2x_1)e) \geq (I)_\xi(x_2x_1) \geq (I)_\xi(x_2),$$

$$(F)_\xi((x_1x_2)x_3) \leq (F)_\xi(x_1x_2) = (F)_\xi((ex_1)x_2) = (F)_\xi((x_2x_1)e) \leq (F)_\xi(x_2x_1) \leq (F)_\xi(x_2).$$

So $\mathscr{M}_\xi \in \mathscr{M}_n$.

$(ii) \Rightarrow (i)$ Let $\mathscr{M}_\xi \in \mathscr{M}_n$. For any $x_1, x_3 \in \mathscr{M}$, we can have $(T)_\xi(x_1x_3) = (T)_\xi((ex_1)x_3) \leq (T)_\xi(x_1)$, $(I)_\xi(x_1x_3) = (I)_\xi((ex_1)x_3) \geq (I)_\xi(x_1)$, $(F)_\xi(x_1x_3) = (F)_\xi((ex_1)x_3) \leq (F)_\xi(x_1)$. So $\mathscr{M}_\xi \in \mathscr{M}_r$. □

**Lemma 3.23.** *Let $\mathscr{M}_\xi \in \mathscr{M}_l$ such that $e \in \mathscr{M}$ as left identity. If $\mathscr{M}_\xi \in \mathscr{M}_n$, then $\mathscr{M}_\xi \in \mathscr{M}_b$.*

*Proof.* Since $\mathscr{M}_\xi \in \mathscr{M}_l$, $(T)_\xi(x_1x_2) \leq (T)_\xi(x_2)$, $(I)_\xi(x_1x_2) \geq (I)_\xi(x_2)$ and $(F)_\xi(x_1x_2) \leq (F)_\xi(x_2)$ for any $x_1, x_2 \in \mathscr{M}$. As $e \in \mathscr{M}$, $ex_1 = x_1 \ \forall \ x_1 \in \mathscr{M}$. Now for any $x_1, x_2 \in \mathscr{M}$, $(T)_\xi(x_1x_2) = (T)_\xi((ex_1)x_2) \leq (T)_\xi(x_1)$, $(I)_\xi(x_1x_2) = (I)_\xi((ex_1)x_2) \geq (I)_\xi(x_1)$, $(F)_\xi(x_1x_2) = (F)_\xi((ex_1)x_2) \leq (F)_\xi(x_1)$ which imply $(T)_\xi(x_1x_2) \leq (T)_\xi(x_1) \vee (T)_\xi(x_2)$, $(I)_\xi(x_1x_2) \geq (I)_\xi(x_1) \wedge (I)_\xi(x_2)$, $(F)_\xi(x_1x_2) \leq (F)_\xi(x_1) \vee (F)_\xi(x_2)$. So $\mathscr{M}_\xi \in \mathscr{M}_s$.

For any $x_1, x_2, x_3 \in \mathscr{M}$,

$$(T)_\xi((x_1x_2)x_3) = (T)_\xi((x_1(ex_2))x_3) = (T)_\xi((e(x_1x_2))x_3) \leq (T)_\xi(x_1x_2) = (T)_\xi((ex_1)x_2) \leq (T)_\xi(x_1),$$

$$(I)_\xi((x_1x_2)x_3) = (I)_\xi((x_1(ex_2))x_3) = (I)_\xi((e(x_1x_2))x_3) \geq (I)_\xi(x_1x_2) = (I)_\xi((ex_1)x_2) \geq (I)_\xi(x_1),$$

$$(F)_\xi((x_1x_2)x_3) = (F)_\xi((x_1(ex_2))x_3) = (F)_\xi((e(x_1x_2))x_3) \leq (F)_\xi(x_1x_2) = (F)_\xi((ex_1)x_2) \leq (F)_\xi(x_1).$$

Also, $(T)_\xi((x_1x_2)x_3) = (T)_\xi((x_3x_2)x_1) = (T)_\xi((x_3(ex_2))x_1) = (T)_\xi((e(x_3x_2))x_1) \leq (T)_\xi(x_3x_2) = (T)_\xi((ex_3)x_2) \leq (T)_\xi(x_3)$. Hence $(T)_\xi((x_1x_2)x_3) \leq (T)_\xi(x_1) \vee (T)_\xi(x_3)$. Now, $(I)_\xi((x_1x_2)x_3) = (I)_\xi((x_3x_2)x_1) = (I)_\xi((x_3(ex_2))x_1) = (I)_\xi((e(x_3x_2))x_1) \geq (I)_\xi(x_3x_2) = (I)_\xi((ex_3)x_2) \geq (I)_\xi(x_3)$. Hence $(I)_\xi((x_1x_2)x_3) \geq (I)_\xi(x_1) \wedge (I)_\xi(x_3)$. Now, $(F)_\xi((x_1x_2)x_3) = (F)_\xi((x_3x_2)x_1) = (F)_\xi((x_3(ex_2))x_1) = (F)_\xi((e(x_3x_2))x_1) \leq (F)_\xi(x_3x_2) = (F)_\xi((ex_3)x_2) \leq (F)_\xi(x_3)$. Hence $(F)_\xi((x_1x_2)x_3) \leq (F)_\xi(x_1) \vee (F)_\xi(x_3)$. Therefore $\mathscr{M}_\xi \in \mathscr{M}_b$. □

**Proposition 3.24.** *Let $e \in \mathscr{M}$ be left identity. If $\mathscr{M}_\xi \in \mathscr{M}_l$ (resp., $\mathscr{M}_\xi \in \mathscr{M}_r$, $\mathscr{M}_\xi \in \mathscr{M}_i$), then $\mathscr{M}_\xi \odot \mathscr{M}_\xi \in \mathscr{M}_i$.*

*Proof.* Since $\mathscr{M}_\xi \in \mathscr{M}_l$, then for $\mathscr{M}_\zeta \in \mathscr{M}$, by Theorem 3.4, $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$. By Lemma 3.6 and Corollary 3.2, $\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot (\mathscr{M}_\xi \odot \mathscr{M}_\xi) = (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot (\mathscr{M}_\xi \odot \mathscr{M}_\xi) = (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \odot (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \subseteq \mathscr{M}_\xi \odot \mathscr{M}_\xi$. Also by Theorem 3.1, $(\mathscr{M}_\xi \odot \mathscr{M}_\xi) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) = (\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi \odot \mathscr{M}_\xi$. Thus $\mathscr{M}_\xi \odot \mathscr{M}_\xi \in \mathscr{M}_i$. □

**Corollary 3.25.** *Let $\mathscr{M}_\xi \in \mathscr{M}$ has left identity. If $\mathscr{M}_\xi \in \mathscr{M}_l$, then $\mathscr{M}_\xi \odot \mathscr{M}_\xi \in \mathscr{M}_b$ and $\mathscr{M}_\xi \odot \mathscr{M}_\xi \in \mathscr{M}_n$.*

*Proof.* By Proposition 3.24, $\mathscr{M}_\xi \odot \mathscr{M}_\xi \in \mathscr{M}_i$. Now by Lemmas 3.22 and 3.23, $\mathscr{M}_\xi \odot \mathscr{M}_\xi \in \mathscr{M}_b$ and $\mathscr{M}_\xi \odot \mathscr{M}_\xi \in \mathscr{M}_n$. □

**Theorem 3.26.** *If $\mathscr{M}_\xi \in \mathscr{M}_i$, then $\mathscr{M}_\xi \in \mathscr{M}_b$ and $\mathscr{M}_\xi \in \mathscr{M}_n$.*

*Proof.* Let $\mathscr{M}_\xi \in \mathscr{M}_i$. Then $\mathscr{M}_\xi$ is neutrosophic $\varkappa$-$AG$-subgroupoid since $\mathscr{M}_\xi \odot \mathscr{M}_\xi \subseteq \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$. Now, $(\chi_{\mathscr{M}}(\mathscr{M}_\zeta) \odot \mathscr{M}_\xi) \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \subseteq \mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta) \subseteq \mathscr{M}_\xi$ and $(\mathscr{M}_\xi \odot \chi_{\mathscr{M}}(\mathscr{M}_\zeta)) \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi \odot \mathscr{M}_\xi \subseteq \mathscr{M}_\xi$ which imply $\mathscr{M}_\xi \in \mathscr{M}_b$ and $\mathscr{M}_\xi \in \mathscr{M}_n$. □

## 4. Conclusion

We presented the ideas of neutrosophic $\varkappa$-ideal structures in an $AG$-groupoid and proved that the product of two neutrosophic $\varkappa$-right-ideal is a neutrosophic $\varkappa$-bi-ideal, and neutrosophic $\varkappa$-right-ideal is equivalent to neutrosophic $\varkappa$-interior-ideal, under certain condition. In future, we will define neutrosophic $\varkappa$-structures over an ordered $AG$-groupoid and investigate the features of an ordered $AG$-groupoid using the results in an $AG$-groupoid.

**Conflict of interest:**

The authors declare that they have no conflict of interest.

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# On Statistical Convergence of Order $\alpha$ in Neutrosophic Normed Spaces

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**Abstract**. In this paper, we introduce the notion of statistical convergent of order $\alpha$ in the neutrosophic normed spaces. We investigate a few properties of the newly introduced notion and examine the relationship with statistical convergence in the neutrosophic normed spaces. Finally, we introduce the concept of statistical Cauchy sequence of order $\alpha$ and show that statistical Cauchy sequences of order $\alpha$ are equivalent to statistical convergent of order $\alpha$ sequences in the neutrosophic normed spaces.

**Keywords:** $\alpha-$density; statistical convergent of order $\alpha$; neutrosophic normed space.

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## 1. Introduction

In 1951, statistical convergence was introduced independently by Fast [12] and Steinhaus [33] to provide deeper insights into summability theory. Subsequently, researchers like Fridy [15], Salat [29], Connor [8], and others explored it from the sequence space perspective. This led to investigations by Hazarika and Esi [17], Altinok et al. [1], Mursaleen [25], Tripathy and Sen [34], Savas and Gurdal [30]. Statistical convergence found applications in number theory, mathematical analysis, probability theory, and other areas.

In 2010, Colak [6] introduced the notion of statistical convergence with order $\alpha$ by means of $\alpha$-density. Later, it was generalized to $\tau$-statistical convergence with a degree of order $\alpha$ [7] and lacunary statistical convergence with a degree of order $\alpha$ [31]. For additional information regarding statistical convergence with a degree of order $\alpha$ and its generalizations, one may refer to [5, 10, 11], which provide many more references.

The notion of fuzzy sets was proposed by Zadeh [35] in 1965, extending classical set theory. To address membership degree uncertainties, in 1986, Atanassov [4] introduced intuitionistic fuzzy sets, which have since found utility in decision-making contexts. Smarandache [32] introduced neutrosophic sets in 2005 as a generalization of fuzzy and intuitionistic fuzzy sets. Neutrosophic sets consist of truth-membership (T), indeterminacy-membership (F), and falsity-membership (I) functions, suitable for tricomponent outcomes in uncertain scenarios.

Felbin [13] introduced fuzzy normed spaces in 1992, followed by the exploration into Saadati and Park [28] in 2006 conducted research on intuitionistic fuzzy normed spaces. Subsequently, Karakus et al. [18] delved into the realm of statistical convergence within intuitionistic fuzzy normed spaces in 2008. Further investigation into convergence of sequences in neutrosophic normed spaces was pursued by Kirisci and Simsek [22]. The research in this area is still developing, showing analogies to conventional normed spaces.

## 2. **Definition and Background**

**Definition 2.1.** [6] Consider $M \subseteq \mathbb{N}$, and let $M_n$ represent the consisting of elements in $M$ that are less than or equal to $n$. The concept of natural density of order $\alpha$ $(0 < \alpha \leq 1)$ associated with the set $M$ is denoted and defined as follows:

$$\delta^\alpha(M) = \lim_{n\to\infty} \frac{|M_n|}{n^\alpha},$$

provided that the limit exists. Here, the symbol $|M_n|$ denotes the cardinality or number of elements in the set $M_n$.

**Definition 2.2.** [6] A sequence $(y_k)$ is considered statistically convergent of order $\alpha$ $(0 < \alpha \leq 1)$ to $l$ if, for every $\zeta > 0$, the following condition holds:

$$\delta^\alpha(M(\zeta)) = 0,$$

where $M(\zeta) = \{k \in \mathbb{N} : |y_k - \ell| \geq \zeta\}$. In this context, $l$ is referred to as the statistical limit of order $\alpha$ for the sequence $(y_k)$, denoted as $y_k \xrightarrow{st^\alpha} l$.

In particular, if $\alpha = 1$, then Definition 2.1 and Definition 2.2 reduces to the definitions of natural density [14] and statistical convergence [15] respectively.

**Definition 2.3.** [24] A binary operation $\circ$ that takes two values from the interval $[0, 1]$ and produces a result within the same interval is referred to as a continuous triangular norm, given that it satisfies the following set of criteria:

(1) The operation $\circ$ possesses the properties of associativity and commutativity.
(2) The operation $\circ$ remains continuous.
(3) For any $s$ belonging to the interval $[0, 1]$, the operation $s \circ 1$ results in $s$.

(4) If $p$ is less than or equal to $q$ and $r$ is less than or equal to $s$, where $p$, $q$, $r$, and $s$ fall within the range of values from 0 to 1, then it follows that the inequality $p \circ r \leq q \circ s$ is satisfied.

**Definition 2.4.** [24] An operation $\bullet$ that takes two values from the interval $[0, 1]$ and returns a value within the same interval is termed a continuous triangular co-norm, provided it meets the following set of criteria:

(1) The operation $\bullet$ possesses the properties of associativity and commutativity.
(2) The operation $\bullet$ maintains continuity.
(3) For all $s \in [0, 1]$, the operation $s \bullet 0$ results in $s$.
(4) If values $p$, $q$, $r$, and $s$ are such that $p \leq r$ and $q \leq s$, with $p$, $q$, $r$, and $s$ belonging to the interval $[0, 1]$, then the outcome of the operation $p \bullet q$ is no greater than the result of $r \bullet s$.

**Definition 2.5.** [32] Consider a universe of discourse denoted as $X$. The subset $K_{NS}$ of $X$ is defined as:

$$K_{NS} = \{< \nu, \mathfrak{R}_K(\nu), \mathfrak{T}_K(\nu), \mathfrak{W}_K(\nu) >: \nu \in X\}$$

This defined set is known as a neutrosophic set. In this context, $\mathfrak{R}_K(\nu)$, $\mathfrak{T}_K(\nu)$, and $\mathfrak{W}_K(\nu)$ are functions mapping $X$ to the interval $[0, 1]$, Expressing the levels of truth-membership, uncertainty-membership, and falsehood-membership, respectively. It is important to satisfy the constraint $0 \leq \mathfrak{R}_K(\nu) + \mathfrak{T}_K(\nu) + \mathfrak{W}_K(\nu) \leq 3$.

**Definition 2.6.** [22] Imagine a vector space denoted as $F$, and consider a normed space $\{< \nu, \mathfrak{R}(\nu), \mathfrak{T}(\nu), \mathfrak{W}(\nu) >: \nu \in F\}$. In this normed space, $\mathfrak{R}$, $\mathfrak{T}$, and $\mathfrak{W}$ are functions mapping $F \times \mathbb{R}^+$ to the interval $[0, 1]$. Furthermore, consider $\circ$ to represent a continuous triangular norm operation, and $\bullet$ to signify a continuous triangular co-norm operation.. If the four-tuple $V = (F, \mathfrak{N}, \circ, \bullet)$ satisfies the following conditions, for any $\nu, v \in F$ and $\tau, \mu > 0$, and for every $\sigma \neq 0$:

(1) The values $\mathfrak{R}(\nu, \tau)$, $\mathfrak{T}(\nu, \tau)$, and $\mathfrak{W}(\nu, \tau)$ are restricted to the range $[0, 1]$.
(2) The sum of $\mathfrak{R}(\nu, \tau)$, $\mathfrak{T}(\nu, \tau)$, and $\mathfrak{W}(\nu, \tau)$ is bounded by $[0, 3]$.
(3) $\mathfrak{R}(\nu, \tau) = 1$ $(for\ \tau > 0)$ iff $\nu = 0$.
(4) $\mathfrak{R}(\sigma\nu, \tau) = \mathfrak{R}(\nu, \frac{\tau}{|\sigma|})$.
(5) $\mathfrak{R}(\nu, \tau) \circ \mathfrak{R}(v, \mu) \leq \mathfrak{R}(\nu + v, \tau + \mu)$.
(6) The function $\mathfrak{R}(\nu, .)$ is both continuous and non-decreasing.
(7) As $\tau$ approaches infinity, $\lim_{\tau \to \infty} \mathfrak{R}(\nu, \tau) = 1$.
(8) $\mathfrak{T}(\nu, \tau) = 0$ $(for\ \tau > 0)$ iff $\nu = 0$.
(9) $\mathfrak{T}(\sigma\nu, \tau) = \mathfrak{T}(\nu, \frac{\tau}{|\sigma|})$.
(10) $\mathfrak{T}(\nu, \tau) \bullet \mathfrak{T}(\nu, \mu) \geq \mathfrak{T}(\nu + v, \tau + \mu)$.

(11) The function $\mathfrak{T}(\nu, .)$ is continuous and non-increasing.
(12) As $\tau$ goes to infinity, $\lim_{\tau\to\infty} \mathfrak{T}(\nu, \tau) = 0$.
(13) $\mathfrak{W}(\nu, \tau) = 0$ $(for\ \tau > 0)$ iff $\nu = 0$.
(14) $\mathfrak{W}(\sigma\nu, \tau) = \mathfrak{W}(\nu, \frac{\tau}{|\sigma|})$.
(15) $\mathfrak{W}(\nu, \tau) \bullet \mathfrak{W}(v, \mu) \geq \gamma(\nu + v, \tau + \mu)$.
(16) The function $\mathfrak{W}(\nu, .)$ is continuous and non-increasing.
(17) As $\tau$ approaches infinity, $\lim_{\tau\to\infty} \mathfrak{W}(\nu, \tau) = 0$.
(18) If $\tau \leq 0$, then we have $\mathfrak{R}(\nu, \tau) = 0, \mathfrak{T}(\nu, \tau) = 1$, and $\mathfrak{W}(\nu, \tau) = 1$.

Furthermore, $\mathfrak{N} = (\mathfrak{R}, \mathfrak{T}, \mathfrak{W})$ forms a neutrosophic norm (NN).

**Example 2.7.** [22] Let's consider a normed space denoted as $(F, ||\cdot||)$. In this context, take any two values, denoted as $s$ and $t$, from the interval $[0, 1]$. We establish the triangular norm operation $\circ$ as the product of $s$ and $t$, denoted by $s \circ t = st$, and the triangular co-norm operation $\bullet$ as $s$ added to $t$ minus their product, given by $s \bullet t = s + t - st$. Now, let's delve into the construction of the functions $\mathfrak{R}(u, \tau)$, $\mathfrak{T}(u, \tau)$, and $\mathfrak{W}(u, \tau)$ for any $\tau$ that surpasses $||u||$:

$$\mathfrak{R}(u, \tau) = \frac{\tau}{\tau + ||u||}, \quad \mathfrak{T}(u, \tau) = \frac{||u||}{\tau + ||u||}, \quad \mathfrak{W}(u, \tau) = \frac{||u||}{\tau}$$

These definitions stand valid for all elements $u \in F$ and $\tau > 0$. However, if $\tau$ is less than or equal to $||u||$, then we redefine the functions as follows: $\mathfrak{R}(u, \tau) = 0$, $\mathfrak{T}(u, \tau) = 1$, and $\mathfrak{W}(u, \tau) = 1$. With these specified conditions, we can confidently assert that the arrangement $(F, \mathfrak{N}, \circ, \bullet)$ establishes itself as a neutrosophic normed space (NNS).

**Definition 2.8.** [22] Let $V$ represent a NNS (Neutrosophic Normed Space). We define a sequence $(y_k)$ to exhibit statistical convergence towards $l$ with respect to the (NN) if, for any $0 < \zeta < 1$, the set $M_\zeta = k \in \mathbb{N} : \mathfrak{R}(y_k - \ell, \tau) \leq 1 - \zeta \text{ or } \mathfrak{T}(y_k - \ell, \tau) \geq \zeta\ ,\ \mathfrak{W}(y_k - \ell, \tau) \geq \zeta$ satisfies the property that $\delta(M_\zeta) = 0$.

This notion can be denoted symbolically as $st - \mathfrak{N} - \lim y_k = l$ or equivalently $y_k \to l(st - \mathfrak{N})$.

**Example 2.9.** Consider a normed space denoted as $(F, ||\cdot||)$. Let any pair of values, denoted as $s$ and $t$, lie within the interval $[0, 1]$. We establish the continuous triangular norm as $s \circ t = st$, and the continuous triangular co-norm as $s \bullet t$ is defined as the minimum of the sum of $s$ and $t$ or 1. Drawing inspiration from the functions $\mathfrak{R}$, $\mathfrak{T}$, and $\mathfrak{W}$ showcased in Example 2.7, and under the constraint of $\tau > 0$, we have: Let's explore the sequence $(y_k)$ defined as follows within a centered environment:

$$y_k = \begin{cases} 2, & \text{if } k = p^4 \text{ where } p \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}.$$

As a result, we establish the convergence $y_k \to 0(st - \mathfrak{N})$.

**Rationale**: For any $0 < \zeta < 1$, we delve into the composition of the set $M$:

$$M = \{k \leq n : \mathfrak{R}(y_k - 0, \tau) \leq 1 - \zeta \text{ or } \mathfrak{T}(y_k - 0, \tau) \geq \zeta\ ,\ \mathfrak{W}(y_k - 0, \tau) \geq \zeta\}.$$

This exploration uncovers:

$$\begin{aligned} M &= \{k \leq n : \frac{\tau}{\tau + |y_k|} \leq 1 - \zeta \text{ or } \frac{|y_k|}{\tau + |y_k|} \geq \zeta\ ,\ \frac{|y_k|}{\tau} \geq \zeta\} \\ &= \{k \leq n : |y_k| \geq \frac{\tau\zeta}{1-\zeta} \text{ or } |y_k| \geq \tau\zeta\} \\ &= \{k \leq n : y_k = 2\}. \end{aligned}$$

Thus, we conclude that $\delta(M) = \lim_{n\to\infty} \frac{|M|}{n} \leq \lim_{n\to\infty} \frac{\sqrt[4]{n}}{n} = 0$. As a result, the sequence $y_k \to 0(st - \mathfrak{N})$ holds true.

**Definition 2.10.** [22] Consider the sequence $(y_k)$ within a NNS $V$. We deem $(y_k)$ to be statistically Cauchy if, for any $0 < \zeta < 1$, there exists an associated natural number $N = N(\zeta)$ satisfying the condition:

$$\delta(MC_\zeta) = 0, \text{ where}$$
$$MC_\zeta = \{k \in \mathbb{N} : \mathfrak{R}(y_k - s_N, \tau) \leq 1 - \zeta \text{ or } \mathfrak{T}(y_k - s_N, \tau) \geq \zeta, \mathfrak{W}(y_k - s_N, \tau) \geq \zeta\}.$$

## 3. Key Findings

**Definition 3.1.** Within the domain of NNS, denoted as $V$, consider a value $0 < \alpha \leq 1$. A sequence $(y_k)$ is classified as statistical convergence with a degree of order $\alpha$ to the value $l$ concerning the neutrosophic norm (NN). This categorization is established when, for all $0 < \zeta < 1$, the following condition holds:

$\delta^\alpha(M_\zeta) = 0$, where $M_\zeta = \{k \in \mathbb{N} : \mathfrak{R}(y_k - \ell, \tau) \leq 1 - \zeta \text{ or } \mathfrak{T}(y_k - \ell, \tau) \geq \zeta\ ,\ \mathfrak{W}(y_k - \ell, \tau) \geq \zeta\}$.

In this context, we symbolize the statement as $st^\alpha - \mathfrak{N} - \lim y_k = l$ or $y_k \to l(st^\alpha - \mathfrak{N})$.

Of particular note, should we select $\alpha = 1$, the above definition aligns with the notion of statistical convergence in NNS, as previously explored by Kirisci and Simsek [22].

**Example 3.2.** Consider a normed space denoted by $(F, ||\cdot||)$. For all $s$ and $t$ within the range of $[0, 1]$, The continuous triangular norm is characterized by the equation $s \circ t = st$, while the continuous triangular co-norm is expressed as the operation $s \bullet t$ as taking the minimum of either the sum of $s$ and $t$ or 1. Utilizing the functions $\mathfrak{R}$, $\mathfrak{T}$, and $\mathfrak{W}$ presented in Example 2.7 for all $\tau > 0$, we confidently affirm the characterization of $V$ as a NNS.

Let us now introduce the sequence $(y_k)$:

$$y_k = \begin{cases} 1, & k = p^3\ (p \in \mathbb{N}) \\ 0, & \text{otherwise} \end{cases}.$$

Then, $y_k \to 0(st^\alpha - \mathfrak{N}),\ for\ \alpha \in (\frac{1}{3}, 1]$.

**Justification**: Any $0 < \zeta < 1$, we have

$$M = \{k \leq n : \mathfrak{R}(y_k - 0, \tau) \leq 1 - \zeta \ or\ \mathfrak{T}(y_k - 0, \tau) \geq \zeta \ , \ \mathfrak{W}(y_k - 0, \tau) \geq \zeta\}.$$

This implies that,

$$\begin{aligned} M &= \{k \leq n : \frac{\tau}{\tau + |y_k|} \leq 1 - \zeta \ or\ \frac{|y_k|}{\tau + |y_k|} \geq \zeta \ \ \frac{|y_k|}{\tau} \geq \zeta\} \\ &= \{k \leq n : |y_k| \geq \frac{\tau\zeta}{1-\zeta} \ \ or\ \ |y_k| \geq \tau\zeta\} \\ &= \{k \leq n : y_k = 1\}. \end{aligned}$$

Then, $\delta^\alpha(M) = \lim_{n\to\infty} \frac{|M|}{n^\alpha} \leq \lim_{n\to\infty} \frac{\sqrt[3]{n}}{n^\alpha} = 0,\ for\ \alpha \in (\frac{1}{3}, 1]$. Hence, $y_k \to 0(st^\alpha - \mathfrak{N})$.

**Lemma 3.3.** *Considering $V$ as a NNS (Neutrosophic Normed Space), and for all $0 < \zeta < 1$, and $0 < \alpha \leq 1$, as well as $\tau > 0$, we unveil the equivalence of the following propositions:*

***(i)*** $y_k \to l(st^\alpha - \mathfrak{N})$;

***(ii)*** $\delta^\alpha(\{k \in \mathbb{N} : \mathfrak{R}(y_k - \ell, \tau) \leq 1 - \zeta\}) = \delta^\alpha(\{k \in \mathbb{N} : \mathfrak{T}(y_k - \ell, \tau) \geq \zeta\}) = \delta^\alpha(\{k \in \mathbb{N} : \mathfrak{W}(y_k - \ell, \tau) \geq \zeta\}) = 0$;

***(iii)*** $\delta^\alpha(\{k \in \mathbb{N} : \mathfrak{R}(y_k - \ell, \tau) > 1 - \zeta \ or\ \mathfrak{T}(y_k - \ell, \tau) < \zeta \ , \ \mathfrak{W}(y_k - \ell, \tau) < \zeta\}) = 1$;

***(iv)*** $\delta^\alpha(\{k \in \mathbb{N} : \mathfrak{R}(y_k - \ell, \tau) > 1 - \zeta\}) = \delta^\alpha(\{k \in \mathbb{N} : \mathfrak{T}(y_k - \ell, \tau) < \zeta\}) = \delta^\alpha(\{k \in \mathbb{N} : \mathfrak{W}(y_k - \ell, \tau) < \zeta\}) = 1$;

***(v)*** $\mathfrak{R}(y_k - \ell, \tau) \to 1(st^\alpha - \mathfrak{N})\ \ or\ \ \mathfrak{T}(y_k - \ell, \tau) \to 0(st^\alpha - \mathfrak{N})\ ,\ \mathfrak{W}(y_k - \ell, \tau) \to 0(st^\alpha - \mathfrak{N})$.

**Theorem 3.4.** *Let $V$ be a NNS and let $(y_k)$ be a sequence such that $y_k \to l(st^\alpha - \mathfrak{N})$. Then $l$ is uniquely determined.*

*Proof.* Assume, for the sake of contradiction, that $y_k \to l_1(st^\alpha - \mathfrak{N})$ and $y_k \to l_2(st^\alpha - \mathfrak{N})$ where $l_1 \neq l_2$. We select a constant $0 < \zeta < 1$ and designate $\mu > 0$ such that $(1-\zeta) \circ (1-\zeta) > 1 - \mu$ and $\zeta \bullet \zeta < \mu$. In the context of any positive $\tau$, we introduce the subsequent sets:

$$\begin{aligned} K_{\mathfrak{R}_1}(\zeta, \tau) &= \{k \in \mathbb{N} : \mathfrak{R}(y_k - \ell_1, \frac{\tau}{2}) \leq 1 - \zeta\} \\ K_{\mathfrak{R}_2}(\zeta, \tau) &= \{k \in \mathbb{N} : \mathfrak{R}(y_k - \ell_2, \frac{\tau}{2}) \leq 1 - \zeta\} \\ K_{\mathfrak{T}_1}(\zeta, \tau) &= \{k \in \mathbb{N} : \mathfrak{T}(y_k - \ell_1, \frac{\tau}{2}) \geq \zeta\} \\ K_{\mathfrak{T}_2}(\zeta, \tau) &= \{k \in \mathbb{N} : \mathfrak{T}(y_k - \ell_2, \frac{\tau}{2}) \geq \zeta\} \\ K_{\mathfrak{W}_1}(\zeta, \tau) &= \{k \in \mathbb{N} : \mathfrak{W}(y_k - \ell_1, \frac{\tau}{2}) \geq \zeta\} \\ K_{\mathfrak{W}_2}(\zeta, \tau) &= \{k \in \mathbb{N} : \mathfrak{W}(y_k - \ell_2, \frac{\tau}{2}) \geq \zeta\}. \end{aligned}$$

Since $y_k \to l_1(st^\alpha - \mathfrak{N})$, we apply Lemma 3.3 to conclude that for any $\tau > 0$,

$$\delta^\alpha(K_{\mathfrak{R}_1}(\zeta, \tau)) = \delta^\alpha(K_{\mathfrak{T}_1}(\zeta, \tau)) = \delta^\alpha(K_{\mathfrak{W}_1}(\zeta, \tau)) = 0.$$

Similarly, since $y_k \to l_2(st^\alpha - \mathfrak{N})$, we again apply Lemma 3.3 to deduce that for any $\tau > 0$,

$$\delta^\alpha(K_{\mathfrak{R}_2}(\zeta,\tau)) = \delta^\alpha(K_{\mathfrak{T}_2}(\zeta,\tau)) = \delta^\alpha(K_{\mathfrak{W}_2}(\zeta,\tau)) = 0.$$

Let's define $K(\zeta,\tau) = (K_{\mathfrak{R}_1}(\zeta,\tau) \cup K_{\mathfrak{R}_2}(\zeta,\tau)) \cap (K_{\mathfrak{T}_1}(\zeta,\tau) \cup K_{\mathfrak{T}_2}(\zeta,\tau)) \cap (K_{\mathfrak{W}_1}(\zeta,\tau) \cup K_{\mathfrak{W}_2}(\zeta,\tau))$. Consequently, $\delta^\alpha(K(\zeta,\tau)) = 0$, which implies $\delta^\alpha(\mathbb{N} \setminus K(\zeta,\tau)) = 1$ and hence The set of natural numbers excluding those in $K(\zeta,\tau)$ is non-empty. Let's pick $p$ is an element of $\mathbb{N}$ that is not in $K(\zeta,\tau)$. We proceed with three cases:

**(i)** If $p \in (\mathbb{N} \setminus (K_{\mathfrak{R}_1}(\zeta,\tau)) \cup (\mathbb{N} \setminus (K_{\mathfrak{R}_2}(\zeta,\tau))$;
**(ii)** If $p \in (\mathbb{N} \setminus (K_{\mathfrak{T}_1}(\zeta,\tau)) \cup (\mathbb{N} \setminus (K_{\mathfrak{T}_2}(\zeta,\tau))$;
**(iii)** If $p \in (\mathbb{N} \setminus (K_{\mathfrak{W}_1}(\zeta,\tau)) \cup (\mathbb{N} \setminus (K_{\mathfrak{W}_2}(\zeta,\tau))$.

For Case (i), we possess:

$$\mathfrak{R}(l_1 - l_2, \tau) \geq \mathfrak{R}(y_k - \ell_1, \frac{\tau}{2}) \circ \mathfrak{R}(y_k - \ell_2, \frac{\tau}{2}) > (1-\zeta) \circ (1-\zeta) > 1 - \mu. \tag{1}$$

Since $\mu$ can take any value,, Equation (1) implies that for any $\tau > 0$, $\mathfrak{R}(l_1 - l_2, \tau) = 1$, leading to $l_1 = l_2$.

For Case (ii), we have:

$$\mathfrak{T}(l_1 - l_2, \tau) \leq \mathfrak{T}(y_k - \ell_1, \frac{\tau}{2}) \bullet \mathfrak{T}(y_k - \ell_2, \frac{\tau}{2}) < \zeta \bullet \zeta < \mu. \tag{2}$$

Since $\mu$ can take any value, Equation (2) implies that for any $\tau > 0$, $\mathfrak{T}(l_1 - l_2, \tau) = 0$, leading to $l_1 = l_2$.

For Case (iii), we have:

$$\mathfrak{W}(l_1 - l_2, \tau) \leq \mathfrak{W}(y_k - \ell_1, \frac{\tau}{2}) \bullet \mathfrak{W}(y_k - \ell_2, \frac{\tau}{2}) < \zeta \bullet \zeta < \mu. \tag{3}$$

Since $\mu$ can take any value, Equation (3) implies that for any $\tau > 0$, $\mathfrak{W}(l_1 - l_2, \tau) = 0$, leading to $l_1 = l_2$.

In all cases, we consistently arrive at $l_1 = l_2$, which contradicts our assumption. Hence, the assumption that $l_1 \neq l_2$ must be false, and therefore $l_1 = l_2$. This concludes the proof. □

**Remark 3.5.** The notion of statistical convergence with a parameter $\alpha$ within neutrosophic normed spaces is precisely defined for values of $0 < \alpha \leq 1$. However, for $\alpha > 1$, a counterexample can be provided as follows:

**Example 3.6.** Consider a normed space $(F, ||\cdot||)$. For any $s, t \in [0,1]$, we introduce the definitions of the continuous triangular norm as $s \circ t = st$ and the continuous triangular conorm as The operation $s \cdot t$ is defined as the minimum of $s + t$ and 1. By adopting the $\mathfrak{R}, \mathfrak{T}, \mathfrak{W}$ functions illustrated in Example 2.7, valid for all $\tau > 0$, the space $V$ is established as a NNS. Let us now examine the sequence $x = (y_k)$, defined as:

$$y_k = \begin{cases} 1, & k = 2n \\ 0, & k \neq 2n \end{cases},$$

where $n \in \mathbb{N}$.

For $\alpha > 1$, we observe that

$\lim_{n\to\infty} \frac{1}{n^\alpha}|\{k \in \mathbb{N} : \mathfrak{R}(y_k - 0, \tau) \leq 1 - \zeta \, or \, \mathfrak{T}(y_k - 0, \tau) \geq \zeta, \mathfrak{W}(y_k - 0, \tau) \geq \zeta\}| \leq \lim_{n\to\infty} \frac{n}{2n^\alpha}$ is zero,

and similarly,

$\lim_{n\to\infty} \frac{1}{n^\alpha}|\{k \in \mathbb{N} : \mathfrak{R}(y_k - 1, \tau) \leq 1 - \zeta \, or \, \mathfrak{T}(y_k - 1, \tau) \geq \zeta, \mathfrak{W}(y_k - 1, \tau) \geq \zeta\}| \leq \lim_{n\to\infty} \frac{n}{2n^\alpha}$ is zero.

This implies that the statements $y_k \to 0(st^\alpha - \mathfrak{N})$ and $y_k \to 1(st^\alpha - \mathfrak{N})$ hold simultaneously, leading to a contradiction.

**Theorem 3.7.** *Let us consider two sequences, denoted as $(t_k)$ and $(s_k)$, within the NNS $V$. such that $t_k \to t_1(st^\alpha - \mathfrak{N})$* and *$x_k \to t_2(st^\alpha - \mathfrak{N})$. Then,*
*(i) $y_k + x_k \to t_1 + t_2(st^\alpha - \mathfrak{N})$* and *(ii) $r.t_k \to r.t_1(st^\alpha - \mathfrak{N})$ where $r \in \mathbb{R}$.*

*Proof.* (i) Suppose $t_k \to t_1(st^\alpha - \mathfrak{N})$and $s_k \to t_2(st^\alpha - \mathfrak{N})$. Now by definition for any $0 < \zeta < 1$,

$\delta^\alpha(M) = 0,$ then we have $M = \{k \in \mathbb{N} : \mathfrak{R}(t_k - t_1, \frac{\tau}{2}) \leq 1 - \zeta$ or $\mathfrak{T}(t_k - t_1, \frac{\tau}{2}) \geq \zeta, \mathfrak{W}(t_k - t_1, \frac{\tau}{2}) \geq \zeta\}$

and

$\delta^\alpha(M') = 0,$ then we have $M' = \{k \in \mathbb{N} : \mathfrak{R}(s_k - t_2, \frac{\tau}{2}) \leq 1 - \zeta or \mathfrak{T}(s_k - t_2, \frac{\tau}{2}) \geq \zeta, \mathfrak{W}(s_k - t_2, \frac{\tau}{2}) \geq \zeta\}$.

Now as the inclusion

$$(\mathbb{N} \setminus M) \cap (\mathbb{N} \setminus M') \subseteq \{k \in \mathbb{N} : \mathfrak{R}(t_k + s_k - t_1 - t_2, \tau) > 1 - \zeta \; or \; \mathfrak{T}(t_k + s_k - t_1 - t_2, \tau) < \zeta, \mathfrak{W}(t_k + s_k - t_1 - t_2, \tau) < \zeta\}$$

holds, so we must have

$$M'' = \{k \in \mathbb{N} : \mathfrak{R}(t_k + s_k - t_1 - t_2, \tau) \leq 1 - \zeta \; or \; \mathfrak{T}(t_k + s_k - t_1 - t_2, \tau) \geq \zeta, \mathfrak{W}(t_k + s_k - t_1 - t_2, \tau) \geq \zeta\} \subseteq M \cup M'$$

and consequently, $\delta^\alpha(M'') = 0$ i.e., $t_k + s_k \to t_1 + t_2(st^\alpha - \mathfrak{N})$.

(ii) If $r = 0$, then there is nothing to prove. So let us assume $r \neq 0$. Then since $t_k \to t_1(st^\alpha - \mathfrak{N})$, we have for any $0 < \zeta < 1$, $\delta^\alpha(M_\zeta) = 0$, where

$$M = \{k \in \mathbb{N} : \mathfrak{R}(t_k - t_1, \tfrac{\tau}{|r|}) \leq 1 - \zeta \; or \; \mathfrak{T}(t_k - t_1, \tfrac{\tau}{|r|}) \geq \zeta \, , \, \mathfrak{W}(t_k - t_1, \tfrac{\tau}{|r|}) \geq \zeta\}.$$

Now let $M'$ denote the set

$$\{k \in \mathbb{N} : \mathfrak{R}(r.t_k - r.t_1, \tau) \leq 1 - \zeta \; or \; \mathfrak{T}(r.t_k - r.t_1, \tau) \geq \zeta \, , \, \mathfrak{W}(r.t_k - r.t_1, \tau) \geq \zeta\}.$$

Then $M' \subseteq M$ holds and consequently, $\delta^{\alpha}(M') = 0$. Hence, $r.t_k \to r.l_1(st^{\alpha} - \mathfrak{N})$. □

**Theorem 3.8.** *Let us consider two sequences, denoted as $(t_k)$ and $(s_k)$, within the NNS $V$. such that $t_k \to l(\mathfrak{N})$ and $\delta^{\alpha}(\{k \in \mathbb{N} : t_k \neq s_k\}) = 0$. Then, $s_k \to l(st^{\alpha} - \mathfrak{N})$.*

*Proof.* Suppose $\delta^{\alpha}(\{k \in \mathbb{N} : t_k \neq s_k\}) = 0$ holds and $t_k \to l(\mathfrak{N})$. So, as per the definition, for any $0 < \zeta < 1$, the set $M_\zeta = \{k \in \mathbb{N} : \mathfrak{R}(t_k - \ell, \tau) \leq 1 - \zeta \text{ or } \mathfrak{T}(t_k - \ell, \tau) \geq \zeta, \mathfrak{W}(t_k - \ell, \tau) \geq \zeta\}$ has a maximum number of finite elements. Consequently, $\delta^{\alpha}(M_\zeta) = 0$. Now, since the inclusion

$$M'_\zeta = \{k \in \mathbb{N} : \mathfrak{R}(s_k - \ell, \tau) \leq 1 - \zeta or \mathfrak{T}(s_k - \ell, \tau) \geq \zeta, \mathfrak{W}(s_k - \ell, \tau) \geq \zeta\} \subseteq M_\zeta \cap \{k \in \mathbb{N} : t_k \neq s_k\}$$

holds, so we must have, $\delta^{\alpha}(M') = 0$. Hence, $y_k \to l(st^{\alpha} - \mathfrak{N})$. □

**Definition 3.9.** Let $(y_k)$ be a sequence in a NNS $V$. We say that $(y_k)$ is a statistical Cauchy sequence of order $\alpha$ if, For any $0 < \zeta < 1$, there is a positive integer $N = N(\zeta)$. such that the $\alpha$-order statistical deviation of the set $MC_\zeta$ satisfies $\delta^{\alpha}(MC_\zeta) = 0$, where:

$$MC_\zeta = \{k \in \mathbb{N} : \mathfrak{R}(y_k - s_N, \tau) \leq 1 - \zeta \text{ or } \mathfrak{T}(y_k - s_N, \tau) \geq \zeta, \mathfrak{W}(y_k - s_N, \tau) \geq \zeta\}.$$

**Theorem 3.10.** *Consider a NNS $V$. Then, for the sequence $(y_k)$ within this NNS $V$, it holds that $(y_k)$ is a statistical convergent sequence of order $\alpha$ if and only if it is a statistical Cauchy sequence of order $\alpha$.*

*Proof.* Assuming that $y_k$ converges towards $l(st^{\alpha} - \mathfrak{N})$, let's select a value of $\mu > 0$ for a given $0 < \zeta < 1$ in such a manner that $(1 - \zeta) \circ (1 - \zeta) > 1 - \mu$, and also $\zeta \bullet \zeta < \mu$. Now, according to the definition, for any $0 < \zeta < 1$, we find that $\delta^{\alpha}(\mathbb{N} \setminus M_\zeta) = 1$, where

$$M_\zeta = \{k \in \mathbb{N} : \mathfrak{R}(y_k - \ell, \frac{\tau}{2}) \leq 1 - \zeta \; or \; \mathfrak{T}(y_k - \ell, \frac{\tau}{2}) \geq \zeta \;, \; \mathfrak{W}(y_k - \ell, \frac{\tau}{2}) \geq \zeta\}.$$

Hence, the set the element of an element $\mathbb{N}$ that is not in $M_\zeta$ contains at least one element. Let us consider $N \in \mathbb{N} \setminus M_\zeta$. Subsequently, we obtain,

$$\mathfrak{R}(s_N - \ell, \frac{\tau}{2}) > 1 - \zeta \; or \; \mathfrak{T}(s_N - \ell, \frac{\tau}{2}) < \zeta \;, \; \mathfrak{W}(s_N - \ell, \frac{\tau}{2}) < \zeta.$$

Now suppose

$$MC_\zeta = \{k \in \mathbb{N} : \mathfrak{R}(y_k - s_N, \tau) \leq 1 - \mu \; or \; \mathfrak{T}(y_k - s_N, \tau) \geq \mu \;, \; \mathfrak{W}(y_k - s_N, \tau) \geq \mu\}.$$

We assert that $MC_\zeta$ is a subset of $M_\zeta$ because if this inclusion does not hold, it implies that there must be some $N_0 \in MC_\zeta \setminus M_\zeta$ which immediately yields $\mathfrak{R}(s_{N_0} - s_N, \tau) \leq 1 - \mu$ and $\mathfrak{R}(s_{N_0} - \ell, \frac{\tau}{2}) > 1 - \zeta$. Especially, $\mathfrak{R}(s_N - \ell, \frac{\tau}{2}) > 1 - \zeta$. But then,

$$1 - \mu \geq \mathfrak{R}(s_{N_0} - s_N, \tau) \geq \mathfrak{R}(s_{N_0} - \ell, \frac{\tau}{2}) \circ \mathfrak{R}(s_N - \ell, \frac{\tau}{2}) > (1 - \zeta) \circ (1 - \zeta) > 1 - \mu,$$

this leads to a contradiction. Moreover, we can observe that, $\mathfrak{T}(s_{N_0} - s_N, \tau) \geq \mu$ and $\mathfrak{T}(s_{N_0} - \ell, \frac{\tau}{2}) < \zeta$. Especially, $\mathfrak{T}(s_N - \ell, \frac{\tau}{2}) < \zeta$. But then,

$$\mu \leq \mathfrak{T}(s_{N_0} - s_N, \tau) \leq \mathfrak{T}(s_{N_0} - \ell, \frac{\tau}{2}) \bullet \mathfrak{T}(s_N - \ell, \frac{\tau}{2}) < \zeta \bullet \zeta < \mu,$$

this leads to a contradiction. Moreover, we can observe that, $\mathfrak{W}(s_{N_0} - s_N, \tau) \geq \mu$ and $\mathfrak{W}(s_{N_0} - \ell, \frac{\tau}{2}) < \zeta$. Especially, $\mathfrak{W}(s_N - \ell, \frac{\tau}{2}) < \zeta$. But then,

$$\mu \leq \mathfrak{W}(s_{N_0} - s_N, \tau) \leq \mathfrak{W}(s_{N_0} - \ell, \frac{\tau}{2}) \bullet \mathfrak{W}(s_N - \ell, \frac{\tau}{2}) < \zeta \bullet \zeta < \mu,$$

This situation presents a contradiction. Therefore, all potential scenarios are in conflict with the presence of an element $N_0 \in MC_\zeta \setminus M_\zeta$. As a result, it becomes evident that $MC_\zeta \subseteq M_\zeta$, and consequently, $\delta^\alpha(MC_\zeta) = 0$. This ultimately establishes that $(y_k)$ is, in fact, a statistical Cauchy sequence of order $\alpha$.

To establish the converse aspect of the argument, we start by assuming that $(y_k)$ is a statistical Cauchy sequence of order $\alpha$ but does not qualify as a statistical convergent sequence of the same order $\alpha$. Given a specific value $0 < \zeta < 1$, we can select a positive parameter $\mu > 0$ such that the composition $(1-\zeta) \circ (1-\zeta) > 1-\mu$ holds, and simultaneously $\zeta \bullet \zeta < \mu$.

Due to the fact that $(y_k)$ does not satisfy the criteria for being statistically convergent of order $\alpha$, we proceed with the assumption.

$$\mathfrak{R}(y_k - s_N, \tau) \geq \mathfrak{R}(y_k - \ell, \frac{\tau}{2}) \circ \mathfrak{R}(s_N - \ell, \frac{\tau}{2}) > (1-\zeta) \circ (1-\zeta) > 1-\mu,$$
$$\mathfrak{T}(y_k - s_N, \tau) \leq \mathfrak{T}(y_k - \ell, \frac{\tau}{2}) \bullet \mathfrak{T}(s_N - \ell, \frac{\tau}{2}) < \zeta \bullet \zeta < \mu,$$
$$\mathfrak{W}(y_k - s_N, \tau) \leq \mathfrak{W}(y_k - \ell, \frac{\tau}{2}) \bullet \mathfrak{W}(s_N - \ell, \frac{\tau}{2}) < \zeta \bullet \zeta < \mu,$$

This property is valid for $P(\zeta, \mu)$ holds true for values of k within the range of k from 1 to N. $N : \mathfrak{T}(y_k - s_N, \tau) \leq 1 - \mu$.

As a result, $\delta^\alpha(P(\zeta, \mu)) = 1$, which contradicts the assumption that $(y_k)$ is a statistical Cauchy sequence of order $\alpha$. Therefore, it is evident that $(y_k)$ must indeed be a statistical convergent sequence of order $\alpha$. This conclusion serves to finalize the entirety of the proof. □

## 4. **Conclusions**

In this study, our primary focus was on exploring fundamental characteristics of statistical convergence with a degree of $\alpha$. We also established a relationship between this type of convergence and the Statistical convergence in neutrosophic normed spaces has been newly introduced by Kirisci and Simsek. Additionally, we introduced the concept of statistical Cauchy sequences of order $\alpha$ and demonstrated that, in a neutrosophic normed space, every sequence that converges statistically with a degree of $\alpha$. also qualifies as a statistical Cauchy sequence of the same order, and vice versa.

In the future, there's potential to extend this research to encompass multisequences, enabling a deeper exploration of the resulting sequence space's structure.

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# Finite Difference Method for Neutrosophic Fuzzy Second Order Differential Equation under Generalized Hukuhara Differentiability

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**Abstract**

This paper solves the boundary value problem of the second-order differential equation under the neutrosophic fuzzy boundary condition. The proposed solution is approximated using the finite difference method but deneutrosophication utilizing strongly generalized Hukuhara differentiability. The error calculation is then processed by comparing the proposed numerical solution with the analytical solution presented in our earlier work [20].

**Keywords:** Fuzzy Differential Equations (FDEs); Neutrosophic Boundary Conditions; Hukuhara Differentiability; Finite Difference Method (FDM).

## 1. Introduction

Modeling real problems as ideal mathematical cases cannot help in applying or finding real solutions, and this was the reason for initiating the concept of fuzzy sets and derivatives with Zadeh [1]. Several later works aimed to widen the concept of differentiability, especially Hukuhara [2-6] who developed the concept of the difference between fuzzy sets, then he initiated Hukuhara differentiability. Following that, both Stefanoni and Bede in [4,5,6] developed the generalized Hukuhara differentiability.

On the other hand, several researchers applied many numerical methods to solve fuzzy problems. The numerical solutions of the fuzzy initial value problem (FIVP) by Euler's method were studied by Ma [17]. Abbasbandy and Allahviranloo [14, 15] used the Taylor and fourth-order Runge-Kutta methods to solve FIVPs. The author of [19] applied the Runge-Kutta method for more general problems and proved the convergence for the n-stage Runge-Kutta method. In [16], numerical solutions of fuzzy differential equations (FDEs) were presented by employing the predictor-corrector method. The dependency problem in fuzzy computation was discussed by Ahmadi and Hasan [17], where Euler's method based on Zadeh's extension principle was used for finding the numerical solution of FIVPs. Omar and Hasan [18] adopted the same computation method to derive the fourth-order Runge-Kutta method for FIVP.

Uncertainty, vagueness, and incompleteness in data increase the requirement for more generalized fuzzy ideas. That motivated Smarandache [10,11,12,13] to initiate a neutrosophic concept, which depends not only on membership, but also on three different memberships Truth, False, and Indeterminacy which facilitates modelling more complicated uncertain data, and this concept is called the neutrosophic fuzzy set.

In this paper, a numerical solution for a neutrosophic fuzzy second-order differential equation is proposed under strongly generalized differentiability by using the finite difference method. The results are compared with the analytical solutions of the same numerical examples presented in our previous results [20].

## 2. Preliminaries

Some relevant basic concepts are presented in this section to preface the mathematical tools of neutrosophic theory which are essential for the issues of this essay.

### 2.1 Definition [12].

Let $X$ be a universe set and a neutrosophic set $A$ on $X$ is defined as $A = \{(T_A(x), I_A(x), F_A(x)): x\epsilon X\}$ represents the degree of membership $T_A(x)$, the degree of indeterministic $I_A(x)$ and the degree of non-membership $F_A(x)$. Such that $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

### 2.2 Definition [11]

The $(\alpha, \beta, \gamma)$-cuts are fixed values on the set $A$ where $A_{\alpha,\beta,\gamma} = \{(T_A(x), I_A(x), F_A(x)): x\epsilon X, T_A(x) \ge \alpha, I_A(x) \le \beta, F_A(x) \le \gamma\}$ which define each of $T_A(x), I_A(x), F_A(x)$ in terms of lower and upper functions of $(\alpha, \beta, \gamma)$-cuts.

### 2.3 Definition [13]

A neutrosophic set $\boldsymbol{A}$ defined on a universal set of real numbers $\boldsymbol{R}$ is said to be a neutrosophic number and:

i) $A$ is normal if $x_a \epsilon R$, $T_A(x_a) = 1, I_A(x_a) = F_A(x_a) = 0$.

ii) $A$ is a convex set on truth function $T_A(x)$ where $T_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(T_A(x_1), T_A(x_2))$ ; $\forall\ x_1, x_2 \in R, \lambda \in [0,1]$

iii) $A$ is a concave set on indeterministic and falsity functions $I_A(x), F_A(x)$, satisfying

$I_A(\lambda x_1 + (1-\lambda)x_2) \geq \max(I_A(x_1), I_A(x_2))$ , $F_A(\lambda x_1 + (1-\lambda)x_2) \geq \max(F_A(x_1), F_A(x_2))$, this is also $\forall\ x_1, x_2 \in R, \lambda \in [0,1]$.

**2.4 Definition [7]**

Let $A$ be a Generalized triangular neutrosophic number $A_{GTN} = (a, b, c: \omega, \eta, \xi)$

$$T_A(x) = \begin{cases} \frac{x-a}{b-a}\omega & a \leq x < b \\ \omega & x = b \\ \frac{c-x}{c-b}\omega & b < x \leq c \\ 0 & \text{otherwise} \end{cases},$$

$$I_A(x) = \begin{cases} \frac{b-x}{b-a}\eta & a \leq x < b \\ \eta & x = b \\ \frac{b-x}{b-c}\eta & b < x \leq c \\ 1 & \text{otherwise} \end{cases}$$

$$F_A(x) = \begin{cases} \frac{b-x}{b-a}\xi & a \leq x < b \\ \xi & x = b \\ \frac{b-x}{b-c}\xi & b < x \leq c \\ 1 & \text{otherwise} \end{cases}$$

The $(\alpha, \beta, \gamma)$-cuts on the generalized triangular neutrosophic number can be written as

$$A_{\alpha,\beta,\gamma} = \left[\underline{A(\alpha)}, \overline{A(\alpha)}\right], \left[\underline{A(\beta)}, \overline{A(\beta)}\right], \left[\underline{A(\gamma)}, \overline{A(\gamma)}\right],$$

$$\left[\underline{A(\alpha)}, \overline{A(\alpha)}\right] = \left[\left(a + \frac{\alpha}{\omega}(b-a)\right), \left(c - \frac{\alpha}{\omega}(c-b)\right)\right],$$

$$\left[\underline{A(\beta)}, \overline{A(\beta)}\right] = \left[\left(a + \frac{1-\beta}{1-\eta}(b-a)\right), \left(c - \frac{1-\beta}{1-\eta}(c-b)\right)\right],$$

$$\left[\underline{A(\gamma)}, \overline{A(\gamma)}\right] = \left[\left(a + \frac{1-\gamma}{1-\xi}(b-a)\right), \left(c - \frac{1-\gamma}{1-\xi}(c-b)\right)\right].$$

**2.5 Definition 5 [8]**

Let $A$ be a generalized trapezoidal neutrosophic number $A_{GTRN} = (a, b, c, d: \omega, \eta, \xi)$ with

$$T_A(x) = \begin{cases} \frac{x-a}{b-a}\omega & a \le x \le b \\ \omega & b \le x \le c \\ \frac{d-x}{d-c}\omega & c \le x \le d \\ 0 & \text{otherwise} \end{cases}$$

$$I_A(x) = \begin{cases} \frac{b-x}{b-a}\eta & a \le x \le b \\ \eta & b \le x \le c \\ \frac{c-x}{c-d}\eta & c \le x \le d \\ 1 & \text{otherwise} \end{cases}$$

$$F_A(x) = \begin{cases} \frac{b-x}{b-a}\xi & a \le x \le b \\ \xi & b \le x \le c \\ \frac{c-x}{c-d}\xi & c \le x \le d \\ 1 & \text{otherwise} \end{cases}$$

The $(\alpha, \beta, \gamma)$-cuts on the generalized trapezoidal neutrosophic number can be represented as

$$A_{\alpha,\beta,\gamma} = \left[\underline{A(\alpha)}, \overline{A(\alpha)}\right], \left[\underline{A(\beta)}, \overline{A(\beta)}\right], \left[\underline{A(\gamma)}, \overline{A(\gamma)}\right],$$

$$\left[\underline{A(\alpha)}, \overline{A(\alpha)}\right] = \left[\left(a + \frac{\alpha}{\omega}(b-a)\right), \left(d - \frac{\alpha}{\omega}(d-c)\right)\right],$$

$$\left[\underline{A(\beta)}, \overline{A(\beta)}\right] = \left[\left(a + \frac{1-\beta}{1-\eta}(b-a)\right), \left(d - \frac{1-\beta}{1-\eta}(d-c)\right)\right],$$

$$\left[\underline{A(\gamma)}, \overline{A(\gamma)}\right] = \left[\left(a + \frac{1-\gamma}{1-\xi}(b-a)\right), \left(d - \frac{1-\gamma}{1-\xi}(d-c)\right)\right].$$

**2.6 Definition [6]**

Let $F: (a, b) \to \mathcal{F}(R)$, if the next limits

$$\lim_{h \to 0+} \frac{F(x_0 + h) \ominus_H F(x_0)}{h}, \qquad \lim_{h \to 0+} \frac{F(x_0) \ominus_H F(x_0 - h)}{h}$$

exist and equal some elements $F'_H(x_0) \in \mathcal{F}(R)$, then $F$ is Hukuhara differentiable at $x_0 \in (a, b)$, and $F'_H(x_0)$ is its derivative at $x_0 \in (a, b)$.

**2.7 Theorem [5]**

Let $F: (a, b) \to \mathcal{F}(R)$ be a generalized Hukuhara differentiable function if and only if one of the following conditions is satisfied

i. $\underline{f'}_\alpha(x)$ is increasing and $\overline{f'}_\alpha(x)$ is decreasing

ii. $\underline{f'}_\alpha(x)$ is decreasing and $\overline{f'}_\alpha(x)$ is increasing

Implies to,

$$\left[F'_{gH}(x)\right]_{\alpha} = \left[min\left(\underline{f'}_{\alpha}(x), \overline{f'}_{\alpha}(x)\right), max\left(\underline{f'}_{\alpha}(x), \overline{f'}_{\alpha}(x)\right)\right].$$

This concept is closer to the generalized differentiability, but it probably focuses on the cases of the function and both of these differentiabilities change the concept of derivatives, so let us show these changes in the next definition of generalized Hukuhara derivatives.

**2.8 Definition [6]**

The generalized Hukuhara first derivative of a fuzzy parametric function is defined as:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) \ominus_{gh} f(x_0)}{h},$$

From this definition, two classes can be defined:

$(i)$-differentiable at $x_0$ : $[f'(x_0)]_{\alpha} = [\underline{f'}_{\alpha}(x_0), \overline{f'}_{\alpha}(x_0)]$

$(ii)$-differentiable at $x_0$ : $[f'(x_0)]_{\alpha} = [\overline{f'}_{\alpha}(x_0), \underline{f'}_{\alpha}(x_0)]$

**2.9 Definition [6]**

The generalized Hukuhara second derivative of the fuzzy function is defined as:

$$f''(x_0) = \lim_{h \to 0} \frac{f'(x_0+h) \ominus_{gh} f'(x_0)}{h},$$

According to the last two definitions, the following classes can be selected:

$f'(x_0)$ is $(i)$-differentiable if:

$$f''(x_0) = \begin{Bmatrix} \left[\underline{f''}_{\alpha}(x_0), \overline{f''}_{\alpha}(x_0)\right] \ if\ f\ is\ (i) - differentiable\ of\ class(1,1) \\ \left[\overline{f''}_{\alpha}(x_0), \underline{f''}_{\alpha}(x_0)\right] \ if\ f\ is\ (ii) - differentiable\ \ of\ class\ (2.2) \end{Bmatrix}.$$

$f'(x_0)$ is $(ii)$-differentiable if:

$$f''(x_0) = \begin{Bmatrix} \left[\overline{f''}_{\alpha}(x_0), \underline{f''}_{\alpha}(x_0)\right] \ if\ f\ is\ (i) - differentiable\ of\ class(1,2) \\ \left[\underline{f''}_{\alpha}(x_0), \overline{f''}_{\alpha}(x_0)\right] \ if\ f\ is\ (ii) - differentiable\ of\ class(2,1) \end{Bmatrix}.$$

**2.10 Definition [7]**

The solution$\tilde{y}(t, \alpha, \beta, \gamma)$ of the neutrosophic fuzzy differential equation is strong only if,

$\frac{\partial \underline{y}}{\partial \alpha} > 0, \frac{d\bar{y}}{d\alpha} < 0$ but $\frac{\partial \underline{y}}{\partial \beta} < 0, \frac{d\bar{y}}{d\beta} > 0$ and $\frac{\partial \underline{y}}{\partial \gamma} < 0, \frac{d\bar{y}}{d\gamma} > 0$.

## 3. Finite Difference Method Under Generalized Hukuhara Differentiability

The generalized Hukuhara differentiability is first utilized on the neutrosophic differential problem, which in turn will transform it into classes of differential equations. Then, the solution can be obtained by applying the finite difference method to these classes of the differential equation.

As an example and for a second-order differential equation that has neutrosophic input adopting the generalized Hukuhara differentiability, this problem is split into four classes, each class contains two certain equations. Then, the normal finite difference method (FDM) $y'' = \frac{y_{i+1}-2y_i+y_{i-1}}{h^2}$, $y' = \frac{y_{i+1}-y_{i-1}}{2h}$

is considered on each certain first and second derivatives of the function.

Under generalized Hukuhara differentiability

$\widetilde{y''}(t,\alpha,\beta,\gamma) = p\widetilde{y}'(t,\alpha,\beta,\gamma) + q\tilde{y}(t,\alpha,\beta,\gamma)$,

$\tilde{y}(t_0) = \tilde{a}$ ; $\tilde{y}(T) = \tilde{b}$.

Where the parameters are positive constants.

**A. Class (1,1)**

$\overline{y''}(t,\alpha,\beta,\gamma) = p.\overline{y'}(t,\alpha,\beta,\gamma) + q.\bar{y}(t,\alpha,\beta,\gamma)$ ,

$\bar{y}(t_0,\alpha,\beta,\gamma) = \bar{a}$ $\bar{y}(T,\alpha,\beta,\gamma) = \bar{b}$,

$\underline{y''}(t,\alpha,\beta,\gamma) = p.\underline{y'}(t,\alpha,\beta,\gamma) + q.\underline{y}(t,\alpha,\beta,\gamma)$ ,

$\underline{y}(t_0,\alpha,\beta,\gamma) = \underline{a}$ $\underline{y}(T,\alpha,\beta,\gamma) = \underline{b}$.

By using the traditional finite difference method we have the following results

$\bar{y}(t,\alpha,\beta,\gamma) = v \text{ and } \underline{y}(t,\alpha,\beta,\gamma) = u$

$$\underline{y''}(t,\alpha,\beta,\gamma) = p.\underline{y'}(t,\alpha,\beta,\gamma) + q.\underline{y}(t,\alpha,\beta,\gamma) \quad (1)$$

$$\frac{u_{i+1}-2u_i+u_{i-1}}{h^2} = p\frac{u_{i+1}-u_{i-1}}{2h} + qu_i \quad (2)$$

$$u_{i+1} - 2u_i + u_{i-1} = \frac{ph}{2}u_{i+1} - \frac{ph}{2}u_{i-1} + qh^2u_i \quad (3)$$

$$\left(1-\frac{ph}{2}\right)u_{i+1} - (2+qh^2)u_i + \left(1+\frac{ph}{2}\right)u_{i-1} = 0 \quad (4)$$

Where $h = \Delta x$ and $i = 1, 2, 3$

$$\begin{bmatrix} -(2+qh^2) & \left(1-\frac{ph}{2}\right) & 0 \\ \left(1+\frac{ph}{2}\right) & -(2+qh^2) & \left(1-\frac{ph}{2}\right) \\ 0 & \left(1+\frac{ph}{2}\right) & -(2+qh^2) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -\left(1+\frac{ph}{2}\right)u_0 \\ 0 \\ -\left(1-\frac{ph}{2}\right)u_4 \end{bmatrix}$$

Where, $u_0 = \underline{a}$ and $u_4 = \underline{b}$

$$\overline{y''}(t,\alpha,\beta,\gamma) = p.\overline{y'}(t,\alpha,\beta,\gamma) + q.\bar{y}(t,\alpha,\beta,\gamma) \quad (5)$$

$\frac{v_{i+1}-2v_i+v_{i-1}}{h^2} = p\frac{v_{i+1}-v_{i-1}}{2h} + qv_i$ (6)

$v_{i+1} - 2v_i + v_{i-1} = \frac{ph}{2}v_{i+1} - \frac{ph}{2}v_{i-1} + qh^2v_i$ (7)

$\left(1-\frac{ph}{2}\right)v_{i+1} - (2+qh^2)v_i + \left(1+\frac{ph}{2}\right)v_{i-1} = 0$ (8)

$$\begin{bmatrix} -(2+qh^2) & \left(1-\frac{ph}{2}\right) & 0 \\ \left(1+\frac{ph}{2}\right) & -(2+qh^2) & \left(1-\frac{ph}{2}\right) \\ 0 & \left(1+\frac{ph}{2}\right) & -(2+qh^2) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\left(1+\frac{ph}{2}\right)v_0 \\ 0 \\ -\left(1-\frac{ph}{2}\right)v_4 \end{bmatrix}$$

Where, $v_0 = \overline{a}$ and $v_4 = \overline{b}$

**B. Class (1,2)**

$\underline{y''}(t,\alpha,\beta,\gamma) = p.\overline{y'}(t,\alpha,\beta,\gamma) + q.\overline{y}(t,\alpha,\beta,\gamma)$

$\overline{y}(t_0,\alpha,\beta,\gamma) = \overline{a} \quad \overline{y}(T,\alpha,\beta,\gamma) = \overline{b}$

$\overline{y''}(t,\alpha,\beta,\gamma) = p.\underline{y'}(t,\alpha,\beta,\gamma) + q.\underline{y}(t,\alpha,\beta,\gamma)$

$\underline{y}(t_0,\alpha,\beta,\gamma) = \underline{a} \quad \underline{y}(T,\alpha,\beta,\gamma) = \underline{b}$

Where, $\overline{y}(t,\alpha) = v \; and \; \underline{y}(t,\alpha) = u$

$\underline{y''}(t,\alpha,\beta,\gamma) = p.\overline{y'}(t,\alpha,\beta,\gamma) + q.\overline{y}(t,\alpha,\beta,\gamma)$ (9)

$\frac{u_{i+1}-2u_i+u_{i-1}}{h^2} = p\frac{v_{i+1}-v_{i-1}}{2h} + qv_i$ (10)

$u_{i+1} - 2u_i + u_{i-1} = \frac{ph}{2}v_{i+1} + qh^2v_i - \frac{ph}{2}v_{i-1}$ (11)

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} u_0 \\ 0 \\ u_4 \end{bmatrix} = \begin{bmatrix} qh^2 & \frac{ph}{2} & 0 \\ -\frac{ph}{2} & qh^2 & \frac{ph}{2} \\ 0 & -\frac{ph}{2} & qh^2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} -\frac{ph}{2}v_0 \\ 0 \\ \frac{ph}{2}v_4 \end{bmatrix} \quad (12)$$

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} qh^2 & \frac{ph}{2} & 0 \\ -\frac{ph}{2} & qh^2 & \frac{ph}{2} \\ 0 & -\frac{ph}{2} & qh^2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} -\frac{ph}{2}v_0 - u_0 \\ 0 \\ \frac{ph}{2}v_4 - u_4 \end{bmatrix} \quad (13)$$

$Au = Bv + C$ (14)

$A^{-1}Au = A^{-1}Bv + A^{-1}C$ (15)

$u = A^{-1}Bv + A^{-1}C$ (16)

For the second equation

$\overline{y''}(t,\alpha,\beta,\gamma) = p.\underline{y'}(t,\alpha,\beta,\gamma) + q.\underline{y}(t,\alpha,\beta,\gamma)$ (17)

$$\frac{v_{i+1}-2v_i+v_{i-1}}{h^2} = p\frac{u_{i+1}-u_{i-1}}{2h} + qu_i \tag{18}$$

$$v_{i+1} - 2v_i + v_{i-1} = \frac{ph}{2}u_{i+1} + qh^2u_i - \frac{ph}{2}u_{i-1} \tag{19}$$

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} v_0 \\ 0 \\ v_4 \end{bmatrix} = \begin{bmatrix} qh^2 & \frac{ph}{2} & 0 \\ -\frac{ph}{2} & qh^2 & \frac{ph}{2} \\ 0 & -\frac{ph}{2} & qh^2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} -\frac{ph}{2}u_0 \\ 0 \\ \frac{ph}{2}u_4 \end{bmatrix} \tag{20}$$

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} qh^2 & \frac{ph}{2} & 0 \\ -\frac{ph}{2} & qh^2 & \frac{ph}{2} \\ 0 & -\frac{ph}{2} & qh^2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} -\frac{ph}{2}u_0 - v_0 \\ 0 \\ \frac{ph}{2}u_4 - v_4 \end{bmatrix} \tag{21}$$

$$Av = Bu + D \tag{22}$$

$$A^{-1}Av = A^{-1}Bu + A^{-1}D \tag{23}$$

$$v = A^{-1}Bu + A^{-1}D \tag{24}$$

By substituting the value of $u$

$$v = A^{-1}B(A^{-1}Bv + A^{-1}C) + A^{-1}D \tag{25}$$

$$(I - (A^{-1}B)^2)v = A^{-1}BA^{-1}C + A^{-1}D \tag{26}$$

$$v = [v_1 \quad v_2 \quad v_3]^T \tag{27}$$

$$u = [u_1 \quad u_2 \quad u_3]^T \tag{28}$$

**C. Class (2,1)**

$\overline{y''}(t,\alpha,\beta,\gamma) = p.\overline{y}'(t,\alpha,\beta,\gamma) + q.\underline{y}(t,\alpha,\beta,\gamma),$

$\underline{y}(t_0,\alpha,\beta,\gamma) = \underline{a} \qquad \underline{y}(T,\alpha,\beta,\gamma) = \underline{b},$

$\underline{y''}(t,\alpha,\beta,\gamma) = p.\underline{y'}(t,\alpha,\beta,\gamma) + q.\bar{y}(t,\alpha,\beta,\gamma),$

$\bar{y}(t_0,\alpha,\beta,\gamma) = \bar{a} \qquad \bar{y}(T,\alpha,\beta,\gamma) = \bar{b}.$

$$\overline{y''}(t,\alpha,\beta,\gamma) = p.\overline{y}'(t,\alpha,\beta,\gamma) + q.\underline{y}(t,\alpha,\beta,\gamma) \tag{29}$$

$$\frac{v_{i+1}-2v_i+v_{i-1}}{h^2} = p\frac{v_{i+1}-v_{i-1}}{2h} + qu_i \tag{30}$$

$$\left(1-\frac{ph}{2}\right)v_{i+1} - 2v_i + \left(1+\frac{ph}{2}\right)v_{i-1} = qh^2u_i \tag{31}$$

$$\begin{bmatrix} -2 & \left(1-\frac{ph}{2}\right) & 0 \\ \left(1+\frac{ph}{2}\right) & -2 & \left(1-\frac{ph}{2}\right) \\ 0 & \left(1+\frac{ph}{2}\right) & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = qh^2 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} -\left(1+\frac{ph}{2}\right) v_0 \\ 0 \\ -\left(1-\frac{ph}{2}\right) v_4 \end{bmatrix} \quad (32)$$

$$Av = qh^2 u + B \quad (33)$$

$$v = A^{-1} qh^2 u + A^{-1} B \quad (34)$$

For the second equation

$$\underline{y}''(t,\alpha,\beta,\gamma) = p.\underline{y}'(t,\alpha,\beta,\gamma) + q.\bar{y}(t,\alpha,\beta,\gamma) \quad (35)$$

$$\frac{u_{i+1}-2u_i+u_{i-1}}{h^2} = p\frac{u_{i+1}-u_{i-1}}{2h} + qv_i \quad (36)$$

$$\left(1-\frac{ph}{2}\right) u_{i+1} - 2u_i + \left(1+\frac{ph}{2}\right) u_{i-1} = qh^2 v_i \quad (37)$$

$$\begin{bmatrix} -2 & \left(1-\frac{ph}{2}\right) & 0 \\ \left(1+\frac{ph}{2}\right) & -2 & \left(1-\frac{ph}{2}\right) \\ 0 & \left(1+\frac{ph}{2}\right) & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = qh^2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} -\left(1+\frac{ph}{2}\right) u_0 \\ 0 \\ -\left(1-\frac{ph}{2}\right) u_4 \end{bmatrix} \quad (38)$$

$$Au = qh^2 v + C \quad (39)$$

$$u = A^{-1} qh^2 v + A^{-1} C \quad (40)$$

By substitution the value of $v$

$$u = A^{-1} qh^2 (A^{-1} qh^2 u + A^{-1} B) + A^{-1} C \quad (41)$$

$$(1 - (qh^2)^2 (A^{-1})^2) u = qh^2 (A^{-1})^2 B + A^{-1} C \quad (42)$$

Then,

$$v = [v_1 \quad v_2 \quad v_3]^T \quad (43)$$

$$u = [u_1 \quad u_2 \quad u_3]^T \quad (44)$$

**D. Class (2,2)**

$$\overline{y}''(t,\alpha,\beta,\gamma) = p.\underline{y}'(t,\alpha,\beta,\gamma) + q.\bar{y}(t,\alpha,\beta,\gamma)$$

$$\bar{y}(t_0,\alpha,\beta,\gamma) = \bar{a} \qquad \bar{y}(T,\alpha,\beta,\gamma) = \bar{b}$$

$$\underline{y}''(t,\alpha,\beta,\gamma,\alpha) = p.\overline{y}'(t,\alpha,\beta,\gamma) + q.\underline{y}(t,\alpha,\beta,\gamma)$$

$$\underline{y}(t_0,\alpha,\beta,\gamma) = \underline{a} \qquad \underline{y}(T,\alpha,\beta,\gamma) = \underline{b}$$

$$\overline{y}''(t,\alpha,\beta,\gamma) = p.\underline{y}'(t,\alpha,\beta,\gamma) + q.\bar{y}(t,\alpha,\beta,\gamma) \quad (45)$$

$$\frac{v_{i+1}-2v_i+v_{i-1}}{h^2} = p\frac{u_{i+1}-u_{i-1}}{2h} + qv_i \quad (46)$$

$$v_{i+1} - (2 + qh^2)v_i + v_{i-1} = \frac{ph}{2}u_{i+1} - \frac{ph}{2}u_{i-1} \quad (47)$$

$$\begin{bmatrix} -(2+qh^2) & 1 & 0 \\ 1 & -(2+qh^2) & 1 \\ 0 & 1 & -(2+qh^2) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{ph}{2} & 0 \\ -\frac{ph}{2} & 0 & \frac{ph}{2} \\ 0 & -\frac{ph}{2} & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} -\frac{ph}{2}u_0 - v_0 \\ 0 \\ \frac{ph}{2}u_4 - v_4 \end{bmatrix} \quad (48)$$

$$Av = Bu + C \quad (49)$$

$$A^{-1}Av = A^{-1}Bu + A^{-1}C \quad (50)$$

$$v = A^{-1}Bu + A^{-1}C \quad (51)$$

For the second equation,

$$\underline{y}''(t,\alpha,\beta,\gamma) = p.\overline{y}'(t,\alpha,\beta,\gamma) + q.\underline{y}(t,\alpha,\beta,\gamma) \quad (52)$$

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = p\frac{v_{i+1} - v_{i-1}}{2h} + qu_i \quad (53)$$

$$u_{i+1} - (2 + qh^2)u_i + u_{i-1} = \frac{ph}{2}v_{i+1} - \frac{ph}{2}v_{i-1} \quad (54)$$

$$\begin{bmatrix} -(2+qh^2) & 1 & 0 \\ 1 & -(2+qh^2) & 1 \\ 0 & 1 & -(2+qh^2) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{ph}{2} & 0 \\ -\frac{ph}{2} & 0 & \frac{ph}{2} \\ 0 & -\frac{ph}{2} & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} -\frac{ph}{2}v_0 - u_0 \\ 0 \\ \frac{ph}{2}v_4 - u_4 \end{bmatrix} \quad (55)$$

$$Au = Bv + C \quad (56)$$

$$A^{-1}Au = A^{-1}Bv + A^{-1}C \quad (57)$$

$$u = A^{-1}Bv + A^{-1}C \quad (58)$$

## 4. Numerical Simulation

In this section, the utility of the proposed method is examined by presenting an example, whose analytical solutions are provided in the previous work [19]. To validate the efficiency of the proposed technique, the numerical results are compared with the exact solution [19]. The absolute error can be formulated as:

- The absolute error $\boldsymbol{E_N}$ of the solution is defined by:

$|\boldsymbol{e_N}| = |\boldsymbol{Exact - Approximation}|$.

### 4.1 Example 1

$\boldsymbol{\widetilde{y''}(t) = 4\widetilde{y}'(t) + 5\widetilde{y}(t)}$

$\tilde{y}(t_0) = \tilde{a} = (0.8,1,1.4; 0.8,0.2,0.3)$

$\tilde{y}(T) = \tilde{b} = (2.6,3,3.1,;0.8,0.2,0.3)$

p = 4 & q =5

- **Case (1, 1)**

**Analytical Solution**

Table (1): *The Analytical Results of Upper and Lower for $(\alpha, \beta, \gamma)$ at $t = 0.7$*

| $\alpha$ | $\underline{y}(0.7,\alpha)$ | $\overline{y}(0.7,\alpha)$ | $\beta$ | $\underline{y}(0.7,\beta)$ | $\overline{y}(0.7,\beta)$ | $\gamma$ | $\underline{y}(0.7,\gamma)$ | $\overline{y}(0.7,\gamma)$ |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.884039 | 1.287469 | 0.2 | 1.060290 | 1.060290 | 0.3 | 1.060290 | 1.060290 |
| 0.2 | 0.928101 | 1.230675 | 0.4 | 1.016227 | 1.117085 | 0.5 | 1.009932 | 1.125198 |
| 0.4 | 0.972164 | 1.173880 | 0.6 | 0.972164 | 1.173880 | 0.7 | 0.959575 | 1.190107 |
| 0.6 | 1.016227 | 1.117085 | 0.8 | 0.928101 | 1.230675 | 0.9 | 0.934396 | 1.255015 |
| 0.8 | 1.060290 | 1.060290 | 1.0 | 0.884039 | 1.287469 | 1.0 | 0.884039 | 1.287469 |

**Approximation Solution and the Related Errors**

Table (2): *The Approximation Results of Upper and Lower for $(\alpha, \beta, \gamma)$ at $t = 0.7$ at N = 50.*

| $\alpha$ | $\underline{y}(0.7,\alpha)$ | $\overline{y}(0.7,\alpha)$ | $\beta$ | $\underline{y}(0.7,\beta)$ | $\overline{y}(0.7,\beta)$ | $\gamma$ | $\underline{y}(0.7,\gamma)$ | $\overline{y}(0.7,\gamma)$ |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.883563 | 1.286953 | 0.2 | 1.059750 | 1.059750 | 0.3 | 1.060290 | 1.060290 |
| 0.2 | 0.927609 | 1.230153 | 0.4 | 1.015703 | 1.116551 | 0.5 | 1.009411 | 1.124665 |
| 0.4 | 0.971656 | 1.173352 | 0.6 | 0.971656 | 1.173352 | 0.7 | 0.959072 | 1.189581 |
| 0.6 | 1.015703 | 1.116551 | 0.8 | 0.927609 | 1.230153 | 0.9 | 0.908732 | 1.254496 |
| 0.8 | 1.059750 | 1.059750 | 1.0 | 0.883563 | 1.286953 | 1.0 | 0.884039 | 1.287469 |

Table (3): *The Errors Results of Upper and Lower for $(\alpha, \beta, \gamma)$ at $t = 0.7$ at N = 50.*

| $\alpha$ | $\underline{e}(0.7,\alpha)$ | $\overline{e}(0.7,\alpha)$ | $\beta$ | $\underline{e}(0.7,\beta)$ | $\overline{e}(0.7,\beta)$ | $\gamma$ | $\underline{e}(0.7,\gamma)$ | $\overline{e}(0.7,\gamma)$ |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 4.76 E-04 | 5.16 E-04 | 0.2 | 5.40 E-04 | 5.40 E-04 | 0.3 | 5.40 E-04 | 5.40 E-04 |
| 0.2 | 4.92 E-04 | 522 E-04 | 0.4 | 5.24 E-04 | 5.34 E-04 | 0.5 | 5.22 E-04 | 5.33 E-04 |
| 0.4 | 5.08 E-04 | 5.28 E-04 | 0.6 | 5.08 E-04 | 5.28 E-04 | 0.7 | 5.03 E-04 | 5.26 E-04 |
| 0.6 | 5.24 E-04 | 5.34 E-04 | 0.8 | 4.92 E-04 | 5.22 E-04 | 0.9 | 4.85 E-04 | 5.19 E-04 |
| 0.8 | 5.40 E-04 | 5.40 E-04 | 1.0 | 4.76 E-04 | 5.16 E-04 | 1.0 | 4.76 E-04 | 5.16 E-04 |

The analytical results for t=0.7 are shown in Table 1, followed by the approximation results for various values of N in Tables 2, 4, and 6, and finally the error between analytical and numerical results in Tables 3, 5, and 7.

*Table* (4): *The Approximation Results of Upper and Lower for* $(\alpha, \beta, \gamma)$ *at* $t = 0.7$*at N = 100.*

| $\alpha$ | $\underline{y}(0.7, \alpha)$ | $\overline{y}(0.7, \alpha)$ | $\beta$ | $\underline{y}(0.7, \beta)$ | $\overline{y}(0.7, \beta)$ | $\gamma$ | $\underline{y}(0.7, \gamma)$ | $\overline{y}(0.7, \gamma)$ |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.883920 | 1.287341 | 0.2 | 1.060155 | 1.060155 | 0.3 | 1.060155 | 1.060155 |
| 0.2 | 0.927979 | 1.230544 | 0.4 | 1.016096 | 1.116951 | 0.5 | 1.009802 | 1.125065 |
| 0.4 | 0.972037 | 1.173748 | 0.6 | 0.972037 | 1.173748 | 0.7 | 0.959449 | 1.189975 |
| 0.6 | 1.016096 | 1.116951 | 0.8 | 0.927979 | 1.230544 | 0.9 | 0.909096 | 1.254885 |
| 0.8 | 1.060155 | 1.060155 | 1.0 | 0.883920 | 1.287341 | 1.0 | 0.883920 | 1.287341 |

*Table* (5): *The Errors Results of Upper and Lower for* $(\alpha, \beta, \gamma)$ *at* $t = 0.7$*at N = 100.*

| $\alpha$ | $\underline{e}(0.7, \alpha)$ | $\overline{e}(0.7, \alpha)$ | $\beta$ | $\underline{e}(0.7, \beta)$ | $\overline{e}(0.7, \beta)$ | $\gamma$ | $\underline{e}(0.7, \gamma)$ | $\overline{e}(0.7, \gamma)$ |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.19 E-04 | 1.29 E-04 | 0.2 | 1.35 E-04 | 1.35 E-04 | 0.3 | 1.35 E-04 | 1.35 E-04 |
| 0.2 | 1.23 E-04 | 1.30 E-04 | 0.4 | 1.31 E-04 | 1.33 E-04 | 0.5 | 1.30 E-04 | 1.33 E-04 |
| 0.4 | 1.27 E-04 | 1.32 E-04 | 0.6 | 1.27 E-04 | 1.32 E-04 | 0.7 | 1.26 E-04 | 1.31 E-04 |
| 0.6 | 1.31 E-04 | 1.33 E-04 | 0.8 | 1.23 E-04 | 1.31 E-04 | 0.9 | 1.21 E-04 | 1.30 E-04 |
| 0.8 | 1.35 E-04 | 1.35 E-04 | 1.0 | 1.19 E-04 | 1.29 E-04 | 1.0 | 1.19 E-04 | 1.29 E-04 |

N = 200 & h = 0.005

*Table* (6): *The Approximation Results of Upper and Lower for* $(\alpha, \beta, \gamma)$ *at* $t = 0.7$*at N = 200.*

| $\alpha$ | $\underline{y}(0.7, \alpha)$ | $\overline{y}(0.7, \alpha)$ | $\beta$ | $\underline{y}(0.7, \beta)$ | $\overline{y}(0.7, \beta)$ | $\gamma$ | $\underline{y}(0.7, \gamma)$ | $\overline{y}(0.7, \gamma)$ |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.884009 | 1.287437 | 0.2 | 1.060256 | 1.060256 | 0.3 | 1.060256 | 1.060256 |
| 0.2 | 0.928071 | 1.230642 | 0.4 | 1.016194 | 1.117052 | 0.5 | 1.009900 | 1.125165 |
| 0.4 | 0.972133 | 1.173847 | 0.6 | 0.972133 | 1.173847 | 0.7 | 0.959544 | 1.190074 |
| 0.6 | 1.016194 | 1.117052 | 0.8 | 0.928071 | 1.230642 | 0.9 | 0.909187 | 1.254983 |
| 0.8 | 1.060256 | 1.060256 | 1.0 | 0.884009 | 1.287437 | 1.0 | 0.884009 | 1.287437 |

*Table* (7): *The Errors Results of Upper and Lower for* $(\alpha, \beta, \gamma)$ *at* $t = 0.7$*at N = 200.*

| $\alpha$ | $\underline{e}(0.7, \alpha)$ | $\overline{e}(0.7, \alpha)$ | $\beta$ | $\underline{e}(0.7, \beta)$ | $\overline{e}(0.7, \beta)$ | $\gamma$ | $\underline{e}(0.7, \gamma)$ | $\overline{e}(0.7, \gamma)$ |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 2.97 E-05 | 3.22 E-05 | 0.2 | 3.37 E-05 | 3.37 E-05 | 0.3 | 3.37 E-05 | 3.37 E-05 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0.2 | 3.07 E-05 | 3.26 E-05 | 0.4 | 3.27E-05 | 3.33 E-05 | 0.5 | 3.26 E-05 | 3.33 E-05 |
| 0.4 | 3.17 E-05 | 3.30 E-05 | 0.6 | 3.17 E-05 | 3.30 E-05 | 0.7 | 3.14 E-05 | 3.29 E-05 |
| 0.6 | 3.27E-05 | 3.33 E-05 | 0.8 | 3.07 E-05 | 3.26 E-05 | 0.9 | 3.03 E-05 | 3.24 E-05 |
| 0.8 | 3.37 E-05 | 3.37 E-05 | 1.0 | 2.97 E-05 | 3.22 E-05 | 1.0 | 2.97 E-05 | 3.22 E-05 |

***Figure 1: Surface of Absolute Error for Lower Case (1,1) Alpha***

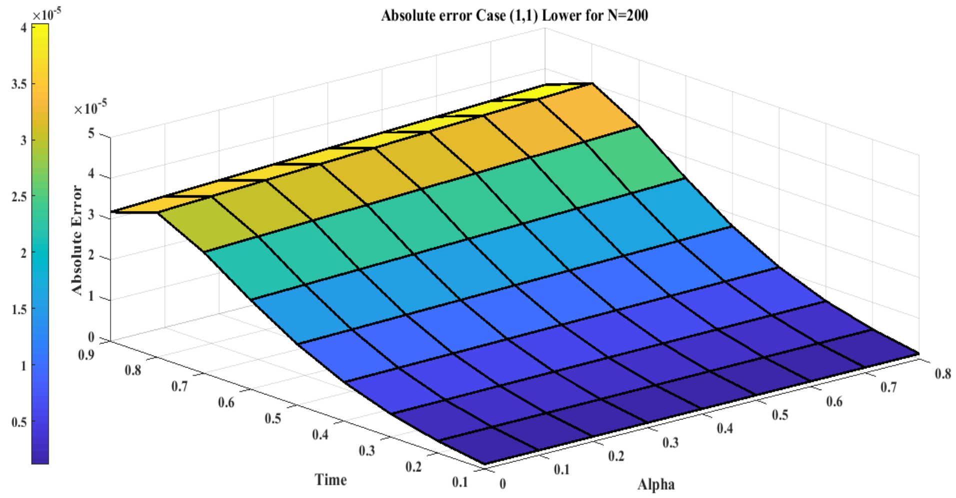



***Figure 2: Surface of Absolute Error for Upper Case (1,1) Alpha***

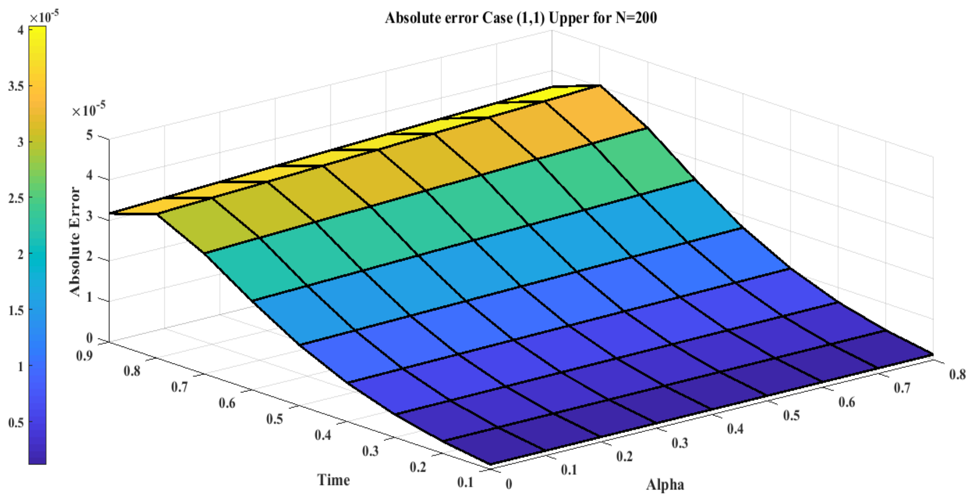


*Figure 3: Surface of Absolute Error for Lower Case (1,1) Beta*

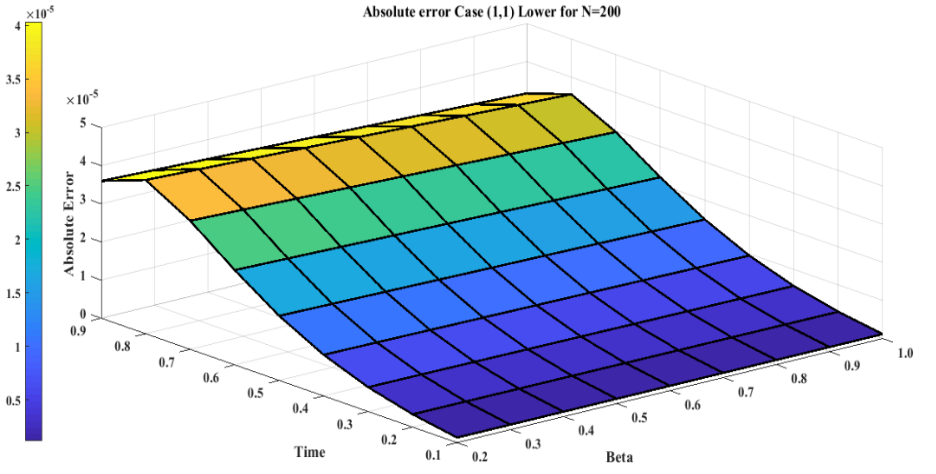



*Figure 4: Surface of Absolute error for Upper Case (1,1) Beta*

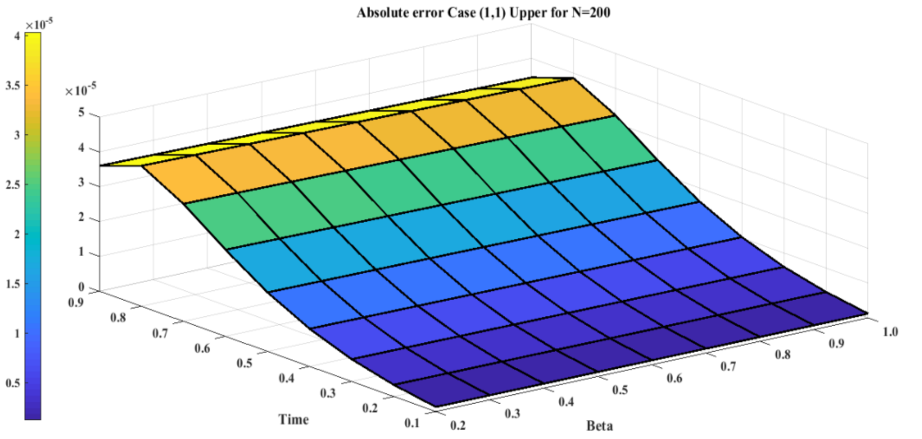



*Figure 5: Surface of Absolute Error for Lower Case (1,1) Gamma*

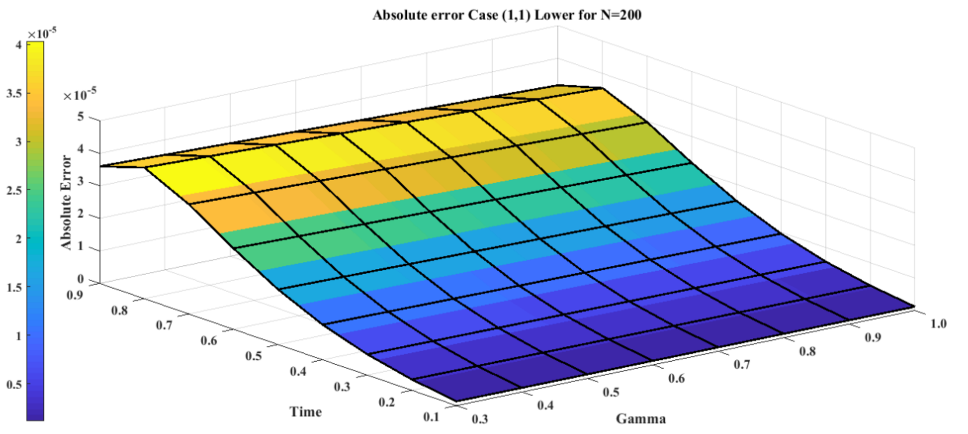
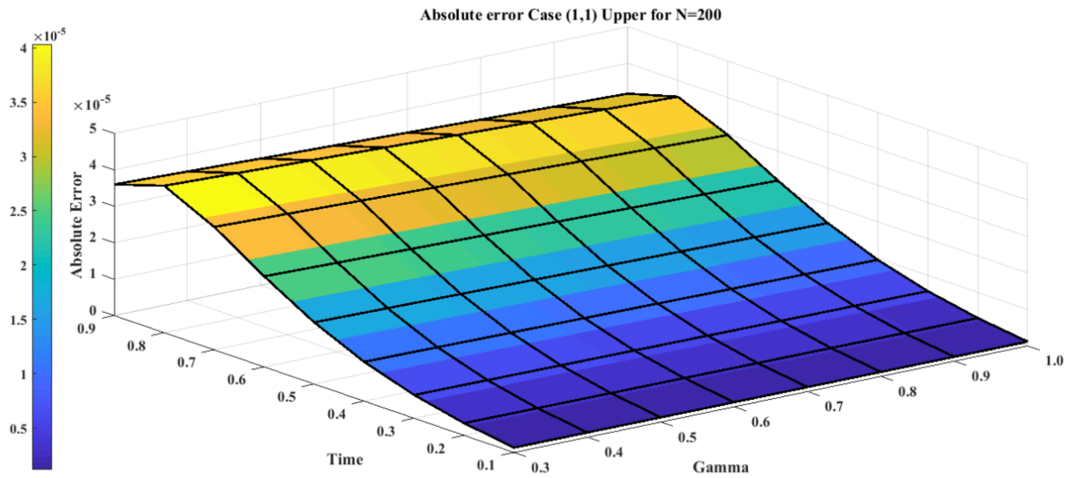


*Figure 6: Surface of Absolute Error for Upper Case (1,1) Gamma*




Figures 1 to 6 show the surface of errors at N=200, $t \in [0.1{:}\,0.9]$, with values of $(\alpha, \beta, \gamma)$.

**Case (1, 2)**

**Analytical Solution**

Table (8): *The Analytical Results of Upper and Lower for $(\alpha, \beta, \gamma)$ at $t = 0.7$*

| $\alpha$ | $\underline{y}(0.7,\alpha)$ | $\bar{y}(0.7,\alpha)$ | $\beta$ | $\underline{y}(0.7,\beta)$ | $\bar{y}(0.7,\beta)$ | $\gamma$ | $\underline{y}(0.7,\gamma)$ | $\bar{y}(0.7,\gamma)$ |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.762704 | 1.408804 | 0.2 | 1.060290 | 1.060290 | 0.3 | 1.060290 | 1.060290 |
| 0.2 | 0.837101 | 1.321675 | 0.4 | 0.985893 | 1.147418 | 0.5 | 0.975265 | 1.159865 |
| 0.4 | 0.911497 | 1.234547 | 0.6 | 0.911497 | 1.234547 | 0.7 | 0.890241 | 1.259441 |
| 0.6 | 0.985893 | 1.147418 | 0.8 | 0.837101 | 1.321675 | 0.9 | 0.805217 | 1.359016 |
| 0.8 | 1.060290 | 1.060290 | 1.0 | 0.762704 | 1.408804 | 1.0 | 0.762704 | 1.408804 |

**Approximation Solution at N = 200 & h = 0.005**

Table (9): *The Approximation Results of Upper and Lower for $(\alpha, \beta, \gamma)$ at $t = 0.7$ for $N = 200$*

| $\alpha$ | $\underline{y}(0.7,\alpha)$ | $\bar{y}(0.7,\alpha)$ | $\beta$ | $\underline{y}(0.7,\beta)$ | $\bar{y}(0.7,\beta)$ | $\gamma$ | $\underline{y}(0.7,\gamma)$ | $\bar{y}(0.7,\gamma)$ |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.762672 | 1.408774 | 0.2 | 1.060256 | 1.060256 | 0.3 | 1.060256 | 1.060256 |
| 0.2 | 0.837068 | 1.321645 | 0.4 | 0.985860 | 1.147386 | 0.5 | 0.975232 | 1.159833 |
| 0.4 | 0.911464 | 1.234515 | 0.6 | 0.911464 | 1.234515 | 0.7 | 0.890208 | 1.259409 |
| 0.6 | 0.985860 | 1.147386 | 0.8 | 0.837068 | 1.321645 | 0.9 | 0.805184 | 1.358986 |
| 0.8 | 1.060256 | 1.060256 | 1.0 | 0.762672 | 1.408774 | 1.0 | 0.762672 | 1.408774 |

Error at N = 200

Table (10): The Error Results of Upper and Lower for $(\alpha,\beta,\gamma)$ at $t = 0.7$ at N = 200.

| $\alpha$ | $\underline{e}(0.7,\alpha)$ | $\bar{e}(0.7,\alpha)$ | $\beta$ | $\underline{e}(0.7,\beta)$ | $\bar{e}(0.7,\beta)$ | $\gamma$ | $\underline{e}(0.7,\gamma)$ | $\bar{e}(0.7,\gamma)$ |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 3.258E-05 | 2.935E-05 | 0.2 | 3.372E-05 | 3.372E-05 | 0.3 | 3.372E-05 | 3.372E-05 |
| 0.2 | 3.285E-05 | 3.044E-05 | 0.4 | 3.344E-05 | 3.263E-05 | 0.5 | 3.340E-05 | 3.247E-05 |
| 0.4 | 3.315E-05 | 3.153E-05 | 0.6 | 3.315E-05 | 3.153E-05 | 0.7 | 3.307E-05 | 3.122E-05 |
| 0.6 | 3.344E-05 | 3.263E-05 | 0.8 | 3.285E-05 | 3.044E-05 | 0.9 | 3.274E-05 | 2.997E-05 |
| 0.8 | 3.372E-05 | 3.372E-05 | 1.0 | 3.258E-05 | 2.935E-05 | 1.0 | 3.258E-05 | 2.935E-05 |

- **Case (2, 1)**

**Analytical Solution**

Table (11): The Analytical Results of Upper and Lower for $(\alpha,\beta,\gamma)$ at $t = 0.7$

| $\alpha$ | $\underline{y}(0.7,\alpha)$ | $\bar{y}(0.7,\alpha)$ | $\beta$ | $\underline{y}(0.7,\beta)$ | $\bar{y}(0.7,\beta)$ | $\gamma$ | $\underline{y}(0.7,\gamma)$ | $\bar{y}(0.7,\gamma)$ |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.638923 | 1.532585 | 0.2 | 1.060290 | 1.060290 | 0.3 | 1.060290 | 1.060290 |
| 0.2 | 0.744265 | 1.414511 | 0.4 | 0.954948 | 1.178364 | 0.5 | 0.939899 | 1.195231 |
| 0.4 | 0.849607 | 1.296438 | 0.6 | 0.849607 | 1.296438 | 0.7 | 0.819509 | 1.330173 |
| 0.6 | 0.954948 | 1.178364 | 0.8 | 0.744265 | 1.414511 | 0.9 | 0.699119 | 1.465114 |
| 0.8 | 1.060290 | 1.060290 | 1.0 | 0.638923 | 1.532585 | 1.0 | 0.638923 | 1.532585 |

**Approximation Solution**

Table (12): The Approximation Results of Upper and Lower for $(\alpha,\beta,\gamma)$ at $t = 0.7$ at N = 150.

| $\alpha$ | $\underline{y}(0.7,\alpha)$ | $\bar{y}(0.7,\alpha)$ | $\beta$ | $\underline{y}(0.7,\beta)$ | $\bar{y}(0.7,\beta)$ | $\gamma$ | $\underline{y}(0.7,\gamma)$ | $\bar{y}(0.7,\gamma)$ |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.638851 | 1.532546 | 0.2 | 1.060230 | 1.060230 | 0.3 | 1.060230 | 1.060230 |
| 0.2 | 0.744196 | 1.414467 | 0.4 | 0.954885 | 1.178309 | 0.5 | 0.939836 | 1.195178 |
| 0.4 | 0.849541 | 1.296388 | 0.6 | 0.849541 | 1.296388 | 0.7 | 0.819442 | 1.330125 |
| 0.6 | 0.954885 | 1.178309 | 0.8 | 0.744196 | 1.414467 | 0.9 | 0.699048 | 1.465073 |
| 0.8 | 1.060230 | 1.060230 | 1.0 | 0.638851 | 1.532546 | 1.0 | 0.638851 | 1.532546 |

Error at N = 150

Table (13): *The Error Results of Upper and Lower for* $(\alpha,\beta,\gamma)$ *at* $t = 0.7$ *at N = 150.*

| $\alpha$ | $\underline{e}(0.7,\alpha)$ | $\bar{e}(0.7,\alpha)$ | $\beta$ | $\underline{e}(0.7,\beta)$ | $\bar{e}(0.7,\beta)$ | $\gamma$ | $\underline{e}(0.7,\gamma)$ | $\bar{e}(0.7,\gamma)$ |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 7.18 E-05 | 3.83 E-05 | 0.2 | 6.00 E-05 | 5.60 E-05 | 0.3 | 6.00 E-05 | 5.60 E-05 |
| 0.2 | 6.88 E-05 | 4.37 E-05 | 0.4 | 6.29 E-05 | 5.45 E-05 | 0.5 | 6.33 E-05 | 5.37 E-05 |
| 0.4 | 6.58 E-05 | 4.91 E-05 | 0.6 | 6.58 E-05 | 4.91 E-05 | 0.7 | 6.67 E-05 | 4.76 E-05 |
| 0.6 | 6.29 E-05 | 5.45 E-05 | 0.8 | 6.88 E-05 | 4.37 E-05 | 0.9 | 7.00 E-05 | 4.14 E-05 |
| 0.8 | 6.00 E-05 | 5.60 E-05 | 1.0 | 7.18 E-05 | 3.83 E-05 | 1.0 | 7.18 E-05 | 3.83 E-05 |

- **Case (2, 2)**

**Analytical Solution**

Table (14): *The Analytical Results of Upper and Lower for* $(\alpha,\beta,\gamma)$ *at* $t = 0.7$

| $\alpha$ | $\underline{y}(0.7,\alpha)$ | $\bar{y}(0.7,\alpha)$ | $\beta$ | $\underline{y}(0.7,\beta)$ | $\bar{y}(0.7,\beta)$ | $\gamma$ | $\underline{y}(0.7,\gamma)$ | $\bar{y}(0.7,\gamma)$ |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.880422 | 1.291087 | 0.2 | 1.060290 | 1.060290 | 0.3 | 1.060290 | 1.060290 |
| 0.2 | 0.925389 | 1.233387 | 0.4 | 1.015323 | 1.117989 | 0.5 | 1.008899 | 1.126231 |
| 0.4 | 0.970356 | 1.175688 | 0.6 | 0.970356 | 1.175688 | 0.7 | 0.957508 | 1.192174 |
| 0.6 | 1.015323 | 1.117989 | 0.8 | 0.925389 | 1.233387 | 0.9 | 0.906117 | 1.258116 |
| 0.8 | 1.060290 | 1.060290 | 1.0 | 0.880422 | 1.291087 | 1.0 | 0.880422 | 1.291087 |

**Approximation Solution at N = 200 & h = 0.005**

Table (15): *The Approximation Results of Upper and Lower for* $(\alpha,\beta,\gamma)$ *at* $t = 0.7$ *at N = 200.*

| $\alpha$ | $\underline{y}(0.7,\alpha)$ | $\bar{y}(0.7,\alpha)$ | $\beta$ | $\underline{y}(0.7,\beta)$ | $\bar{y}(0.7,\beta)$ | $\gamma$ | $\underline{y}(0.7,\gamma)$ | $\bar{y}(0.7,\gamma)$ |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.880391 | 1.291055 | 0.2 | 1.060256 | 1.060256 | 0.3 | 1.060256 | 1.060256 |
| 0.2 | 0.925357 | 1.233355 | 0.4 | 1.015290 | 1.117956 | 0.5 | 1.008866 | 1.126198 |
| 0.4 | 0.970324 | 1.175656 | 0.6 | 0.970324 | 1.175656 | 0.7 | 0.957476 | 1.192141 |
| 0.6 | 1.015290 | 1.117956 | 0.8 | 0.925357 | 1.233355 | 0.9 | 0.906086 | 1.258084 |
| 0.8 | 1.060256 | 1.060256 | 1.0 | 0.880391 | 1.291055 | 1.0 | 0.880391 | 1.291055 |

Error at N = 200

*Table* (16): *The Error Results of Upper and Lower for* $(\alpha, \beta, \gamma)$ *at* $t = 0.7$*at N = 200.*

| $\alpha$ | $\underline{e}(0.7,\alpha)$ | $\bar{e}(0.7,\alpha)$ | $\beta$ | $\underline{e}(0.7,\beta)$ | $\bar{e}(0.7,\beta)$ | $\gamma$ | $\underline{e}(0.7,\gamma)$ | $\bar{e}(0.7,\gamma)$ |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 3.06 E-05 | 3.14 E-05 | 0.2 | 3.37 E-05 | 3.37 E-05 | 0.3 | 3.37 E-05 | 3.37 E-05 |
| 0.2 | 3.14 E-05 | 3.20 E-05 | 0.4 | 3.29 E-05 | 3.31 E-05 | 0.5 | 3.28 E-05 | 3.30 E-05 |
| 0.4 | 3.21 E-05 | 3.25 E-05 | 0.6 | 3.21 E-05 | 3.25 E-05 | 0.7 | 3.19 E-05 | 3.23 E-05 |
| 0.6 | 3.29 E-05 | 3.31 E-05 | 0.8 | 3.14 E-05 | 3.20 E-05 | 0.9 | 3.10 E-05 | 3.17 E-05 |
| 0.8 | 3.37 E-05 | 3.37 E-05 | 1.0 | 3.06 E-05 | 3.14 E-05 | 1.0 | 3.06 E-05 | 3.14 E-05 |

**Conclusion:**

In this paper, the numerical solution of the second-order differential equation under the effect of a neutrosophic environment in the boundary condition has been developed. The finite difference method and the generalized Hukuhara differentiability are applied to present the approximate solution of each one of the four classes of the solution. The results of the proposed solution are compared with the analytical solution of the same problem even after applying generalized Hukuhara differentiability. It is found that by decreasing the step size, the error is decreased, and it can be classified as an acceptable error when it reaches 10^-6 at h = 0.005. In future work, new numerical progressed methods will be presented and applied for numerical application to investigate if it will enhance the error in such an uncertain environment.

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# Neutrosophic-based correlation analysis for fingerprint image pattern recognition and matching

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**Abstract**. A sophisticated mathematical model known as neurosophic extends the vague concept to tackle real-world issues and applications. The neutrosophic connection of the image can be harnessed to ascertain its correlation pattern. This paper aims to employ the neutrosophic method for detecting correlations among fingerprint images using neutrosophic-based pattern analysis. Additionally, it will propose four strategies for identifying relationships within image data by employing the neutrosophic approach. The study explores the four fundamental forms of fingerprint images and experiments with various $\alpha$ values. An $\alpha$ value of 0.99 or higher is favored for image matching.

**Keywords:** Neutrosophic computer vision; neutrosophic image processing; neutrosophic correlation; fingerprint image matching; neutrosophic biometric

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## 1. Introduction

About Biometric. Biometric is a human recognition system that refers to characteristics of their biological and behavioral. The types of biometrics are facial recognition, fingerprints, finger geometry, iris recognition, vein recognition, retina scanning, voice recognition, DNA matching, and digital signatures. Generally, biometric recognition contains two steps which are enrollment and recognition. First, the system captures the person's physiognomy to create a digital representation, and it becomes a template to be compared by the enrollment systems. The recognition system distinguishes the person's characteristics converting this into the same digital format as the template. Fingerprint identification is based on the analysis of the ridge patterns on the tips of fingers. Sensors generate images of the ridges, and these were scanned their for structural features (called minutiae) such as branches or terminations [10].
Origin of Neutrosophic. The fuzzy concept was first introduced by Zadeh [11] which deals with membership functions, and this concept solves real life problems successfully. The Fuzzy Logic

System (FLS) was first implemented in pattern recognition by Mendel [14]. Zimmermann evaluates the efficiency of fuzzy concepts in computer vision [31] , Atanassov [1] proposed the Intuitionistic Fuzzy Sets ($IFS$) in 1986 the extension of $IFS$ is known as Neutrosophic Set ($NS$) is a advanced mathematical concept and also a branch of philosophy which was introduced by Smarandache [23] in 1998 which discuss about membership function ($T$), non-membership function ($F$), indeterminacy membership function ($I$). The correlation for neutrosophic data initially originated from Hanafy et al., [9]. Later the correlation coefficient for interval neutrosophic set formulated by Broumi et al [32]. Image matching measures the degree of similarity between two image sets that are superimposed on one another plays an important role in many areas, such as pattern recognition, image analysis, and computer vision [29]. The similarity was measured by various operations: feature extraction, distance transformation, and similarity measurement.

This article, section 1 gives a brief introduction about fuzzy sets and neutrosophic sets. Then, section 2 focuses on the related work based on image processing and image similarities with the concept of fuzzy sets and neutrosophic sets. The following section 3 defines some basic definitions of neutrosophic sets for the image domain. Then, section 4 explains our contribution in this article to find the relationship between the images. The experimental analysis is done in section 5, here we do three types of analysis methods. Methodology-1 finds the relationship between the two images. Methodology-2 is based on relationship finding the matching for multiple images. To calculate the performance quality of proposal methodology-3 experiments with a combined image database. Finally, section 6 gives the conclusion of the proposal based on the getting results.

## 2. **Related work**

I.K.Vlachos [27] used the Intuitionistic fuzzy cross-entropy and discrimination approach to determine distance, similarity, dissimilarity, and correlation. This allows us to obtain image discrimination information. Guo, et al. [6] find neutrosophic similarity scores this approach can find two different criteria which are the local mean intensity criterion and local homogeneity of the images. R.M.Zulqarnain, et al. [32] used the Multipolar interval-valued neutrosophic soft set (mPIVNSS) method to find the similarity measurement in medical diagnoses. In this proposal, Euclidean distance and Hamming distance are used for finding the similarity distances. Gao Q. [7] invented a comparison of fake and real fingerprint images based on different types of sensors. The author used the NIST database and ATVS-FFp database for analysis. This proposal distinguished between authentic and fake fingerprint images using the matching score. Jun Ye [28] introduced simplified neutrosophic sets (SNSs) for Cosine similarity measurements. This method is very useful in medical image diagnosis problems.

M.Baskar, et al. [2] introduced Region Centric Minutiae Propagation Measure (RCMPM) which removes noise from the image based on a forgery detection algorithm and classifies the minutiae, loop, whorl is some basic patterns to classify the fingerprint images. R.P.Sharma, and S.Dey [21] used together Fuzzy c-means technique, Gabor filter, Fourier transforms, and a diffusion filter to extract classification features such as mean, moisture, variance, uniformity, contrast, ridge valley area uniformity, and ridge valley for the fingerprint images. Two-stage quality adaptive fingerprint image enhancement method can classify the images as the classes dry, wet, normally dry, normal wet, and good [30]. The identification of gender also relied on fingerprints. Based on fingerprints, Gustisyaf et al. [8] classified gender using a convolutional neural network. The classification accuracy value for 49270 images is 99.9667%. For the SDUMLA-HMT database, gender determination is achieved with 99.8% accuracy [17] which used feature-level fusion. The Linear Discriminant Analysis, K-Nearest Neighbor classifier, and Support Vector Machine algorithms were addressed in this proposal. The accuracy for the KNN correlation method was 89.8%. Narwal et al. [16] proposed an algorithm that is not restricted to the local fingerprint image features, such as a ridge, location, and direction. In order to select the best principal features, the author used the PCA algorithm. The Gabour filter made it much smoother to match the templates.

## 3. **Preliminaries**

**Definition 3.1.** Let $A$ be an universe of data, the element in $A$ denoted by $a$, then the neutrosophic set $(NS)$, of the object $A$ is in the form [3, 22]

$$A = \{(a, T_A(a), I_A(a), F_A(a)\}$$

where the neutrosophic membership functions $T, I, F : A \rightarrow ]^-0, 1^+[$ define respectively the degrees of truth, indeterminacy and the falsity of the element $a \in A$ to the set condition.

$$^-0 \leq T_A(a) + I_A(a) + F_A(a) \leq 3^+$$

**Definition 3.2.** A neutrosophic image $P_{NS}$ is charecterized with neutrosophic membership functions which are $T, I, F$ where $P_{NS}$ are the intensities of the image. Universally for neutrosophic image approach is gray intensities of the image. The image neutrosophic set is defined as [3, 5]

$$P_{NS}A(i,j) = \{T_A(i,j), I_A(i,j), F_A(i,j)\} \tag{1}$$

In general the arithmetic mean is consider as truth membership values and the standard deviation of the image is consider as indeterminacy membership. The neutrosophic transformation intensity of the image is define by the following formulae

$$\begin{aligned} T_A(i,j) &= \frac{\bar{p}(i,j) - \bar{p}\min}{\bar{p}\max - \bar{p}\min} \\ \bar{p}_A(i,j) &= \frac{1}{w*w}\sum_{m=i-\frac{w}{2}}^{m=i+\frac{w}{2}}\sum_{n=j-\frac{w}{2}}^{n=j+\frac{w}{2}} p(m,n) \\ I_A(i,j) &= \frac{\delta(i,j) - \delta\min}{\delta\max - \delta\min} \\ \delta_A(i,j) &= abs(p(i,j) - \bar{p}_A(i,j)) \\ F_A(i,j) &= 1 - T_A(i,j) \end{aligned}$$

where $\bar{p}_A(i,j)$ denotes the pixel mean in the region $w*w$ and $w$ is generally $w = 2n+1, \ (n \geq 1)$.

## 4. Proposed method

**Definition 4.1** (Neutrosophic correlaltion)**.** Let $A = [a_{i,j}]_{m\times n}$ and $B = [b_{i,j}]_{m\times n}$ be two images with $m$ rows and $n$ columns then the image's neutrosophic correlation is defined as follows

$$K(A,B) = \frac{\frac{1}{3}C(A,B)}{\delta_A\delta_B} \tag{2}$$

where

$$\begin{aligned} C(A,B) &= \left(\begin{array}{c} \max(T_A(i\pm\Delta i, j\pm\Delta j), T_B(i\pm\Delta i, j\pm\Delta j)) - \\ \left(\begin{array}{c}\max(I_A(i\pm\Delta i, j\pm\Delta j), I_B(i\pm\Delta i, j\pm\Delta j)) + \\ \max(F_A(i\pm\Delta i, j\pm\Delta j), F_B(i\pm\Delta i, j\pm\Delta j))\end{array}\right)\end{array}\right) \\ \delta(A_{ij}) &= \frac{1}{3}\sqrt{T_A(i,j)^2 + I_A(i,j)^2 + F_A(i,j)^2} \\ \delta(B_{ij}) &= \frac{1}{3}\sqrt{T_B(i,j)^2 + I_B(i,j)^2 + F_B(i,j)^2} \end{aligned}$$

Then the correlation pattern for the matching level value $\alpha$ is defined as

$$\Im[K(A,B)]_{m\times n} = \begin{cases} K(A,B) & \text{if } K(A,B) \geq \alpha \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

If we need to find the correlation pattern for the image $B$ then

$${}_B\Im[K(A,B)]_{m\times n} = \begin{cases} b_{ij} & \text{if } \Im[K(A,B)] = K(A,B) \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

**Definition 4.2** (**Neutrosophic Based Spearman Rank correlation**)**.** Let $A = [a_{i,j}]_{m\times n}$ and $B = [b_{i,j}]_{m\times n}$ be two images with $m$ rows and $n$ columns. Let $A's$ truth membership subset $T[a(i, i\pm\Delta i, j, j\pm\Delta j)$ contains $X_{T_i} = (x_{T_1}, x_{T_2}...x_{T_n})$ data and $B's$ truth membership subset $T[b(i, i\pm\Delta i, j, j\pm\Delta j)$ contains $Y_{T_i} = (y_{T_1}, y_{T_2}...y_{T_n})$ data. The image's neutrosophic based Spearman Rank correlation is defined as follows

The truth membership's correlation is

$$r_T(a_{ij}, b_{ij}) = \frac{\sum_{i=1}^{n}(r(X_{T_i}) - \overline{r(X_T)}).(r(Y_{T_i}) - \overline{r(Y_T)})}{\sqrt{[\sum_{i=1}^{n}(r(X_{T_i}) - \overline{r(X_T)})^2][\sum_{i=1}^{n}(r(Y_{T_i}) - \overline{r(Y_T)})^2]}} \tag{5}$$

where

$$\begin{aligned} r(X_{T_i}) &= \text{rank of } X_{T_i} \\ r(Y_{T_i}) &= \text{rank of } Y_{T_i} \\ \overline{r(X_T)} &= \frac{\sum_{i=1}^{n} r(X_{T_i})}{N} \\ \overline{r(Y_T)} &= \frac{\sum_{i=1}^{n} r(Y_{T_i})}{N} \\ N &= \text{Total number of the data} \end{aligned}$$

Similarly, indeterminacy rank correlation is

$$r_I(a_{ij}, b_{ij}) = \frac{\sum_{i=1}^{n}(r(X_{I_i}) - \overline{r(X_I)}).(r(Y_{I_i}) - \overline{r(Y_I)})}{\sqrt{[\sum_{i=1}^{n}(r(X_{I_i}) - \overline{r(X_I)})^2][\sum_{i=1}^{n}(r(Y_{I_i}) - \overline{r(Y_I)})^2]}}$$

Then the neutrosophic based Spearman Rank correlation of universal image data is

$$r_{NR}(A, B)_{m\times n} = \max\{P((r_T(a_{ij}, b_{ij})_{m\times n}), P(r_I(a_{ij}), b_{ij})_{m\times n})\}$$

For the correlation pattern matching value of $\alpha$ then the image is as follows

$$_{\alpha}r_{NR}(A, B)_{m\times n} = \begin{cases} r_{NR}(A, B) & \text{if } r_{NR}(A, B) > \alpha \\ 0 & \text{if } r_{NR}(A, B) \leq \alpha \end{cases} \tag{6}$$

The image $B$'s correlation pattern is

$$\Im({}_{\alpha B}r_{NR}(A, B)_{m\times n}) = \begin{cases} b_{ij} & \text{if } {}_{\alpha}r_{NR}(A, B) > \alpha \\ 0 & \text{if } {}_{\alpha}r_{NR}(A, B) \leq \alpha \end{cases} \tag{7}$$

**Definition 4.3** (**Neutrosophic Based Kendall Rank Correlation**). Let $A = [a_{i,j}]_{m\times n}$ and $B = [b_{i,j}]_{m\times n}$ be two images with $m$ rows and $n$ columns. Let $A's$ truth membership subset $T[a(i, i \pm \Delta i, j, j \pm \Delta j)$ contains $X_{T_i} = (x_{T_1}, x_{T_2}...x_{T_n})$ data and $B's$ truth membership subset $T[b(i, i \pm \Delta i, j, j \pm \Delta j)$ contains $Y_{T_i} = (y_{T_1}, y_{T_2}...y_{T_n})$ data. The image's neutrosophic based Kendall Rank correlation is defined as follows

The truth membership's correlation is

$$\tau_T(a_{ij}, b_{ij}) = \frac{\sum_1^n \sum_1^n sign(X_{T_{ki}} - X_{T_{kj}})sign(Y_{T_{ki}} - Y_{T_{kj}})}{n(n-1)}$$

The indeterminacy membership rank correlation is

$$\tau_I(a_{ij}, b_{ij}) = \frac{\sum_1^n \sum_1^n sign(X_{I_{ki}} - X_{I_{kj}})sign(Y_{I_{ki}} - Y_{I_{kj}})}{n(n-1)}$$

Then the neutrosophic based Kendall rank correlation for global image data is

$$\tau_{NR}(A,B)_{m\times n} = \{\max(P(\tau_{T(a_{ij},b_{ij})}), \max(P(\tau_{I(a_{ij},b_{ij})})\}$$

For the correlation pattern matching value of $\alpha$ then the image is as follows

$$_{\alpha}\tau_{NR}(A,B)_{m\times n} = \begin{cases} \tau_{NR}(A,B) & \text{if } \tau_{NR}(A,B) > \alpha \\ 0 & \text{if } \tau_{NR}(A,B) \leq \alpha \end{cases} \quad (8)$$

The image $B$'s correlation pattern is

$$\Im({}_{\alpha B}\tau_{NR}(A,B)_{m\times n}) = \begin{cases} b_{ij} & \text{if } {}_{\alpha}\tau_{NR}(A,B) > \alpha \\ 0 & \text{if } {}_{\alpha}\tau_{NR}(A,B) \leq \alpha \end{cases} \quad (9)$$

**Definition 4.4 (Neutrosophic Based Karl Pearson Correlation).** Let $A = [a_{i,j}]_{m\times n}$ and $B = [b_{i,j}]_{m\times n}$ be two images with $m$ rows and $n$ columns. Let $A's$ truth membership subset $T[a(i,i\pm\Delta i,j,j\pm\Delta j)]$ contains $X_{T_i} = (x_{T_1}, x_{T_2}...x_{T_n})$ data and $B's$ truth membership subset $T[b(i,i\pm\Delta i,j,j\pm\Delta j)$ contains $Y_{T_i} = (y_{T_1}, y_{T_2}...y_{T_n})]$ data. The image's neutrosophic based Karl Pearson correlation is defined as follows
The truth membership's rank correlation is

$$\rho_T(a_{ij},b_{ij}) = \frac{\sum_{i=1}^{n}(X_{T_k} - \overline{X_{T_k}})(Y_{T_k} - \overline{Y_{T_k}})}{\sqrt{\sum_{i=1}^{n}(X_{T_k} - \overline{X_{T_k}})^2}\sqrt{\sum_{i=1}^{n}(Y_{T_k} - \overline{Y_{T_k}})^2}}$$

The indeterminacy membership rank correlation is

$$\rho_I(a_{ij},b_{ij}) = \frac{\sum_{i=1}^{n}(X_{I_k} - \overline{X_{I_k}})(Y_{I_k} - \overline{Y_{I_k}})}{\sqrt{\sum_{i=1}^{n}(X_{I_k} - \overline{X_{I_k}})^2}\sqrt{\sum_{i=1}^{n}(Y_{I_k} - \overline{Y_{I_k}})^2}}$$

Then the neutrosophic based Karl pearson correlation for global image data is

$$\rho_{NR}(A,B)_{m\times n} = \max\{P(\rho_T(a_{ij},b_{ij})_{m\times n}), P(\rho_I(a_{ij},b_{ij})_{m\times n})\}$$

For the correlation pattern matching value of $\alpha$ then the image is as follows

$$_{\alpha}\rho_{NR}(A,B)_{m\times n} = \begin{cases} \rho_{NR}(A,B) & \text{if } \rho_{NR}(A,B) > \alpha \\ 0 & \text{if } \rho_{NR}(A,B) \leq \alpha \end{cases} \quad (10)$$

The image $B$ 's correlation pattern is

$$\Im({}_{\alpha B}\rho_{NR}(A,B)_{m\times n}) = \begin{cases} b_{ij} & \text{if } {}_{\alpha}\rho_{NR}(A,B) > \alpha \\ 0 & \text{if } {}_{\alpha}\rho_{NR}(A,B) \leq \alpha \end{cases} \quad (11)$$

**Algorithm 1** The correlation pattern

**Require:** $T, I, F$ values of Image $A$ & $B$

  **for** $i = 1 : m,\ j = 1 : n$ **do**

    $c_1 = \Im[K(A,B)]_{m\times n}$

    $c_2 =_\alpha \tau_{NR}(A,B)_{m\times n}$

    $c_3 =_\alpha r_{NR}(A,B)_{m\times n}$

    $c_4 =_\alpha \rho_{NR}(A,B)_{m\times n}$

    **if** mean$(c_1 + c_2 + c_3 + c_4 < \alpha)$ **then**

      Output (correlation pattern exist)

    **else if** mean$(c_1 + c_2 + c_3 + c_4 \geq \alpha)$ **then**

      Output (matching pattern exist)

    **end if**

  **end for**

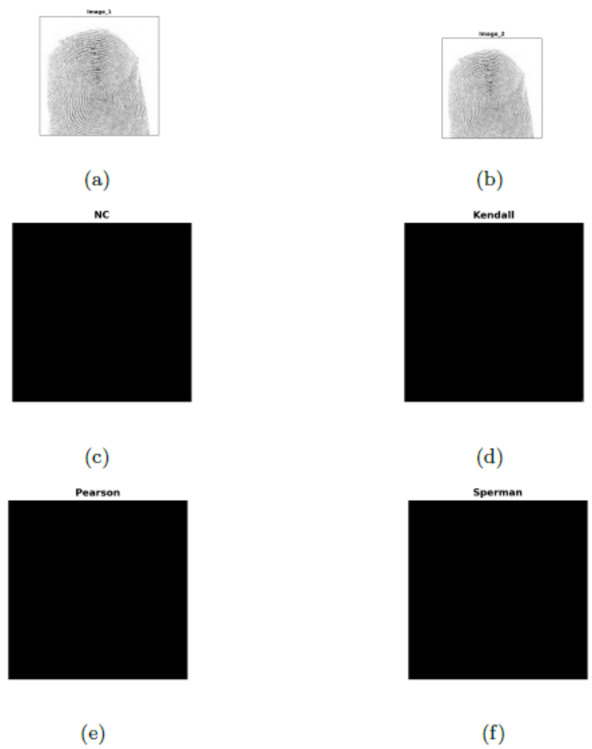


FIGURE 1. Proposed methods output for similar images. $\alpha$= 0.9, $h$= 3

## 5. Experiments and discussions

Finding the image's relationship is performed in the image matching task. Here we took fingerprint images as our application. The fingerprint image data is collected by the National Institute of Standards and Technology (NIST) database [4] named as SD302 which was organized by the Inteligence Advanced Research Projects Activity (IARPA), in September 2017.

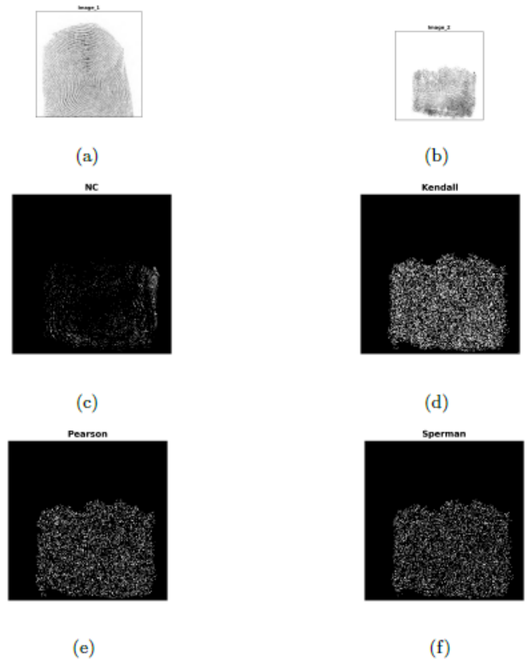


FIGURE 2. Proposed methods output for non-similar images. $\alpha$= 0.9, $h$= 3

Initially, the information pertaining to each latent fingerprint image was obtained by isolating the enclosed section from a comprehensive scene capture. The bounding coordinates were of non-rectangular shape, and pixels beyond this confined region were white. The dimensions of the image were approximated to be 1000 pixels per inch in terms of width and height. NIST supplied the image in Portable Network Graphics (PNG) format to facilitate utilization within conventional image processing software.

Arches: The finger's ridges form an uninterrupted pattern from end to end without any backward turns. Generally, an arch doesn't have a delta, but if it does, there shouldn't be a curving ridge between the core and delta points.

Loops: Loops involve ridges that make a backward turn without twisting. The loop's direction on the hand, rather than the card used for impressions, characterizes this backward rotation. The fingerprint impression resembles a mirror's reverse image. A loop pattern typically has only one delta.

Whorls: In a whorl pattern, at least one set of ridges forms a circuit. Therefore, any pattern with two or more deltas is classified as a whorl.

Tented arch: Similar to the plain arch, the tented arch pattern starts on one side of the finger and flows consistently to the other side.

The sample visualization of similar image is shown in Figure 1. We can determine that there are no patterns between the images because of their similarity. The $\alpha$ value in this case is fixed at 0.9. We get a matching score of 0.99 on average for images that are similar. We can deduce

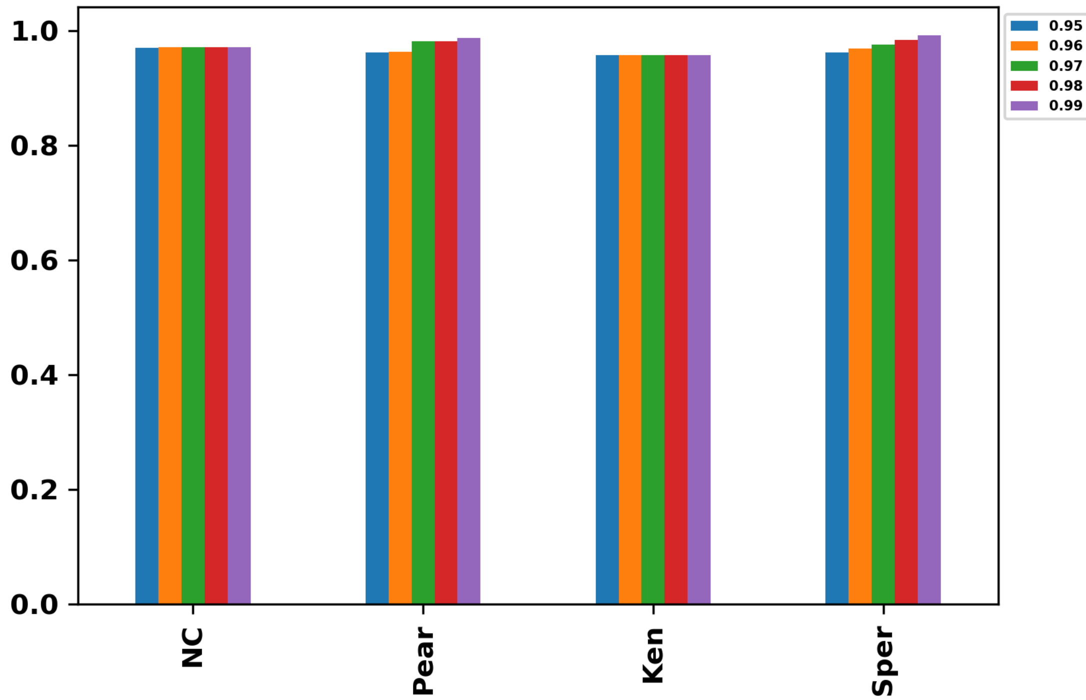


Figure 3. Analysis for multiple $\alpha$ values

that there is a perfect match between the images if the average matching score is 0.99 or higher. Whether the images are the same or not can be determined with clarity in this way. When the images are not similar to one another, Figure 2 illustrates the correlation between the images. The proposed approach reaches each type correlation pattern for non-similar images. Different variations exist for each method's matching score. We can determine the correlation pattern level of the images cumulatively based on the average matching score. When there are no similarities between the images, the proposed methods assist us in identifying correlation patterns.

The $\alpha$ factor is used in the next analysis. Our first task is to adjust the $\alpha$ value for better correlation patterns. We are able to obtain each correlation pattern and matching score for every $\alpha$ value. Therefore, the better $\alpha$ values must be fixed. Figure 3 illustrates the matching score level for the non-similar images. Here, we focus on values between 0.95 and 0.99. The NC, Pearson, Kendall, and Spearman methods exhibit the best matching score for the similar pictures for the taken values when $\alpha = 0.99$. If the images have the same patterns, this method of analysis helps us find the matching; otherwise, when the images have different patterns, we can experience correlation patterns. The matching score for each type of image is listed in Table 1 below. According to the table, perfect matching images have an average matching

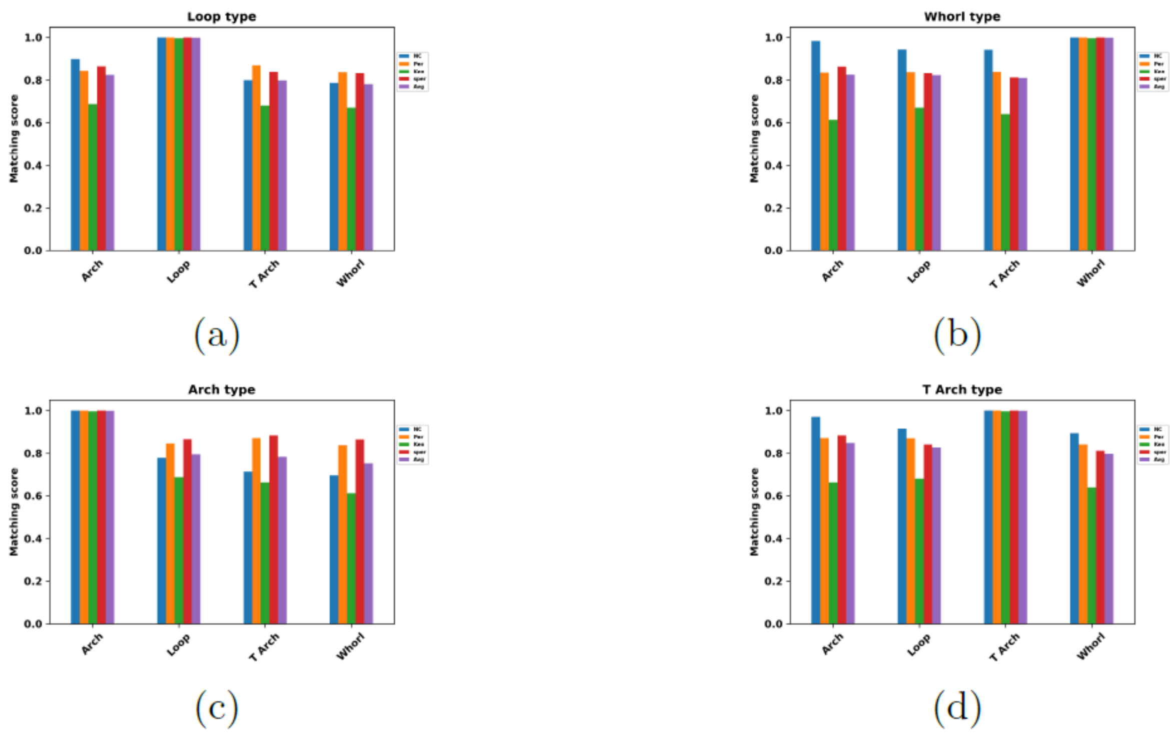


Figure 4. Fingerprint image type matching score analysis. (a) loop, (b) whorl, (c) arch, (d) tented arch

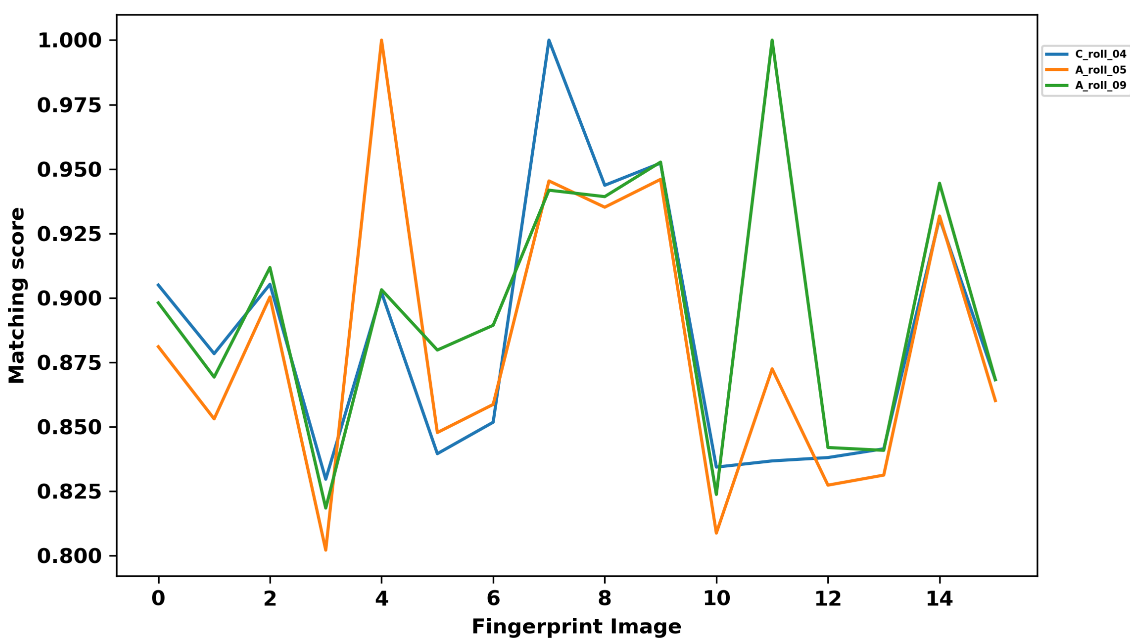


Figure 5. Result of random images

Table 1. Fingerprint type wise analysis

| Image Type | NC | Pearson | Kendall | Spearman | Average |
|---|---|---|---|---|---|
| Loop | 0.8976 | 0.8464 | 0.688 | 0.8656 | 0.8244 |
| | 1 | 1 | 0.9968 | 1 | 0.9992 |
| | 0.8 | 0.8704 | 0.6816 | 0.8416 | 0.7984 |
| | 0.7872 | 0.84 | 0.672 | 0.832 | 0.7828 |
| Whorl | 0.984 | 0.8384 | 0.6128 | 0.864 | 0.8248 |
| | 0.9456 | 0.84 | 0.672 | 0.832 | 0.8224 |
| | 0.944 | 0.8416 | 0.64 | 0.8128 | 0.8096 |
| | 1 | 1 | 0.9968 | 1 | 0.9992 |
| Arch | 1 | 1 | 0.9968 | 1 | 0.9992 |
| | 0.7792 | 0.8464 | 0.688 | 0.8656 | 0.7948 |
| | 0.7136 | 0.872 | 0.664 | 0.8848 | 0.7836 |
| | 0.696 | 0.8384 | 0.6128 | 0.864 | 0.7528 |
| T Arch | 0.9712 | 0.872 | 0.664 | 0.8848 | 0.848 |
| | 0.9152 | 0.8704 | 0.6816 | 0.8416 | 0.8272 |
| | 1 | 1 | 0.9968 | 1 | 0.9992 |
| | 0.8944 | 0.8416 | 0.64 | 0.8128 | 0.7972 |

score greater than 0.99, indicating that the images were similar. If not, there are differences between the images. From the previous analysis we obtain the best matching score alpha should be greater or equal to 0.99 and the $h$ is should be 3.

Based on these values we will analysis the further analysis. We choose two random images from each SD302 dataset types for the final analysis. Since we have already captured, the images could be any kind of fingerprint image. The image's dimensions are fixed at 250 by 250. $h = 3$, $\alpha$ =0.99. We selected three images at random from the generated dataset. Since we used three sample images for matching, the average matching score for the remaining three images was below 0.99. This means that the alpha value for those three images should be greater than 0.99. The data set includes plain, rolled, and touch-free impression fingerprints as well as numerous sets of plain palm impressions that were taken from a variety of devices. All these types were chosen by the article under analysis for analysis. The results of the proposed methods matching scores are listed in Table 2. From the table, we observed that only three images the random image attained the highest level of matching score. This proves how well the proposed technique worked with both similar and dissimilar images. Additionally found the accurate matching for images chosen at random. This demonstrates the method's duality.

TABLE 2. Proposed methods result for random image matching

| Random image ID | | |
|---|---|---|
| **C_roll_04** | **A_roll_05** | **A_roll_09** |
| $0.905 \pm 0.0764$ | $0.8811 \pm 0.0288$ | $0.8981 \pm 0.0195$ |
| $0.8784 \pm 0.1085$ | $0.8531 \pm 0.0642$ | $0.8693 \pm 0.022$ |
| $0.9053 \pm 0.0902$ | $0.9004 \pm 0.0291$ | $0.9118 \pm 0.0294$ |
| $0.8297 \pm 0.1706$ | $0.8022 \pm 0.1155$ | $0.8185 \pm 0.0599$ |
| $0.9021 \pm 0.0731$ | $1.0 \pm 0.0$ | $0.9032 \pm 0.0653$ |
| $0.8396 \pm 0.1595$ | $0.8478 \pm 0.0692$ | $0.8798 \pm 0.0376$ |
| $0.8518 \pm 0.1524$ | $0.8587 \pm 0.047$ | $0.8894 \pm 0.0466$ |
| $1.0 \pm 0.0$ | $0.9454 \pm 0.0126$ | $0.9418 \pm 0.0271$ |
| $0.9437 \pm 0.0251$ | $0.9352 \pm 0.0186$ | $0.9393 \pm 0.032$ |
| $0.9523 \pm 0.0268$ | $0.946 \pm 0.0155$ | $0.9527 \pm 0.0268$ |
| $0.8344 \pm 0.162$ | $0.8088 \pm 0.1126$ | $0.8238 \pm 0.0609$ |
| $0.8368 \pm 0.1617$ | $0.8725 \pm 0.035$ | $1.0 \pm 0.0$ |
| $0.8381 \pm 0.1475$ | $0.8274 \pm 0.0734$ | $0.842 \pm 0.0447$ |
| $0.8415 \pm 0.1445$ | $0.8313 \pm 0.0677$ | $0.8409 \pm 0.0436$ |
| $0.9309 \pm 0.0609$ | $0.9318 \pm 0.0249$ | $0.9445 \pm 0.0332$ |

## 6. Conclusion

We proposed four different types of correlation methods based on neutrosophic sets in this article. We have performed analysis for a number of fingerprint image types, including the arch, loop, whorl, and tented arch. According to the analysis, it is very hard to identify the matching between the images without first fixing the matching pattern. Therefore, we tried to look at the matching task for different $\alpha$ values. Finally, an $\alpha$ value of 0.99 results in better image matching patterns. The findings suggest that there is matching if the matching score is greater than or equal to the $\alpha$ . If it is less, we can recognise the image's correlation patterns. We can perform the matching task locally for the fingerprint images since it went well. We will expand the concept in the feature to include gradient-level image matching.

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# An Introduction to NeutroHyperVector Spaces

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**Abstract**. The aim of this paper is to combine the notion of NeutroAlgebra, that includes the partiality and indeterminacy in the operations and axioms of algebraic structures, with algebraic hyperstructures. In this regard, a NeutroHyperVector space is introduced and various examples are given. Next, some types of linear transformations between NeutroHyperVector spaces are presented and some properties of mentioned concepts are studied. Finally, by giving a suitable field, the Cartesian product of NeutroHyperVector spaces over a common field is constructed, under certain conditions, and supported by interesting examples.

**Keywords:** Hypervector space; NeutroHyperoperation; NeutroAxiom; NeutroHyperVector space; SubNeutro-Hyperspace; NeutroTransformation

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## 1. **Introduction**

There are various mathematical tools for modeling the real facts. For example, the theory of fuzzy sets introduced by Zadeh [1] in 1934 expresses the vague and uncertain properties, and the theory of intuitionistic fuzzy sets introduced by Atanassov [2] in 1983 adds the non-membership information. Neutrosophic theory introduced by Smarandache in 1995 as a generalization of fuzzy sets and intuitionistic fuzzy set, is another way to this goal. In this idea, information related to the truth (T), falsity (F) and indeterminacy (I) of the problem is considered. In fact, the indeterminacy distinguishes neutrosophy from the other philosophy. Then some linked methods have been studied, such as neutrosophic rough set [3], complex neutrosophic set [4], neutrosophic soft set [5, 6]. The theory of neutrosophic set applied in various branches such as in medical diagnosis [7], decision-making process [8], pattern recognition [9], economics [10] and operation research [11].

In 2019 Smarandache [12] introduced the notion of NeutroAlgebra, as a generalization of the classic algebraic structures, where it has operations and axioms that are partially well-defined, partially indeterminate, and partially false. By applying the new idea to another algebraic structures, many NeutroAlgebras have been studied, for example NeutroGroups [13], NeutroRings [14] and NeutroOrderedAlgebra [15].

On the other hand, the theory of algebraic hyperstructures was born in 1934 by Marty [16] as a generalization of algebraic structures, where the hyperoperation of two elements is a non-empty set. This theory has been extended in many branches of mathematics such as fields, lattices, rings, quasigroups, semigroups, ordered structures, combinatorics, topology, geometry, graphs, codes, etc.; for example, see the books [17–19]. The notion of hypervector space was introduced by Scafati-Tallini [20] in 1990 and has been investigated by herself, Ameri [21], Sedghi [22] and the author [23–27].

Recently, NeutroAlgebraic structures have been extended to NeutroHyperalgebraic structures; Ibrahim [28] defined NeutroHypergroups and Al-Tahan [29] and Rezaei [30] studied some properties of NeutroSemihypergroups. Now in this paper, we apply the theory of neutrosophy in hypervector spaces and introduce NeutroHyperVector spaces as an alternative structure and a type of generalization for hypervector spaces. In Section 3, we introduce the notion of NeutroHyperVector spaces, present some interesting examples and shortly study the concept of SubNeutroHyperspaces. In Section 4, we investigate the relation of two NeutroHyperVector spaces by using of different types of transformations, especially, the behavior of SubNeutroHyperspaces under transformations and their inverse, again supported by some examples. Finally, we make new NeutroHyperVector spaces by Cartesian product of NeutroHyperVector spaces and give some examples.

## 2. Preliminaries

In this section, we present some definitions and propositions that we shall use in later.

A hyperoperation over a non-empty set $S$ is a mapping "$\circ : S \times S \to P_*(S)$", where $P_*(S)$ is the set of non-empty subsets of $S$. If $A, B \subseteq S$, then $A \circ B = \bigcup\limits_{a\in A, b\in B} a \circ b$. Especially, $A \circ b = A \circ \{b\} = \bigcup\limits_{a\in A} a \circ b$ and $a \circ B = \{a\} \circ B = \bigcup\limits_{b\in B} a \circ b$.

**Definition 2.1.** [20] Let $K$ be a field, $(V, +)$ be an Abelian group and $P_*(V)$ be the set of all non-empty subsets of $V$. We define a hypervector space over $K$ to be the quadruplet $(V, +, \circ, K)$, where "$\circ$" is an external hyperoperation

$$\circ : K \times V \longrightarrow P_*(V), \tag{1}$$

such that for all $a, b \in K$ and $x, y \in V$ the following conditions hold:

($HV_1$) $a \circ (x + y) \subseteq a \circ x + a \circ y$, right distributive law, (HV$_2$) $(a+b)\circ x \subseteq a\circ x + b\circ x$, left distributive law,

(HV$_3$) $a\circ(b\circ x) = (ab)\circ x$,

(HV$_4$) $a\circ(-x) = (-a)\circ x = -(a\circ x)$,

(HV$_5$) $x \in 1\circ x$,

where in (HV$_1$), $a\circ x + a\circ y = \{p+q : p\in a\circ x, q\in a\circ y\}$. Similarly, it is in (HV$_2$). Also, in (HV$_3$), $a\circ(b\circ x) = \bigcup\limits_{t\in b\circ x} a\circ t$.

$V$ is called strongly right distributive, if we have equality in (HV$_1$). In a similar way we define the strongly left distributive hypervector spaces.

In the sequel of this paper, $V$ denotes a hypervector space over the field $K$, unless otherwise is specified.

**Example 2.2.** Let $K = \mathbb{Z}_2 = \{0,1\}$ be the field of two numbers with the following operations:

| $+$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

and

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

Also, let $V = \mathbb{Z}_3 = \{0,1,2\}$ be an Abelian group with the following operation:

| $+$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 0 |
| 2 | 2 | 0 | 1 |

Then $(\mathbb{Z}_3, +, \circ_1, \mathbb{Z}_2)$ and $(\mathbb{Z}_3, +, \circ_2, \mathbb{Z}_2)$ are hypervector spaces over the field $\mathbb{Z}_2$ with the following external hyperoperations, where they are not strongly left or right hypervector spaces:

| $\circ_1$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{0\}$ | $\{0,1,2\}$ | $\{0,1,2\}$ |

and

| $\circ_2$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{0\}$ | $\{1,2\}$ | $\{1,2\}$ |

(2)

## 3. NeutroHyperVector Spaces

Let $U$ be the universe of discourse. Then a *neutrosophic set* is an object having the form

$$A = \{(x, \mu_A(x), \omega_A(x), \nu_A(x)), x\in U\}, \tag{3}$$

where the functions $\mu, \omega, \nu : U \to [0,1]$ define respectively the degree of membership, the degree of indeterminacy and the degree of non-membership of $x$ to the set $A$.

For example, the following functions define a neutrosophic set in $U = \{1, 2, 3, 4\}$:

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\mu_A(x)$ | 0.4 | 0 | 0.7 | 0.7 |
| $\omega_A(x)$ | 0.6 | 0.2 | 0 | 0.1 |
| $\nu_A(x)$ | 0.5 | 0.8 | 0.3 | 0.2 |

A hyperoperation "$\circ : S \times S \to P_*(U)$", where $U$ is the universe of discourse containing $S$, is called a *NeutroHyperoperation* on $S$, if $x \circ y \subseteq S$, for some $x, y \in S$ (the degree of truth "T") and some (or all) of the following conditions hold:

(1) $x \circ y \nsubseteq S$, for some $x, y \in S$ (the degree of falsity "F");

(2) $x \circ y$ is indeterminate in $S$, for some $x, y \in S$ (the degree of indeterminacy "I").

For example, if $U = \{1, 2, \ldots, 10\}$ and $S = \{2, 5, 8\}$, then the followings are NeutroHyperoperations on $S$:

| $\circ$ | 2 | 5 | 8 |
|---|---|---|---|
| 2 | $\{3\}$ | $\{2, 4\}$ | $\{2, 8\}$ |
| 5 | $\{2, 5\}$ | $\{2, 5, 8\}$ | $\{5\}$ |
| 8 | $\{2\}$ | $\{2\}$ | $\{2\}$ |

and

| $\circ$ | 2 | 5 | 8 |
|---|---|---|---|
| 2 | $\{8\}$ | $\{2, 5\}$ | |
| 5 | $\{2, 5\}$ | $\{2, 5, 8\}$ | $\{5\}$ |
| 8 | $\{2\}$ | | $\{2\}$ |

The hyperoperation "$\circ$" is called an *AntiHyperoperation* on $S$, if $x \circ y \nsubseteq S$, for all $x, y \in S$. The following example is an AntiHyperoperation on $S = \{2, 5, 8\}$:

| $\circ$ | 2 | 5 | 8 |
|---|---|---|---|
| 2 | $\{1, 2\}$ | $\{1, 5\}$ | $\{4\}$ |
| 5 | $\{10\}$ | $\{2, 4, 8\}$ | $\{3\}$ |
| 8 | $\{1, 2\}$ | $U$ | $\{2, 3\}$ |

A hyperoperation "$\circ : S \times S \to P_*(U)$", where $U$ is the universe of discourse containing $S$, is called *NeutroAssociative* on $S$, if $x \circ (y \circ z) = (x \circ y) \circ z$, for some $x, y, z \in S$ (the degree of truth "T") and some (or all) of the following conditions hold:

(1) $x \circ (y \circ z) \neq (x \circ y) \circ z$, for some $x, y, z \in S$ (the degree of falsity "F");

(2) for some $x, y, z \in S$, $x \circ (y \circ z)$ is indeterminate in $S$, or $(x \circ y) \circ z$ is indeterminate in $S$, or we can not find if $x \circ (y \circ z)$ and $(x \circ y) \circ z$ are equal (the degree of indeterminacy "I").

For example, if $U = \{1, 2, \ldots, 10\}$ and $S = \{2, 5, 8\}$, then the following hyperoperations are NeutroAssociative on $S$:

| $\diamond_1$ | 2 | 5 | 8 |
|---|---|---|---|
| 2 | $\{2, 5\}$ | $\{2, 5\}$ | $\{5\}$ |
| 5 | $\{2, 5\}$ | $\{5, 8\}$ | $\{2\}$ |
| 8 | $\{8\}$ | $\{8\}$ | $\{5\}$ |

| $\diamond_2$ | 2 | 5 | 8 |
|---|---|---|---|
| 2 | $\{2, 5\}$ | $\{2, 5\}$ | $\{5\}$ |
| 5 | $\{2, 5\}$ | $\{5, 8\}$ | $\{2\}$ |
| 8 | $\{8\}$ | | $\{5\}$ |

| $\diamond_3$ | 2 | 5 | 8 |
|---|---|---|---|
| 2 | $\{2, 5\}$ | $\{2, 5\}$ | $\{5\}$ |
| 5 | $\{2, 5\}$ | $\{5, 8\}$ | $\{2\}$ |
| 8 | | $\{8\}$ | $\{5\}$ |

More precisely, $\diamond_1$ is NeutroAssociative on $S$, since $2 \diamond_1 (2 \diamond_1 2) = (2 \diamond_1 2) \diamond_1 2$ and $5 \diamond_1 (5 \diamond_1 5) \neq (5 \diamond_1 5) \diamond_1 5$. $\diamond_2$ is NeutroAssociative on $S$, since $2 \diamond_2 (2 \diamond_2 2) = (2 \diamond_2 2) \diamond_2 2$ and $(5 \diamond_2 5) \diamond_2 5$ is indeterminate in $S$. $\diamond_3$ is NeutroAssociative on $S$, since $2 \diamond_3 (2 \diamond_3 2) = (2 \diamond_3 2) \diamond_3 2$, $5 \diamond_3 (5 \diamond_3 5) \neq (5 \diamond_3 5) \diamond_3 5$ and $(8 \diamond_3 2) \diamond_3 5$ is indeterminate in $S$.
The hyperoperation " $\circ$ " is called *AntiAssociative* on $S$, if $x \circ (y \circ z) \neq (x \circ y) \circ z$, for all $x, y, z \in S$. The following hyperoperation is AntiAssociative on $S$:

| $\diamond$ | 2 | 5 | 8 |
|---|---|---|---|
| 2 | $\{2,5\}$ | $\{2,8\}$ | $\{2\}$ |
| 5 | $\{5\}$ | $\{8\}$ | $\{2\}$ |
| 8 | $\{5\}$ | $\{8\}$ | $\{5,8\}$ |

**Definition 3.1.** Let $K$ be a field and $(G,+)$ be a group. Then an external hyperoperation "$\circ : K \times G \to P_*(U)$", where $U$ is the universe of discourse containing $G$, is called an *external NeutroHyperoperation* on $G$, if $a \circ x \subseteq G$, for some $a \in K,\ x \in G$ (the degree of truth "T") and at least one of the following conditions hold:

(1) $a \circ x \nsubseteq G$, for some $a \in K,\ x \in G$ (the degree of falsity "F");

(2) $a \circ x$ is indeterminate in $G$, for some $a \in K,\ x \in G$ (the degree of indeterminacy "I").

The external hyperoperation " $\circ$ " is called an *external AntiHyperoperation* on $G$, if $a \circ x \nsubseteq G$, for all $a \in K,\ x \in G$.

**Example 3.2.** Consider the field $K = \mathbb{Z}_2 = \{0,1\}$ and the Abelian group $G = \mathbb{Z}_3 = \{0,1,2\}$ defined in Example 2.2. Then the followings are external NeutroHyperoperations on $G$, where $U = \{0,1,2,3,4,5\}$:

| $\circ_3$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $\{0,4\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{1\}$ | $\{1,2\}$ | $\{0,1,2\}$ |

| $\circ_4$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $\{0,1\}$ | $\{1,2\}$ | $\{1,2\}$ |
| 1 | $\{1\}$ | | $\{0,1,2\}$ |

| $\circ_5$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $\{0,2\}$ | $\{1,2\}$ | $\{1,2\}$ |
| 1 | $\{2,4\}$ | | $\{0,1,2\}$ |

Note that in the first table, $0 \circ_3 0 = \{0,4\}$ is not a subset of $\mathbb{Z}_3$; in the second table, $1 \circ_4 1$ is indeterminate in $\mathbb{Z}_3$ and in the third table, $0 \circ_5 1 = \{2,4\}$ is not a subset of $\mathbb{Z}_3$ and $1 \circ_5 1$ is indeterminate in $\mathbb{Z}_3$. Also, the following is an external AntiHyperoperation on $G$:

| $\circ$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $\{0,4\}$ | $\{0,3\}$ | $\{5\}$ |
| 1 | $\{1,4,5\}$ | $\{1,3\}$ | $\{0,5\}$ |

A NeutroHyperVector space is an alternative of a hypervector space $(V,+,\circ,K)$ such that "$+ : V \times V \to P_*(V)$" is a NeutroHyperoperation, or "$\circ : K \times V \to P_*(V)$" is an external NeutroHyperoperation, or at least it has one NeutroAxiom. Thus, there are several types of NeutroHyperVector spaces, based on the number of NeutroOperation, NeutroHyperoperation and NeutroAxioms.

In this paper, we consider the following definition for a NeutroHyperVector space:

**Definition 3.3.** Let $(V, +, \circ, K)$ be a hypervector space over the field $K$ such that "$+ : V \times V \to V$" is an operation on $V$ and "$\circ : K \times V \to P_*(U)$" is an external NeutroHyperoperation, where $U$ is the universe of discourse containing $V$. Then $V$ is called a *NeutroHyperVector space* over the field $K$, if at least one of the following NeutroAxioms hold:

$NHV_1$) " $\circ$ " is *right NeutroDistributive* on " $+$ ", i.e. $a \circ (x + y) \subseteq a \circ x + a \circ y$, for some $a \in K,\ x, y \in V$ (the degree of truth "T") and at least one of the following conditions hold:

- $a \circ (x + y) \nsubseteq a \circ x + a \circ y$, for some $a \in K,\ x, y \in V$ (the degree of falsity "F");
- for some $a \in K,\ x, y \in V$, $a \circ (x + y)$ is indeterminate in $V$, or $a \circ x + a \circ y$ is indeterminate in $V$, or we can not find if $a \circ (x + y)$ is a subset of $a \circ x + a \circ y$ (the degree of indeterminacy "I").

$NHV_2$) " $\circ$ " is *left NeutroDistributive* on " $+$ ", i.e. $(a + b) \circ x \subseteq a \circ x + b \circ x$, for some $a, b \in K,\ x \in V$ (the degree of truth "T") and at least one of the following conditions hold:

- $(a + b) \circ x \nsubseteq a \circ x + b \circ x$, for some $a, b \in K,\ x \in V$ (the degree of falsity "F");
- for some $a, b \in K,\ x \in V$, $(a + b) \circ x$ is indeterminate in $V$, or $a \circ x + b \circ x$ is indeterminate in $V$, or we can not find if $(a + b) \circ x$ is a subset of $a \circ x + b \circ x$ (the degree of indeterminacy "I").

$NHV_3$) $a \circ (b \circ x) = (ab) \circ x$, for some $a, b \in K,\ x \in V$ (the degree of truth "T") and at least one of the following conditions hold:

- $a \circ (b \circ x) \neq (ab) \circ x$, for some $a, b \in K,\ x \in V$ (the degree of falsity "F");
- for some $a, b \in K,\ x \in V$, $a \circ (b \circ x)$ is indeterminate in $V$, or $(ab) \circ x$ is indeterminate in $V$, or we can not find if $a \circ (b \circ x)$ and $(ab) \circ x$ are equal (the degree of indeterminacy "I").

$NHV_4$) $a \circ (-x) = (-a) \circ x = -(a \circ x)$, for some $a \in K,\ x \in V$ (the degree of truth "T") and at least one of the following conditions hold:

- $a \circ (-x) \neq (-a) \circ x$ or $(-a) \circ x \neq -(a \circ x)$ or $a \circ (-x) \neq -(a \circ x)$, for some $a \in K,\ x \in V$ (the degree of falsity "F");
- for some $a \in K,\ x \in V$, $a \circ (-x)$ is indeterminate in $V$, or $(-a) \circ x$ is indeterminate in $V$, or $-(a \circ x)$ is indeterminate in $V$, or we can not find if $a \circ (-x)$, $(-a) \circ x$ and $-(a \circ x)$ are equal (the degree of indeterminacy "I").

$NHV_5$) $x \in 1 \circ x$ and ($y \notin 1 \circ y$ or $1 \circ z$ is indeterminate in $V$), for some $x, y, z \in V$.

We say that $(V, +, \circ, K)$ is strongly right distributive NeutroHyperVector space, if $a \circ (x + y) = a \circ x + a \circ y$, for some $a \in K,\ x, y \in V$. In a similar way, the strongly left distributive

NeutroHyperVector space is defined. $V$ is said to be strongly distributive, if it is both strongly left distributive and strongly right distributive.

**Example 3.4.** Consider the external NeutroHyperoperations $\circ_3, \circ_4, \circ_5 : \mathbb{Z}_2 \times \mathbb{Z}_3 \to P_*(U)$ defined in Example 3.2. Then $V_3 = (\mathbb{Z}_3, +, \circ_3, \mathbb{Z}_2)$, $V_4 = (\mathbb{Z}_3, +, \circ_4, \mathbb{Z}_2)$ and $V_5 = (\mathbb{Z}_3, +, \circ_5, \mathbb{Z}_2)$ are NeutroHyperVector spaces over the field $\mathbb{Z}_2$, since they are satisfied in the NeutroAxioms ($NHV_1$)-($NHV_5$) of Definition 3.3, as follows:

$V_3 = (\mathbb{Z}_3, +, \circ_3, \mathbb{Z}_2)$ is a strongly distributive NeutroHyperVector space:

$NHV_1$) $a \circ_3 (x+y) = a \circ_3 x + a \circ_3 y$, for $a = 1,\ x = 0,\ y = 2$, $a \circ_3 (x+y) \nsubseteq a \circ_3 x + a \circ_3 y$, for $a = 1,\ x = 0,\ y = 1$, and $a \circ_3 x + a \circ_3 y$ is indeterminate in $V$, for $a = 0,\ x = 0,\ y = 0$.

$NHV_2$) $(a+b) \circ_3 x = a \circ_3 x + b \circ_3 x$, for $a = 0,\ b = 1,\ x = 2$ and $a \circ_3 x + b \circ_3 x$ is indeterminate in $V$, for $a = b = 0,\ x = 0$.

$NHV_3$) $a \circ_3 (b \circ_3 x) = (ab) \circ_3 x$, for $a = b = 1,\ x = 2$, $a \circ_3 (b \circ_3 x) \neq (ab) \circ_3 x$, for $a = b = 1,\ x = 0$, and $1 \circ_3 (0 \circ_3 0)$ and $(01) \circ_3 0$ are indeterminate in $V$.

$NHV_4$) $a \circ_3 (-x) = (-a) \circ_3 x = -(a \circ_3 x)$, for $a = 0,\ x = 1$, but $1 \circ_3 (-1) = 1 \circ_3 2 = \{0,1,2\} \neq (-1) \circ_3 1 = 1 \circ_3 1 = \{1,2\} = -(1 \circ_3 1) = -(\{1,2\})$ and $-(0 \circ_3 0)$ is indeterminate in $V$.

$NHV_5$) $1 \in 1 \circ_3 1$ and $2 \in 1 \circ_3 2$, but $0 \notin 1 \circ_3 0$.

Similarly, $V_4 = (\mathbb{Z}_3, +, \circ_4, \mathbb{Z}_2)$ is a strongly distributive NeutroHyperVector space:

$NHV_1$) $1 \circ_4 (0+2) = 1 \circ_4 2 = \{0,1,2\} = 1 \circ_4 0 + 1 \circ_4 2 = \{1\} + \{0,1,2\}$, $1 \circ_4 (0+0) = 1 \circ_4 0 = \{1\} \nsubseteq 1 \circ_4 0 + 1 \circ_4 0 = \{1\} + \{1\} = \{2\}$, and $1 \circ_4 0 + 1 \circ_4 1$ is indeterminate in $V$.

$NHV_2$) $(0+1) \circ_4 2 = 1 \circ_4 2 = \{0,1,2\} = 0 \circ_4 2 + 1 \circ_4 2 = \{0,2\} + \{0,2\}$, $(1+1) \circ_4 0 = 0 \circ_4 0 = \{0,1\} \nsubseteq 1 \circ_4 0 + 1 \circ_4 1 = \{1\} + \{1\} = \{2\}$, and $0 \circ_4 1 + 1 \circ_4 1$ is indeterminate in $V$.

$NHV_3$) $0 \circ_4 (0 \circ_4 1) = 0 \circ_4 (\{1,2\}) = \{1,2\} = (00) \circ_4 1$, $0 \circ_4 (0 \circ_4 0) = 0 \circ_4 (\{0,1\}) = \{0,1,2\} \neq (00) \circ_4 0 = \{0,1\}$, and $1 \circ_4 (1 \circ_4 1)$ is indeterminate in $V$.

$NHV_4$) $0 \circ_4 (-1) = 0 \circ_4 2 = \{1,2\} = (-0) \circ_4 1 = -(0 \circ_4 1) = -(\{1,2\})$, while $1 \circ_4 (-0) = 1 \circ_4 0 = \{1\} = (-1) \circ_4 0 = 1 \circ_4 0 \neq -(1 \circ_4 0) = -(\{1\}) = \{2\}$, and $-(1 \circ_4 1)$ and $1 \circ_4 (-2)$ are indeterminate in $V$.

$NHV_5$) $2 \in 1 \circ_4 2$, $0 \notin 1 \circ_4 0$, and we can not find $1 \in 1 \circ_4 1$, since $1 \circ_4 1$ is indeterminate in $V$.

The NeutroHyperVector space $V_5 = (\mathbb{Z}_3, +, \circ_5, \mathbb{Z}_2)$ is strongly left distributive and it is not strongly right distributive:

$NHV_1$) $0 \circ_5 (2+2) = 0 \circ_5 1 = \{0,2\} \subseteq 0 \circ_5 2 + 0 \circ_5 2 = \{1,2\} + \{1,2\} = \{0,1,2\}$, $1 \circ_5 (2+2)$ is indeterminate in $V$ and we can not find $1 \circ_5 (2+2)$ and $1 \circ_5 2 + 1 \circ_5 2$ are equal. There don't exist $a \in K,\ x, y \in V$, such that $a \circ_5 (x+y) = a \circ_5 x + a \circ_5 y$.

$NHV_2$) $(0+1) \circ_5 2 = 1 \circ_5 2 = \{0,1,2\} = 0 \circ_5 2 + 1 \circ_5 2 = \{1,2\} + \{0,1,2\}$, and $1 \circ_5 1 + 1 \circ_5 1$ is indeterminate in $V$, so we can not find $(1+1) \circ_5 1$ and $1 \circ_5 1 + 1 \circ_5 1$ are equal.

NHV$_3$) $0 \circ_5 (0 \circ_5 1) = 0 \circ_5 (\{1,2\}) = 0 \circ_5 1 \cup 0 \circ_5 2 = \{1,2\} = (00) \circ_5 1$, and $0 \circ_5 (0 \circ_5 0) = 0 \circ_5 (\{0,2\}) = 0 \circ_5 0 \cup 0 \circ_5 2 = \{0,1,2\} \neq (00) \circ_5 0 = \{0,2\}$.

NHV$_4$) $0 \circ_5 (-1) = 0 \circ_5 1 = \{1,2\} = (-0) \circ_5 1 = -(0 \circ_5 1)$ and $0 \circ_5 (-0) = 0 \circ_5 0 = \{0,2\} \neq -(0 \circ_5 0) = \{0,1\}$.

NHV$_5$) $2 \in 1 \circ_5 2$, $0 \notin 1 \circ_5 0$, and we can not find $1 \in 1 \circ_5 1$, since $1 \circ_5 1$ is indeterminate in $V$.

**Example 3.5.** Consider the field $K = \mathbb{Z}_2 = \{0,1\}$ defined in Example 2.2. Let $(\mathbb{Z},+)$ be the Abelian group of integer numbers and $t$ be an arbitrary nonzero element of $\mathbb{Z}$. Define a mapping "$\circ_6 : \mathbb{Z}_2 \times \mathbb{Z} \to P_*(\mathbb{R})$" by:

$$\forall x \in \mathbb{Z},\ 0 \circ_6 x = \{0\}, \qquad \text{and} \qquad 1 \circ_6 x = \begin{cases} \{1,x\} & x \in \mathbb{Z}\backslash\{t\}, \\ \{t+1,\pi\} & x = t. \end{cases} \tag{4}$$

Then $V_6 = (\mathbb{Z},+,\circ_6,\mathbb{Z}_2)$ is a NeutroHyperVector space over the field $K$. Note that "$\circ_6$" is an external NeutroHyperoperation, since $1 \circ_6 t \nsubseteq \mathbb{Z}$. Also, all axioms (HV$_1$)-(HV$_5$) are replaced by the NeutroAxioms (NHV$_1$)-(NHV$_5$); more details are listed below, choosing $t = 6$:

NHV$_1$) $1 \circ_6 (0+0) = 1 \circ_6 0 = \{0,1\} \subseteq 1 \circ_6 0 + 1 \circ_6 0 = \{0,1\} + \{0,1\} = \{0,1,2\}$ and $1 \circ_6 (2+3) = 1 \circ_6 5 = \{1,5\} \nsubseteq 1 \circ_6 2 + 1 \circ_6 3 = \{1,2\} + \{1,3\} = \{2,3,4,5\}$.

NHV$_2$) $(0+1) \circ_6 2 = 1 \circ_6 2 = \{1,2\} = 0 \circ_6 2 + 1 \circ_6 2 = \{0\} + \{1,2\}$, and $(1+1) \circ_6 2 = 0 \circ_6 2 = \{0\} \nsubseteq 1 \circ_6 2 + 1 \circ_6 2 = \{1,2\} + \{1,2\} = \{2,3,4\}$. Thus, $V$ is strongly left distributive.

NHV$_3$) $0 \circ_6 (0 \circ_6 x) = 0 \circ_6 0 = \{0\} = (0) \circ_6 x$, for all $x \in V$, and $1 \circ_6 (0 \circ_6 2) = 1 \circ_6 0 = \{0,1\} \neq 0 \circ_6 2 = \{0\}$.

NHV$_4$) $0 \circ_6 (-x) = (-0) \circ_6 x = -(0 \circ_6 x) = \{0\}$, for all $x \in V$, but $1 \circ_6 (-2) = \{1,-2\} \neq (-1) \circ_6 2 = \{-1,2\}$, $(-1) \circ_6 2 = \{-1,2\} \neq -(1 \circ_6 2) = \{-1,-2\}$ and $1 \circ_6 (-2) = \{1,-2\} \neq -(1 \circ_6 2) = \{-1,-2\}$.

NHV$_5$) $x \in 1 \circ_6 x$, for all $x \in V\backslash\{t\}$ and $t \notin 1 \circ_6 t$.

**Example 3.6.** Consider the field $K = \mathbb{Z}_2 = \{0,1\}$ defined in Example 2.2. Define the operation "$+ : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}^2$" by $(x,y) + (m,n) = (x+m, y+n)$. Then $(\mathbb{Z}^2,+)$ is an Abelian group. Choose $(x_0,y_0) \in \mathbb{Z}^2$ and define "$\circ_7 : \mathbb{Z}_2 \times \mathbb{Z}^2 \to P_*(\mathbb{R}^2)$" by

$$a \circ_7 (x,y) = \begin{cases} \{(0,0)\} & a = 0, \\ \{(1,1),(x,y)\} & a = 1, (x,y) \in \mathbb{Z}^2\backslash\{(x_0,y_0)\}, \\ \{(\pi,\pi),(x_0+1,y_0+1)\} & a = 1, (x,y) = (x_0,y_0). \end{cases} \tag{5}$$

Then $V_7 = (\mathbb{Z}^2,+,\circ_7,\mathbb{Z}_2)$ is a strongly distributive NeutroHyperVector space over the field $\mathbb{Z}_2$. In fact, "$\circ_7$" is an external NeutroHyperoperation, since $1 \circ_7 (x_0,y_0) \nsubseteq \mathbb{Z}^2$ and all NeutroAxioms (NHV$_1$)-(NHV$_5$) are satisfied:

NHV$_1$) $0 \circ_7 ((x,y) + (m,n)) = 0 \circ_7 (x+m, y+n) = \{(0,0)\} = 0 \circ_7 (x,y) + 0 \circ_7 (m,n) = \{(0,0)\} + \{(0,0)\}$, and $1 \circ_7 ((x,y) + (m,n)) = 1 \circ_7 (x+m, y+n) = \{(1,1),(x+m,y+n)\} \nsubseteq$

$1 \circ_7 (x, y) + 1 \circ_7 (m, n) = \{(1, 1), (x, y)\} + \{(1, 1), (m, n)\} = \{(2, 2), (x + 1, y + 1), (m + 1, n + 1), (x + m, y + n)\}$.

NHV$_2$) $(1+0) \circ_7 (x, y) = 1 \circ_7 (x, y) = \{(1, 1), (x, y)\} = 1 \circ_7 (x, y) + 0 \circ_7 (x, y) = \{(1, 1), (x, y)\} + \{(0, 0)\}$, for all $(x, y) \in \mathbb{Z}^2$ and $(1+1) \circ_7 (x, y) = 0 \circ_7 (x, y) = \{(0, 0)\} \nsubseteq 1 \circ_7 (x, y) + 1 \circ_7 (x, y) = \{(1, 1), (x, y)\} + \{(1, 1), (x, y)\} = \{(2, 2), (x + 1, y + 1), (2x, 2y)\}$, for $(x, y) \in \mathbb{Z}^2 \backslash \{(0, 0), (-1, -1)\}$.

NHV$_3$) $1 \circ_7 (1 \circ_7 (x, y)) = 1 \circ_7 \{(1, 1), (x, y)\} = \{(1, 1), (x, y)\} = (1 \cdot 1) \circ_7 (x, y)$, and $1 \circ_7 (0 \circ_7 (x, y)) = 1 \circ_7 \{(0, 0)\} = \{(1, 1), (0, 0)\} \neq (1 \cdot 0) \circ_7 (x, y) = \{(0, 0)\}$.

NHV$_4$) $0 \circ_7 (-(x, y)) = (-0) \circ_7 (x, y) = -(0 \circ_7 (x, y)) = \{(0, 0)\}$, for all $(x, y) \in \mathbb{Z}^2$, but $1 \circ_7 (-(x, y)) = \{(1, 1), (-x, -y)\} \neq (-1) \circ_7 (x, y) = \{(1, 1), (x, y)\} \neq -(0 \circ_7 (x, y)) = \{(-1, -1), (-x, -y)\}$, for all $(x, y) \in \mathbb{Z}^2 \backslash \{(0, 0)\}$.

NHV$_5$) $(x, y) \in 1 \circ_7 (x, y)$, for all $(x, y) \in \mathbb{Z}^2 \backslash \{(x_0, y_0)\}$ and $(x_0, y_0) \notin 1 \circ_7 (x_0, y_0)$.

**Definition 3.7.** Let $(V, +, \circ, K)$ be a NeutroHyperVector space and $H$ be a nonempty subset of V. Then $H$ is called a *SubNeutroHyperspace* of $V$, if $(H, +, \circ, K)$ is itself a NeutroHyperVector space. In other words, $H$ is a SubNeutroHyperspace of $V$ if and only if the following conditions hold:

(1) $x - y \in H$, for all $x, y \in H$;
(2) $a \circ x \subseteq H$, for some $a \in K,\ x \in H$ and ($b \circ y \nsubseteq H$ or $c \circ z$ is indeterminate in $H$, for some $b, c \in K,\ y, z \in H$);
(3) at least one of the NeutroAxioms of the Definition 3.3, is satisfied for $H$.

It is clear that every NeutroHyperVector space is a SubNeutroHyperspace of itself. The NeutroHyperVector spaces $(\mathbb{Z}_3, +, \circ_3, \mathbb{Z}_2)$, $(\mathbb{Z}_3, +, \circ_4, \mathbb{Z}_2)$ and $(\mathbb{Z}_3, +, \circ_5, \mathbb{Z}_2)$ defined in Example 3.4, do not have any proper SubNeutroHyperspace. If we define $0 \circ 0 = \{0\}$ or $0 \circ 0 = \mathbb{Z}_3$, then $\{0\}$ is the only proper SubNeutroHyperspace of $(\mathbb{Z}_3, +, \circ_3, \mathbb{Z}_2)$. In the following examples, some nontrivial SubNeutroHyperspaces are presented:

**Example 3.8.** Consider the field $K = \mathbb{Z}_2 = \{0, 1\}$ defined in Example 2.2, $V = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ and $U = \{0, 1, 2, 3, 4, 5\}$. Then $V_8 = (\mathbb{Z}_4, +, \circ_8, \mathbb{Z}_2)$ is a strongly distributive NeutroHyperVector space over the field $\mathbb{Z}_2$, where the operation "$+ : V \times V \to V$" and the external NeutroHyperoperation "$\circ_8 : K \times V \to P_*(U)$" are defined by

| $+$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

| $\circ_8$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | $\{0\}$ | $\{0,1\}$ | $\{0,2\}$ | $\{3,5\}$ |
| 1 | $\{1,2,3\}$ | $\{1\}$ | $\{1,2\}$ | $\{2,3\}$ |

One can see that, $H = \{0, 2\}$ is a SubNeutroHyperspace of $V$, since $x - y \in H$, for all $x, y \in H$, and $0 \circ_8 2 = \{0, 2\} \subseteq H$, and $1 \circ_8 0 = \{1, 2, 3\} \nsubseteq H$. Also, $(H, +, \circ_8, K)$ is a NeutroHyperVector space over the field $K$, where the operation "$+ : H \times H \to H$" and the external NeutroHyperoperation "$\circ_8 : K \times H \to P_*(U)$" are defined by

| $+$ | 0 | 2 |
|---|---|---|
| 0 | 0 | 2 |
| 2 | 2 | 0 |

| $\circ_8$ | 0 | 2 |
|---|---|---|
| 0 | $\{0\}$ | $\{0, 2\}$ |
| 1 | $\{1, 2, 3\}$ | $\{1, 2\}$ |

In fact, $0 \circ_8 (0+0) = 0 \circ_8 0 = \{0\} = 0 \circ_8 0 + 0 \circ_8 0 = \{0\} + \{0\}$, and $1 \circ_8 2 + 1 \circ_8 2$ is indeterminate in $H$. $(0+0) \circ_8 0 = 0 \circ_8 0 = \{0\} = 0 \circ_8 0 + 0 \circ_8 0 = \{0\} + \{0\}$, and $1 \circ_8 0 + 0 \circ_8 0$ is indeterminate in $H$. $0 \circ_8 (0 \circ_8 0) = 0 \circ_8 (\{0\}) = \{0\} = (00) \circ_8 0$, and $1 \circ_8 (0 \circ_8 2) = 1 \circ_8 (\{0, 2\}) = \{1, 2, 3\} \neq (1 \cdot 0) \circ_8 2 = \{0, 2\}$. $0 \circ_8 (-2) = 0 \circ_8 2 = \{0, 2\} = (-0) \circ_8 2 = -(0 \circ_8 2)$, and $-(1 \circ_8 2)$ is indeterminate in $H$. $2 \in 1 \circ_8 2$ and $0 \notin 1 \circ_8 0$.

**Example 3.9.** For every $m \in \mathbb{Z} \setminus \{\pm 1\}$, the set $m\mathbb{Z} = \{mn : n \in \mathbb{Z}\}$ is a proper SubNeutroHyperspace of $(\mathbb{Z}, +, \circ_6, \mathbb{Z}_2)$, defined in Example 3.5, such that $0 \circ_6 mn = \{0\} \subseteq m\mathbb{Z}$, $1 \circ_6 mn = \{1, mn\} \nsubseteq m\mathbb{Z}$, for all $mn \in m\mathbb{Z}$ and so the restriction " $\circ_6$ " into $m\mathbb{Z}$ is a NeutroHyperoperaion. Also, all NeutroAxioms ($NHV_1$)-($NHV_5$) are satisfied:

$NHV_1$) $0 \circ_6 (mn + m\acute{n}) = \{0\} = 0 \circ_6 mn + 0 \circ_6 m\acute{n}$ and $1 \circ_6 (mn + m\acute{n}) = \{1, m(n + \acute{n})\} \nsubseteq 1 \circ_6 mn + 1 \circ_6 m\acute{n} = \{1, mn\} + \{1, m\acute{n}\} = \{2, mn + 1, m\acute{n} + 1, m(n + \acute{n})\}$, for all $mn, m\acute{n} \in m\mathbb{Z}$. Then $m\mathbb{Z}$ is strongly right distributive.

$NHV_2$) $(0 + 0) \circ_6 mn = 0 \circ_6 mn = \{0\} = 0 \circ_6 mn + 0 \circ_6 mn$, and $(1 + 1) \circ_6 mn = 0 \circ_6 mn = \{0\} \nsubseteq 1 \circ_6 mn + 1 \circ_6 mn = \{1, mn\} + \{1, mn\} = \{2, mn + 1, 2mn\}$, for all $mn \in m\mathbb{Z}$. So $m\mathbb{Z}$ is strongly left distributive.

$NHV_3$) $0 \circ_6 (0 \circ_6 mn) = 0 \circ_6 0 = \{0\} = (0) \circ_6 mn$, and $1 \circ_6 (0 \circ_6 mn) = 1 \circ_6 0 = \{0, 1\} \neq 0 \circ_6 mn = \{0\}$, for all $mn \in m\mathbb{Z}$.

$NHV_4$) $0 \circ_6 (-mn) = (-0) \circ_6 mn = -(0 \circ_6 mn) = \{0\}$, but $1 \circ_6 (-mn) = \{1, -mn\} \neq (-1) \circ_6 mn = \{-1, mn\}$, $(-1) \circ_6 mn = \{-1, mn\} \neq -(1 \circ_6 mn) = \{-1, -mn\}$ and $1 \circ_6 (-mn) = \{1, -mn\} \neq -(1 \circ_6 mn) = \{-1, -mn\}$, for all $mn \in m\mathbb{Z}$.

$NHV_5$) $mn \in 1 \circ_6 mn$, for all $mn \in m\mathbb{Z} \setminus \{t\}$ and $t \notin 1 \circ_6 t$.

**Example 3.10.** The sets $H = \{(x, 0); x \in \mathbb{Z}\}$ and $L = \{(0, y); y \in \mathbb{Z}\}$ are proper SubNeutroHyperspaces of $(\mathbb{Z}^2, +, \circ_7, \mathbb{Z}_2)$, defined in Example 3.6. The restrictions of " $\circ_7$ " into $H$ and $L$, are NeutroHyperoperaions, since $0 \circ_7 (x, 0) = \{(0, 0)\} \subseteq H$ and $0 \circ_7 (0, y) = \{(0, 0)\} \subseteq L$, but $1 \circ_7 (x, 0) = \{(1, 1), (x, 0)\} \nsubseteq H$ and $1 \circ_7 (0, y) = \{(1, 1), (0, y)\} \nsubseteq L$. The NeutroAxioms ($NHV_1$)-($NHV_4$) and the axiom ($HV_5$) for $(H, +, \circ_7, \mathbb{Z}_2)$ are given in the following (details for $(L, +, \circ_7, \mathbb{Z}_2)$ are similar):

NHV$_1$) $0 \circ_7 ((x,0)+(m,0)) = 0 \circ_7 (x+m,0) = \{(0,0)\} = 0 \circ_7 (x,0) + 0 \circ_7 (m,0) = \{(0,0)\} + \{(0,0)\}$, and $1 \circ_7 ((x,0)+(m,0))$ is indeterminate in $H$.

NHV$_2$) $(0+0) \circ_7 (x,0) = \{(0,0)\} = 0 \circ_7 (x,0) + 0 \circ_7 (x,0)$ and $1 \circ_7 (x,0) + 1 \circ_7 (x,0)$ is indeterminate in $H$, for $x \in \mathbb{Z} \backslash \{-1\}$.

NHV$_3$) $1 \circ_7 (1 \circ_7 (x,0)) = 1 \circ_7 \{(1,1),(x,0)\} = \{(1,1),(x,0)\} = (1 \cdot 1) \circ_7 (x,y)$, and $1 \circ_7 (0 \circ_7 (x,0)) = 1 \circ_7 \{(0,0)\}$ is indeterminate in $H$.

NHV$_4$) $0 \circ_7 (-(x,0)) = (-0) \circ_7 (x,0) = -(0 \circ_7 (x,0)) = \{(0,0)\}$, for all $(x,y) \in \mathbb{Z}^2$, but $1 \circ_7 (-(x,0))$, $(-1) \circ_7 (x,0)$ and indeterminate in $H$.

HV$_5$) $(x,0) \in 1 \circ_7 (x,0)$, for all $(x,0) \in H$.

It is well-known that if $H, L$ are subgroups of $(V,+)$, then $H \cap L$ is a subgroup of $V$, but $H \cup L$ is a subgroup of $V$ if and only if $H \subseteq L$ or $L \subseteq H$. Now, if $H, L$ are SubNeutroHyperspaces of $V$, then $H \cap L$ may be a SubNeutroHyperspace of $V$. For example, the intersection of two arbitrary SubNeutroHyperspaces $m\mathbb{Z}, n\mathbb{Z}$ of the NeutroHyperVector space $(\mathbb{Z},+,\circ_6,\mathbb{Z}_2)$, presented in Example 3.9, is the SubNeutroHyperspace $[m,n]\mathbb{Z}$ of $(\mathbb{Z},+,\circ_6,\mathbb{Z}_2)$, where $[m,n]$ is the smallest common multiplication of $m,n$. Also, the intersection of the SubNeutroHyperspaces $H = \{(x,0);\ x \in \mathbb{Z}\}$ and $L = \{(0,y);\ y \in \mathbb{Z}\}$ of $(\mathbb{Z}^2,+,\circ,\mathbb{Z}_2)$, defined in Example 3.10, is the SubNeutroHyperspace $\{(0,0)\}$. Moreover, similar to the groups, $H \cup L$ is a SubNeutroHyperspace of $V$ if and only if $H \subseteq L$ or $L \subseteq H$.

## 4. NeutroLinearTransformations

In this section, some types of transformations between NeutroHyperVector spaces are introduced and some properties of mentioned concepts are studied, supported by some examples.

**Definition 4.1.** Let $(V,+,\circ,K)$ and $(W,\acute{+},\acute{\circ},K)$ be NeutroHyperVector spaces over the field $K$. Then a mapping $T : V \longrightarrow W$ is called

(1) *NeutroLinearTransformation*, if $T(x+y) = T(x)\acute{+}T(y)$, for all $x,y \in V$ and $T(a \circ x) \subseteq a\acute{\circ}T(x)$, for some $a \in K,\ x \in V$;

(2) *NeutroGoodTransformation*, if $T(x+y) = T(x)\acute{+}T(y)$, for all $x,y \in V$ and $T(a \circ x) = a\acute{\circ}T(x)$, for some $a \in K,\ x \in V$;

(3) *NeutroStrongLinearTransformation*, if $T(x+y) = T(x)\acute{+}T(y)$, for all $x,y \in V$, $T(a \circ x) \subseteq a\acute{\circ}T(x)$ when $a \circ x \subseteq V$, $a\acute{\circ}T(x) \nsubseteq W$ when $a \circ x \nsubseteq V$, and $a\acute{\circ}T(x)$ is indeterminate in $W$ when $a \circ x$ is indeterminate in $V$;

(4) *NeutroStrongGoodLinearTransformation*, if $T(x+y) = T(x)\acute{+}T(y)$, for all $x,y \in V$, $T(a \circ x) = a\acute{\circ}T(x)$ when $a \circ x \subseteq V$, $a\acute{\circ}T(x) \nsubseteq W$ when $a \circ x \nsubseteq V$, and $a\acute{\circ}T(x)$ is indeterminate in $W$ when $a \circ x$ is indeterminate in $V$;

(5) *NeutroStrongGoodIsomorphism*, if $T$ is a bijective NeutroStrongGoodLinearTransformation. In this case, $V$ and $W$ are called NeutroIsomorphic and it is denoted by $V \underset{NS}{\cong} W$.

**Proposition 4.2.** *If $(V, +, \circ, K)$ is a NeutroHyperVector space over the field $K$, then the identity function $i_V : V \to V$ is a NeutroStrongGoodIsomorphism.*

*Proof.* It is clear that $i_V$ is a bijection and $i_V(x + y) = i_V(x) + i_V(y)$, for all $x, y \in V$. Now, if $a \circ x \subseteq V$, then $T(a \circ x) = \{T(t); t \in a \circ x\} = a \circ x = a \circ T(x)$, and if $a \circ x \nsubseteq V$, then $a \circ T(x) \nsubseteq V$. Also, $a \circ T(x)$ is indeterminate in $V$, when $a \circ x$ is indeterminate in $V$. □

**Example 4.3.** Consider the NeutroHyperVector spaces $V_3 = (\{0, 1, 2\}, +, \circ_3, K)$, $V_4 = (\{0, 1, 2\}, +, \circ_4, K)$ defined in Example 3.4. Then the mappings $T_{34} : V_3 \to V_4$ and $S_{34} : V_3 \to V_4$ with $T_{34}(x) = x$ and $S_{34}(x) = 2x$ are NeutroGoodTransformations, since $T_{34}(x + y) = x + y = T_{34}(x) + T_{34}(y)$ and $S_{34}(x + y) = 2(x + y) = 2x + 2y = S_{34}(x) + S_{34}(y)$, for all $x, y \in V_3$. Also, $T_{34}(1 \circ_3 0) = 1 \circ_4 T_{34}(0)$ and $S_{34}(0 \circ_3 1) = 0 \circ_4 T_{34}(1)$. But both of them are not NeutroStrongLinearTransformations, because $0 \circ_3 0 \nsubseteq V_3$, while $0 \circ_4 T_{34}(0) \subseteq V_4$ and $0 \circ_4 S_{34}(0) \subseteq V_4$.

**Example 4.4.** Consider the field $K = \{0, 1\}$ and the Abelian group $V = \mathbb{Z}_3 = \{0, 1, 2\}$ defined in Example 2.2. Then similar to the Example 3.4, one can see that $V_9 = (\mathbb{Z}_3, +, \circ_9, K)$ and $V_{10} = (\mathbb{Z}_3, +, \circ_{10}, K)$ are NeutroHyperVector spaces over the field $\mathbb{Z}_2$, where the external NeutroHyperoperations $\circ_9, \circ_{10} : \mathbb{Z}_2 \times \mathbb{Z}_3 \to P_*(U)$ are defined by the following tables:

| $\circ_9$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $\{0\}$ | $\{1, 2\}$ | $\{1, 2\}$ |
| 1 | $\{1\}$ | | $\{0, 2\}$ |

| $\circ_{10}$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $\{0\}$ | $\{0, 1, 2\}$ | $\{0, 1, 2\}$ |
| 1 | $\{0, 1\}$ | | $\{0, 2\}$ |

Now, define the mapping $\acute{T} : V_9 \to V_{10}$ by $\acute{T}(x) = x$. Then $\acute{T}$ is a NeutroStrongLinearTransformation which is not a NeutroStrongGoodLinearTransformation, since $\acute{T}(x+y) = \acute{T}(x)+\acute{T}(y)$, for all $x, y \in V_9$. Also, $\acute{T}(0 \circ_9 0) = \{0\} = 0 \circ_{10} \acute{T}(0)$, $\acute{T}(0 \circ_9 1) = \acute{T}(\{1, 2\}) = \{1, 2\} \subseteq \{0, 1, 2\} = 0 \circ_{10} \acute{T}(1)$, $\acute{T}(0 \circ_9 2) = \acute{T}(\{1, 2\}) = \{1, 2\} \subseteq \{0, 1, 2\} = 0 \circ_{10} \acute{T}(2)$, $\acute{T}(1 \circ_9 0) = \acute{T}(\{1\}) = \{1\} \subseteq \{0, 1\} = 1 \circ_{10} \acute{T}(0)$ and $\acute{T}(1 \circ_9 2) = \acute{T}(\{0, 2\}) = \{0, 2\} = 1 \circ_{10} \acute{T}(2)$, Moreover, $1 \circ_9 1$ and $1 \circ_{10} \acute{T}(1)$ are indeterminate in $V$;

**Theorem 4.5.** *$\underset{NS}{\cong}$ is an equivalence relation on the set of NeutroHyperVector spaces.*

*Proof.* By Proposition 4.2, the identity function $i_V : V \to V$ is a NeutroStrongGoodIsomorphism, i.e. $V \underset{NS}{\cong} V$ and so $\underset{NS}{\cong}$ is reflexive. If $V \underset{NS}{\cong} W$, where $V = (V, +, \circ, K)$ and $W = (W, \acute{+}, \acute{\circ}, K)$, then there exists a NeutroStrongGoodIsomorphism $T : V \to W$. We show that $T^{-1} : W \to V$ is a NeutroStrongGoodIsomorphism. It is clear that $T^{-1}$ is bijective and

$T^{-1}(w_1 \acute{+} w_2) = T^{-1}(w_1) + T^{-1}(w_2)$, for all $w_1, w_2 \in W$. For any $w \in W$, there exist $v \in V$ such that $T(v) = w$, so $T^{-1}(a\acute{\circ}w) = T^{-1}(a\acute{\circ}T(v))$, for all $a \in K$. We must check the following three cases:

(1) $a\acute{\circ}w \subseteq W$: In this case, $a \circ v \subseteq V$ (if $a \circ v \nsubseteq V$, then $a\acute{\circ}w = a\acute{\circ}T(v) \nsubseteq W$) and so

$$T^{-1}(a\acute{\circ}w) = T^{-1}(a\acute{\circ}T(v)) = T^{-1}(T(a \circ v)) = a \circ v = a \circ T^{-1}(w).$$

(2) $a\acute{\circ}w \nsubseteq W$: Let $a \circ v = a \circ T^{-1}(w) \subseteq V$ or $a \circ v = a \circ T^{-1}(w)$ is indeterminate in $V$. If $a \circ v \subseteq V$, then $T(a \circ v) = a\acute{\circ}T(v) = a\acute{\circ}w \subseteq T(v) \subseteq W$, which is a contradiction. Also, if $a \circ v$ is indeterminate in $V$, then $a\acute{\circ}T(v) = a\acute{\circ}w$ is indeterminate in $W$, which is a contradiction, too. Thus in this case, $a \circ T^{-1}(w) \nsubseteq V$.

(3) $a\acute{\circ}w$ is indeterminate in $W$: If $a \circ v \subseteq V$, then $a\acute{\circ}w = a\acute{\circ}T(v) = T(a \circ v) \subseteq T(V) \subseteq W$, which is a contradiction. Also, if $a \circ v \nsubseteq V$, then $a\acute{\circ}T(v) \nsubseteq W$, which is a contradiction, too. Hence in this case, $a \circ T^{-1}(w)$ is indeterminate in $V$.

Consequently, $T^{-1}$ is a NeutroStrongGoodIsomorphism and so $\underset{NS}{\cong}$ is symmetric.
Now let $V \underset{NS}{\cong} W$ and $W \underset{NS}{\cong} U$. Then there exist NeutroStrongGoodIsomorphisms $T : V \to W$ and $S : W \to U$. We shall prove that $S \circ T : V \to U$ is a NeutroStrongGoodIsomorphism. It is clear that, $S \circ T$ is bijective and $(S \circ T)(x + y) = S(T(x + y)) = S(T(x) + T(y)) = S(T(x)) + S(T(y))$, for all $x, y \in V$. Suppose $a \in K$ and $x \in V$. If $a \circ x \subseteq V$, then by using the hypothesis that $T$ and $S$ are NeutroStrongGoodIsomorphisms, $a \circ T(x) = T(a \circ x) \subseteq T(V) \subseteq W$ and $S(a \circ T(x)) = a \circ S(T(x)) = a \circ (S \circ T)(x)$. If $a \circ x \nsubseteq V$, then $a \circ T(x) \nsubseteq W$, and so $a \circ (S \circ T)(x) \nsubseteq U$. If $a \circ x$ is indeterminate in $V$, then $a \circ T(x)$ is indeterminate in $W$ and so $a \circ (S \circ T)(x)$ is indeterminate in $U$. Hence, $V \underset{NS}{\cong} U$. Therefore, $\underset{NS}{\cong}$ is transitive.
Consequentely, $\underset{NS}{\cong}$ is an equivalence relation. □

**Theorem 4.6.** *Let $V = (V, +, \circ, K)$ and $W = (W, \acute{+}, \acute{\circ}, K)$ be NeutroHyperVector spaces over the field $K$ and $T : V \to W$ be an injective NeutroStrongGoodTransformation. If $H$ is a SubNeutroHyperspace of $V$, then $T(H)$ is a SubNeutroHyperspace of $W$.*

*Proof.* For all $T(x), T(y) \in T(H)$, $x - y \in H$ and so $T(x) \acute{-} T(y) = T(x - y) \in T(H)$. Also, there exist $a \in K$ and $x \in H$ such that $a \circ x \subseteq H$. Then $T(x) \in T(H)$ and $a\acute{\circ}T(x) = T(a \circ x) \subseteq T(H)$. Moreover, if $b \circ y \nsubseteq H$, for some $b \in K$, $y \in H$, then by injectivity of $T$, it follows that $b \circ T(y) \nsubseteq T(H)$. Also, if $c \circ z$ is indeterminate in $V$, for some $c \in K$, $z \in H$, then $c\acute{\circ}T(z)$ is indeterminate in $W$. Now we show that if $H$ is satisfied in every NeutroAxiom of Definition 3.3, then $T(H)$ is satisfied in the same NeutroAxiom:

NHV$_1$) $a \circ (x+y) \subseteq a \circ x + a \circ y$, for some $a \in K,\ x, y \in V$. Thus

$$\begin{aligned} a\acute{\circ}(T(x)\acute{+}T(y)) &= a\acute{\circ}T(x+y) \\ &= T(a \circ (x+y)) \\ &\subseteq T(a \circ x + a \circ y) \\ &= T(a \circ x)\acute{+}T(a \circ y) \\ &= a\acute{\circ}T(x)\acute{+}a\acute{\circ}T(y). \end{aligned}$$

Also, if $\acute{a} \circ (\acute{x}+\acute{y}) \nsubseteq \acute{a} \circ \acute{x} + \acute{a} \circ \acute{y}$, for some $\acute{a} \in K$, $\acute{x}, \acute{y} \in V$, then by injectivity of $T$, it follows that $\acute{a}\acute{\circ}(T(\acute{x})\acute{+}T(\acute{y})) \nsubseteq \acute{a}\acute{\circ}T(\acute{x})\acute{+}\acute{a}\acute{\circ}T(\acute{y})$. Moreover, if for some $\hat{a} \in K,\ \hat{x}, \hat{y} \in V$, $\hat{a} \circ (\hat{x}+\hat{y})$ is indeterminate in $H$, or $\hat{a} \circ \hat{x} + \hat{a} \circ \hat{y}$ is indeterminate in $H$, or we can not find if $\hat{a} \circ (\hat{x}+\hat{y})$ is a subset of $\hat{a} \circ \hat{x} + \hat{a} \circ \hat{y}$, then $\hat{a}\acute{\circ}(T(\hat{x})\acute{+}T(\hat{y}))$ is indeterminate in $T(H)$, or $\hat{a}\acute{\circ}T(\hat{x})\acute{+}\hat{a}\acute{\circ}T(\hat{y})$ is indeterminate in $T(H)$, or we can not find if $\hat{a}\acute{\circ}(T(\hat{x})\acute{+}T(\hat{y}))$ is a subset of $\hat{a}\acute{\circ}T(\hat{x})\acute{+}\hat{a}\acute{\circ}T(\hat{y})$, respectively.

NHV$_2$) $(a+b) \circ x \subseteq a \circ x + b \circ x$, for some $a, b \in K,\ x \in H$. Then

$$\begin{aligned} (a+b)\acute{\circ}T(x) &= T((a+b) \circ x) \\ &\subseteq T(a \circ x + b \circ x) \\ &= T(a \circ x)\acute{+}T(b \circ x) \\ &= a\acute{\circ}T(x)\acute{+}b\acute{\circ}T(x). \end{aligned}$$

Also, if $(\acute{a}+\acute{b}) \circ \acute{x} \nsubseteq \acute{a} \circ \acute{x} + \acute{b} \circ \acute{x}$, for some $\acute{a}, \acute{b} \in K$, $\acute{x} \in H$, then $(\acute{a}+\acute{b})\acute{\circ}T(\acute{x}) \nsubseteq \acute{a}\acute{\circ}T(\acute{x})\acute{+}\acute{b}\acute{\circ}T(\acute{x})$. Moreover, if $(\hat{a}+\hat{b}) \circ \hat{x}$ is indeterminate in $H$, or $\hat{a} \circ \hat{x} + \hat{b} \circ \hat{x}$ is indeterminate in $H$, or we can not find if $(\hat{a}+\hat{b}) \circ \hat{x}$ is a subset of $\hat{a} \circ \hat{x} + \hat{b} \circ \hat{x}$, for some $\hat{a}, \hat{b} \in K,\ \hat{x} \in H$, then $(\hat{a}+\hat{b})\acute{\circ}T(\hat{x})$ is indeterminate in $T(H)$, or $\hat{a}\acute{\circ}T(\hat{x}) + \hat{b}\acute{\circ}T(\hat{x})$ is indeterminate in $T(H)$, or we can not find if $(\hat{a}+\hat{b})\acute{\circ}T(\hat{x})$ is a subset of $\hat{a}\acute{\circ}T(\hat{x})\acute{+}\hat{b}\acute{\circ}T(\hat{x})$, respectively.

NHV$_3$) $a \circ (b \circ x) = (ab) \circ x$, for some $a, b \in K,\ x \in H$. Thus

$$a\acute{\circ}(b\acute{\circ}T(x)) = a\acute{\circ}(T(b \circ x)) = T(a \circ (b \circ x)) = T((ab) \circ x) = (ab)\acute{\circ}T(x).$$

Also, if $a \circ (b \circ x) \neq (ab) \circ x$, for some $a, b \in K,\ x \in H$, then $a\acute{\circ}(b\acute{\circ}T(x)) \neq (ab)\acute{\circ}T(x)$. Moreover, if $a \circ (b \circ x)$ is indeterminate in $H$, or $(ab) \circ x$ is indeterminate in $H$, or we can not find if $a \circ (b \circ x) = (ab) \circ x$ or $a \circ (b \circ x) \neq (ab) \circ x$, for some $a, b \in K,\ x \in H$, then $a\acute{\circ}(b\acute{\circ}T(x))$ is indeterminate in $T(H)$, or $(ab)\acute{\circ}T(x)$ is indeterminate in $T(H)$, or we can not find if $a\acute{\circ}(b\acute{\circ}T(x)) = (ab)\acute{\circ}T(x)$ or $a\acute{\circ}(b\acute{\circ}T(x)) \neq (ab)\acute{\circ}T(x)$.

NHV$_4$) $a \circ (-x) = (-a) \circ x = -(a \circ x)$, for some $a \in K,\ x \in H$. Thus

$$\begin{aligned} a\acute{\circ}(-T(x)) &= a\acute{\circ}(T(-x)) = T(a \circ (-x)) = T((-a) \circ x) = (-a)\acute{\circ}T(-x) \\ &= T(-(a \circ x)) = -(T(a \circ x)) = -(a\acute{\circ}T(x)). \end{aligned}$$

Also, if $a \circ (-x) \neq (-a) \circ x$ or $(-a) \circ x \neq -(a \circ x)$ or $a \circ (-x) \neq -(a \circ x)$, for some $a \in K,\ x \in H$, then $a\acute{\circ}(-T(x)) \neq (-a)\acute{\circ}T(x)$ or $(-a)\acute{\circ}T(x) \neq -(a\acute{\circ}T(x))$ or $a\acute{\circ}(-T(x)) \neq -(a\acute{\circ}T(x))$, respectively. Moreover, if $a \circ (-x)$ or $(-a) \circ x$ or $-(a \circ x)$ is indeterminate in $H$, for some $a \in K,\ x \in H$, or we can not find if two of them are equal, then $a\acute{\circ}(-T(x))$ or $(-a)\acute{\circ}T(x)$ or $-(a\acute{\circ}T(x))$ is indeterminate in $T(H)$, or we can not find if they are equal, respectively.

$NHV_5$) $x \in 1 \circ x$, for some $x \in H$. Thus $T(x) \in T(1 \circ x) = 1\acute{\circ}T(x)$. Also, if $y \notin 1 \circ y$, for some $y \in H$, then $T(y) \notin 1\acute{\circ}T(y)$. Moreover, if $1 \circ z$ is indeterminate or we can not find if $z \in 1 \circ z$ or $z \notin 1 \circ z$, then $1\acute{\circ}T(z)$ is indeterminate or we can not find if $T(z) \in 1\acute{\circ}T(z)$ or $T(z) \notin 1\acute{\circ}T(z)$.

Therefore, by Definition 3.7, $T(H)$ is a SubNeutroHyperspace of $W$. □

**Theorem 4.7.** *Let $V = (V, +, \circ, K)$ and $W = (W, \acute{+}, \acute{\circ}, K)$ be NeutroHyperVector spaces over the field $K$ and $T : V \to W$ be a NeutroStrongGoodIsomorphism. If $L$ is a SubNeutroHyperspace of $W$, then $T^{-1}(L)$ is a SubNeutroHyperspace of $V$.*

*Proof.* For all $x, y \in T^{-1}(L)$, $T(x), T(y) \in L$ and so $T(x - y) = T(x) - T(y) \in L$. Then $x - y \in T^{-1}(L)$. Also, there exist $a \in K$, $y \in L$ such that $a \circ y \subseteq L$. Since $T$ is surjective, $y = T(x)$, for some $x \in V$. Thus $T(a \circ x) = a \circ T(x) = a \circ y \subseteq L$ and so $a \circ x \subseteq T^{-1}(L)$, where $x \in T^{-1}(L)$. Moreover, if $a \circ y \nsubseteq L$, for some $a \in K$, $y \in L$, then $a \circ x \nsubseteq T^{-1}(L)$, for some $x \in V$ such that $T(x) = y$. Next, if $a \circ y$ is indeterminate, for some $a \in K$, $y \in L$, then $a \circ x$ is indeterminate, for some $x \in T^{-1}(y)$. One can similar to the proof of Theorem 4.6, show that if $L$ is satisfied in each NeutroAxiom of Definition 3.3, then $T^{-1}(L)$ is satisfied in the same NeutroAxiom. Therefore, $T^{-1}(L)$ is a SubNeutroHyperspace of $V$. □

**Corollary 4.8.** *Let $V = (V, +, \circ, K)$ and $W = (W, \acute{+}, \acute{\circ}, K)$ be NeutroHyperVector spaces over the field $K$ and $T : V \to W$ be a NeutroStrongGoodIsomorphism. Then $H$ is a SubNeutroHyperspace of $V$, if and only if $T(H)$ is a SubNeutroHyperspace of $W$.*

*Proof.* It follows from Theorems 4.5, 4.6, and 4.7. □

**Corollary 4.9.** *If $V = (V, +, \circ, K)$ and $W = (W, \acute{+}, \acute{\circ}, K)$ are NeutroHyperVector spaces over the field $K$ such that $V \underset{NS}{\cong} W$, then $V$ has no proper SubNeutroHyperspace, if and only if $W$ has no proper SubNeutroHyperspace.*

*Proof.* It follows from Corollary 4.8. □

## 5. **Cartesian Product of NeutroHyperVector Spaces**

In order to make the Cartesian product of NeutroHyperVector spaces over a common field $K$, we need a suitable field, that is the Cartesian product $K \times K$ with the following operations:

$$(a,b)+(c,d)=(a+c,b+d), \quad (a,b)(c,d)=(ac-bd,ad-bc). \tag{6}$$

The zero and the multiplicative identity of the field $K \times K$ are $(0,0)$ and $(1,0)$, respectively.

**Theorem 5.1.** *If $V_1=(V_1,+_1,\circ_1,K)$ and $V_2=(V_2,+_2,\circ_2,K)$ are NeutroHyperVector spaces over the field $K$ such that are satisfied in the same NeutroAxioms, $k\circ_1(0\circ_1 v_1)=0\circ_1 v_1$, $1\circ_2(0\circ_2 v_2)=0\circ_2 v_2$ and $0_{V_2}\in 0\circ_2 \acute{v}_2$, for some $k\in K$, $v_1\in V_1$ and $v_2,\acute{v}_2\in V_2$, then $V_1\times V_2=(V_1\times V_2,+,\circ,K\times K)$ is a NeutroHyperVector space over the field $K\times K$, where*

$$\begin{aligned}(x_1,x_2)+(y_1,y_2)&=(x_1+_1 y_1,x_2+_2 y_2),\\ (a_1,a_2)\circ(x_1,x_2)&=\{(r,s);\ r\in a_1\circ_1 x_1, s\in a_2\circ_2 x_2\}.\end{aligned} \tag{7}$$

*Proof.* It is easy to see that $(V_1\times V_2,+)$ is an Abelian group. Since "$\circ_1$" and "$\circ_2$" are external NeutroHyperoperations, there exist $a_1,a_2\in K$, $x_1\in V_1$, $x_2\in V_2$, such that $a_1\circ_1 x_1\subseteq V_1$ and $a_2\circ_2 x_2\subseteq V_2$. Then

$$(a_1,a_2)\circ(x_1,x_2)=\{(r,s);\ r\in a_1\circ_1 x_1, s\in a_2\circ_2 x_2\}\subseteq V_1\times V_2.$$

If $a_1\circ_1 x_1\nsubseteq V_1$, for some $a_1\in K$, $x_1\in V_1$, then $(a_1,a_2)\circ(x_1,x_2)\nsubseteq V_1\times V_2$, for all $a_2\in K$, $x_2\in V_2$. If $a_1\circ_1 x_1$ is indeterminate in $V_1$, for some $a_1\in K$, $x_1\in V_1$, then $(a_1,a_2)\circ(x_1,x_2)$ is indeterminate in $V_1\times V_2$, for all $a_2\in K$, $x_2\in V_2$. Similarly, If $a_2\circ_2 x_2\nsubseteq V_2$, for some $a_2\in K$, $x_2\in V_2$, then $(a_1,a_2)\circ(x_1,x_2)\nsubseteq V_1\times V_2$, for all $a_1\in K$, $x_1\in V_1$ and if $a_2\circ_2 x_2$ is indeterminate in $V_2$, for some $a_2\in K$, $x_2\in V_2$, then $(a_1,a_2)\circ(x_1,x_2)$ is indeterminate in $V_1\times V_2$, for all $a_1\in K$, $x_1\in V_1$. Hence "$\circ$" is an external NeutroHyperoperation on $V_1\times V_2$. Now we show that if $V_1,V_2$ are satisfied in each NeutroAxioms of Definition 3.3, then $V_1\times V_2$ is satisfied in the same NeutroAxiom.

NHV$_1$) If $a_1\circ_1(x_1+_1 y_1)\subseteq a_1\circ_1 x_1+_1 a_1\circ_1 y_1$, and $a_2\circ_2(x_2+_2 y_2)\subseteq a_2\circ_2 x_2+_2 a_2\circ_2 y_2$, for some $a_1,a_2\in K$, $x_1,y_1\in V_1$, $x_2,y_2\in V_2$, then

$$\begin{aligned}(a_1,a_2)\circ((x_1,x_2)+(y_1,y_2)) &= (a_1,a_2)\circ(x_1+_1 y_1,x_2+_2 y_2)\\ &= \{(r,s);\ r\in a_1\circ_1(x_1+_1 y_1), s\in a_2\circ_2(x_2+_2 y_2)\}\\ &\subseteq \{(r,s);\ r\in a_1\circ_1 x_1+_1 a_1\circ_1 y_1, s\in a_2\circ_2 x_2+_2 a_2\circ_2 y_2\}\\ &= \left\{\begin{array}{c}(r_1+_1\acute{r}_1,s_2+_2\acute{s}_2);\ r_1\in a_1\circ_1 x_1,\acute{r}_1\in a_1\circ_1 y_1,\\ s_2\in a_2\circ_2 x_2,\acute{s}_2\in a_2\circ_2 y_2\end{array}\right\}\\ &= \{(r_1,s_2);\ r_1\in a_1\circ_1 x_1, s_2\in a_2\circ_2 x_2\}\\ &\quad +\{(\acute{r}_1,\acute{s}_2);\ \acute{r}_1\in a_1\circ_1 y_1,\acute{s}_2\in a_2\circ_2 y_2\}\\ &= (a_1,a_2)\circ(x_1,x_2)+(a_1,a_2)\circ(y_1,y_2).\end{aligned}$$

If $a_1 \circ_1 (x_1 +_1 y_1) \nsubseteq a_1 \circ_1 x_1 +_1 a_1 \circ_1 y_1$, and $a_2 \circ_2 (x_2 +_2 y_2) \nsubseteq a_2 \circ_2 x_2 +_2 a_2 \circ_2 y_2$, for some $a_1, a_2 \in K$, $x_1, y_1 \in V_1$, $x_2, y_2 \in V_2$, then

$$(a_1, a_2) \circ ((x_1, x_2) + (y_1, y_2)) \nsubseteq (a_1, a_2) \circ (x_1, x_2) + (a_1, a_2) \circ (y_1, y_2).$$

If $a_1 \circ_1 (x_1 +_1 y_1)$ and $a_2 \circ_2 (x_2 +_2 y_2)$ are indeterminate in $V_1$ and $V_2$, respectively, for some $a_1, a_2 \in K$, $x_1, y_1 \in V_1$, $x_2, y_2 \in V_2$, then $(a_1, a_2) \circ ((x_1, x_2) + (y_1, y_2))$ is indeterminate in $V_1 \times V_2$.

If $a_1 \circ_1 x_1 +_1 a_1 \circ_1 y_1$ and $a_2 \circ_2 x_2 +_2 a_2 \circ_2 y_2$ are indeterminate in $V_1$ and $V_2$, respectively, for some $a_1, a_2 \in K$, $x_1, y_1 \in V_1$, $x_2, y_2 \in V_2$, then $(a_1, a_2) \circ (x_1, x_2) + (a_1, a_2) \circ (y_1, y_2)$ is indeterminate in $V_1 \times V_2$.

If we can not find if $a_1 \circ_1 (x_1 +_1 y_1)$ is a subset of $a_1 \circ_1 x_1 +_1 a_1 \circ_1 y_1$ and we can not find if $a_2 \circ_2 (x_2 +_2 y_2)$ is a subset of $a_2 \circ_2 x_2 +_2 a_2 \circ_2 y_2$, for some $a_1, a_2 \in K$, $x_1, y_1 \in V_1$, $x_2, y_2 \in V_2$, then we can not find if $(a_1, a_2) \circ ((x_1, x_2) + (y_1, y_2))$ is a subset of $(a_1, a_2) \circ (x_1, x_2) + (a_1, a_2) \circ (y_1, y_2)$.

NHV$_2$) It is similar to the NeutroAxiom (NHV$_1$).

NHV$_3$) By the hypothesis, there exist $k \in K$, $v_1 \in V_1$ and $v_2, \acute{v}_2 \in V_2$, such that $c \circ_1 (0 \circ_1 v_1) = 0 \circ_1 v_1$ and $1 \circ_2 (0 \circ_2 v_2) = 0 \circ_2 v_2$. Thus

$$\begin{aligned}
(k,1) \circ ((0,0) \circ (v_1, v_2)) &= \bigcup_{(r,s) \in (0,0) \circ (v_1, v_2)} (k,1) \circ (r,s) \\
&= \bigcup_{\substack{p \in k \circ_1 r, q \in 1 \circ_2 s, \\ r \in 0 \circ_1 v_1, s \in 0 \circ_2 v_2}} (p,q) \\
&= \bigcup_{p \in k \circ_1 (0 \circ_1 v_1), q \in 1 \circ_2 (0 \circ_2 v_2)} (p,q) \\
&= \bigcup_{p \in 0 \circ_1 v_1, q \in 0 \circ_2 v_2} (p,q) \\
&= (0,0) \circ (v_1, v_2) \\
&= ((k,1)(0,0)) \circ (v_1, v_2).
\end{aligned}$$

If $a \circ_1 (b \circ_1 x_1) \neq (ab) \circ_1 x_1$ and $c \circ_2 (d \circ_2 x_2) \neq (cd) \circ_2 x_2$, for some $a, b, c, d \in K$, $x_1 \in V_1$, $x_2 \in V_2$, then $(a, c) \circ ((b, d) \circ (x_1, x_2)) \neq ((a, c)(b, d)) \circ (x_1, x_2)$.

If $a \circ_1 (b \circ_1 x_1)$ and $c \circ_2 (d \circ_2 x_2)$ are indeterminate in $V_1$ and $V_2$, respectively, for some $a, b, c, d \in K$, $x_1 \in V_1$, $x_2 \in V_2$, then $(a, c) \circ ((b, d) \circ (x_1, x_2))$ is indeterminate in $V_1 \times V_2$.

If $(ab) \circ_1 x_1$ and $(cd) \circ_2 x_2$ are indeterminate in $V_1$ and $V_2$, respectively, for some $a, b, c, d \in K$, $x_1 \in V_1$, $x_2 \in V_2$, then $((a, c)(b, d)) \circ (x_1, x_2)$ is indeterminate in $V_1 \times V_2$.

If we can not find if $a \circ_1 (b \circ_1 x_1) = (ab) \circ_1 x_1$ and $c \circ_2 (d \circ_2 x_2) = (cd) \circ_2 x_2$, for some $a_1, a_2 \in K$, $x_1, y_1 \in V_1$, $x_2, y_2 \in V_2$, then we can not find if $(a, c) \circ ((b, d) \circ (x_1, x_2)) = ((a, c)(b, d)) \circ (x_1, x_2)$.

NHV$_4$) If $a \circ_1 (-x_1) = (-a) \circ_1 x_1 = -(a \circ_1 x_1)$ and $b \circ_2 (-x_2) = (-b) \circ_2 x_2 = -(b \circ_2 x_2)$, for some $a, b \in K$, $x_1 \in V_1$, $x_2 \in V_2$, then

$$\begin{aligned}(a,b) \circ (-(x_1,x_2)) &= (a,b) \circ (-x_1,-x_2)\\ &= \{(r,s); r \in a \circ_1 (-x_1), s \in b \circ_2 (-x_2)\}\\ &= \{(r,s); r \in (-a) \circ_1 x_1, s \in (-b) \circ_2 x_2\}\\ &= (-a,-b) \circ (x_1,x_2)\\ &= (-(a,b)) \circ (x_1,x_2),\end{aligned}$$

and

$$\begin{aligned}(a,b) \circ (-(x_1,x_2)) &= \{(r,s); r \in a \circ_1 (-x_1), s \in b \circ_2 (-x_2)\}\\ &= \{(r,s); r \in -(a \circ_1 x_1), s \in -(b \circ_2 x_2)\}\\ &= \{(-\acute{r},-\acute{s}); \acute{r} \in a \circ_1 x_1, \acute{s} \in b \circ_2 x_2\}\\ &= \{-(\acute{r},\acute{s}); \acute{r} \in a \circ_1 x_1, \acute{s} \in b \circ_2 x_2\}\\ &= -((a,b) \circ (x_1,x_2)).\end{aligned}$$

If $a \circ_1 (-x_1) \neq (-a) \circ_1 x_1$ and $b \circ_2 (-x_2) \neq (-b) \circ_2 x_2$, for some $a, b \in K$, $x_1 \in V_1$, $x_2 \in V_2$, then $(a,b) \circ (-(x_1,x_2)) \neq (-(a,b)) \circ (x_1,x_2)$. Similarly, if $a \circ_1 (-x_1) \neq -(a \circ_1 x_1)$ and $b \circ_2 (-x_2) \neq -(b \circ_2 x_2)$, then $(a,b) \circ (-(x_1,x_2)) \neq -((a,b) \circ (x_1,x_2))$. Also, if $(-a) \circ_1 x_1 \neq -(a \circ_1 x_1)$ and $(-b) \circ_2 x_2 \neq -(b \circ_2 x_2)$, then $(-(a,b)) \circ (x_1,x_2) \neq -((a,b) \circ (x_1,x_2))$.
If $a \circ_1 (-x_1)$ and $b \circ_2 (-x_2)$ are indeterminate in $V_1$ and $V_2$, for some $a, b \in K$, $x_1 \in V_1$, $x_2 \in V_2$, then $(a,b) \circ (-(x_1,x_2))$ is indeterminate in $V_1 \times V_2$. Similarly, if $(-a) \circ_1 x_1$ and $(-b) \circ_2 x_2$ are indeterminate in $V_1$ and $V_2$, then $(-(a,b)) \circ (x_1,x_2)$ is indeterminate in $V_1 \times V_2$. Also, if $-(a \circ_1 x_1)$ and $-(b \circ_2 x_2)$ are indeterminate in $V_1$ and $V_2$, then $-((a,b) \circ (x_1,x_2))$ is indeterminate in $V_1 \times V_2$.

NHV$_5$) If $x_1 \in 1 \circ_1 x_1$ and $x_2 \in 1 \circ_2 x_2$, for some $x_1 \in V_1$, $x_2 \in V_2$, then by the hypothesis it follows that, $(x_1, \acute{v}_2) \in (1,0) \circ (x_1, \acute{v}_2)$, where $(1,0)$ is the identity of $K \times K$.
If $x_1 \notin 1 \circ_1 x_1$ and $x_2 \notin 1 \circ_2 x_2$, for some $x_1 \in V_1$, $x_2 \in V_2$, then $(x_1,x_2) \notin (1,0) \circ (x_1,x_2)$.
If $1 \circ_1 x_1$ and $1 \circ_2 x_2$ are indeterminate in $V_1$ and $V_2$, respectively, for some $x_1 \in V_1$, $x_2 \in V_2$, then $(1,0) \circ (x_1,x_2)$ is indeterminate in $V_1 \times V_2$.
If we can not fine if $x_1 \in 1 \circ_1 x_1$ and $x_2 \in 1 \circ_2 x_2$, for some $x_1 \in V_1$, $x_2 \in V_2$, then we can not find $(x_1,x_2) \in (1,0) \circ (x_1,x_2)$.

Therefore, by Definition 3.3, $(V_1 \times V_2, +, \circ, K \times K$ is a NeutroHyperVector space over the field $K \times K$. □

It is easy to see that, the NeutroHyperVector space $V_3 = (\mathbb{Z}_3, +, \circ_3, \mathbb{Z}_2)$ was defined in Example 3.4, does not satisfy in the hypothesis of the Theorem 5.1, then we can not construct $V_3 \times W$ or $U \times V_3$ for any NeutroHyperVector spaces $W$ and $U$. Also, the NeutroHyperVector spaces $V_4 = (\mathbb{Z}_3, +, \circ_4, \mathbb{Z}_2)$, $V_5 = (\mathbb{Z}_3, +, \circ_5, \mathbb{Z}_2)$ were defined in Example 3.4, $V_6 = (\mathbb{Z}, +, \circ_6, \mathbb{Z}_2)$ was defined in Example 3.5, $V_7 = (\mathbb{Z}^2, +, \circ_7, \mathbb{Z}_2)$ was defined in Example 3.6, $V_8 = (\mathbb{Z}_4, +, \circ_8, \mathbb{Z}_2)$ was defined in Example 3.8, and $V_9 = (\mathbb{Z}_3, +, \circ_9, \mathbb{Z}_2)$, $V_{10} = (\mathbb{Z}_3, +, \circ_{10}, \mathbb{Z}_2)$, were defined in Example 4.4, satisfy in the first hypothesis of the Theorem 5.1, (i.e. $k \circ_i (0 \circ_i v) = 0 \circ_i v$, for some $k \in K$, $v \in V_i$, $4 \leq i \leq 10$), thus we can construct the Cartesian product $V_i \times W$, for suitable NeutroHyperVector space $W$ over the field $K = \{0, 1\}$, where $W = (W, +, \circ, K)$ satisfies in the necessary conditions of the Theorem 5.1, (i.e. $1 \circ (0 \circ w_1) = 0 \circ w_1$ and $0_W \in 0 \circ w_2$, for some $w_1, w_2 \in W$). In the following example, such a suitable NeutroHyperVector space is given:

**Example 5.2.** Consider the field $K = \{0, 1\}$ and the Abelian group $\mathbb{Z}_3 = \{0, 1, 2\}$ defined in Example 2.2. Define an external NeutroHyperoperation $\circ_{11} : \mathbb{Z}_2 \times \mathbb{Z}_3 \to P_*(U)$ by

| $\circ_{11}$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $\{0, 4\}$ | $\{1, 2\}$ | $\{1, 2\}$ |
| 1 | $\{1\}$ | $\{1\}$ | $\{2\}$ |

Then, similar to the Examples 3.4, and 4.4, it follows that $V_{11} = (\mathbb{Z}_3, +, \circ_{11}, K)$ is a strongly right distributive NeutroHyperVector space (it is not strongly left distributive) over the field $\mathbb{Z}_2$, such that $1 \circ_{11} (0 \circ_{11} 1) = 1 \circ_{11} (\{1, 2\}) = 1 \circ_{11} 1 \cup 1 \circ_{11} 2 = \{1, 2\} = 0 \circ_{11} 1$ and $0 \in 0 \circ_{11} 0$. Thus, by Theorem 5.1, $V_i \times V_{11}$, $4 \leq i \leq 11$, is a NeutroHyperVector space over the field $\mathbb{Z}_2 \times \mathbb{Z}_2$. Note that, $V_i = (V_i, +_i, \circ_i, K)$ and $V_{11} = (V_{11}, +_{11}, \circ_{11}, K)$ are satisfied in the same NeutroAxioms of Definition 3.3.

For instance, in the following, we check the NeutroAxioms of Definition 3.3, for the NeutroHyperVector space $V_4 \times V_{11} = (\mathbb{Z}_3 \times \mathbb{Z}_3, +, \circ, \mathbb{Z}_2 \times \mathbb{Z}_2)$, which is a strongly right distributive NeutroHyperVector space, but it is not strongly left distributive:

NHV$_1$) $1 \circ_4 (0+2) = 1 \circ_4 0 + 1 \circ_4 2$, and $1 \circ_{11} (1+1) = 1 \circ_{11} 1 + 1 \circ_{11} 1$, so $(1, 1) \circ ((0, 1) + (2, 1)) = (1, 1) \circ (0, 1) + (1, 1) \circ (2, 1)$, since

$$
\begin{aligned}
(1,1) \circ ((0,1) + (2,1)) &= (1,1) \circ (2,2) \\
&= \{(r,s); r \in 1 \circ_4 2, s \in 1 \circ_{11} 2\} \\
&= \{(0,2), (1,2), (2,2)\},
\end{aligned}
$$

and

$$\begin{aligned}(1,1)\circ(0,1)+(1,1)\circ(2,1) &= \{(r,s); r\in 1\circ_4 0, s\in 1\circ_{11} 1\}\\ &\quad +\{(p,q); p\in 1\circ_4 2, q\in 1\circ_{11} 1\}\\ &= \{(1,1)\}+\{(0,1),(1,1),(2,1)\}\\ &= \{(0,2),(1,2),(2,2)\}.\end{aligned}$$

Also, $1\circ_4(0+0)\nsubseteq 1\circ_4 0+1\circ_4 0$ and $0\circ_{11}(1+2)\nsubseteq 1\circ_{11} 1+0\circ_{11} 2$, thus $(1,0)\circ((0,1)+(0,2))\nsubseteq (1,0)\circ(0,1)+(1,0)\circ(0,2)$, since $(1,0)\circ((0,1)+(0,2))=(1,0)\circ(0,0)=\{(1,0),(1,4)\}$ and

$$\begin{aligned}(1,0)\circ(0,1)+(1,0)\circ(0,2) &= \{(1,1),(1,2)\}+\{(1,1),(1,2)\}\\ &= \{(2,0),(2,1),(2,2)\}.\end{aligned}$$

NHV$_2$) $(0+1)\circ_4 2=1\circ_4 2=\{0,1,2\}=0\circ_4 2+1\circ_4 2=\{0,2\}+\{0,2\}$ and $(0+0)\circ_{11} 1\subseteq 0\circ_{11} 1+0\circ_{11} 1$, so $((0,0)+(1,0))\circ(2,1)\subseteq(0,0)\circ(2,1)+(1,0)\circ(2,1)$, since

$$\begin{aligned}((0,0)+(1,0))\circ(2,1) &= (1,0)\circ(2,1)\\ &= \{(0,1),(0,2),(1,1),(1,2),(2,1),(2,2)\},\end{aligned}$$

and

$$\begin{aligned}(0,0)\circ(2,1)+(1,0)\circ(2,1) &= \{(1,1),(1,2),(2,1),(2,2)\}\\ &\quad +\{(0,1),(0,2),(1,1),(1,2),(2,1),(2,2)\}\\ &= \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}.\end{aligned}$$

Also, $(1+1)\circ_4 0\nsubseteq 1\circ_4 0+1\circ_4 1$ and $(0+1)\circ_{11} 1\nsubseteq 0\circ_{11} 1+1\circ_{11} 1$, thus $((1,0)+(1,1))\circ(0,1)\nsubseteq (1,0)\circ(0,1)+(1,1)\circ(0,1)$, since $((1,0)+(1,1))\circ(0,1)=(0,1)\circ(2,1)=\{(0,1),(1,1)\}$ and

$$\begin{aligned}(1,0)\circ(0,1)+(1,1)\circ(0,1) &= \{(1,1),(1,2)\}+\{(1,1)\}\\ &= \{(2,0),(2,2)\}.\end{aligned}$$

NHV$_3$) $0\circ_4(0\circ_4 1)=(00)\circ_4 1$ and $0\circ_{11}(0\circ_{11} 1)=(00)\circ_{11} 1$, so $(0,1)\circ((0,0)\circ(1,1))=((0,1)(0,0))\circ(1,1)$, since

$$\begin{aligned}(0,1)\circ((0,0)\circ(1,1)) &= (0,1)\circ\{(1,1),(1,2),(2,1),(2,2)\}\\ &= ((0,1)\circ(1,1))\cup((0,1)\circ(1,2))\\ &\quad \cup((0,1)\circ(2,1))\cup((0,1)\circ(2,2))\\ &= \{(1,1),(2,1)\}\cup\{(1,2),(2,2)\}\\ &\quad \cup\{(1,1),(2,1)\}\cup\{(1,2),(2,2)\}\\ &= \{(1,1),(1,2),(2,1),(2,2)\},\end{aligned}$$

and $((0,1)(0,0)) \circ (1,1) = (0,0) \circ (1,1) = \{(1,1),(1,2),(2,1),(2,2)\}$.
Also, $(1,1) \circ ((1,0) \circ (1,0))$ is indeterminate in $V_4 \times V_{11}$.

NHV$_4$) $0 \circ_4 (-1) = (-0) \circ_4 1 = -(0 \circ_4 1)$ and $0 \circ_{11} (-1) = (-0) \circ_{11} 1 = -(0 \circ_{11} 1)$, so $(0,0) \circ (-(1,1)) = (-(0,0)) \circ (1,1) = -((0,0) \circ (1,1))$, since

$$(0,0) \circ (-(1,1)) = (0,0) \circ (2,2) = \{(1,1),(1,2),(2,1),(2,2)\},$$

$$(-(0,0)) \circ (1,1) = (0,0) \circ (1,1) = \{(1,1),(1,2),(2,1),(2,2)\},$$

and

$$-((0,0) \circ (1,1)) = -\{(1,1),(1,2),(2,1),(2,2)\} = \{(1,1),(1,2),(2,1),(2,2)\}.$$

Also, $(-1) \circ_4 0 \neq -(1 \circ_4 0)$ and $(-1) \circ_{11} 1 \neq -(1 \circ_{11} 1)$, thus $(-(1,1)) \circ (0,1) \neq -((1,1) \circ (0,1))$, since $(-(1,1)) \circ (0,1) = (1,1) \circ (0,1) = \{(1,1)\}$ and $-((1,1) \circ (0,1)) = -\{(1,1)\} = \{(2,2)\}$.

NHV$_5$) $2 \in 1 \circ_4 2$ and $1 \in 1 \circ_{11} 1$, so $(2,0) \in (1,0) \circ (2,0)$. Moreover, $(0,0) \notin (1,0) \circ (0,0)$.

**Theorem 5.3.** *Let $V_1 = (V_1, +_1, \circ_1, K)$ be a NeutroHyperVector space over the field $K$ such that $k \circ_1 (0 \circ_1 v_1) = 0 \circ_1 v_1$, for some $k \in K$, $v_1 \in V_1$, and let $V_2 = (V_2, +_2, \circ_2, K)$ be a hypervector space over the field $K$ such that $0_{V_2} \in 0 \circ_2 \acute{v}_2$, for some $\acute{v}_2 \in V_2$, then $(V_1 \times V_2, +, \circ, K \times K)$ with operation "$+$" and the external NeutroHyperoperation "$\circ$" defined in Theorem 5.1, is a NeutroHyperVector space over the field $K \times K$.*

*Proof.* It is similar to the proof of Theorem 5.1. □

**Example 5.4.** Let $V$ be the hypervector spaces $(\mathbb{Z}_3, +, \circ_1, \mathbb{Z}_2)$ or $(\mathbb{Z}_3, +, \circ_2, \mathbb{Z}_2)$ over the field $K = \mathbb{Z}_2$, were defined in Example 2.2. Then for every $4 \leq i \leq 11$, the Cartesian product $V_i \times V$, is a NeutroHyperVector space over the field $K \times K$, where the NeutroHyperVector spaces $V_i$ were defined in Examples 3.4, 3.5, 3.6, 3.8, 4.4, and 5.2.

## 6. **Conclusions**

A NeutroHyperVector space is an alternative of a hypervector space $(V, +, \circ, K)$ such that "$+ : V \times V \to P_*(V)$" is a NeutroHyperoperation, or "$\circ : K \times V \to P_*(V)$" is an external NeutroHyperoperation, or at least it has one NeutroAxiom. Thus, there are several types of NeutroHyperVector spaces, based on the number of NeutroOperation, NeutroHyperoperation and NeutroAxioms.

In this paper, we considered an specific and expected type of NeutroHyperVector spaces and studied some of their basic properties, such as SubNeutroHyperspace, NeutroLinearTransformation and Cartesian product of NeutroHyperVector spaces. Throughout the paper, a variety of examples are provided for each concept. For future research, one can investigate another basic properties of vector spaces, fuzzy vector spaces, hypervector spaces and fuzzy hypervector spaces in NeutroHyperVector spaces. It is also possible to change the basic definition of NeutroHyperVector space by changing the number of operations/hyperoperations/Neutrohyperoperations and axioms/NeutroAxioms and check the features of the new structure. Finding the applications of the defined structures can be the subject of valuable research works.

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# Neutrosophic Model for Evaluation Healthcare Security Criteria for Powerful and Lightweight Secure Storage System in Cloud-Based E-Healthcare Services

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**Abstract:** Large data volumes make manually keeping and preserving health records for future reference problematic. Most unusual is the coronavirus epidemic's overcrowding of hospitals. The Secure Pattern Electronic Healthcare Records (SPEHR) scheme provides a powerful and lightweight secure storage system for cloud-based E-healthcare services that meets remote healthcare security criteria. So we proposed a neutrosophic model with a multi-criterion decision-making (MCDM) method for the analysis and evaluation of these criteria. The neutrosophic model is used to deal with vague and uncertain data. Then we integrated the neutrosophic model with the AHP method to rank and compute the weights of the criteria. Offering stakeholders a secure interface and avoiding unwanted access to cloud-stored data mitigates E-healthcare risks. The key derivation ensures end-to-end data. ciphering to prevent unauthorized use (KDF). This work provides robust security solutions for various environments. This work can meet security needs for people and secure-communicating organizations. We found the data privacy is the best criterion.

**Keywords:** Neutrosophic Set, Multi-Criteria Decision Making (MCDM), Data Security, Privacy-preserving, Authentication, Access Control, Uncertainty.

## 1. Introduction

The Large data volumes make manually storing and maintaining health records for future reference difficult [1]. Most unusual is the coronavirus epidemic's overcrowding of hospitals. Due to its inability to accommodate more patients, healthcare systems in the US, Brazil, and India are under strain [2]. The global pandemic's impact on healthcare is incomprehensible [3]. Manually recording data makes it hard to find a patient's information in a record room full of health records. Finding a patient's medical record is time-consuming. Disasters can also wipe data. Since the data is in plain text, it can be hacked and read, written, or modified [4].

Recently, IoT medical record storage. E-healthcare security is crucial because health data is sensitive. Attackers exploit open wireless channels [5]. Attackers exploit open wireless channels [5]. These attacks could cause several E-healthcare difficulties. Consider a patient who is admitted to a hospital in a different city for treatment and then returned home. Later, he falls ill and is admitted to a local hospital, but he cannot access his previous data or files. A lack of understanding may postpone his treatment fatally. If the patient's information was saved on devices that could connect to the cloud, the new medical staff may start treatment immediately [6]. Health care can store encrypted data in the cloud using secure cryptographic techniques. These algorithms allow only authorized individuals to view the information from any remote place with internet connectivity, wired or wireless [7]. "Cloud" servers allow users to host a variety of applications and databases that can be viewed online. Cloud platforms exist worldwide [8]. Sponsors, health insurers, and healthcare organizations no longer need to maintain servers or administer applications [9].

E-healthcare can adapt in institutions without cloud-based services [10]. Paper can keep general hospital data, but the cloud can save essential data. Authorized users worldwide can securely share information [11]. Selected stakeholders can access, remove, and change data [12]. Protecting patient privacy and data integrity is difficult [13,14].

We evaluated the healthcare security by analyzing its criteria. We used the MCDM method to evaluate the criteria [25]. The healthcare security has many component as shown in Figure 1.

For a fresh take on ambiguity, imprecision, inconsistency, and fuzziness, see Florantin Smarandache's neutrosophic sets, which expand on Atanassov's intuitionistic fuzzy sets (IFSs). Smarandache defined a neutrosophic set with three parts: truth membership, indeterminacy membership, and falsity membership, and he established the degree of indeterminacy/neutrality as a new and independent component of fuzzy sets. The use of neutrosophic sets in decision-making may improve outcomes due to the indeterminacy parameter's contribution to a more precise formulation of membership functions [26,27]. A neutrosophic set, on the other hand, is more difficult to implement in actual scientific and technical domains. The distinction between absolute truth and relative truth in logic, as well as between absolute membership and relative non-membership, is a particularly valuable application of neutrosophic logic. A decision maker is relieved of the burden of ensuring that the sum of the items in a membership function for a given event is at most 1. If they're unrelated, the amount might go up to 3 [28, 29].

Saaty's AHP is a well-known approach for multi-criteria decision-making. Researchers may easily determine how important certain factors are by using this method. Several fuzzy variations of the conventional AHP approach have been developed as a response to inadequate data and unpredictability [30 ,31].

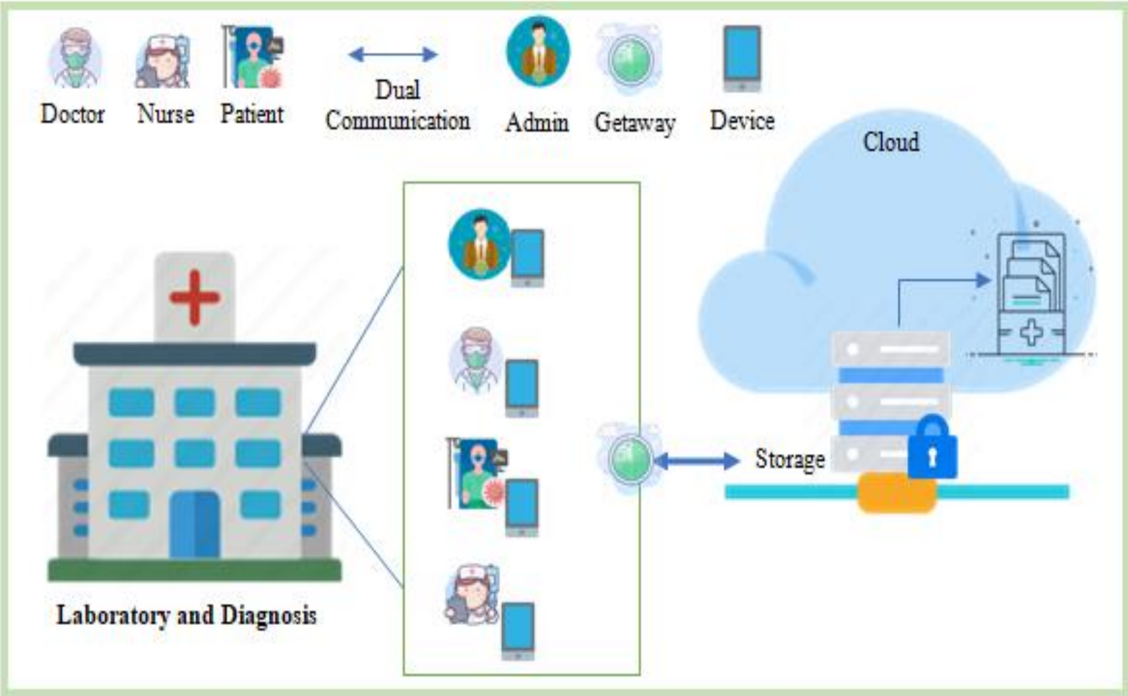


Figure 1. The component of the healthcare security.

## 2. Literature Review

In this sub-section, the most recent work relevant to the healthcare security is reviewed. It examines the literature and identifies the strengths and flaws of pertinent models. This subparagraph highlights academic attempts to develop new E-Healthcare security approaches [15]. Using Olutayo et al. [16] web-based solution. physicians, pharmacists, and nurses can access patients' health records. The local cloud holds patient information. Remotely editable data facilitates collaboration between hospitals and physicians through the sharing of patient records. This proposal limits patient access to their medical records.

A data sharing and profile pairing mechanism for Mobile Healthcare Social Networks (MHSN) in cloud computing for EHR is detailed in the approach [17]. Using an identity-based encryption system, the scheme enables the encryption of medical records. This approach permits conditional attribute-based re-encryption of data. This approach prevents eavesdropping on sensitive information. Using identity-based encryption and an equality test, MHSN's method for matching profiles is adaptable and secure.

A trust negotiation framework facilitates authentication, privacy, and user access for health services [18, 19]. Digital credentials enable secure transaction feature disclosure. This method cannot defend the E-healthcare system against all significant threats. Modular exponentiation-based group key agreement was proposed by S. Paliwal et al. [20]. The proposed method is secure against DOS assaults, but it is susceptible to replay attacks (lack of timestamp or random number), requires the identities of all communicating parties, loses anonymity, and cannot withstand impersonation.

The authors of [21] focused on privacy and E-Healthcare platform access control techniques. According to the authors, their access control technique is superior to others. The proposed access control technique relies on communication confidence. User behavior determines trust. If the user and service have mutual trust, the request is permitted. The methodology confines record access to authorized, reliable personnel, according to the author. Li et al. [22] introduced a biometric-based authentication approach and emphasized that incorrectly developed biohashing-based protocols are susceptible to insider assaults. Comparing protocols for E-Healthcare security.

Existing systems are susceptible to numerous E-Healthcare security flaws. The vulnerabilities identified could be used by cybercriminals to execute cyberattacks against various medical devices. In terms of identity anonymity, authenticity, confidentiality, and communication integrity, the majority of provided techniques do not provide absolute security. Normal schemes are inadequate for mission-critical e-healthcare applications because they lack particular security features. Existing system frameworks contain vulnerabilities that allow unauthorized access to resources. In addition, older systems require extensive processing.

**3 Neutrosophic AHP Method**

Zadeh contributes to the literature by introducing the fuzzy theory, a method for dealing with ambiguity and uncertainty. To begin, the membership function is the single component of a fuzzy set. After that, a wide variety of fuzzy sets are created to help with uncertainty and ambiguity. Atanassov introduced a kind of fuzzy sets called intuitionistic fuzzy sets (IFSs) [32, 33]. The membership and non-membership functions of a fuzzy set generalize into IFSs. Smarandache extends fuzzy logic with a new function called "uncertainty" to create the neutrosophic logic as a more sophisticated form of IFSs that can more effectively handle ambiguity [34,35].

Neutrosophic set has three membership functions as truth, indeterminacy and falsity membership functions $(X_T, X_I, X_F)$.

**Definition 1**

Interval valued neutrosophic number can be represented as:

$$X = \{x, [X_T^L, X_T^U], [X_I^L, X_I^U], [X_F^L, X_F^U]\} \quad (1)$$

**Definition 2**

We can convert the interval valued neutrosophic set into a crisp value by:

$$S(X) = \left(\frac{X_T^L + X_T^U}{2}\right) + \left(1 - \frac{X_I^L + X_I^U}{2}\right) X_I^U - \left(\frac{X_F^L + X_F^U}{2}\right)(1 - X_F^U) \quad (2)$$

**Definition 3**

We can compute some mathematical operations on interval valued neutrosophic numbers as:

$$x_1 = [x_{1T}^{\,L}, x_{1T}^{\,U}], [x_{1I}^{\,L}, x_{1I}^{\,U}], [x_{1F}^{\,L}, x_{1F}^{\,U}]; \quad x_2 = [x_{2T}^{\,L}, x_{2T}^{\,U}], [x_{2i}^{\,L}, x_{2I}^{\,U}], [x_{2F}^{\,L}, x_{2F}^{\,U}]$$

$$x_1 \oplus x_2 = \ < \begin{pmatrix} [x_{1T}^{\,L} + x_{2T}^{\,L} - x_{1T}^{\,L} x_{2T}^{\,L}, x_{1T}^{\,U} + x_{2T}^{\,U} - x_{1T}^{\,U} x_{2T}^{\,U}], \\ [x_{1I}^{\,L} x_{2I}^{\,L}, x_{1I}^{\,U} x_{2I}^{\,U}], \\ [x_{1F}^{\,L} x_{2F}^{\,L}, x_{1F}^{\,U} x_{2F}^{\,U}] \end{pmatrix} > \quad (3)$$

$$x_1 \ominus x_2 =< \begin{pmatrix} [x_{1T}^{L} - x_{2F}^{U}, x_{1T}^{U} - x_{2F}^{L}], \\ [\max(x_{1I}^{L}, x_{2I}^{L}), \max(x_{1I}^{U} x_{2I}^{U})], \\ [x_{1F}^{L} - x_{2T}^{U}, x_{1F}^{U} - x_{2T}^{L}] \end{pmatrix} > \quad (4)$$

$$x_1 \otimes x_2 = < \begin{pmatrix} [x_{1T}^{L} x_{2T}^{L}, x_{1T}^{U} x_{2T}^{U}], \\ [x_{1I}^{L} + x_{2I}^{L} - x_{1I}^{L} x_{2I}^{L}, x_{1I}^{U} + x_{2I}^{U} - x_{1I}^{U} x_{2I}^{U}], \\ [x_{1F}^{L} + x_{2F}^{L} - x_{1F}^{L} x_{2F}^{L}, x_{1F}^{U} + x_{2F}^{U} - x_{1F}^{U} x_{2F}^{U}] \end{pmatrix} > \quad (5)$$

$$x_1^{\curlywedge} =< \begin{pmatrix} [1 - (1 - x_{1T}^{L})^{\curlywedge}, 1 - (1 - x_{1T}^{U})^{\curlywedge}], \\ [(x_{1I}^{L})^{\curlywedge}, (x_{1I}^{U})^{\curlywedge}], \\ [1 - (1 - x_{1F}^{L})^{\curlywedge}, 1 - (1 - x_{1F}^{U})^{\curlywedge}] \end{pmatrix} > \quad (6)$$

$$\curlywedge x_1 =< \begin{pmatrix} [1 - (1 - x_{1T}^{L})^{\curlywedge}, 1 - (1 - x_{1T}^{U})^{\curlywedge}], \\ [(x_{1I}^{L})^{\curlywedge}, (x_{1I}^{U})^{\curlywedge}], \\ [(x_{1F}^{L})^{\curlywedge}, (x_{1F}^{U})^{\curlywedge}] \end{pmatrix} > \quad (7)$$

Based on the logic of arranging issues in hierarchical and then assessing each element in the order via pairwise comparisons, the AHP technique was created and systematized by Thomas Saaty and introduced to the literature through systematization. Although AHP is widely utilized, there are situations when it does not accurately represent human reasoning when solving MCDM issues. IVN-AHP is an improvement over conventional AHP because it incorporates human cognition more effectively into the process of making choices and allows for a powerful expression of uncertainty using three variables.

Step 1. Build the comparison matrix

$$A = \begin{bmatrix} < [x_{11T}^{L}, x_{11T}^{U}], [x_{11I}^{L}, x_{11I}^{U}], [x_{11F}^{L}, x_{11F}^{U}] > & \cdots & < [x_{1mT}^{L}, x_{1mT}^{U}], [x_{1mI}^{L}, x_{1mI}^{U}], [x_{1mF}^{L}, x_{1mF}^{U}] > \\ \vdots & \ddots & \vdots \\ < [x_{m1T}^{L}, x_{m1T}^{U}], [x_{m1I}^{L}, x_{m1I}^{U}], [x_{m1F}^{L}, x_{m1F}^{U}] > & \cdots & < [x_{mmT}^{L}, x_{mmT}^{U}], [x_{mmI}^{L}, x_{mmI}^{U}], [x_{mmF}^{L}, x_{mmF}^{U}] > \end{bmatrix} \quad (8)$$

Step 2. Add the numbers in every column

$$Ad = < \begin{pmatrix} [\sum_{k=1}^{m} x_{kjT}^{L}, \sum_{k=1}^{m} x_{kjT}^{U},], \\ [\sum_{k=1}^{m} x_{kjI}^{L}, \sum_{k=1}^{m} x_{kjI}^{U},] \\ [\sum_{k=1}^{m} x_{kjF}^{L}, \sum_{k=1}^{m} x_{kjF}^{U},] \end{pmatrix} > \quad (9)$$

Step 3. Normalize the pairwise comparison matrix

$$Z_{ij} = < \begin{pmatrix} \left[\frac{x_{kj}{}_T^L}{\sum_{k=1}^m x_{kj}{}_T^U}, \frac{x_{kj}{}_T^U}{\sum_{k=1}^m x_{kj}{}_T^U}\right], \\ \left[\frac{x_{kj}{}_I^L}{\sum_{k=1}^m x_{kj}{}_I^U}, \frac{x_{kj}{}_I^U}{\sum_{k=1}^m x_{kj}{}_I^U}\right] \\ \left[\frac{x_{kj}{}_F^L}{\sum_{k=1}^m x_{kj}{}_F^U}, \frac{x_{kj}{}_F^U}{\sum_{k=1}^m x_{kj}{}_F^U}\right] \end{pmatrix} > \tag{10}$$

$$A = \begin{bmatrix} < \begin{pmatrix} \left[\frac{x_{11}{}_T^L}{\sum_{k=1}^m x_{kj}{}_T^U}, \frac{x_{11}{}_T^U}{\sum_{k=1}^m x_{kj}{}_T^U}\right], \\ \left[\frac{x_{11}{}_I^L}{\sum_{k=1}^m x_{kj}{}_I^U}, \frac{x_{11}{}_I^U}{\sum_{k=1}^m x_{kj}{}_I^U}\right], \\ \left[\frac{x_{11}{}_F^L}{\sum_{k=1}^m x_{kj}{}_F^U}, \frac{x_{11}{}_F^U}{\sum_{k=1}^m x_{kj}{}_F^U}\right] \end{pmatrix} > & \cdots & < \begin{pmatrix} \left[\frac{x_{1m}{}_T^L}{\sum_{k=1}^m x_{kj}{}_T^U}, \frac{x_{1m}{}_T^U}{\sum_{k=1}^m x_{kj}{}_T^U}\right], \\ \left[\frac{x_{1m}{}_I^L}{\sum_{k=1}^m x_{kj}{}_I^U}, \frac{x_{1m}{}_I^U}{\sum_{k=1}^m x_{kj}{}_I^U}\right], \\ \left[\frac{x_{1m}{}_F^L}{\sum_{k=1}^m x_{kj}{}_F^U}, \frac{x_{1m}{}_F^U}{\sum_{k=1}^m x_{kj}{}_F^U}\right] \end{pmatrix} > \\ \vdots & \ddots & \vdots \\ < \begin{pmatrix} \left[\frac{x_{m1}{}_T^L}{\sum_{k=1}^m x_{kj}{}_T^U}, \frac{x_{m1}{}_T^U}{\sum_{k=1}^m x_{kj}{}_T^U}\right], \\ \left[\frac{x_{m1}{}_I^L}{\sum_{k=1}^m x_{kj}{}_I^U}, \frac{x_{m1}{}_I^U}{\sum_{k=1}^m x_{kj}{}_I^U}\right], \\ \left[\frac{x_{m1}{}_F^L}{\sum_{k=1}^m x_{kj}{}_F^U}, \frac{x_{m1}{}_F^U}{\sum_{k=1}^m x_{kj}{}_F^U}\right] \end{pmatrix} > & \cdots & < \begin{pmatrix} \left[\frac{x_{mm}{}_T^L}{\sum_{k=1}^m x_{kj}{}_T^U}, \frac{x_{mm}{}_T^U}{\sum_{k=1}^m x_{kj}{}_T^U}\right], \\ \left[\frac{x_{mm}{}_I^L}{\sum_{k=1}^m x_{kj}{}_I^U}, \frac{x_{mm}{}_I^U}{\sum_{k=1}^m x_{kj}{}_I^U}\right], \\ \left[\frac{x_{mm}{}_F^L}{\sum_{k=1}^m x_{kj}{}_F^U}, \frac{x_{mms}{}_F^U}{\sum_{k=1}^m x_{kj}{}_F^U}\right] \end{pmatrix} > \end{bmatrix} \tag{11}$$

Step 4. Compute the weights of criteria

$$W = \begin{pmatrix} \frac{\left[\sum_{k=1}^m \frac{x_1{}_T^L}{\sum_{k=1}^m x_{kj}{}_T^U}, \sum_{k=1}^m \frac{x_1{}_T^U}{\sum_{k=1}^m x_{kj}{}_T^U}\right]}{m}, \\ \frac{\left[\sum_{k=1}^m \frac{x_1{}_I^L}{\sum_{k=1}^m x_{kj}{}_I^U}, \sum_{k=1}^m \frac{x_1{}_I^U}{\sum_{k=1}^m x_{kj}{}_I^U}\right]}{m}, \\ \frac{\left[\sum_{k=1}^m \frac{x_1{}_F^L}{\sum_{k=1}^m x_{kj}{}_F^U}, \sum_{k=1}^m \frac{x_1{}_F^U}{\sum_{k=1}^m x_{kj}{}_F^U}\right]}{m} \end{pmatrix} \tag{12}$$

Step 5. Rank the healthcare security criteria.

## 4. Analysis of criteria

We analysis the healthcare security criteria by the interval valued neutrosophic numbers. We collected nine criteria in this paper.

Security is of the utmost importance for healthcare providers because of the sensitive nature of patient data and the importance of the services they offer [23, 24]. The following factors should be taken into account while assessing healthcare security criteria:

Safeguarding sensitive patient information is of utmost importance. HIPAA (Health Insurance Portability and Accountability Act) and other data protection rules necessitate that healthcare providers take strong precautions to secure their patients' personal information. Secure access restrictions, frequent security audits, and breach response strategies are all things to consider.

Authentication and access controls are essential for protecting private data and computer systems. Strong access control techniques, such as multi-factor authentication, role-based access restrictions, and user provisioning procedures, should be implemented in organizations. It's crucial to conduct regular checks of user access privileges and promptly revoke access for dismissed workers or contractors.

To avoid hacking and other forms of data leaks, it is crucial to keep the network's infrastructure secure. Firewalls, intrusion detection/prevention systems, and secure network segmentation are all tools that may help businesses protect their most vital information. Potential security flaws must be assessed regularly, and patches must be applied promptly.

Data centers, server rooms, and secure storage locations all need to be protected with stringent physical security protocols. It is important to have security measures in place including access limits, video monitoring, and intrusion detection. Secure destruction procedures should be followed when disposing of paper documents or electronic storage devices.

Management of Security problems: Effectively identifying, responding to, and recovering from security problems requires dependable incident response and management procedures. Clear roles and responsibilities, incident reporting processes, and frequent testing and training exercises are all essential components of an organization's incident response strategy.

Employees have a crucial part in security, thus it's important to raise awareness and provide training on the topic. Security concerns, best practices for data management, and spotting and reporting security problems are all topics that employees should be trained on regularly.

In the healthcare industry, third-party service providers and suppliers play a vital role. To guarantee the safety of shared data and systems, it is essential to assess their security procedures. Third-party risk evaluations, as well as continuous monitoring of an organization's security practices, should all be included in contractual agreements.

HIPAA, the General Data Protection Regulation, and the Health Information Trust Alliance are just a few of the regulations that healthcare providers must follow. Evaluation of security procedures should heavily weigh how well they conform to these rules.

The availability and data integrity of vital systems and information are dependent on healthcare organizations having solid business continuity and disaster recovery strategies. It is crucial to regularly test and update these strategies to lessen the effect of any possible interruptions.

These healthcare security criteria may be used to provide a safe space for patients' information, keep services running smoothly, and prevent breaches.

Step 1. We build the comparison matrix between criteria by using Eq. (8). We collected the nine criteria to be used in this study. We used the linguistic scale of interval valued neutrosophic set to evaluate the criteria in the comparison matrix as shown in Table 1.

Table 1. The interval valued neutrosophic comparison matrix.

| | $HSER_1$ | $HSER_2$ | $HSER_3$ | $HSER_4$ | $HSER_5$ | $HSER_6$ | $HSER_7$ | $HSER_8$ | $HSER_9$ |
|---|---|---|---|---|---|---|---|---|---|

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $HSER_1$ | 1 | ⟨[0.95,1.0],[0.0,0.0],[0.0,0.10]⟩ | ⟨[0.80,0.90],[0.05,0.10],[0.10,0.20]⟩ | ⟨[0.55,0.65],[0.30,0.40],[0.35,0.45]⟩ | ⟨[0.90,0.95],[0,0.05],[0.05,0.15]⟩ | ⟨[0.90,0.95],[0,0.05],[0.05,0.15]⟩ | ⟨[0.80,0.90],[0.05,0.10],[0.10,0.20]⟩ | ⟨[0.95,1.0],[0.0,0.0],[0.0,0.10]⟩ | ⟨[0.95,1.0],[0.0,0.0],[0.0,0.10]⟩ |
| $HSER_2$ | ⟨[0.95,1.0],[0.0,0.0],[0.0,0.10]⟩ | 1 | ⟨[0.70,0.80],[0.15,0.25],[0.20,0.30]⟩ | ⟨[0.90,0.95],[0,0.05],[0.05,0.15]⟩ | ⟨[0.55,0.65],[0.30,0.40],[0.35,0.45]⟩ | ⟨[0.80,0.90],[0.05,0.10],[0.10,0.20]⟩ | ⟨[0.90,0.95],[0,0.05],[0.05,0.15]⟩ | ⟨[0.55,0.65],[0.30,0.40],[0.35,0.45]⟩ | ⟨[0.80,0.90],[0.05,0.10],[0.10,0.20]⟩ |
| $HSER_3$ | ⟨[0.80,0.90],[0.05,0.10],[0.10,0.20]⟩ | ⟨[0.70,0.80],[0.15,0.25],[0.20,0.30]⟩ | 1 | ⟨[0.80,0.90],[0.05,0.10],[0.10,0.20]⟩ | ⟨[0.95,1.0],[0.0,0.0],[0.0,0.10]⟩ | ⟨[0.90,0.95],[0,0.05],[0.05,0.15]⟩ | ⟨[0.90,0.95],[0,0.05],[0.05,0.15]⟩ | ⟨[0.65,0.75],[0.20,0.30],[0.25,0.35]⟩ | ⟨[0.65,0.75],[0.20,0.30],[0.25,0.35]⟩ |
| $HSER_4$ | ⟨[0.55,0.65],[0.30,0.40],[0.35,0.45]⟩ | ⟨[0.90,0.95],[0,0.05],[0.05,0.15]⟩ | ⟨[0.80,0.90],[0.05,0.10],[0.10,0.20]⟩ | 1 | ⟨[0.70,0.80],[0.15,0.25],[0.20,0.30]⟩ | ⟨[0.90,0.95],[0,0.05],[0.05,0.15]⟩ | ⟨[0.80,0.90],[0.05,0.10],[0.10,0.20]⟩ | ⟨[0.90,0.95],[0,0.05],[0.05,0.15]⟩ | ⟨[0.80,0.90],[0.05,0.10],[0.10,0.20]⟩ |
| $HSER_5$ | ⟨[0.90,0.95],[0,0.05],[0.05,0.15]⟩ | ⟨[0.55,0.65],[0.30,0.40],[0.35,0.45]⟩ | ⟨[0.95,1.0],[0.0,0.0],[0.0,0.10]⟩ | ⟨[0.70,0.80],[0.15,0.25],[0.20,0.30]⟩ | 1 | ⟨[0.70,0.80],[0.15,0.25],[0.20,0.30]⟩ | ⟨[0.95,1.0],[0.0,0.0],[0.0,0.10]⟩ | ⟨[0.65,0.75],[0.20,0.30],[0.25,0.35]⟩ | ⟨[0.65,0.75],[0.20,0.30],[0.25,0.35]⟩ |
| $HSER_6$ | ⟨[0.90,0.95],[0,0.05],[0.05,0.15]⟩ | ⟨[0.80,0.90],[0.05,0.10],[0.10,0.20]⟩ | ⟨[0.90,0.95],[0,0.05],[0.05,0.15]⟩ | ⟨[0.90,0.95],[0,0.05],[0.05,0.15]⟩ | ⟨[0.70,0.80],[0.15,0.25],[0.20,0.30]⟩ | 1 | ⟨[0.70,0.80],[0.15,0.25],[0.20,0.30]⟩ | ⟨[0.95,1.0],[0.0,0.0],[0.0,0.10]⟩ | ⟨[0.65,0.75],[0.20,0.30],[0.25,0.35]⟩ |
| $HSER_7$ | ⟨[0.80,0.90],[0.05,0.10],[0.10,0.20]⟩ | ⟨[0.90,0.95],[0,0.05],[0.05,0.15]⟩ | ⟨[0.90,0.95],[0,0.05],[0.05,0.15]⟩ | ⟨[0.80,0.90],[0.05,0.10],[0.10,0.20]⟩ | ⟨[0.95,1.0],[0.0,0.0],[0.0,0.10]⟩ | ⟨[0.70,0.80],[0.15,0.25],[0.20,0.30]⟩ | 1 | ⟨[0.70,0.80],[0.15,0.25],[0.20,0.30]⟩ | ⟨[0.95,1.0],[0.0,0.0],[0.0,0.10]⟩ |
| $HSER_8$ | ⟨[0.95,1.0],[0.0,0.0],[0.0,0.10]⟩ | ⟨[0.55,0.65],[0.30,0.40],[0.35,0.45]⟩ | ⟨[0.65,0.75],[0.20,0.30],[0.25,0.35]⟩ | ⟨[0.90,0.95],[0,0.05],[0.05,0.15]⟩ | ⟨[0.65,0.75],[0.20,0.30],[0.25,0.35]⟩ | ⟨[0.95,1.0],[0.0,0.0],[0.0,0.10]⟩ | ⟨[0.70,0.80],[0.15,0.25],[0.20,0.30]⟩ | 1 | ⟨[0.80,0.90],[0.05,0.10],[0.10,0.20]⟩ |
| $HSER_9$ | ⟨[0.95,1.0],[0.0,0.0],[0.0,0.10]⟩ | ⟨[0.80,0.90],[0.05,0.10],[0.10,0.20]⟩ | ⟨[0.65,0.75],[0.20,0.30],[0.25,0.35]⟩ | ⟨[0.80,0.90],[0.05,0.10],[0.10,0.20]⟩ | ⟨[0.65,0.75],[0.20,0.30],[0.25,0.35]⟩ | ⟨[0.65,0.75],[0.20,0.30],[0.25,0.35]⟩ | ⟨[0.95,1.0],[0.0,0.0],[0.0,0.10]⟩ | ⟨[0.80,0.90],[0.05,0.10],[0.10,0.20]⟩ | 1 |

Step 2. Add the numbers in every column by using Eq. (9).

Step 3. Normalize the pairwise comparison matrix by using Eqs. (10 and 11) as shown in Table 2.

Table 2. The Normalization matrix.

| | $HSER_1$ | $HSER_2$ | $HSER_3$ | $HSER_4$ | $HSER_5$ | $HSER_6$ | $HSER_7$ | $HSER_8$ | $HSER_9$ |
|---|---|---|---|---|---|---|---|---|---|
| $HSER_1$ | 0.093892 | 0.081637 | 0.079446 | 0.059066 | 0.09331 | 0.098941 | 0.093786 | 0.124938 | 0.131339 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $HSER_2$ | 0.098259 | 0.085434 | 0.069515 | 0.095696 | 0.057593 | 0.086762 | 0.105509 | 0.072877 | 0.11011 |
| $HSER_3$ | 0.117365 | 0.122048 | 0.099307 | 0.085063 | 0.099064 | 0.097607 | 0.105509 | 0.085996 | 0.090104 |
| $HSER_4$ | 0.169023 | 0.094926 | 0.124134 | 0.106329 | 0.072574 | 0.097607 | 0.093786 | 0.118234 | 0.109959 |
| $HSER_5$ | 0.104325 | 0.153796 | 0.103932 | 0.151898 | 0.103678 | 0.075917 | 0.111957 | 0.085577 | 0.090097 |
| $HSER_6$ | 0.102918 | 0.106792 | 0.110341 | 0.118143 | 0.148111 | 0.108453 | 0.082062 | 0.125361 | 0.089646 |
| $HSER_7$ | 0.117365 | 0.094926 | 0.110341 | 0.132911 | 0.108563 | 0.154932 | 0.117232 | 0.091834 | 0.131339 |
| $HSER_8$ | 0.098592 | 0.153796 | 0.151498 | 0.117982 | 0.158942 | 0.113498 | 0.167474 | 0.131192 | 0.109959 |
| $HSER_9$ | 0.09826 | 0.106645 | 0.151487 | 0.132911 | 0.158166 | 0.166282 | 0.122685 | 0.16399 | 0.137448 |

Step 4. Compute the weights of criteria by using Eq. (12) as shown in Figure 2.

Step 5. Rank the healthcare security criteria as shown in Figure 2. Data privacy is the best criterion.

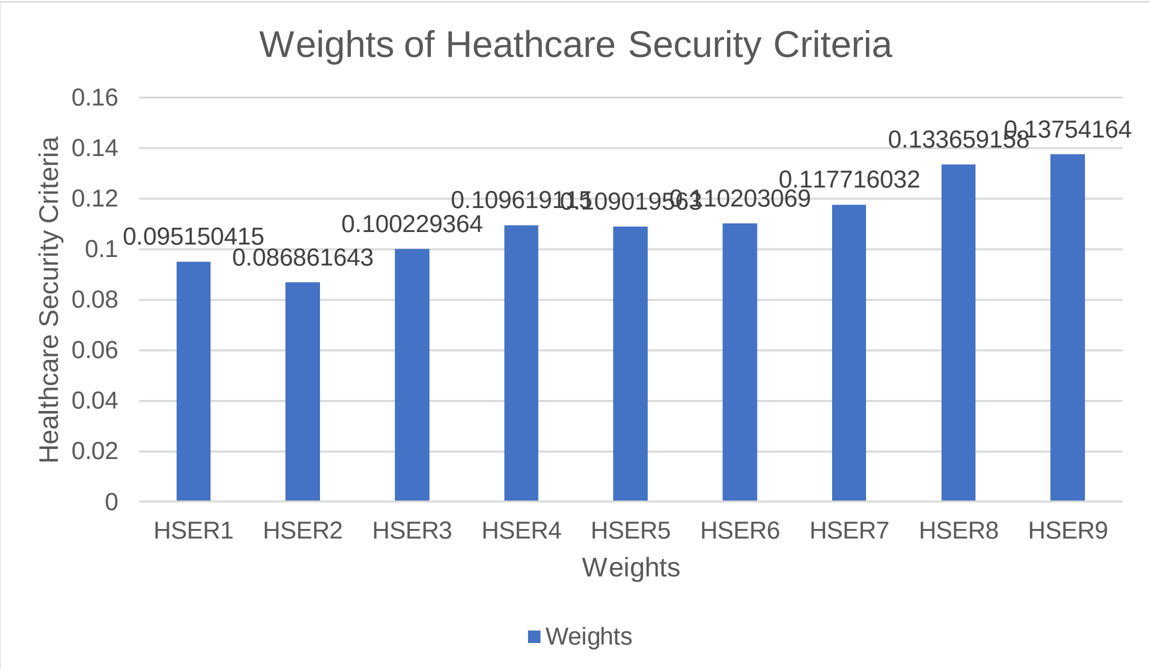


Figure 2. The weights of healthcare security criteria.

**5. Conclusions**

E-healthcare services are becoming increasingly popular as a result of the ease with which patients' medical records may be accessed and moved from one location to another. Because of their ability to supply solutions at a lower overall cost, cloud service providers have made it possible to practice telemedicine. Despite the many benefits it offers, the framework for storing information in the cloud and retrieving that information through the cloud is extremely susceptible since it makes use of open channels. Only authorized actors will be able to access and keep patient data under the secure interface. A full end-to-end encryption service that makes use of several KDF-derived keys is given to protect confidential patient data. The hospital is relieved of the responsibility of maintaining patient records, and improvements are made to both access to and storage of medical records.

We evaluate the criteria of healthcare security by using the interval-valued neutrosophic AHP method. The neutrosophic set is used to overcome the uncertain information. We used the AHP method to compute the weights of the criteria and evaluate them. We used the nine criteria. The main results show that data privacy is the best criterion.

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# A Total Ordering on $n$ - Valued Refined Neutrosophic Sets using Dictionary Ranking based on Total ordering on $n$ - Valued Neutrosophic Tuplets

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**Abstract**. The notion of fuzzy subsets was first introduced by Zadeh in 1965, and was later extended to intuitionistic fuzzy subsets by Atanassov in 1983. Since the inception of fuzzy set theory, we have encountered a number of generalizations of sets, one of which is neutrosophic sets introduced by Smarandache [15]. Later neutrosophic sets was generalized into interval valued neutrosophic, triangular valued neutrosophic, trapezoidal valued neutrosophic and $n$ - valued refined neutrosophic sets in the literature [19, 31, 33, 35]. Further, the ordering on single-valued neutrosophic triplets and interval valued neutrosophic triplets have been proposed by Smarandache in [16] and they are further extended to total ordering on interval valued neutrosophic triplets in [32].The total ordering of $n$ - valued neutrosophic tuplets is very significant in multi-criteria decision making (MCDM) involving $n$ - valued neutrosophic tuplets. Hence, in this paper, different methods for ordering $n$ - valued neutrosophic tuplets (NVNT) are developed with the goal of achieving a total ordering on $n$ - valued neutrosophic tuplets and the applicability of the proposed methods is shown by illustrative examples in MCDM problems involving $n$ - valued neutrosophic tuplets. Further, a total ordering algorithm for $n$ - valued refined neutrosophic sets by following dictionary ranking method at the final stage is developed using those proposed total ordering methods on $n$ - valued neutrosophic tuplets.

Keywords: n - Valued Refined Neutrosophic Sets, Dictionary, Neutrosophic Tuplets, Uncertainty

## 1. Introduction

Our daily life is full of uncertain situations and we need to make better decisions based on their volatility. Despite this, Zadeh established the concept of fuzzy sets in 1965 to handle such ambiguity [18]. Though this idea of fuzzy sets was reluctantly acknowledged initially, researchers believed that analyzing this concept might bring a tremendous revolution in the future with real-life MCDM and MADM problems with with uncertainty or vagueness. Hence, a great progress has been made in the research of fuzzy set generalisation, resulting in numerous forms of fuzzy sets such as intuitionistic fuzzy sets, neutrosophic sets, picture fuzzy sets, bipolar fuzzy sets, and so on [3–5, 15, 20]. These versions of fuzzy sets were widely used in a variety of real-world issues. The multi-criteria decision making (MCDM) problem is a rising topic of research due to its importance in most real-world challenges [12, 14, 17, 19].

The neutrosophic sets are introduced by Florentine smarandache [15], as a generalization of intuitionistic fuzzy sets. In intuitionistic fuzzy sets, we usually consider membership, non membership values. But, in neutrosophic sets, we consider membership, non membership values and an indeterminacy value which differentiates neutrosophic sets from intuitionistic fuzzy sets. Later, $n$ - valued redefined neutrosophic sets were introduced by Florentine smarandache as further generalization and some MCDM problems have been studied in real-world scenarios using $n$ - valued redefined neutrosophic sets in [33]. To solve such MCDM problems, we need total ordering on $n$ - valued neutrosophic tuplets. For each fuzzy MCDM problem, there are several total ordering on fuzzy numbers in the literature [7–9, 11]. Furthermore, the decision maker selects the total ordering strategy that best suits his needs. For a fuzzy MCDM, the total order does not have to be unique. The ranking of single valued neutrosophic triplets has been analysed in [13,16] and further extended to total ordering on interval valued neutrosophic triplets in [32].

Dictionary ordering is usually followed to rank totally the elements of $\mathbb{Y}^X$ using the total order $<$ on $Y$ for any countable set $X$. In detail, to compare $(a_1, a_2, \ldots a_n, \ldots)$ and $(b_1, b_2, \ldots b_n, \ldots)$, we first compare $a_1$ and $b_1$ using total order $<$ on $Y$. If $a_1 < b_1$(or $b_1 < a_1$) then $(a_1, a_2, \ldots a_n, \ldots) < (b_1, b_2, \ldots b_n, \ldots)$(or $(b_1, b_2, \ldots b_n, \ldots) < (a_1, a_2, \ldots a_n, \ldots)$). If $a_1 = b_1$, then we follow the same procedure for comparing $a_2$ and $b_2$ using total order $<$ on $Y$ and so on. The same method only indirectly followed in any ranking of fuzzy numbers, intuitionistic fuzzy numbers, single valued neutrosophic numbers and interval valued neutrosophic numbers using score functions [7, 8, 11, 16, 32].

In this paper, we aim to achieve a total ordering on $n$ - valued neutrosophic tuplets and $n$ - valued refined neutrosophic sets. To derive total ordering on $n$ - valued refined neutrosophic sets, we need a total ordering on $n$ - valued neutrosophic tuplets. First, we derive total ordering on $n$ - valued neutrosophic tuplets for which we introduce two algorithms. In the first stage of the both algorithms, we convert $n$ - valued neutrosophic tuplets into single valued neutrosophic triplets and then we try to rank them. In the next stage, first method follows a reverse dictionary order and second method follows a method of ranking based on the fluctuations on truth, falsity and indeterminacy values. To rank the $n$ -valued neutrosophic sets, we develop a total ordering algorithm for $n$ - valued refined neutrosophic sets by following dictionary ranking method at the final stage using those proposed total ordering methods on $n$ - valued neutrosophic tuplets.

## 2. Preliminaries

This section contains all of the necessary definitions to move deeper into the concept of total ordering on $n$ - valued neutrosophic tuplets.

**Definition 2.1.** [16] Let $\mathcal{M} = \{(T, I, F), \text{where } T, I, F \in [0,1], 0 \leq T+I+F \leq 3\}$ be the set of single valued neutrosophic triplet (SVNT) numbers. Let $N = (T, I, F) \in \mathcal{M}$ be a generic SVNT number, where $T$ denotes grade of membership ; $I$ denotes indeterminacy grade ; $F$ denotes grade of non-membership.

**Definition 2.2.** [32] A SVNT membership score $S^+ : \mathcal{M} \to [0,1]$ is defined by

$$S^+(T, I, F) = \frac{2 + (T - F)(2 - I) - I}{4}.$$

**Definition 2.3.** [32] A SVNT non-membership score $S^- : \mathcal{M} \to [0,1]$ is defined by

$$S^-(T, I, F) = \frac{2 + (F - T)(2 - I) - I}{4}.$$

**Definition 2.4.** [32] A SVNT average score $C : \mathcal{M} \to [0,1]$ is defined by

$$C(T, I, F) = \frac{T + F}{2}$$

**Definition 2.5.** [33] Let $(T, I, F)$ be a $n$ - valued neutrosophic triplet number, where $T$ can be split into many types of truths as $T_1, T_2 \ldots T_p$, $I$ can be split into many types of indeterminacies as $I_1, I_2 \ldots I_q$ and $F$ can be split into many types of falsities as $F_1, F_2 \ldots F_r$ where $T_i, I_j, F_k \in [0,1]$ for $i \in \{1, \ldots p\}, j \in \{1, \ldots q\}$ and $k \in \{1, \ldots r\}$ and $p + q + r = n$. Therefore we have $0 \leq \sum_{i=1}^{p} T_i + \sum_{j=1}^{q} I_j + \sum_{k=1}^{r} F_k \leq n$.

**Definition 2.6.** Let $N_1 = (T_1, T_2, \ldots T_p, I_1, I_2, \ldots I_q, F_1, F_2, \ldots F_r)$ and $N_2 = (T_1', T_2', \ldots T_p', I_1', I_2', \ldots I_q', F_1', F_2', \ldots F_r')$ be two $n$ - valued neutrosophic triplet numbers, where $p + q + r = n$. Then we define $N_1 + N_2 = (T_1 + T_1', T_2 + T_2', \ldots T_p + T_p', I_1 + I_1', I_2 + I_2', \ldots I_q + I_q', F_1 + F_1', F_2 + F_2', \ldots F_r + F_r')$ and $\alpha N_1 = (\alpha T_1, \ldots \alpha T_p, \alpha I_1, \ldots \alpha I_q, \alpha F_1, \ldots \alpha F_r)$ where $\alpha \in \mathbb{R}$.

## 3. A Total order on $n$-Valued Neutrosophic tuplets

In this section, we present a ranking technique for $n$ - valued neutrosophic tuplets that inherits total ordering.

3.1. *Ranking algorithm for $n$ - valued neutrosophic tuplets*

Let $A = (T, I, F)$ and $B = (T', I', F')$ be two $n$ - valued neutrosophic tuplets such that $A \neq B$, where $T$ can be split into many types of truths in ascending order as $T_1, T_2 \dots T_{p_1}$, $I$ can be split into many types of indeterminacies in ascending order as $I_1, I_2 \dots I_{q_1}$ and $F$ can be split into many types of falsities in ascending order as $F_1, F_2 \dots F_{r_1}$ where $T_i, I_j, F_k \in [0,1]$ and $p_1+q_1+r_1 = n$. Therefore we have $0 \leq \sum_{i=1}^{p_1} T_i + \sum_{j=1}^{q_1} I_j + \sum_{k=1}^{r_1} F_k \leq n$. Similarly $T'$ can be split into many types of truths in ascending order as $T'_1, T'_2 \dots T'_{p_2}$, $I'$ can be split into many types of indeterminacies in ascending order as $I'_1, I'_2 \dots I'_{q_2}$ and $F'$ can be split into many types of falsities in ascending order as $F'_1, F'_2 \dots F'_{r_2}$ where $T'_i, I'_j, F'_k \in [0,1]$ and $p_2 + q_2 + r_2 = n$. Therefore we have $0 \leq \sum_{i=1}^{p_2} T_i + \sum_{j=1}^{q_2} I_j + \sum_{k=1}^{r_2} F_k \leq n$.

**Step 1:** Choose $k = lcm\{p_1, p_2, q_1, q_2, r_1, r_2\}$. Now convert both $n$ - valued neutrosophic tuplets $A$ and $B$ by rewriting $T, I, F$ and $T', I', F'$ as follows;

Suppose $k = x_1 p_1$ then $T = (\underbrace{T_1, \dots T_1}_{x_1 \text{ times}}, \underbrace{T_2, \dots T_2}_{x_1 \text{ times}}, \dots \underbrace{T_{p_1}, \dots T_{p_1}}_{x_1 \text{ times}}) = (T_1, T_2, \dots T_k)$

Suppose $k = y_1 q_1$ then $I = (\underbrace{I_1, \dots I_1}_{y_1 \text{ times}}, \underbrace{I_2, \dots I_2}_{y_1 \text{ times}}, \dots \underbrace{I_{q_1}, \dots I_{q_1}}_{y_1 \text{ times}}) = (I_1, I_2, \dots I_k)$

Suppose $k = z_1 r_1$ then $F = (\underbrace{F_1, \dots F_1}_{z_1 \text{ times}}, \underbrace{F_2, \dots F_2}_{z_1 \text{ times}}, \dots \underbrace{F_{r_1}, \dots F_{r_1}}_{z_1 \text{ times}}) = (F_1, F_2, \dots F_k)$

Suppose $k = x_2 p_2$ then $T' = (\underbrace{T'_1, \dots T'_1}_{x_2 \text{ times}}, \underbrace{T'_2, \dots T'_2}_{x_2 \text{ times}}, \dots \underbrace{T'_{p_2}, \dots T_{p_2}}_{x_2 \text{ times}}) = (T'_1, T'_2, \dots T'_k)$

Suppose $k = y_2 q_2$ then $I' = (\underbrace{I'_1, \dots I'_1}_{y_2 \text{ times}}, \underbrace{I'_2, \dots I'_2}_{y_2 \text{ times}}, \dots \underbrace{I'_{q_2}, \dots I'_{q_2}}_{y_2 \text{ times}}) = (I'_1, I'_2, \dots I'_k)$

Suppose $k = z_2 r_2$ then $F = (\underbrace{F'_1, \dots F'_1}_{z_2 \text{ times}}, \underbrace{F'_2, \dots F'_2}_{z_2 \text{ times}}, \dots \underbrace{F'_{r_2}, \dots F'_{r_2}}_{z_2 \text{ times}}) = (F'_1, F'_2, \dots F'_k)$

Now we have $A = (T_1, \dots T_k, I_1, \dots I_k, F_1, \dots F_k)$ and $B = (T'_1, \dots T'_k, I'_1, \dots I'_k, F'_1, \dots F'_k)$ as a $3k$ valued neutrosophic tuplets where truth, falsity and indeterminacy values are k tuple.

Let $(T_0, I_0, F_0) = (\frac{\sum_{i=1}^{k} T_i}{k}, \frac{\sum_{i=1}^{k} I_i}{k}, \frac{\sum_{i=1}^{k} F_i}{k})$ and $(T'_0, I'_0, F'_0) = (\frac{\sum_{i=1}^{k} T'_i}{k}, \frac{\sum_{i=1}^{k} I'_i}{k}, \frac{\sum_{i=1}^{k} F'_i}{k})$.

**Step 2:** We compare $(T_0, I_0, F_0)$ and $(T'_0, I'_0, F'_0)$ using score functions. Apply neutrosophic membership score function $S^+$. Suppose $S^+(T_0, I_0, F_0) > S^+(T'_0, I'_0, F'_0)$, then we have $A > B$. Suppose $S^+(T_0, I_0, F_0) < S^+(T'_0, I'_0, F'_0)$, then we have $A < B$. Suppose $S^+(T_0, I_0, F_0) = S^+(T'_0, I'_0, F'_0)$, go to step 3.

**Step 3:** Apply neutrosophic non-membership score function $S^-$. Suppose $S^-(T_0, I_0, F_0) > S^-(T'_0, I'_0, F'_0)$, then we have $A < B$. Suppose $S^-(T_0, I_0, F_0) < S^-(T'_0, I'_0, F'_0)$, then we have $A > B$. Suppose $S^-(T_0, I_0, F_0) = S^-(T'_0, I'_0, F'_0)$, go to step 4.

**Step 4:** Apply neutrosophic average function $C$. Suppose $C(T_0, I_0, F_0) > C(T'_0, I'_0, F'_0)$, then we have $A > B$. Suppose $C(T_0, I_0, F_0) < C(T'_0, I'_0, F'_0)$, then we have $A < B$. Suppose $C(T_0, I_0, F_0) = C(T'_0, I'_0, F'_0)$, then go to step 5.

**Step 5:** Now we compare $(T_m, I_m, F_m)$ and $(T'_m, I'_m, F'_m)$ for $m = k$ by considering

$(T_0, I_0, F_0) = (T_m, I_m, F_m)$ and $(T'_0, I'_0, F'_0) = (T'_m, I'_m, F'_m)$ using steps 2, 3 and 4. If we are not still able to differentiate $A$ and $B$, then we compare for $m = m - 1$ by applying step 5 till ranking $A$ and $B$.

**Theorem 3.1.** *Proposed ranking algorithm inherits a total order on set of all $n$ - valued neutrosophic tuplets.*

*Proof.* We show that for any two $n$ - valued neutrosophic sets $(T, I, F)$ and $(T', I', F')$, either $(T, I, F) < (T', I', F')$ or $(T, I, F) > (T', I', F')$ or $(T, I, F) = (T', I', F')$. Let $A = (T, I, F) = (T_1, T_2 \ldots T_{p_1}, I_1, I_2 \ldots I_{q_1}, F_1, F_2 \ldots F_{r_1})$ and $B = (T', I', F') = (T'_1, T'_2 \ldots T'_{p_2}, I'_1,$ $I'_2 \ldots I'_{q_2}, F'_1, F'_2 \ldots F'_{r_2})$ be two $n$ - valued neutrosophic tuplets such that $A \neq B$, where $p_1 + q_1 + r_1 = p_2 + q_2 + r_2 = n$. Now we show that either $A < B$ or $B < A$. By applying step 1, we have $A = (T_1, \ldots T_k, I_1, \ldots I_k, F_1, \ldots F_k,)$ and $B = (T'_1, \ldots T'_k, I'_1, \ldots I'_k, F'_1, \ldots F'_k,)$ where $k = lcm\{p_1, q_1, r_1, p_2, q_2, r_2\}$.

Now,

let $(T_0, I_0, F_0) = (\sum_{i=1}^{k} T_i, \sum_{i=1}^{k} I_i, \sum_{i=1}^{k} F_i)$ and $(T'_0, I'_0, F'_0) = (\sum_{i=1}^{k} T'_i, \sum_{i=1}^{k} I'_i, \sum_{i=1}^{k} F'_i)$. First we apply membership score function $S^+$. Suppose $S^+ \ (T_0, I_0, F_0) > S^+ \ (T'_0, I'_0, F'_0)$ ( or $S^+(T_0, I_0, F_0) < S^+(T'_0, I'_0, F'_0)$, then we have $A > B$( or $A < B$), which is done. When $S^+ \ (T_0, I_0, F_0) = S^+ \ (T'_0, I'_0, F'_0)$, we have to go to next step. So, suppose $\frac{2+(T_0-F_0)(2-I_0)-I_0}{4} = \frac{2+(T_0-F_0)(2-I_0)-I_0}{4}$, equivalently, if $(T_0 - F_0)(2 - I_0) - I_0 = (T'_0 - F'_0)(2 - I'_0) - I'_0$, we apply non-membership score function. Hence, if $S^- \ (T_0, I_0, F_0) > S^- \ (T'_0, I'_0, F'_0)$ $(S^- \ (T_0, I_0, F_0) < S^- \ (T'_0, I'_0, F'_0))$, then $A < B(A > B)$, which is done. When $S^- \ (T_0, I_0, F_0) = S^- \ (T'_0, I'_0, F'_0)$, equivalently, if $(F_0 - T_0)(2 - I_0) - I_0 = (F'_0 - T'_0)(2 - I'_0) - I'_0$, we have to go to next step by using average score function. Hence, suppose $C \ (T_0, I_0, F_0) > C \ (T'_0, I'_0, F'_0)$ ( or $C \ (T_0, I_0, F_0)$ $< C \ (T'_0, I'_0, F'_0))$, then we have $A > B$( or $A < B$), which is done. When $C \ (T_0, I_0, F_0) = C \ (T'_0, I'_0, F'_0)$, we have $T_0 + F_0 = T'_0 + F'_0$. At this stage, we have triplets $(T_0, I_0, F_0)$ and $(T'_0, I'_0, F'_0)$ satisfying following system of 3 equations.

$$(T_0 - F_0)(2 - I_0) - I_0 = (T'_0 - F'_0)(2 - I'_0) - I'_0 \tag{1}$$

$$(F_0 - T_0)(2 - I_0) - I_0 = (F'_0 - T'_0)(2 - I'_0) - I'_0 \tag{2}$$

$$T_0 + F_0 = T'_0 + F'_0 \tag{3}$$

Now, we solve this system of equations. By adding equations 1 and 2, we get $I_0 = I'_0$ which makes equation 1 into

$$T_0 - F_0 = T'_0 - F'_0$$

now, by adding the above equation with equation 3, we get $F_0 = F_0'$ and $T_0 = T_0'$ .
Thus, we get

$$(T_0, I_0, F_0) = (T_0', I_0', F_0').$$

As a result, we have

$$T_1 + T_2 + \ldots T_k = T_1' + T_2' + \ldots T_k' \tag{4}$$

$$I_1 + I_2 + \ldots I_k = I_1' + I_2' + \ldots I_k' \tag{5}$$

$$F_1 + F_2 + \ldots F_k = F_1' + F_2' + \ldots F_k' \tag{6}$$

Let us compare $(T_k, I_k, F_k)$ and $(T_k', I_k', F_k')$. First we apply membership score function $S^+$. Suppose $S^+$ $(T_k, I_k, F_k) > S^+$ $(T_k', I_k', F_k')$( or $S^+(T_k, I_k, F_k) < S^+(T_k', I_k', F_k')$, then we have $A > B$( or $A < B$), which is done. When $S^+$ $(T_k, I_k, F_k) = S^+$ $(T_k', I_k', F_k')$, we have to go to next step. So, suppose $\frac{2+(T_k-F_k)(2-I_k)-I_k}{4} = \frac{2+(T_k'-F_k')(2-I_k')-I_k'}{4}$, equivalently, if $(T_k - F_k)(2 - I_k) - I_k = (T_k' - F_k')(2 - I_k) - I_k$, we apply non-membership score function. Hence, if $S^-$ $(T_k, I_k, F_k) > S^-$ $(T_k', I_k', F_k')$ ($S^-$ $(T_k, I_k, F_k) < S^-$ $(T_k', I_k', F_k')$, then $A < B(A > B)$, which is done. When $S^-$ $(T_k, I_k, F_k) = S^-$ $T_k', I_k', F_k')$, equivalently, if $(F_k - T_k)(2 - I_k) - I_k = (F_k' - T_k')(2 - I_k') - I_k'$, we have to go to average score function. Hence, suppose $C$ $(T_k, I_k, F_k)$ $> C$ $(T_k', I_k', F_k')$ ( or $C$ $(T_k, I_k, F_k) < C$ $(T_k', I_k', F_k')$), then we have $A > B$ ( or $A < B$), which is done. When $C$ $(T_k, I_k, F_k) = C$ $(T_k', I_k', F_k')$, we have $T_k + F_k = T_k' + F_k'$. At this stage, we have triplets $(T_k, I_k, F_k)$ and $(T_k', I_k', F_k')$ satisfying following system of 3 equations.

$$(T_k - F_k)(2 - I_k) - I_k = (T_k' - F_k')(2 - I_k') - I_k' \tag{7}$$

$$(F_k - T_k)(2 - I_k) - I_k = (F_k' - T_k)(2 - I_k) - I_k \tag{8}$$

$$T_k + F_k = T_k' + F_k' \tag{9}$$

Now, we solve this system of equations. By adding equations 7 and 8, we get $I_k = I_k'$ which makes equation 7 into

$$T_k - F_k = T_k' - F_k'$$

now, by adding the above equation with equation 9, we get $F_k = F_k'$ and $T_k = T_k'$ .
Thus, we get

$$(T_k, I_k, F_k) = (T_k', I_k', F_k').$$

Similarly, by continuing the above process for $m = k - 1, \ldots 2, 1$, till we get $A < B$ or $B < A$. If we have $(T_m, I_m, F_m) = (T_m', I_m', F_m')$ for $m = \{k, k-1, k-2 \ldots 2, 1\}$. By solving with equations 3.7, 3.8, and 3.9, we get $A = B$, a contradiction. Thus we have proved the proposed ranking algorithm inherits a total order on set of all $n$ - valued neutrosophic tuplets. □

The following statement's proofs are direct applications of definitions, hence proofs are omitted.

**Proposition 3.2.** *Let* $N_1 = (T_1, T_2, \dots T_p, I_1, I_2, \dots I_q, F_1, F_2, \dots F_r)$ *and* $N_2 = (T'_1, T'_2, \dots T'_p, I'_1, I'_2, \dots I'_q, F'_1, F'_2, \dots F'_r)$ *be two* $n$ *- valued neutrosophic tuplets, where* $p + q + r = n$.

(1) *If* $\sum_{n=1}^{p} T_n = \sum_{n=1}^{p} T'_n, \sum_{n=1}^{r} F_n = \sum_{n=1}^{r} F'_n$ *and* $\sum_{n=1}^{q} I_n > \sum_{n=1}^{q} I'_n$*, then we get* $N_1 < N_2$.

(2) *If* $\sum_{n=1}^{p} T_n = \sum_{n=1}^{p} T'_n, \sum_{n=1}^{r} F_n = \sum_{n=1}^{r} F'_n$ *and* $\sum_{n=1}^{q_1} I_n < \sum_{n=1}^{q_2} I'_n$*, then* $N_1 > N_2$.

**Proposition 3.3.** *Let* $N_1 = (T_1, T_2, \dots T_p, I_1, I_2, \dots I_q, F_1, F_2, \dots F_r)$ *and* $N_2 = (T'_1, T'_2, \dots T'_p, I'_1, I'_2, \dots I'_q, F'_1, F'_2, \dots F'_r)$ *be two* $n$ *- valued neutrosophic tuplets, where* $p + q + r = n$.

(1) *If* $\sum_{n=1}^{p} T_n = \sum_{n=1}^{p} T'_n, \sum_{n=1}^{q} I_n = \sum_{n=1}^{q} I'_n$ *and* $\sum_{n=1}^{r} F_n > \sum_{n=1}^{r} F'_n$*, then we get* $N_1 < N_2$.

(2) *If* $\sum_{n=1}^{p} T_n = \sum_{n=1}^{p} T'_n, \sum_{n=1}^{q_1} I_n = \sum_{n=1}^{q_2} I'_n$ *and* $\sum_{n=1}^{r} F_n < \sum_{n=1}^{r} F'_n$*, then we get* $N_1 > N_2$.

**Proposition 3.4.** *Let* $N_1 = (T_1, T_2, \dots T_p, I_1, I_2, \dots I_q, F_1, F_2, \dots F_r)$ *and* $N_2 = (T'_1, T'_2, \dots T'_p, I'_1, I'_2, \dots I'_q, F'_1, F'_2, \dots F'_r)$ *be two* $n$ *- valued neutrosophic tuplets, where* $p + q + r = n$.

(1) *If* $\sum_{n=1}^{r} F_n = \sum_{n=1}^{r} F'_n, \sum_{n=1}^{q} I_n = \sum_{n=1}^{q} I'_n$ *and* $\sum_{n=1}^{p} T_n > \sum_{n=1}^{p} T'_n$*, then we get* $N_1 > N_2$.

(2) *If* $\sum_{n=1}^{r} F_n = \sum_{n=1}^{r} F'_n, \sum_{n=1}^{q} I_n = \sum_{n=1}^{q} I'_n$ *and* $\sum_{n=1}^{p} T_n < \sum_{n=1}^{p} T'_n$*, then we get* $N_1 < N_2$.

**Remark 3.5.** Let $N_1 = (T_1, T_2, \dots T_p, I_1, I_2, \dots I_q, F_1, F_2, \dots F_r)$ and $N_2 = (T'_1, T'_2, \dots T'_p, I'_1, I'_2, \dots I'_q, F'_1, F'_2, \dots F'_r)$ be two $n$ - valued neutrosophic tuplets, where $p + q + r = n$. We suppose that $\sum_{n=1}^{p} T_n = \sum_{n=1}^{p} T'_n, \sum_{n=1}^{r} F_n = \sum_{n=1}^{r} F'_n$ , $\sum_{n=1}^{q} I_n = 0$ and $\sum_{n=1}^{q} I'_n = q$ in which collective membership and non membership grades of $N_1$ and $N_2$ are equal, whereas $N_1$ has no indeterminacy and $N_2$ has full indeterminacy. Then we get $N_1 > N_2$ which favours our intuition.

**Remark 3.6.** Let $N_1 = (T_1, T_2, \dots T_p, I_1, I_2, \dots I_q, F_1, F_2, \dots F_r)$ and $N_2 = (T'_1, T'_2, \dots T'_p, I'_1, I'_2, \dots I'_q, F'_1, F'_2, \dots F'_r)$ be two $n$ - valued neutrosophic tuplets, where $p + q + r = n$. We suppose that $\sum_{n=1}^{p} T_n = \sum_{n=1}^{p} T'_n, \sum_{n=1}^{q} I_n = \sum_{n=1}^{q} I'_n$ , $\sum_{n=1}^{r} F_n = 0$ and $\sum_{n=1}^{r} F'_n = r$ in which collective membership and indeterminacy grades of $N_1$ and $N_2$ are equal, whereas $N_1$ has no non membership grade and $N_2$ has full non membership grade. Then we get $N_1 > N_2$ which favours our intuition.

**Remark 3.7.** Let $N_1 = (T_1, T_2, \ldots T_p, I_1, I_2, \ldots I_q, F_1, F_2, \ldots F_r)$ and $N_2 = (T_1', T_2', \ldots T_p', I_1', I_2', \ldots I_q', F_1', F_2', \ldots F_r')$ be two $n$ - valued neutrosophic tuplets, where $p + q + r = n$. We suppose that $\sum_{n=1}^{q} I_n = \sum_{n=1}^{q} I_n', \sum_{n=1}^{r} F_n = \sum_{n=1}^{r} F_n'$ , $\sum_{n=1}^{p} T_n = 0$ and $\sum_{n=1}^{p} T_n' = p$ in which collective hesitancy and non membership grades of $N_1$ and $N_2$ are equal, whereas $N_1$ has no membership grade and $N_2$ has full membership grade. Then we get $N_1 < N_2$ which favours our intuition.

**Remark 3.8.** We know that $n$-valued neutrosophic tuplets are generalization of single valued neutrosophic triplets and hence we can apply our ranking method also to them which will be a total ordering on single valued neutrosophic triplets.

## 4. Numerical examples

Let us consider the following example as a brief example for the proposed total ordering algorithm. Assume that $(T; I; F) = (0.8, 0.7, 0.9; 0.4; 0.2, 0.7, 0.6)$ and $(T'; I'; F') = (0.9, 0.7; 0.2, 0.8; 0.2, 0.4, 0.6)$ be 7 - valued neutrosophic tuplets.

First we rearrange these two 7 - valued neutrosophic tuplets in ascending order as follows $(T; I; F) = (0.7, 0.8, 0.9; 0.4; 0.2, 0.6, 0.7)$ and $(T'; I'; F') = (0.7, 0.9; 0.2, 0.8; 0.2, 0.4, 0.6)$

Now $k = lcm\{3, 1, 3, 2, 2, 3\} = 6$. Since $T$ has 3 elements and $k = 6 = 2(3)$, we rewrite $T = (0.7, 0.8, 0.9)$ as $T = (0.7, 0.7, 0.8, 0.8, 0.9, 0.9)$. In similar manner, we rewrite $I, F, T', I', F'$ as follows;

$I = (0.4, 0.4, 0.4, 0.4, 0.4, 0.4), F = (0.2, 0.2, 0.6, 0.6, 0.7, 0.7), T' = (0.7, 0.7, 0.7, 0.9, 0.9, 0.9)$, $I' = (0.2, 0.2, 0.2, 0.6, 0.6, 0.6)$. Now we take $(a, b, c) = (\frac{\sum_{i=1}^{6} T_i}{6}, \frac{\sum_{i=1}^{6} I_i}{6}, \frac{\sum_{i=1}^{6} F_i}{6})$ and $(d, e, f) = (\frac{\sum_{i=1}^{6} T_i'}{6}, \frac{\sum_{i=1}^{6} I_i'}{6}, \frac{\sum_{i=1}^{6} F_i'}{6})$, which implies $(a, b, c) = (0.8, 0.4, 0.5)$ and $(d, e, f) = (0.8, 0.4, 0.5)$. By applying steps 2, 3 and 4, we cannot rank $(T, I, F)$ and $(T', I', F')$. Now we go to step 5. In step 5, we take $(T_6, I_6, F_6)$ and $(T_6', I_6', F_6')$. Since $S^+(0.9, 0.4, 0.7) = 0.48 > 0.3 = S^+(0.9, 0.8, 0.9)$, we get the ranking as $(T, I, F) > (T', I', F')$.

## 5. Application to MCDM Problem

Consider the following MCDM problem based on 6 - valued neutrosophic numbers. Now we have to find the ranking between the alternatives $A_1, A_2, A_3, A_4$ with respect to criteria $C_1, C_2, C_3$. The ratings of the alternatives with respect to the criteria are given in the form of6 - valued neutrosophic number as shown in table 1. We are given that the respective weights of the criteria $C_1, C_2, C_3$ are $0.3, 0.3, 0.4$.

Now we rearrange the truth, indeterminacy, and false membership grades in the table 1 as ascending order which is given in table 2. Next we multiply corresponding weights of the criteria into the decision table which results table 3.

| | $C_1$ (Criteria 1) | $C_2$ (Criteria 2) | $C_3$ (Criteria 3) |
|---|---|---|---|
| $A_1$ (Alternative 1) | (0.3;0.6,0.2;0.1,0.5,0.2) | (0.4,0.2;0.3;0.7,0.8,0.3) | (0.6,0.5;0.1,0.6;0.2,0.4) |
| $A_2$ (Alternative 2) | (0.9,0.7,0.8;0,0.2;0.2) | (0.5,0.1;0.2,0.6;0.4,0.5) | (0.1,0.4;0.5,0.6,0.1;0.5) |
| $A_3$ (Alternative 3) | (0.3,0.6,0.4;0.2;0.1,0.4) | (0.7,0.1;0.1,0.2;0.4,0.8) | (0.3,0.5,0.7;0.2,0.5;0.2) |
| $A_4$ (Alternative 4) | (0.7,0.3,0.2;0.2,0.3;0.1) | (0.5,0.3.0.1;0.2,0.6;0.4) | (0.5,0.9;0.6,1,0.1;0.6) |

TABLE 1. MCDM decision matrix

| | $C_1$ (Criteria 1) | $C_2$ (Criteria 2) | $C_3$ (Criteria 3) |
|---|---|---|---|
| $A_1$ (Alternative 1) | (0.3;0.2,0.6;0.1,0.2,0.5) | (0.2,0.4;0.3;0.3,0.7,0.8) | (0.5,0.6;0.1,0.6;0.2,0.4) |
| $A_2$ (Alternative 2) | (0.7,0.8,0.9;0,0.2;0.2) | (0.1,0.5;0.2,0.6;0.4,0.5) | (0.1,0.4;0.1,0.5,0.6;0.5) |
| $A_3$ (Alternative 3) | (0.3,0.4,0.6;0.2;0.1,0.4) | (0.1,0.7;0.1,0.2;0.4,0.8) | (0.3,0.5,0.7;0.2,0.5;0.2) |
| $A_4$ (Alternative 4) | (0.2,0.3,0.7;0.2,0.3;0.1) | (0.1,0.3.0.5;0.2,0.6;0.4) | (0.5,0.9;0.1,0.6,1;0.6) |

TABLE 2. MCDM decision matrix in an rearranged form

| | $C_1$ (Criteria 1) | $C_2$ (Criteria 2) | $C_3$ (Criteria 3) |
|---|---|---|---|
| $A_1$ (Alternative 1) | (.09;.06,.18;.03,.06,.15) | (.06,.12;.09;.09,.21,.24) | (.2,.24;.04,.26;.08,.16) |
| $A_2$ (Alternative 2) | (.21,.24,.27;0,.06;.06) | (.03,.15;.06,.18;.12,.15) | (.04,.16;.04,.2,.24;.2) |
| $A_3$ (Alternative 3) | (.09,.12,.18;.06;.03,.12) | (.03,.21;.03,.06;.12,.24) | (.12,.2,.28;.08,.2;.08) |
| $A_4$ (Alternative 4) | (.06,.09,.21;.06,.09;.03) | (.03,.09,.15;.06,.18;.12) | (.2,.36;.04,.24,.4;.18) |

TABLE 3. Weighted MCDM decision matrix

From table 3, we find $k = lcm\{1, 2, 3\} = 6$. Thus we rewrite each entries of MCDM table as follows

$A_1C_1 = (.09, .09, .09, .09, .09, .09; .06, .06, .06, .18, .18, .18; .03, .03, .06, .06, .15, .15)$

$A_1C_2 = (.2, .2, .2, .24, .24, .24; .04, .04, .04, .26, .26, .26; .08, .08, .08, .16, .16, .16)$

$A_1C_3 = (.06, .06, .06, .12, .12, .12; .09, .09, .09, .09, .09, .09; .09, .09, .21, .21, .24, .24)$

$A_2C_1 = (.21, .21, .24, .24, .27, .27; 0, 0, 0, .3, .3, .3; .06, .06, .06, .06, .06, .06)$

$A_2C_2 = (.03, .03, .03, .15, .15, .15; .06, .06, .06, .18, .18, .18; .12, .12, .12, .15, .15, .15)$

$A_2C_3 = (.04, .04, .04, .16, .16, .16; .04, .04, .2, .2, .24, .24; .2, .2, .2, .2, .2, .2)$

$A_3C_1 = (.09, .09, .12, .12, .18, .18; .06, .06, .06, .06, .06, .06; .03, .03, .03, .12, .12, .12)$

$A_3C_2 = (.03, .03, .03, .21, .21, .21; .03, .03, .03, .06, .06, .06; .12, .12, .12, .24, .24, .24)$

$A_3C_3 = (.12, .12, .2, .2, .28, .28; .08, .08, .08, .2, .2, .2; .08, .08, .08, .08, .08, .08)$

$A_4C_1 = (.06, .06, .09, .09, .21, .21; .06, .06, .06, .09, .09, .09; .03, .03, .03, .03, .03, .03)$

$A_4C_2 = (.03, .03, .09, .09, .15, .15; .06, .06, .06, .18, .18, .18; .12, .12, .12, .12, .12, .12)$

$A_4C_3 = (.2, .2, .2, .36, .36, .36; .04, .04, .24, .24, .4, .4; .18, .18, .18, .18, .18, .18)$. Now we have the following weighted arithmetic neutrosophic scores for each $A_i$ as $A_i = A_iC_1 + A_iC_2 + A_iC_3, i =$

1 to 4. $A_1 = (.35, .35, .35, .45, .45, .45; .19, .19, .19, .53, .53, .53; .21, .21, .36, .36, .54, .54)$
$A_2 = .27, .27, .3, .54, .57, .57; .09, .09, .27, .69, .72, .72; .4, .4, .4, .42, .42, .42)$
$A_3 = (.24, .24, .36, .54, .66, .66; .18, .18, .18, .33, .33, .33; .24, .24, .24, .45, .45, .45)$
$A_4 = (.3, .3, .4, .54, .72, .72; .15, .15, .36, .5, .66, .66; .33, .33, .33, .33, .33, .33)$
Now, by applying proposed ranking algorithm, we have $(a_1, b_1, c_1) = (\frac{.35+.35+.35+.45+.45+.45}{6}, \frac{.19+.19+.19+.53+.53+.53}{6}, \frac{21+.21+.36+.36+.54+.54}{6}) = (0.4, 0.36, 0.37)$. Similarly, we find $(a_2, b_2, c_2) = (0.42, 0.43, 0.37), (a_3, b_3, c_3) = (0.45, 0.26, 0.35)$ and $(a_4, b_4, c_4) = (0.5, 0.41, 0.33)$. Now $S^+(a_1, b_1, c_1) = 0.422, S^+(a_2, b_2, c_2) = 0.412, S^+(a_3, b_3, c_3) = 0.479$ and $S^+(a_4, b_4, c_4) = 0.465$. Therefore, we get the ranking as $A_3 > A_4 > A_1 > A_2$.

### 5.1. *Limitations of the proposed method*

In the proposed ranking method, summation of the collective membership, non membership, indeterminacy grades are first taken into account and then highest to lowest membership, non membership, indeterminacy grades are used to rank in the next stages. In some cases, when there is a fluctuation between membership, non membership, and indeterminacy grades, the proposed ranking method may rank differently to intuition of some decision maker. For example, take the following two 7 - valued neutrosophic tuplets $A = (0.3, 0.34, 0.36, 0.6; 0.15, 0.25; 0.3), B = (0.4; 0.15, 0.25; 0.15, 0.25, 0.35, 0.45)$.
Now, we rewrite $A = (0.3, 0.34, 0.36, 0.6; 0.15, 0.15, 0.25, 0.25; 0.3, 0.3, 0.3, 0.3), B = (0.4, 0.4, 0.4, 0.4; 0.15, 0.15, 0.25, 0.25; 0.15, 0.25, 0.35, 0.45)$. Then $(a, b, c) = (\frac{\sum_{i=1}^{k} T_i}{k}, \frac{\sum_{i=1}^{k} I_i}{k}, \frac{\sum_{i=1}^{k} F_i}{k}) = (0.4, 0.2, 0.3)$ and $(d, e, f) = (\frac{\sum_{i=1}^{k} T'_i}{k}, \frac{\sum_{i=1}^{k} I'_i}{k}, \frac{\sum_{i=1}^{k} F'_i}{k}) = (0.4, 0.2, 0.3)$. Therefore, we go to next step, which implies $S^+(T_4, I_4, F_4) = S^+(0.6, 0.25, 0.3) = 0.569 > 0.416 = S^+(0.4, 0.25, 0.45) = S^+(T'_4, I'_4, F'_4)$. Thus, we get the ranking as $A > B$.
But, we have the membership value as a single element 0.4 in $B$ where as there is a fluctuation between membership grades in $A$ and three of them are lesser than the membership grade of $B$. But, non membership and hesitancy information are same for $A$ and $B$. Since there is more fluctuation in $A$, we expect the ranking as $A < B$ intuitively. To overcome this, we have given the improved ranking algorithm in the next section.

## 6. Improved ranking algorithm for $n$ - valued neutrosophic tuplets

In this section, we present an improved ranking algorithm for n-valued neutrosophic tuplets that inherits total ordering.

Let $A = (T, I, F) = (T_1, T_2 \dots T_{p_1}, I_1, I_2 \dots I_{q_1}, F_1, F_2 \dots F_{r_1})$ and $B = (T', I', F') = (T'_1, T'_2 \dots T'_{p_2}, I'_1, I'_2 \dots I'_{q_2}, F'_1, F'_2 \dots F'_{r_2})$ be two $n$ - valued neutrosophic tuplets such that $A \neq B$, where $p_1 + q_1 + r_1 = p_2 + q_2 + r_2 = n$.

**Step 1:** We follow step 1 to step 4 in the previous ranking algorithm in section 3.1. If $A$ and

$B$ are not ranked at this stage and if step 4 fails to rank, then we go to step 2.

**Step 2:** Now let neutrosophic triplets $(a_m, b_m, c_m) = (T_m - (\frac{\sum_{i=1}^{m-1} T_i}{m-1}), I_m - (\frac{\sum_{i=1}^{m-1} I_i}{m-1}), F_m - (\frac{\sum_{i=1}^{m-1} F_i}{m-1}))$ and $(d_m, e_m, f_m) = (T'_m - (\frac{\sum_{i=1}^{m-1} T'_i}{m-1}), I'_m - (\frac{\sum_{i=1}^{m-1} I'_i}{m-1}), F'_m - (\frac{\sum_{i=1}^{m-1} F'_i}{m-1}))$. For $m = k$, by applying step 2, 3, and 4 of proposed algorithm in 3.1by considering $(T_0, I_0, F_0) = (a_m, b_m, c_m)$, we will have either $(a_m, b_m, c_m) < (d_m, e_m, f_m)$ or $(d_m, e_m, f_m) < (a_m, b_m, c_m)$ and hence either $A < B$ or $B < A$. If step 4 fails to rank, then we go to step 3.

**Step 3:** By successive application of step 2 for $m = m - 1$, we will have either $A < B$ or $B < A$.

**Theorem 6.1.** *Proposed ranking algorithm inherits a total order on set of all $n$ valued neutrosophic tuplets.*

*Proof.* Let $A = (T, I, F) = (T_1, T_2 \dots T_{p_1}, I_1, I_2 \dots I_{q_1}, F_1, F_2 \dots F_{r_1})$ and $B = (T', I', F') = (T'_1, T'_2 \dots T'_{p_2}, I'_1,$
$I'_2 \dots I'_{q_2}, F'_1, F'_2 \dots F'_{r_2})$ be two $n$ - valued neutrosophic tuplets such that $A \neq B$, where $p_1 + q_1 + r_1 = p_2 + q_2 + r_2 = n$. Now we show that either $A < B$ or $B < A$.

By applying step 1 in the previous ranking algorithm in section 3.1, we have $A = (T_1, \dots T_k, I_1, \dots I_k, F_1, \dots F_k,)$ and $B = (T'_1, \dots T'_k, I'_1, \dots I'_k, F'_1, \dots F'_k,)$ where $k = lcm\{p_1, q_1, r_1,$
$p_2, q_2, r_2\}$.

By applying step 2 to step 4 in the previous ranking algorithm in section 3.1, we get either $A < B$ or $B < A$. If $A$ and $B$ are not ranked at this stage and if step 4 fails to rank, then we go to step 2 of the improved ranking algorithm.

$$T_1 + T_2 + \dots T_k = T'_1 + T'_2 + \dots T'_k \tag{10}$$

$$I_1 + I_2 + \dots I_k = I'_1 + I'_2 + \dots I'_k \tag{11}$$

$$F_1 + F_2 + \dots F_k = F'_1 + F'_2 + \dots F'_k \tag{12}$$

We apply step 2 of proposed algorithm for $m = k$ by letting neutrosophic triplets $(a_m, b_m, c_m) = (T_m - (\frac{\sum_{i=1}^{m-1} T_i}{m-1}), I_m - (\frac{\sum_{i=1}^{m-1} I_i}{m-1}), F_m - (\frac{\sum_{i=1}^{m-1} F_i}{m-1}))$ and $(d_m, e_m, f_m) = (T'_m - (\frac{\sum_{i=1}^{m-1} T'_i}{m-1}), I'_m - (\frac{\sum_{i=1}^{m-1} I'_i}{m-1}), F'_m - (\frac{\sum_{i=1}^{m-1} F'_i}{m-1}))$. we will have either $(a_m, b_m, c_m) < (d_m, e_m, f_m)$ or $(d_m, e_m, f_m) < (a_m, b_m, c_m)$ and hence either $A < B$ or $B < A$. Otherwise, we go to step 3. At this stage, we have $(a_m, b_m, c_m) = (d_m, e_m, f_m)$ and hence,

$$T_k - (\frac{\sum_{i=1}^{k-1} T_i}{k-1}) = T'_k - (\frac{\sum_{i=1}^{k-1} T'_i}{k-1}) \tag{13}$$

$$I_k - (\frac{\sum_{i=1}^{k-1} I_i}{k-1}) = I'_k - (\frac{\sum_{i=1}^{k-1} I'_i}{k-1}) \tag{14}$$

$$F_k - (\frac{\sum_{i=1}^{k-1} F_i}{k-1}) = F'_k - (\frac{\sum_{i=1}^{k-1} F'_i}{k-1}) \tag{15}$$

From equations 10, 11, 12, 13, 14 and 15, we get $T_k = T'_k, I_k = I'_k$ and $F_k = F'_k$.

Now we apply step 3 of proposed improved algorithm for $m = m - 1$. So we have to apply step 2 by considering $m = k - 1$ and hence we get either $A < B$ or $B < A$. Otherwise, we go to step 3. At this stage, we have $(a_m, b_m, c_m) = (d_m, e_m, f_m)$ for $m = k - 1$ and hence at this stage, we have $(T_{k-1} - (\frac{\sum_{i=1}^{k-2} T_i}{k-2}), I_{k-1} - (\frac{\sum_{i=1}^{k-2} I_i}{k-2}), F_{k-1} - (\frac{\sum_{i=1}^{k-2} F_i}{k-2}))$ and $(T'_{k-1} - (\frac{\sum_{i=1}^{k-2} T'_i}{k-2}), I'_{k-1} - (\frac{\sum_{i=1}^{k-2} I'_i}{k-2}), F'_{k-1} - (\frac{\sum_{i=1}^{k-2} F'_i}{k-2}))$, then continue the same process. As a result, we get $T_{k-1} = T'_{k-1}, I_{k-1} = I'_{k-1}$ and $F_{k-1} = F'_{k-1}$. By repeating step 3 for $m = m - 1$ again and again, we will have either $A < B$ or $B < A$ or otherwise $T_k = T'_k, I_k = I'_k$ and $F_k = F'_k$, for every $m = k, k - 1, \ldots, 1$, a contradiction to $A \neq B$. Thus we have shown that proposed improved ordering algorithm is a total order on $n$ - valued neutrosophic tuplets. □

**Remark 6.2.** For example, take the following two 7 - valuedneutrosophic tuplets $A = (0.3, 0.34, 0.36, 0.6; 0.15, 0.25; 0.3), B = (0.4; 0.15, 0.25; 0.15, 0.25, 0.35, 0.45)$.

Now this can be rewritten as $A = (0.3, 0.34, 0.36, 0.6; 0.15, 0.15, 0.25, 0.25; 0.3, 0.3, 0.3, 0.3), B = (0.4, 0.4, 0.4, 0.4; 0.15, 0.15, 0.25, 0.25; 0.15, 0.25, 0.35, 0.45)$. Then $(T_0, I_0, F_0) = (\frac{\sum_{i=1}^{k} T_i}{k}, \frac{\sum_{i=1}^{k} I_i}{k}, \frac{\sum_{i=1}^{k} F_i}{k}) = (0.4, 0.2, 0.3)$ and $(T'_0, I'_0, F'_0) = (\frac{\sum_{i=1}^{k} T'_i}{k}, \frac{\sum_{i=1}^{k} I'_i}{k}, \frac{\sum_{i=1}^{k} F'_i}{k}) = (0.4, 0.2, 0.3)$. Therefore we go to step 2 of the improved algorithm. So we have to apply step 2, followed by step 3 and step 4 of algorithm in section 3.1 if needed by letting $(a_4, b_4, c_4) = (T_4 - \frac{T_1+T_2+T_3}{3}, I_4 - \frac{I_1+I_2+I_3}{3}, F_4 - \frac{F_1+F_2+F_3}{3}$ and $(d_m, e_m, f_m) = (T'_4 - \frac{T'_1+T'_2+T'_3}{3}, I'_4 - \frac{I'_1+I'_2+I'_3}{3}, F'_4 - \frac{F'_1+F'_2+F'_3}{3})$.

Now by step 2 of algorithm in section 3.1, $S^+(T_4 - \frac{T_1+T_2+T_3}{3}, I_4 - \frac{I_1+I_2+I_3}{3}, F_4 - \frac{F_1+F_2+F_3}{3}) = S^+(0.27, 0.07, 0) = 0.61 > 0.39 = S^+(0, 0.07, 0.2) = S^+(T'_4 - \frac{T'_1+T'_2+T'_3}{3}, I'_4 - \frac{I'_1+I'_2+I'_3}{3}, F'_4 - \frac{F'_1+F'_2+F'_3}{3})$. Thus we get the ranking as $A < B$. As we stated in remark 5.1, since there is more fluctuation in $A$, as an intuition we expect the ranking as $A < B$ which coincide with our ranking.

## 7. Comparision between proposed ranking method and improved ranking method via MCDM problem

Consider the following MCDM problem based on 5 - valued neutrosophic numbers. Now we rank alternatives $A_1, A_2, A_3, A_4$ with respect to criteria $C_1, C_2, C_3$. The ratings of the alternatives with respect to the criteria are given in the form of5 - valued neutrosophic number as shown in table 4. We assume that the respective weights of the criteria $C_1, C_2, C_3$ are $0.3, 0.3, 0.4$.

Next we multiply corresponding weights of the criteria into the table 4 and we rewrite

| | $C_1$ (Criteria 1) | $C_2$ (Criteria 2) | $C_3$ (Criteria 3) |
|---|---|---|---|
| $A_1$ (Alternative 1) | (0.3,0.4;0.2,0.6;0.6) | (0.5,0.51;0.3,0.2;0.7) | (0.4;0.3,0.6;0.2,0.3) |
| $A_2$ (Alternative 2) | (0.2,0.5;0.4;0.8,0.4) | (0.6,0.4;0.25;0.4,1) | (0.1,0.7;0.45;0.1,0.4) |
| $A_3$ (Alternative 3) | (0.6,0.1;0.35,0.45;0.6) | (0.3,0.7;0.1,0.4;0.7) | (0.2,0.6;0.2,0.7;0.25) |
| $A_4$ (Alternative 4) | (0.35;0.2,0.6;0.3,0.9) | (0.1,0.9;0.25;0.6,0.8) | (0.4;0.3,0.6;0.15,0.35) |

TABLE 4. MCDM decision matrix

according to our algorithm by taking $k = 2$ which results table 5.

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $A_1$ | (0.09,0.12;0.06,0.18;0.18,0.18) | (0.15,0.153;0.06,0.09;0.0.21,0.21) | (0.16,0.16;0.12,0.24;0.08,0.12) |
| $A_2$ | (0.06,0.15;0.12,0.12;0.12,0.24) | (0.12,0.18;0.075,0.075;0.12,0.3) | (0.04,0.28;0.18,0.18;0.04,0.16) |
| $A_3$ | (0.18,0.03;0.105,0.135;0.18,0.18) | (0.09,0.21;0.03,0.12;0.21,0.21) | (0.08,0.24;0.08,0.28;0.1,0.1) |
| $A_4$ | (0.105,0.105;0.06,0.18;0.09,0.27) | (0.03,0.27;0.075,0.075;0.18,0.24) | (0.16,0.16;0.12,0.24;0.06,0.14) |

TABLE 5. weighted MCDM decision matrix in an rearranged form

Now we have the following weighted arithmetic neutrosophic scores for each $A_i, i = 1 \text{ to } 4$. $A_1 = A_1C_1 + A_1C_2 + A_1C_3 = (0.4, 0.43; 0.24, 0.51; 0.47, 0.51)$. Similarly we get $A_2 = (0.28, 0.55; 0.375, 0.375; 0.4, 0.58)$, $A_3 = (0.35, 0.48; 0.215, 0.535; 0.489, 0.489)$, $A_4 = (0.294, 0.534; 0.255, 0.495; 0.33, 0.65)$. Now we go to next step, $(a_1, b_1, c_1) = (\frac{0.4+0.43}{2}, \frac{0.24+0.51}{2}, \frac{0.47+0.51}{2}) = (0.415, 0.375, 0.49)$. Similarly we find $(a_2, b_2, c_2) = (0.415, 0.375, 0.49)$, $(a_3, b_3, c_3) = (0.415, 0.375, 0.49)$ and $(a_4, b_4, c_4) = (0.415, 0.375, 0.49)$. Now $(a_1, b_1, c_1) = (a_2, b_2, c_2) = (a_3, b_3, c_3) = (a_4, b_4, c_4)$. Therefore we go to next step.

| | $C_1$ (Criteria 1) | $C_2$ (Criteria 2) | $C_3$ (Criteria 3) |
|---|---|---|---|
| $A_1$ (Alternative 1) | (0.1;0.4;0) | (0;0.1;0) | (0;0.3;0.1) |
| $A_2$ (Alternative 2) | (0.3;0;0.4) | (0.2;0;0.6) | (0.6;0;0.3) |
| $A_3$ (Alternative 3) | (0.5;0.10;0) | (0.4;0.3;0) | (0.4;0.5;0) |
| $A_4$ (Alternative 4) | (0;0.4;0.6) | (0.8;0;0.2) | (0;0.3;0.2) |

TABLE 6. fluctuation of MCDM decision matrix

Now we have $A_1 = A_1C_1 + A_1C_2 + A_1C_3 = (0.03, 0.27, 0.04)$. Similarly we find $A_2 = (0.39, 0, 0.42)$, $A_3 = (0.43, 0.32, 0)$ and $A_4 = (0.24, 0.24, 0.32)$. Therefore $S^+(A_1) = 0.429$, $S^+(A_2) = 0.485$, $S^+(A_3) = 0.6$ and $S^+(A_4) = 0.41$. We get the ranking as $A_3 < A_2 < A_1 < A_4$. And as a comparison purpose suppose we apply previous proposed ranking algorithm we get the ranking as $A_3 > A_2 > A_1 > A_4$ for this problem.

| | $C_1$ (Criteria 1) | $C_2$ (Criteria 2) | $C_3$ (Criteria 3) |
|---|---|---|---|
| $A_1$ (Alternative 1) | (0.03;0.12;0) | (0;0.03;0) | (0;0.12;0.04) |
| $A_2$ (Alternative 2) | (0.09;0;0.12) | (0.06;0;0.18) | (0.24;0;0.12) |
| $A_3$ (Alternative 3) | (0.15;0.03;0) | (0.12;0.09;0) | (0.16;0.2;0) |
| $A_4$ (Alternative 4) | (0;0.12;0.18) | (0.24;0;0.06) | (0;0.12;0.08) |

Table 7. Weighted fluctuation of MCDM decision matrix

**Remark 7.1.** From our proposed ranking methods, we have shown that we can rank any two $n$ - valued neutrosophic tuplets. As an extension, we can rank any two $m_1$ valued and $n_1$ valued neutrosophic tuplets where $m_1 \neq n_1$. In detail, suppose that $A = (T, I, F) = (T_1, T_2 \dots T_{p_1}, I_1, I_2 \dots I_{q_1}, F_1, F_2 \dots F_{r_1})$ be a $m_1$ - valued neutrosophic triplet and $B = (T', I', F') = (T'_1, T'_2 \dots T'_{p_2}, I'_1, I'_2 \dots I'_{q_2}, F'_1, F'_2 \dots F'_{r_2})$ be $n_1$ - valued neutrosophic tuplets, where $p_1 + q_1 + r_1 = m_1, p_2 + q_2 + r_2 = n_1$. To rank $A$ and $B$, we rewrite using $n = lcm\{m_1, n_1\}$ as follows. Suppose $n = x_1 m_1$, then we rewrite $(T, I, F)$ as $A = (\underbrace{T_1, \dots T_1}_{x_1 \text{ times}}, \dots, \underbrace{T_{p_1}, \dots T_{p_1}}_{x_1 \text{ times}}, \underbrace{I_1, \dots I_1}_{x_1 \text{ times}}, \dots \underbrace{I_{q_1}, \dots I_{q_1}}_{x_1 \text{ times}}, \underbrace{F_1, \dots F_1}_{x_1 \text{ times}}, \dots \underbrace{F_{r_1}, \dots F_{r_1}}_{x_1 \text{ times}})$

Suppose $n = y_1 n_1$, then we rewrite $(T', I', F')$ as follows

$B = (\underbrace{T_1, \dots T_1}_{y_1 \text{ times}}, \dots, \underbrace{T_{p_2}, \dots T_{p_2}}_{y_1 \text{ times}}, \underbrace{I_1, \dots I_1}_{y_1 \text{ times}}, \dots \underbrace{I_{q_2}, \dots I_{q_2}}_{y_1 \text{ times}}, \underbrace{F_1, \dots F_1}_{y_1 \text{ times}}, \dots \underbrace{F_{r_2}, \dots F_{r_2}}_{y_1 \text{ times}})$

Now in this stage, we have two $n$ - valued neutrosophic tuplets $A$ and $B$ which can be ordered by our proposed algorithms.

## 8. Total ordering on $n$ - valued refined neutrosophic sets

In this section, we derive an algorithm to rank any two n-valued refined neutrosophic sets by using the proposed total ordering method on $n$ - valued neutrosophic tuplets.

Let $X = \{x_1, x_2, \dots x_m\}$ be a universe of discourse. Let $N_1$ and $N_2$ be two arbitrary n-valued refined neutrosophic sets. Hence $N_1 = (N_1(x_1), N_1(x_2), \dots N_1(x_m)), N_2 = (N_2(x_1), N_2(x_2), \dots N_2(x_m))$ are ordered $m$-tuples of $n$ - valued neutrosophic tuplets. Now to prove the total ordering, if $N_1 \neq N_2$, then we need to show that either $N_1 > N_2$ or $N_1 < N_2$. We assume that all the elements of $X$ are equally important. Let $m_1, m_2$ be the number of elements in $x$ for which $N_1(x) > N_2(x)$ and $N_1(x) < N_2(x)$ respectively using the proposed total ordering algorithm for $n$ - valued neutrosophic tuplets.

**Step 1:** If $m_1 > m_2\,(m_1 < m_2)$, then $N_1 > N_2\,(N_1 < N_2)$. If $m_1 = m_2$, then go to step 2.

**Step 2:** Apply dictionary order on $m$-tuples using proposed total ordering algorithm for $n$ - valued neutrosophic tuplets. That is, if $N_1(x_1) > N_2(x_1)$ by proposed total ordering method, then $N_1 > N_2$. If $N_1(x_1) = N_2(x_1)$, then go to next step.

**Step 3:** If $N_1(x_{j+1}) > N_2(x_{j+1})\,(N_1(x_{j+1}) < N_2(x_{j+1}))$ for $j = 1$, then $N_1 > N_2\,(N_1 < N_2)$.

If $N_1(x_{j+1}) = N_2(x_{j+1})$, then go to step 4.

**Step 4:** Repeat the step 3 for $j = j + 1$ up to $j = m - 1$ till we reach $N_1 < N_2$ or $N_1 > N_2$.

**Remark 8.1.** This proposed algorithm derives total ordering algorithm on $n$ - valued neutrosophic sets. Let $X = \{x_1, x_2, \ldots x_m\}$ be an universe of discourse. To prove that, take any two distinct $n$ - valued neutrosophic sets with $N_1 = (N_1(x_1), N_1(x_2), \ldots N_1(x_m)), N_2 = (N_2(x_1), N_2(x_2), \ldots N_2(x_m))$ are ordered $m$-tuples of $n$ - valued neutrosophic tuplets. Let $m_1, m_2$ be the number of elements in $X$ for which $N_1(x) > N_2(x)$ and $N_1(x) < N_2(x)$ respectively using the proposed total ordering algorithm for $n$ - valued neutrosophic tuplets. By step 1, if $m_1 > m_2\,(m_1 < m_2)$, then $N_1 > N_2\,(N_1 < N_2)$ and hence the ordering is done. If $m_1 = m_2$, we go to step 2. We apply dictionary order on $m$-tuples of $n$ - valued neutrosophic tuplets using proposed total ordering algorithm for $n$ - valued neutrosophic tuplets. If $N_1(x_1) > N_2(x_1)$ by proposed total ordering method, then $N_1 > N_2$. If $N_1(x_1) = N_2(x_1)$, then we go to next step. By applying step 3 and step 4, we get if $N_1(x_{j+1}) > N_2(x_{j+1})\,(N_1(x_{j+1}) < N_2(x_{j+1}))$ for some $j$. Otherwise we get $N_1(x_i) = N_2(x_i)$ for every $i \in \{1, \ldots m\}$ which implies $N_1 = N_2$, a contradiction to $N_1 \neq N_2$. Thus we have proved the total ordering.

**Example 8.2.** Let $X = \{x_1, x_2, x_3\}$ be a universe of discourse. Let us take three $n$ - valued refined neutrosophic sets $(n = 5)$ $N_1, N_2$ and $N_3$, where

$N_1 = \{((x_1, (0.3, 0.4; 0.2, 0.6; 0.6)), (x_2, (0.4, 0.6; 0.3, 0.2; 0.7)), (x_3, (0.3; 0.3, 0.6; 0.2, 0.3))\}$

$N_2 = \{((x_1, (0.2, 0.4; 0.2, 0.6; 0.6)), (x_2, (0.2, 0.5; 0.4; 0.8, 0.4)), (x_3, (0.6, 0.4; 0.25; 0.4, 1))\}$

$N_3 = \{((x_1, (0.6, 0.8; 0.2, 0.6; 0.4)), (x_2, (0.4, 0.6; 0.3, 0.2; 0.7)), (x_3, (0.1; 0.4, 0.5; 0.5, 0.6))\}$

Now we find the ordering between $N_1, N_2$ and $N_3$. Now we compare the $n$ - valued neutrosophic sets $N_1$ and $N_2$. Now we get $S^+(N_1(x_1)) = 0.3, S^+(N_2(x_1)) = 0.28$ using proposed total ordering method, which implies $N_1(x_1)) > N_2(x_1)$. In similar manner, we find that $N_1(x_2) > N_2(x_2)$, $N_1(x_2) > N_2(x_2)$. Thus we find that $m_1 = 3 > 0 = m_2$. By step 1, we get $N_2 < N_1$.

Now we compare $N_3$ and $N_1$. We get $S^+(N_1(x_1)) = 0.3, S^+(N_3(x_1)) = 0.52$ using proposed total ordering method, which implies $N_3(x_1)) > N_1(x_1)$. In similar manner, we find that $N_3(x_2) = N_1(x_2)$, $N_3(x_3) < N_1(x_3)$. Hence we find that $m_1 = 1 = m_2$. Since step 1 fails to rank them, we go to step 2. By dictionary order, we compare $N_1(x_1)$ and $N_3(x_1)$. Thus we get $N_1 < N_3$. Finally our ordering for these three $n$ - valued $(n = 5)$ refined neutrosophic sets is $N_3 > N_1 > N_2$.

## 9. Conclusion and future scope

We have proposed two ranking algorithms for total ordering $n$ - valued neutrosophic tuplets. The proposed first ranking method accounts summation of the collective membership, non membership, indeterminacy grades in the first stage and then highest to lowest membership, non membership, indeterminacy grades are used to rank in the next stages. In some cases, when there is a fluctuation between highest and lowest membership, non membership, and indeterminacy grades, the proposed ranking method may rank differently to decision maker's intuition. To overcome this, we have proposed improved ranking method which also first accounts summation of the collective membership, non membership, indeterminacy grades. But, it considers the fluctuation between the membership values, non membership values and indeterminacy values in the next stages. Further, the score functions used in the both the ranking approaches takes into account not only membership, non-membership, and indeterminacy values, but also the portion of membership and non-membership value that is contained within the hesitance value. Through the proposed ranking algorithms for total ordering $n$ - valued neutrosophic tuplets using score functions, we develop a total ordering algorithm for $n$ - valued refined neutrosophic sets using dictionary order at the final stage. In near future, a total order on $n$ - valued refined neutrosophic sets may be developed by defining more number of score and accuracy functions on $n$ - valued neutrosophic tuplets.

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# A Novel Approach of Residue Neutrosophic Technique for Threshold Based Image Segmentation

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**Abstract**. The Residue Neutrosophic Set *(RNS)* is a new idea in image additional pixel level. Our idea is to make an analysis based on the additional pixel amount. In recent decades, computer vision has revolutionized image analysis by researchers. Image segmentation is a more investigated topic in the science of computer vision. Neutrosophic is a sophisticated mathematical idea to solve a myriad of challenges. The objective is to invent a neutrosophic technique to execute image thresholding. In the article, the residue methodology was applied, which denotes the residual values of neutrosophic membership intensities. This article will explore a novel idea for image thresholding termed *RNS*. There will be three types of *RNS* techniques: minimum, average, and maximum. The concepts of existing thresholding techniques in neutrosophic solvation are considered in this proposal. This article adopts novel methodologies to provide an integrated visionary path segmentation methodology. Furthermore, the proposed technique reaches a better average accuracy score.

**Keywords:** Image segmentation, neutrosophic sets, neutrosophic image, residue neutrosophic, thresholding

## 1. Introduction

Massive investigative study challenges in computer vision and image analysis have emerged during this technological era. The mathematical principles entice everyone to strive to tackle difficulties and challenges. Lotfi Zadeh created the notion of fuzzy in 1965. It is a remarkable mathematical approach for solving most research difficulties. After a few decades, Krassimir Atanassov developed the extended fuzzy idea known as Intuitionistic fuzzy sets ($IFS$) in 1983. Smarandache later developed the more advanced concept of neutrosophic set ($NS$) in 1998 [22]. Obstacles must be approached with developed methods in our modernistic universe. In this sense, the neutrosophic theory should be the preferable strategy for discovering the difficulties hidden aspects. A neutrosophic set has three components: truth membership($T$), indeterminacy membership($I$), and falsity membership($F$), according to the neutrosophic conception. In 2009, the first image segmentation in the neutrosophic domain was beginning it is one of the most challenging tasks in image processing and pattern recognition, it is used in a variety of applications including robot vision, object recognition, medical imaging, computer vision, etc [8]. Authors [19] classifies image segmentation approaches into three categories: threshold, edge, and region-based methods. The best segmentation results are usually obtained with a gray-level image.

In various scenario approaches, such as clustering, Support Vector Machine ($SVM$), segmentation made with fuzzy sets, $IFS$, and $NS$ set. In this manner, image segmentation occurs in stages, but as the image has become more informative, the segmentation may exclude lower pieces of data. Considering these timeframes, we may lose the image's information. If the image is a fingerprint application, it is essential to give preference to every little feature. Instead, it is critical not to lose any of the image's information when executing image analysis. The proposal focuses on reducing the loss of information in the threshold images. We approached the $NS$ domain intending to segment the aim of segmenting images based on the residue value of the neutrosophic intensities, so we developed a novel approach called the $RNS$ idea to try to accomplish the objective. Existing image segmentation algorithms, such as binary threshold, binary inverse threshold, TRUNC threshold, To zero threshold, and To zero inverse threshold, will be applied to the modified $RNS$ image. When dealing with the specified neutrosophic collection, the amalgamation results of any image should have more information than the others, according to $RNS$ segmentation. Precision, recall score, and $F1$ score is used to evaluate performance.

The article section 1 provides a brief overview of the concepts in this section of the article. The section 2 collects research on image segmentation and neutrosophic concept development from the literature. The essential preliminaries of the neutrosophic set carry in section 3. Perhaps the next section 4, expands on the definitions of the proposed approaches and discusses the algorithm. In section 5, we perform an experimental study of the concept and produce a result. Finally, section 6 examines the suggested methodologies conclusion and future scope.

## 2. Related Work

Sengur et al. [20] achieved the neutrosophic strategies for color texture image segmentation in 2011. Neutrosophic similarity clustering made a significant contribution by [9] Yanhui Guo et al. Later the same author suggested a new method called Breast ultrasound image segmentation in the article [10]. Koundal [13] in 2017 successfully performed neutrosophic clustering segmentation with noisy data on many instances. Many academics are still aiming to enhance the effectiveness of K-means clustering, which is still a universal approach for unsupervised classifications. In 2018, the authors [16] discovered a better method for picture segmentation in the neutrosophic image domain. In the same year, authors [15] suggested neutrosophic c-means clustering kernel metric-based image segmentation local information. The article [6] focuses on computer vision of neutrosophic images, which performs using nine distinct classifiers for image classification with an accuracy of 98.4%. Elays et al. [18] was implementing the cost function to boost the accuracy of degrees of the membership clustering algorithms. Yanhui et al. [11] implemented neutrosophic c-means clustering and an adaptive region expanding technique. The objective function was solved using the Lagrange multiplier approach in the paper [14], which focused on a single-valued neutrosophic set as a restricting minimization issue. The neutrosophic clustering method Amira et al. [1] approaches skin lesion detection histogram-based clustering estimation algorithm. Romualdas et al. [3] introduced a weighted aggregated sum product assessment approach in 2019 that was ranked using an edge detection algorithm. In digital image processing, Samarandachec [24] proposed using offsets and off uniforms for segmentation and edge detection. Jing et al. [28] provide a novel particle swarm optimization method for neutrosophic images based on fuzzy c-means. By including single valued (SV) trapezoidal neutrosophic numbers in all of the objective function and constraint parameters, the neutrosophic complex programming *(NCP)* process is classified. The objective of the difficult programming challenge is to improve the applicability of SV trapezoidal neutrosophic numbers in new decision-making scenarios [12]. This method advantage of more adaptable and realistic in a real-world situation. In the realm of psychological research, the concepts of single-valued neutrosophic N soft set (SVNNSS) and quasi-hyperbolic discounting intertemporal single-valued neutrosophic N soft set (QHDISVNNSS) are employed to demonstrate student's mental state through observation. Neutrosophic numbers are utilized to represent values in counseling sessions, ensuring no loss of information. The focus of the counselor is on individuals with mental health issues, using SVNNSS and QHDISVNNSS due to their higher susceptibility to emotional distress [25].

## 3. **Preliminaries**

### 3.1. *Neutrosophic set*

**Definition 3.1.** Let $A$ be an universe of data, the element in $A$ denoted by $a$, then the neutrosophic set $(NS)$, of the object $A$ is in the form [5, 22]

$$A = \{(a, T_A(a), I_A(a), F_A(a)\}$$

where the neutrosophic membership functions $T, I, F : A \rightarrow ]^{-}0, 1^{+}[$ define respectively the degrees of truth, indeterminacy and the falsity of the element $a \in A$ to the set condition.

$$^{-}0 \leq T_A(a) + I_A(a) + F_A(a) \leq 3^{+}$$

**Definition 3.2. (Basic properties)** [21, 23]

(1) **Complement**

The complement of $A$ is represented as $A^c$ where

$$A^c = \{a, F_A(a), 1 - I_A(a), T_A(a)\}$$

(2) **Intersection**

Let $A, B$ be two sets then the neutrosophic intersection is represented as $A \cap B$ where

$$A \cap B = \{a, \min(T_A(a), T_B(a)), \max(I_A(a), I_B(a)), \max(F_A(a), F_B(a))\}$$

(3) **Union**

Let $A, B$ be two sets then the neutrosophic union is represented as $A \cup B$ where

$$A \cup B = \{a, \max(T_A(a), T_B(a)), \min(I_A(a), I_B(a)), \min(F_A(a), F_B(a))\}$$

**Definition 3.3.** Let $A$ be a nonempty set, the single valued neutrosophic set $(SVNS)$ is defined as [5, 26]

$$A = \{(a, T_A(a), I_A(a), F_A(a)\}$$

where their membership functions are $T, I, F : A \rightarrow ]^{-}0, 1^{+}[$ denotes respectively the degrees of truth, indeteminacy and falsity of the element $a \in A$ to the set values.

$$0 \leq T_A(a) + I_A(a) + F_A(a) \leq 3$$

### 3.2. *Image Neutosophic sets*

**Definition 3.4.** A neutrosophic image $P_{NS}$ is charecterized with neutrosophic membership functions which are $T, I, F$ where $P_{NS}$ are the intensities of the image. Universally for neutrosophic image approach is gray intensities of the image. The image neutrosophic set is defined as [5, 7]

$$P_{NS}A(i, j) = \{T_A(i, j), I_A(i, j), F_A(i, j)\} \tag{1}$$

In general the arithmetic mean is consider as truth membership values and the standard deviation of the image is consider as indeterminacy membership. The neutrosophic transformation intensity of the image is define by the following formulae

$$
\begin{aligned}
T_A(i,j) &= \frac{\bar{p}(i,j) - \bar{p}\min}{\bar{p}\max - \bar{p}\min} \\
\bar{p}_A(i,j) &= \frac{1}{w * w} \sum_{m=i-\frac{w}{2}}^{m=i+\frac{w}{2}} \sum_{n=j-\frac{w}{2}}^{n=j+\frac{w}{2}} p(m,n) \\
I_A(i,j) &= \frac{\delta(i,j) - \delta\min}{\delta\max - \delta\min} \\
\delta_A(i,j) &= abs(p(i,j) - \bar{p}(i,j)) \\
F_A(i,j) &= 1 - T_A(i,j)
\end{aligned}
$$

where $\bar{p}_A(i,j)$ denotes the pixel mean in the region $w*w$ and $w$ is generally $w = 2n+1, \ (n \geq 1)$.

**Definition 3.5.** The entropy of the neutrosophic image is defined as follows [7]

$$En_{NS} = En_T + En_I + En_F \tag{2}$$

where

$$
\begin{aligned}
En_T &= -\sum_{i=\min(T)}^{\max(T)} p_T(i)\ln p_T(i) \\
En_I &= -\sum_{i=\min(I)}^{\max(I)} p_I(i)\ln p_I(i) \\
En_F &= -\sum_{i=\min(F)}^{\max(F)} p_F(i)\ln p_F(i)
\end{aligned}
$$

$p$ refers that the probability of the membership functions.

## 4. **Proposed method**

We will approach a different technique for the neutrosophic image segmentation which is based on the value of global neutrosophic image data. This contains three types of approaches which are the minimum value of universal image set based, the maximum value of image set based, and the average value of global image set. In the classical method generally, the piece-wise linear transformation means is used to transform the set into the neutrosophic domain. This proposal focused on the minimum or maximum or mean values of the entire image data. While approaching this technique it may possibly the image data is turned from *SVNS* to *NS*. Entering deal with this scenario proposal recommend that the residue technique to handle an affirmative concept. Here *SVNS* can be denoted as *NS* for comprehensive understanding while approaching the method. The complete architecture of the proposed method is shown in Figure 1.

**Definition 4.1.** Let $\wp(m, n)$ be an image data with $m, n$ dimensions, the neutrosophic membership components of the image data $T, I, F$ are modified as the classical method with the alteration of $\wp_{\min}$, $\wp_{\max}$ which are known as minimum, maximum value of the global image data.

$$_{\min}NS_{\wp(i,j)} = \{T_{\wp(i,j)} + I_{\wp(i,j)} + F_{\wp(i,j)}\} \tag{3}$$

$$\begin{aligned} \text{where } T_{\wp(i,j)} &= \frac{\wp_{(i,j)} - \wp_{\min}}{\wp_{\max} - \wp_{\min}} \\ I_{\wp(i,j)} &= \sqrt{1 - (T^2_{\wp(i,j)} + F^2_{\wp(i,j)})} \\ F_{\wp(i,j)} &= 1 - T_{\wp(i,j)} \end{aligned}$$

The residue neutrosophic set's minimum method is defined as

$$_{\min}RNS_{\wp(i,j)} = (\tau_{\min}) \mod (L) \tag{4}$$

$$\tau_{\min} = {}_{\min}NS_{\wp(i,j)} \times L$$

**Definition 4.2.** Let $\wp(m, n)$ be an image data with $m, n$ dimensions, the neutrosophic membership components of the image data $T, I, F$ are modified as the classical method with the alteration of $\wp_{\min}$, $\wp_{\max}$ which are known as minimum, maximum value of the global image data.

$$_{\max}NS_{\wp(i,j)} = \{T_{\wp(i,j)} + I_{\wp(i,j)} + F_{\wp(i,j)}\} \tag{5}$$

$$\begin{aligned} \text{where } T_{\wp(i,j)} &= \frac{\wp_{\max} - \wp_{(i,j)}}{\wp_{\max} - \wp_{\min}} \\ I_{\wp(i,j)} &= \sqrt{1 - (T^2_{\wp(i,j)} + F^2_{\wp(i,j)})} \\ F_{\wp(i,j)} &= 1 - T_{\wp(i,j)} \end{aligned}$$

The residue neutrosophic set's maximum method is defined as

$$_{\max}RNS_{\wp(i,j)} = (\tau_{\max}) \mod (L) \tag{6}$$

$$\tau_{\max} = {}_{\max}NS_{\wp(i,j)} \times L$$

**Definition 4.3.** Let $\wp(m, n)$ be an image data with $m, n$ dimensions, the neutrosophic membership components of the image data $T, I, F$ are modified as the classical method with the alteration of $\wp_{\min}$, $\wp_{\max}$, $\wp_{avg}$ which are known as minimum, maximum value, average value of the global image data.

$$_{\text{avg}}NS_{\wp(i,j)} = \{T_{\wp(i,j)} + I_{\wp(i,j)} + F_{\wp(i,j)}\} \tag{7}$$

$$\text{where } T_{\wp(i,j)} = \frac{\wp_{(i,j)} - \wp_{\text{avg}}}{\wp_{\max} - \wp_{\min}}$$
$$I_{\wp(i,j)} = \sqrt{1 - (T^2_{\wp(i,j)} + F^2_{\wp(i,j)})}$$
$$F_{\wp(i,j)} = 1 - T_{\wp(i,j)}$$

The residue neutrosophic set's average method is defined as

$$_{\text{avg}}RNS_{\wp(i,j)} = (\tau_{\text{avg}}) \mod (L) \tag{8}$$

$$\tau_{\text{avg}} = {}_{\text{avg}}NS_{\wp(i,j)} \times L$$

**Algorithm:**

**Step 1:** Convert the image as $L$ gray scale image.

**Step 2:** Make that $L$ image to neutosophic domain with any one of the equation 3 or 5 or 7 using the membership formulae.

**Step 3:** Transform the $NS$ to any one of $RNS$ domain

**Step 4:** Apply the segmentation methods for the transformed $RNS$.

**Step 5:** Detect the hidden pattern of the image

A single intensity values contain in standard image analysis. The proposed method, on the other contrary, can analyze the three membership intensity values. As a result, the analysis is more trustworthy than the traditional way. The image characteristics were retrieved at a consistent level using these thresholding methods. As a result, we may lower the image's indeterminacy for various thresholding values. This character will aid us in classifying the image with convincing processing and analysis in the future. There is no need to find or fix a threshold value.

## 5. **Expirement and Result**

TABLE 1. The employed metrics for quantitative evaluation.

| **Metrics** | **Formula** |
|---|---|
| Sensitivity | True Positive/(True Positive/False Negative) |
| Specificity | True Negative/(True Negative + False Positive) |
| Precision | True Positive/(True Positive + False Positive) |
| Recall | True Positives / (True positive + False negative) |
| $F1$ score | (2 * Precision * Recall) / (Precision + Recall) |
| Accuracy | (True Positive + True Negative)/Total samples |

For the experiment, we used an Intel(R) Core(TM) i5 processor with 16GB of RAM and a 64-bit operating system. The programing tool to be used is Python. A fingerprint image was

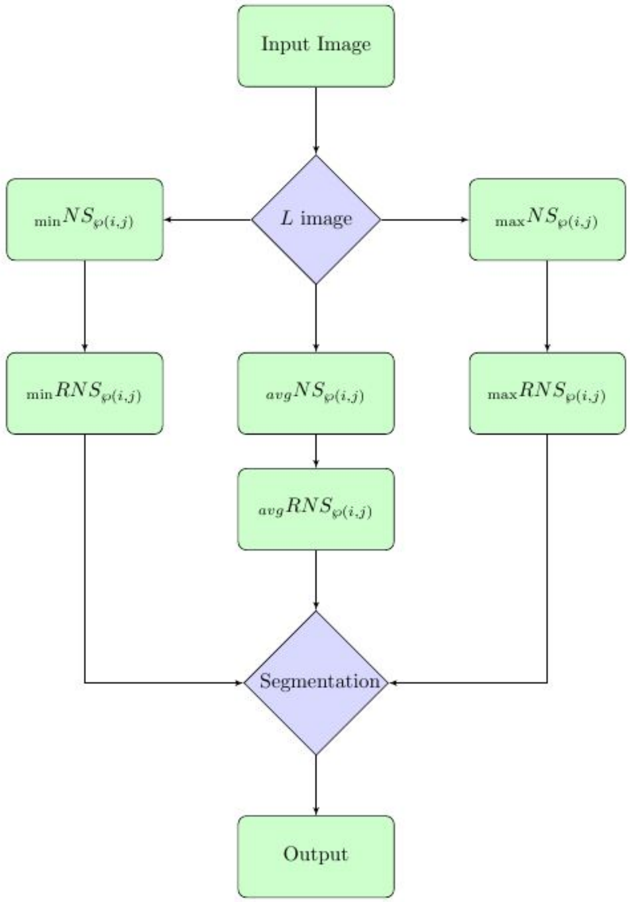


FIGURE 1. Residue neutrosophic set image segmentation architecture

collected from the [2] for the experiment analysis suggestion. Because the primary purpose of acquiring a fingerprint image contains more information than the image itself, it is critical to analyze it without losing the data. In this regard, the suggested method compares favorably to existing segmentation methods. Each image was resized to $250 \times 250$ for the segmentation analysis, and the segmentation pixel was set to $\alpha = 128$. Figure 2 shows the images that resulted. Apart from the example, some segmentation fails in $NS$ and $RNS$ however, if we emphasize avoiding the loss of image information for segmentation, any of the three given approaches should be a superior proposal for different scenarios. The proposal's further tasks will be based on the failed scenario. The Table 1 calculates the performance evaluation of the methods. For the evaluation of the resulted image and measurements, a single sample image is shown, The performance evaluation is shown in Figure 3. Table 2 tabulates the results of each segmentation method resulted for the metrics. It is calculating by metric formulae with concept of confusion matrix.

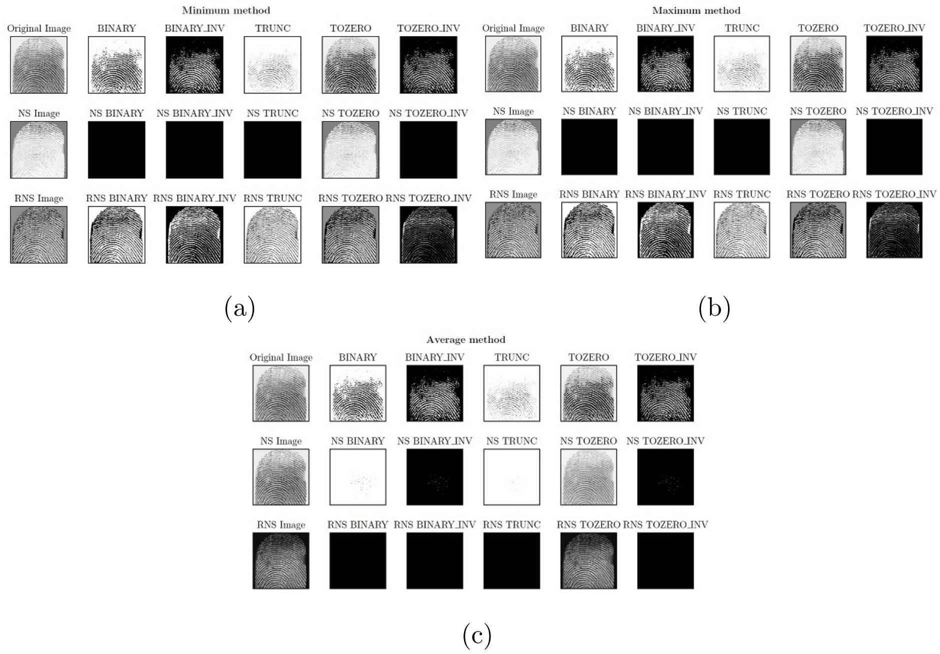


FIGURE 2. The resulted images of the proposed methods: (a) $_{min}RNS$ method image segmentation; (b) $_{max}RNS$ method image segmentation; (c)$_{avg}RNS$ method image segmentation.

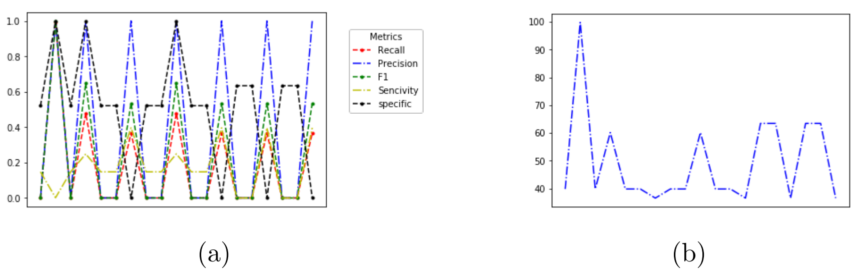


FIGURE 3. This is a figure of evaluation metrics: (a) Recall, precision, $F1$-score, sensivity, specific; (b) Accuracy.

TABLE 2. Table of performance evaluation.

| **Method** | **Recall** | **Precision** | **F1-score** | **Sensitivity** | **Specificity** | **Accuracy** |
|---|---|---|---|---|---|---|
| $_{min}RNS$ | 0.000068 | 0.000044 | 0.000053 | 0.14753 | 0.521426 | 39.8352 |
| $_{min}RNS$ Binary | 0.994040 | 1.000000 | 0.997011 | 0.00137 | 1.000000 | 99.7808 |
| $_{min}RNS$ Binary INV | 0.000068 | 0.000044 | 0.000053 | 0.14753 | 0.521426 | 39.8352 |
| $_{min}RNS$ TRUNC | 0.478574 | 0.999956 | 0.647336 | 0.24896 | 0.999932 | 60.1648 |
| $_{min}RNS$ ToZero | 0.000068 | 0.000044 | 0.000053 | 0.14753 | 0.521426 | 39.8352 |
| $_{min}RNS$ ToZero INV | 0.000068 | 0.000044 | 0.000053 | 0.14753 | 0.521426 | 39.8352 |
| $_{max}RNS$ | 0.365616 | 1.000000 | 0.535459 | 0.39649 | 0.000000 | 36.5616 |
| $_{max}RNS$ Binary | 0.000068 | 0.000044 | 0.000053 | 0.14753 | 0.521426 | 39.8352 |
| $_{max}RNS$ Binary INV | 0.000068 | 0.000044 | 0.000053 | 0.14753 | 0.521426 | 39.8352 |
| $_{max}RNS$ TRUNC | 0.478574 | 0.999956 | 0.647336 | 0.24896 | 0.999932 | 60.1648 |
| $_{max}RNS$ ToZero | 0.000068 | 0.000044 | 0.000053 | 0.14753 | 0.521426 | 39.8352 |
| $_{max}RNS$ ToZero INV | 0.000068 | 0.000044 | 0.000053 | 0.14753 | 0.521426 | 39.8352 |
| $_{avg}RNS$ | 0.365616 | 1.000000 | 0.535459 | 0.39649 | 0.000000 | 36.5616 |
| $_{avg}RNS$ Binary | 0.000000 | 0.000000 | 0.000000 | 0.00000 | 0.634384 | 63.4384 |
| $_{avg}RNS$ Binary INV | 0.000000 | 0.000000 | 0.000000 | 0.00000 | 0.634384 | 63.4384 |
| $_{avg}RNS$ TRUNC | 0.365616 | 1.000000 | 0.535459 | 0.39649 | 0.000000 | 36.5616 |
| $_{avg}RNS$ ToZero | 0.000000 | 0.000000 | 0.000000 | 0.00000 | 0.634384 | 63.4384 |
| $_{avg}RNS$ ToZero INV | 0.000000 | 0.000000 | 0.000000 | 0.00000 | 0.634384 | 63.4384 |

## 6. Conclusion

The article proposed three kinds of novel approaches to the neutrosophic image segmentation technology in this proposal. Our focus was to use neutrosophic sets to segment without losing information, and the idea succeeded with a better result. The use of these procedures is preferred solvation, especially for fingerprint images. Researchers can see from the analysis that the ${}_{min}RNS$ Binary segmentation method works exceptionally well in comparison to the other approaches. The TRUNC segmentation methodology, which focuses on the ${}_{max}RNS$ method, is preferred. When compared to the ${}_{min}RNS$ and ${}_{max}RNS$ methods, ${}_{avg}RNS$ performs poorly. Binary segmentation is a great way to approach neutrosophic thresholding binarization ${}_{min}RNS$. Since it can able to generalize the results for a collection of image sets if the image features are constant. These qualities are extremely beneficial when it comes to image processing and categorization. We will use this knowledge feature to apply these principles to machine learning approaches in order to achieve better outcomes. Because of the residue approach, the segmented picture features are always constant. In this way, this article produces the neutrosophic thresholding with better results for the samples.

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# Neutrosophic Intelligence Approach Safeguarding Patient Data in Blockchain-based Smart Healthcare

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**Abstract:** In the ever-evolving landscape of blockchain-based smart healthcare, ensuring the security and integrity of patient data stands as an utmost priority. This paper is dedicated to unveiling the pivotal role of Neutrosophic sets theory within our methodology as we tackle the multifaceted challenges of safeguarding patient data. Central to our approach is the innovative integration of Neutrosophic sets theory, a mathematical framework adept at handling the inherent uncertainties and imprecisions often encountered in healthcare decision-making. Leveraging Neutrosophic sets, we construct a comprehensive evaluation model that accommodates the complexities of the smart healthcare environment. Our methodology harnesses Multi-Criteria Decision Making (MCDM) techniques, notably the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), to systematically assess and rank service providers based on their proximity to ideal solutions. The Neutrosophic aspect comes to the forefront as we apply Neutrosophic sets in representing and managing decision-makers' judgments, which often exhibit varying degrees of truth, indeterminacy, and falsity. Furthermore, the Ordered Weighted Averaging (OWA) operator is strategically employed to aggregate these Neutrosophic judgments, accentuating the role of Neutrosophic sets in our decision fusion process. Our empirical study, firmly rooted in Neutrosophic sets theory, showcases the efficacy of this innovative model. It offers a structured and robust framework for healthcare organizations to fortify patient data security in the realm of blockchain-based smart healthcare systems. This paper advances the understanding of the indispensable role played by Neutrosophic sets in enhancing data security, thereby facilitating the adoption of blockchain technology within healthcare, while contributing to the burgeoning field of Neutrosophic Intelligence.
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**Keywords:** Neutrosophic Intelligence; Blockchain Technology; Smart Healthcare; Data Privacy; Trustworthiness, Confidentiality, Neutrosophic Sets.

## 1. Introduction

Neutrosophic sets, a mathematical framework introduced by Smarandache in 1995, have gained significant attention in recent years due to their unique ability to model and manage uncertainty, vagueness, and imprecision. This innovative concept extends traditional set theory by accommodating indeterminate elements—those elements whose membership, non-membership, and neutral status are all concurrently possible within a set [1]. Such inherent flexibility makes Neutrosophic sets a powerful tool for capturing and quantifying complex, real-world phenomena, particularly in decision-making processes. The core premise of Neutrosophic sets is rooted in the notion of "neutrality," signifying the coexistence of opposites within a set. This neutrality allows for the representation of incomplete, vague, or conflicting information, mirroring the inherent ambiguity frequently encountered in various domains, including healthcare [2].

In the context of blockchain-based smart healthcare, where the security and privacy of patient data are paramount, Neutrosophic sets emerge as a promising solution to address the inherent uncertainties that permeate this dynamic environment. Blockchain technology, known for its transparency and immutability, has transformed the healthcare sector by offering secure and decentralized data storage and management. However, the adoption of blockchain in healthcare introduces multifaceted uncertainties, including fluctuating regulatory landscapes, evolving technological challenges, and variable patient data requirements [3-5]. Neutrosophic sets offer a means to navigate these complexities effectively. By enabling the representation of vague or imprecise data within the blockchain, healthcare stakeholders can make informed decisions, even when faced with incomplete or conflicting information. For instance, patient consent, a critical component of healthcare data management, often exhibits varying degrees of consent, uncertainty, or ambiguity. Neutrosophic sets can elegantly capture and quantify these nuanced consent dynamics, facilitating more accurate decision-making [6-7]. Moreover, within the realm of Multi-Criteria Decision Making (MCDM) for evaluating service providers, Neutrosophic sets provide a structured framework to handle the diverse judgments of experts. Decision-makers' assessments, often characterized by differing degrees of truth, falsity, and indeterminacy, can be systematically integrated, allowing for a more holistic evaluation [8-9].

This paper introduces a pioneering Neutrosophic Intelligence approach, placing Neutrosophic sets theory at the forefront of our methodology. Neutrosophic sets provide a mathematical foundation capable of handling the inherent uncertainties, vagueness, and imprecisions often encountered in healthcare decision-making. As the healthcare industry navigates the complexities of blockchain technology, the integration of Neutrosophic sets theory emerges as a powerful means to enhance patient data security and privacy. Our work revolves around the utilization of MCDM techniques, particularly the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), to systematically evaluate and rank service providers within blockchain-based smart healthcare systems. However, what sets our methodology apart is the pervasive role of Neutrosophic sets in the representation and management of decision-makers' judgments. These judgments inherently possess varying degrees of truth, indeterminacy, and falsity, a challenge that Neutrosophic sets adeptly address. Furthermore, we employ the Ordered Weighted Averaging (OWA) operator to aggregate these Neutrosophic judgments, emphasizing the integral role of Neutrosophic sets in our decision fusion process. Through this approach, we strive to provide a structured and robust framework for healthcare organizations to fortify patient data security while embracing the potential of blockchain technology in smart healthcare [10].

Our paper is organized as follows: In Section 2, we provide a comprehensive review of related work in the fields of healthcare data security, blockchain technology, and Neutrosophic Intelligence, highlighting the current state of research and identifying the gaps that motivate our study. Section 3 outlines our methodology, detailing the Neutrosophic Intelligence approach we have developed for safeguarding patient data within blockchain-based smart healthcare systems. In Section 4, we present the results of our empirical evaluation and provide a thorough analysis of the data collected during our study. Section 5 offers a discussion of our findings, emphasizing their implications for healthcare data security, privacy, and the broader adoption of blockchain in healthcare. In Section 6, we conclude our research, summarizing the key insights and contributions of our study.

## 2. Related Works

This section provides a comprehensive overview of the existing research and literature relevant to our study, setting the stage for the development of a neutrosophic approach for anomaly detection in smart agriculture systems using edge intelligence. Yaqoob et al. [11] conducted a comprehensive review of skin cancer detection and classification using federated learning, emphasizing the importance of privacy in healthcare applications. Their work highlights the relevance of advanced machine-learning techniques in medical contexts, which resonates with our exploration of anomaly detection in smart agriculture. Alaba et al. [12] explored the security applications and challenges of smart contracts, which is an area of interest when considering the secure execution of algorithms, especially in edge computing environments. Their insights into the security aspects of smart contracts provide valuable context for our work. Kumar et al. [13] presented an approach to region-of-interest detection in COVID-19 CT images using neutrosophic logic. While their focus is on medical imaging, their utilization of neutrosophic logic is relevant to our proposed methodology, as it demonstrates the applicability of this logic in image analysis and pattern recognition. Thillaigovindan et al. [14] developed an integrated model for heart disease prediction using cryptographic and machine learning methods. Their work showcases the potential benefits of integrating different technologies for enhanced accuracy and security, which aligns with our approach to integrating edge intelligence and neutrosophic logic. Saha et al. [15] discussed AI-enabled human and machine activity monitoring in industrial IoT systems. Their exploration of AI in the context of IoT is relevant to our study, as it highlights the broader implications of AI and edge computing in monitoring and managing complex systems. Fernandez-Vazquez et al. [16] investigated the use of blockchain in sustainable supply chain management, applying analytical hierarchical process (AHP) methodology. Their work demonstrates the versatility of blockchain technology, which can be considered in the context of data security and integrity in smart agriculture systems. Mohammed et al. [17] proposed a Bitcoin network-based anonymity and privacy model for metaverse implementation in Industry 5.0. Their research touches upon privacy and security concerns in decentralized systems, which can provide insights into securing data in agricultural edge environments. Singh et al. [18] discussed transfer fuzzy learning for security in industrial IoT, which can be relevant to our study's focus on secure anomaly detection. Their work showcases advanced cryptographic techniques that can be considered in securing edge devices. Morhaim [19] provided a comprehensive overview of blockchain and cryptocurrency technologies. While not focused on agriculture, this reference offers foundational knowledge about blockchain technology, which can be useful when discussing potential applications in the context of smart agriculture systems.

## 3. Methodology

In this section, we elucidate the rigorous methodology employed in our study, outlining the systematic steps undertaken to develop and implement our Neutrosophic Intelligence approach for safeguarding patient data in the context of blockchain-based smart healthcare systems. In this section, we elucidate the systematic methodology employed to comprehensively assess the performance of blockchain technology as a fundamental safeguard for patient data within blockchain-based smart healthcare systems. The integration of blockchain technology in healthcare settings is driven by a multitude of factors and challenges by various complexities. To shed light on this intricate landscape, our study integrates expert surveys to construct a holistic understanding of blockchain technology's impact on patient data security in smart healthcare environments. We specifically focus on the perspectives of industry experts, decision-makers, and stakeholders, who offer valuable insights into the drivers, barriers, and risks associated with blockchain implementation. It is important to note that the opinions of decision-makers and experts are inherently diverse and may lack clear consensus, reflecting the multifaceted nature of this field. Therefore, our study places a particular emphasis on the decision-making processes that underpin the safeguarding of patient data within blockchain-based smart healthcare, categorizing them according to three critical dependent factors.

***Stakeholder Proficiency and Attitudes***

a. Decision-Maker Expertise: In the context of smart healthcare, decision-maker expertise is crucial for understanding the intricacies of integrating blockchain technology. Their familiarity with healthcare data security, blockchain protocols, and the nuances of smart healthcare systems can significantly impact the success of implementation. Expertise allows decision-makers to make informed choices, anticipate potential challenges, and devise strategies for maximizing the benefits of blockchain.
b. Risk Appetite: Smart healthcare introduces innovative approaches to patient care through technologies like IoT devices, AI-driven diagnostics, and telemedicine. Decision-makers' risk appetite plays a critical role in adopting blockchain, as it often involves a departure from conventional data security methods. Embracing blockchain may require a willingness to accept initial uncertainties and invest in new technologies that promise long-term security and efficiency gains.
c. Organizational Culture: The culture within healthcare institutions can either facilitate or hinder the integration of blockchain. An organizational culture that values innovation, collaboration, and a patient-centric approach is more likely to embrace the changes brought about by smart healthcare and the implementation of blockchain for enhanced patient data security. Conversely, a resistant or risk-averse culture may present challenges to adoption.

***Environmental Conditions***

a. Regulatory Landscape: Smart healthcare operates within a complex regulatory environment that varies by region. Decision-makers must navigate these regulations to ensure compliance when implementing blockchain solutions. A supportive regulatory landscape can encourage blockchain adoption by providing clear guidelines for data security and patient privacy in smart healthcare.
b. Technological Infrastructure: The readiness of the technological infrastructure in smart healthcare settings is essential. Decision-makers need to assess whether the existing IT infrastructure can seamlessly integrate with blockchain. Compatibility with electronic health record (EHR) systems, IoT devices, and other smart healthcare components is critical for successful implementation.
c. Industry Collaboration: Collaboration within the healthcare industry and with blockchain solution providers can foster innovation in smart healthcare. Decision-makers must evaluate the extent to which partnerships and collaborations can accelerate the adoption of blockchain for patient data security. Partnerships may involve healthcare providers, technology companies, research institutions, and regulatory bodies working together to develop standards and best practices.

***Multiple Criteria and Alternatives (MCDM)***

a. Evaluation Criteria: Decision-makers in smart healthcare need to define and prioritize evaluation criteria. These criteria may include data privacy, scalability, cost-effectiveness, interoperability with IoT devices, and the ability to ensure data integrity in real time. The selection of these criteria influences the decision-making process.

b. Alternative Solutions: Decision-makers should consider a spectrum of alternatives alongside blockchain. This might encompass traditional data security methods, cloud-based solutions, or hybrid approaches. Evaluating the strengths and weaknesses of each alternative helps decision-makers make well-informed choices aligned with smart healthcare goals.
c. Decision-Making Methodology: Formal decision-making methodologies, such as Multi-Criteria Decision Analysis (MCDA) or Analytic Hierarchy Process (AHP), can aid decision-makers in systematically evaluating criteria and alternatives. These methodologies provide a structured approach to assessing the suitability of blockchain and other options, ensuring that the final choice aligns with the unique requirements of smart healthcare.

Incorporating these considerations into the decision-making process for implementing blockchain in smart healthcare ensures a comprehensive approach that accounts for the complexities and nuances of this evolving field. By addressing personality conditions, environmental conditions, and multiple criteria and alternatives, decision-makers can navigate the dynamic landscape of smart healthcare to enhance patient data security effectively.

In our work, we adopted a comprehensive approach to collect and aggregate specialists' perspectives on the adoption of blockchain technology for safeguarding patient data in smart healthcare environments. The data collection process involved a series of discussions, meetings, and qualitative discrete choice experiments [19], which allowed us to capture a diverse range of decision-making experts' insights and opinions. To aggregate the experts' perspectives effectively, we employed the OWA [20], a method that enables the combination of individual expert viewpoints into a consensus representation. This approach helps mitigate potential biases and ensures a more comprehensive assessment of criteria and alternatives relevant to our research. Furthermore, we applied the MCDM method to analyze the decision-makers' perspectives on criteria and alternatives. MCDM provides a structured framework for evaluating and prioritizing multiple criteria and alternatives, aiding in the systematic assessment of the suitability of blockchain technology for patient data security in smart healthcare. To assess the robustness and reliability of our results, we conducted a Monte Carlo simulation. This simulation involved iteratively generating random variations in the input parameters and evaluating their impact on the decision outcomes.

The process of aggregating the specialists' perspectives was executed using the OWA operator, which serves as a valuable tool to harmonize the diverse judgments of decision-making experts and mitigate the impact of inconsistent assessments. To elaborate on its application, let's consider a scenario involving $q$ specialists, denoted as $D_1, D_2, \dots, D_q$, participating in a decision-making problem. Each expert's viewpoint contributes to the overall decision process, with $1 \leq k \leq q$ representing the index of each expert. The OWA operator is employed to calculate a consensus outcome by considering the ordered weights assigned to individual experts' judgments. Specifically, it follows a defined formula that systematically combines these perspectives, considering their relative importance: Figure 1 shows the steps of the proposed method.

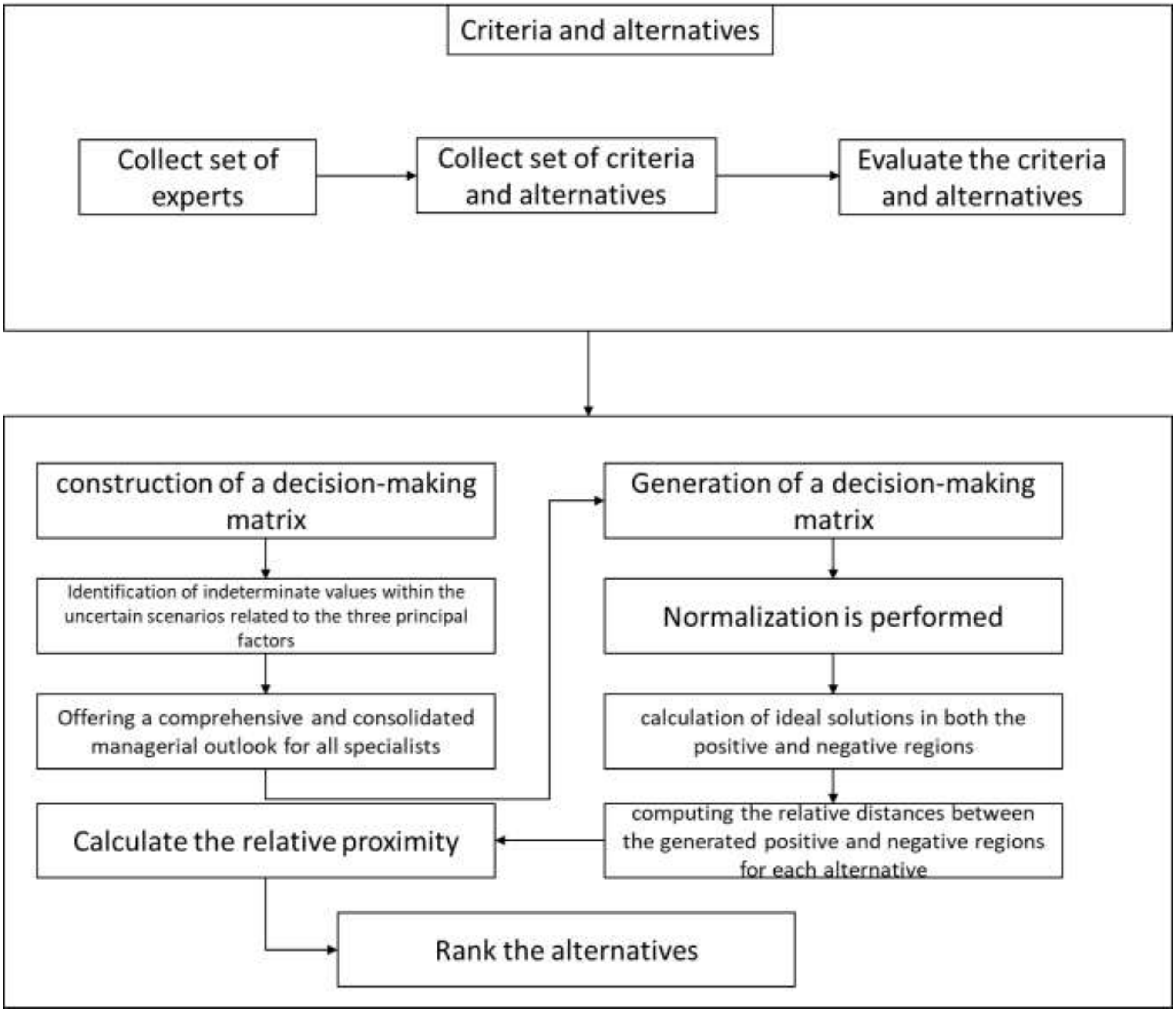


**Figure 1.** The steps of the proposed method.

Some definitions show the mathematical operations of triangular neutrosophic numbers.

***Definition* 1**

The truth, indeterminacy, and falsity membership functions can be computed as:

$$T_X(y) = \begin{cases} \alpha\left(\dfrac{y - x_1}{y_2 - y_1}\right) & (x_1 \le y \le x_2) \\ \alpha & (y = x\_2) \\ \alpha\left(\dfrac{x_3 - y}{y_3 - y_2}\right) & (x_2 \le y \le x_3) \\ 0 & otherwise \end{cases} \quad (1)$$

$$I_X(y) = \begin{cases} \left(\frac{x_{2-y+\theta-(y-x_1)}}{y_2 - y_1}\right) & (x_1 \le y \le x_2) \\ \theta \;\; (y = x\_2) \\ \left(\frac{y_{-x_2+\theta-(x_3-y)}}{y_3 - y_2}\right) & (x_2 \le y \le x_3) \\ 0 \;\; otherwise \end{cases} \tag{2}$$

$$F_X(y) = \begin{cases} \left(\frac{x_{2-y+\beta-(y-x_1)}}{y_2 - y_1}\right) & (x_1 \le y \le x_2) \\ \beta \;\; (y = x\_2) \\ \left(\frac{y_{-x_2+\beta-(x_3-y)}}{y_3 - y_2}\right) & (x_2 \le y \le x_3) \\ 0 \;\; otherwise \end{cases} \tag{3}$$

***Definition* 2**

The arithmetic operations can be computed as:

$let\ x =< (x_1, x_2, x_3); \alpha_x, \theta_x, \beta_x >, z =< (z_1, z_2, z_3); \alpha_z, \theta_z, \beta_z >$

$$x \oplus z =< (x_1 + z_1, x_2 + z_2, x_3 + z_3 ); \alpha_x \wedge \alpha_z, \theta_x \vee \theta_z, \beta_x \vee \beta_z > \tag{4}$$

$$x \ominus z =< (x_1 - z_1, x_2 - z_2, x_3 - z_3 ); \alpha_x \wedge \alpha_z, \theta_x \vee \theta_z, \beta_x \vee \beta_z > \tag{5}$$

$$x \otimes z = \begin{cases} < (x_1 z_1, x_2 z_2, x_3 z_3); \alpha_x \wedge \alpha_z, \theta_x \vee \theta_z, \beta_x \vee \beta_z, > if\ (x_3 > 0, z_3 > 0) \\ < (x_1 z_3, x_2 z_2, x_3 z_1); \alpha_x \wedge \alpha_z, \theta_x \vee \theta_z, \beta_x \vee \beta_z, > if\ (x_3 < 0, z_3 > 0) \\ < (x_3 z_3, x_2 z_2, x_3 z_3); \alpha_x \wedge \alpha_z, \theta_x \vee \theta_z, \beta_x \vee \beta_z, > if\ (x_3 > 0, z_3 < 0) \end{cases} \tag{6}$$

$$x \oslash z = \begin{cases} < \left(\frac{x_1}{z_3}, \frac{x_2}{z_2}, \frac{x_3}{z_1}\right); \alpha_x \wedge \alpha_z, \theta_x \vee \theta_z, \beta_x \vee \beta_z > if\ (x_3 > 0, z_3 > 0) \\ < \left(\frac{x_3}{z_3}, \frac{x_2}{z_2}, \frac{x_1}{z_1}\right); \alpha_x \wedge \alpha_z, \theta_x \vee \theta_z, \beta_x \vee \beta_z, > if\ (x_3 < 0, z_3 > 0) \\ < \left(\frac{x_3}{z_1}, \frac{x_2}{z_2}, \frac{x_1}{z_3}\right); \alpha_x \wedge \alpha_z, \theta_x \vee \theta_z, \beta_x \vee \beta_z, > if\ (x_3 > 0, z_3 < 0) \end{cases} \tag{7}$$

$$\curlyvee \oslash x = \begin{cases} < \left(\frac{x_1}{\curlyvee}, \frac{x_2}{\curlyvee}, \frac{x_3}{\curlyvee}\right); \alpha_x, \theta_x, \beta_x if\ \curlyvee > 0 > \\ < \left(\frac{x_3}{\curlyvee}, \frac{x_2}{\curlyvee}, \frac{x_1}{\curlyvee}\right); \alpha_x, \theta_x, \beta_x if\ \curlyvee < 0 > \end{cases} \tag{8}$$

$$x^{-1} =< \left(\frac{1}{x_3}, \frac{1}{x_2}, \frac{1}{x_1}\right); \alpha_x, \theta_x, \beta_x > \tag{9}$$

Step 1 involves the construction of a decision-making matrix, denoted as $DM_{ij}^K$, to represent the viewpoints of experts (indexed as $D_k$) and model their perspectives regarding blockchain technology as criteria. The $DM_{ij}^K$ matrix adopts a neutrosophic triangular scale [2] for its structure, and its definition is as follows [6]:

$$DM_{ij}^{K} = \begin{matrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{matrix} \begin{bmatrix} x_{11}^{k} & x_{12}^{k} & \dots & x_{1m}^{k} \\ x_{21}^{k} & x_{22}^{k} & \dots & x_{2m}^{k} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1}^{k} & x_{n2}^{k} & \dots & x_{nm}^{k} \end{bmatrix} \quad (10)$$

In the above formula, the $x_{ij}^{k}$ denote the performance ranking of the constituent of the $i-th$ criterion regarding $f_1, f_2, \dots, f_n$. it is worth noting that the category of $x_{ij}^{k}$ symbolizes the viewpoint of experts based on the neutrosophic scale.

Step 2 involves the identification of indeterminate values within the uncertain scenarios related to the three principal factors. During this step, the neutrosophic scale can be transformed into tangible numerical values using the score function outlined in [18]. The resultant values for the de-neutrosophic specialists' viewpoint matrix, denoted as $DM_{ij}^{K}$, are presented in as follows:

$$DM_{ij}^{K} = \begin{matrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{matrix} \begin{bmatrix} x_{11}^{k} & x_{12}^{k} & \dots & x_{1m}^{k} \\ x_{21}^{k} & x_{22}^{k} & \dots & x_{2m}^{k} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1}^{k} & x_{n2}^{k} & \dots & x_{nm}^{k} \end{bmatrix} \quad (11)$$

Step 3 involves offering a comprehensive and consolidated managerial outlook for all specialists, indexed as $k = 1, 2, \dots, q$, using the $DM_{ij}$ matrix in conjunction with OWA operators. The resulting outcome is then presented in Form (3) as follows:

$$DM_{ij} = \begin{matrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{matrix} \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1m} \\ x_{21} & x_{22} & \dots & x_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nm} \end{bmatrix} \quad (12)$$

Then, the TOPSIS method, which stands for Technique for Order Preference by Similarity to Ideal Solution, is applied to order potential alternatives based on the generation of principle solutions. In our case, let's consider a situation in which, we have $p$ substitutions denoted as $O_1, O_2, \dots, O_p$, and these substitutes are evaluated across $m$ criteria represented as $x_1, x_2, \dots, x_m$. Moreover, there are $q$ experts participating in this decision-making endeavor, as previously mentioned. To facilitate this process, we utilize a weighting vector consisting of $w_1, w_2, \dots, w_m$ for the $m$ criteria, following the condition $1 \leq j \leq m$. These weights, denoted as $w_j$, satisfy the conditions $w_j \geq 0$, and their sum $\sum_{j=1}^{m} w_j = 1$.

Step 4 involves the generation of a decision-making matrix denoted as $Y_{rt}^{K}$. This matrix is designed to capture the viewpoints of experts indexed as $D_k$ regarding blockchain solutions as criteria and their impact on smart healthcare systems. The $Y_{rt}^{K}$ matrix, presented in Form 6, is subsequently transformed into a numerical format through the application of the score function as expressed below:

$$Y_{rt}^{K} = \begin{matrix} O_1 \\ O_2 \\ \vdots \\ O_p \end{matrix} \begin{bmatrix} y_{11}^{k} & y_{12}^{k} & \dots & y_{1m}^{k} \\ y_{21}^{k} & y_{22}^{k} & \dots & y_{2m}^{k} \\ \vdots & \vdots & \vdots & \vdots \\ y_{p1}^{k} & y_{p2}^{k} & \dots & y_{pm}^{k} \end{bmatrix} \quad (13)$$

The consolidation of judgments from decision-makers is accomplished as follows:

$$y_{rt} = \frac{\sum_{t=1}^{m} (y_{rt}^{k})}{D_q} \quad (14)$$

Here, $y_{rt}$ denotes the evaluations provided by decision-makers for the alternatives, and $D_q$ signifies the count of decision-makers involved in the process. The resulting outcome is expressed as follows:

$$Y_{rt} = \begin{matrix} O_1 \\ O_2 \\ \vdots \\ O_p \end{matrix} \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1m} \\ y_{21} & y_{22} & \cdots & y_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ y_{p1} & y_{p2} & \cdots & y_{pm} \end{bmatrix} \tag{15}$$

TOPSIS is structured around three fundamental steps to aid in decision-making:

**Normalization**: In the first step, normalization is performed to standardize the data. This step ensures that all the criteria are on the same scale and avoids biases that could result from the use of different units or measurement scales. By normalizing the data, TOPSIS makes it possible to directly compare the importance of various criteria, regardless of their original units or magnitudes. This step transforms the raw data into a format that can be uniformly analyzed, which is essential for effective multi-criteria decision-making.

$$z_{rt} = w_j * \frac{y_{r\ t}}{\sqrt{\sum_{t=1}^{m}\ x_{rt}^2}}; r = 1,2,3 \ldots p; t = 1,2,3 \ldots m \tag{16}$$

**Calculating the Ideal Solution in Positive and Negative Regions**: The second step involves the calculation of ideal solutions in both the positive and negative regions. The ideal solutions represent the best possible outcomes based on the selected criteria. In the positive region, the ideal solution is characterized by having the highest values for beneficial criteria and the lowest values for non-beneficial criteria. Conversely, in the negative region, the ideal solution has the lowest values for beneficial criteria and the highest values for non-beneficial criteria. By identifying these ideal solutions, TOPSIS establishes a reference point for evaluating and ranking the alternatives.

$$z_t^+ = \begin{Bmatrix} \langle \max(z_{rt} | r = 1,2, \ldots, p) \ | j \in j^+ \rangle, \\ \langle \min(z_{rt} | r = 1,2, \ldots, p) \ | j \in \ j^- \rangle \end{Bmatrix} \tag{17}$$

$$z_t^- = \begin{Bmatrix} \langle \min(z_{rt} | r = 1,2, \ldots, p) \ | j \in j^+ \rangle, \\ \langle \max(z_{rt} | r = 1,2, \ldots, p) \ | j \in j^- \rangle \end{Bmatrix} \tag{18}$$

**Computing the Relative Distances between the Generated Positive and Negative Regions**: The third step focuses on computing the relative distances between the generated positive and negative regions for each alternative. This distance calculation is performed to determine how closely each alternative aligns with the ideal solutions. Alternatives that are closer to the positive ideal solution and farther from the negative ideal solution are considered more favorable and receive higher rankings. Conversely, alternatives that are closer to the negative ideal solution and farther from the positive ideal solution are considered less desirable and receive lower rankings.

$$d_r^+ = \sqrt{\sum_{t=1}^{m} (z_{rt} - z_t^+)^2}, r = 1,2, \ldots, p \tag{19}$$

$$d_r^- = \sqrt{\sum_{t=1}^{m} (z_{rt} - z_t^-)^2}, r = 1,2, \ldots, p \tag{20}$$

Finally, we calculate the relative proximity by combining the positive and negative regions of the solutions to attain the ideal solutions, as outlined below:

$$c_r = \frac{d_r^-}{d_r^+ + d_r^-}; r = 1,2, \dots, p \quad (21)$$

## 4. Results and Analysis

This section delves into the pivotal phase of our research, where we present the results of our Neutrosophic Intelligence approach in action and offer a comprehensive analysis of the data obtained during our study. Through meticulous experimentation and evaluation, we assess the performance, security, and adaptability of our approach within the complex landscape of blockchain-based smart healthcare systems. To substantiate the practical applicability of our proposed model for safeguarding healthcare patient data, we conducted an empirical study encompassing a comprehensive case study analysis. The chosen case study revolves around the evaluation of fifteen critical criteria for the security of healthcare patient data, as outlined in Table 1.

**Table 1.** Criteria for Patient Data Security in Healthcare Systems

| Group | Criteria | ID | Description |
|---|---|---|---|
| **Drivers** | Regulatory Compliance | D1 | The extent of compliance with data privacy regulations (e.g., HIPAA, GDPR) |
| | Technology Adoption | D2 | The level of adoption of advanced security technologies (e.g., blockchain, encryption) |
| | Interoperability | D3 | The ability of systems to communicate and share data securely |
| | Cybersecurity Training | D4 | The level of training provided to staff and employees on cybersecurity best practices |
| | Patient Engagement | D5 | The degree to which patients actively participate in maintaining the security of their health data |
| **Barriers** | Legacy Systems | B1 | The presence of outdated legacy systems that lack modern security features |
| | Budget Constraints | B2 | Financial limitations that hinder investments in robust cybersecurity measures |
| | Resistance to Change | B3 | Organizational resistance to adopting new security protocols and technologies |
| | Third-party Vulnerabilities | B4 | Risks associated with reliance on third-party vendors and service providers |
| | Data Volume and Complexity | B5 | The challenges posed by growing volumes and complexity of patient data, requiring scalable solutions |
| **Risks** | Data Breach Risk | R1 | Likelihood and potential impact of data breaches, including unauthorized exposure of patient data |
| | Malware and Ransomware Risk | R2 | Risk associated with malware and ransomware attacks that can compromise or encrypt patient data |
| | Insider Threat Risk | R3 | Risk of insider threats, including employees mishandling data intentionally or unintentionally |
| | IoT Vulnerabilities | R4 | Vulnerabilities introduced by the use of IoT devices in healthcare, which can be exploited |
| | Regulatory Non-compliance Risk | R5 | The risk of failing to comply with data privacy regulations, leading to penalties and legal consequences |

To comprehensively assess the security of patient data within healthcare systems, we conducted an illuminating case study that drew upon the aforementioned fifteen criteria, categorized into three pivotal groups: Driver Criteria, Barrier Criteria, and Risk Criteria. The case study unfolded within the dynamic landscape of a modern healthcare institution, where the critical importance of patient data security is magnified. Leveraging a multidisciplinary team of experts, we embarked on a meticulous examination of the organization's data security ecosystem. This encompassed evaluating the impact of regulatory compliance, technology adoption, and patient engagement as drivers that underpin a robust data security framework. Simultaneously, we scrutinized the hurdles posed by legacy systems, budget constraints, and resistance to change as barriers that often necessitate strategic solutions. Moreover, we probed the intricacies of data breach risks, malware vulnerabilities, insider threats, IoT-related concerns, and the looming specter of regulatory non-compliance as formidable risks to be mitigated. By systematically evaluating these criteria, we aim to identify vulnerabilities, strengths, and opportunities for improving data security. The case study involves interviews with key stakeholders, a thorough examination of existing security measures, and the application of our proposed model to assess the overall data security posture.

In this section, we leveraged the OWA operator to systematically aggregate the perspectives and evaluations of experts. To maintain transparency and consistency in our analysis, we adopted predefined weights for the OWA operator, as detailed in Table 2. These weights were thoughtfully determined to reflect the relative importance of various criteria within the decision-making process.

**Table 2.** Predefined Weights for OWA Operator

| **Criteria** | **Weights** |
|---|---|
| Regulatory Compliance | 0.217744 |
| Technology Adoption | 0.304067 |
| Interoperability | 0.251275 |
| Cybersecurity Training | 0.288399 |
| Patient Engagement | 0.186048 |
| Legacy Systems | 0.146608 |
| Budget Constraints | 0.229047 |
| Resistance to Change | 0.224132 |
| Third-party Vulnerabilities | 0.237017 |
| Data Volume and Complexity | 0.084615 |
| Data Breach Risk | 0.171464 |
| Malware and Ransomware Risk | 0.144085 |
| Insider Threat Risk | 0.147436 |
| IoT Vulnerabilities | 0.112711 |
| Regulatory Non-compliance Risk | 0.222641 |

We present the outcomes of the decision-making process, which were meticulously generated using the OWA operator. To achieve this, we harnessed the collective expertise of our decision-makers, each of whom provided valuable insights and evaluations across a spectrum of criteria. The OWA operator, guided by predefined weights as detailed in Table 3, played a pivotal role in harmonizing these diverse perspectives and aggregating them into a coherent whole. This process facilitated the creation of OWA general and aggregated decision-makers decisions, offering a holistic view of the evaluation outcomes. These judgments reflect the consensus reached by our expert panel, embodying their collective wisdom and expertise in assessing the critical aspects of patient data security within the healthcare system.

**Table 3**. The ultimate matrix produced employs OWA for the evaluation of decision-maker judgments regarding driver, barrier, and risk criteria

| | D1 | D2 | D3 | D4 | D5 | B1 | B2 | B3 | B4 | B5 | R1 | R2 | R3 | R4 | R5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| D1 | 0.32877 | 0.32877 | 0.32877 | 0.32877 | 0.32877 | 0.32877 | 0.32877 | 0.32877 | 0.32877 | 0.32877 | 0.32877 | 0.32877 | 0.32877 | 0.32877 | 0.32877 |
| D2 | 0.36401 | 0.36401 | 0.36401 | 0.36401 | 0.36401 | 0.36401 | 0.36401 | 0.36401 | 0.36401 | 0.36401 | 0.36401 | 0.36401 | 0.36401 | 0.36401 | 0.36401 |
| D3 | 0.26230 | 0.26230 | 0.26230 | 0.26230 | 0.26230 | 0.26230 | 0.26230 | 0.26230 | 0.26230 | 0.26230 | 0.26230 | 0.26230 | 0.26230 | 0.26230 | 0.26230 |
| D4 | 0.34992 | 0.34992 | 0.34992 | 0.34992 | 0.34992 | 0.34992 | 0.34992 | 0.34992 | 0.34992 | 0.34992 | 0.34992 | 0.34992 | 0.34992 | 0.34992 | 0.34992 |
| D5 | 0.40303 | 0.40303 | 0.40303 | 0.40303 | 0.40303 | 0.40303 | 0.40303 | 0.40303 | 0.40303 | 0.40303 | 0.40303 | 0.40303 | 0.40303 | 0.40303 | 0.40303 |
| B1 | 0.28105 | 0.28105 | 0.28105 | 0.28105 | 0.28105 | 0.28105 | 0.28105 | 0.28105 | 0.28105 | 0.28105 | 0.28105 | 0.28105 | 0.28105 | 0.28105 | 0.28105 |
| B2 | 0.22335 | 0.22335 | 0.22335 | 0.22335 | 0.22335 | 0.22335 | 0.22335 | 0.22335 | 0.22335 | 0.22335 | 0.22335 | 0.22335 | 0.22335 | 0.22335 | 0.22335 |
| B3 | 0.23423 | 0.23423 | 0.23423 | 0.23423 | 0.23423 | 0.23423 | 0.23423 | 0.23423 | 0.23423 | 0.23423 | 0.23423 | 0.23423 | 0.23423 | 0.23423 | 0.23423 |
| B4 | 0.13997 | 0.13997 | 0.13997 | 0.13997 | 0.13997 | 0.13997 | 0.13997 | 0.13997 | 0.13997 | 0.13997 | 0.13997 | 0.13997 | 0.13997 | 0.13997 | 0.13997 |
| B5 | 0.24324 | 0.24324 | 0.24324 | 0.24324 | 0.24324 | 0.24324 | 0.24324 | 0.24324 | 0.24324 | 0.24324 | 0.24324 | 0.24324 | 0.24324 | 0.24324 | 0.24324 |
| R1 | 0.21704 | 0.21704 | 0.21704 | 0.21704 | 0.21704 | 0.21704 | 0.21704 | 0.21704 | 0.21704 | 0.21704 | 0.21704 | 0.21704 | 0.21704 | 0.21704 | 0.21704 |
| R2 | 0.07627 | 0.07627 | 0.07627 | 0.07627 | 0.07627 | 0.07627 | 0.07627 | 0.07627 | 0.07627 | 0.07627 | 0.07627 | 0.07627 | 0.07627 | 0.07627 | 0.07627 |
| R3 | 0.13483 | 0.13483 | 0.13483 | 0.13483 | 0.13483 | 0.13483 | 0.13483 | 0.13483 | 0.13483 | 0.13483 | 0.13483 | 0.13483 | 0.13483 | 0.13483 | 0.13483 |
| R4 | 0.21613 | 0.21613 | 0.21613 | 0.21613 | 0.21613 | 0.21613 | 0.21613 | 0.21613 | 0.21613 | 0.21613 | 0.21613 | 0.21613 | 0.21613 | 0.21613 | 0.21613 |
| R5 | 0.04277 | 0.04277 | 0.04277 | 0.04277 | 0.04277 | 0.04277 | 0.04277 | 0.04277 | 0.04277 | 0.04277 | 0.04277 | 0.04277 | 0.04277 | 0.04277 | 0.04277 |

## 5. Discussion and implications

In this pivotal section, we embark on a thoughtful examination of the results presented in the preceding section and delve into their broader significance. Our discussion transcends the numerical findings, focusing on the qualitative insights, the implications of our Neutrosophic Intelligence approach, and the potentially transformative effects on the landscape of healthcare data security within blockchain-based smart healthcare systems.

To facilitate decision-making in uncertain conditions, our case study incorporates the use of the TOPSIS methodology. This method was applied to assess and rank five distinct service providers operating within the realm of blockchain-based smart healthcare systems. These service providers were meticulously evaluated based on the predefined weights assigned to the criteria established in our study. To maintain confidentiality and impartiality, we have anonymized these providers, referring to them as "Alternative A," "Alternative B," "Alternative C," Alternative D." and "Alternative E.". To ensure a comprehensive analysis, we employed a triangular neutrosophic scale to collect the judgments of our decision-makers. This scale effectively captures the nuances and uncertainties inherent in decision-making processes, allowing our experts to express their evaluations with precision. Subsequently, these neutrosophic judgments were transformed into numerical values, which served as the foundation for our TOPSIS-based assessments. This robust approach allowed us to methodically evaluate the service providers' performance across various criteria while accommodating the inherent complexities of real-world decision-making in the context of blockchain-based smart healthcare systems.

The ultimate ranking of the alternatives, derived from the TOPSIS analysis based on their relative closeness to the ideal solutions, has been methodically compiled and is presented in Table 4. This table serves as a comprehensive summary of our evaluation process, showcasing the performance of each alternative in comparison to the others. The relative closeness values in Table 4 provide valuable insights into how well each alternative aligns with the predefined ideal solutions, which were carefully determined following the specified criteria. Alternatives that exhibit higher relative closeness values are positioned at the top of the ranking, indicating their superior alignment with the ideal solutions across the considered criteria. Conversely, those with lower relative closeness values are situated lower in the ranking, reflecting comparatively weaker performance in meeting the specified criteria. This ranking, displayed in Table 4, offers a clear and concise visualization of the outcomes of our analysis, enabling stakeholders and decision-makers to make informed choices based on the relative strengths and weaknesses of each alternative within the context of blockchain-based smart healthcare systems. It serves as a valuable reference point for identifying the most suitable service providers for specific healthcare scenarios and requirements.

**Table 4.** Results of Alternatives Based on Relative Closeness to Ideal Solutions

| | $d_r^+$ | $d_r^-$ | $c_r$ |
|---|---|---|---|
| Alternative A | 0.251449 | 0.298566 | 0.817441 |
| Alternative B | 0.137375 | 0.189318 | 0.719279 |
| Alternative C | 0.103608 | 0.086685 | 0.458436 |
| Alternative D | 0.169257 | 0.16279 | 0.558562 |
| Alternative C | 0.288908 | 0.282441 | 0.678213 |

The findings of this research carry profound implications for the healthcare industry and its ongoing transformation towards blockchain-based smart healthcare systems. By developing a Neutrosophic Intelligence approach and integrating it with established decision-making methodologies, this study contributes significantly to the enhancement of patient data security. As blockchain technology gains prominence in healthcare, our model empowers decision-makers with a robust framework to navigate the complexities of this dynamic landscape. The ability to rank service providers objectively based on a multitude of criteria, while accounting for uncertainty, offers healthcare organizations a vital tool to strengthen their data security strategies. This, in turn, can bolster patient trust, facilitate data sharing among healthcare providers, and pave the way for more efficient and secure healthcare services.

Beyond its immediate applications, our research underscores the potential for Neutrosophic Intelligence to stimulate innovation and adaptation within healthcare and other data-sensitive domains. The successful integration of Neutrosophic theory with established decision-making techniques opens doors to novel approaches for handling uncertainty and ambiguity. By embracing this paradigm, organizations can better address the ever-evolving challenges in safeguarding sensitive data. Moreover, our study encourages further exploration of Neutrosophic Intelligence's applicability in diverse contexts where decision-making involves intricate, multifaceted criteria. As healthcare and technology continue to evolve, the principles and methodologies advanced in this research may find broader utility in addressing complex problems, fostering resilience in decision-making, and shaping a more secure and efficient future for various industries.

## 6. Conclusions

This study presents a novel Neutrosophic Intelligence approach that leverages advanced methodologies like OWA and TOPSIS to address the intricate challenge of safeguarding patient data in blockchain-based smart healthcare systems. By integrating the insights and evaluations of decision-makers across a spectrum of driver, barrier, and risk criteria, we have formulated a comprehensive framework that enhances decision-making in uncertain conditions. The results demonstrate the effectiveness of our approach in systematically assessing and ranking service providers, ultimately aiding in the selection of the most suitable alternatives. Furthermore, the study underscores the significance of factors such as regulatory compliance, technology adoption, and data breach risk in shaping the security landscape of healthcare systems. As the smart healthcare ecosystem continues to evolve, our research offers valuable guidance for stakeholders, enabling them to make informed choices and fortify patient data security in an increasingly dynamic and technology-driven healthcare landscape.

In light of the ever-expanding role of blockchain technology in healthcare, our findings emphasize the pivotal importance of adopting advanced security measures and fostering a culture of data privacy. We conclude by highlighting the potential of Neutrosophic Intelligence as a robust decision-making tool in smart healthcare, not only for patient data security but also for addressing broader challenges in this transformative field. As the healthcare industry continues to embrace innovation, our research provides a significant step toward ensuring the confidentiality, integrity, and availability of patient data, ultimately enhancing the quality and safety of healthcare services for all.

Data Availability: All data generated or analyzed during this study are included in this article.

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# SOME RESULTS IN NEUTROSOPHIC SOFT METRIC SPACES

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**Abstract**. In this research, we define a novel concept termed neutrosophic soft metric space as well as investigate its fundamental characteristics. Additionally, in neutrosophic soft metric spaces, we present various topological features of this newly developed space, such as the soft open sphere and the soft closed sphere. The neutrosophic soft metric space has already been studied in depth, and its topological as well as structural features have been mapped out.

**Keywords:** Metric spaces, Soft set, Open ball, Closed set, Neutrosophic soft metric space.

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## 1. **Introduction**

Since its development in 1965, Zadeh's [19] fuzzy set has made a significant impact throughout all of logical thought. A lot of real-world issues can be solved thanks to this notion, however it is not sufficient for other difficulties. For this purpose, Atanassov [1] developed Intuitionistic Fuzzy Sets (IFS). After establishing the IFS, it generalises findings from research on Fuzzy Sets. Smarandache [12] defines a subclass of the crisp set called the Neutrosophic Set (NS). Neutrosophy is a theoretical framework that made its way into print in 1998. Fuzzy set was incorporated into probabilistic metric space to create Fuzzy Metric Space (FMS) [11]. A proof of the Fuzzy Sets version of Baire's Category Theorem in FMS form is presented and several fundamental ideas of Fuzzy Sets are analysed in [6]. Since then, FMS have gained widespread use in fields including medical imaging, data processing, and decision making. Molodtsov [9] first proposed soft set theory as a problem-free mathematical method for dealing with uncertainties. Parameterizing the universal set, we get the soft set, a collection of subsets. Any collection of phrases, natural integers, etc., may be used as the parameter set. Therefore, soft sets theory has appealing uses in a wide variety of contexts.

Neutrosophic soft metric space is a novel concept we developed, and its fundamental characteristics were investigated. Additionally, in neutrosophic soft metric spaces, we present various topological structures of this newly discovered space, such as the soft open ball and the soft closed ball. Neutrosophic soft metric space has been studied for its many topological and structural characteristics.

## 2. PRELIMINARIES

**Definition 2.1.** [13] A mapping given by $S : X \to P(U)$, then a pair $(S, X)$ is called a soft set over $U$. Then a soft set is characterized by a class of subsets of $U$.

**Definition 2.2.** A 6-tuple $(\mathfrak{X}, \Lambda, \Omega, \Upsilon, \odot, \oplus)$ is known to be an Neutrosophic Soft Metric Space (shortly NSMS), $\mathfrak{X}$ is an arbitrary non empty set, $\odot$ and $\oplus$ , a neutrosophic CTN and CTCN and $\Lambda, \Omega$ and $\Upsilon$ are neutrosophic on $SP(\mathfrak{X}^2) \times (0, \infty)SP \to [0, 1]SP$ satisfying the following conditions: For all $\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \ddot{\iota}_{\alpha_3} \in SP(\mathfrak{X}), \vartheta, \hat{\nu} > 0$.

(i) $0 \leq \Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) \leq 1; 0 \leq \Omega(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) \leq 1; 0 \leq \Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) \leq 1$;
(ii) $\Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) + \Omega(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) + \Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) \leq 3$;
(iii) $\Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) > 0$;
(iv) $\Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) = 1$ if and only if $\ddot{\xi}_{\alpha_1} = \ddot{\kappa}_{\alpha_2}$;
(v) $\Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) = \Lambda(\ddot{\kappa}_{\alpha_2}, \ddot{\xi}_{\alpha_1}, \vartheta)$;
(vi) $\Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) \odot \Lambda(\ddot{\kappa}_{\alpha_2}, \ddot{\iota}_{\alpha_3}, \hat{\nu}) \leq \Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\iota}_{\alpha_3}, \vartheta + \hat{\nu}),$ for all$\vartheta, \hat{\nu} > 0$;
(vii) $\Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, .) : (0, \infty) \to (0, 1](E)$ is neutrosophic continuous (NC);
(viii) $\lim_{\vartheta \to \infty} \Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) = 1$ for all $\vartheta > 0$;
(ix) $\Omega(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) < 1$;
(x) $\Omega(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) = 0$ if and only if $\ddot{\xi}_{\alpha_1} = \ddot{\kappa}_{\alpha_2}$;
(xi) $\Omega(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) = \Omega(\ddot{\kappa}_{\alpha_2}, \ddot{\xi}_{\alpha_1}, \vartheta)$;
(xii) $\Omega(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) \oplus \Omega(\ddot{\kappa}_{\alpha_2}, \ddot{\iota}_{\alpha_3}, \hat{\nu}) \geq \Omega(\ddot{\xi}_{\alpha_1}, \ddot{\iota}_{\alpha_3}\vartheta + \hat{\nu})$, for all $\vartheta, \hat{\nu} > 0$;
(xiii) $\Omega(\ddot{\xi}, \ddot{\kappa}, .) : (0, \infty) \to (0, 1](E)$ is NC;
(xiv) $\lim_{\vartheta \to \infty} \Omega(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) = 0$ for all $\vartheta > 0$;
(xv) $\Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) < 1$;
(xvi) $\Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) = 0$ if and only if $\ddot{\xi}_{\alpha_1} = \ddot{\kappa}_{\alpha_2}$;
(xvii) $\Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) = \Upsilon(\ddot{\kappa}_{\alpha_2}, \ddot{\xi}_{\alpha_1}, \vartheta)$;
(xviii) $\Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) \oplus \Upsilon(\ddot{\kappa}_{\alpha_2}, \ddot{\iota}_{\alpha_3}, \hat{\nu}) \geq \Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\iota}_{\alpha_3}, \vartheta + \hat{\nu})$, for all $\vartheta, \hat{\nu} > 0$;

(xix) $\Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, .) : (0, \infty) \to (0, 1](E)$ is NC;
(xx) $\lim_{\vartheta \to \infty} \Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) = 0$ for all $\vartheta > 0$;
(xxi) If $\vartheta > 0$ then $\Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) = 0; \Omega(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) = 1; \Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) = 1$.
Then, $(\Lambda, \Omega, \Upsilon)$ is called an NMS on $SP(\mathfrak{X})$. The functions $\Lambda, \Upsilon$ and $\Omega$ denote degree of nearness, inconclusiveness and non-nearness between $\ddot{\xi}_{\alpha_1}$ and $\ddot{\kappa}_{\alpha_2}$ with respect to $\vartheta$ respectively.

**Example 2.3.** Define $S : SP(\mathfrak{X}) \times SP(\mathfrak{X}) \times (0, \infty) \to [0, 1]$ by
$\Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) = \frac{\vartheta}{\vartheta + d(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2})}; \Omega(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) = \frac{d(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2})}{\vartheta + d(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2})}; \Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) = \frac{d(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2})}{\vartheta}$ for all $\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2} \in \mathfrak{X}$ and $\vartheta > 0$ and $\omega \odot \tau = \min\{\omega, \tau\}$ and $\omega \oplus \tau = \max\{\omega, \tau\}$ for all $\omega, \tau \in [0, 1](E)$.
Then $(\mathfrak{X}, \Lambda, \Omega, \Upsilon, \odot, \oplus)$ is an NSMS.

## 3. MAIN RESULTS

**Definition 3.1.** An NSMS $(\mathfrak{X}, \Lambda, \Omega, \Upsilon, \odot, \oplus)$ and $\varsigma \in (0, 1)P, \vartheta > 0$, the set $\breve{B}(\ddot{\xi}_{e_1}, \varsigma, \vartheta) = \{\ddot{\kappa}_{e_2} \in \mathfrak{X} : \Lambda(\ddot{\xi}_{e_1}, \ddot{\kappa}_{e_2}, \vartheta) > 1 - \varsigma, \Omega(\ddot{\xi}_{e_1}, \ddot{\kappa}_{e_2}, \vartheta) < \varsigma, \Upsilon(\ddot{\xi}_{e_1}, \ddot{\kappa}_{e_2}, \vartheta) < \varsigma\}$ is known to be Neutrosophic Soft Open Ball (NSOB).

**Theorem 3.2.** *Let $\breve{B}(\ddot{\xi}_{e_1}, \varsigma, \vartheta)$ be an NSOB, hence it is a Open Set (OS).*

*Proof.* Consider $\breve{B}(\ddot{\xi}_{e_1}, \varsigma, \vartheta)$ is an OB. Let $\ddot{\kappa}_{e_2} \in \breve{B}(\ddot{\xi}_{e_1}, \varsigma, \vartheta)$ .
Then $\Lambda(\ddot{\xi}_{e_1}, \ddot{\kappa}_{e_2}, \vartheta) > 1 - \varsigma, \Omega(\ddot{\xi}_{e_1}, \ddot{\kappa}_{e_2}, \vartheta) < \varsigma, \Upsilon(\ddot{\xi}_{e_1}, \ddot{\kappa}_{e_2}, \vartheta) < \varsigma$ .
Since $\Lambda(\ddot{\xi}_{e_1}, \ddot{\kappa}_{e_2}, \vartheta) > 1 - \varsigma$. There exists $\vartheta_0 \in (0, \vartheta)$ such that $\Lambda(\ddot{\xi}_{e_1}, \ddot{\kappa}_{e_2}, \vartheta) > 1 - \varsigma, \Omega(\ddot{\xi}_{e_1}, \ddot{\kappa}_{e_2}, \vartheta) < \varsigma$ and $\Upsilon(\ddot{\xi}_{e_1}, \ddot{\kappa}_{e_2}, \vartheta) < \varsigma$.
If take $\varsigma_0 = \Lambda(\ddot{\xi}_{e_1}, \ddot{\kappa}_{e_2}, \vartheta_0)$ then for $\varsigma_0 > 1 - \varsigma, \rho \in (0, 1)$ such that $\varsigma_0 > 1 - \rho > 1 - \varsigma$.
Now for given $\varsigma$ and $\rho$ such that $\varsigma_0 > 1 - \rho$, there exist $\varsigma_1, \varsigma_2 \in (0, 1)$ so that $\varsigma_0 \odot \varsigma_1 > 1 - \rho$ and $(1 - \varsigma_0) \oplus (1 - \varsigma_2) \leq \rho$ and $(1 - \varsigma_0) \oplus (1 - \varsigma_3) \leq \rho$.
Choose $\varsigma_4 = \max\{\varsigma_1, \varsigma_2, \varsigma_3\}$. Consider an OB $\breve{B}(\ddot{\kappa}_{e_2}, 1 - \varsigma_4, \vartheta - \vartheta_0)$.
We will show that $\breve{B}(\ddot{\kappa}_{e_2}, 1 - \varsigma_4, \vartheta - \vartheta_0) \subset \breve{B}(\ddot{\xi}_{e_1}, \varsigma, \vartheta)$.
Consider $\ddot{\iota}_{e_3} \in \breve{B}(\ddot{\kappa}_{e_2}, 1 - \varsigma_4, \vartheta - \vartheta_0)$, then $\Lambda(\ddot{\kappa}_{e_2}, \ddot{\iota}_{e_3}, \vartheta - \vartheta_0) > \varsigma_4, \Omega(\ddot{\kappa}_{e_2}, \ddot{\iota}_{e_3}, \vartheta - \vartheta_0) < \varsigma_4$ and $\Upsilon(\ddot{\kappa}_{e_2}, \ddot{\iota}_{e_3}, \vartheta - \vartheta_0) < \varsigma_4$.
Hence,

$$\Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\iota}_{\alpha_3}, \vartheta) \geq \Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta_0) \odot \Lambda(\ddot{\kappa}_{\alpha_2}, \ddot{\iota}_{\alpha_3}, \vartheta - \vartheta_0) \geq \varsigma_0 \odot \varsigma_4 \geq \varsigma_0 \odot \varsigma_1 \geq 1 - \rho > 1 - \varsigma,$$

$$\Omega(\ddot{\xi}_{\alpha_1}, \ddot{\iota}_{\alpha_3}, \vartheta) \leq \Omega(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta_0) \oplus \Omega(\ddot{\kappa}_{\alpha_2}, \ddot{\iota}_{\alpha_3}, \vartheta - \vartheta_0) \leq (1 - \varsigma_0) \oplus (1 - \varsigma_4) \leq (1 - \varsigma_0) \oplus (1 - \varsigma_2) \leq \rho < \varsigma,$$

$$\Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\iota}_{\alpha_3}, \vartheta) \leq \Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta_0) \oplus \Upsilon(\ddot{\kappa}_{\alpha_2}, \ddot{\iota}_{\alpha_3}, \vartheta - \vartheta_0) \leq (1 - \varsigma_0) \oplus (1 - \varsigma_4) \leq (1 - \varsigma_0) \oplus (1 - \varsigma_2) \leq \rho < \varsigma$$

It shows that $\ddot{\iota}_{\alpha_3} \in \breve{B}(\ddot{\xi}_{\alpha_1}, \varsigma, \vartheta)$ and $\breve{B}(\ddot{\kappa}_{\alpha_2}, 1 - \varsigma_4, \vartheta - \vartheta_0) \subset \breve{B}(\ddot{\xi}_{\alpha_1}, \varsigma, \vartheta)$. □

**Remark 3.3.** Consider an NSMS $(\mathfrak{X}, \Lambda, \Omega, \Upsilon, \odot, \oplus)$. Define $\hat{\nu}(\Lambda, \Omega, \Upsilon) = \{K \subset \mathfrak{X} :$ for each $\ddot{\xi}_{\alpha_1} \in K,$ there exists$\vartheta > 0\}$ and $\varsigma \in (0,1)$ such that $\breve{B}(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) \subset K\}$. Hence, $\hat{\nu}(\Lambda, \Omega, \Upsilon)$ is a topology on $\mathfrak{X}$.

**Theorem 3.4.** *Every NSMS is Hausdorff.*

*Proof.* Let $(\mathfrak{X}, \Lambda, \Omega, \Upsilon, \odot, \oplus)$ be a NSMS. Choose $\ddot{\xi}_{\alpha_1}$ and $\ddot{\kappa}_{\alpha_2}$ as two distinct points in $\mathfrak{X}$. Hence, $0 < \Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) < 1, 0 < \Omega(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) < 1, 0 < \Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) < 1$.
Take $\varsigma_1 = \Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta), \varsigma_2 = \Omega(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta), \varsigma_3 = \Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta)$ and $\varsigma = \max\{\varsigma_1, 1-\varsigma_2, 1-\varsigma_3\}$. If we take $\varsigma_0 \in (\varsigma, 1)$, then there exist $\varsigma_4, \varsigma_5, \varsigma_6$ such that $\varsigma_4 \odot \varsigma_4 \geq \varsigma_0, (1-\varsigma_5) \oplus (1-\varsigma_5) \leq 1-\varsigma_0$ and $(1-\varsigma_6) \oplus (1-\varsigma_6) \leq 1-\varsigma_0$.
Let $\varsigma_7 = \max\{\varsigma_4, \varsigma_5, \varsigma_6\}$. If we consider the $\breve{B}\left(\ddot{\xi}_{\alpha_1}, 1-\varsigma_7, \frac{\vartheta}{2}\right)$ and $\breve{B}\left(\ddot{\kappa}_{\alpha_2}, 1-\varsigma_7, \frac{\vartheta}{2}\right)$, then clearly $\breve{B}\left(\ddot{\xi}_{\alpha_1}, 1-\varsigma_7, \frac{\vartheta}{2}\right) \bigcap \breve{B}\left(\ddot{\kappa}_{\alpha_2}, 1-\varsigma_7, \frac{\vartheta}{2}\right) = \emptyset$. From here, if we choose $\ddot{\iota}_{\alpha_3} \in \breve{B}\left(\ddot{\xi}_{\alpha_1}, 1-\varsigma_7, \frac{\vartheta}{2}\right) \bigcap \breve{B}\left(\ddot{\kappa}_{\alpha_2}, 1-\varsigma_7, \frac{\vartheta}{2}\right)$ then

$$\begin{aligned}
\varsigma_1 = \Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) &\geq \Lambda\left(\ddot{\xi}_{\alpha_1}, \ddot{\iota}_{\alpha_3}, \frac{\vartheta}{2}\right) \odot \Lambda\left(\ddot{\iota}_{\alpha_3}, \ddot{\kappa}_{\alpha_2}, \frac{\vartheta}{2}\right) \\
&\geq \varsigma_7 \odot \varsigma_7 \geq \varsigma_4 \odot \varsigma_4 \geq \varsigma_0 > \varsigma_1, \\
\varsigma_2 = \Omega(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) &\leq \Omega\left(\ddot{\xi}_{\alpha_1}, \ddot{\iota}_{\alpha_3}, \frac{\vartheta}{2}\right) \oplus \Omega\left(\ddot{\iota}_{\alpha_3}, \ddot{\kappa}_{\alpha_2}, \frac{\vartheta}{2}\right) \\
&\leq (1-\varsigma_7) \oplus (1-\varsigma_7) \leq (1-\varsigma_5) \oplus (1-\varsigma_5) \leq (1-\varsigma_0) < \varsigma_2, \\
\varsigma_3 = \Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) &\leq \Upsilon\left(\ddot{\xi}_{\alpha_1}, \ddot{\iota}_{\alpha_3}, \frac{\vartheta}{2}\right) \oplus \Upsilon\left(\ddot{\iota}_{\alpha_3}, \ddot{\kappa}_{\alpha_2}, \frac{\vartheta}{2}\right) \\
&\leq (1-\varsigma_7) \oplus (1-\varsigma_7) \leq (1-\varsigma_6) \oplus (1-\varsigma_6) \leq (1-\varsigma_0) < \varsigma_3.
\end{aligned}$$

which is a contradiction. Therefore, we say that NSMS is Hausdorff. □

**Definition 3.5.** Let $(\mathfrak{X}, \Lambda, \Omega, \Upsilon, \odot, \oplus)$ be a NSMS. A subset A of $\mathfrak{X}$ is called Neutrosophic Bounded (NB), if there exist $\vartheta > 0$ and $\varsigma \in (0,1)$ such that $\Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) > 1-\varsigma, \Omega(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) < \varsigma$ and $\Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2}, \vartheta) < \varsigma$ , for all $\ddot{\xi}_{\alpha_1}, \ddot{\kappa}_{\alpha_2} \in A$.

**Definition 3.6.** If $A \subseteq \bigcup_{U \in C_N} U$ , a collection $C_N$ of OSs is said to be an Open Cover(OC) of A. A subspace A of a NSMS is compact, if every OC of A has a finite subcover. If every sequence in A has a convergent subsequence to a point in A, then it is called sequential compact.

**Theorem 3.7.** *If A is a compact member of an NSMS* $(\mathfrak{X}, \Lambda, \Omega, \Upsilon, \odot, \oplus)$ *then it is NSMS bounded.*

*Proof.* Consider A is a compact member of an NSMS $\mathfrak{X}$. Let $\vartheta > 0$ and $0 < \varsigma < 1$. Let $\{\breve{B}(\ddot{\xi}_{\alpha_1}, \varsigma, \vartheta) : \ddot{\xi}_{\alpha_1} \in A\}$ be an open cover of A. Since A is compact, there exists $\ddot{\xi}_{\alpha_1}, \ddot{\xi}_{\alpha_2}, \ddot{\xi}_{\alpha_3} \ldots \ddot{\xi}_{\alpha_n} \in A$ such that $A \subseteq \bigcup_{i=1}^{n} \breve{B}(\ddot{\xi}_{\alpha_i}, \varsigma, \vartheta)$. Let $\ddot{\xi}_{\alpha_1}, \ddot{\varkappa}_{\alpha_2} \in A$ . Then $\ddot{\xi}_{\alpha_1} \in \breve{B}(\ddot{\xi}_{\alpha_i}, \varsigma, \vartheta)$ and $\ddot{\varkappa}_{\alpha_2} \in \breve{B}(\ddot{\xi}_{\alpha_j}, \varsigma, \vartheta)$ for some $i, j$.

Then $\Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\xi}_{\alpha_i}, \vartheta) > 1 - \varsigma, \Omega(\ddot{\xi}_{\alpha_1}, \ddot{\xi}_{\alpha_i}, \vartheta) < \varsigma, \Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\xi}_{\alpha_i}, \vartheta) < \varsigma$ and $\Lambda(\ddot{\varkappa}_{\alpha_1}, \ddot{\xi}_{\alpha_j}, \vartheta) > 1 - \varsigma, \Omega(\ddot{\varkappa}_{\alpha_1}, \ddot{\xi}_{\alpha_j}, \vartheta) < \varsigma, \Upsilon(\ddot{\varkappa}_{\alpha_1}, \ddot{\xi}_{\alpha_j}, \vartheta) < \varsigma$

Now, let $\alpha = \min\{\Lambda(\ddot{\xi}_{\alpha_i}, \ddot{\xi}_{\alpha_j}, \vartheta) : 1 \leq i, j \leq n\}, \beta = \max\{\Omega(\ddot{\xi}_{\alpha_i}, \ddot{\xi}_{\alpha_j}, \vartheta) : 1 \leq i, j \leq n\}$ and $\gamma = \max\{\Upsilon(\ddot{\xi}_{\alpha_i}, \ddot{\xi}_{\alpha_j}, \vartheta) : 1 \leq i, j \leq n\}$.

Then $\alpha, \beta, \gamma > 0$, from here, for $0 < \ddot{\xi}_{\alpha_1}, \ddot{\xi}_{\alpha_2}, \ddot{\xi}_{\alpha_3} < 1$.

Next, we have

$$\begin{aligned}
\Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\varkappa}_{\alpha_2}, 3\vartheta) &\geq \Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\xi}_{\alpha_i}, \vartheta) \odot \Lambda(\ddot{\xi}_{\alpha_i}, \ddot{\xi}_{\alpha_j}, \vartheta) \odot \Lambda(\ddot{\xi}_{\alpha_i}, \ddot{\varkappa}_{\alpha_1}, \vartheta) \\
&\geq (1 - \varsigma) \odot (1 - \varsigma) \odot \mu \geq 1 - \varsigma_1, \text{for some} \quad 0 < \varsigma_1 < 1 \\
\Omega(\ddot{\xi}_{\alpha_1}, \ddot{\varkappa}_{\alpha_2}, 3\vartheta) &\leq \Omega(\ddot{\xi}_{\alpha_1}, \ddot{\xi}_{\alpha_i}, \vartheta) \oplus \Omega(\ddot{\xi}_{\alpha_i}, \ddot{\xi}_{\alpha_j}, \vartheta) \oplus \Omega(\ddot{\xi}_{\alpha_i}, \ddot{\varkappa}_{\alpha_1}, \vartheta) \\
&\leq \varsigma \oplus \varsigma \oplus \beta \leq \varsigma_2, \text{ for some} \quad 0 < \varsigma_2 < 1 \\
\Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\varkappa}_{\alpha_2}, 3\vartheta) &\leq \Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\xi}_{\alpha_i}, \vartheta) \oplus \Upsilon(\ddot{\xi}_{\alpha_i}, \ddot{\xi}_{\alpha_j}, \vartheta) \oplus \Upsilon(\ddot{\xi}_{\alpha_i}, \ddot{\varkappa}_{\alpha_1}, \vartheta) \\
&\leq \varsigma \oplus \varsigma \oplus \gamma \leq \varsigma_3, \text{for some} \quad 0 < \varsigma_3 < 1.
\end{aligned}$$

Taking $\varsigma = \max\{\varsigma_1, \varsigma_2, \varsigma_3\}$ and $\vartheta_0 = 3\vartheta$ , we have $\Lambda(\ddot{\xi}_{\alpha_1}, \ddot{\varkappa}_{\alpha_2}, \vartheta_0) > 1 - \varsigma, \Omega(\ddot{\xi}_{\alpha_1}, \ddot{\varkappa}_{\alpha_2}, \vartheta_0) < \varsigma$ and $\Upsilon(\ddot{\xi}_{\alpha_1}, \ddot{\varkappa}_{\alpha_2}, \vartheta_0) < \varsigma$, for all $\ddot{\xi}_{\alpha_1}, \ddot{\varkappa}_{\alpha_2} \in A$. Hence A is NSMS is bounded. □

**Theorem 3.8.** *If $(\mathfrak{X}, \Lambda, \Omega, \Upsilon, \odot, \oplus)$ is an NSMS and $\tau_{(\Lambda,\Omega,\Upsilon)}$ is a topology on $\mathfrak{X}$. Then $\ddot{\xi}_{\alpha_n} \to \ddot{\xi}_{\alpha_1}$ iff $\Lambda(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) \to 1, \Omega(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) \to 0$ and $\Upsilon(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) \to 0$ as $n \to \infty$ for $\{\ddot{\xi}_{\alpha_n}\}$ in $\mathfrak{X}$.*

*Proof.* Let $\vartheta > 0$. Consider $\ddot{\xi}_{\alpha_n} \to \ddot{\xi}_{\alpha_1}$. There exist $n_0 \in \mathbb{N}$ such that $\ddot{\xi}_{\alpha_n} \in \breve{B}(\ddot{\xi}_{\alpha_1}, \varsigma, \vartheta)$ for all $n \geq n_0, \varsigma \in (0, 1)$.

Then $1 - \Lambda(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) < \varsigma, \Omega(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) < \varsigma$ and $\Upsilon(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) < \varsigma$.

Hence $\Lambda(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) \to 1, \Omega(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) \to 0$ and $\Upsilon(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) \to 0$ as $n \to \infty$.

Conversely, $\Lambda(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) \to 1, \Omega(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) \to 0$ and $\Upsilon(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) \to 0$, as $n \to \infty$,

Then for $\varsigma \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $1 - \Lambda(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) < \varsigma, \Omega(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) < \varsigma$ and $\Upsilon(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) < \varsigma$, for each $n \geq n_0$.

It follows that $\Lambda(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) > 1 - \varsigma, \Omega(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) < \varsigma$ and $\Upsilon(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) < \varsigma$, for each $n \geq n_0$.

Thus $\ddot{\xi}_{\alpha_n} \in \breve{B}(\ddot{\xi}_{\alpha_1}, \varsigma, \vartheta)$ for each $n \geq n_0$. Hence $\ddot{\xi}_{\alpha_n} \to \ddot{\xi}_{\alpha_1}$. □

**Theorem 3.9.** *If $(\mathfrak{X}, \Lambda, \Omega, \Upsilon, \odot, \oplus)$ is an NSMS and Cauchy sequence in $\mathfrak{X}$ has a convergent sequence. Then $(\mathfrak{X}, \Lambda, \Omega, \Upsilon, \odot, \oplus)$ is a complete NSMS.*

*Proof.* Consider $\{\ddot{\xi}_{\alpha_n}\}$ is a Cauchy sequence and $\{\ddot{\xi}_{\alpha_{n_i}}\}$ is a member of $\{\ddot{\xi}_{\alpha_n}\}$ that converge to $\ddot{\xi}_{\alpha_1}$. We have to prove $\{\ddot{\xi}_{\alpha_n}\} \to \ddot{\xi}_{\alpha_1}$. Let $\vartheta > 0$ and $\hat{\nu} \in (0,1)$. Consider $\varsigma \in (0,1)$ such that $(1-\varsigma) \odot (1-\varsigma) \geq 1-\hat{\nu}$ and $\varsigma \oplus \varsigma \leq \hat{\nu}$. Since $\{\ddot{\xi}_{\alpha_n}\}$ is Cauchy sequence, there is $\alpha_{n_0} \in \mathbb{N}$ such that $\Lambda\left(\ddot{\xi}_{\alpha_m}, \ddot{\xi}_{\alpha_n}, \frac{\vartheta}{2}\right) > 1-\varsigma, \Omega\left(\ddot{\xi}_{\alpha_m}, \ddot{\xi}_{\alpha_n}, \frac{\vartheta}{2}\right) < \varsigma$ and $\Upsilon\left(\ddot{\xi}_{\alpha_m}, \ddot{\xi}_{\alpha_n}, \frac{\vartheta}{2}\right) < \varsigma$, for all $\alpha_m, \alpha_n \geq \alpha_{n_0}$. Since $\ddot{\xi}_{\alpha_{n_i}} \to \ddot{\xi}_{\alpha_1}$, there is positive integer $\alpha_{i_p}$ such that $\alpha_{i_p} > \alpha_{n_0}, \Lambda\left(\ddot{\xi}_{\alpha_{i_p}}), \ddot{\xi}_{\alpha_1}, \frac{\vartheta}{2}\right) > 1-\varsigma, \Omega\left(\ddot{\xi}_{\alpha_{i_p}}), \ddot{\xi}_{\alpha_1}, \frac{\vartheta}{2}\right) < \varsigma$ and $\Upsilon\left(\ddot{\xi}_{\alpha_{i_p}}, \ddot{\xi}_{\alpha_1}, \frac{\vartheta}{2}\right) < \varsigma$.
Thus if $\alpha_n \geq \alpha_{n_0}, \Lambda(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) \geq \Lambda\left(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_{i_p}}, \frac{\vartheta}{2}\right) \odot \Lambda\left(\ddot{\xi}_{\alpha_{i_p}}, \ddot{\xi}_{\alpha_1}, \frac{\vartheta}{2}\right) > (1-\varsigma) \odot (1-\varsigma) \geq 1-\hat{\nu}, \Omega(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) \leq \Omega\left(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_{i_p}}, \frac{\vartheta}{2}\right) \oplus \Omega\left(\ddot{\xi}_{\alpha_{i_p}}, \ddot{\xi}_{\alpha_1}, \frac{\vartheta}{2}\right) < \varsigma \oplus \varsigma \leq \hat{\nu}$ and $\Upsilon(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_1}, \vartheta) \leq \Upsilon\left(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_{i_p}}, \frac{\vartheta}{2}\right) \oplus \Upsilon\left(\ddot{\xi}_{\alpha_{i_p}}, \ddot{\xi}_{\alpha_1}, \frac{\vartheta}{2}\right) < \varsigma \oplus \varsigma \leq \hat{\nu}$
Thus $\ddot{\xi}_{\alpha_{i_p}}) \to \ddot{\xi}_{\alpha_1}$ and hence $(\mathfrak{X}, \Lambda, \Omega, \Upsilon, \odot, \oplus)$ is complete. □

**Theorem 3.10.** *If a sequence $\{\Upsilon n : n \in \mathbb{N}\}$ is dense open members of a complete NSMS $(\mathfrak{X}, \Lambda, \Omega, \Upsilon, \odot, \oplus)$. Then $\bigcap_{n\in\mathbb{N}} \upsilon_n$ is dense in $\mathfrak{X}$.*

*Proof.* Consider $\varrho$ is a nonempty open set in $\mathfrak{X}$. Then we have $\varrho \cap \upsilon_1 \neq \emptyset$. Let $\ddot{\xi}_{\alpha_1} \in \varrho \cap \upsilon_1$ (Since $\upsilon_1$ is dense in $\mathfrak{X}$ )
Then $\breve{B}(\ddot{\xi}_{\alpha_1}, \varsigma_1, \vartheta_1) \subset \varrho \cap \upsilon_1$ (Since $\varrho \cap \upsilon_1$ is open) for $\varsigma_1 \in (0,1)$ and $\vartheta_1 > 0$,
Choose $\varsigma_1^{'} < \varsigma_1$ and $\vartheta_1^{'} = \min\{\vartheta_1, 1\}$ such that $\breve{B}(\ddot{\xi}_{\alpha_1}, \varsigma_1^{'}, \vartheta_1^{'}) \subset \varrho \cap \upsilon_1$
Since $\upsilon_2$ is dense in $\mathfrak{X}, \breve{B}(\ddot{\xi}_{\alpha_1}, \varsigma_1^{'}, \vartheta_1) \subset \varrho \cap \upsilon_2 \neq \emptyset$. Let $\ddot{\xi}_{\alpha_2} \in \breve{B}(\ddot{\xi}_{\alpha_1}, \varsigma_1^{'}, \vartheta_1^{'}) \cap \upsilon_2$.
Then there exist $\varsigma_2 \in (0, \frac{1}{2})$ and $\vartheta_2 > 0$ such that
$\breve{B}(\ddot{\xi}_{\alpha_2}, \varsigma_2, \vartheta_2) \subset \breve{B}(\ddot{\xi}_{\alpha_1}, \varsigma_1^{'}, \vartheta_1^{'}) \cap \upsilon_2$ (Since $\breve{B}(\ddot{\xi}_{\alpha_1}, \varsigma_1^{'}, \vartheta_1^{'}) \cap \upsilon_2$ is open).
Choose $\varsigma_2^{'} < \varsigma_2$ and $\vartheta_2^{'} = \min\left(\vartheta_2, \frac{1}{2}\right)$ such that $\breve{B}(\ddot{\xi}_{\alpha_2}, \varsigma_2^{'}, \vartheta_2^{'}) \subset \breve{B}(\ddot{\xi}_{\alpha_1}, \varsigma_1^{'}, \vartheta_1^{'}) \cap \upsilon_2$
By repeating this procedure, we obtain a sequence $\{\ddot{\xi}_{\alpha_n}\}$ in $\mathfrak{X}$ and a sequence $\{\vartheta_n^{'}\}$ such that $0 < \vartheta_n^{'} < \frac{1}{n}$ and $\breve{B}(\ddot{\xi}_{\alpha_n}, \varsigma_n^{'}, \vartheta_n^{'}) \subset \breve{B}(\ddot{\xi}_{\alpha_{n-1}}), \varsigma_{n-1}^{'}, \vartheta_{n-1}^{'}) \cap \upsilon_n$.
Now, we have to prove $\{\ddot{\xi}_{\alpha_n}\}$ is a Cauchy sequence.
Consider $\alpha_{n_0} \in \mathbb{N}$ such that $\frac{1}{\alpha_{n_0}} < \vartheta$ and $\frac{1}{\alpha_{n_0}} < \hat{\nu}$ for $\vartheta > 0$ and $\hat{\nu} > 0$.
Then $\Lambda(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_m}, \vartheta) \geq \Lambda\left(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_m}, \frac{1}{\alpha_n}\right) \geq 1 - \frac{1}{\alpha_n} > 1 - \hat{\nu}$ for $\alpha_n \geq \alpha_{n_0}$ and $\alpha_m \geq \alpha_n \Omega(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_m}, \vartheta) \leq \Omega\left(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_m}, \frac{1}{\alpha_n}\right) \leq \frac{1}{\alpha_n} < \hat{\nu}, \Upsilon(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_m}, t) \leq \Upsilon\left(\ddot{\xi}_{\alpha_n}, \ddot{\xi}_{\alpha_m}, \frac{1}{\alpha_n}\right) \leq \frac{1}{\alpha_n} < \hat{\nu}$.
There fore $\{\ddot{\xi}_{\alpha_n}\}$ is a Cauchy sequence.
Then there exist $\ddot{\xi}_{\alpha_1} \in \mathfrak{X}$ such that $\ddot{\xi}_{\alpha_n} \to \ddot{\xi}_{\alpha_1}$ (since $\mathfrak{X}$ is complete).
Since $\ddot{\xi}_{\alpha_k} \in \breve{B}(\ddot{\xi}_{\alpha_n}, \varsigma_n^{'}, \vartheta_n^{'})$ for $k \geq n$, we obtain $\ddot{\xi}_{\alpha_1} \in \breve{B}(\ddot{\xi}_{\alpha_n}, \varsigma_n^{'}, \vartheta_n^{'})$.
Hence $\breve{B}(\ddot{\xi}_{\alpha_n}, \varsigma_n^{'}, \vartheta_n^{'}) \subset \breve{B}(\ddot{\xi}_{\alpha_{n-1}}), \varsigma_{n-1}^{'}, \vartheta_{n-1}^{'}) \cap \upsilon_n$ for all $n$.
Therefore $\varrho \cap (\cap_{n\in\mathbb{N}} \upsilon_n) \neq \emptyset$. Thus $\cap_{n\in\mathbb{N}} \upsilon_n$ is dense in $\mathfrak{X}$. □

## 4. CONCLUSION

Neutrosophic soft metric space was introduced in this study, and it is distinct from the fuzzy soft metric spaces defined in [3]. In this novel setting, we analysed a number of topological configurations. To define an NSMS and investigate its characteristics is the focus of this research. Open ball, open set, Hausdorffness, compactness, completeness, and nowhere dense in NSMS are some of the structural characteristic features that have been identified.

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# Some Characterizations of Linguistic Neutrosophic Topological Spaces

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**Abstract**. Many properties of linguistic neutrosophic cl-open spaces and linguistic neutrosophic semi spaces are characterized. Furthermore some conditions are studied which are essential for the existence of certain linguistic neutrosophic spaces. As well the linkage among the spaces are examined and exemplified with precise counter cases. Also, the notion of linguistic neutrosophic quasi-semi component and contra-semi closed graph are introduced and some results are discussed.

**Keywords:** Ultra hausdorff space; cl-open normal space; cl-open regular space; urysohn space; Quasi semi-component space; in Linguistic Neutrosophic Spaces.

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## 1. Introduction

People's needs have changed with the advancement of technology and topology has become inefficient in real-life situations as a result. Separation axioms played a critical role in different kinds of topological spaces that were later discovered. Chang [2] discovered fuzzy topological spaces based on fuzzy sets [12]. Coker [3] developed a hybrid topological space by utilizing intuitionistic fuzzy sets [1]. A new set called neutrosophic set is described by Smarandache [11] by combining indeterminacy membership functions with truth and falsity memberships. Further neutrosophic topological space has been found by Salama and Alblowi [10]. Meanwhile, Gayathri and Helen [6] instigated the notion linguistic neutrosophic topology. The purpose of this article is to examine the inter-linkage between linguistic neutrosophic cl-open spaces. Studies are also conducted on the properties of linguistic neutrosophic semi spaces.

## 2. Preliminaries

**Definition 2.1.** [11] Let S be a space of points (objects), with a generic element in x denoted by S. A neutrosophic set A in S is characterized by a truth-membership function $T_A$, an indeterminacy membership function $I_A$ and a falsity-membership function $F_A$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$. That is
$T_A : S \to ]0^-, 1^+[, I_A : S \to ]0^-, 1^+[, F_A : S \to ]0^-, 1^+[$
There is no restriction on the sum of $T_A(x), I_A(x)$ and $F_A(x)$, so $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

**Definition 2.2.** [11] Let S be a space of points (objects), with a generic element in x denoted by S. A single valued neutrosophic set (SVNS) A in S is characterized by truth-membership function $T_A$, indeterminacy-membership function $I_A$ and falsity-membership function $F_A$. For each point S in S, $T_A(x), I_A(x), F_A(x) \in [0, 1]$.
When S is continuous, a SVNS A can be written as $A = \int \langle T(x), I(x), F(x)\rangle / x \in S$.
When S is discrete, a SVNS A can be written as $A = \sum \langle T(x_i), I(x_i), F(x_i)\rangle / x_i \in S$.

**Definition 2.3.** [9] Let $S = \{s_\theta | \theta = 0, 1, 2, ....., \tau\}$ be a finite and totally ordered discrete term set, where $\tau$ is the even value and $s_\theta$ represents a possible value for a linguistic variable.

**Definition 2.4.** [9] Let $Q = \{s_0, s_1, s_2, ..., s_t\}$ be a linguistic term set (LTS) with odd cardinality $t+1$ and $\overline{Q} = \{s_h / s_0 \leq s_h \leq s_t, h \in [0, t]\}$. Then, a linguistic single valued neutrosophic set A is defined by, $A = \{\langle x, s_\theta(x), s_\psi(x), s_\sigma(x)\rangle | x \in S\}$, where $s_\theta(x), s_\psi(x), s_\sigma(x) \in \overline{Q}$ represent the linguistic truth, linguistic indeterminacy and linguistic falsity degrees of S to A, respectively, with condition $0 \leq \theta + \psi + \sigma \leq 3t$. This triplet $(s_\theta, s_\psi, s_\sigma)$ is called a linguistic single valued neutrosophic number.

**Definition 2.5.** [6] For a linguistic neutrosophic topology $\tau_{LN}$, the collection of linguistic neutrosophic sets should obey,

(1) $0_{LN}, 1_{LN} \in \tau_{LN}$
(2) $K_1 \bigcap K_2 \in \tau_{LN}$ for any $K_1, K_2 \in \tau_{LN}$
(3) $\bigcup K_i \in \tau_{LN}, \forall \{K_i : i \in J\} \subseteq \tau_{LN}$

We call, the pair $(S_{LN}, \tau_{LN})$, a linguistic neutrosophic topological space.

**Definition 2.6.** A topological space $(S_{LN}, \tau_{LN})$ is said to be

(1) LN semi-$T_0$ [7] if for each pair of distinct linguistic neutrosophic points in $S_{LN}$, there exists a LN semi-open set containing one but not the other.
(2) LN semi-$T_1$ [7](resp. LN cl-open-$T_1$ [5]) if for each pair of distinct linguistic neutrosophic points $s_1$ and $s_2$ in $S_{LN}$, there exist LN semi-open(resp. cl-open) sets $E_{LN}$ and $F_{LN}$ containing $s_1$ and $s_2$ such that $s_1 \in E_{LN}, s_2 \notin F_{LN}$ and $s_2 \notin E_{LN}, s_2 \in F_{LN}$.

(3) LN semi-$T_2$ [7](resp. LN ultra-hausdorff [12]) if every two linguistic neutrosophic points can be separated by disjoint LN semi-open(resp. LN cl-open) sets.

(4) LN semi-normal [4](resp. LN cl-open-normal [5]) if for each pair of distinct LN semi-closed(resp. LN cl-open) sets $E_{LN}$ and $F_{LN}$ of $S_{LN}$, there exist two disjoint LN semi-open(resp. LN open) sets $G_{LN}$ and $H_{LN}$ such that $E_{LN} \subset G_{LN}$ and $F_{LN} \subset H_{LN}$.

(5) LN semi-regular [4](resp. cl-open-regular [5]) if for each LN semi-closed(resp. LN cl-open) set $K_{LN}$ of $S_{LN}$ and each $s \notin K_{LN}$, there exist two disjoint LN semi-open(resp. LN open) sets $E_{LN}$ and $F_{LN}$ such that $K_{LN} \subset E_{LN}$ and $s \in F_{LN}$.

(6) The LN quasi-component [9] of $s_1$ is such that the set of all linguistic neutrosophic points $s_2$ in $S_{LN}$ such that $s_1$ and $s_2$ cannot be separated by LN semi-separation of $S_{LN}$.

**Definition 2.7.** [7] A LNS $P_{LN} = \{\langle s_1, T_{P_{LN}}(s_1), I_{P_{LN}}(s_1), F_{P_{LN}}(s_1)\rangle : s_1 \in S_{LN}\}$ is called a linguistic neutrosophic point(LNP in short) if and only if for any element $s_2 \in S_{LN}$,
$\langle T_{P_{LN}}(s_1), I_{P_{LN}}(s_1), F_{P_{LN}}\rangle = \langle l_p, l_q, l_r\rangle$ for $s_2 = s_1$,
$\langle T_{P_{LN}}(s_1), I_{P_{LN}}(s_1), F_{P_{LN}}\rangle = \langle 0, 0, 1\rangle$ for $s_2 \neq s_1$.
where $0 < p \leq t, 0 \leq q < t, 0 \leq r < t$.

**Definition 2.8.** [7] A LNP $P_{LN} = \{\langle s, T_{P_{LN}}(s), I_{P_{LN}}(s), F_{P_{LN}}(s)\rangle : s \in S_{LN}\}$ will be denoted by $s_{\langle l_p, l_q, l_r\rangle}$. The complement of the LNP $s_{\langle l_p, l_q, l_r\rangle}$ will be denoted by ${s^c}_{\langle l_p, l_q, l_r\rangle}$.

**Definition 2.9.** [7] A LNTS $(S_{LN}, \tau)$ is semi-$R_0$ if for every LNSO set $K_{LN}$, $s \in K_{LN}$ implies $LNSCl(\{s\}) \subseteq K_{LN}$.

## 3. Some Characterization of Linguistic Neutrosophic Spaces

**Definition 3.1.** A LNTS $(S_{LN}, \tau_{LN})$ is said to be

(1) LNCOS-$T_1$ if for every pair of distinct points in $S_{LN}$, there exist LNCOSs $E_{LN}$ and $F_{LN}$ containing two points respectively such that $E_{LN} \cap F_{LN} = \phi$.

(2) LN ultra-hausdorff if every two distinct points of $S_{LN}$ can be separated by disjoint LNCOSs.

(3) LNCOS-normal if for each pair of disjoint LNCOS sets $E_{LN}$ and $F_{LN}$ of $S_{LN}$, there exist two disjoint LNOSs $K_{LN}$ and $H_{LN}$ such that $E_{LN} \subset G_{LN}$ and $F_{LN} \subset H_{LN}$.

(4) LNCOS-regular if for each LNCOS $E_{LN}$ of $S_{LN}$ and each $s \notin E_{LN}$, there exist disjoint LNOSs $K_{LN}$ and $H_{LN}$ such that $E_{LN} \subset G_{LN}$ and $s \in H_{LN}$.

(5) LN locally-indiscrete if each LNOS of $S_{LN}$ is LNCS in $S_{LN}$.

**Definition 3.2.** A LNTS $(S_{LN}, \tau_{LN})$ is said to be LNS-regular if for each LNSCS $E_{LN}$ of $S_{LN}$ and each $s \notin E_{LN}$, there exist disjoint LNSOSs $K_{LN}$ and $H_{LN}$ such that $E_{LN} \subset K_{LN}$ and $s \in H_{LN}$.

**Example 3.3.** Let the universe of discourse be $\mathcal{U} = \{x, y, z, w\}$ and let $S_{LN} = \{x, y, z\}$.
The set of all LTS be L= { very poor ($l_0$), poor($l_1$), very weak($l_2$), weak($l_3$), below average ($l_4$), average ($l_5$), above average ($l_6$), good ($l_7$), very good ($l_8$), excellent ($l_9$), outstanding ($l_{10}$)}.
And let $A_{LN} = \{\langle(x, l_4, l_5, l_2), (y, l_3, l_2, l_1), (z, l_9, l_6, l_8)\rangle\}$ and
$B_{LN} = \{\langle(x, l_2, l_4, l_5), (y, l_1, l_1, l_2), (z, l_6, l_5, l_8)\rangle\}$.
Let LNP be $s_{\langle(x,l_2,l_4,l_6),(y,l_1,l_4,l_5),(z,l_3,l_3,l_9)\rangle}$.
The LNSCS and the LNSOSs are given by $E_{LN} = \langle(x, l_1, l_3, l_4),\ (y, l_1, l_6, l_5), (z, l_5, l_8, l_9)$ and
$K_{LN} = \langle(x, l_5, l_6, l_1), (y, l_4, l_6, l_1), (z, l_9, l_8, l_5)\rangle$ ,
$H_{LN} = \langle(x, l_3, l_5, l_4), (y, l_1, l_6, l_2), (z, l_7, l_5, l_8)\rangle$ respectively with $s \notin E_{LN}$.
Now, $s \in H_{LN}$ and $E_{LN} \subset K_{LN}$ and thus $S_{LN}$ is LNS-regular.

**Definition 3.4.** A LNTS $(S_{LN}, \tau_{LN})$ is said to be LNS-normal if for each pair of disjoint LNCSs $A_{LN}$ and $B_{LN}$, there exist two distinct LNSOSs $K_{LN}$ and $H_{LN}$ with $A_{LN} \subseteq K_{LN}$, $B_{LN} \subseteq H_{LN}$.

**Example 3.5.** Let the universe of discourse $\mathcal{U}$ and LTS be as in Example (3.3).
Let the LNCS be $A_{LN} = \langle(x, l_1, l_3, l_4),\ (y, l_1, l_6, l_5),\ (z, l_5, l_8, l_9)\rangle$ ,
$B_{LN} = \langle(x, l_1, l_1, l_5),\ (y, l_0, l_1, l_6),\ (z, l_6, l_5, l_8)\rangle$.
The LNSOSs are given by $K_{LN} = \langle(x, l_5, l_6, l_1), (y, l_4, l_6, l_1), (z, l_9, l_8, l_5)\rangle$
and $H_{LN} = \langle(x, l_3, l_5, l_4), (y, l_1, l_6, l_2), (z, l_7, l_5, l_8)\rangle$.
Now, $A_{LN} \subseteq K_{LN}, B_{LN} \subseteq H_{LN}$ and hence $S_{LN}$ is LNS-normal.

**Definition 3.6.** A LNTS $(S_{LN}, \tau_{LN})$ is said to be LN-urysohn space if there exist two disjoint LNnbds $V_{t_1}$ and $V_{t_2}$, containing $t_1$ and $t_2$ in $(T_{LN}, \eta)$ such that $LNCl(V_{t_1}) \cap LNCl(V_{t_2}) = \phi$.

**Definition 3.7.** Let $(S_{LN}, \tau_{LN})$ be a LNTS and $s \in (S_{LN}, \tau_{LN})$. Then the set of all points $t$ in $(S_{LN}, \tau_{LN})$ such that $s$ and $t$ cannot be separated by LNS separation of $S_{LN}$ is called as the LN quasi-semi-component of $s$.

**Remark 3.8.** A LN quasi semi-component of $s$ in a LNTS $(S_{LN}, \tau_{LN})$ is the intersection of all LNSO sets containing $s$.

**Theorem 3.9.** *If a LNTS is LN semi-regular and LN-$T_0$, then the space is LN semi-$T_2$.*

Proof: Since $S_{LN}$ is LN-$T_0$, there lies a LNO set $U_{LN}$ containing either of the points $s_1$ or $s_2$ in $S_{LN}$. Thus, $S_{LN} \backslash U_{LN}$ is LNC set such that $s_1 \notin S_{LN} \backslash U_{LN}$. Then, there lie disjoint LNSO $E_{LN}$, $F_{LN}$ with $S_{LN} \backslash U_{LN} \subseteq E_{LN}$, $s_1 \in F_{LN}$.

**Remark 3.10.** A LN semi-$T_2$ space need not be LN semi-regular space.

**Example 3.11.** Let the universe of discourse be $U = \{a, b, c\}$. The set of all linguistic term is, $L = \{$ very salt($l_0$), salt($l_1$), very sour($l_2$), sour($l_3$), bitter($l_4$), sweet($l_5$), very sweet($l_6$)$\}$. Let

$S_{LN} = \{a\}$. Let $s_{1\langle a,l_0,l_2,l_6\rangle}$ be a LN point in $S_{LN}$ and let $F_{LN} = \langle a, (l_1, l_0, l_7)\rangle$ be a LNSCS in $S_{LN}$ such that $s_1 \notin F_{LN}$. Also, $A_{LN} = \langle a, (l_1, l_1, l_7)\rangle$ and $B_{LN} = \langle a, (l_2, l_0, l_6)\rangle$ are the LNSOSs in $S_{LN}$. Then, $s_1 \notin A_{LN}$ but $F_{LN} \subseteq B_{LN}$ respectively, which proves that $S_{LN}$ is not LNS-regular.

**Theorem 3.12.** *For each LN set $E_{LN}$ such that $s \notin E_{LN}$, where $s \in S_{LN}$, there lies a LNSO set $U_{LN}$ in $S_{LN}$ containing $s$ with $LNSCl(U_{LN}) \cap E_{LN} = \phi$ if and only if the LN space $S_{LN}$ is semi-regular.*

Proof: Necessity Part: Let $s \in S_{LN}$ be arbitrary and $s \notin E_{LN}$, where $E_{LN}$ is any LNC set in $S_{LN}$. Then, there exists $U_{LN} \in LNSO(S_{LN}, s)$ such that $LNSCl(U_{LN}) \cap E_{LN} = \phi$. Thus, $E_{LN} \subseteq S_{LN} \backslash LNSCl(U_{LN})$.
Sufficiency Part: Let $s \in S_{LN}$ be arbitrary and $s \notin E_{LN}$, where $E_{LN}$ is any LNC set in $S_{LN}$.Then, $S_{LN} \backslash E_{LN}$ is LNO set containing $s$. From the hypothesis, there lies a LNSO set $U_{LN}$ containing $s$ with $LNSCl(U_{LN}) \subseteq S_{LN} \backslash E_{LN}$.

**Theorem 3.13.** *A LN space $S_{LN}$ is semi-regular if and only if for each LNC set $E_{LN}$ and for $s \notin E_{LN}$, there lies a LNSO subsets $C_{LN}$ and $D_{LN}$ in $S_{LN}$ such that $s \in C_{LN}$ and $E_{LN} \subseteq D_{LN}$. Also, $LNCl(C_{LN}) \cap LNCl(D_{LN}) = \phi$.*

Proof: Necessity Part: Let $E_{LN}$ be any LNC set that is not containing the point $s$ in $S_{LN}$. Then, there exists two disjoint LNSO sets $(C_s)_{LN}$ and $F_{LN}$ in $S_{LN}$ such that $E_{LN} \subseteq F_{LN}, s \in C_{LN}$. Then, there lies two LNSO sets $U_{LN}$ and $V_{LN}$ with zero inter section in $S_{LN}$ such that $LNCl(F_{LN}) \subseteq V_{LN}$ and $s \in U_{LN}$, since $LNCl(F_{LN})$ is a LNC subset in $S_{LN}$ that is not containing $s$. Then, $LNCl(U_{LN}) \cap V_{LN} = \phi$. Now, $(C_s)_{LN} \cap U_{LN}$ is a LNSO set different from $F_{LN}$ such that $s \in C_{LN}$ and $E_{LN} \subseteq F_{LN}$.
Sufficiency Part: For any LNC set $E_{LN}$ that is not containing the point $s$ of $S_{LN}$, there exists LNSO subsets $C_{LN}$ and $D_{LN}$ in $S_{LN}$ such that $s \in C_{LN}$ and $E_{LN} \subseteq D_{LN}$. Moreover, $LNCl(C_{LN}) \cap LNCl(D_{LN}) = \phi$.

**Theorem 3.14.** *If $f_{LN} : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ is injective, LN closed and LN irresolute function and $(T_{LN}, \eta_{LN})$ is LN semi-regular space, then $S_{LN}$ is LN semi-regular space.*

Proof: For any $s \in S_{LN}$ and for any LN subset $E_{LN}$, we have $s \notin E_{LN}$. Thus, there lie LNSO sets $U_{LN}$ and $V_{LN}$ with zero intersection in $T_{LN}$ with $f_{LN}(s) \in U_{LN}$ and $f_{LN}(E_{LN}) \in V_{LN}$. For any LN irresolute function, the LNSO sets $(f_{LN})^{-1}(U_{LN})$ and $(f_{LN})^{-1}(V_{LN})$ are disjoint, so that $s \in (f_{LN})^{-1}(U_{LN})$ and $E_{LN} \subseteq (f_{LN})^{-1}(V_{LN})$.

**Theorem 3.15.** *For any LN space $(S_{LN}, \tau_{LN})$ the following are equivalent.*

(a) *the space $(S_{LN}, \tau_{LN})$ is LN semi-regular.*

(b) *For each* $s \in S_{LN}$ *and for each LN* $U_{LN}$ *containing s, there lies a LNSO set* $V_{LN}$ *containing s such that* $LNSCl(V_{LN}) \subseteq U_{LN}$.

(c) *For each non-void set* $A_{LN}$*, different from a LNO set* $U_{LN}$*, there lies a LNSO set* $V_{LN}$ *such that* $A_{LN} \cap V_{LN} \notin \phi$ *and* $LNSCl(V_{LN}) \subseteq U_{LN}$.

(d) *For each non-void set* $A_{LN}$*, different from a LNC set* $E_{LN}$*, there lie two LNSO sets* $U_{LN}$ *and* $V_{LN}$ *such that* $A_{LN} \cap V_{LN} \notin \phi$ *and* $E_{LN} \subseteq U_{LN}$.

Proof: $(a) \Rightarrow (b)$: Let $U_{LN}$ be any LNO set $U_{LN}$ such that $s \in U_{LN}$. Then, $S_{LN} \backslash U_{LN}$ is LNC set that is not containing s. Then, there lie LNSO sets $V_{LN}$ and $W_{LN}$ with zero intersection so that $S_{LN} \backslash U_{LN} \subseteq W_{LN}, s \in V_{LN} \subseteq LNSCl(V_{LN})$. Suppose that $s_1 \in S_{LN}$ such that $s_1 \notin U_{LN}$, then $W_{LN}$ is LNSO set containing $s_1$ such that $V_{LN} \cap W_{LN} = \phi$. Now, $s_1 \in LNSCl(V_{LN})$ and thus $LNSCl(V_{LN}) \subseteq U_{LN}$.
$(b) \Rightarrow (c)$: Let $A_{LN}$ be a non-void LN set which has zero intersection with a LNO subset $U_{LN}$ of $S_{LN}$. Let $s \in A_{LN} \cap U_{LN}$. Then, $U_{LN}$ is a LNSO subset $V_{LN}$ of $S_{LN}$ such that $s \in V_{LN} \subseteq LNSCl(V_{LN}) \subseteq U_{LN}$.
$(c) \Rightarrow (d)$: Let $A_{LN}$ be a non-void LN set which has zero intersection with a LNC subset $E_{LN}$ of $S_{LN}$, then $S_{LN} \backslash E_{LN}$ is a LNC set such that $A_{LN} \cap (S_{LN} \backslash E_{LN}) \neq \phi$. By the assumption, there lies a LNSO set $V_{LN}$ with $A_{LN} \cap V_{LN} \neq \phi$ and $LNSCl(V_{LN}) \subseteq S_{LN} \backslash E_{LN}$.
$(d) \Rightarrow (a)$: Suppose let $E_{LN}$ be any LNC set in $S_{LN}$ and $s \notin S_{LN}$. If $A_{LN} = \{s\}$, then $A_{LN} \cap E_{LN} = \phi$. Then, there lies disjoint LNSO sets $U_{LN}$ and $V_{LN}$ such that $A_{LN} \cap V_{LN} \neq \phi$ and $E_{LN} \subseteq U_{LN}$.

**Theorem 3.16.** *Every LN semi-regular space* $(S_{LN}, \tau_{LN})$ *is LN semi-normal.*

Proof: Let $A_{LN}$ and $B_{LN}$ be any two LNC sets that has void intersection and $s \in A_{LN}$, then $s \notin B_{LN}$. Then, there lie two different LNSO sets $U_s$, $V_s$ with $s \in U_s$, $B_{LN} \subseteq V_s$. Thus, $U_{LN} = \bigcup_{s \in A_{LN}} U_s$ is a LNSO set in $S_{LN}$ such that $A_{LN} \subseteq U_{LN}$. Moreover, $U_{LN} \cap V_s = \phi$, (i.e) $(S_{LN}, \tau_{LN})$ is LN semi normal.

**Remark 3.17.** A LN semi-normal space need not be LN semi-regular in general, which is clear from the following example.

**Example 3.18.** Let the universe of discourse be $\mathcal{U} = \{x, y, z, w\}$ and let $S_{LN} = \{x, y, z\}$. And LTS be as in Example (3.3). The space $S_{LN}$ is LNS-normal, by Example (3.5). The LNSCS $E_{LN}$ is given by, $E_{LN} = \langle (x, l_1, l_2, l_5),\ (y, l_0, l_1, l_6),\ (z, l_6, l_5, l_9) \rangle$ and let the point $s$ be $s_{\langle (x,l_2,l_6,l_2),(y,l_3,l_4,l_2),(z,l_6,l_4,l_7) \rangle}$. Now, for the LNSOS's $K_{LN} = \langle (x, l_5, l_6, l_1), (y, l_4, l_6, l_1), (z, l_9, l_8, l_5) \rangle$, and $H_{LN} = \langle (x, l_3, l_5, l_4), (y, l_1, l_6, l_2), (z, l_7, l_5, l_8) \rangle$, the inclusion relationship $s \notin H_{LN}$ does not hold. Thus the space is not LNS-regular.

**Remark 3.19.** A LN semi-normal space is a LN semi-regular if and only if the LN space is LN semi-$R_0$.

**Example 3.20.** Let the universe of discourse $\mathcal{U}$ and LTS be as in Example (3.3). Let LNP be $s_{\langle(x,l_1,l_0,l_2),\ (y,l_1,l_3,l_3),\ (z,l_4,l_5,l_6)\rangle}$, which is different from the LNSCS $E_{LN} = \langle(x, l_1, l_3, l_4),\ (y, l_1, l_6, l_5),\ (z, l_5, l_8, l_9)\rangle$.
The LNSOSs are given by $K_{LN} = \langle(x, l_5, l_6, l_1),\ (y, l_4, l_6, l_1),\ (z, l_9, l_8, l_5)\rangle$, $H_{LN} = \langle(x, l_3, l_5, l_4),\ (y, l_1, l_6, l_2),\ (z, l_7, l_5, l_8)\rangle$.
Now, $s \notin H_{LN}$ and $E_{LN} \subset K_{LN}$ and thus $S_{LN}$ is LNS-regular.

**Theorem 3.21.** *If $(S_{LN}, \tau_{LN})$ is LN semi-$R_0$ and semi-normal then LN space is LN semi-regular.*

Proof: Let $s \notin K_{LN} \in LNC(S_{LN}, \tau_{LN})$. As the space is LN semi-$R_0$, we have $LNCl(\{s\}) \subseteq S_{LN}\backslash K_{LN}$ and $LNCl(\{s\}) \cap K_{LN} = \phi$. Also, there lie LNSO sets $U_{LN}$ and $V_{LN}$ with $LNCl(\{s\}) \subseteq U_{LN}$, $K_{LN} \subseteq V_{LN}$ with $s \in U_{LN}, K_{LN} \subseteq V_{LN}$ and $U_{LN} \cap V_{LN} = \phi$.

**Theorem 3.22.** *For any two LNC sets $C_{LN}$, $D_{LN}$ of $(S_{LN}, \tau_{LN})$, there lies a LNSO set $U_{LN} \subseteq S_{LN}$ containing $A_{LN}$ and $LNCl(U_{LN}) \cap D_{LN} = \phi$ holds if and only if the space $(S_{LN}, \tau_{LN})$ is LN semi-normal.*

Proof: Necessity Part: Let $S_{LN}$ be a LN semi-normal space and suppose that $C_{LN}$ and $D_{LN}$ be any two disjoint LNC sets in $S_{LN}$, then $C_{LN} \subseteq S_{LN}\backslash D_{LN}$. Then, there lies a LNSO set $U_{LN}$ with $A_{LN} \subseteq U_{LN} \subseteq LNCl(U_{LN}) \subseteq S_{LN}\backslash D_{LN}$.
Sufficiency Part: Suppose $C_{LN}$ and $D_{LN}$ be any two disjoint LNC sets in $S_{LN}$. From the hypothesis, there lies a LNSO set $U_{LN}$ in $S_{LN}$ containing $C_{LN}$ and $LNScl(U_{LN}) \cap D_{LN} = \phi$.

**Theorem 3.23.** *If $f_{LN} : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ is injective, closed and LN irresolute function and $(T_{LN}, \eta_{LN})$ is LN semi-normal space, then $S_{LN}$ is LN semi-normal space.*

Proof: Let $A_{LN}$ and $B_{LN}$ be any two LNC sets in $S_{LN}$ such that $A_{LN} \cap B_{LN} = \phi$. Now, $f_{LN}(A_{LN})$ and $f_{LN}(B_{LN})$ are also LNC in $T_{LN}$. Moreover, $f_{LN}$ is injective, $f(A_{LN})$ and $f(B_{LN})$ are disjoint LNC in $T_{LN}$. Now, there lies a LNSO $U_{LN} \subseteq S_{LN}$ with $f_{LN}(A_{LN}) \subseteq U_{LN}$ and $f_{LN}(B_{LN}) \subseteq V_{LN}$, as the space $T_{LN}$ is LN semi-normal. As the function $f_{LN}$ is LN irresolute, the reverse images $(f_{LN})^{-1}(U_{LN})$ and $(f_{LN})^{-1}(V_{LN})$ are disjoint LNSO in $S_{LN}$ with $A_{LN} \subseteq (f_{LN})^{-1}(U_{LN})$ and $B_{LN} \subseteq (f_{LN})^{-1}(V_{LN})$ respectively.

**Definition 3.24.** A function $f_{LN} : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ is LN semi-totally continuous if the reverse image of every LNSO set is a LNCO subset of $(S_{LN}, \tau_{LN})$.

**Theorem 3.25.** *Let $f_{LN} : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ be LN semi-totally continuous and $f_{LN}$ be injective. Also, the LN space $(T_{LN}, \eta_{LN})$ LN semi-$T_1$, then the LN space $(S_{LN}, \tau_{LN})$ is LNCO-$T_1$.*

Proof: As $f_{LN}$ is injective, $f_{LN}(s_1) \neq f_{LN}(s_2)$, where $f_{LN}(s_1), f_{LN}(s_2) \in T_{LN}$. As $T_{LN}$ is LN semi-$T_1$, there lie LNSOS's $E_{LN}$ and $F_{LN}$ with $f_{LN}(s_1) \in E_{LN}, f_{LN}(s_2) \notin E_{LN}$ and $f_{LN}(s_2) \in F_{LN}, f_{LN}(s_1) \notin F_{LN}$. Thus, $s_1 \in (f_{LN})^{-1}(E_{LN}), s_2 \notin f^{-1}(E_{LN})$ and $s_2 \in (f_{LN})^{-1}(F_{LN}), s_1 \notin (f_{LN})^{-1}(F_{LN})$, where $(f_{LN})^{-1}(E_{LN})$ and $(f_{LN})^{-1}(F_{LN})$ are LNCO subsets of $S_{LN}$.

**Theorem 3.26.** *Let $f_{LN} : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ be LN semi-totally continuous and $f_{LN}$ be injective. Also, the LN space $(T_{LN}, \eta_{LN})$ LN semi-$T_0$, then the LN space $(S_{LN}, \tau_{LN})$ is LN ultra-hausdorff.*

Proof: Let $s_1$ and $s_2$ be any two points in $S_{LN}$. As $f_{LN}$ is injective, $f_{LN}(s_1) \neq f_{LN}(s_2)$, where $f_{LN}(s_1), f_{LN}(s_2) \in T_{LN}$. As $T_{LN}$ is LN semi-$T_0$, there lies a LNSO set $E_{LN}$ containing $f_{LN}(s_1)$ but not $f_{LN}(s_2)$. Then, $s_1 \in (f_{LN})^{-1}(E_{LN})$ and $s_2 \notin (f_{LN})^{-1}(E_{LN})$. As $f_{LN}$ is LN semi-totally continuous, $(f_{LN})^{-1}(E_{LN})$ is LNCOs in $S_{LN}$. Moreover, $s_1 \in (f_{LN})^{-1}(E_{LN})$ and $s_2 \in S_{LN} \backslash (f_{LN})^{-1}(E_{LN})$.

**Theorem 3.27.** *Let $f_{LN} : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ be LN semi-totally continuous and $f_{LN}$ be injective. Also, the LN space $(T_{LN}, \eta_{LN})$ LN semi-$T_2$, then the LN space $(S_{LN}, \tau_{LN})$ is LN ultra-hausdorff.*

Proof: Let $s_1, s_2 \in S_{LN}$ with $s_1 \neq s_2$. As $f_{LN}$ is injective, we have, $f_{LN}(s_1) \neq f_{LN}(s_2)$. In addition, $(T_{LN}, \eta_{LN})$ is LN semi $T_2$, there lie LNSOS's $K_{LN}$ and $H_{LN}$ with $f_{LN}(s_1) \in K_{LN}, f_{LN}(s_2) \in H_{LN}$ and $K_{LN} \cap H_{LN} = \phi$. Then, $s_1 \in (f_{LN})^{-1}(K_{LN})$ and $s_2 \in (f_{LN})^{-1}(HLN)$. As $f_{LN}$ is LN semi-totally continuous, $(f_{LN})^{-1}(K_{LN})$ and $(f_{LN})^{-1}(H_{LN})$ are LNCO in $S_{LN}$. Moreover, $(f_{LN})^{-1}(K_{LN}) \cap (f_{LN})^{-1}(H_{LN}) = (f_{LN})^{-1}(K_{LN} \cap H_{LN}) = \phi$.

**Theorem 3.28.** *Let $f_{LN} : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ be LN semi-totally continuous, injective and LN semi-open mapping from LNCO regular topological space $(S_{LN}, \tau_{LN})$ into a LN space $(T_{LN}, \eta_{LN})$. Then $(T_{LN}, \eta_{LN})$ is LN semi-regular.*

Proof: Let $K_{LN}$ be a LNSC set in $T_{LN}$ and $t \notin K_{LN}$. Since $f$ is LN semi-totally continuous, $(f_{LN})^{-1}(K_{LN})$ is LNCO set in $S_{LN}$. Then, $(f_{LN})^{-1}(t) \notin (f_{LN})^{-1}(K_{LN})$. As $S_{LN}$ is LNCO regular, there lie distinct LNO sets $A_{LN}$ and $B_{LN}$ such that ${f^{-1}}_{LN}(K_{LN}) \subset A_{LN}$ and $(f_{LN})^{-1}(t) \in B_{LN}$. Thus, $K_{LN} \subset f_{LN}(A_{LN})$ and $t \in f_{LN}(B_{LN})$. Also, as the map $f_{LN}$ is LNSO and injective, we have, $f_{LN}(A_{LN})$ and $f_{LN}(B_{LN})$ are LNSO sets and $f_{LN}(A_{LN}) \cap f_{LN}(B_{LN}) = f_{LN}(A_{LN} \cap B_{LN}) = \phi$.

**Theorem 3.29.** *Let $f_{LN} : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ be LN semi-totally continuous, injective and LN semi-closed function. If $T_{LN}$ is LN semi-regular, then $(S_{LN}, \eta_{LN})$ is LN ultra-regular.*

Proof: Let $H_{LN}$ be a LNC set and $s \notin H_{LN}$ in $(S_{LN}, \tau_{LN})$. As $f_{LN}$ is LNSC, $f_{LN}(H_{LN})$ is LNSC set in $T_{LN}$, not containing $f_{LN}(s)$. As $T_{LN}$ is LN semi-regular, there lie distinct LNSOS's

$A_{LN}$, $B_{LN}$ with $f_{LN}(s) \in A_{LN}$, $f_{LN}(H_{LN}) \subset B_{LN}$. Then, we have, $s \in (f_{LN})^{-1}(A_{LN})$ and $H_{LN} \subset (f_{LN})^{-1}(B_{LN})$. Because the function $f_{LN}$ is LN semi-totally continuous, the LN sets $(f_{LN})^{-1}(A_{LN})$ and $(f_{LN})^{-1}(B_{LN})$ are LNCOS's. As $f_{LN}$ is injective, $(f_{LN})^{-1}(A_{LN}) \cap (f_{LN})^{-1}(B_{LN}) = (f_{LN})^{-1}(A_{LN} \cap B_{LN}) = (f_{LN})^{-1}(\phi) = \phi$.

**Theorem 3.30.** *Let $f_{LN} : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ be LN semi-totally continuous, injective and LNSO function from LNCO normal topological space $(S_{LN}, \tau_{LN})$ into a LN space $(T_{LN}, \eta_{LN})$, then $(T_{LN}, \eta_{LN})$ is LN semi-normal.*

Proof: As $f_{LN}$ is LN semi-totally continuous, $(f_{LN})^{-1}(U_{LN})$ and $(f_{LN})^{-1}(V_{LN})$ are LNCOS's in $S_{LN}$, where $U_{LN}$ and $V_{LN}$ be two different LNSC sets in $T_{LN}$. As $f_{LN}$ is injective, $(f_{LN})^{-1}(U_{LN}) \cap (f_{LN})^{-1}(V_{LN}) = (f_{LN})^{-1}(\phi) = \phi$. Then, there lies disjoint LNO's $A_{LN}$ and $B_{LN}$ with $U_{LN} \subset A_{LN}$ and $V_{LN} \subset B_{LN}$, (i.e)$f_{LN}(U_{LN}) \subset f_{LN}(A_{LN})$ and $f_{LN}(V_{LN}) \subset f_{LN}(B_{LN})$. As $f_{LN}$ is injective and LNSO, $f_{LN}(A_{LN})$ and $f_{LN}(B_{LN})$ are disjoint LNSOS's.

**Theorem 3.31.** *Let any collection of LNSO sets be a LNSO set. And let $(T_{LN}, \eta_{LN})$ be a LN urysohn space. If for two different points $s_1$ and $s_2$ of $(S_{LN}, \tau_{LN})$, there exists a LN function $f_{LN} : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ with $f_{LN}(s_1) \neq f_{LN}(s_2)$ and the map $f_{LN}$ is LN contra-semi continuous at $s_1$, $s_2$, ergo $(S_{LN}, \tau_{LN})$ is LN semi-$T_2$ space.*

Proof: Let $(T_{LN}, \eta_{LN})$ be a LN urysohn space and $s_1$ and $s_2$ be two distinct points in $S_{LN}$. Then $f_{LN}(s_1) \neq f_{LN}(s_2)$ in $T_{LN}$. Also, since $(T_{LN}, \eta_{LN})$ is a LN urysohn space, there lie open neighborhoods $V_{f_{LN}(s_1)}$ and $V_{f_{LN}(s_2)}$ in $(T_{LN}, \eta_{LN})$ containing $f_{LN}(s_1)$ and $f_{LN}(s_2)$ such that $LNCl(V_{f_{LN}(s_1)}) \cap LNCl(V_{f_{LN}(s_2)}) = \phi$. There lie LNSOS's $U_{s_1}$, $U_{s_2}$ containing respectively $s_1$, $s_2$ with $f_{LN}(U_{s_1}) \subseteq LNCl(V_{f_{LN}(s_1)})$ and $f_{LN}(U_{s_2}) \subseteq LNCl(V_{f_{LN}(s_2)})$. Then, $f_{LN}(U_{s_1} \cap U_{s_2}) \subseteq f_{LN}(U_{s_1}) \cap f_{LN}(U_{s_2}) \subseteq LNCl(V_{f_{LN}(s_1)}) \cap LNCl(V_{f_{LN}(s_2)}) = \phi$, (i.e) $f_{LN}(U_{s_1} \cap U_{s_2}) = \phi$.

**Theorem 3.32.** *Let the collection of any number of LNSO's be a LNSO set. Then, if the map $f_{LN}$ is LN contra-semi continuous and injective in $(S_{LN}, \tau_{LN})$ and $(T_{LN}, \eta_{LN})$ is a LN ultra-hausdorff space, then $(S_{LN}, \tau_{LN})$ is LN semi-$T_2$ space.*

Proof: For any two points $s_1$ and $s_2$ in $S_{LN}$, we have $f_{LN}(s_1) \neq f_{LN}(s_2)$ as the map $f_{LN}$ is injective. Then, there lie two LNCOS's $E_1$, $E_2$ with $f_{LN}(s_1) \in E_1$, $f_{LN}(s_2) \in E_2$ and $E_1 \cap E_2 = \phi$ and there lie LNOS's $H_1$, $H_2$ with $f_{LN}(H_1) \subseteq E_1$ and $f_{LN}(H_2) \subseteq E_2$ respectively. Then, $H_1 \subseteq (f_{LN})^{-1}(E_1)$ and $H_2 \subseteq (f_{LN})^{-1}(E_2)$,(i.e) $H_1 \cap H_2 \subseteq (f_{LN})^{-1}(E_1) \cap (f_{LN})^{-1}(E_2) = (f_{LN})^{-1}(E_1 \cap E_2) = (f_{LN})^{-1}(\phi)$. Therefore, $H_1 \cap H_2 = \phi$.

**Theorem 3.33.** *Let LN function $f_{LN} : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ be LN contra-semicontinuous, injective and LN closed function. Then the LN space $(S_{LN}, \tau_{LN})$ is LN semi-normal if $(T_{LN}, \eta_{LN})$ is LN-ultra normal.*

Proof: Let $B_1$ and $B_2$ be two different LNC sets in $(S_{LN}, \tau_{LN})$, then $f_{LN}(B_1)$ and $f_{LN}(B_2)$ are different LNC sets in $(T_{LN}, \eta_{LN})$, as the mapping $f_{LN}$ is injective and LN closed. There lie two LNCOS's $U_1$, $U_2$ which separates $f_{LN}(B_1)$ and $f_{LN}(B_2)$ in $T_{LN}$ respectively. Thus, $f_{LN}(B_1) \subseteq U_1$ and $f_{LN}(B_2) \subseteq U_2$,(i.e)$B_1 \subseteq (f_{LN})^{-1}(U_1)$ and $B_2 \subseteq (f_{LN})^{-1}(U_2)$ such that $(f_{LN})^{-1}(U_1) \cap (f_{LN})^{-1}(U_2) = \phi$. It is shown that $(f_{LN})^{-1}(U_1)$ and $(f_{LN})^{-1}(U_2)$ are two different LNSO with $B_1 \subseteq (f_{LN})^{-1}(U_1)$ and $B_2 \subseteq (f_{LN})^{-1}(U_2)$ in $S_{LN}$.

**Definition 3.34.** A LN graph $LNGR(f_{LN})$ of a function $f_{LN} : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ is LN contra-semi closed graph if a LNSO set $E_{LN}$ and a LNC set $K_{LN}$ lie with the property $(E_{LN} \cap K_{LN}) \cap LNGR(f_{LN}) = \phi$ for all $(s,t) \in (S_{LN} \times T_{LN})$ $LNGR(f_{LN})$.

**Theorem 3.35.** *Let any union of LNSO sets be a LNSO set and* $f_{LN} : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ *be a function and* $g_{LN} : (S_{LN}, \tau_{LN}) \to (S_{LN}, \tau_{LN}) \times (T_{LN}, \eta_{LN})$ *be the LN graph function given by* $g(s) = (s, f_{LN}(s))$ *for each* $s \in S_{LN}$. *Then* $g_{LN}$ *is LN contra-semi continuous if and only if the map* $f_{LN}$ *is LN contra-semi continuous.*

Proof: Necessity Part: Let $V_{LN}$ be any LNC set of $(S_{LN}, \tau_{LN}) \times (T_{LN}, \eta_{LN})$ containing $g_{LN}(s)$ where $s \in S_{LN}$. Then $V_{LN} \cap (\{s\} \times \{T_{LN}\})$ containing $g_{LN}(s)$. Moreover $\{s\} \times \{T_{LN}\}$ is homeomorphic to $T_{LN}$ and hence $\{t : (s,t) \in V_{LN}\}$ is a LNC subset of $T_{LN}$. As $f_{LN}$ is LN contra-semi continuous, $\bigcup\{(f_{LN})^{-1}(t) : (s,t) \in V_{LN}\}$ is LNSO in $S_{LN}$ such that $s \in \bigcup\{(f_{LN})^{-1}(t) : (s,t) \in V_{LN}\} \subseteq (g_{LN})^{-1}(V_{LN})$. Thus, $(g_{LN})^{-1}(V_{LN})$ is LNSO.
Sufficiency Part: Let $U_{LN}$ be any LNC subset of $T_{LN}$. Then, $S_{LN} \times U_{LN} = S_{LN} \times LNCl(U_{LN}) = LNCl(S_{LN} \times U_{LN}) \subseteq S_{LN} \times T_{LN}$ and $S_{LN} \times U_{LN}$ is LNC. Then, $(g_{LN})^{-1}(S_{LN} \times U_{LN})$ is LNSO in $S_{LN}$ as LN contra- semicontinuous. Moreover, $(g_{LN})^{-1}(S_{LN} \times U_{LN}) = (f_{LN})^{-1}(U_{LN})$.

**Lemma 3.36.** *A LN graph* $LNGR(f_{LN})$ *of a function* $f_{LN} : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ *is LN contra-semi closed graph if there exist a LNSO set* $E_{LN}$ *and a LN closed set* $K_{LN}$ *such that* $f_{LN}(E_{LN}) \cap K_{LN} = \phi$.

**Theorem 3.37.** *Let* $f_{LN} : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ *be an injective function and LN contra-semi closed graph, then* $S_{LN}$ *is LN semi-*$T_1$ *space.*

Proof: Since $f_{LN}$ is LN contra-semi closed graph, $(s, f(t)) \in (S_{LN} \times T_{LN})$ $LNGR(f_{LN})$, where $s$ and $t$ are different points of $S_{LN}$. Then by lemma, (3.25), a LNSO set $U_{LN}$ lies in $S_{LN}$ that contain $s$ and a LNC set $V_{LN}$ lies in $T_{LN}$ that contain $f_{LN}(t)$ with $f_{LN}(U_{LN}) \cap V_{LN} = \phi$. Ergo, $S_{LN}$ is LN semi-$T_1$ space, as $t \notin U_{LN}$.

**Theorem 3.38.** *Let* $f_{LN} : (S_{LN}, \tau_{LN}) \to (T_{LN}, \eta_{LN})$ *be LN contra-semi continuous where* $(T_{LN}, \eta_{LN})$ *is LN-urysohn space, then the graph of* $f_{LN}$ *is LN contra-semi closed in* $(S_{LN} \times T_{LN})$.

Proof: Let $(s,t) \in (S_{LN} \times T_{LN})\ LNGR(f_{LN})$, then $f_{LN}(s) \neq t$. Now, there lie LNO sets $E_{LN}$ and $F_{LN}$ in $T_{LN}$ such that $f_{LN}(s) \in E_{LN}$ and $t \in F_{LN}$ such that $LNCl(E_{LN}) \cap LNCl(F_{LN}) = \phi$. Now, there exists a LNSO set $K_{LN} \in (S_{LN}, s)$ such that $f_{LN}(K_{LN}) \subseteq LNCl(E_{LN})$, as the function $f_{LN}$ is LN contra-semi continuous. Thus, $f_{LN}(E_{LN}) \cap LNCl(F_{LN}) \subseteq LNCl(E_{LN}) \cap LNCl(F_{LN}) = \phi$. Then, $LNGR(f_{LN})$ is LN contra-semi closed, by lemma (3.25).

## Conclusion:

In this study, the characterization of linguistic neutrosophic spaces and cl-open spaces are discussed. The inter connections among these also have studied. Appropriate examples are given to explicate the results and connections. We hope that these inception works will be useful for scholars to progress the research in linguistic neutrosophic topology.

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# Interpretation of Neutrosophic Soft cubic T-ideal in the Environment of PS-Algebra

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**Abstract**. This study provides an innovative approach to neutrosophic algebraic structures by introducing a new structure called Neutrosophic Soft Cubic T-ideal (NSCTID), which combines T-ideal (TID) and Neutrosophic Soft Cubic Sets (NSCSs) within the framework of PS-Algebra. Within the already-existing neutrosophic cubic structures, the addition of soft sets with the characteristics of TID makes this structure more desirable. The theoretical development of the proposed structure includes the application of fundamental ideas as union, intersection, the Cartesian product, and homomorphism. We also introduce the notions of NSCTID-translation and NSCTID-multiplication to further enhance the structure of NSCTID. By applying the idea of translation and multiplication, we offer improved algorithm for neutrosophic cubic sets to deal with different parameters that are satisying the TID's distinctive characteristics. Through this thorough research, we offer an elementary understanding of NSCTID and its capabilities, providing the way to new algebraic structures.

**Keywords:** Neutrosophic soft cubic set; T-ideal; PS-algebra; Cartesian product; Homomorphism; Translation and Multiplication.

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## 1. Introduction

Zadeh was the first who put up the notion of Fuzzy Sets (FSs) in 1965, which contained a membership degree for each element say "t" [1,2]. The Intuitionistic FSs (IFSs) was established by Atanassov [3], which is a general form of FS on a universe U in which non-membership degree was taken into consideration and presented Interval-Valued IFSs which are undoubtedly both

IVFS and IFS extensions. Rahman et al. [4] employed a fresh approach of a refined intuitionistic fuzzy set to conceptualize its fundamental characteristics through set theoretic procedures such as extended union as well as intersection and same for restricted. The Neutrosophic Sets (NSs) was developed by Smarandache [5] by proposing the concept of the indeterminate degree of an element as an independent element in his 1995 manuscript, which was later published in 1998. The Interval -Valued NSs was first given by Wang et al. [6]. A new methodology for simulating fuzziness and uncertainty was established by Molodtsov in 1999 [7], which is called "soft set theory". Saeed et al. [8] conducted an exhaustive investigation of the idea of soft elements as well as soft members in the respect of soft sets (SSs). Maji et al. [9, 10] expanded SSs to IFSSs and NSSs. Smarandache [11] generalized the soft set to the hypersoft set by transforming the function F into a multi-attribute function and then introduced the hybrids of Crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic Hypersoft Set. In [12], Distance and similarity measures using Max-Min operators were proposed in the environment of neutrosophic hypersoft sets with application in MCDM. The concept of cubic sets considers only the intervals of membership and ignores the segments of non-membership information. However, expressing the degree of membership in fuzzy sets with exact values can be difficult in real-world situations. In such cases, it may be simple to represent ambiguity both interval and exact values instead of just one. Therefore, Jun et al. [13] proposed the cubic ISs which combines two sets to represent membership degrees: an interval-valued IFS and an IFS. This hybrid approach can be useful in decision-making when dealing with uncertain judgments. A theoretical development of cubic pythagorean fuzzy soft set was established with its application in MADM. Based on the established TOPSIS method and aggregation operators, the decision-making algorithm is proposed under an intuitionistic fuzzy hypersoft environment to resolve uncertain and confusing information [14, 15]. By combining the NS and IVNS, Ali et al. [16] proposed the definition of NCS. In [17], Mumtaz et al. made an adjustable approach to NCS based decision making by similarity measure and is employed in pattern recognition to show that they can be successfully applied to problems that contain uncertainties. The NCSS was first introduced and some of its characteristics were proved by Chinnadurai et al. [18] in 2016. Gulistan et al. [19] developed a more general approach of neutrosophic cubic soft matrices and used it in MCDM-problems. In 1978, Iseki et al. were the first who established the idea of BCK-algebra [20]. In 1980, BCI-algebras were proposed [21]. The algebra named as BCK is a subclass of BCI-algebras. PS-algebras is a generalization of algebras like BCI, BCK, Q and KU and was first explored in [22] by Priya et al.. Shah et al. investigated images as well as pre images of anti-homomorphism for semiprime, strongly prime, irreducible, and highly irreducible fuzzy ideals in rings [23, 24]. They also established the ideas of strongly primary fuzzy ideals and strongly irreducible fuzzy ideals in ring with unity as well as commutativity and examined their importance in a Leskerian ring. The research conducted by Senapati et al. [25] examined the cubic intuitionistic q-ideals within BCI-algebras. Kandasamy et al. [26] were the first who introduced T- ideals, IFT-ideals, and IF closed T-ideals. Priya et al. [27] investigated the concept of fuzzy translation and fuzzy multiplication in the context of PS-algebra, In [28, 29], Khalid et al. established the MBJ-neutrosophic T-ideal in B-algebra and described several of its properties. The concept of NC translation and multiplication of BF-subalgebra and BF-ideal was also introduced. In [30], they presented the ideas of IF Alpha-Translation as well as multiplication and IF Magnified Beta-Alpha-Translation in the context of PS-algebra. This paper introduces the Neutrosophic Soft Cubic Set (NSCS) and its application in PS-algebra through the concept of T-ideal. Section two provides relevant definitions, while section three presents the concept of NSCTID in PS algebra. Section four examines NSCTID's fundamental properties, followed by an exploration of its translation and multiplication in section five. Finally, section six summarizes the results and suggests future directions for research.

## 2. **Preliminaries**

To provide a clear understanding of the proposed work, some essential definitions that will be utilized throughout the explanation are given.

**Definition 2.1.** [22]:Let P be a set with the constant $'0'$ and the binary operation $'*'$ which is nonempty. If P satisfies the axioms below, it is regarded to be PS-algebra.

i. $\vartheta_1 * \vartheta_1 = 0$
ii. $\vartheta_1 * 0 = 0$
iii. $\vartheta_1 * \vartheta_2 = 0$, and $\vartheta_2 * \vartheta_1 = 0$ implies $\vartheta_1 = \vartheta_2$ , for all $\vartheta_1, \vartheta_2 \in$ P.

In any PS-Algebra $(\mathrm{P}, *, 0)$, the following characteristics also satisfy for every $\vartheta_1, \vartheta_2 \in$ P.

iv. $\vartheta_1 * (\vartheta_2 * \vartheta_1) = \vartheta_2 * (\vartheta_1 * \vartheta_1)$
v. $\vartheta_2 * (\vartheta_1 * (\vartheta_2 * \vartheta_1)) = 0$
vi. $\vartheta_1 * (\vartheta_1 * (\vartheta_1 * \vartheta_2)) = \vartheta_1 * \vartheta_2$
vii. $\vartheta_2 * (\vartheta_1 * (\vartheta_1 * \vartheta_2)) = 0$

**Definition 2.2.** [26]:Let I be a nonempty subset of P. I be an ideal of P if:

i. $0 \in$ I,
ii. $\vartheta_1 * \vartheta_2 \in$ I and $\vartheta_2 \in$ I implies $\vartheta_1 \in$ I,

A nonempty subset I be the T-ideal if it satisfies (i) with

iii. $(\vartheta_1 * \vartheta_2) * \vartheta_3 \in$ I and $\vartheta_2 \in$ I implies $\vartheta_1 * \vartheta_3 \in$ I for all $\vartheta_1, \vartheta_2, \vartheta_3 \in$ P.

**Definition 2.3.** [26]:An IFS is said to be an IFT-ideal of P if it these three conditions holds:

i. $\alpha_{\mathrm{A}}(0) \geq \alpha_{\mathrm{A}}(\vartheta_1), \beta_{\mathrm{A}}(0) \leq \beta_{\mathrm{A}}(\vartheta_1)$,

ii. $\alpha_{\mathrm{A}}(\vartheta_1 * \vartheta_3) \geq \min\{\alpha_{\mathrm{A}}((\vartheta_1 * \vartheta_2) * \vartheta_3), \alpha_{\mathrm{A}}(\vartheta_2)\}$,

iii. $\beta_{\mathrm{A}}(\vartheta_1 * \vartheta_3) \leq \max\{\beta_{\mathrm{A}}((\vartheta_1 * \vartheta_2) * \vartheta_3), \beta_A(\vartheta_2)\}$, for all $\vartheta_1, \vartheta_2 \in \mathrm{P}$.

**Definition 2.4.** [5]:A set in P represented by

$$\mathrm{A} = \{\langle \vartheta_1, \alpha_{\mathrm{A}}(\vartheta_1), \gamma_{\mathrm{A}}(\vartheta_1), \beta_{\mathrm{A}}(\vartheta_1)\rangle \mid \vartheta_1 \in \mathrm{P}\}$$

is called NS in which the mapping $\alpha_{\mathrm{A}}(\vartheta_1) : \mathrm{P} \to ]0^-, 1^+[ \gamma_{\mathrm{A}}(\vartheta_1) : \mathrm{P} \to ]0^-, 1^+[$ and $\beta_{\mathrm{A}}(\vartheta_1) : \mathrm{P} \to ]0^-, 1^+[$ denotes the functions of truth, indeterminate, and falsity membership respectively, and satisfy

$$0^- \leq \alpha_{\mathrm{A}}(\vartheta_1) + \gamma_{\mathrm{A}}(\vartheta_1) + \beta_{\mathrm{A}}(\vartheta_1) \leq 3^+.$$

For an interval-valued NS, $\widehat{\mathrm{A}}$ in P, the mappings $\widehat{\alpha}_{\mathrm{A}}(\vartheta_1), \widehat{\gamma}_{\mathrm{A}}(\vartheta_1), \widehat{\beta}_{\mathrm{A}}(\vartheta_1) \subset [0,1]$ with $\widehat{\alpha}_{\mathrm{A}}(\vartheta_1) = [\widehat{\alpha}_{\mathrm{A_L}}(\vartheta_1), \widehat{\alpha}_{\mathrm{A_U}}(\vartheta_1)], \hat{\beta}_{\mathrm{A}}(\vartheta_1) = [\hat{\beta}_{\mathrm{A_L}}(\vartheta_1), \hat{\beta}_{\mathrm{A_U}}(\vartheta_1)]$ and $\hat{\gamma}_{\mathrm{A}}(\vartheta_1) = [\hat{\gamma}_{\mathrm{A_L}}(\vartheta_1), \hat{\gamma}_{\mathrm{A_U}}(\vartheta_1)]$.

**Definition 2.5.** [16]:A NCS in P is defined as $\mathrm{C} = \langle \widehat{\mathrm{A}}, \mathrm{A}\rangle$ in which $\widehat{\mathrm{A}}$ is an interval-valued NS and A is the NS. The collection of all NCSs in P is represented by C(P).

**Definition 2.6.** [7]:Let E be the collection of parameters and V be the universal set. The set $\mathrm{S_K}$ which is defined as $\hat{\mathrm{S}}_{\mathrm{K}} = \{\langle \mathrm{v}, \mathrm{S(e)}\rangle, \mathrm{v} \in \mathrm{V}, \mathrm{e} \in \mathrm{K}, \mathrm{S(e)} \in \mathrm{P(V)}\}$ is said to be a soft set over V. Where $\mathrm{S} : \mathrm{E} \to \mathrm{P(V)}$ in which the P(V) represents the power set of V and $\mathrm{K} \subset \mathrm{E}$.

**Definition 2.7.** [18]:A SS denoted by $\widehat{\mathrm{S}}_{\mathrm{K}}$ is said to be a NCSS in P if $\widehat{\mathrm{S}}_{\mathrm{K}}$ is the mapping from E to the set C(P). i.e. $\widehat{\mathrm{S}}_{\mathrm{K}} : \mathrm{E} \to \mathrm{C(P)}$ where $\mathrm{K} \subset \mathrm{E}$. The NCSS is denoted by $\widehat{\mathrm{S}}_{\mathrm{K}} = \langle \widehat{\mathrm{B}}, \mathrm{B}\rangle$. Where $\widehat{\mathrm{B}} = \{\langle \vartheta_1, \alpha_{\widehat{\mathrm{B}}_{\mathrm{e_i}}}(\vartheta_1), \gamma_{\widehat{\mathrm{B}}_{\mathrm{e_i}}}(\vartheta_1), \beta_{\widehat{\mathrm{B}}_{\mathrm{e_i}}}(\vartheta_1)\rangle \mid \vartheta_1 \in \mathrm{P}, \mathrm{e_i} \in \mathrm{E}\}$ is the interval-valued NSS and $\mathrm{B} = \{\langle \vartheta_1, \alpha_{\mathrm{B_{e_i}}}(\vartheta_1), \gamma_{\mathrm{B_{e_i}}}(\vartheta_1), \beta_{\mathrm{B_i}}(\vartheta_1)\rangle \mid \vartheta_1 \in \mathrm{P}, \mathrm{e_i} \in \mathrm{E}\}$ is the NSS.

**Definition 2.8.** [29]:Let $\mathrm{C} = \langle \widehat{\mathrm{A}}, \mathrm{A}\rangle$ be an NCS of P and for the set $\widehat{\mathrm{A}}$, the $\mu, v \in [[0,0], \Theta]$ and $\lambda \in [[0,0], \mathrm{I}]$, where for the set $\mathrm{A}, \mu, v \in [0, \Gamma]$ and $\lambda \in [0, \epsilon]$. An object of the form $\mathrm{C}^{\mathrm{T}}_{\mu,v,\lambda} = \langle\ (\widehat{\mathrm{A}})^{\mathrm{T}}_{\mu,v,\lambda}, (\mathrm{A})^{\mathrm{T}}_{\mu,v,\lambda}\rangle$ is called an NC-Translation of C, when

$$(\widehat{\mathrm{A}}_{\mathrm{T}})^{\mathrm{Tr}}_{\mu}(\mathrm{t}_1) = \widehat{\mathrm{A}}_{\mathrm{T}}(\mathrm{t}_1) + \mu, (\widehat{\mathrm{A}}_{\mathrm{I}})^{\mathrm{Tr}}_{v}(\mathrm{t}_1) = \widehat{\mathrm{A}}_{\mathrm{I}}(\mathrm{t}_1) + v, (\widehat{\mathrm{A}}_{\mathrm{F}})^{\mathrm{Tr}}_{\lambda}(\mathrm{t}_1) = \widehat{\mathrm{A}}_{\mathrm{F}}(\mathrm{t}_1) - \lambda$$

$$(\mathrm{A}_{\mathrm{T}})^{\mathrm{Tr}}_{\mu}(\mathrm{t}_1) = \mathrm{A}_{\mathrm{T}}(\mathrm{t}_1) + \mu, (\mathrm{A}_{\mathrm{I}})^{\mathrm{Tr}}_{v}(\mathrm{t}_1) = \mathrm{A}_{\mathrm{I}}(\mathrm{t}_1) + v, (\mathrm{A}_{\mathrm{F}})^{\mathrm{Tr}}_{\lambda}(\mathrm{t}_1) = \mathrm{A}_{\mathrm{F}}(\mathrm{t}_1) - \lambda$$

for all $\mathrm{t}_1 \in \mathrm{P}$.

**Definition 2.9.** [29]:Let $\mathrm{C} = \langle \widehat{\mathrm{A}}, \mathrm{A}\rangle$ be an NCS of P and $\sigma \in [0,1]$. A set having the representation as $\mathrm{C}^{\mathrm{Mp}}_{\sigma} = \langle((\kappa_{\mathrm{T}})^{\mathrm{Mp}}_{\sigma}, (\kappa_{\mathrm{I}})^{\mathrm{Mp}}_{\sigma}, (\kappa_{\mathrm{F}})^{\mathrm{Mp}}_{\sigma}), ((\upsilon_{\mathrm{T}})^{\mathrm{Mp}}_{\sigma}, (\upsilon_{\mathrm{I}})^{\mathrm{Mp}}_{\sigma}, (\upsilon_{\mathrm{F}})^{\mathrm{Mp}}_{\sigma})\rangle$ is called an NC-Multiplication of C, when

$$(\kappa_{\mathrm{T}})^{\mathrm{Mp}}_{\delta}(\mathrm{t}_1) = \delta \cdot \kappa_{\mathrm{T}}(\mathrm{t}_1), (\kappa_{\mathrm{I}})^{\mathrm{Mp}}_{\delta}(\mathrm{t}_1) = \delta \cdot \kappa_{\mathrm{I}}(\mathrm{t}_1), (\kappa_{\mathrm{F}})^{\mathrm{Mp}}_{\delta}(\mathrm{t}_1) = \delta \cdot \kappa_{\mathrm{F}}(\mathrm{t}_1)$$

$$(\upsilon_{\mathrm{T}})^{\mathrm{Mp}}_{\delta}(\mathrm{t}_1) = \delta \cdot \upsilon_{\mathrm{T}}(\mathrm{t}_1), (\upsilon_{\mathrm{I}})^{\mathrm{Mp}}_{\delta}(\mathrm{t}_1) = \delta \cdot \upsilon_{\mathrm{I}}(\mathrm{t}_1), (\upsilon_{\mathrm{F}})^{\mathrm{Mp}}_{\delta}(\mathrm{t}_1) = \delta \cdot \upsilon_{\mathrm{F}}(\mathrm{t}_1)$$

for all $\mathrm{t}_1 \in \mathrm{P}$.

## 3. Neutrosophic soft cubic T-ideal

This section aims to present the notion of a NSCTID, accompanied by an illustrative example. Additionally, we explore several properties that are pertinent to this concept.

**Definition 3.1.** Let an NSC-set which is denoted as $K = \langle (\widehat{N}_K)_{e_i}^{T,I,F}, (A_K)_{e_i}^{T,F} \rangle$, where $\widehat{N}_{K_{e_i}}$ and $A_{K_{e_i}}$ represents an interval-valued NSS and NSS respectively in PS-algebra P. The set K is termed as Neutrosophic soft cubic T-ideal (NSCTID) in P if it achieves the following assertions:

**i.** $(\widehat{N}_K)_{e_i}^T(0) \geq (\widehat{N}_K)_{e_i}^T(t_1), (\widehat{N}_K)_{e_i}^I(0) \geq (\widehat{N}_K)_{e_i}^I(t_1), (\widehat{N}_K)_{e_i}^F(0) \geq (\widehat{N}_K)_{e_i}^F(t_1),$
$(A_K)_{e_i}^T(0) \leq (A_K)_{e_i}^T(t_1), (A_K)_{e_i}^I(0) \leq (A_K)_{e_i}^I(t_1), (A_K)_{e_i}^F(0) \leq (A_K)_{e_i}^F(t_1).$

**ii.** $(\widehat{N}_K)_{e_i}^T(t_1 * t_3) \geq \text{rmin}\{(\widehat{N}_K)_{e_i}^T((t_1 * t_2) * t_3), (\widehat{N}_K)_{e_i}^T(t_2)\},$
$(\widehat{N}_K)_{e_i}^I(t_1 * t_3) \geq \text{rmin}\{(\widehat{N}_K)_{e_i}^I((t_1 * t_2) * t_3), (\widehat{N}_K)_{e_i}^I(t_2)\},$
$(\widehat{N}_K)_{e_i}^F(t_1 * t_3) \geq \text{rmin}\{(\widehat{N}_K)_{e_i}^F((t_1 * t_2) * t_3), (\widehat{N}_K)_{e_i}^F(t_2)\},$

**iii.** $(A_K)_{e_i}^T(t_1 * t_3) \leq \max\{(A_K)_{e_i}^T((t_1 * t_2) * t_3), (A_K)_{e_i}^T(t_2)\}$
$(A_K)_{e_i}^I(t_1 * t_3) \leq \max\{(A_K)_{e_i}^I((t_1 * t_2) * t_3), (A_K)_{e_i}^I(t_2)\},$
$(A_K)_{e_i}^F(t_1 * t_3) \leq \max\{(A_K)_{e_i}^F((t_1 * t_2) * t_3), (A_K)_{e_i}^F(t_2)\},$

For all $t_1, t_2, t_3 \in P$. Where $(\widehat{N}_K)_{e_i}^{T,I,F}$ and $(A_K)_{e_i}^{T,I,F} \in [0, 1]$.

**Example 3.2.** Let $P = \{0, t_1, t_2, t_3\}$ be a PS-algebra with the following Cayley table. We

Table 1. Cayley table of $(P, *, 0)$.

| $*$ | 0 | $t_1$ | $t_2$ | $t_3$ |
|---|---|---|---|---|
| 0 | 0 | $t_2$ | $t_1$ | $t_3$ |
| $t_1$ | 0 | 0 | 0 | $t_2$ |
| $t_2$ | 0 | 0 | 0 | $t_2$ |
| $t_3$ | 0 | $t_2$ | $t_2$ | 0 |

define a NSCS represented as $K = \langle \widehat{N}_{e_i}, A_{e_i} \rangle$ in P as in Table **2** and Table **3**. The set K with the aforementioned values satisfies all of the requirements of the definition **3.1** above.

The calculations below show a few outcomes.

$[0.5, 0.7] = (\widehat{N}_K)_{e_i}^T(t_1 * t_3) = (\widehat{N}_K)_{e_i}^T(t_2) \geq \text{rmin}\{(\widehat{N}_K)_{e_i}^T(t_3), (\widehat{N}_K)_{e_i}^T(t_2)\} = \text{rmin}\{[0.4, 0.6], [0.5, 0.7]\},$

$[0.4, 0.6] = (\widehat{N}_K)_{e_i}^I(t_1 * t_3) = (\widehat{N}_K)_{e_i}^I(t_2) \geq \text{rmin}\{(\widehat{N}_K)_{e_i}^I(t_3), (\widehat{N}_K)_{e_i}^I(t_2)\} = \text{rmin}\{[0.3, 0.5], [0.4, 0.6]\},$

$[0.4, 0.5] = (\widehat{N}_K)_{e_i}^F(t_1 * t_3) = (\widehat{N}_K)_{e_i}^F(t_2) \geq \text{rmin}\{(\widehat{N}_K)_{e_i}^F(t_3), (\widehat{N}_K)_{e_i}^F(t_2)\} = \text{rmin}\{[0.2, 0.4], [0.4, 0.5]\},$

TABLE 2. Interval-valued NSS ($\widehat{N}_{e_i}$).

| $*$ | 0 | $t_1$ | $t_2$ | $t_3$ |
|---|---|---|---|---|
| $(\widehat{N}_K)^T_{e_i}$ | $[0.6, 0.8]$ | $[0.3, 0.6]$ | $[0.5, 0.7]$ | $[0.4, 0.6]$ |
| $(\widehat{N}_K)^I_{e_i}$ | $[0.5, 0.7]$ | $[0.3, 0.4]$ | $[0.4, 0.6]$ | $[0.3, 0.5]$ |
| $(\widehat{N}_K)^F_{e_i}$ | $[0.4, 0.9]$ | $[0.3, 0.6]$ | $[0.4, 0.5]$ | $[0.2, 0.4]$ |

TABLE 3. NSS ($A_{e_i}$).

| $*$ | 0 | $t_1$ | $t_2$ | $t_3$ |
|---|---|---|---|---|
| $(A_K)^T_{e_i}$ | 0.2 | 0.3 | 0.5 | 0.6 |
| $(A_K)^I_{e_i}$ | 0.5 | 0.6 | 0.8 | 1 |
| $(A_K)^F_{e_i}$ | 0.3 | 0.7 | 0.4 | 0.5 |

$$0.5 = (A_K)^T_{e_i}(t_1 * t_3) = (A_K)^T_{e_i}(t_2) \leq \max\{(A_K)^T_{e_i}(t_3), (A_K)^T_{e_i}(t_2)\} = 0.6,$$
$$0.8 = (A_K)^T_{e_i}(t_1 * t_3) = (A_K)^T_{e_i}(t_2) \leq \max\{(A_K)^T_{e_i}(t_3), (A_K)^T_{e_i}(t_2)\} = 1,$$
$$0.4 = (A_K)^T_{e_i}(t_1 * t_3) = (A_K)^T_{e_i}(t_2) \leq \max\{(A_K)^T_{e_i}(t_3), (A_K)^T_{e_i}(t_2)\} = 0.5$$

Hence $K = \langle(\widehat{N}_K)^{T,I,F}_{e_i}, A_K)^{T,I,F}_{e_i}\rangle$ is NSCTID in P.

**Theorem 3.3.** *Every NSCTID* $K = \langle(\widehat{N}_K)^{T,I,F}_{e_i}, (A_K)^{T,I,F}_{e_i}\rangle$ *of* P *fulfills the following inequalities for all* $x, t_1, t_2, t_3 \in P$.

***i.*** *If* $t_1 * t_2 \leq t_3$, *then*

$$(\widehat{N}_K)^{T,I,F}_{e_i}(t_1 * x) \geq \text{rmin}\{(\widehat{N}_K)^{T,I,}_{e_i}(t_3), (\widehat{N}_K)^{T,I,F}_{e_i}(t_2)\}$$
$$(A_K)^{T,I,F}_{e_i}(t_1 * x) \leq \max\{(A_K)^{T,I,F}_{e_i}(t_3), (A_K)^{T,I,F}_{e_i}(t_2)\}.$$

*Proof.* Let $x, t_1, t_2, t_3 \in P$ such that $t_1 * t_2 = t_3$. Now, $(\widehat{N}_K)^{T,I,F}_{e_i}(t_1 * x) \geq \{\text{rmin}\{(\widehat{N}_K)^{T,I,F}_{e_i}((t_1 * t_2) * x), (\widehat{N}_K)^{T,I,F}_{e_i}(t_2)\} \geq \text{rmin}\{\text{rmin}\{(\widehat{N}_K)^{T,I,F}_{e_i}(((t_1 * t_3) * t_2) * x), (\widehat{N}_K)^{T,I,F}_{e_i}(t_3)\}, (\widehat{N}_K)^{T,F}_{e_i}(t_2)\}$ $= \text{rmin}\{\text{rmin}\{(\widehat{N}_K)^{T,I,F}_{e_i}(0), (\widehat{N}_K)^{T,I,F}_{e_i}(t_3)\}, (\widehat{N}_K)^{T,I,F}_{e_i}(t_2)\} = \text{rmin}\{(\widehat{N}_K)^{T,I}_{e_i}(t_3), (\widehat{N}_K)^{T,I,F}_{e_i}(t_2)\}$ $(A_K)^{T,I,F}_{e_i}(t_1 * x) \leq \{\max\{(A_K)^{T,I}_{e_i}((t_1 * t_2) * x), (A_K)^{T,IF}_{e_i}(t_2)\} \leq \max\{\max\{(A_K)^{T,I,F}_{e_i}(((t_1 * t_3) * t_2) * x)), (A_K)^{T,I,F}_{e_i}(t_3)\}, (A_K)^{T,F}_{e_i}(t_2)\} = \max\{\max\{(A_K)^{T,I,F}_{e_i}(0), (A_K)^{T,I,F}_{e_i}(t_3)\}, (A_K)^{T,F}_{e_i}(t_2)\} = \max\{(A_K)^{T,,F}_{e_i}(t_3), (A_K)^{T,F}_{e_i}(t_2)\}$.

Hence

$$(\widehat{N}_K)^{T,I,F}_{e_i}(t_1 * x) \geq \text{rmin}\{(\widehat{N}_K)^{T,IF}_{e_i}(t_3), (\widehat{N}_K)^{T,I,F}_{e_i}(t_2)\}$$
$$(A_K)^{T,IF}_{e_i}(t_1 * x) \leq \max\{(A_K)^{T,IF}_{e_i}(t_3), (A_K)^{T,I,F}_{e_i}(t_2)\}$$

forall $t_1, t_2, t_3 \in P$. □

***ii.*** *If* $t_1 \leq t_2$*, then*

$$(\widehat{N}_K)^{T,I,F}_{e_i}(t_1 * x) \geq (\widehat{N}_K)^{T,I,F}_{e_i}(t_2)$$
$$(A_K)^{T,I,F}_{e_i}(t_1 * x) \leq (A_K)^{T,I,F}_{e_i}(t_2)$$

*for all* $x, t_1, t_2, t_3 \in P$.

*Proof.* Let $x, t_1, t_2, t_3 \in P$ such that $t_1 \leq t_2 \to t_1 * t_2 = 0$, Now, $(\widehat{N}_K)^{T,I,F}_{e_i}(t_1 * x) \geq \{\text{rmin}\{(\widehat{N}_K)^{T,I,F}_{e_i}((t_1 * t_2) * x), (\widehat{N}_K)^{T,I,F}_{e_i}(t_2)\} = \{\text{rmin}\{(\widehat{N}_K)^{T,F}_{e_i}(0), (\widehat{N}_K)^{T,F}_{e_i}(t_2)\} = (\widehat{N}_K)^{T,F}_{e_i}(t_2)$.
And
$(A_K)^{T,I,F}_{e_i}(t_1 * x) \leq \{\max\{(A_K)^{T,I,F}_{e_i}((t_1 * t_2) * x), (A_K)^{T,I,F}_{e_i}(t_2)\}. = \{\max\{(A_K)^{T,I,F}_{e_i}(0), (A_K)^{T,I,F}_{e_i}(t_2)\}\} = (A_K)^{T,IF}_{e_i}(t_2)$.
Hence $(\widehat{N}_K)^{T,I,F}_{e_i}(t_1 * x) \geq (\widehat{N}_K)^{T_i,IF}_{e_i}(t_2)$ and $(A_K)^{T,I,F}_{e_i}(t_1 * x) \leq (A_K)^{T,I,F}_{e_i}(t_2)$ for all $t_1, t_2, t_3 \in P$.
□

***iii.***

$$((\widehat{N}_K)^{T,I,F}_{e_i}(t_1 * (t_2 * t_1)) * x) \geq (\widehat{N}_K)^{T,I,F}_{e_i}(t_2)$$
$$((A_K)^{T,I,F}_{e_i}(t_1 * (t_2 * t_1)) * x) \leq (A_K)^{T,I,F}_{e_i}(t_2).$$

*Proof.* Let K is an NSCTID. Then, $(\widehat{N}_K)^{T,I,F}_{e_i}((t_1 * (t_2 * t_1)) * x) \geq \text{rmin}\{(\widehat{N}_K)^{T,I}_{e_i}(((t_1 * (t_2 * t_1)) * t_2) * x), (\widehat{N}_K)^{T,I,F}_{e_i}(t_2)\} = \text{rmin}\{(\widehat{N}_K)^{T,I}_{e_i}(0), (\widehat{N}_K)^{T,IF}_{e_i}(t_2)\} = (\widehat{N}_K)^{T,I}_{e_i}(t_2)$, $(A_K)^{T,I,F}_{e_i}((t_1 * (t_2 * t_1)) * x) \leq \max\{(A_K)^{T,I,F}_{e_i}(((t_1 * (t_2 * t_1)) * t_2) * x), (A_K)^{T,I,F}_{e_i}(t_2)\} = \max\{(A_K)^{T,F}_{e_i}(0), (A_K)^{T,F}_{e_i}(t_2)\} = (A_K)^{T,I,F}_{e_i}(t_2)$.
Hence

$$(\widehat{N}_K)^{T,I,F}_{e_i}((t_1 * (t_2 * t_1)) * x) \geq (\widehat{N}_K)^{T,I,F}_{e_i}(t_2)$$
$$(A_K)^{T,I,F}_{e_i}((t_1 * (t_2 * t_1)) * x) \leq (A_K)^{T,F}_{e_i}(t_2),$$

for all $x, t_1, t_2, t_3 \in P$. □

**Theorem 3.4.** *Suppose* $K = \langle(\widehat{N}_K)^{T,I,F}_{e_i}, (A_K)^{T,,F}_{e_i}\rangle$ *and* $L = \langle(\widehat{N}_L)^{T,I,F}_{e_i}, (A_L)^{T,I,F}_{e_i}\rangle$ *are NSCTIDs of* $P$*. Then their union* $K \cup L$ *is also an NSCTID of* $P$*.*

*Proof.* If $t_1, t_2 \in K \cup L$, then $t_1, t_2 \in K$ and $t_1, t_2 \in L$. Then, $(\widehat{N}_{e_i})^{T,I,F}_{K \cup L}(0) = (\widehat{N}_{e_i})^{T,I,F}_{K \cup L}(t_1 * t_1) = \text{rmax}\{(\widehat{N}_K)^{T,IF}_{e_i}(t_1 * t_1), (\widehat{N}_L)^{T,I,F}_{e_i}(t_1 * t_1)\} \geq \text{rmax}\{\text{rmin}\{(\widehat{N}_K)^{T,I,F}_{e_i}(t_1), (\widehat{N}_K)^{T,I,F}_{e_i}(t_1)\}, \text{rmin}\{(\widehat{N}_L)^{T,I,F}_{e_i}(t_1), (\widehat{N}_L)^{T,I,F}_{e_i}(t_1)\}\} = \text{rmax}\{(\widehat{N}_K)^{T,I,F}_{e_i}(t_1), (\widehat{N}_L)^{T,I,F}_{e_i}(t_1)\} = (\widehat{N}_{e_i})^{T,I,F}_{K \cup L}(t_1)$, And $(A_{e_i})^{T,I,F}_{K \cup L}(0) = (A_{e_i})^{T,I,F}_{K \cup L}(t_1 * t_1) = \min\{(A_K)^{T,I,F}_{e_i}(t_1 * t_1), (A_L)^{T,I,F}_{e_i}(t_1 * t_1)\} \leq \min\{\max\{(A_K)^{T,I,F}_{e_i}(t_1), (A_K)^{T,I,F}_{e_i}(t_1)\}, \max\{(A_L)^{T,I,F}_{e_i}(t_1), (A_L)^{T,I,F}_{e_i}(t_1)\}\} = \min\{(A_K)^{T,I,F}_{e_i}(t_1), (A_L)^{T,I,F}_{e_i}(t_1)\} = (A_{e_i})^{T,I,F}_{K \cup L}(t_1)$. Thus $(\widehat{N}_{e_i})^{T,I,F}_{K \cup L}(0) \geq (\widehat{N}_{e_i})^{T,I,F}_{K \cup L}(t_1)$ and $(A_{e_i})^{T,I,F}_{K \cup L}(0) \leq (A_{e_i})^{TI,F}_{K \cup L}(t_1)$.
Now, $(\widehat{N}_{e_i})^{T,I,F}_{K \cup L}(t_1 * t_3) = \text{rmax}\{(\widehat{N}_K)^{T,I,F}_{e_i}(t_1 * t_3), (\widehat{N}_L)^{T,I,F}_{e_i}(t_1 * t_3)\} \geq \text{rmax}\{\text{rmin}\{(\widehat{N}_K)^{T,I,F}_{e_i}((t_1 * t_2) * t_3), (\widehat{N}_K)^{T,I,F}_{e_i}(t_2)\}, \text{rmin}\{(\widehat{N}_L)_{e_i}((t_1 * t_2) * t_3), (\widehat{N}_L)^{T,I,F}_{e_i}(t_2)\}\} = \text{rmin}\{\text{rmax}\{(\widehat{N}_K)^{T,I,F}_{e_i}((t_1 * t_2) * t_3), (\widehat{N}_L)^{T,I,F}_{e_i}((t_1 * t_2) * t_3)\} \text{rmax}\{(\widehat{N}_K)^{T,IF}_{e_i}(t_2), (\widehat{N}_L)^{T,I,F}_{e_i}(t_2)\}\} = \text{rmin}\{(\widehat{N}_{e_i})^{T,I,F}_{K \cup L}((t_1 * t_2) * t_3), (\widehat{N}_{e_i})^{T,I,F}_{K \cup L}(t_2)\}$, And $(A_{e_i})^{T,I,F}_{K \cup L}(t_1 * t_3) = \min\{(A_K)^{T,I,F}_{e_i}(t_1 * t_3), (A_L)^{T,I,F}_{e_i}(t_1 * t_3)\}$

$\leq \min\{\max\{(A_K)_{e_i}^{T,F}((t_1 * t_2) * t_3), (A_K)_{e_i}^{T,I,F}(t_2)\}, \max\{(A_L)_{e_i}^{T,F}((t_1 * t_2) * t_3), (A_L)_{e_i}^{T,I,}(t_2)\}\}$
$= \max\{\min\{(A_K)_{e_i}^{T,I,F}((t_1 * t_2) * t_3), (A_L)_{e_i}^{T,I,F}((t_1 * t_2) * t_3)\}, \min\{(A_K)_{e_i}^{T,I,F}(t_2), (A_L)_{e_i}^{T,I,F}(t_2)\}\}$
$= \max\{(A_{e_i})_{KUL}^{T,I,F}((t_1 * t_2) * t_3), (A_{e_i})_{KUL}^{T,I,}(t_2)\}$
Hence the union " $K \cup L$ " is also an NSCTID of P. □

**Theorem 3.5.** *Suppose* $K = \langle(\widehat{N}_K)_{e_i}^{T,I,F}, (A_K)_{e_i}^{T,I,F}\rangle$ *and* $L = \langle(\widehat{N}_L)_{e_i}^{T,I,F}, (A_L)_{e_i}^{T,I,F}\rangle$ *are two NSCTIDs of a PS-algebra* P*. Then the intersection* $K \cap L$ *is also NSCTID of* P*.*

*Proof.* Let $t_1, t_2 \in K \cap L$, then $t_1, t_2 \in K$ and $t_1, t_2 \in L$. As $(\widehat{N}_{e_i})_{K\cap L}^{T,I,F}(0) = (\widehat{N}_{e_i})_{K\cap L}^{T,I,F}(t_1 * t_1)$ $= \text{rmin}\{(\widehat{N}_K)_{e_i}^{T,I,F}(t_1 * t_1), (\widehat{N}_L)_{e_i}^{T,I,F}(t_1 * t_1)\} \geq \text{rmin}\{\text{rmin}\{(\widehat{N}_K)_{e_i}^{T,I,F}(t_1), (\widehat{N}_K)_{e_i}^{T,I,F}(t_1)\}, \text{rmin}\{(\widehat{N}_L)_{e_i}^{T,I,F}(t_1), (\widehat{N}_L)_{e_i}^{T,I,F}(t_1)\}\} = \text{rmin}\{(\widehat{N}_K)_{e_i}^{T,I,F}(t_1), (\widehat{N}_L)_{e_i}^{T,I,F}(t_1)\} = (\widehat{N}_{e_i})_{K\cap L}^{T,I,F}(t_1)$. And $(A_{e_i})_{K\cap L}^{T,I,F}(0) = (A_{e_i})_{K\cap L}^{T,I,F}(t_1 * t_1) = \max\{(A_K)_{e_i}^{T,I,F}(t_1 * t_1), (A_L)_{e_i}^{T,I,F}(t_1 * t_1)\} \leq \max\{\max\{(A_K)_{e_i}^{T,I,F}(t_1), (A_K)_{e_i}^{T,I,F}(t_1)\}, \max\{(A_L)_{e_i}^{T,I,F}(t_1), (A_L)_{e_i}^{T,I,F}(t_1)\}\} = \max\{(A_K)_{e_i}^{T,I,F}(t_1), (A_L)_{e_i}^{T,I,F}(t_1)\} = (A_{e_i})_{K\cap L}^{T,I,F}(t_1)$. Thus$(\widehat{N}_{e_i})_{K\cap L}^{T,I,F}(0) \geq (\widehat{N}_{e_i})_{K\cap L}^{T,I,F}(t_1)$ and $(A_{e_i})_{K\cap L}^{T,I,F}(0) \leq (A_{e_i})_{K\cap L}^{T,IF}(t_1)$.
Now, $(\widehat{N}_{e_i})_{K\cap L}^{T,I,F}(t_1 * t_3) = \text{rmin}\{(\widehat{N}_K)_{e_i}^{T,I,F}(t_1 * t_3), (\widehat{N}_L)_{e_i}^{T,I,F}(t_1 * t_3)\} \geq \text{rmin}\{\text{rmin}\{(\widehat{N}_K)_{e_i}^{T,I,F}((t_1 * t_2) * t_3), (\widehat{N}_K)_{e_i}^{T,I,F}(t_2)\}, \text{rmin}\{(\widehat{N}_L)_{e_i}^{T,F}((t_1 * t_2) * t_3), (\widehat{N}_L)_{e_i}^{T,I,F}(t_2)\}\} = \text{rmin}\{\text{rmin}\{(\widehat{N}_K)_{e_i}^{T,IF}((t_1 * t_2) * t_3), (\widehat{N}_L)_{e_i}^{T,I,F}((t_1 * t_2) * t_3)\}\ \text{rmin}\{(\widehat{N}_K)_{e_i}^{T,IF}(t_2), (\widehat{N}_L)_{e_i}^{T,I,F}(t_2)\}\} = \text{rmin}\{(\widehat{N}_{e_i})_{K\cap L}^{T,I,F}((t_1 * t_2) * t_3), (\widehat{N}_{e_i})_{K\cap L}^{T,I,F}(t_2)\}$, And $A_{e_{i_{K\cap L}}}(t_1 * t_3) = \max\{(A_K)_{e_i}^{T,I,F}(t_1 * t_3), (A_L)_{e_i}^{T,I,F}(t_1 * t_3)\} \leq$ $\max\{\max\{(A_K)_{e_i}^{T,I,F}((t_1 * t_2) * t_3), (A_K)_{e_i}^{T,I,F}(t_2)\}, \max\{(A_L)_{e_i}^{T,I,F}((t_1 * t_2) * t_3), (A_L)_{e_i}^{T,I,F}(t_2)\}\}$
$= \max\{\max\{(A_K)_{e_i}^{T,I,F}((t_1 * t_2) * t_3), (A_L)_{e_i}^{T,I,F}((t_1 * t_2) * t_3)\}, \max\{(A_K)_{e_i}^{T,I,F}(t_2), (A_L)_{e_i}^{T,I,F}(t_2)\}\}$
$= \max\{(A_{e_i})_{K\cap L}^{T,I,F}((t_1 * t_2) * t_3), (A_{e_i})_{K\cap L}^{T,I,F}(t_2)\}$
Hence, $K \cap L$ is an NSCTID of P. □

## 4. Cartesian product and Homomorphism of NSCTID

In this section, the interpretation of the cartesian product and homomorphism of NSCTID is given by proving some theorems.

**Definition 4.1.** Let $K = ((\widehat{N}_K)_{e_i}^{T,I,F}, (A_K)_{e_i}^{T,I,F})$ and $L = ((\widehat{N}_L)_{e_i}^{T,I,F}, (A_L)_{e_i}^{T,I,F})$ are two NSCTIDs of R and P respectively. The cartesian product $K \times L = (R \times P, (\widehat{N}_K)_{e_i}^{T,I,F} \times \widehat{N}_L)_{e_i}^{T,I,F}, (A_K)_{e_i}^{T,I,F} \times (A_L)_{e_i}^{T,I,F})$ is defined as:

$$(\widehat{N}_K)_{e_i}^{T,I,F} \times (\widehat{N}_L)_{e_i}^{T,I,F}(t_1, t_2) \geq \text{rmin}\{(\widehat{N}_K)_{e_i}^{T,IF}(t_1), (\widehat{N}_L)_{e_i}^{T,I,F}(t_2)\}$$
$$(A_K)_{e_i}^{T,I,F} \times (A_L)_{e_i}^{T,I,F}(t_1, t_2) \leq \max\{(A_K)_{e_i}^{T,I,F}(t_1), (A_L)_{e_i}^{T,I,F}(t_2)\}.$$

Where $(\widehat{N}_K)_{e_i}^{T,I,F} \times (\widehat{N}_L)_{e_i}^{T,I,F} : R \times P \to [0, 1]$ and $(A_K)_{e_i}^{T,I,F} \times (A_L)_{e_i}^{T,I,F} : R \times P \to [0, 1]$ for all $t_1 \in R$ and $t_2 \in P$.

**Theorem 4.2.** *Let* K *and* L *be two NSCTIDs of* P*, then* $K \times L$ *is an NSCTID of* $R \times P$*.*

*Proof.* Let R and P be two PS-algebras.For any $t_1, t_2 \in R \times P$, We have $((\widehat{N}_K)^{T,I,F}_{e_i} \times (\widehat{N}_L)^{T,I,F}_{e_i})(0,0) = rmin\{(\widehat{N}_K)^{T,I,F}_{e_i}(0), (\widehat{N}_L)^{T,I,F}_{e_i}(0)\} \geq rmin\{(\widehat{N}_K)^{T,F}_{e_i}(t_1), (\widehat{N}_L)^{T,F}_{e_i}(t_2)\} = ((\widehat{N}_K)^{T,F}_{e_i} \times (\widehat{N}_L)^{T,I,F}_{e_i})(t_1, t_2)((A_K)^{T,I,F}_{e_i} \times (A_L)^{T,I,F}_{e_i})(0,0) = max\{(A_K)^{T,F}_{e_i}(0), (A_L)^{T,I,F}_{e_i}(0)\} \leq max\{(A_K)^{T,I,F}_{e_i}(t_1), (A_L)^{T,I,F}_{e_i}(t_2)\} = ((A_K)^{T,F}_{e_i} \times (A_L)^{T,I,F}_{e_i})(t_1, t_2)$ That is $((\widehat{N}_K)^{T,I,F}_{e_i} \times (\widehat{N}_L)^{T,F}_{e_i})(0,0) \geq ((\widehat{N}_K)^{T,I,F}_{e_i} \times (\widehat{N}_L)^{T,I}_{e_i})(t_1, t_2)$ $((A_K)^{T,I,F}_{e_i} \times (A_L)^{T,F}_{e_i})(0,0) \leq ((A_K)^{T,I,F}_{e_i} \times (A_L)^{T,F}_{e_i})(t_1, t_2)$.

Now, Let $(x_1, x_2), (t_1, t_2)$ and $(y_1, y_2) \in R \times P$.Then, $((\widehat{N}_K)^{T,I}_{e_i} \times (\widehat{N}_L)^{T,I}_{e_i})(t_1 * x_1, t_2 * x_2) = rmin\{(\widehat{N}_K)^{T,I,F}_{e_i}(t_1 * x_1), (\widehat{N}_L)^{T,IF}_{e_i}(t_2 * x_2)\} \geq rmin\{rmin\{(\widehat{N}_K)^{T,I}_{e_i}((t_1 * y_1) * x_1), (\widehat{N}_K)^{T,IF}_{e_i}(y_1)\}, rmin\{(\widehat{N}_L)^{T,I,F}_{e_i}((t_2 * y_2) * x_2), (\widehat{N}_L)^{T,I,F}_{e_i}(y_2)\}\} = rmin\{rmin\{(\widehat{N}_K)^{T,I,F}_{e_i}((t_1 * y_1) * x_1), (\widehat{N}_L)^{T,I,F}_{e_i}((t_2 * y_2) * x_2)\}\ rmin\{(\widehat{N}_K)^{T,I,F}_{e_i}(y_1), (\widehat{N}_L)^{T,I,F}_{e_i}(y_2)\}\} = rmin\{((\widehat{N}_K)^{T,I,F}_{e_i} \times (\widehat{N}_L)^{T,I,F}_{e_i})(((t_1 * y_1)x_1)\ ((t_2 * y_2) * x_2)), ((\widehat{N}_K)^{T,IF}_{e_i} \times (\widehat{N}_L)^{T,IF}_{e_i})(y_1, y_2)\} = rmin\{((\widehat{N}_K)^{T,I,F}_{e_i} \times (\widehat{N}_L)^{T,I,F}_{e_i})((t_1 * x_1, t_2 * x) * (y_1, y_2)), ((\widehat{N}_K)^{T,I,F}_{e_i} \times (\widehat{N}_L)^{T,I,F}_{e_i})(y_1, y_2)$.

And $(A_K)^{T,I,F}_{e_i} \times (A_L)^{T,I,F}_{e_i})(t_1 * x_1, t_2 * x_2) = max\{(A_K)^{T,I,F}_{e_i}(t_1 * x_1), (A_L)^{T,I,F}_{e_i}(t_2 * x_2)\} \leq max\{max\{(A_K)^{T,I,F}_{e_i}((t_1 * y_1) * x_1), (A_K)^{T,F}_{e_i}(y_1)\}, max\{(A_L)^{T,I}_{e_i}((t_2 * y_2) * x), (A_L)^{T,I,F}_{e_i}(y_2)\}\} = max\{max\{(A_K)^{T,I,F}_{e_i}((t_1 * y_1) * x_1), (A_L)^{T,I,F}_{e_i}((t_2 * y_2) * x_2)\}, max\{(A_K)^{T,I,F}_{e_i}(y_1), (A_L)^{T,I,F}_{e_i}(y_2)\} = max\{((A_K)^{T,I,F}_{e_i} \times (A_L)^{T,I,F}_{e_i})(((t_1 * y_1) * x_1), ((t_2 * y_2) * x_2)), (A_K)^{T,I,F}_{e_i} \times (A_L)^{T,I,F}_{e_i})(y_1, y_2)\} = max\{(A_K)^{T,I,F}_{e_i} \times (A_L)^{T,I,F}_{e_i})((t_1 * x_1, t_2 * x_2) * (y_1, y_2)), (A_K)^{T,I,F}_{e_i} \times (A_L)^{T,I,F}_{e_i})(y_1, y_2)$.

Thus $((\widehat{N}_K)^{T,I,F}_{e_i} \times (\widehat{N}_L)^{T,I,F}_{e_i})(t_1 * x_1, t_2 * x_2) \geq rmin\{((\widehat{N}_K)^{T,I,F}_{e_i} \times (\widehat{N}_L)^{T,I,F}_{e_i})((t_1 * x_1, t_2 * x_2) * (y_1, y_2)), ((\widehat{N}_K)^{T,I,F}_{e_i} \times (\widehat{N}_L)^{T,I,F}_{e_i})(y_1, y_2)$. $(A_K)^{T,I,F}_{e_i} \times (A_L)^{T,I,F}_{e_i})(t_1 * x_1, t_2 * x_2) \leq max\{(A_K)^{T,I,F}_{e_i} \times (A_L)^{T,I,F}_{e_i})((t_1 * x_1, t_2 * x_2) * (y_1, y_2)), (A_K)^{T,I,F}_{e_i} \times (A_L)^{T,I,F}_{e_i})(y_1, y_2)$. □

**Theorem 4.3.** *Let* K *and* L *are two NSCSs of* R *and* P *such that* $K \times L$ *is an NSCTID of* $R \times P$*, Then*

***i.*** *For all* $t_1 \in R$ *and* $t_2 \in P$,

$$(\widehat{N}_K)^{T,I,F}_{e_i}(0) \geq (\widehat{N}_K)^{T,I,F}_{e_i}(t_1), (A_K)^{T,I,F}_{e_i}(0) \leq (A_K)^{T,I,F}_{e_i}(t_2),$$
$$(\widehat{N}_L)^{T,I}_{e_i}, F(0) \geq (\widehat{N}_L)^{T,I,F}_{e_i}(t_1), (A_L)^{T,I,F}_{e_i}(0) \leq (A_L)^{T,I,F}_{e_i}(t_2).$$

***ii.*** *For all* $t_1 \in P$*, If* $(\widehat{N}_K)^{T,I,F}_{e_i}(0) \geq (\widehat{N}_K)^{T,I,F}_{e_i}(t_1)$ *then,*

$$(\widehat{N}_L)^{T,I,F}_{e_i}(0) \geq (\widehat{N}_K)^{T,I,F}_{e_i}(t_1) \text{ and } (\widehat{N}_L)^{T,I,F}_{e_i}(0) \geq (\widehat{N}_L)^{T,I,F}_{e_i}(t_1),$$

*Also if* $(A_K)e^{T,F}_{e_i}(0) \leq (A_K)^{T,I,F}_{e_i}(t_1)$ *then*

$$(A_L)^{T,I,F}_{e_i}(0) \leq (A_K)^{T,I,F}_{e_i}(t_1) \text{ and } (A_L)^{T,I,F}_{e_i}(0) \leq (A_L)^{T,I,F}_{e_i}(t_1).$$

***iii.*** *For all* $t_1 \in P$*, If* $(\widehat{N}_L)^{T,I,F}_{e_i}(0) \geq (\widehat{N}_L)^{T,I,F}_{e_i}(t_1)$ *then,*

$$(\widehat{N}_K)^{T,I,F}_{e_i}(0) \geq (\widehat{N}_K)^{T,I,F}_{e_i}(t_1) \text{and } (\widehat{N}_K)^{T,I,F}_{e_i}(0) \geq (\widehat{N}_L)^{T,I,F}_{e_i}(t_1),$$

*Also if* $(\mathrm{A_L})_{e_i}^{T,I,F}(0) \leq (\mathrm{A_L})_{e_i}^{T,I,F}(t_1)$ *then*

$$(\mathrm{A_K})_{e_i}^{T,I,F}(0) \leq (\mathrm{A_K})_{e_i}^{T,I,F}(t_1) and (\mathrm{A_K})_{e_i}^{T,I,F}(0) \leq (\mathrm{A_L})_{e_i}^{T,I,F}(t_1).$$

*Proof.* ***i.*** Suppose $(\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F}(t_1) > (\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F}(0)$ or $(\widehat{\mathrm{N}}_L)_{e_i}^{T,I,F}(t_2) > (\widehat{\mathrm{N}}_L)_{e_i}^{T,I,F}(0)$ for all $t_1 \in R$ and $t_2 \in P$. Then, $(\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F} \times (\widehat{\mathrm{N}}_L)_{e_i}^{T,I,F})(t_1, t_2) = \mathrm{rmin}\{(\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F}(t_1), (\widehat{\mathrm{N}}_L)_{e_i}^{T,I,F}(t_2)\} > \mathrm{rmin}\{(\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F}(0), (\widehat{\mathrm{N}}_L)_{e_i}^{T,I,F}(0)\} = ((\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F} \times (\widehat{\mathrm{N}}_L)_{e_i}^{T,I,F})(0,0)$. Thus $((\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F} \times (\widehat{\mathrm{N}}_L)_{e_i}^{T,I,F})(t_1, t_2) > ((\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F} \times (\widehat{\mathrm{N}}_L)_{e_i}^{T,I,F})(0,0)$ for all $t_1 \in \mathrm{R}$ and $t_2 \in \mathrm{P}$, which is the contradiction to $((\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F} \times (\widehat{\mathrm{N}}_L)_{e_i}^{T,I,F})$ is a NSCTID of $\mathrm{R} \times \mathrm{P}$.
Also, if $(\mathrm{A_K})_{e_i}^{T,I,F}(t_1) < (\mathrm{A_K})_{e_i}^{T,I,F}(0)$ or $(\mathrm{A_L})_{e_i}^{T,I,F}(t_2) < (\mathrm{A_L})_{e_i}^{T,I,F}(0)$ for all $t_1 \in \mathrm{R}$ and $t_2 \in \mathrm{P}$. Then, $(\mathrm{A_K})_{e_i}^{T,I,F} \times (\mathrm{A_L})_{e_i}^{T,I,F})(t_1, t_2) = \max\{(\mathrm{A_K}))_{e_i}^{T,I,F}(t_1), (\mathrm{A_L})_{e_i}^{T,I,F}(t_2)\} > \max\{(\mathrm{A_K})_{e_i}^{T,I,F}(0), (\mathrm{A_L})_{e_i}^{T,I,F}(0)\} = ((\mathrm{A_K})_{e_i}^{T,I,F} \times (\mathrm{A_L})_{e_i}^{T,I,F})(0,0)$.
Thus $(\mathrm{A_K})_{e_i}^{T,I,F} \times (\mathrm{A_L})_{e_i}^{T,I,F})(t_1, t_2) < (\mathrm{A_K})_{e_i}^{T,I,F} \times (\mathrm{A_L})_{e_i}^{T,I,F})(0,0)$ for all $t_1 \in \mathrm{R}$ and $t_2 \in \mathrm{P}$, which is the contradiction to $(\mathrm{A_K})_{e_i}^{T,I,F} \times (\mathrm{A_L})_{e_i}^{T,I,F})$ is an NSCTID of $\mathrm{R} \times \mathrm{P}$. □

*Proof.* ***ii.*** Suppose $(\widehat{\mathrm{N}}_L)_{e_i}^{T_iI,F}(0) < (\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F}(t_1)$ or $(\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F}(0) < (\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F}(t_1)$ for all $t_1 \in \mathrm{R}, \mathrm{P}$. Then, $((\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F} \times (\widehat{\mathrm{N}}_L)_{e_i}^{T,I,F})(0,0) = \mathrm{rmin}\{(\widehat{\mathrm{N}}_K)_{e_i}^{T,I}(0), (\widehat{\mathrm{N}}_L)_{e_i}^{T,IF}(0)\} = (\widehat{\mathrm{N}}_L)_{e_i}^{T,I,F}(0)$, And $((\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F} \times (\widehat{\mathrm{N}}_L)_{e_i}^{T,I})(t_1, t_1) = \mathrm{rmin}\{(\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F}(t_1), (\widehat{\mathrm{N}}_L)_{e_i}^{T,I,F}(t_1)\} > (\widehat{\mathrm{N}}_L)_{e_i}^{T,F}(0) = ((\widehat{\mathrm{N}}_K)_{e_i}^{T,F} \times (\widehat{\mathrm{N}}_L)_{e_i}^{T,I,F})(0,0)$. This implies $((\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F} \times (\widehat{\mathrm{N}}_L)_{e_i}^{T,IF})(t_1, t_1) = ((\widehat{\mathrm{N}}_K)_{e_i}^{T,IF} \times (\widehat{\mathrm{N}}_L)_{e_i}^{T,I,F})(0,0)$. Which is the contradiction to $((\widehat{\mathrm{N}}_K)_{e_i}^{TI,F} \times (\widehat{\mathrm{N}}_L)_{e_i}^{T,F,F})$ is an NSCTID of $\mathrm{R} \times \mathrm{P}$. Hence if $(\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F}(0) \geq (\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F}(t_1)$ then $(\widehat{\mathrm{N}}_L)_{e_i}^{T,I,F}(0) \geq (\widehat{\mathrm{N}}_L)_{e_i}^{T,I,F}(t_1)$ and $(\widehat{\mathrm{N}}_L)_{e_i}(0) \geq (\widehat{\mathrm{N}}_L)_{e_i}(t_1)$ for all $t_1 \in \mathrm{R}, \mathrm{P}$. Now suppose $(\mathrm{A_L})_{e_i}^{T,I,F}(0) > (\mathrm{A_K})_{e_i}^{T,I,F}(t_1)$ or $(\mathrm{A_K})_{e_i}^{T,I,F}(0) > (\mathrm{A_K})_{e_i}^{T,I,F}(t_1)$ for all $t_1 \in \mathrm{R}, \mathrm{P}$. Then, $(\mathrm{A_K})_{e_i}^{T,I,F} \times (\mathrm{A_L})_{e_i}^{T,I,F})(0,0) = \max\{(\mathrm{A_K})_{e_i}^{T,I,F}(0), (\mathrm{A_L})_{e_i}^{T,I}(0)\} = (\mathrm{A_L})_{e_i}(0)$.
And $(\mathrm{A_K})_{e_i}^{T,IF} \times (\mathrm{A_L})_{e_i}^{T,I,F})(t_1, t_1) = \max\{(\mathrm{A_K})_{e_i}^{T,I,F}(t_1), (\mathrm{A_L})_{e_i}^{T,I,F}(t_1)\} > (\mathrm{A_L})_{e_i}^{T,I,F}(0) = (\mathrm{A_K})_{e_i}^{T,IF} \times (\mathrm{A_L})_{e_i}^{T,I,F})(0,0)$.
This implies $(\mathrm{A_K})_{e_i}^{T,IF} \times (\mathrm{A_L})_{e_i}^{T,I,F})(t_1, t_1) = (\mathrm{A_K})_{e_i}^{T,IF} \times (\mathrm{A_L})_{e_i}^{T,I,F})(0,0)$. Which is a contradiction to $(\mathrm{A_K})_{e_i}^{T,I,F} \times (\mathrm{A_L})_{e_i^{T,I,F}})$ is an NSCTID of $\mathrm{R} \times \mathrm{P}$. □

*Proof.* ***iii.*** The proof is quite the same as ***ii***. □

**Theorem 4.4.** *Let* $\Sigma : \mathrm{P} \to \mathrm{R}$ *is a homomorphism of PS -algebra. If* $\mathrm{K} = \langle (\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F}, (\mathrm{A_K})_{e_i}^{T,I,F} \rangle$ *be an NSCTID of* R *then the pre-image* $\Sigma^{-1}(\mathrm{K}) = (\Sigma^{-1}(\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F}), \Sigma^{-1}((\mathrm{A_K})_{e_i}^{T,I,F}))$ *of* P *under* $\Sigma$ *is an NSCTID in* P.

*Proof.* For any $t_1 \in \mathrm{P}$, we have
$\Sigma^{-1}((\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F})(t_1) = (\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F}(\Sigma(t_1)) \leq (\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F}(0) = (\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F}(\Sigma(0)) = \Sigma^{-1}(\widehat{\mathrm{N}}_K)_{e_i}^{T,I,F}(0)$,
$\Sigma^{-1}((\mathrm{A_K})_{e_i}^{T,I,F})(t_1) = (\mathrm{A_K})_{e_i}^{T,I,F}(\Sigma(t_1)) \geq (\mathrm{A_K})_{e_i}^{T,I,F}(0) = (\mathrm{A_K})_{e_i}^{T,I,F}(\Sigma(0)) = \Sigma^{-1}(\mathrm{A_K})_{e_i}^{T,I,F}(0)$.
Also,

$\Sigma^{-1}(\widehat{N}_K)^{T,I,F}_{e_i})(t_1 * x) = (\widehat{N}_K)^{T,I,F}_{e_i}(\Sigma(t_1 * x)) \geq \mathrm{rmin}\{(\widehat{N}_K)^{T,F}_{e_i}((\Sigma(t_1) * \Sigma(t_2)) * \Sigma(x)), (\widehat{N}_K)^{T,I,F}_{e_i}$ $(\Sigma(t_2))\} \quad = \mathrm{rmin}\{(\widehat{N}_K)^{T,I,F}_{e_i}(\Sigma((t_1 * t_2) * x), (\widehat{N}_K)^{T,F}_{e_i}(\Sigma(t_2))\} = \mathrm{rmin}\{\Sigma^{-1}((\widehat{N}_K)^{T,I,F}_{e_i})((t_1 * t_2)$ $x), \Sigma^{-1}((\widehat{N}_K)^{T,I,F}_{e_i})(t_2)\}$ ,$\Sigma^{-1}((A_K)^{T,I,F}_{e_i})(t_1 * x) = (A_K)^{T,I,F}_{e_i}(\Sigma(t_1 * x)) \leq \max\{(A_K)^{T,I,F}_{e_i}(\Sigma(((t_1)$ $\Sigma(t_2)) * \Sigma(x)), (A_K)^{T,F}_{e_i}(\Sigma(t_2))\} = \max\{(A_K)^{T,I,F}_{e_i}(\Sigma((t_1 * t_2) * x), (A_K)^{T,F}_{e_i}(\Sigma(t_2))\} = \max\{\Sigma^{-1}$ $((A_K)^{T,F}_{e_i})((t_1 * t_2) * x), \Sigma^{-1}((A_K)^{T,I,F}_{e_i})(t_2)\}$. □

**Theorem 4.5.** *Let* $\Sigma : P \to R$ *be an epimorphism of PS-algebra. Then* $K = \langle(\widehat{N}_K)^{T,I,F}_{e_i}, (A_K)^{T,I,F}_{e_i}\rangle$ *is an NSCTID of* P, *if* $\Sigma^{-1}(K) = ((\Sigma^{-1}((\widehat{N}_K)^{T,I,F}_{e_i}), \Sigma^{-1}((A_K)^{T,I,F}_{e_i}$ $))$ *of* P *under* $\Sigma$ *is an NSCTID in* P.

*Proof.* For any $t_1 \in R$, there exist $a \in P$ such that $\Sigma(a) = t_1$. Then, $(\widehat{N}_K)^{T,I,F}_{e_i}(t_1) = (\widehat{N}_K)^{T,I,F}_{e_i}(\Sigma(a)) = \Sigma^{-1}((\widehat{N}_K)^{T,I,F}_{e_i})(a) \leq \Sigma^{-1}((\widehat{N}_K)^{T,I,F}_{e_i})(0) = (\widehat{N}_K)^{T,I,F}_{e_i}(\Sigma(0)) = (\widehat{N}_K)^{T,I,F}_{e_i}(0)$, And $(A_K)^{T,I,F}_{e_i}(t_1) = (A_K)^{T,I,F}_{e_i}(\Sigma(a)) = \Sigma^{-1}((A_K)^{T,I,F}_{e_i})(a) \geq \Sigma^{-1}((A_K)^{T,F}_{e_i})(0) = (A_K)^{T,IF}_{e_i}(\Sigma(0)) = (A_K)^{T,I,F}_{e_i}(0)$.

Let $y, t_1, t_2 \in R$ then $\Sigma(x) = y, \Sigma(a) = t_1$ and $\Sigma(b) = t_2$ for some $a, b, x \in P$.

Now, $(\widehat{N}_K)^{T,I,F}_{e_i}(t_1 * y) = (\widehat{N}_K)^{T,I,}_{e_i}(\Sigma(a * x)) = \Sigma^{-1}((\widehat{N}_K)^{T,I,F}_{e_i})(a * x) \geq \mathrm{rmin}\{\Sigma^{-1}((\widehat{N}_K)^{T,F}_{e_i}((a * b) * x)), \Sigma^{-1}((\widehat{N}_K)^{T,I,F}_{e_i}(b))\} = \mathrm{rmin}\{(\widehat{N}_K)^{T,F}_{e_i}(\Sigma((a * b) * x)), (\widehat{N}_K)^{T,I}_{e_i}(\Sigma(b))\} = \mathrm{rmin}\{(\widehat{N}_K)^{T,I,F}_{e_i}$ $(\Sigma(a) * \Sigma(b)) * \Sigma(x)), (\widehat{N}_K)_{e_i}(\Sigma(b))\} = \mathrm{rmin}\{(\widehat{N}_K)^{T,I,F}_{e_i}((t_1 * t_2) * y), (\widehat{N}_K)^{T,I,F}_{e_i}(t_2)\}$.

And $(A_K)^{T,I,F}_{e_i}(t_1 * y) = (A_K)^{T,I,F}_{e_i}(\Sigma(a * x)) = \Sigma^{-1}((A_K)^{T,F}_{e_i}(a * x) \leq \max\{\Sigma^{-1}((A_K)^{T,I,F}_{e_i}((a * b) * x)), \Sigma^{-1}((A_K)^{T,I}_{e_i}(b))\} = \max\{(A_K)^{TI,F}_{e_i}(\Sigma((a * b) * x)), (A_K)^{T,I,F}_{e_i}(\Sigma(b))\} = \max\{(A_K)^{T,I,F}_{e_i}$ $(\Sigma(a) * \Sigma(b)) * \Sigma(x))\}, (A_K)^{T,I,F}_{e_i}(\Sigma(b))\} = \max\{(A_K)^{T,I,F}_{e_i}((t_1 * t_2) * y), (A_K)^{T,I,F}_{e_i}(t_2)\}$ □

## 5. Translation and Multiplication of Neutrosophic Soft Cubic T-Ideal

In this section, the interpretation of NSCTID-translation and NSCTID-multiplication is given. For the simplicity, the notation $K =< t_1, (\widehat{N}_K)^{T,I,F}_{e_i}(t_1), (A_K)^{T,I}_{e_i}(t_1) \mid t_1 \in P >$ for the NSCS is used.

In this paper, we use $\xi = [1,1] - \mathrm{rsup}\{(\widehat{N}_K)^{T,I,F}_{e_i}(t_1) \mid t_1 \in P\}$, $\Psi = \mathrm{r}\inf\{(\widehat{N}_K)^{F}_{e_i}(t_1) \mid t_1 \in P\}$, $\zeta = 1 - \sup\{(A_K)^{T,I}_{e_i}(t_1) \mid t_1 \in P\}, \Phi = \inf\{(A_K)^{F}_{e_i}(t_1)\} \mid t_1 \in P$ for any NSCS $K = \langle(\widehat{N}_K)^{T,I,F}_{e_i}, (A_K)^{T,I,F}_{e_i}\rangle$ of P.

**Definition 5.1.** Let $K = \langle(\widehat{N}_K)^{T,I,,F}_{e_i}, (A_K)^{T,IF}_{e_i}\rangle$ be an NSCS of P. For $(\widehat{N}_K)^{T,I,F}_{e_i}, \alpha, \beta \in [[0,0], \xi]$ and $\gamma \in [[0,0], \Psi]$, where for $(A_K)^{T,I,F}_{e_i}, \alpha, \beta \in [0, \zeta]$ and $\gamma \in [0, \Phi]$. A set of the form $\widetilde{K}^{Tr}_{\alpha,\beta,\gamma} = \langle((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}\rangle$ is called an Neutrosophic Soft Cubic Translation **NSCTr** of K, As for all $t_1 \in P$.

$((\widehat{N}_K)^{T}_{e_i})^{Tr}_{\alpha}(t_1) = (\widehat{N}_K)^{T}_{e_i}(t_1) + \alpha, ((\widehat{N}_K)^{I}_{e_i})^{Tr}_{\beta}(t_1) = (\widehat{N}_K)^{I}_{e_i}(t_1) + \beta, ((\widehat{N}_K)^{F}_{e_i})^{Tr}_{\gamma}(t_1) = (\widehat{N}_K)^{F}_{e_i}(t_1) - \gamma,$

$((A_K)^{T}_{e_i})^{Tr}_{\alpha}(t_1) = (A_K)^{T}_{e_i}(t_1) + \alpha, ((A_K)^{I}_{e_i})^{Tr}_{\beta}(t_1) = (A_K)^{I}_{e_i}(t_1) + \beta, ((A_K)^{F}_{e_i})^{Tr}_{\gamma}(t_1) = (A_K)^{F}_{e_i}(t_1) - \gamma.$

**Definition 5.2.** Let K be an NSCS of P and $\eta \in [0,1]$. An object having the form $\widetilde{K}_{\eta}^{Mp} = \langle(((\widehat{N}_K)_{e_i}^{T})_{\eta}^{Mp}, ((\widehat{N}_K)_{e_i}^{I})_{\eta}^{Mp}, ((\widehat{N}_K)_{e_i}^{F})_{\eta}^{Mp}), (((A_K)_{e_i}^{T})_{\eta}^{Mp}, ((A_K)_{e_i}^{I})_{\eta}^{Mp}, ((A_K)_{e_i}^{F})_{\eta}^{Mp})\rangle$ is called a Neutrosophic Soft Cubic Translation **NSCMp** of K, Where

$((\widehat{N}_K)_{e_i}^{T})_{\eta}^{Mp}(t_1) = \eta\cdot(\widehat{N}_K)_{e_i}^{T}(t_1), ((\widehat{N}_K)_{e_i}^{I})_{\eta}^{Mp}(t_1) = \eta\cdot(\widehat{N}_K)_{e_i}^{I}(t_1), ((\widehat{N}_K)_{e_i}^{F})_{\eta}^{Mp}(t_1) = \eta\cdot(\widehat{N}_K)_{e_i}^{F}(t_1),$
$((A_K)_{e_i}^{T})_{\eta}^{Mp}(t_1) = \eta\cdot A_K)_{e_i}^{T}(t_1), (A_K)_{e_i}^{I})_{\eta}^{Mp}(t_1) = \eta\cdot(A_K)_{e_i}^{I}(t_1), ((A_K)_{e_i}^{F})_{\eta}^{Mp}(t_1) = \eta\cdot(A_K)_{e_i}^{F}(t_1)$
for all $t_1 \epsilon P$.

### 5.1. *Neutrosophic Soft Cubic T-Ideal Translation*

This section defines neutrosophic soft cubic T-Ideal translation with a theorem and example.

**Theorem 5.3.** *If* K *is an NSCTID of* P*, then NSCTr* $\widetilde{K}_{\alpha,\beta,\gamma}^{Tr}$ *of K is an NSCTID of* P.

*Proof.* Let $K = \langle(\widehat{N}_K)_{e_i}^{T,I,F}, (A_K)_{e_i}^{T,I,F}\rangle$ of P be an NSCTID of P. Then we have $((\widehat{N}_K)_{e_i}^{T})_{\alpha}^{Tr}(0) = (\widehat{N}_K)_{e_i}^{T}(0)+\alpha \geq (\widehat{N}_K)_{e_i}^{T}(t_1)+\alpha = ((\widehat{N}_K)_{e_i}^{T})_{\alpha}^{Tr}(t_1)$, $((\widehat{N}_K)_{e_i}^{I})_{\beta}^{Tr}(0) = (\widehat{N}_K)_{e_i}^{I}(0)+\beta \geq (\widehat{N}_K)_{e_i}^{I}(t_1)+\beta = ((\widehat{N}_K)_{e_i}^{I})_{\beta}^{Tr}(t_1), ((\widehat{N}_K)_{e_i}^{F})_{\gamma}^{Tr}(0) = (A_K)_{e_i}^{F}(0)+\gamma \geq (A_K)_{e_i}^{F}(t_1)+\gamma = ((\widehat{N}_K)_{e_i}^{F})_{\gamma}^{Tr}(t_1),$
And $((A_K)_{e_i}^{T})_{\alpha}^{Tr}(0) = (A_K)_{e_i}^{T}(0)+\alpha \leq ((A_K)_{e_i}^{T}(t_1)+\alpha = ((A_K)_{e_i}^{T})_{\alpha}^{Tr}(t_1)((A_K)_{e_i}^{I})_{\beta}^{Tr}(0) = (A_K)_{e_i}^{I}(0)+\beta \leq (A_K)_{e_i}^{I}(t_1)+\beta = ((A_K)_{e_i}^{I})_{\beta}^{Tr}(t_1)$ $((A_K)_{e_i}^{F})_{\gamma}^{Tr}(0) = (A_K)_{e_i}^{F}(0)+\gamma \leq (A_K)_{e_i}^{F}(t_1)+\gamma = ((A_K)_{e_i}^{F})_{\gamma}^{Tr}(t_1)$ Now $((\widehat{N}_K)_{e_i}^{T})_{\alpha}^{Tr}(t_1{*}t_3) = \widehat{N}_{e_i}(t_1{*}t_3)+\alpha \geq \text{rmin}\{\widehat{N}_{e_i}((t_1{*}t_2){*}t_3), \widehat{N}_{e_i T}(t_2)\}+\alpha = \text{rmin}\{\widehat{N}_{e_i}((t_1*t_2)*t_3)+\alpha, \widehat{N}_{e_i}(t_2)+\alpha\} = \text{rmin}\{((\widehat{N}_K)_{e_i}^{T})_{\alpha}^{Tr}((t_1*t_2)*t_3), ((\widehat{N}_K)_{e_i}^{T})_{\alpha}^{Tr}(t_2)\},$
$((\widehat{N}_K)_{e_i}^{I})_{\beta}^{Tr}(t_1*t_3) = (\widehat{N}_K)_{e_i}^{I}(t_1*t_3)+\beta \geq \text{rmin}\{(\widehat{N}_K)_{e_i}^{I}((t_1*t_2)*t_3), (\widehat{N}_K)_{e_i}^{I}(t_2)\}+\beta = \text{rmin}\{(\widehat{N}_K)_{e_i}^{I}((t_1*t_2)*t_3)+\beta, (\widehat{N}_K)_{e_i}^{I}(t_2)+\beta\} = \text{rmin}\{((\widehat{N}_K)_{e_i}^{I})_{\beta}^{Tr}((t_1*t_2)*t_3), ((\widehat{N}_K)_{e_i}^{I})_{\beta}^{Tr}(t_2)\}((\widehat{N}_K)_{e_i}^{F})_{\gamma}^{Tr}(t_1*t_3) = (\widehat{N}_K)_{e_i}^{F}(t_1*t_3)+\gamma \geq \text{rmin}\{(\widehat{N}_K)_{e_i}^{F}((t_1*t_2)*t_3), (\widehat{N}_K)_{e_i}^{F}(t_2)\}+\gamma = \text{rmin}\{(\widehat{N}_K)_{e_i}^{F}((t_1*t_2)*t_3)+\gamma, (\widehat{N}_K)_{e_i}^{F}(t_2)+\gamma\} = \text{rmin}\{((\widehat{N}_K)_{e_i}^{F})_{\gamma}^{Tr}((t_1*t_2)*t_3), ((\widehat{N}_K)_{e_i}^{F})_{\gamma}^{Tr}(t_2)\}$ And $((A_K)_{e_i}^{T})_{\alpha}^{Tr}(t_1*t_3) = (A_K)_{e_i}^{T}(t_1*t_3)+\alpha \leq \max\{(A_K)_{e_i}^{T}((t_1*t_2)*t_3), (A_K)_{e_i}^{T}(t_2)\}+\alpha = \max\{(A_K)_{e_i}^{T}((t_1*t_2)*t_3)+\alpha, (A_K)_{e_i}^{T}(t_2)+\alpha\} = \max\{((A_K)_{e_i}^{T})_{\alpha}^{Tr}((t_1*t_2)*t_3), ((A_K)_{e_i}^{T})_{\alpha}^{Tr}(t_2)\}, ((A_K)_{e_i}^{I})_{\beta}^{Tr}(t_1*t_3) = (A_K)_{e_i}^{I}(t_1*t_3)+\beta \leq \max\{(A_K)_{e_i}^{I}((t_1*t_2)*t_3), (A_K)_{e_i}^{I}(t_2)\}+\beta = \max\{(A_K)_{e_i}^{I}((t_1*t_2)*t_3)+\beta, (A_K)_{e_i}^{I}(t_2)+\beta\} = \max\{((A_K)_{e_i}^{I})_{\beta}^{Tr}((t_1*t_2)*t_3), ((A_K)_{e_i}^{I})_{\beta}^{Tr}(t_2)\}, ((A_K)_{e_i}^{F})_{\gamma}^{Tr}(t_1*t_3) = (A_K)_{e_i}^{F}(t_1*t_3)+\gamma \leq \max\{(A_K)_{e_i}^{F}((t_1*t_2)*t_3), (A_K)_{e_i}^{F}(t_2)\}+\gamma = \max\{(A_K)_{e_i}^{F}((t_1*t_2)*t_3)+\gamma, (A_K)_{e_i}^{F}(t_2)+\gamma\} = \max\{((A_K)_{e_i}^{F})_{\gamma}^{Tr}((t_1*t_2)*t_3), ((A_K)_{e_i}^{F})_{\gamma}^{Tr}(t_2)\}$, Hence $\widetilde{K}_{\alpha,\beta,\gamma}^{Tr}$ of K is an NSCTID of P. □

**Example 5.4.** Let $P = \{0, t_1, t_2, t_3\}$ be a PS-algebra with the cayley's table as shown in **Table 1**. The NSCS $K = \langle(\widehat{N}_K)_{e_i}^{T,I,F}, (A_K)_{e_i}^{T,I,F}\rangle$ of P is defined as

$$(\widehat{N}_K)_{e_i}^{T}(t_i) = \{[0.7, 0.9] \text{ if } t_i = 0 \text{ and } [0.3, 0.6] \text{ if otherwise}$$

$$(\widehat{N}_K)_{e_i}^{I}(t_i) = \{[0.6, 0.8] \text{ if } t_i = 0 \text{ and } [0.4, 0.5] \text{ if otherwise}$$

$$(\widehat{N}_K)_{e_i}^{F}(t_i) = \{[0.5, 1] \text{ if } t_i = 0 \text{ and } [0.2, 0.7] \text{ if otherwise}$$

And

$$(\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t}_i) = \{0.2 \text{ if } \mathrm{t}_i = 0 \text{ and } 0.7 \text{ if otherwise}$$

$$(\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}}(\mathrm{t}_i) = \{0.5 \text{ if } \mathrm{t}_i = 0 \text{ and } 0.9 \text{ if otherwise}$$

$$(\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t}_i) = \{0.4 \text{ if } \mathrm{t}_i = 0 \text{ and } 1 \text{ if otherwise.}$$

The set K is an NSCTID of P as it satisfies (i) but fulfills (ii) and (iii) with the condition: If $\mathrm{i} = \mathrm{j}$ then $\mathrm{t_i} * \mathrm{t_j} = 0$ with

$$(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_i} * \mathrm{t_j}) = [0.7, 0.9], (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}}(\mathrm{t_i} * \mathrm{t_j}) = [0.6, 0.8], (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t_i} * \mathrm{t_j}) = [0.5, 1],$$

Otherwise

$$(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_i} * \mathrm{t_j}) = [0.3, 0.6], (\mathrm{t_i} * \mathrm{t_j}) = [0.4, 0.5], (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t_i} * \mathrm{t_j}) = [0.2, 0.7]$$

And

$$(\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_i}) = 0.2, (\ \mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}}(\mathrm{t_i}) = 0.5, (\ \mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t_i}) = 0.4,$$

Otherwise $(\mathrm{A_K})^{T}_{\mathrm{e_i}}(\mathrm{t}_i) = 0.7, (\mathrm{A_K})^{I}_{\mathrm{e_i}}(\mathrm{t}_i) = 0.9, (\mathrm{A_K})^{F}_{\mathrm{e_i}}(\mathrm{t}_i) = 1$ given in definition 3.1. Now, for $(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}$ we choose $\alpha = [0.04, 0.08], \beta = [0.05, 0.09], \gamma = [0.03, 0.07]$ and for $(\mathrm{A})^{\mathrm{T,I,F}}_{\mathrm{e_i}}, \alpha = 0.03, \beta = 0.04, \gamma = 0.05$. Then the mapping $\widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha,\beta,\gamma} \mid \mathrm{P} \to [0, 1]$ is given by

$$((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{[0.04,0.08]}(0) = (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}}(0) + [0.04, 0.08] = [0.74, 0.98]$$

$$((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{[0.05,0.09]}(0) = (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}}(0) + [0.05, 0.09] = [0.65, 0.89]$$

$$((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{[0.03,0.07]}(0) = (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}}(0) - [0.03, 0.07] = [0.47, 0.93]$$

$$((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{[0.04,0.08]}(\mathrm{t_i}) = (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_i}) + [0.04, 0.08] = [0.34, 0.68]$$

$$((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{[0.05,0.09]}(\mathrm{t_i}) = (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_i}) + [0.05, 0.09] = [0.45, 0.59]$$

$$((\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{[0.03,0.07]}(\mathrm{t_i}) = (\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{F}}_{\mathrm{e_i}}(\mathrm{t_i}) - [0.03, 0.07] = [0.17, 0.63]$$

$$((\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{0.03}(0) = (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(0) + 0.03 = [0.23]$$

$$((\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{0.04}(0) = (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(0) + 0.04 = [0.54]$$

$$((\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{0.05}(0) = (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(0) - 0.05 = [0.35]$$

$$((\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{0.03}(\mathrm{t_i}) = (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_i}) + 0.03 = [0.73]$$

$$((\mathrm{A_K})^{\mathrm{I}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{0.04}(\mathrm{t_i}) = (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_i}) + 0.04 = [0.94]$$

$$((\mathrm{A_K})^{\mathrm{F}}_{\mathrm{e_i}})^{\mathrm{Tr}}_{0.05}(\mathrm{t_i}) = (\mathrm{A_K})^{\mathrm{T}}_{\mathrm{e_i}}(\mathrm{t_i}) - 0.05 = [0.95]$$

Hence $\widetilde{K}^{\mathrm{Tr}}_{\alpha,\beta,\gamma}$ is an NSCTID of P.

**Theorem 5.5.** *The union of any two NSC-translations of an NSCTID is an NSCTID of P.*

*Proof.* Suppose $\widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha,\beta,\gamma}$ and $\widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}$ are two NSC-translations of NSCTID of P respectively. In $\widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha,\beta,\gamma}$ for $(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I}}_{\mathrm{e_i}}, \alpha, \beta \in [[0, 0], \xi]$ and $\gamma \in [[0, 0], \Psi]$, where for $(\mathrm{A_K})^{\mathrm{T,I,F}}_{\mathrm{e_i}}, \alpha, \beta \in [0, \zeta]$ and $\gamma \in [0, \Phi]$, and in $\widetilde{\mathrm{K}}^{\mathrm{Tr}}_{\alpha',\beta',\gamma'}$, for $(\widehat{\mathrm{N}}_{\mathrm{K}})^{\mathrm{T,I,F}}_{\mathrm{e_i}}, \alpha', \beta' \in [[0, 0], \xi]$ and $\gamma' \in [[0, 0], \Psi]$, where for

$(A_K)^{T,I,F}_{e_i}, \alpha', \beta' \in [0,\zeta]$ and $\gamma' \in [0,\Phi]$. and $\alpha \geq \alpha', \beta \geq \beta', \gamma \geq \gamma'$ as we know that, $\widetilde{K}^{Tr}_{\alpha,\beta,\gamma}$ and $\widetilde{K}^{Tr}_{\alpha',\beta',\gamma'}$ are NSCTID of P. Then

$(((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha} \cup ((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha'})(0) = (((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha} \cup ((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha'})(t_1 * t_1) = \mathrm{rmax}\{((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha}(t_1 * t_1), ((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha'}(t_1 * t_1)\} \geq \mathrm{rmax}\{\mathrm{rmin}\{((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha}(t_1), ((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha}(t_1)\}, \mathrm{rmin}\{((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha'}(t_1), ((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha'}(t_1)\}\} = \mathrm{rmax}\{((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha}(t_1), ((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha'}(t_1)\} = \mathrm{rmax}\{(\widehat{N}_K)^T_{e_i}(t_1) + \alpha, (\widehat{N}_K)^T_{e_i}(t_1) + \alpha'\} = ((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha} \cup (\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha'})(t_1)$, $(((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta} \cup ((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta'})(0) = (((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta} \cup ((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta'})(t_1 * t_1) = \mathrm{rmax}\{((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta}(t_1 * t_1), ((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta'}(t_1 * t_1)\} \geq \mathrm{rmax}\{\mathrm{rmin}\{((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta}(t_1), ((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta}(t_1)\}, \mathrm{rmin}\{((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta'}(t_1), ((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta'}(t_1)\}\} = \mathrm{rmax}\{((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta}(t_1), ((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta'}(t_1)\} = \mathrm{rmax}\{(\widehat{N}_K)^I_{e_i}(t_1) + \beta, (\widehat{N}_K)^I_{e_i}(t_1) + \beta'\} = ((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta} \cup (\widehat{N}_K)^I_{e_i})^{Tr}_{\beta'})(t_1)$, $(((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma} \cup ((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma'})(0) = (((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma} \cup ((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma'})(t_1 * t_1) = \mathrm{rmax}\{((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma}(t_1 * t_1), ((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma'}(t_1 * t_1)\} \geq \mathrm{rmax}\{\mathrm{rmin}\{((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma}(t_1), ((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma}(t_1)\}, \mathrm{rmin}\{((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma'}(t_1), ((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma'}(t_1)\}\} = \mathrm{rmax}\{((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma}(t_1), ((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma'}(t_1)\} = \mathrm{rmax}\{(\widehat{N}_K)^F_{e_i}(t_1) + \gamma, (\widehat{N}_K)^F_{e_i}(t_1) + \gamma'\} = ((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma} \cup (\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma'})(t_1)$, $(((A_K)^T_{e_i})^{Tr}_{\alpha} \cup ((A_K)^T_{e_i})^{Tr}_{\alpha'})(0) = (((A_K)^T_{e_i})^{Tr}_{\alpha} \cup ((A_K)^T_{e_i})^{Tr}_{\alpha'})(t_1 * t_1) = \min\{((A_K)^T_{e_i})^{Tr}_{\alpha}(t_1 * t_1), ((A_K)^T_{e_i})^{Tr}_{\alpha'}(t_1 * t_1)\} \leq \min\{\max\{((A_K)^T_{e_i})^{Tr}_{\alpha}(t_1), ((A_K)^T_{e_i})^{Tr}_{\alpha}(t_1)\} \max\{((A_K)^T_{e_i})^{Tr}_{\alpha'}(t_1), ((A_K)^T_{e_i})^{Tr}_{\alpha'}(t_1)\}\} = \min\{((A_K)^T_{e_i})^{Tr}_{\alpha}(t_1), ((A_K)^T_{e_i})^{Tr}_{\alpha'}(t_1)\} = \min\{(A_K)^T_{e_i}(t_1) + \alpha, (A_K)^T_{e_i}(t_1) + \alpha'\} = ((A_K)^T_{e_i})^{Tr}_{\alpha} \cup (A_K)^T_{e_i})^{Tr}_{\alpha'})(t_1)$. $(((A_K)^I_{e_i})^{Tr}_{\beta} \cup ((A_K)^I_{e_i})^{Tr}_{\beta'})(0) = (((A_K)^I_{e_i})^{Tr}_{\beta} \cup ((A_K)^I_{e_i})^{Tr}_{\beta'})(t_1 * t_1) = \min\{((A_K)^I_{e_i})^{Tr}_{\beta}(t_1 * t_1), ((A_K)^I_{e_i})^{Tr}_{\beta'}(t_1 * t_1)\} \leq \min\{\max\{((A_K)^I_{e_i})^{Tr}_{\beta}(t_1), ((A_K)^I_{e_i})^{Tr}_{\beta}(t_1)\}, \max\{((A_K)^I_{e_i})^{Tr}_{\beta'}(t_1), ((A_K)^I_{e_i})^{Tr}_{\beta'}(t_1)\}\} = \min\{((A_K)^I_{e_i})^{Tr}_{\beta}(t_1), ((A_K)^I_{e_i})^{Tr}_{\beta'}(t_1)\} = \min\{(A_K)^I_{e_i}(t_1) + \beta, (A_K)^I_{e_i}(t_1) + \beta'\} = ((A_K)^I_{e_i})^{Tr}_{\beta} \cup (A_K)^I_{e_i})^{Tr}_{\beta'})(t_1)$ $(((A_K)^F_{e_i})^{Tr}_{\gamma} \cup ((A_K)^F_{e_i})^{Tr}_{\gamma'})(0) = (((A_K)^F_{e_i})^{Tr}_{\gamma} \cup ((A_K)^F_{e_i})^{Tr}_{\gamma'})(t_1 * t_1) = \min\{((A_K)^F_{e_i})^{Tr}_{\gamma}(t_1 * t_1), ((A_K)^F_{e_i})^{Tr}_{\gamma'}(t_1 * t_1)\} \leq \min\{\max\{((A_K)^F_{e_i})^{Tr}_{\gamma}(t_1), ((A_K)^F_{e_i})^{Tr}_{\gamma}(t_1)\} \max\{((A_K)^F_{e_i})^{Tr}_{\gamma'}(t_1), ((A_K)^F_{e_i})^{Tr}_{\gamma'}(t_1)\}\} = \min\{((A_K)^F_{e_i})^{Tr}_{\gamma}(t_1), ((A_K)^F_{e_i})^{Tr}_{\gamma'}(t_1)\} = \min\{(A_K)^F_{e_i}(t_1) + \gamma, (A_K)^F_{e_i}(t_1) + \gamma'\} = ((A_K)^F_{e_i})^{Tr}_{\gamma} \cup (A_K)^F_{e_i})^{Tr}_{\gamma'})(t_1)$.

Now

$(((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma} \cup ((\widehat{N}_K)^{T,IF}_{e_i})^{Tr}_{\alpha',\beta',\gamma'})(t_1 * t_3) = \mathrm{rmax}\{((\widehat{N}_K)^{TI,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}(t_1 * t_3), ((\widehat{N}_K)^{T,I}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}(t_1 * t_3)\} \geq \mathrm{rmax}\{\mathrm{rmin}\{((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}((t_1 * t_2) * t_3), ((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}(t_2)$, $\mathrm{rmin}\{((\widehat{N}_K)^{TI,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}((t_1 * t_2) * t_3), ((\widehat{N}_K)^{T,IF}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}(t_2)\}, = \mathrm{rmin}\{\mathrm{rmax}\{((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}((t_1 * t_2) * t_3), ((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}((t_1 * t_2) * t_3)\}$, $\mathrm{rmax}\{((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}(t_2), ((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}(t_2)\}\} = \mathrm{rmin}\{(((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma} \cup ((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'})((t_1 * t_2) * t_3)$, $(((\widehat{N}_K)^{T,IF}_{e_i})^{Tr}_{\alpha,\beta,\gamma} \cup ((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'})(t_2)\}$.

And

$((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma} \cup ((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'})(t_1 * t_3) = \min\{((A_K)^{T,I}_{e_i})^{Tr}_{\alpha,\beta,\gamma}(t_1 * t_3), ((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}(t_1 * t_3)\} \leq \min\{\max\{((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}((t_1 * t_2) * t_3), ((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}(t_2)\}, \max\{((A_K)^{T,IF}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}((t_1 * t_2) * t_3), ((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}(t_2)\}\} = \max\{\min\{((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}((t_1 * t_2) * t_3), ((A_K)^{T,IF}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}((t_1 * t_2) * t_3)\}, \min\{((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}(t_2), ((A_K)^{TI,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}(t_2)\}\} = \max\{(((A_K)^{T,IF}_{e_i})^{Tr}_{\alpha,\beta,\gamma} \cup ((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'})((t_1 * t_2) * t_3)$, $(((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma} \cup ((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'})(t_2)\}$.

Hence $\widetilde{K}^{Tr}_{\alpha,\beta,\gamma} \cup \widetilde{K}^{Tr}_{\alpha',\beta',\gamma'}$ is an NSCTID of P. □

**Theorem 5.6.** *The intersection of any two NSC-translations of an NSCTID is an NSCTID of P.*

*Proof.* Suppose $\widetilde{K}^{Tr}_{\alpha,\beta,\gamma}$ and $\widetilde{K}^{Tr}_{\alpha',\beta',\gamma'}$ are two NSC-translations of NSCTID of P respectively. Where in $\widetilde{K}^{Tr}_{\alpha,\beta,\gamma}$ for $(\widehat{N}_K)^{T,I,F}_{e_i}, \alpha, \beta \in [[0,0],\xi]$ and $\gamma \in [[0,0],\Psi]$, where for $(A_K)^{T,I,F,F}_{e_i}, \alpha, \beta \in [0,\zeta]$ and $\gamma \in [0,\Phi]$, and in $\widetilde{K}^{Tr}_{\alpha',\beta',\gamma'}$, for $(\widehat{N}_K)^{T,F}_{e_i}, \alpha', \beta' \in [[0,0],\xi]$ and $\gamma' \in [[0,0],\Psi]$, where for $(A_K)^{T,I,F}_{e_i}, \alpha', \beta' \in [0,\zeta]$ and $\gamma' \in [0,\Phi]$. and $\alpha \leq \alpha', \beta \leq \beta', \gamma \leq \gamma'$ as we know that, $\widetilde{K}^{Tr}_{\alpha,\beta,\gamma}$ and $\widetilde{K}^{Tr}_{\alpha',\beta',\gamma'}$ are NSCTID of P.Then
$(((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha} \cap ((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha'})(0) = (((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha} \cap ((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha'})(t_1 * t_1) = rmin\{((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha}(t_1 * t_1), ((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha'}(t_1 * t_1)\} \geq rmin\{rmin\{((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha}(t_1), ((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha}(t_1)\}, rmin\{((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha'}(t_1), ((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha'}(t_1)\}\} = rmin\{((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha}(t_1), ((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha'}(t_1)\} = rmin\{(\widehat{N}_K)^T_{e_i}(t_1) + \alpha, (\widehat{N}_K)^T_{e_i}(t_1) + \alpha'\} = ((\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha} \cap (\widehat{N}_K)^T_{e_i})^{Tr}_{\alpha'})(t_1)$, $(((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta} \cap ((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta'})(0) = (((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta} \cap ((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta'})(t_1 * t_1) = rmin\{((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta}(t_1 * t_1), ((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta'}(t_1 * t_1)\} \geq rmin\{rmin\{((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta}(t_1), ((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta}(t_1)\}, rmin\{((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta'}(t_1), ((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta'}(t_1)\}\} = rmin\{((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta}(t_1), ((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta'}(t_1)\} = rmin\{(\widehat{N}_K)^I_{e_i}(t_1) + \beta, (\widehat{N}_K)^I_{e_i}(t_1) + \beta'\} = ((\widehat{N}_K)^I_{e_i})^{Tr}_{\beta} \cap (\widehat{N}_K)^I_{e_i})^{Tr}_{\beta'})(t_1)$, $(((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma} \cap ((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma'})(0) = (((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma} \cap ((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma'})(t_1 * t_1) = rmin\{((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma}(t_1 * t_1), ((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma'}(t_1 * t_1)\} \geq rmin\{rmin\{((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma}(t_1), ((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma}(t_1)\}, rmin\{((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma'}(t_1), ((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma'}(t_1)\}\} = rmin\{((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma}(t_1), ((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma'}(t_1) = rmin\{(\widehat{N}_K)^F_{e_i}(t_1) + \gamma, (\widehat{N}_K)^F_{e_i}(t_1) + \gamma'\} = ((\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma} \cap (\widehat{N}_K)^F_{e_i})^{Tr}_{\gamma'})(t_1)$, $(((A_K)^T_{e_i})^{Tr}_{\alpha} \cap ((A_K)^T_{e_i})^{Tr}_{\alpha'})(0) = (((A_K)^T_{e_i})^{Tr}_{\alpha} \cap ((A_K)^T_{e_i})^{Tr}_{\alpha'})(t_1 * t_1) = \max\{((A_K)^T_{e_i})^{Tr}_{\alpha}(t_1 * t_1), ((A_K)^T_{e_i})^{Tr}_{\alpha'}(t_1 * t_1)\} \leq \max\{\max\{((A_K)^T_{e_i})^{Tr}_{\alpha}(t_1), ((A_K)^T_{e_i})^{Tr}_{\alpha}(t_1)\}, \max\{((A_K)^T_{e_i})^{Tr}_{\alpha'}(t_1), ((A_K)^T_{e_i})^{Tr}_{\alpha'}(t_1)\}\} = \max\{((A_K)^T_{e_i})^{Tr}_{\alpha}(t_1), ((A_K)^T_{e_i})^{Tr}_{\alpha'}(t_1)\} = \max\{(A_K)^T_{e_i}(t_1) + \alpha, (A_K)^T_{e_i}(t_1) + \alpha'\} = ((A_K)^T_{e_i})^{Tr}_{\alpha} \cap (A_K)^T_{e_i})^{Tr}_{\alpha'})(t_1)$. $(((A_K)^I_{e_i})^{Tr}_{\beta} \cap ((A_K)^I_{e_i})^{Tr}_{\beta'})(0) = (((A_K)^I_{e_i})^{Tr}_{\beta} \cap ((A_K)^I_{e_i})^{Tr}_{\beta'})(t_1 * t_1) = \max\{((A_K)^I_{e_i})^{Tr}_{\beta}(t_1 * t_1), ((A_K)^I_{e_i})^{Tr}_{\beta'}(t_1 * t_1)\} \leq \max\{\max\{((A_K)^I_{e_i})^{Tr}_{\beta}(t_1), ((A_K)^I_{e_i})^{Tr}_{\beta}(t_1)\}, \max\{((A_K)^I_{e_i})^{Tr}_{\beta'}(t_1), ((A_K)^I_{e_i})^{Tr}_{\beta'}(t_1)\}\} = \max\{((A_K)^I_{e_i})^{Tr}_{\beta}(t_1), ((A_K)^I_{e_i})^{Tr}_{\beta'}(t_1)\} = \max\{(A_K)^I_{e_i}(t_1) + \beta, (A_K)^I_{e_i}(t_1) + \beta'\} = ((A_K)^I_{e_i})^{Tr}_{\beta} \cap (A_K)^I_{e_i})^{Tr}_{\beta'})(t_1)$ $(((A_K)^F_{e_i})^{Tr}_{\gamma} \cap ((A_K)^F_{e_i})^{Tr}_{\gamma'})(0) = (((A_K)^F_{e_i})^{Tr}_{\gamma} \cap ((A_K)^F_{e_i})^{Tr}_{\gamma'})(t_1 * t_1) = \max\{((A_K)^F_{e_i})^{Tr}_{\gamma}(t_1 * t_1), ((A_K)^F_{e_i})^{Tr}_{\gamma'}(t_1 * t_1)\} \leq \max\{\max\{((A_K)^F_{e_i})^{Tr}_{\gamma}(t_1), ((A_K)^F_{e_i})^{Tr}_{\gamma}(t_1)\}, \max\{((A_K)^F_{e_i})^{Tr}_{\gamma'}(t_1), ((A_K)^F_{e_i})^{Tr}_{\gamma'}(t_1)\}\} = \max\{((A_K)^F_{e_i})^{Tr}_{\gamma}(t_1), ((A_K)^F_{e_i})^{Tr}_{\gamma'}(t_1)\} = \max\{(A_K)^F_{e_i}(t_1) + \gamma, (A_K)^F_{e_i}(t_1) + \gamma'\} = ((A_K)^F_{e_i})^{Tr}_{\gamma} \cap (A_K)^F_{e_i})^{Tr}_{\gamma'})(t_1)$.

Now
$(((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma} \cap ((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'})(t_1 * t_3) = rmin\{((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}(t_1 * t_3) \geq ((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}(t_1 * t_3)\} \geq rmin\{rmin\{((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}((t_1 * t_2) * t_3), ((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}(t_2)\}, rmin\{((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}((t_1 * t_2) * t_3), ((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}(t_2)\}\} = rmin\{rmin\{((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}((t_1 * t_2) * t_3), ((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}((t_1 * t_2) * t_3)\}, rmin\{((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}(t_2), ((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}(t_2)\}\} = \{rmin\{(((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma} \cap ((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'})((t_1 * t_2) * t_3), (((\widehat{N}_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma} \cap ((\widehat{N}_K)^{Tr,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'})(t_2)\}$.

And
$(((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma} \cap ((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'})(t_1 * t_3) = \max\{((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}(t_1 * t_3), ((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}(t_1 * t_3)\} \leq \max\{\max\{((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}((t_1 * t_2) * t_3), ((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}(t_2)\} \max\{((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}$

$((t_1 * t_2) * t_3), ((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}(t_2)\}\} = \max\{\max\{((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}((t_1 * t_2) * t_3), ((A_K)^{T,I,F}_{e_i})$ $^{Tr}_{\alpha',\beta',\gamma'}((t_1 * t_2)\ t_3)\}, \max\{((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma}(t_2), ((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'}(t_2)\ \{\max\{(((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma} \cap$ $((A_K)^{T,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'})((t_1 * t_2) * t_3),\ (((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha,\beta,\gamma} \cap ((A_K)^{T,I,F}_{e_i})^{Tr}_{\alpha',\beta',\gamma'})(t_2)\}$.

Hence $\widetilde{K}^{Tr}_{\alpha,\beta,\gamma} \cap \widetilde{K}^{Tr}_{\alpha',\beta',\gamma'}$ is an NSCTID of P. □

5.2. *Neutrosophic Soft Cubic T-Ideal Multiplication*

This section defines neutrosophic soft cubic T-Ideal multiplication with a theorem and example.

**Theorem 5.7.** *If* K *is an NSCTID of* P*, then NSCMp* $\widetilde{K}^{Mp}_{\eta}$ *of* K *is an NSCTID of* P*, for all* $\eta\epsilon\ [0,1]$.

*Proof.* Let K be an NSCTID of P and $\eta \in [0,1]$. Then we have $((\widehat{N}_K)^T_{e_i})^{Mp}_{\eta}(0) = \eta \cdot (\widehat{N}_K)^T_{e_i}(0) \geq \eta \cdot (\widehat{N}_K)^T_{e_i}(t_1) \rightarrow ((\widehat{N}_K)^T_{e_i})^{Mp}_{\eta}(0) \geq ((\widehat{N}_K)^T_{e_i})^{Mp}_{\eta}(t_1)$, $((\widehat{N}_K)^I_{e_i})^{Mp}_{\eta}(0) = \eta \cdot (\widehat{N}_K)^I_{e_i}(0) \geq \eta \cdot (\widehat{N}_K)^I_{e_i}(t_1) \rightarrow ((\widehat{N}_K)^I_{e_i})^{Mp}_{\eta}(0) \geq ((\widehat{N}_K)^I_{e_i})^{Mp}_{\eta}(t_1)$, $((\widehat{N}_K)^F_{e_i})^{Mp}_{\eta}(0) = \eta \cdot (\widehat{N}_K)^F_{e_i}(0) \geq \eta \cdot (\widehat{N}_K)^F_{e_i}(t_1) \rightarrow ((\widehat{N}_K)^F_{e_i})^{Mp}_{\eta}(0) \geq ((\widehat{N}_K)^F_{e_i})^{Mp}_{\eta}(t_1)$,

And $((A_K)^T_{e_i})^{Mp}_{\eta}(0) = \eta \cdot (A_K)^T_{e_i}(0) \leq \eta \cdot (A_K)^T_{e_i}(t_1) \rightarrow ((A_K)^T_{e_i})^{Mp}_{\eta}(0) \leq ((A_K)^T_{e_i})^{Mp}_{\eta}(t_1)$, $((A_K)^I_{e_i})^{Mp}_{\eta}(0) = \eta \cdot (A_K)^I_{e_i}(0) \leq \eta \cdot (A_K)^I_{e_i}(t_1) \rightarrow ((A_K)^I_{e_i})^{Mp}_{\eta}(0) \leq ((A_K)^I_{e_i})^{Mp}_{\eta}(t_1)$, $((A_K)^F_{e_i})^{Mp}_{\eta}(0) = \eta \cdot (A_K)^F_{e_i}(0) \leq \eta \cdot (A_K)^F_{e_i}(t_1) \rightarrow ((A_K)^F_{e_i})^{Mp}_{\eta}(0) \leq ((A_K)^F_{e_i})^{Mp}_{\eta}(t_1)$, Now $((\widehat{N}_K)^T_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \eta \cdot (\widehat{N}_K)^T_{e_i}(t_1 * t_3) \geq \eta \cdot \mathrm{rmin}\{(\widehat{N}_K)^T_{e_i}((t_1 * t_2) * t_3), (\widehat{N}_K)^T_{e_i}(t_2)\} = \mathrm{rmin}\{\eta \cdot (\widehat{N}_K)^T_{e_i}((t_1 * t_2) * t_3), \eta \cdot (\widehat{N}_K)^T_{e_i}(t_2)\}$ $((\widehat{N}_K)^T_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \mathrm{rmin}\{((\widehat{N}_K)^T_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((\widehat{N}_K)^T_{e_i})^{Mp}_{\eta}(t_2)\}$ $((\widehat{N}_K)^T_{e_i})^{Mp}_{\eta}(t_1 * t_3) \geq \mathrm{rmin}\{((\widehat{N}_K)^T_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((\widehat{N}_K)^T_{e_i})^{Mp}_{\eta}(t_2)\}$, $((\widehat{N}_K)^I_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \eta \cdot (\widehat{N}_K)^I_{e_i}(t_1 * t_3) \geq \eta \cdot \mathrm{rmin}\{(\widehat{N}_K)^I_{e_i}((t_1 * t_2) * t_3), (\widehat{N}_K)^I_{e_i}(t_2)\} = \mathrm{rmin}\{\eta \cdot (\widehat{N}_K)^I_{e_i}((t_1 * t_2) * t_3), \eta \cdot (\widehat{N}_K)^I_{e_i}(t_2)\}$ $((\widehat{N}_K)^I_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \mathrm{rmin}\{((\widehat{N}_K)^I_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((\widehat{N}_K)^I_{e_i})^{Mp}_{\eta}(t_2)\}$ $((\widehat{N}_K)^I_{e_i})^{Mp}_{\eta}(t_1 * t_3) \geq \mathrm{rmin}\{((\widehat{N}_K)^I_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((\widehat{N}_K)^I_{e_i})^{Mp}_{\eta}(t_2)\}$, $((\widehat{N}_K)^F_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \eta \cdot (\widehat{N}_K)^F_{e_i}(t_1 * t_3) \geq \eta \cdot \mathrm{rmin}\{(\widehat{N}_K)^F_{e_i}((t_1 * t_2) * t_3), (\widehat{N}_K)^F_{e_i}(t_2)\} = \mathrm{rmin}\{\eta \cdot (\widehat{N}_K)^F_{e_i}((t_1 * t_2) * t_3), \eta \cdot (\widehat{N}_K)^F_{e_i}(t_2)\}$ $((\widehat{N}_K)^F_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \mathrm{rmin}\{((\widehat{N}_K)^F_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((\widehat{N}_K)^F_{e_i})^{Mp}_{\eta}(t_2)\}$ $((\widehat{N}_K)^F_{e_i})^{Mp}_{\eta}(t_1 * t_3) \geq \mathrm{rmin}\{((\widehat{N}_K)^F_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((\widehat{N}_K)^F_{e_i})^{Mp}_{\eta}(t_2)\}$

And $((A_K)^T_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \eta \cdot (A_K)^T_{e_i}(t_1 * t_3) \leq \eta \cdot \max\{(A_K)^T_{e_i}((t_1 * t_2) * t_3), (A_K)^T_{e_i}(t_2)\}$ $= \max\{\eta \cdot (A_K)^T_{e_i}((t_1 * t_2) * t_3), \eta \cdot (A_K)^T_{e_i}(t_2)\}$ $((A_K)^T_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \max\{((A_K)^T_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((A_K)^T_{e_i})^{Mp}_{\eta}(t_2)\}$ $((A_K)^T_{e_i})^{Mp}_{\eta}(t_1 * t_3) \leq \max\{((A_K)^T_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((A_K)^T_{e_i})^{Mp}_{\eta}(t_2)\}$, $((A_K)^{I}_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \eta \cdot (A_K)^I_{e_i}(t_1 * t_3) \leq \eta \cdot \max\{(A_K)^I_{e_i}((t_1 * t_2) * t_3), (A_K)^I_{e_i}(t_2)\} = \max\{\eta \cdot (A_K)^I_{e_i}((t_1 * t_2) * t_3), \eta \cdot (A_K)^I_{e_i}(t_2)\}$ $((A_K)^I_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \max\{((A_K)^I_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((A_K)^I_{e_i})^{Mp}_{\eta}(t_2)\}$ $((A_K)^I_{e_i})^{Mp}_{\eta}(t_1 * t_3) \leq \max\{((A_K)^I_{e_i})^{Mp}_{\eta}((t_1 * t_2) * t_3), ((A_K)^I_{e_i})^{Mp}_{\eta}(t_2)\}$c, $((A_K)^{F}_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \eta \cdot (A_K)^F_{e_i}(t_1 * t_3) \leq \eta \cdot \max\{(A_K)^F_{e_i}((t_1 * t_2) * t_3), (A_K)^F_{e_i}(t_2)\} = \max\{\eta \cdot (A_K)^F_{e_i}((t_1 * t_2) * t_3), \eta \cdot (A_K)^F_{e_i}(t_2)\}$ $((A_K)^F_{e_i})^{Mp}_{\eta}(t_1 * t_3) = \max\{((A_K)^F_{e_i})^{Mp}_{\eta}((t_1 * t_2) *$

$t_3), ((A_K)^F_{e_i})^{Mp}_\eta(t_2)\}$ $((A_K)^F_{e_i})^{Mp}_\eta(t_1 * t_3) \leq \max\{((A_K)^F_{e_i})^{Mp}_\eta((t_1 * t_2) * t_3), ((A_K)^F_{e_i})^{Mp}_\eta(t_2)\}$. Hence $\widetilde{K}^{Mp}_\eta$ of K is an NSCTID of P, for all $\eta \in [0, 1]$. □

**Example 5.8.** Let $P = \{0, t_1, t_2, t_3\}$ be a PS-algebra with the cayley's table as shown in Table 1 . The NSC-set $K = \langle(\widehat{N}_K)^{T,I,F}_{e_i}, (A_K)^{T,I,F}_{e_i}\rangle$ of P is defined as

$$(\widehat{N}_K)^T_{e_i}(t_i) = \{[0.7, 0.9] \text{ if } t_i = 0 \text{ and } [0.3, 0.6]. \text{ if otherwise}$$
$$(\widehat{N}_K)^I_{e_i}(t_i) = \{[0.6, 0.8] \text{ if } t_i = 0 \text{ and } [0.4, 0.5]. \text{ if otherwise}$$
$$(\widehat{N}_K)^F_{e_i}(t_i) = \{[0.5, 1] \text{ if } t_i = 0 \text{ and } [0.2, 0.7]. \text{ if otherwise.}$$

And

$$(A_K)^T_{e_i}(t_i) = \{0.2 \text{ if } t_i = 0 \text{ and } 0.7. \text{ if otherwise}$$
$$(A_K)^I_{e_i}(t_i) = \{0.5 \text{ if } t_i = 0 \text{ and } 0.9. \text{ if otherwise}$$
$$(A_K)^F_{e_i}(t_i) = \{0.4 \text{ if } t_i = 0 \text{ and } 1. \text{ if otherwise.}$$

The set K is an NSCTID of P as it satisfies (i) but fulfills (ii) and (iii) with the condition: If $i = j$ then $t_i * t_j = 0$ with $(\widehat{N}_K)^T_{e_i}(t_i * t_j) = [0.7, 0.9], (\widehat{N}_K)^I_{e_i}(t_i * t_j) = [0.6, 0.8], (\widehat{N}_K)^F_{e_i}(t_i * t_j) = [0.5, 1]$
Otherwise

$$(\widehat{N}_K)^T_{e_i}(t_i * t_j) = [0.3, 0.6], (\widehat{N}_K)^I_{e_i}(t_i * t_j) = [0.4, 0.5], (\widehat{N}_K)^F_{e_i}(t_i * t_j) = [0.2, 0.7],$$

And

$$(A_K)^T_{e_i}(t_i) = 0.2, (\ A_K)^I_{e_i}(t_i) = 0.5, (\ A_K)^F_{e_i}(t_i) = 0.4$$

Otherwise $(A_K)^T_{e_i}(t_i) = 0.7, (A_K)^I_{e_i}(t_i) = 0.9, (A_K)^F_{e_i}(t_i) = 1$ given in definition 3.1. Now, for $(\widehat{N}_K)^{T,I,F}_{e_i}\eta \in [0.2, 0.5]$ and for $(A)^{T,I,F}_{e_i}, \eta = 0.2$. Then the mapping $\widetilde{K}^{Mp}_\eta \mid Parrow[0, 1]$ is given by

$$((\widehat{N}_K)^T_{e_i})^{Mp}_{[0.2,0.5]}(0) = [0.2, 0.5] \cdot (\widehat{N}_K)^T_{e_i}(0) = [0.14, 0.45]$$
$$((\widehat{N}_K)^I_{e_i})^{Mp}_{[0.2,0.5]}(0) = [0.2, 0.5] \cdot (\widehat{N}_K)^I_{e_i}(0) = [0.12, 0.4]$$
$$((\widehat{N}_K)^F_{e_i})^{Mp}_{[0.2,0.5]}(0) = [0.2, 0.5] \cdot (\widehat{N}_K)^F_{e_i}(0) = [0.1, 0.5]$$
$$((\widehat{N}_K)^T_{e_i})^{Mp}_{[0.2,0.5]}(t_i) = [0.2, 0.5] \cdot (\widehat{N}_K)^T_{e_i}(t_i) = [0.06, 0.3]$$
$$((\widehat{N}_K)^I_{e_i})^{Mp}_{[0.2,0.5]}(t_i) = [0.2, 0.5] \cdot (\widehat{N}_K)^I_{e_i}(t_i) = [0.08, 0.25]$$
$$((\widehat{N}_K)^F_{e_i})^{Mp}_{[0.2,0.5]}(t_i) = [0.2, 0.5].(\widehat{N}_K)^F_{e_i}(t_i) = [0.04, 0.35],$$

And

$$((A_K)^T_{e_i})^{Mp}_{0.03}(0) = 0.2.(A_K)^T_{e_i}(0) = [0.04]$$
$$((A_K)^F_{e_i})^{Mp}_{0.2}(0) = 0.2.(A_K)^T_{e_i}(0) = [0.08]$$
$$((A_K)^T_{e_i})^{Mp}_{0.2}(t_i) = 0.2.(A_K)^T_{e_i}(t_i) = [0.14]$$
$$((A_K)^I_{e_i})^{Mp}_{0.2}(t_i) = 0.2.(A_K)^T_{e_i}(t_i) = [0.18]$$
$$((A_K)^F_{e_i})^{Mp}_{0.2}(t_i) = 0.2.(A_K)^T_{e_i}(t_i) = [0.2]$$

Hence $\widetilde{K}_{\eta}^{Mp}$ is an NSCTID of P.

**Theorem 5.9.** *The union of any two NSC-multiplications of an NSCTID is an NSCTID of* P*.*

*Proof.* Suppose $\widetilde{K}_{\eta}^{Mp}$ and $\widetilde{K}_{\eta'}^{Mp}$ are two NSC-multiplications of an NSCTID is an NSCTID of P, where $\eta, \eta' \in (0,1]$ and $\eta \leq \eta'$. As we know that $\widetilde{K}_{\eta}^{Mp}$ and $\widetilde{K}_{\eta'}^{Mp}$ are NSCTIDs of P. Then $(((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta}^{Mp} \cup ((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp})(0) = (((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta}^{Mp} \cup ((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp})(t_1 * t_1) = \text{rmin}\{((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta}^{Mp}(t_1 * t_1), ((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}(t_1 * t_1)\} \geq \text{rmin}\{\text{rmin}\{((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta}^{Mp}(t_1), ((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta}^{Mp}(t_1)\}, \text{rmin}\{((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}(t_1), ((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}(t_1)\}\} = \text{rmin}\{((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta}^{Mp}(t_1), ((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}(t_1)\} = \text{rmin}\{\eta \cdot (\widehat{N}_K)_{e_i}^{T,I,F}(t_1), \eta' \cdot (\widehat{N}_K)_{e_i}^{T,I,F}(t_1)\} = ((\widehat{N}_K)_{e_i}^{TI,F})_{\eta}^{Mp} \cup (\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp})(t_1)$.

And

$(((A_K)_{e_i}^{T,I,F})_{\eta}^{Mp} \cup ((A_K)_{e_i}^{T,I,F})_{\eta'}^{Mp})(0) = (((A_K)_{e_i}^{T,I,F})_{\eta}^{Mp} \cup ((A_K)_{e_i}^{T,I,F})_{\eta'}^{Mp})(t_1 * t_1) = \max\{((A_K)_{e_i}{}^{T,I,F})_{\eta}^{Mp}(t_1 * t_1), ((A_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}(t_1 * t_1)\} \leq \max\{\max\{((A_K)_{e_i}^{T,I,F})_{\eta}^{Mp}(t_1), ((A_K)_{e_i}^{TI,F})_{\eta}^{Mp}(t_1)\}, \max\{((A_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}(t_1), ((A_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}(t_1)\}\} = \max\{((A_K)_{e_i}^{T,IF})_{\eta}^{Mp}(t_1), ((A_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}(t_1)\} = \max\{\eta \cdot (A_K)_{e_i}^{T,I,F}(t_1), \eta' \cdot (A_K)_{e_i}^{T,I,F}(t_1)\} = ((A_K)_{e_i}^{T,I,F})_{\eta}^{Mp} \cup (A_K)_{e_i}^{T,I,F})_{\eta'}^{Mp})(t_1)$. $(((\widehat{N}_K)_{e_i}^{T,IF})_{\eta}^{Mp} \cup ((\widehat{N}_K)_{e_i}{}^{T,I,F})_{\eta'}^{Mp})(t_1 * t_3) = \text{rmin}\{((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta}^{Mp}(t_1 * t_3), ((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}(t_1 * t_3)\} \geq \text{rmin}\{\text{rmin}\{((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta}^{Mp}((t_1 * t_2) * t_3), ((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta}^{Mp}(t_2)\}, \text{rmin}\{((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}((t_1 * t_2) * t_3), ((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}(t_2)\}\} = \text{rmin}\{\text{rmin}\{((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta}^{Mp}((t_1 * t_2) * t_3), ((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}((t_1 * t_2) * t_3)\}, \text{rmin}\{((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta}^{Mp}(t_2), ((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}(t_2)\}\} = \{\text{rmin}\{(((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta}^{Mp} \cup ((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp})((t_1 * t_2) * t_3), (((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta}^{Mp} \cup ((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp})(t_2)\}$.

And

$(((A_K)_{e_i}^{T,I,F})_{\eta}^{Mp} \cup ((A_K)_{e_i}^{T,I,F})_{\eta'}^{Mp})(t_1 * t_3) = \max\{((A_K)_{e_i}^{T,I,F})_{\eta}^{Mp}(t_1 * t_3), ((A_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}(t_1 * t_3)\} \leq \max\{\max\{((A_K)_{e_i}^{T,I,F})_{\eta}^{Mp}((t_1 * t_2) * t_3), ((A_K)_{e_i}^{T,I,F})_{\eta}^{Mp}(t_2)\}. \max\{((A_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}((t_1 * t_2) * t_3), ((A_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}(t_2)\}\} = \max\{\max\{((A_K)_{e_i}^{T,I,F})_{\eta}^{Mp}((t_1 * t_2) * t_3), ((A_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}((t_1 * t_2) * t_3)\}, \max\{((A_K)_{e_i}^{T,I,F})_{\eta}^{Mp}(t_2), ((A_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}(t_2)\}\} = \{\max\{(((A_K)_{e_i}^{T,I,F})_{\eta}^{Mp} U ((A_K)_{e_i}^{T,I,F})_{\eta'}^{Mp})((t_1\ t_2) * t_3), (((A_K)_{e_i}^{T,I,F})_{\eta}^{Mp} \cup ((A_K)_{e_i}^{T,I,F})_{\eta'}^{Mp})(t_2)\}$

Hence $\widetilde{K}_{\eta}^{Mp} \cup \widetilde{K}_{\eta'}^{Mp}$ is an NSCTID of P. □

**Theorem 5.10.** *The intersection of any two NSC-multiplications of an NSCTID is an NSCTID of* P*.*

*Proof.* Suppose $\widetilde{K}_{\eta}^{Mp}$ and $\widetilde{K}_{\eta'}^{Mp}$ are two NSC-multiplications of an NSCTID is an NSCTID of P, where $\eta, \eta' \in (0,1]$ and $\eta \leq \eta'$. As we know that $\widetilde{K}_{\eta}^{Mp}$ and $\widetilde{K}_{\eta'}^{Mp}$ are NSCTIDs of P. Then $(((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta}^{Mp} \cap ((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp})(0) = (((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta}^{Mp} \cap ((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp})(t_1 * t_1) = \text{rmin}\{((\widehat{N}_K)_{e_i}{}^{T,I,F})_{\eta}^{Mp}(t_1 * t_1), ((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}(t_1 * t_1)\} \geq \text{rmin}\{\text{rmin}\{((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta}^{Mp}(t_1), ((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta}^{Mp}(t_1)\}, \text{rmin}\{((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}(t_1), ((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}(t_1)\}\} = \text{rmin}\{((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta}^{Mp}(t_1), ((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp}(t_1)\} = \text{rmin}\{\eta \cdot (\widehat{N}_K)_{e_i}^{T,I,F}(t_1), \eta' \cdot (\widehat{N}_K)_{e_i}^{T,I,F}(t_1)\} = ((\widehat{N}_K)_{e_i}^{T,I,F})_{\eta}^{Mp} \cap (\widehat{N}_K)_{e_i}^{T,I,F})_{\eta'}^{Mp})(t_1)$.

And

$(((\mathrm{A_K})_{\mathrm{e}_i}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \cap ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})(0) = (((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \cap ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})(\mathrm{t_1} * \mathrm{t_1}) = \max\{((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t_1} * \mathrm{t_1}), ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t_1} * \mathrm{t_1})\} \leq \max\{\max\{((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t_1}), ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t_1})\}, \max\{((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t_1}), ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t_1})\}\} = \max\{((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I}})_{\eta}^{\mathrm{Mp}}(\mathrm{t_1}), ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,IF}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t_1})\} = \max\{\eta \cdot (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}}(\mathrm{t_1}), \eta' \cdot (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}}(\mathrm{t_1})\} = ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \cap (\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})(\mathrm{t_1})$. $(((\widehat{\mathrm{N}}_\mathrm{K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \cap ((\widehat{\mathrm{N}}_\mathrm{K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})(\mathrm{t_1} * \mathrm{t_3}) = \mathrm{rmin}\{((\widehat{\mathrm{N}}_\mathrm{K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t_1} * \mathrm{t_3}), ((\widehat{\mathrm{N}}_\mathrm{K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t_1} * \mathrm{t_3})\} \geq \mathrm{rmin}\{\mathrm{rmin}\{((\widehat{\mathrm{N}}_\mathrm{K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\widehat{\mathrm{N}}_\mathrm{K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t_2})\}, \mathrm{rmin}\{((\widehat{\mathrm{N}}_\mathrm{K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\widehat{\mathrm{N}}_\mathrm{K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t_2})\}\} = \mathrm{rmin}\{\mathrm{rmin}\{((\widehat{\mathrm{N}}_\mathrm{K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\widehat{\mathrm{N}}_\mathrm{K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3})\}, \mathrm{rmin}\{((\widehat{\mathrm{N}}_\mathrm{K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t_2}), ((\widehat{\mathrm{N}}_\mathrm{K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t_2})\}\} = \{\mathrm{rmin}\{(((\widehat{\mathrm{N}}_\mathrm{K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \cap ((\widehat{\mathrm{N}}_\mathrm{K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3})\ (((\widehat{\mathrm{N}}_\mathrm{K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \cap ((\widehat{\mathrm{N}}_\mathrm{K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})(\mathrm{t_2})\}$.

And

$(((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \cap ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})(\mathrm{t_1} * \mathrm{t_3}) = \max\{((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t_1} * \mathrm{t_3}), ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t_1} * \mathrm{t_3})\} \leq \max\{\max\{((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t_2})\}, \max\{((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t_2})\}\} = \max\{\max\{((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3})\}, \max\{((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}}(\mathrm{t_2}), ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}}(\mathrm{t_2})\}\} = \{\max\{(((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \cap ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})((\mathrm{t_1} * \mathrm{t_2}) * \mathrm{t_3}), (((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta}^{\mathrm{Mp}} \cap ((\mathrm{A_K})_{\mathrm{e_i}}^{\mathrm{T,I,F}})_{\eta'}^{\mathrm{Mp}})(\mathrm{t_2})\}$.

Hence $\widetilde{\mathrm{K}}_{\eta}^{\mathrm{Mp}} \cap \widetilde{\mathrm{K}}_{\eta'}^{\mathrm{Mp}}$ is an NSCTID of P. □

## 6. **Conclusion**

This paper extensively explores the application of the neutrosophic soft cubic set to investigate specific properties of the t-ideal in a PS-algebra. The derived definitions and fundamental outcomes hold potential for broader use in other algebraic structures like Lie algebras and lattices in the future. Moreover, there are several areas where this proposed structure can be advantageous, such as DNA identification, formalizing procedures in genetic algorithms, genomics, fuzzy logic-based networks, and particularly complex networks. Within the already-existing neutrosophic cubic structures, the addition of soft sets with T-ideal properties makes this structure a better choice for future implementations. This can also be extended to the hypersoft sets and can be advantegeous for MCDM algorithms in various real life applications such as transportation, healthcare etc.

One potential limitation of our proposed structure NSCTID is the ground algebra because the results and properties in this study may or may not be satisfy for other algebras in future. Also the practical implementation and computational efficiency of the NSCTID framework in real-world applications may pose challenges. The translation of theoretical concepts into efficient algorithms and the handling of computational complexities could require further investigation and optimization to fully realize the benefits of NSCTID in practical scenarios.

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# An application of neutrosophic theory on manifolds and their topological transformations

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Abstract. This paper presents an investigation into the mathematical concepts of neutrosophic folding and neuretraction on neutrosophic manifolds, specifically focusing on their application in hyperspace. Through the application of specific transformations on a neutrosophic manifold situated in hyperspace, we can obtain neutrosophic manifolds in lower dimensions. Based on our research, we can accurately establish the connection between neutrosophic folding and neuretraction on a neutrosophic manifold. Furthermore, we can determine the relationship between neuretraction and neutrosophic folding.

**Keywords:** neutrosophic folding; neuretraction; neutrosophic hyperspace; neutrosophic manifold.

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## 1. Introduction

Neutrosophy is a scientific field that combines neutrality and philosophy. Samaransache founded various fields in 1980, such as set theory, probability, and logic, with numerous applications that highlight the deep interaction between mathematics and other scientific disciplines. [19]. The concept of fuzzy sets was introduced by Zadeh as a novel method for elucidating intricate concepts by including the concept of membership. Scholars in the fields of mathematics and computer science developed this theory, which possesses a broad spectrum of expedient applications [22]. Neutrosophy is basically rooted in the fundamental concepts of fuzzy set theory ($NS$) and intuitionistic fuzzy set theory ($IFS$) [8, 10, 13, 22]. The concept of neutrosophic sets was introduced by Smarandache with the aim of representing uncertain or vague information. This is achieved through the utilization of three distinct functions, namely truth, indeterminacy, and falsity. Unlike other theories, the function of indeterminacy is independent of the functions of truth and falsity [12,18,19]. Smarandache's $(NS)$ theory expanded the scope of $(IFS)$, offering novel perspectives on how to effectively manage uncertainty when making decisions based on personal experience, as stated in reference [20]. The values of the truth, indeterminacy, and falsity functions are within $]^{-}0,\ 1^{+}[$, making it difficult to apply to practical problems [17].

Due to this, Wang created the single-valued neutrosophic sets $(SVNS)$, such that the truth, indeterminacy, and falsity maps are real elements of the $[0,\ 1]$ space [12, 21]. Further investigation on a $(SVNG)$ and a neutrosophic topology was debated in [1, 4, 6, 7, 9, 11, 14, 16]. Additional insights on the applications of homotopy theory were provided in [2, 3]. The paper aims to contribute to the field of mathematics by exploring and providing a deeper understanding of the neutrosophic transformation in the context of neutrosophic manifold theory.

## 2. **Preliminaries**

**Definition 2.1.** [19] Assume that $\mathcal{W}$ is a finite set of objects, and that $(\mathfrak{t})$ stands for a generic component in $\mathcal{W}$. A $(NS)$ E in $\mathcal{W}$ is comprised of three membership functions, a truth-membership function $\upsilon_{\mathrm{E}}(\mathfrak{t})$, an indeterminacy-membership function $\rho_{\mathrm{E}}(\mathfrak{t})$ and a falsity-membership function $\sigma_{\mathrm{E}}(\mathfrak{t})$. Also, $\upsilon_{\mathrm{E}}(\mathfrak{t}),\ \rho_{E}\left(\mathfrak{t}\right)$ and $\sigma_{E}\left(\mathfrak{t}\right)$ are the elements of $]^{-}0,\ 1^{+}[$. E can be represented as

E$=\{\mathfrak{t},(\upsilon_{\mathrm{E}}(\mathfrak{t}),\ \rho_{\mathrm{E}}(\mathfrak{t}),\ \sigma_{\mathrm{E}}(\mathfrak{t})) : \mathfrak{t}\in\mathcal{W},\ \upsilon_{\mathrm{E}}(\mathfrak{t}),\ \rho_{\mathrm{E}}(\mathfrak{t}),\ \sigma_{\mathrm{E}}(\mathfrak{t})\in\ ]^{-}0,\ 1^{+}[\}$. Indeed, $^{-}0\leq\upsilon_{\mathrm{E}}(\mathfrak{t})+\rho_{\mathrm{E}}(\mathfrak{t})+\sigma_{\mathrm{E}}(\mathfrak{t})\leq 3^{+}$.

**Definition 2.2.** [21] Assume that $\mathcal{W}$ is a finite set of objects, and that $(\mathfrak{t})$ stands for a generic component in $\mathcal{W}$ . A $(SVNS)$ $E$ in $\mathcal{W}$ is comprised of three membership functions, a truth-membership function $\upsilon_{\mathrm{E}}\left(\mathfrak{t}\right)$, an indeterminacy-membership function $\rho_{\mathrm{E}}\left(\mathfrak{t}\right)$ and a falsity-membership function $\sigma_{\mathrm{E}}\left(\mathfrak{t}\right)$. Also each, $\upsilon_{\mathrm{E}}\left(\mathfrak{t}\right),\rho_{\mathrm{E}}\left(\mathfrak{t}\right)$ and $\sigma_{\mathrm{E}}\left(\mathfrak{t}\right)$ are elements in $]0,\ 1[$. E can be represented as E$=\{\mathfrak{t},(\upsilon_{\mathrm{E}}(\mathfrak{t}),\ \rho_{\mathrm{E}}(\mathfrak{t}),\ \sigma_{\mathrm{E}}(\mathfrak{t})) : \mathfrak{t}\in\mathcal{W},\ \upsilon_{\mathrm{E}}(\mathfrak{t}),\ \rho_{\mathrm{E}}(\mathfrak{t}),\ \sigma_{\mathrm{E}}(\mathfrak{t})\in\ ]0,\ 1[\}$. In this approach, $0\leq\upsilon_{\mathrm{E}}(\mathfrak{t})+\rho_{\mathrm{E}}(\mathfrak{t})+\sigma_{\mathrm{E}}(\mathfrak{t})\leq 3$. In the interest of clarity and concision, we refer to a neutrosophic set $\langle\upsilon_{\mathrm{E}},\ \rho_{\mathrm{E}},\ \sigma_{\mathrm{E}}\rangle$ and $\langle\upsilon_{\mathrm{E}}(\mathfrak{t}),\ \rho_{\mathrm{E}}(\mathfrak{t}),\ \sigma_{\mathrm{E}}(\mathfrak{t})\rangle$ as $\omega$ and $\omega\left(\mathfrak{t}\right)$ respectively.

**Definition 2.3.** [15] A topological space that satisfies the $T_2$ separation axiom and is locally homeomorphic to an open $n$-dimensional disk $U^n$ is referred to as an $n$-dimensional manifold.

**Definition 2.4.** [5] Let $X$ be a topological space and $C$be a subspace of $X$, where $i : C\to X$ is the inclusion. if there exists a continuous map $r : X\to C$ satisfying the condition $r\circ i = 1|_C$. Then, $C$ is referred to a retract of $X$. The existence of a map $r$ is denoted as a retraction of $X$ into $C$.

**Theorem 2.5.** *[5] The $n$-dimensional closed disk $L^n = \{z \in \mathcal{R}^n : |z| \leq 1\}$ is a retract of $\mathcal{R}^n$.*

**Definition 2.6.** [15] Consider two topological spaces $X_1$ and $X_2$, and let $\varphi_0$ and $\varphi_1$denote continuous mappings from $X_1$ to $X_1$. The homotopy between $\varphi_0$ and $\varphi_1$is established when a continuous map $\varphi : X_1 \times I \to X_2$ exists, and satisfying the conditions $\varphi(s, 0) = \varphi_0(s)$ and $\varphi(s, 1) = \varphi_1(s)$ for all $s \in X_1$.

## 3. Neutrosophic manifolds and their transformations

Our study introduces a collection of important concepts that support our paper and enable us to arrive at significant conclusions.

**Definition 3.1.** A neutrosophic $n$-dimensional manifold is characterized as a pair $\langle M^n,\ \omega\rangle$ in which, $M^n$ is $n$ -dimensional manifold.

**Example 3.2.** A neutrosophic Euclidean $n$-space $\langle \mathcal{R}^n,\ \omega\rangle$ can be regarded as a neutrosophic $n$-dimensional manifold. Additionally, a neutrosophic unit $n$-dimensional sphere $\langle S^n,\ \omega\rangle$ can be considered as a neutrosophic $n$-dimensional manifold.

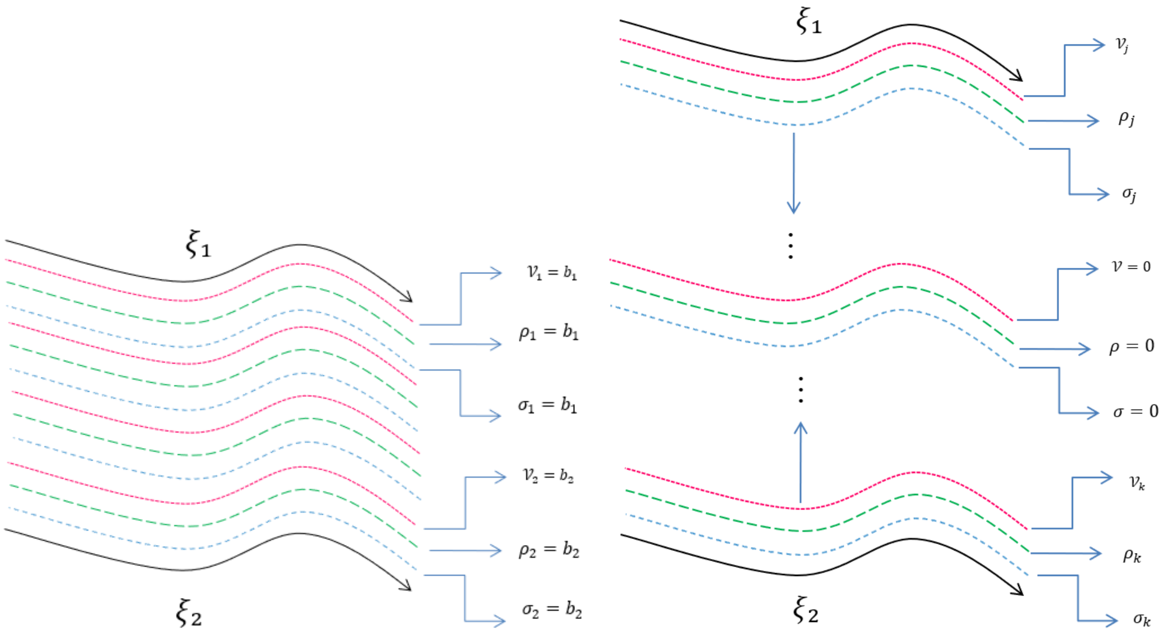
**Definition 3.3.** The neutrosophic arc $\zeta : [0,\ 1] \to \mathcal{R}^3$ is called a simple neutrosophic arc if, for each $\mathrm{z_j}$, $\mathrm{z_k} \in [0,\ 1]$, $\xi((\mathrm{z_j},\ \omega_\mathrm{j}) \neq \xi((\mathrm{z_k},\ \omega_\mathrm{k})$ whenever $(\mathrm{z_j}\ ,\ \omega_\mathrm{j}) \neq (\mathrm{z_k},\ \omega_\mathrm{k})$.

Now, we will delve into the notion of neutrosophy homotopic and describe two types of it.

**Definition 3.4.** A neutrosophic homotopy is a collection of neutrosophic maps $h_t : \langle M,\ \omega\rangle \to \langle N,\ \omega\rangle$, $t \in [0, 1]$, in which the associated neutrosophic map $\Phi : \langle M,\ \omega\rangle \times [0, 1] \to \langle N,\ \omega\rangle$ given by $\Phi((x,\ \omega), t) = h_t(x,\ \omega)$, and the two neutrosophic maps $h_0,\ h_1 : \langle M,\ \omega\rangle \to \langle N,\ \omega\rangle$ are called neutrosophy homotopic if there is a neutrosophic homotopy $h_t$ that connects them and is represented by $h_0 \approx h_1$.

**Theorem 3.5.** *Let $\xi_1$ and $\xi_2$ be two neutrosophic arcs. Then, there are two types of neutrosophy homotopic arcs.*

*Proof.* The initial category encompasses a pair of neutrosophic arcs, $\xi_1$ and $\xi_2$ with specific values for $\omega_1$ and $\omega_2$ namely, $\omega_1 = \mathrm{b}_1$ and $\omega_2 = \mathrm{b}_2$ for all points of the arcs as shown in Fig.1a. The second category encompasses a pair of neutrosophic arcs, $\xi_1$ and $\xi_2$ with specific values for $\omega_1$ and $\omega_2$, where $\xi_1$ is a neutrosophic arc that has values for $\omega_j$ in the form of $\langle \upsilon_j,\ \rho_j,\ \sigma_j\ \rangle$ and $\xi_2$ is a neutrosophic arc that has values for $\omega_k$ in the form of $\langle \upsilon_k,\ \rho_k,\ \sigma_k\ \rangle$ for which max $\omega_j \longrightarrow 0$ or max $\omega_k \longrightarrow 0$ where, $\omega_j,\ \omega_k \in [0,\ 1]$, as shown in Fig.1b. □




(A) Neutrosophy homotopic of type (I) (B) Neutrosophy homotopic of type (II)

FIGURE 1. neutrosophy homotopic

**Definition 3.6.** Let $\langle M,\ \omega\rangle$ be a neutrosophic submanifold $\langle \mathcal{C}\ ,\ \omega\rangle$, and let us consider the existence of a continuous neutrosophic map $\mathfrak{d}$ : $\langle M,\ \omega\rangle \longrightarrow \langle \mathcal{C}\ ,\ \omega\rangle$ for which $\mathfrak{d}\,(\mathfrak{c},\ \omega\,(\mathfrak{c})) = (\mathfrak{c}\ \omega\,(\mathfrak{c}))\,,\ \forall \mathfrak{c} \in \mathcal{C}$ . Then, $\mathfrak{d}$ is called neuretraction.

**Example 3.7.** $\left\langle S^1,\ \omega\right\rangle$ is neuretraction of $\left\langle R^2 - \{0\},\ \omega\right\rangle$.

Based on Definition 3.6, we can conclude that any of the following situations qualify as neuretraction:

**Definition 3.8.** (a) $\mathfrak{d}\,(\mathfrak{c},\ \omega\,(\mathfrak{c})) =\ (\mathfrak{c},\ \min(\upsilon),\ \min(\rho),\ \min(\sigma))$
(b) $\mathfrak{d}\,(\mathfrak{c},\ \omega\,(\mathfrak{c})) = (\mathfrak{c},\ \max(\upsilon),\ \max(\rho),\ \max(\sigma))$
(c) $\mathfrak{d}\,(\mathfrak{c},\ \omega\,(\mathfrak{c})) = (\mathfrak{c},\ \omega \in (0,\ 1))$. Now, for the rest of our discussion, and for simplicity, we shall denote the neutrosophic manifold $\langle M,\ \omega\rangle$ by the symbol $M$.

To show that isometry exists on both the upper and lower neutrosophic hypermanifolds, we shall use the potent framework of neutrosophic theory in the concept that follows.

**Definition 3.9.** A map $\mathcal{F} : \cup M \longrightarrow \cup M$ is said to be an isoneutrosophic folding if $\mathcal{F}\,(M) = M$ and for each member of the upper neutrosophic hypermanifold $\overline{M}_g$, there is a $\underline{M}_{\mathfrak{g}}$ lower $M$ for which $\overline{\omega}_{\mathfrak{g}} = \underline{\omega}_{\mathfrak{g}}$ for any corresponding point, i.e., $\overline{\omega_{\mathfrak{g}}}(\overline{c}) = \underline{\omega}_{\mathfrak{g}}(\ \underline{c})$ as shown in Fig.2

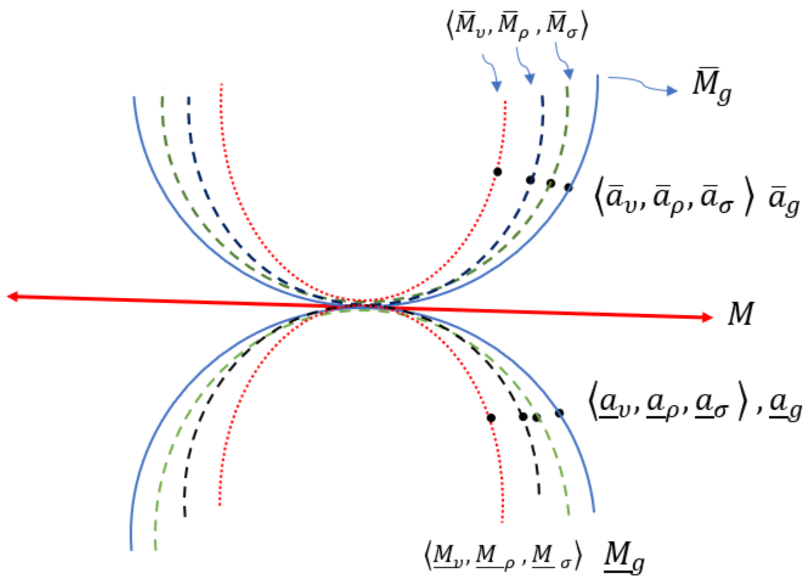



FIGURE 2. Isoneutrosophic folding

**Theorem 3.10.** *Assuming $M$ is a neutrosophic hyperspace in $\mathcal{R}^{m+1}$. Then, we conclude that there are two types of neutrosophication that coincide in $M$.*

(a) For every $\mathfrak{c} \in M$, $\omega(\mathfrak{c}) = \langle 1,\ 1,\ 1 \rangle$. Geometrically, parallel neutrosophic manifolds, which is known as the "crisp property."

(b) For each distinct point $\mathfrak{c}_{t_1},\ \mathfrak{c}_{t_2} \in M$ and, $\omega(\mathfrak{c}_{t_1}) \neq \omega(\mathfrak{c}_{t_2})$, there is a chain of homeomorphic neutrosophic manifolds connected at a common point.

*Proof.* (a) In "a crisp property." For all $y_{t_1},\ y_{t_2} \in M$, we have $\omega(y_{t_1}) = \omega(y_{t_2}) = \langle 1,\ 1,\ 1 \rangle$ also, all neutrosophic hypermanifolds $M_s$ are parallel, $\forall \mathfrak{c}_s \in \underline{M}_s,\ \omega(\mathfrak{c}_s) < \langle 1,\ 1,\ 1 \rangle$ and $\forall \mathfrak{c}_1, \mathfrak{c}_2 \in M_s$, $\omega(\mathfrak{c}_1) = \omega(\mathfrak{c}_2)$, $M_s = \overline{M}_s$ or $M_s = \underline{M}_s$ as shown in Fig.3. In this situation, we can define $\omega$ as

$\omega = \langle \upsilon,\ \rho,\ \sigma \rangle$ where

$$\langle \upsilon,\ \rho,\ \sigma \rangle = \left\langle \begin{cases} \frac{1}{1+l_1} & \text{if} \quad l_1 > 0 \\ \frac{1}{1-l_1} & \text{if} \quad l_1 < 0 \end{cases}, \begin{cases} \frac{1}{1+l_2} & \text{if} \quad l_2 > 0 \\ \frac{1}{1-l_2} & \text{if} \quad l_2 < 0 \end{cases}, \begin{cases} \frac{1}{1+l_3} & \text{if} \quad l_3 > 0 \\ \frac{1}{1-l_3} & \text{if} \quad l_3 < 0 \end{cases} \right\rangle,$$

the list $(l_i,\ i = 1,\ 2,\ 3)$ can be represented in Fig.3. Moreover, we have $\omega = \langle 0,\ 0,\ 0 \rangle$ whenever $l_i \longrightarrow \pm\infty$. However, this illustrates the degree of neutrosophication in the crisp case of $M$. In fact, $\forall \gamma$ there is a neutrosophic strip at $\gamma$, specifically $\zeta_\gamma$ for which $\omega(\mathfrak{c}) < \langle 1,\ 1,\ 1 \rangle$ whenever $\mathfrak{c} \in \zeta_\gamma$ and decreases if $l_i \longrightarrow \pm\infty$.

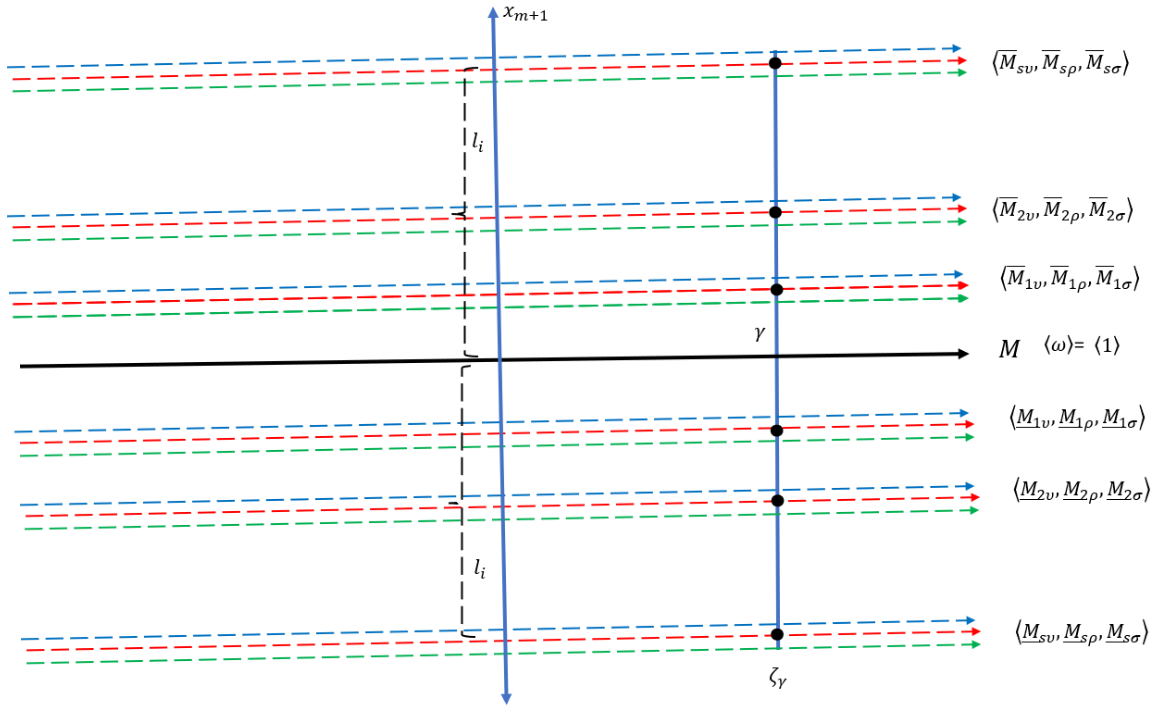



FIGURE 3. parallel hyperspaces and their isoneutrosophic folding

(b) Let $M$ be a neutrosophic hyperspace for which $\omega(\mathfrak{c}_s) \neq \omega(\mathfrak{c}_t)$, $\mathfrak{c}_s \neq \mathfrak{c}_t$ in $M$, and suppose that $\mathfrak{q}$ is a point at which $\omega(\mathfrak{q}) = (\max \omega_s,\ s \in \mathbb{N})$. For all point $\gamma \in M$, $\exists$ neutrosophic strip $\zeta_\gamma$ such that $\omega(\mathfrak{c}_1) < \omega(\mathfrak{c}_2) < \omega(\gamma)$, whenever $\mathfrak{c}_1,\ \mathfrak{c}_2 \in \zeta_\gamma$. If there is no other common neutrosophic point than $\mathfrak{q}$, then $\exists$ a point $\mathfrak{c}_j \in M_j,\ j = 1,\ 2,\ 3$. For all horizontal neutrosophic strips, $q$ has a maximum value (neutrosophic point) in the neutrosophic strip. □

The sequence of neuretraction within a neutrosophic hyperspace will be inferred from the data that follow.

**Theorem 3.11.** *If $M$ is a neutrosophic hyperspace in $\mathcal{R}^{m+1}$, and $\mathfrak{d} : M \longrightarrow \mathcal{C}$, is a neuretraction. Then, there exists a sequence $\langle \mathfrak{d}_i : \cup M \longrightarrow \mathcal{C}_i,\ i = 1, 2, \ldots m \rangle$ of a neuretraction. Also, if we consider $dim\,(\cup M) = dim\ \mathcal{C}_i$, then all $\mathfrak{d}_i$ are special types of isoneutrosophic folding.*

*Proof.* Let $\mathcal{F}$ be a isoneutrosophic folding of $\cup \underline{M}$ into $\cup \overline{M}$ such that $\mathcal{F}\,(M) = M\ \ \omega\,(\ \underline{M}) = \omega(\overline{M})$. Thus, we conclude $\mathcal{F}\,(\underline{M}) = \overline{M}$ as shown in Fig.4. Now for each neuretraction $\mathfrak{d} : M \longrightarrow \mathcal{C}$ (in a case of no common point) we obtain the induced neuretractions $\mathfrak{d}_i : M \longrightarrow \mathcal{C}_i$, $dim\ \mathcal{C}_i = dim M$. But if $\mathfrak{d} : M \longrightarrow p$, there are induced neuretractions $\underline{\mathfrak{d}_i} : \underline{M_i} \longrightarrow \underline{p_i}$ and $\overline{\mathfrak{d}_i} : \overline{M_i} \longrightarrow \overline{p_i}$. However, these neuretractions are not types of neutrosophic folding, because $dim\ \mathcal{C}_i \neq dim M_i$. For example, in Fig.5, $\exists$ an isoneutrosophic folding, whereas there is no isneutrosophic folding as a type of neuretraction in $\mathfrak{d} : \cup M \longrightarrow p$, since $dim\ p \neq dim M_i$. □

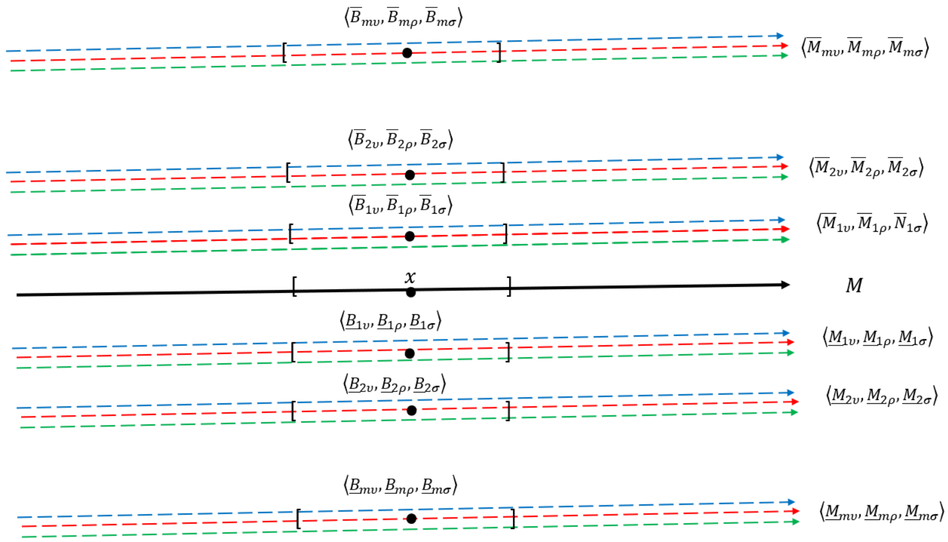



Figure 4. Isoneutrosophic folding on parallel hyperspaces

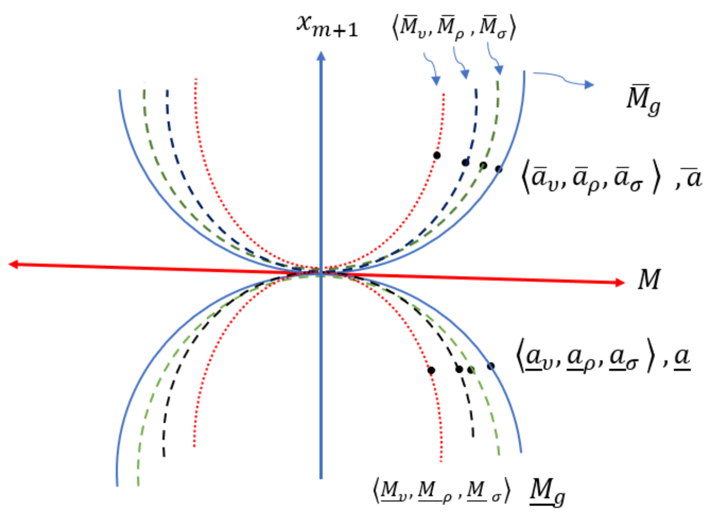



Figure 5. Neuretraction on hyperspaces with common point

The advanced results in this study reveal multiple occurrences of neuretractions corresponding to a set of neutrosophic manifolds that exhibit homeomorphism to a set of neutrosophic unit spheres with n dimensions, all of which possess a shared center.

**Theorem 3.12.** *Suppose that $M$ is a neutrosophic manifold of dimension $m$, which is homeomorphic to a neutrosophic unit sphere, with $\omega(y_i) = 1$ for each $y_j \in \mathcal{S}^m$, else $\omega = \langle \upsilon,\ \rho,\ \sigma \rangle$ where*

$$\langle \upsilon,\ \rho,\ \sigma \rangle = \left\langle \begin{cases} l_1, & 0 < l_1 < 1 \\ \frac{1}{l_1}, & l_1 > 1 \end{cases}, \begin{cases} l_2, & 0 < l_2 < 1 \\ \frac{1}{l_2}, & l_2 > 1 \end{cases}, \begin{cases} l_3, & 0 < l_3 < 1 \\ \frac{1}{l_3}, & l_3 > 1 \end{cases} \right\rangle \text{ where, } y_j \in$$

*$\cup \mathcal{S}_j^m$ as a union of m-dimensional neutrosophic spheres with a common center and let $H = \{(y,\ \omega) : |y| \leq 1\}$ be an n-dimensional neutrosophic closed ball. Then, for every neuretraction of $(H - \mathfrak{q})$ onto $\mathcal{S}^{m-1}$ there are induced neuretractions of $H_j - \mathfrak{q}$ onto $\mathcal{S}_j^{m-1}$. Moreover, under the condition $\mathcal{F}(\mathcal{S}^m) = \mathcal{S}^m$, we get an isoneutrosophic folding $\mathcal{F} : \overline{\mathcal{S}}_j^{\,m} \longrightarrow \underline{\mathcal{S}}_j^m$.*

*Proof.* Assume $M$ is a neutrosophic manifold, $\mathcal{S}^n$ is a neutrosophic unit sphere, and $M$ is homeomorphic to $\mathcal{S}^n$ as shown in Fig.6. If there is a neutrosophic sphere $\mathcal{S}^m$ inside the neutrosophic system, say $\underline{\mathcal{S}}_j^m$ (Neutrosophication will be reduced, $\omega = \langle \upsilon,\ \rho,\ \sigma \rangle \longrightarrow \langle 0,\ 0,\ 0 \rangle$ if $l_i \longrightarrow 0$ and $\omega = \langle \upsilon,\ \rho,\ \sigma \rangle \longrightarrow \langle 0,\ 0,\ 0 \rangle$ if $l_i \longrightarrow \infty$ for $i = 1, 2, 3$. Indeed, for all neutrosophic points (c, $\omega = \langle 1,\ 1,\ 1 \rangle$) $\exists$ (a neutrosophic) strip of neutrosophic points $(\overline{c}_j,\ \overline{\omega}_j < \langle 1,\ 1,\ 1 \rangle) \in \overline{\mathcal{S}}_j^{\,m}$, and $(\underline{c}_j,\ \underline{\omega}_j < \langle 1,\ 1,\ 1 \rangle) \in \underline{\mathcal{S}}_j^m$. However, for the isoneutrosophic folding $\mathcal{F} : \overline{\mathcal{S}}_j^{\,m} \longrightarrow \underline{\mathcal{S}}_j^m$, in which $\overline{\omega_j} = \underline{\omega_j}$ there is an induced isoneutrosophic folding $\overline{\mathcal{F}} : \overline{H}_j \longrightarrow \underline{H}_j$, as well as neuretractions $\overline{\mathfrak{d}}_j : (\overline{H}_j - \mathfrak{q}) \longrightarrow \overline{\mathcal{S}}_j^{\,m-1}$ and $\underline{\mathfrak{d}}_j : (\underline{H}_j - \mathfrak{q}) \longrightarrow \underline{\mathcal{S}}_j^{m-1}$. □

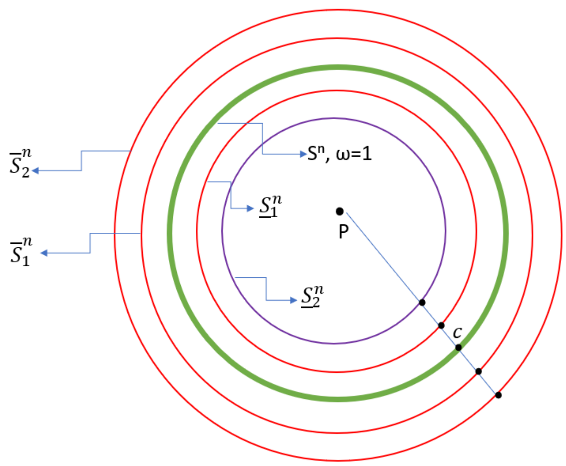



FIGURE 6. Neuretraction and neutrosophic folding on a spheres

**Theorem 3.13.** *Suppose that $\mathcal{N}$ is a neutrosophic manifold with $\mathfrak{d} : \mathcal{N} \longrightarrow \mathcal{M}$ is a neuretraction, then the geometric neuretraction $\mathfrak{d}_{\mathfrak{g}}$ induces a neuretractions $\mathfrak{d}_\upsilon,\ \mathfrak{d}_\rho,\ \mathfrak{d}_\sigma$. On the other hand, the converse is not true.*

*Proof.* Let us consider $\mathfrak{d} : \mathcal{N} \longrightarrow \mathcal{M}$ as a neuretraction, such that $\mathcal{N} = \langle \mathcal{N}_{\mathfrak{g}}, \mathcal{N}_\upsilon,\ \mathcal{N}_\rho, \mathcal{N}_\sigma \rangle$ *and* $\mathcal{M} \subseteq \mathcal{N}$. Now, consider the geometric neuretraction of $\mathfrak{d}_{\mathfrak{g}} : \mathcal{N}_{\mathfrak{g}} \longrightarrow \mathcal{M}_{\mathfrak{g}}$ of $\mathcal{N}_{\mathfrak{g}}$ into $\mathcal{M}_{\mathfrak{g}}$, then we get the induced neuretractions $\mathfrak{d}_\upsilon : \mathcal{N}_\upsilon \longrightarrow \mathcal{M}_\upsilon$, $\mathfrak{d}_\rho : \mathcal{N}_\rho \longrightarrow \mathcal{M}_\rho$, $\mathfrak{d}_\sigma : \mathcal{N}_\sigma \longrightarrow \mathcal{M}_\sigma$ as shown in Fig.7. On the other hand, consider the neuretractions $\mathfrak{d}_\upsilon : \mathcal{N}_\upsilon \longrightarrow \mathcal{M}_\upsilon$, $\mathfrak{d}_\rho : \mathcal{N}_\rho \longrightarrow \mathcal{M}_\rho$, $\mathfrak{d}_\sigma : \mathcal{N}_\sigma \longrightarrow \mathcal{M}_\sigma$ as the identity neuretractions for all membership degrees, which have no impact on the geometric manifold $\mathcal{N}_{\mathfrak{g}}$ as shown in Fig.8 .
□

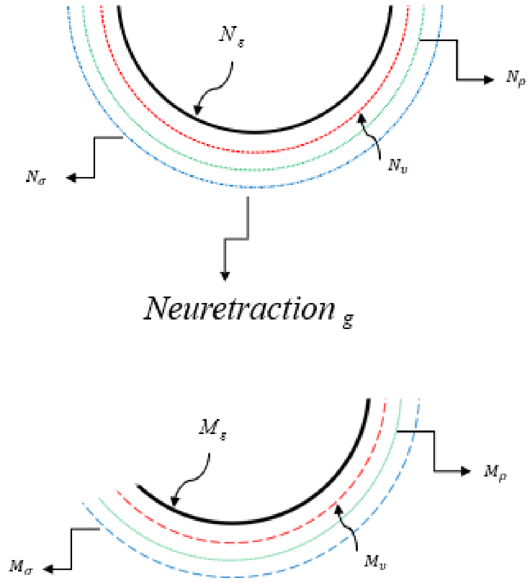



FIGURE 7. A neuretraction of type (I)

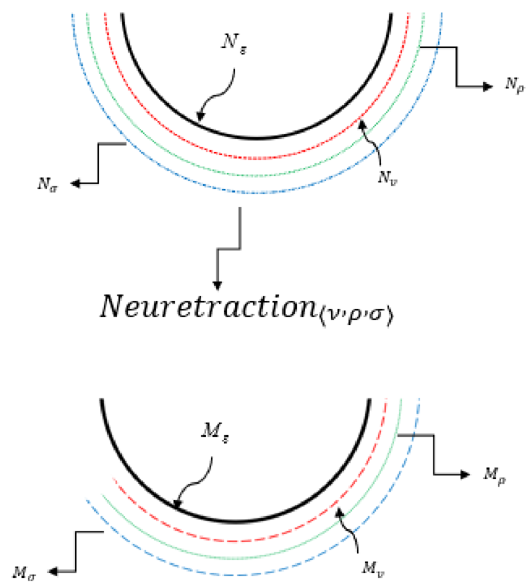



FIGURE 8. A neuretraction of type (II)

## 4. Conclusion

The present study aimed to develop a theoretical basis for neuretraction on a neutrosophic manifold. The neutrosophic folding and neuretraction on a neutrosophic manifold are achieved geometrically and topologically. The sequence of neuretractions in a neutrosophic hyperspace is obtained. The relationship between some types of transformations is deduced. An area that necessitates additional investigation pertains to the establishment and exploration of a fitting notion of neutrosophic homotopy groups, in conjunction with a thorough examination of their consequent neutrosophic homomorphism.

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# Neutrosophic Relative Importance Analysis of the Construction Projects Delay of Mosul City After Its Liberation from ISIS Occupation

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## Abstract

This manuscript was concerned with the reasons for the lateness of completion of building projects, highway projects, and other infrastructures in Mosul City that were hugely damaged due to the occupation of the terrorist ISIS gangs from 2014 to 2018. The questionnaire survey was launched in a very critical period of time (i.e. during the spread of the coronavirus pandemic), It was during a couple of months Oct. and Nov. of the year 2020. The neutrosophic theory has very flexible tools for analyzing the vagueness, hesitancy, and incomplete data, therefore, the author tried to partition the delay reasons into three major causes depending upon the probability bias of occurrence (i.e. truth, indeterminacy, and falsity), We figured out a good innovative approach to sort the importance of thirty-one causes of delay the construction projects.

**Keywords:** Neutrosophic Relative Importance; Construction Projects; Delay Causes; ISIS Occupation; Mosul City; Building Projects; Highway Projects; Infrastructure Projects.

## 1. Introduction

To gain a good insight into the causes and how to therapy the construction delay in the Infrastructure of Mosul city after a hugely damaged city faced by the ISIS terrorist gangs' occupation of the city in 2014 for about three years. The researcher launched 1500 local questionnaires, exactly a national survey. This survey consists of thirty-one scopes of causes (thirty causes already exist in the literature, while the last cause was added by the author which is the delay of finishing the project caused by the Coronavirus pandemic spreading in Iraq for the period from Feb 2020 to Jan. 2022) [1].

The following table illustrates the definition of each delaying cause with its mathematical symbol accompanied by its grade of intense in view of neutrosophic theory, that is there are nine grades of intense (the gradations 9-8-7 have been dedicated to the truth bias, and the grades 6-5-4 went for indeterminate bias, and 3-2-1 have been specified for falsity bias). The survey has been targeted at 1,500 of experience, but unfortunately, the persons responding were just 250 individuals whose expertise is diverse between Project owners, Project designers (i.e. Consulting offices), Contractors…etc.

Table (1): The degrees of the intense in the construction delay reasons from the neutrosophic point of view.

| Problem Math. Code | Definition of the problem | Falsity bias | | | Indeterminate bias | | | Truth bias | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| R1 | Unrealistic schedule (bid duration is too short) | 14 | 0 | 4 | 1 | 6 | 25 | 81 | 26 | 93 |
| R2 | Ineffective delay penalty provisions in the contract | 92 | 86 | 17 | 13 | 5 | 22 | 12 | 1 | 7 |
| R3 | Errors in contract documents | 13 | 2 | 9 | 101 | 18 | 98 | 4 | 2 | 3 |
| R4 | Selecting an inappropriate project delivery method | 65 | 90 | 66 | 3 | 4 | 13 | 3 | 3 | 3 |
| R5 | Excessive change orders by the owner during construction | 19 | 26 | 7 | 43 | 93 | 17 | 10 | 16 | 19 |
| R6 | Delayed payments by the owner | 0 | 1 | 1 | 16 | 15 | 3 | 14 | 100 | 100 |
| R7 | Delay in approving design documents by the owner | 11 | 9 | 15 | 19 | 22 | 27 | 53 | 50 | 44 |
| R8 | Time-consuming decision-making process of the owner | 72 | 30 | 88 | 10 | 0 | 13 | 0 | 12 | 25 |
| R9 | Unnecessary Inference by the owners | 30 | 10 | 60 | 48 | 34 | 20 | 0 | 4 | 39 |
| R10 | Delay to furnish and deliver the site to the contractor | 80 | 20 | 80 | 15 | 10 | 60 | 4 | 50 | 3 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| R11 | Poor communication and coordination of the owner with designer and/ or contractor | 25 | 0 | 146 | 9 | 27 | 31 | 2 | 10 | 0 |
| R12 | Poor Quality Assurance (QA) plan of the owner | 31 | 13 | 37 | 0 | 22 | 59 | 0 | 11 | 77 |
| R13 | Lack of management staffs of the owner | 64 | 79 | 0 | 8 | 74 | 14 | 2 | 4 | 5 |
| R14 | Inappropriate construction methods | 86 | 13 | 41 | 1 | 0 | 0 | 39 | 0 | 70 |
| R15 | Contractor inefficiency (in providing the labor, equipment and material and handling sub-contractors) | 12 | 18 | 51 | 10 | 36 | 29 | 18 | 27 | 49 |
| R16 | Poor communication and coordination of the contractor with owner and/ or designer | 80 | 17 | 4 | 3 | 16 | 9 | 24 | 21 | 76 |
| R17 | Inadequate contractor experience | 15 | 12 | 13 | 29 | 23 | 35 | 21 | 38 | 63 |
| R18 | Financial difficulties and mismanagement by the contractor | 12 | 14 | 15 | 23 | 19 | 31 | 39 | 41 | 56 |
| R19 | Poor site management and Quality Control (QC) by the contractor | 8 | 11 | 19 | 21 | 31 | 29 | 46 | 43 | 42 |
| R20 | Legal disputes between the designer and the owner | 5 | 71 | 13 | 21 | 19 | 12 | 47 | 59 | 67 |
| R21 | Design errors | 10 | 13 | 17 | 17 | 21 | 22 | 45 | 52 | 53 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| R22 | Complexities and ambiguities of project design | 8 | 21 | 14 | 27 | 42 | 39 | 29 | 37 | 33 |
| R23 | Delays in providing the design documents by the designer | 53 | 7 | 26 | 32 | 40 | 47 | 15 | 17 | 13 |
| R24 | Inadequate experience of the designer | 17 | 18 | 0 | 57 | 99 | 12 | 9 | 28 | 8 |
| R25 | Inadequate site assessment by the designer during the design phase | 4 | 39 | 70 | 23 | 10 | 3 | 57 | 33 | 11 |
| R26 | Misunderstandings between owner and designer about the scope of the work | 36 | 64 | 57 | 25 | 15 | 29 | 8 | 16 | 0 |
| R27 | Financial difficulties with the designer | 6 | 3 | 9 | 67 | 51 | 64 | 17 | 10 | 23 |
| R28 | Poor communication and coordination of the designer with the owner and/ or contractor | 12 | 0 | 17 | 53 | 57 | 61 | 0 | 19 | 31 |
| R29 | Legal dispute between the designer and the owner | 56 | 60 | 71 | 7 | 22 | 9 | 12 | 13 | 0 |
| R30 | Delay in getting permits and acquisitions (Environmental, building, right of way, utilities, etc.) | 0 | 5 | 2 | 6 | 8 | 19 | 81 | 37 | 92 |
| R31 | The coronavirus pandemic spreading in Iraq from Feb 2020 to Jan. 2022 | 7 | 10 | 3 | 12 | 3 | 15 | 69 | 48 | 83 |

It is worth mentioning that the grades from 1 to 9 have different neutrosophic linguistic meanings [2-11], as forthcoming table (2) demonstrates the grade linguistic meaning of the impact of the reason on delaying the completion of the project:

Table (2): The Grade Linguistic Meaning of the Impact of the Reason on Delaying the Completion of the Project

| Neutrosophic grade | Neutrosophic bias | Neutrosophic linguistic statement |
|---|---|---|
| 9 | Grade of Truth membership function | The reason always has an impact on delaying the completion of the project |
| 8 | Grade of Truth membership function | The reason usually has an impact on delaying project completion |
| 7 | Grade of Truth membership function | The reason generally has an impact on the delay in completing the project |
| 6 | Grade of Indeterminate membership function | The reason often has an impact on the delay in completing the project |
| 5 | Grade of Indeterminate membership function | The reason sometimes has an impact on the delay in completing the project |
| 4 | Grade of Indeterminate membership function | The reason occasionally has an impact on the delay in completing the project |
| 3 | Grade of Falsity membership function | The reason seldom has an impact on the delay in completing the project |
| 2 | Grade of Falsity membership function | The reason rarely has an impact on the delay in completing the project |
| 1 | Grade of Falsity membership function | The reason never has an impact on the delay in completing the project |

**2.**

## 3. Analyzing the Results of Survey (A)

The local survey (A) attached at the end of this manuscript, achieves some important aims such as investigating the effectiveness of the most popular causes of construction delay, determining the scale of riskiness for every probable cause of delay, and confinement the list of the most critical delay causes.

Because of the situation of quarantine during the period of coronavirus spreading, we tried to avoid doing the paper survey, and an online survey was launched via Google form within a couple of months Oct. and Nov. 2020, This needed to invite 1500 experts, especially the Officials in Nineveh Governorate, and some civil society organizations who had projects for the reconstruction of Nineveh as a UNDP organization, TEKA organization...etc. we gained just 250 completed surveys.

The following subsections contain traditional statistical percentages categorized as the kinds of projects, the ownerships of the projects, how delivery method of the project, the role of the respondents worked in the project, the accumulation experience of the respondents, the impacts of different causes of delay.

### 3.1 Projects' Kinds Subject to Respondents' Answers

The 250 answers gained out of 1,500 issued questionnaires confirm that 117 equivalently to (46.8%) of the projects were building projects, while 100 of them were infrastructure projects (i.e. it is 40% of the projects), wherein the highway projects were 33 projects out of 250 (i.e. 13.2%). The below figure (3.2) illustrates the pie chart of the distributed percentages figured out using MATLABR 2023a.

```
% plot the Pie chart of distributed percentages of projects type
clc;
clear;
close;
numberofbuildingstype=[117 100 33];
percentage=numberofbuildingstype./250;
% Create Pie Chart
```

```
ax1=nexttile;
pie3(ax1,percentage)
title(' Types of Projects')
labels={'buildings projects','infrastructure','highway projects'};
% Create legend
lgd=legend(labels);
lgd.Layout.Tile='east';
```

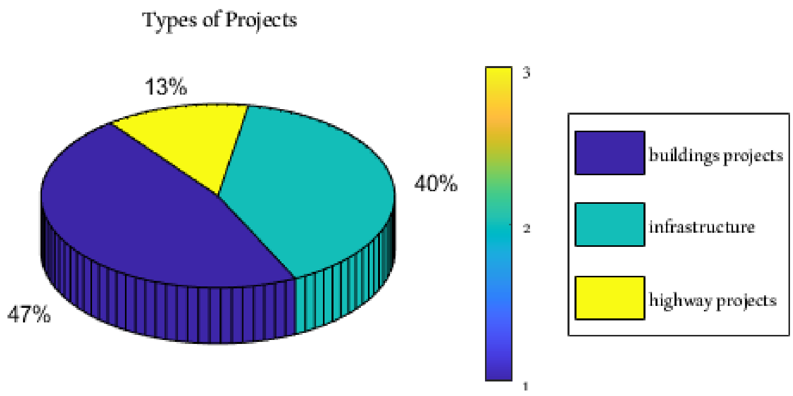


**Figure (1): Percentages of Project Types**

### 3.2 The Ownership Categories

Again, after completing the surveys, we concluded that 170 projects out of 250 belong to government projects (public projects) which represent the percentage (68%), whilst 50 projects are in the private sector (i.e. 20%). Finally, the remaining 30 projects have been done by civil society organizations, that is, 12%. The following figure (3.2.2) demonstrates the distributed percentages figured out by Pie chart using MATLAB R2023a.

```
% plot the Pie chart of distributed percentages of ownerships type
clc;
clear;
```

```
close;

numberofownershipstype=[170 50 30];

percentage=numberofownershipstype./250;

% Create Pie Chart

ax1=nexttile;

pie3(ax1,percentage)

title(' Types of Projects')

labels={'public projects','private sector','civil society organizations'};

% Create legend

lgd=legend(labels);

lgd.Layout.Tile='east';
```

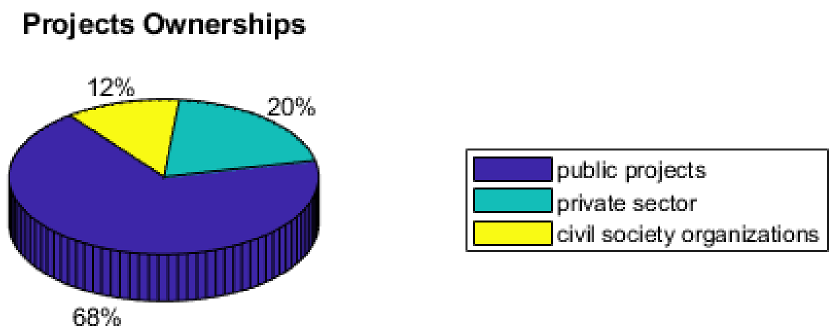


**Figure (2): Types of Ownerships**

## 3.3 Choosing the Method of Delivering the Project to the Beneficiary

It is well known that there are four methods of delivering the projects in Iraq country, they are: [11-13]

1- Traditional Approach (TA).

2- Direct Labor (DL).

3- Design Build (DB).

4- Turn Key (TK).

Hence, we should not forget that our studying is focused on the projects that already completed but they were suffered from delays in the completion in some implementation stages for critical causes [14,15]. Our database of survey results showed that (145) projects were of the type of (TA) delivered method, while (55) projects went in favor of the (DL) method, and (30) projects were delivered by the (DB) method, finally, there were (20) projects of (TK) delivery method. The following MATLAB program and figure (3.2.3) clarify the statistical categories of the above-mentioned information:

```
% plot the Pie chart of the projects delivery method
clc;
clear;
close;
numberofdeliverymethods=[145 55 30 20];
percentage=numberofdeliverymethods./250;
% Create Pie Chart
ax1=nexttile;
pie3(ax1,percentage)
title(' Statistical Categories of Delivery Methods')
labels={'TA approach','Dl approach','DB approach', 'TK approach' };
% Create legend
lgd=legend(labels);
lgd.Layout.Tile='east';
```

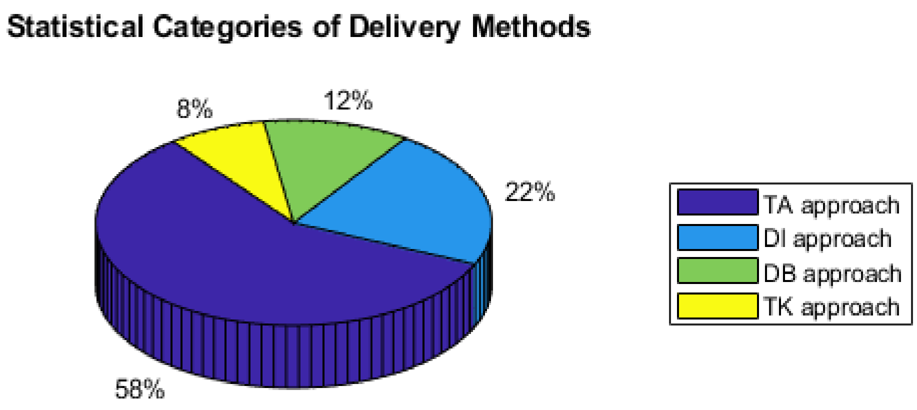


**Figure (3): Categories Percentage of Delivery Methods**

### 3.4 Respondents' Role in Project

In this study, we are dependent on three main kinds of respondents' roles, the owners of the projects are 33 respondents, the contractors are 52, and 79 of the consulting offices, while other kinds of respondents such as architects, safety directors, foremen…etc, all of them counting together were 86. Hence the forthcoming program and figure (4) are simply illustrate the percentage of them:

```
% plot the Pie chart of the respondents' role in the Project
clc;
clear;
close;
numberofrespondentsrole=[33 79 52 86];
percentage=numberofrespondentsrole./250;
% Create Pie Chart
ax1=nexttile;
pie3(ax1,percentage)
title(' Respondents' Role in Project')
labels={'projects owners','Projects designers','Contractors', 'Others' };
```

```
% Create legend

lgd=legend(labels);

lgd.Layout.Tile='east';
```

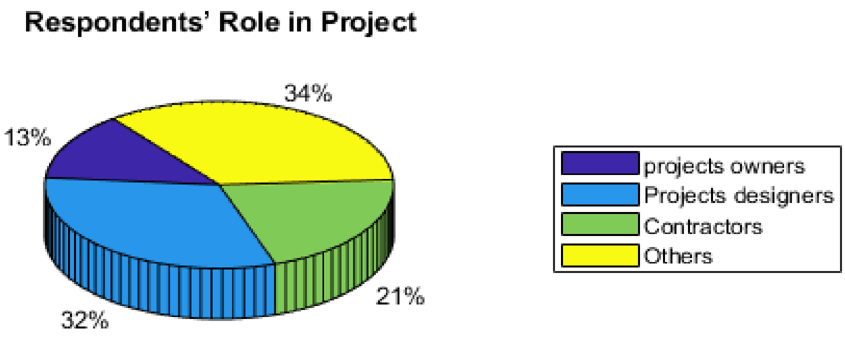


**Figure (4): Categories Respondents’ Role in Projects**

## 4. Neutrosophic Perspective-Based Impact Delay Causes

It is important to determine the impishness of the 31 potential causes of the occurrence of delay. the participants during the launched survey were requested to evaluate the impact of each cause by ranking the effectiveness using numbers from 1 to 9, where the number one shows the least impishness, on the other hand, the no. nine means the severest cause. By using neutrosophic theory, we will cover the vagueness of the data, the hesitancy in the decisions because there are some reasons that cannot be determined with certainty whether they have an impact on delaying the project or not! or they may have some impact on delaying the completion of some projects and not others, and some of the reasons may have an impact on a particular project at certain times, while at other times they have no effect. All this confusion may affect the process of making the correct decision, which consequently leads to the emergence of an urgent need to use neutrosophic logic, which is an ideal generalization of fuzzy logic because in all complicated cases, it has the appropriate mathematical tools to represent all cases of ambiguity, vagueness, and incomplete information [4-7].

This section will be dedicated to analyzing the questions raised by the survey (A) using the newly suggested neutrosophic mathematical concept named as Neutrosophic Relative Importance Index (NRII) which has the ability to partition the causes' effects of delay the projects into three categories:

1-Truth Relative Importance Index (RIItruth).

2- Indeterminate Importance Index (RIIindeterminate).

3- Falsity Importance Index (RIIfalsity).

For the above three neutrosophic components, the authors suggested the following mathematical formulas:

$$RIItruth, RIIindeterminate, RIIfalsity = \frac{\sum_{i=1}^{31}\sum_{j=1}^{3} m_j r_{ij}}{M*250}, \quad (3.1)$$

Where $i = 1,2,3,\dots,31$ , for truth bias $M = 9, m_1 = 9, m_2 = 8, m_3 = 7$, that is mean:

$$RIItruth = \frac{9*r_{i9}+8*r_{i8}+7*r_{i7}}{9*250}, \quad (3.2)$$

Again, for indeterminacy bias, $M = 6, m_1 = 6, m_2 = 5, m_3 = 4$, which implies to

$$RIIindeterminacy = \frac{6*r_{i6}+5*r_{i5}+4*r_{i4}}{6*250}, \quad (3.3)$$

Finally, for falsity bias, $M = 3, m_1 = 3, m_2 = 2, m_3 = 1$, hence

$$RIIfalsity = \frac{3*r_{i3}+2*r_{i2}+1*r_{i1}}{3*250}, \quad (3.4)$$

It should be noticed that the values of $r_{ij}$ represent the intensity of the (31) causes, the index $i$ is the number of the (31) rules, and the index $j$ is the number of the nine columns stated in table (1).

The reader should be notified the number 250 that has been embedded in the denominator of the formulas (3.2, 3.3, 3.4) is the total number of respondents that have been received after launching the survey (A).

The upcoming MATLAB program with its graph consisting of three pie charts represents the severity of each cause in terms of truth-biasing, indeterminacy-biasing, and falsity-biasing:

```
% Three Graphs categorizing the cause's severity according to their relative importance
```

```
clc;
clear;
close;
Z1=[93 26 81;7 1 12;3 2 4;3 3 3;19 16 10;100 100 14;44 50 53;25 12 0;39 4 5;3 50 4;0 10 2;77 11 0;5 4 2;70 0 39;49 27 18;76 21 24;63 38 21;56 41 39;42 43 46;67 59 47;53 52 45;33 37 29;8 28 9;13 17 15;11 33 57;0 16 8;23 10 17;31 19 0;0 13 12;92 37 81;83 48 69];
M1=[9 8 7];
F1=M1';
RIItruth=(Z1*F1)./(9*250)
% Create Pie Chart
ax1=nexttile;
pie3(ax1,RIItruth)
title('Relative Importance Index for Truth')
Z2=[25 6 1;22 5 13;98 18 101;13 4 3;17 93 43;3 15 16;27 22 19;13 0 10;20 34 48;60 10 15;31 27 9;59 22 0;14 74 8;0 0 1;29 36 10;9 16 3;35 23 29;31 19 23;29 31 21;12 19 21;22 21 17;39 42 27;12 99 57;47 40 32;3 10 23;29 15 25;64 51 67;61 57 53;9 22 7;19 8 6;15 3 12];
M2=[6 5 4];
F2=M2';
RIIindeterminacy=(Z2*F2)./(6*250)
% Create Pie Chart
ax2=nexttile;
pie3(ax2,RIIindeterminacy)
title('Relative Importance Index for Indeterminacy')
Z3=[4 0 14;17 86 92;9 2 13;66 90 65;7 26 19;1 1 0;15 9 11;88 30 72;60 10 30;80 20 8;146 0 25;37 13 31;0 79 64;41 13 86;51 18 12;4 17 80;13 12 15;15 14 12;19 11 8;13 7 5;17 13 10;14 21 8;0 18 17;26 7 53;70 39 4;57 64 36;9 3 6;17 0 12;71 60 56;2 5 0;3 10 7];
M3=[3 2 1];
```

```
F3=M3';
RIIfalsity=(Z3*F3)./(3*250)
labels={'R1','R2','R3','R4','R5','R6','R7','R8','R9','R10','R11','R12','R13','R14','R15','R16','R17','R18','R19','R
20','R21','R22','R23','R24','R25','R26','R27','R28', 'R29', 'R30', 'R31'};
% Create Pie Chart
ax3=nexttile;
pie3(ax3,RIIfalsity)
title('Relative Importance Index for Falsity')
% Create legend
lgd=legend(labels);
lgd.Layout.Tile='east';
```

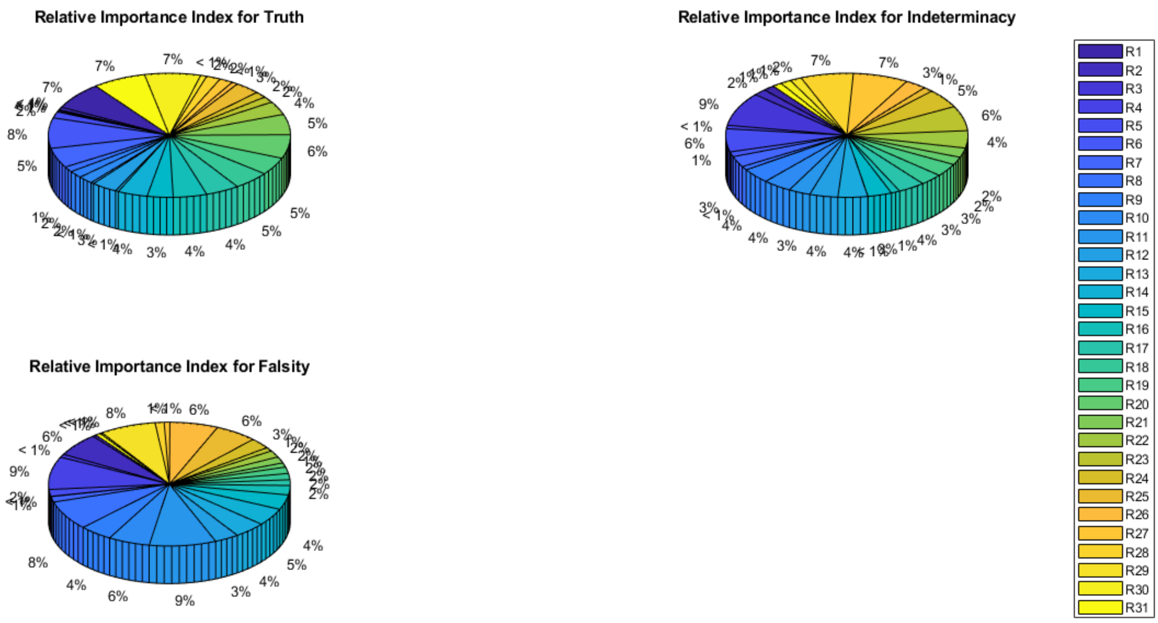


**Figure (5): Neutrosophic Relative Importance Index (NRII)**

Now to make a fair analysis of the most important causes that are expected to be ranked due to their effectiveness, we need to calculate the accumulated Neutrosophic Relative Importance values:

$$RIItotal = RIItruth + RIIindeterminacy + RIIfalsity \qquad (3.5)$$

The following table shows all $RIItotal$ depending upon the number of $Ri$ reason.

**Table (3): Total Neutrosophic Relative Importance**

| $RIItotal$ | Corresponding $Ri$ |
|---|---|
| 0.8738 | $R1$ |
| 0.6282 | $R2$ |
| 0.8116 | $R3$ |
| 0.6960 | $R4$ |
| 0.7793 | $R5$ |
| 0.9104 | $R6$ |
| 0.8493 | $R7$ |
| 0.7493 | $R8$ |

| | |
|---|---|
| 0.8138 | $R9$ |
| 0.8996 | $R10$ |
| 0.8971 | $R11$ |
| 0.8804 | $R12$ |
| 0.6604 | $R13$ |
| 0.7173 | $R14$ |
| 0.8787 | $R15$ |
| 0.7187 | $R16$ |
| 0.8504 | $R17$ |
| 0.8531 | $R18$ |
| 0.8553 | $R19$ |
| 0.8687 | $R20$ |
| 0.8562 | $R21$ |
| 0.8444 | $R22$ |
| 0.7602 | $R23$ |
| 0.7591 | $R24$ |
| 0.8347 | $R25$ |
| 0.7611 | $R26$ |
| 0.8371 | $R27$ |
| 0.8509 | $R28$ |
| 0.7302 | $R29$ |
| 0.8916 | $R30$ |
| 0.8673 | $R31$ |

The above table need to sort the total RII in descent order using the following MATLAB commands.

```
% Sorting the Neutrosophic Relative Importance in Descend Order
clc;
clear;
close;
RIItotal=[0.8738 0.6282 0.8116 0.6960 0.7793 0.9104 0.8493 0.7493 0.8138 0.8996 0.8971 0.8804 0.6604
0.7173 0.8787 0.7187 0.8504 0.8531 0.8553 0.8687 0.8562 0.8444 0.7602 0.7591 0.8347 0.76110.8371 0.8509
0.7302 0.8916 0.8673];
S=sort(RIItotal,'descend');
W=S'
```

**Table (4): Sorting Neutrosophic Relative Importance in descend Order**

| Importance Order | $RIItotal$ | Corresponding $Ri$ |
|---|---|---|
| 1 | *0.9104* | $R6$ |
| 2 | *0.8996* | $R10$ |
| 3 | *0.8971* | $R11$ |
| 4 | *0.8916* | $R30$ |
| 5 | *0.8804* | $R12$ |
| 6 | *0.8787* | $R15$ |
| 7 | *0.8738* | $R1$ |
| 8 | *0.8687* | $R20$ |
| 9 | *0.8673* | $R31$ |
| 10 | *0.8562* | $R21$ |
| 11 | *0.8553* | $R13$ |
| 12 | *0.8531* | $R18$ |
| 13 | *0.8509* | $R28$ |

| | | |
|---|---|---|
| 14 | 0.8504 | $R17$ |
| 15 | 0.8493 | $R7$ |
| 16 | 0.8444 | $R22$ |
| 17 | 0.8371 | $R27$ |
| 18 | 0.8347 | $R25$ |
| 19 | 0.8138 | $R9$ |
| 20 | 0.8116 | $R3$ |
| 21 | 0.7793 | $R5$ |
| 22 | 0.7611 | $R26$ |
| 23 | 0.7602 | $R23$ |
| 24 | 0.7591 | $R24$ |
| 25 | 0.7493 | $R8$ |
| 26 | 0.7302 | $R29$ |
| 27 | 0.7187 | $R16$ |
| 28 | 0.7173 | $R14$ |
| 29 | 0.6960 | $R4$ |
| 30 | 0.6604 | $R13$ |
| 31 | 0.6282 | $R2$ |

## 5. Conclusion and Results Analysis

The descending sorting of the neutrosophic RII shows in table (3.3.2) that the reason R6 (i.e. Delayed payments by the owner) comes in the first order due to its importance, whereas the cause R10 (i.e. delayed preparation and delivery of the site to the contractor) comes in the second order, while the reason of spreading the coronavirus and the quarantine situation that Nineveh province suffered from was in ninth order according to its importance. Consequently, we are convinced that these results are very harmonized with the obstacles and situations of Mosul Province in the period 2020 to the end of 2022 where the federal general budget did not Endorsed in those years, and there was a budget deficit and there was austerity has been conducted by the central government and local governments. We should state that all projects that were studied have been finished and completed but they had suffered from delays.

**Survey (A): Local Survey has been issued to experts in Nineveh Province during the couple of months Oct. and Nov. of the year 2020.**

1- What are the kinds of the projects you are/were enrolled in (you can choose all that applies):

◯ Building Projects. ◯ Highway Projects. ◯ Infrastructure Projects.

◯ Other please mention……………………………………………………………………………..

2- Select the kind of ownership in the Projects, you were involved in (you can choose all that applies):

◯ Government Projects. ◯ Private Sector. ◯ Civil society organization.

◯ Other please mention………………………………………………………………………………..

3- Select of project delivery method you are/were involved with (you can choose all that applies):

◯ Traditional Approach (TA). ◯ Direct Labor (DL).

◯ Design Build (DB). ◯ Turn Key (TK).

◯ Other please mention………………………………………………………………………………..

4- Select which of the following parties you worked for (you can choose all that applies)

◯ Owner. ◯ Designer/ Consulting office. ◯ Contractor.

◯ Other please mention………………………………………………………………………………..

5- Years of experience in construction……………………………………………

6- If you wish, provide us an email, and we will send you the studying results once it is completed.

Kindly, specify the intensity of the occurrence of the following problems that caused a delay in the project construction, where the numbers, 9, 8 and 7 mean the grades of truth's state. The numbers 6, 5 and 4 are of indeterminate bias levels of the delays' causes. While the numbers 3, 2 and 1 are the gradation of the falsity states. The definitions of all numbers (9 to 1) are specified through the table (2).

**Table (5): The Gradations of the 31 Reasons for the Construction Project Delay**

| Definition of the problem | Falsity bias | | | Indet. bias | | | Truth bias | | |
|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| Unrealistic schedule (bid duration is too short) | | | | | | | | | |
| Ineffective delay penalties provisions in contract | | | | | | | | | |
| Errors in contract documents | | | | | | | | | |
| Selecting inappropriate project delivery method | | | | | | | | | |
| Excessive change orders by owner during construction | | | | | | | | | |
| Delayed payments by the owner | | | | | | | | | |
| Delay in approving design documents by the owner | | | | | | | | | |
| Time consuming decision making process of the owner | | | | | | | | | |
| Unnecessary Inference by the owners | | | | | | | | | |
| Delay to furnish and deliver the site to the contractor | | | | | | | | | |
| Poor communication and coordination of the owner with designer and/ or contractor | | | | | | | | | |
| Poor Quality Assurance (QA) plan of the owner | | | | | | | | | |
| Lack of management staffs of the owner | | | | | | | | | |
| Inappropriate construction methods | | | | | | | | | |
| Contractor inefficiency (in providing the labor, equipment and material and handling sub-contractors) | | | | | | | | | |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Poor communication and coordination of the contractor with owner and/ or designer | | | | | | | | | |
| Inadequate contractor experience | | | | | | | | | |
| Financial difficulties and mismanagement by the contractor | | | | | | | | | |
| Poor site management and Quality Control (QC) by the contractor | | | | | | | | | |
| Legal disputes between designer and the owner | | | | | | | | | |
| Design errors | | | | | | | | | |
| Complexities and ambiguities of project design | | | | | | | | | |
| Delays in providing the design documents by the designer | | | | | | | | | |
| Inadequate experience of the designer | | | | | | | | | |
| Inadequate site assessment by the designer during design phase | | | | | | | | | |
| Misunderstandings between owner and designer about work scope | | | | | | | | | |
| Financial difficulties with the designer | | | | | | | | | |
| Poor communication and coordination of the designer with owner and/ or contractor | | | | | | | | | |
| Legal disputed between designer and the owner | | | | | | | | | |
| Delay in getting permits and acquisitions (Environmental, building, Right of way, utilities, etc.) | | | | | | | | | |
| the Coronavirus pandemic spreading in Iraq from Feb 2020 to Jan. 2022 | | | | | | | | | |

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# Distributive Properties of $Q-$neutrosophic Soft Quasigroups

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**Abstract.** The $Q-$neutrosophic soft quasigroup is a mathematical innovation for dealing with indeterminate occurrences. The characterization of quasigroups using the concept of $Q-$neutrosophic soft set is an evolving area of study that, in recent times, has attracted pools of researchers. Different researchers have defined the idea of a $Q$-neutrosophic soft set under associative structures like groups, fields, rings, and modules. The distributive and symmetric properties of the $Q-$neutrosophic soft quasigroup are examined in this study, which extends the idea of a $Q-$neutrosophic soft set to a non-associative behaviour known as a quasigroup. Our findings were quite revealing. In particular, after defining $Q-$neutrosophic soft quasigroup in relation to the three binary operations of product, right, and left division operations, it was found that these operations are distributive over one another. Additionally, these binary operations are distributive over the operations of intersection, union, AND, and OR. It was obtained that, $Q-$neutrosophic soft quasigroup does not obey the key laws, and that the quasigroup is self-distributive with respect to the product, left, and right divisions. The effort which is novel, has advanced the course of study in this emerging field.

**Keywords:** Quasigroup; Distributive; Symmetric properties; Soft set; Q-neutrosophic soft set

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## 1. Introduction

**Definition 1.1.** Suppose that $\hat{G}$ is a non-empty set and the binary operation $(\odot)$ is define on $\hat{G}$ such that $r \odot w \in \hat{G}$ for all $r, w \in \hat{G}$ and if there exist $\alpha, \beta \in \hat{G}$, the pair $(\hat{G}, \odot)$ is called a *groupoid*. If the equations:

$$\alpha \odot r = \beta \text{ and } w \odot \alpha = \beta$$

has unique solutions $r, w \in \hat{G}$ for all $\alpha, \beta \in \hat{G}$, then $(\hat{G}, \odot)$ is called quasigroup. Suppose there is a unique element $1 \in \hat{G}$ called the identity element such that $1 \odot r = r \odot 1 = r$ for all $r \in \hat{G}$, then $(\hat{G}, \odot)$ is a loop.

In this research, we sometime write $rw$ instead of $r \odot w$, when the operation $\odot$ is a multiplication in $\hat{G}$. Suppose that $r$ is a fixed element in a quasigroup $(\hat{G}, \odot)$. Then, the left and right translation maps for all $r \in \hat{G}$, written as $L_r$ and $R_r$ respectively are defined by $wL_r = r \odot w \quad$ and $\quad wR_r = w \odot r$. It is shown that a groupoid $(\hat{G}, \odot)$ is a quasigroup if the left and right translation maps are bijective. Hence, the inverse mappings $L_r^{-1}$ and $R_r^{-1}$ also exist.Thus,

$$r \backslash w = wL_r^{-1} \qquad \text{and} \qquad r/w = rR_w^{-1}$$

Zadeh launched fuzzy set concept for the first time in [15]; Atanassov extended on it in [20] with the definition of intuitionistic fuzzy set. Fuzzification of quasigroup was first introduced in 1998, [9] by Dudek. In 1999, Dudek and Jun [10] extended the results in [9] to fuzzy subquasigroup under t-norm. In 2000, Kyung et.al. [12] studied intuitionistic fuzzy subquasigroups as a way of generalizing the results obtained in [9]. In 2005, intuitionistic fuzzy subquasigroups were further studied by Dudek [13]. In, 2008 Muhammad and Dudek presented fuzzy subquasigroups with different types of $(\alpha, \beta)$- fuzzy quasigroups. Although, these two notions has some limitations and difficulties when dealing with uncertainty and incomplete data stated in [16]. A soft set theory was presented by Molodtsov in [16] as an analytical instrument for addressing uncertainty in order to address some of the aforementioned issues. Over the years, many experts in the field of algebra have applied this mathematical concept to an algebraic structure and studied it through the structural characteristic of the algebraic structure. For example, the algebraic properties of soft sets under a quasigroup were introduced by Oyem. et.al. [18, 19]. It is well known that one of the most beautiful properties of soft set theory is that its parameter set has a capacity to accommodate a wide range of information in terms of decision-making in real-life problems. Although, the characterization of membership degrees present in neutrosophic set are not applicable in the study of soft set theory. Therefore, soft set theory is not applicable when solving problems involving indeterminate data.

To deal with indeterminate real-world data, a mathematical concept called neutrosophy was launched. This mathematical concept was launched in 1998 by Smarandache [23, 24]. It is well known that this unique idea is the only application of classical set theory that has been generalized in the literature to address issues with uncertainty and indeterminacy. The culture of a neutrosophic set is characterized via three independent membership degrees called the true, indeterminate, and falsity which are respectively denoted as $\mathcal{T}^{`}$, $\mathcal{I}^{`}$, and $\mathcal{F}^{`}$. The concept of the neutrosophic set and its method of determining the indeterminate in real-life data are applicable in different fields of study. For example, the authors in [26] used the concept to study the inspection assignment form for product quality control, while characterizations of separation axioms in neutrosophic topological spaces were studied in [27]. The concept of neutrosophy culture is also applicable in the area of operation research in management. In particular, the authors in [28] presented a study on neutrosophic methods of operation research in the management of corporate work.

Recently, the study of a neutrosophic set combined with the concept of soft set theory has received tremendous attention in the field of mathematics [1–4, 17, 22, 30]. This is because the combination of these two mathematical concepts provides a generalized structure for dealing with uncertainties and indeterminacy present in real-world problems. For example, the $Q-$neutrosophic soft set ($Q-$NS) set is an expanded model of the neutrosophic soft set described by two universal sets. Hence, it has the capacity to handle the two universal sets and its indeterminate membership at the same time.

Since the notion of a $Q-$neutrosophic soft set of two universal sets was defined in [6], different authors have applied it to associative behavior such as fields, groups, rings, and modules [6, 7, 21, 25]. In addition. the concept of a $Q-$neutrosophic soft set is long overdue to be extended to the structure of a quasigroup where associative property is not assumed.

This work characterizes the distributive properties of the $Q-$neutrosophic soft set under a non-associative algebra termed quasigroup. In particular, the distributive properties of quasigroups have a very interesting characteristic in the classical study of quasigroup theory. It is well known that quasigroups are not inherently distributive across their binary operations [14]. This serves as motivation to investigate the distributive properties of the $Q-$neutrosophic soft quasigroup.

## 2. **Preliminaries**

**Definition 2.1.** [14] Let $(\hat{G}, \odot)$ be quasigroup and $P \leq \hat{G}$. Then, $P$ is called subgroupoid (subquasigroup) of $\hat{G}$ if $(P, \odot)$ is a quasigroup. Let $V$ and $K$ be non empty subsets of $\hat{G}$, then the product $V \odot K = \{v \odot k \mid v \in K, k \in K\}$, the right division $V/K = \{v/k \mid v \in V, k \in K\}$ and left division $V \backslash K = \{v \backslash k \mid v \in V, k \in K\}$

**Definition 2.2.** [14] Let $(\hat{G}, \odot)$ be a quasigroup. $(\hat{G}, \odot)$ is a left distributive if $f \odot (w^1 \odot z) = (f \odot) \odot (f \odot z)$ and a right distributive if $(f \odot w^1) \odot z = (f \odot z) \odot (w^1 \odot z)$. Whenever right and left distributive properties hold in $(\hat{G}, \odot)$, it is called a distributive quasigroup.

**Definition 2.3.** A groupoid (quasigroup) $(\hat{G}, \odot)$ is

(1) right symmetric if $(\alpha \odot \beta) \odot \beta = \alpha$ for all $\alpha, \beta \in \hat{G}$
(2) left symmetric if $\beta \odot (\beta \odot \alpha) = \alpha$ for all $\alpha, \beta \in \hat{G}$

**Definition 2.4.** A quasigroup $(\hat{G}, \odot)$ is said to obeys key-laws if it satisfies both Definitions 2.3

**Definition 2.5.** Let $W^1$ be a set, if it is a poset in which any two elements have supremum and infimum. Then, it is called a lattice for $\sup\{k^*, m^*\}$ and $\inf\{k^*, m^*\}$ are respectively denoted as $k^* \vee m^*$ and $k^* \wedge m^*$. It called a distributive lattice if $k^* \wedge (m^* \vee n^*) = (k^* \wedge m^*) \vee (k^* \wedge n^*)$ for any $k^*, m^*, n^* \in L$

**Definition 2.6.** [1] Given the two $Q-$NS sets $(\Lambda^Q, \mathfrak{A}^1)$ and $(\Theta^Q, \mathfrak{B}^1)$. Then, the intersection, AND, union and OR operations are defined as follows:

(1) $(\Lambda^Q, \mathfrak{A}^1) \cap (\Theta^Q, \mathfrak{B}^1) = (\triangle_1^Q, \mathfrak{C}^1)$ is a $Q-$NS set, where $\mathfrak{C}^1 = \mathfrak{A}^1 \cap \mathfrak{B}^1$

$$T_{\triangle_1^Q(\alpha)}(w^1, u^1) = \min\{T_{\Lambda^Q(\alpha)}(w^1, u^1), T_{\Theta^Q(\alpha)}(w^1, u^1)\}$$
$$I_{\triangle_1^Q(\alpha)}(w^1, u^1) = \max\{I_{\Lambda^Q(\alpha)}(w^1, u^1), I_{\Theta^Q(\alpha)}(w^1, u^1)\}$$
$$F_{\triangle_1^Q(\alpha)}(w^1, u^1) = \max\{F_{\Lambda^Q(\alpha)}(w^1, u^1), F_{\Theta^Q(\alpha)}(w^1, u^1)\}$$

(2) $(\Lambda^Q, \mathfrak{A}^1) \cup (\Theta^Q, \mathfrak{B}^1) = (\triangle_1^Q, \mathfrak{C}^1)$ is a $Q-$NS, where $\mathfrak{C}^1 = \mathfrak{A}^1 \cup \mathfrak{B}^1$

$$T_{\triangle_1^Q(a)}(w^1, u^1) = \begin{cases} T_{\Lambda^Q(\alpha)}(w^1, u^1), & \text{if } \alpha \in \mathfrak{A}^1 - \mathfrak{B}^1 \\ T_{\Theta^Q(\alpha)}(w^1, u^1), & \text{if } \alpha \in \mathfrak{B}^1 - \mathfrak{A}^1 \\ \max\{T_{\Lambda^Q(\alpha)}(w^1, u^1), T_{\Theta^Q(\alpha)}(w^1, u^1)\}, & \text{if } \alpha \in \mathfrak{B}^1 \cap \mathfrak{A}^1 \end{cases}$$

$$I_{\triangle_1^Q(a)}(w^1, u^1) = \begin{cases} I_{\Lambda^Q(\alpha)}(w^1, u^1), & \text{if } \alpha \in \mathfrak{A}^1 - \mathfrak{B}^1 \\ I_{\Theta^Q(\alpha)}(w^1, u^1), & \text{if } \alpha \in \mathfrak{B}^1 - \mathfrak{A}^1 \\ \min\{I_{\Lambda^Q(\alpha)}(w^1, u^1), I_{\Theta^Q(\alpha)}(w^1, u^1)\}, & \text{if } \alpha \in \mathfrak{B}^1 \cap \mathfrak{A}^1 \end{cases}$$

$$F_{\triangle_1^Q(a)}(w^1, u^1) = \begin{cases} F_{\Lambda^Q(\alpha)}(w^1, u^1), & \text{if } \alpha \in \mathfrak{A}^1 - \mathfrak{B}^1 \\ F_{\Theta^Q(\alpha)}(w^1, u^1), & \text{if } \alpha \in \mathfrak{B}^1 - \mathfrak{A}^1 \\ \min\{F_{\Lambda^Q(\alpha)}(w^1, u^1), F_{\Theta^Q(\alpha)}(w^1, u^1)\}, & \text{if } \alpha \in \mathfrak{B}^1 \cap \mathfrak{A}^1 \end{cases}$$

(3) $(\Lambda^Q, \mathfrak{A}^1) \wedge (\Theta^Q, \mathfrak{B}^1) = (\triangle_1^Q, \mathfrak{C}^1)$ is a $Q-$NS set, where $\triangle_{Q(\alpha,\beta)} = \Lambda_{Q(\alpha)} \cap \Theta_{Q(\beta)}$ and $(\alpha, \beta) \in \mathfrak{A}^1 \times \mathfrak{B}^1, w^1 \in W^1$, and $u^1 \in Q$.

$$T_{\triangle_1^Q(\alpha,\beta)}(w^1, u^1) = \min\{T_{\Lambda^Q(\alpha)}(w^1, u^1), T_{\Theta^Q(\beta)}(w^1, u^1)\}$$
$$I_{\triangle_1^Q(\alpha,\beta)}(w^1, u^1) = \max\{I_{\Lambda^Q(\alpha)}(w^1, u^1), I_{\Theta^Q(\beta)}(w^1, u^1)\}$$
$$F_{\triangle_1^Q(\alpha,\beta)}(w^1, u^1) = \max\{F_{\Lambda^Q(\alpha)}(w^1, u^1), F_{\Theta^Q(\beta)}(w^1, u^1)\}$$

(4) $(\Lambda^Q, \mathfrak{A}^1) \vee (\Theta^Q, \mathfrak{B}^1) = (\triangle_1^Q, \mathfrak{C}^1)$ is a $Q-$NS set, where $\triangle_{Q(\alpha,\beta)} = \Lambda_{Q(\alpha)} \cup \Theta_{Q(\beta)}$ and $(\alpha, \beta) \in \mathfrak{A}^1 \times \mathfrak{B}^1, w^1 \in W^1$, and $u^1 \in Q$.

$$T_{\triangle_1^Q(\alpha,\beta)}(w^1, u^1) = \max\{T_{\Lambda^Q(\alpha)}(w^1, u^1), T_{\Theta^Q(\beta)}(w^1, u^1)\}$$
$$I_{\triangle_1^Q(\alpha,\beta)}(w^1, u^1) = \min\{I_{\Lambda^Q(\alpha)}(w^1, u^1), I_{\Theta^Q(\beta)}(w^1, u^1)\}$$
$$F_{\triangle_1^Q(\alpha,\beta)}(w^1, u^1) = \min\{F_{\Lambda^Q(\alpha)}(w^1, u^1), F_{\Theta^Q(\beta)}(w^1, u^1)\}$$

**Definition 2.7.** [16] Let $W^1$ be set, a pair $(F, \mathfrak{A}^1)$ is called a soft set if $F : \mathfrak{A}^1 \to P(W^1)$, where $P(W^1)$ is power set of $W^1$ and $\mathfrak{A}^1$ is a set of parameters.

**Definition 2.8.** [24] Let $W^1$ be a set. A neutrosophic set $(NS)$ is described as
$\Phi = \{\langle w^1, (T_\Phi(w^1), I_\Phi(w^1), F_\Phi(w^1))\rangle : w^1 \in W^1\}$ such that $T_\Phi, I_\Phi, F_\Phi : W^1 \to ]^-0, 1^+[$.

**Definition 2.9.** [6] A $Q-$neutrosophic set $\Pi_1^Q$ in $W^1$ is described in the form
$\Pi_1^Q = \{\langle (w^1, u^1), T_{\Phi^Q}(w^1, u^1), I_{\Phi^Q}(w^1, u^1), F_{\Phi^Q}(w^1, u^1)\rangle : w^1 \in W^1, u^1 \in Q\}$, where $T_{\Phi^Q}, I_{\Phi^Q}, F_{\Phi^Q} : W^1 \times Q \to ]^-0, 1^+[$ are the membership degrees.

**Definition 2.10.** [17] Let $W^1$ be a set and $\mathfrak{A}^1$ be a parameter sets. A (NS) set $(\Phi, \mathfrak{A}^1)$ is described as $(\Phi, \mathfrak{A}^1) = \{\langle w^1, (T_\Phi(w^1), I_\Phi(w^1), F_\Phi(w^1))\rangle : w^1 \in W^1\}$.

**Definition 2.11.** [6] Let $k$ be any positive integer, $I$ be a unit interval $[0, 1]$, $W^1$ be a universe of discourse and $Q$ be a non-empty sets. A $Q-$neutrosophic set $\Pi_1^Q$ in $W^1$ and $Q$ is described as

$$\Pi_1^Q = \{< (w^1, u^1), T_{\Phi_1}^{Q_i}(w^1, u^1), I_{\Phi_1}^{Q_i}(w^1, u^1), F_{\Phi_1}^{Q_i}(w^1, u^1) >: w^1 \in W^1, u^1 \in Q\, \forall\, i = 1, 2, 3, ..., k\},$$

where $T_{\hat{\Phi_1}^{Q_i}}, I_{\hat{\Phi_1}^{Q_i}}, F_{\hat{\Phi_1}^{Q_i}} : W^1 \times Q \to I^k\ \forall i = 1, 2, ..., k$ are membership degrees.

**Definition 2.12.** [1] Suppose that $W^1$ is a universal set and $Q$ is a non-empty set. Let $\mathfrak{A}^1 \subset E$ be a set of parameters. A pair $(\Pi_1^Q, \mathfrak{A}^1)$ is called a $(Q-$NSS) over $W^1$ and $Q$, where $\Pi_1^Q : \mathfrak{A} \to \rho^l QNS(W^1)$ is a map such that $\Pi_1^Q(a) = \emptyset$ if $a \notin \mathfrak{A}^1$. It is denoted by $(\Pi_1^Q, \mathfrak{A}^1) = \{(a, \Pi_1^Q(a)) : a \in \mathfrak{A}^1, \Pi_1^Q(a) \in \rho^l QNS(W^1)\}$

**Definition 2.13.** The direct product $(\Pi_1^Q, \mathfrak{A}^1) \times (\Psi_1^Q, \mathfrak{B}^1)$ of $(\Pi_1^Q, \mathfrak{A}^1)$ and $(\Psi_1^Q, \mathfrak{B}^1)$ is a $Q-$NS set $(\Pi_1^Q, \mathfrak{C}^1)$ under $\hat{W_1^1} \times \hat{W_2^1}$ such that $\mathfrak{A}^1 \times \mathfrak{B}^1 = \mathfrak{C}^1$.

$$\Pi_1^Q(\alpha, \beta) = \Big\{\langle (((w_1^1, w_2^1), u^1), T_{\Phi_1^Q(\alpha,\beta)}((w_1^1, w_2^1), u^1)), I_{\Phi_1^Q(\alpha,\beta)}((w_1^1, w_2^1), q), F_{\Phi_1^Q(\alpha,\beta)}((w_1^1, w_2^1), q)\rangle : (w_1^1, w_2^1) \in \hat{Q}_1 \times \hat{Q}_2, u^1 \in Q\Big\}$$

The membership degrees are defined as

$$T_{\Pi_1^Q(\alpha,\beta)}((w_1^1,w_2^1),u^1) = \min\{T_{\Pi_1^Q(\alpha)}(w_1^1,u^1), T_{\Psi_1^Q(\beta)}(w_2^1,u^1)\},$$
$$I_{\Pi_1^Q(\alpha,\beta)}((w_1^1,w_2^1),u^1) = \max\{I_{\Pi_1^Q(\alpha)}(w_1^1,u^1), I_{\Psi_1^Q(\beta)}(w_2^1,u^1)\},$$
$$F_{\Pi_1^Q(\alpha,\beta)}((w_1^1,w_2^1),u^1) = \max\{F_{\Pi_1^Q(\alpha)}(w_1^1,u^1), F_{\Psi_1^Q(\beta)}(w_2^1,u^1))\}.$$

## 3. Main Results

**Definition 3.1.** Let $(\nabla_1^Q,\mathfrak{A}^1)$ be a $Q-$NS set defined over a quasigroup $(\hat{G},\odot/,\backslash)$. Then $(\nabla_1^Q,\mathfrak{A}^1)$ is called a $Q-$NS quasigroup over a quasigroup $\hat{G}$ if for all $\alpha\in\mathfrak{A}^1, u^1\in Q, \nabla_1^Q(\alpha)$ is a $Q-$NS quasigroup given a map $\nabla_1^Q(a):\hat{G}\times Q\to[0,1]^3$

**Definition 3.2.** Let $(\nabla_1^Q,\mathfrak{A}^1)$ be a $Q-$NS set defined under a quasigroup $(\hat{G},\odot/,\backslash)$. Then $(\nabla_1^Q,\mathfrak{A}^1)$ is called a $Q-$neutrosphic soft quasigroup if for all $a\in\mathfrak{A}^1, w^1,t^1\in\hat{G}, u^1\in Q$ satisfies the following

(1) $T_{\nabla_1^Q(a)}((w^1*t^1),u^1)\geq\min\{T_{\nabla_1^Q(a)}(w^1,u),T_{\nabla_1^Q(a)}(t^1,u^1)\}$
(2) $I_{\nabla_1^Q(a)}((w^1*t^1),u^1)\leq\max\{I_{\nabla_1^Q(a)}(w^1,u),I_{\nabla_1^Q(a)}(t^1,u^1)\}$
(3) $F_{\nabla_1^Q(a)}((w^1*t^1),u^1)\leq\max\{F_{\nabla_1^Q(a)}(w^1,u),F_{\nabla_1^Q(a)}(t^1,u^1)\}$

where $*\in\{\odot,/,\backslash\}$

**Example 3.3.** Let $\hat{G}=\{i,j,k,l,m,n,o\}$ be quasigroup of order 7 and $\mathfrak{A}^1$ be a subset of $E$ called the parameter sets. Given the quasigroup in Cayley table below.

TABLE 1. Quasigroup of order 7

| ⊙ | i | j | k | l | m | n | o |
|---|---|---|---|---|---|---|---|
| i | i | m | o | n | j | l | k |
| j | m | j | n | o | i | k | l |
| k | o | n | k | m | l | j | i |
| l | n | o | m | k | l | i | j |
| m | j | i | l | k | m | o | n |
| n | l | k | j | i | o | n | m |
| o | k | l | i | j | n | m | o |

Define a $Q-$NS set $(\nabla_1^Q,\mathfrak{A}^1)$, for all $u^1\in Q$ and $w^1,t^1,z^1\in\hat{G}$ such that $z^1=w^1*t^1\in\hat{G}$. Let $\mathfrak{A}^1$ be the set parameters and $n\in\mathbb{N}$ a set of natural numbers.

$$T_{\nabla_1^Q(a)}((w^1*t^1),u^1)=\begin{cases}1-\frac{1}{2n}, & \text{if } z^1=\{j,l,k,m,n,o\}\\ 1, & \text{otherwise.}\end{cases}$$

$$I_{\nabla_1^Q(a)}((w^1*t^1),u^1)=\begin{cases}0, & \text{if } z^1=\{j,l,m,k,n,o\}\\ 1-\frac{1}{2n}, & \text{otherwise.}\end{cases}$$

$$F_{\nabla_1^Q(a)}((w^1 * t^1), u^1) = \begin{cases} 0, & \text{if } z^1 = \{j,k,m,l,n,o\} \\ 1 - \frac{1}{2n}, & \text{otherwise.} \end{cases}$$

Considering
the operation " $\odot$ ", then, $T_{\nabla_1^Q(a)}((w^1 * t^1), u^1) \geq \min\{T_{\nabla_1^Q(a)}(w^1, u^1), T_{\nabla_1^Q(a)}(t^1, u^1)\}$. Put $w^1 = j, t^1 = m$, then we have

$$\begin{aligned} T_{\nabla_1^Q(a)}(j \odot m, u^1) &= T_{\nabla_1^Q(a)}(i, u^1) \\ &\Rightarrow \text{RHS} = 1 \in [0,1] \end{aligned} \tag{1}$$

On the other hand,

$$\min\{T_{\nabla_1^Q(a)}(j, u^1), T_{\nabla_1^Q(a)}(m, u^1)\} =$$
$$\min\{(1 - \frac{1}{2n}, u^1), (1 - \frac{1}{2n}, u^1)\} = 1 - \frac{1}{2n} = 0.5 \in [0,1] \text{ for } n = 1 \tag{2}$$

Hence, from the definition 3.2, we have that $1 \geq \min\{1-\frac{1}{2n}, 1-\frac{1}{2n}\} \Rightarrow 1 \geq 1-\frac{1}{2n}$ for all $n \in \mathbb{N}$. It holds for true membership degree. The results for right and left division operations "/", and "\″ can also be verify in similar way. Also, the results for indeterminate and falsity membership degrees are similar with the result obtained for true membership degree. Hence, $(\nabla_1^Q, \mathfrak{A}^1)$ is a $Q-$neutrosophic soft quasigroup over the quasigroup $(\hat{G}, \odot, /, \backslash)$

**Definition 3.4.** Given the two $Q-$neutrosophic soft quasigroups $(\nabla_1^Q, \mathfrak{A}^1)$ and $(\Psi, \mathfrak{B}^1)$ over a quasigroup $(\hat{G}, \odot, /, \backslash)$, and let $Q$ be a non empty set. Then,

(1) The product $(\nabla_1^Q, \mathfrak{A}^1) \odot (\Psi_1^Q, \mathfrak{B}^1)$ of $(\nabla_1^Q, \mathfrak{A}^1)$ and $(\Psi, \mathfrak{B}^1)$ is a $Q-$NSS $(\Pi_1^Q, \mathfrak{C}^1)$ over $(\hat{G}, \odot)$ such that $\mathfrak{C}^1 = \mathfrak{A}^1 \cap \mathfrak{B}^1$.

$$\Pi_1^Q(\alpha \odot \beta) = \left\{ \langle((w^1, u^1), T_{\nabla_1^Q(a,b)}((w^1, u^1), I_{\nabla_1^Q(a,b)}(w^1, u^1), F_{\nabla_1^Q(a,b)}((t, u^1)\rangle : w^1, f^1 \in \hat{G}, u^1 \in Q \right\}$$

where

$$\begin{aligned} T_{\Pi_1^Q(\alpha\odot\beta)}(w^1 \odot f^1, u^1) &\geq \min\{T_{\nabla_1^Q(\alpha)}(w^1, u^1), T_{\Psi_1^Q(\beta)}(f^1, u^1)\}, \\ I_{\Pi_1^Q(\alpha\odot\beta)}(w^1 \odot f^1, u^1) &\leq \max\{I_{\nabla_1^Q(\alpha)}(w^1, u^1), I_{\Psi_1^Q(\beta)}(f^1, u^1)\}, \\ F_{\Pi_1^Q(\alpha\odot\beta)}(w^1 \odot f^1, u^1) &\leq \max\{F_{\nabla_1^Q(\alpha)}(w^1, u^1), F_{\Psi_1^Q(\beta)}(f^1, u^1))\}, \end{aligned}$$

(2) The right division $(\nabla_1^Q, \mathfrak{A}^1)/(\Psi_1^Q, \mathfrak{B}^1)$ of $(\nabla_1^Q, \mathfrak{A}^1)$ and $(\Psi_1^Q, \mathfrak{A}^1)$ is a $Q-NSS$ $(\Pi_1^Q, \mathfrak{C}^1)$ over $(\hat{G}, /)$ such that $\mathfrak{A}^1 \cap \mathfrak{B}^1 = \mathfrak{C}^1$. Thus,

$$\Pi_1^Q(\alpha/\beta) = \left\{ \langle(w^1/f^1, u^1), T_{\nabla_1^Q(\alpha/\beta)}((w^1/f^1, u^1), I_{\nabla_1^Q(\alpha/\beta)}((w^1/f^1, u^1), F_{\nabla_1^Q(\alpha/\beta)}(w^1/f^1, u^1)\rangle : w^1, f^1 \in \hat{G}, u^1 \in Q \right\}$$

where

$$\begin{aligned} T_{\Pi_1^Q(\alpha/\beta)}(w^1/f^1,u^1) &\geq \max\{T_{\nabla_1^Q(\alpha)}(w^1,u^1),T_{\Psi_1^Q(\beta)}(f^1,u^1)\},\\ I_{\Pi_1^Q(\alpha/\beta)}(w^1/f^1,u^1) &\leq \min\{I_{\nabla_1^Q(\alpha)}(w^1,u^1),I_{\Psi_1^Q(\beta)}(f^1,u^1)\},\\ I_{\Pi_1^Q(\alpha/\beta)}(w^1/f^1,u^1) &\leq \min\{I_{\nabla_1^Q(\alpha)}(w^1,u^1),I_{\Psi_1^Q(\beta)}(f^1,u^1)\} \end{aligned}$$

(3) The left division $(\nabla_1^Q,\mathfrak{A}^1)\backslash(\Psi_1^Q,\mathfrak{B}^1)$ of $(\nabla_1^Q,\mathfrak{A}^1)$ and $(\Psi,\mathfrak{B}^1)$ is a $Q-NSS$ $(\Pi_1^Q,\mathfrak{C}^1)$ over $\hat{G}$ such that $\mathfrak{A}^1\cap\mathfrak{A}^1=\mathfrak{C}^1$. Thus,

$$\Pi_{Q(\alpha\backslash\beta)}=\Big\{\langle(w^1/f^1,u^1),T_{\nabla_1^Q(\alpha\backslash\beta)}((w^1/f^1,u^1),I_{\nabla_1^Q(\alpha\backslash\beta)}((w^1/f^1,u^1),F_{\nabla_1^Q(\alpha\backslash\beta)}(w^1/f^1,u^1)\rangle: w^1,f^1\in\hat{G},u^1\in Q\Big\}$$

where

$$\begin{aligned} T_{\Pi_1^Q(\alpha\backslash\beta)}(w^1\backslash f^1,u^1) &\geq \max\{T_{\Psi_1^Q(\alpha)}(w^1,u^1),T_{\nabla_1^Q(\beta)}(f^1,u^1)\},\\ I_{\Pi_1^Q(\alpha\backslash\beta)}(w^1\backslash f^1,u^1) &\leq \min\{I_{\Psi_1^Q(\alpha)}(w^1,u^1),I_{\nabla_1^Q(\beta)}(f^1,u^1)\},\\ F_{\Pi_1^Q(\alpha\backslash\beta)}(w^1\backslash f^1,u^1) &\leq \min\{F_{\Psi_1^Q(\alpha)}(w^1,u^1),F_{\nabla_1^Q(\beta)}(f^1,u^1)\} \end{aligned}$$

**Theorem 3.5.** *Let $(\Lambda_1^Q,\mathfrak{A}^1),(\Theta_1^Q,\mathfrak{B}^1)$ and $(\triangle_1^Q,\mathfrak{C}^1)$ be $Q-$neutrosophic soft quasigroups over quasigroup $(\hat{G},\odot)$. Then, the following holds*

(1) $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\cap(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\cap\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$
(2) $\big((\Lambda_1^Q,\mathfrak{A}^1)\cap(\Theta_1^Q,\mathfrak{B}^1)\big)\odot(\triangle_1^Q,\mathfrak{C}^1)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)\cap\big((\Theta_1^Q,\mathfrak{B}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$
(3) $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\wedge\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$
(4) $\big((\Lambda_1^Q,\mathfrak{A}^1)\wedge(\Theta_1^Q,\mathfrak{B}^1)\big)\odot(\triangle_1^Q,\mathfrak{C}^1)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)\wedge\big((\Theta_1^Q,\mathfrak{B}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$
(5) $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\cup(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\cup\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$
(6) $\big((\Lambda_1^Q,\mathfrak{A}^1)\cup(\Theta_1^Q,\mathfrak{B}^1)\big)\odot(\triangle_1^Q,\mathfrak{C}^1)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)\cup\big((\Theta_1^Q,\mathfrak{B}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$
(7) $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\vee(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\vee\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$
(8) $\big((\Lambda_1^Q,\mathfrak{A}^1)\vee(\Theta_1^Q,\mathfrak{B}^1)\big)\odot(\triangle_1^Q,\mathfrak{C}^1)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)\vee\big((\Theta_1^Q,\mathfrak{B}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$

*Proof:*

(1) We shall show that $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\cap(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\cap\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$

Let

$$(\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)=(\Psi_1^Q,\mathfrak{C}^1)\text{ such that}$$
$$T_{\Psi_1^Q(a\odot b)}(w\odot t,u^1)=\min\underbrace{\{T_{\Lambda_1^Q(a)}(w,u^1),T_{\Theta_1^Q(b)}(t,u^1)\}}_{(a\odot b)\in\mathfrak{A}^1\cap\mathfrak{B}^1} \tag{3}$$

for all $w,t\in\hat{G}$ and $u^1\in Q$. And, let

$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\triangle_1^Q, \mathfrak{C}^1) = (\mho_1^Q, \mathfrak{F}^1)$ such that for all $w, t \in \hat{G}, u^1 \in Q$ we have

$$T_{\mho_1^Q(a \odot c)}(w \odot t, u^1) = \min \underbrace{\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\triangle_1^Q(c)}(t, u^1)\}}_{(a \odot c) = f \in \mathfrak{A}^1 \cap \mathfrak{C}^1} \tag{4}$$

Let $(\Psi_1^Q, \mathfrak{E}^1) \cap (\mho_1^Q, \mathfrak{F}^1) = (\Pi_1^Q, \mathfrak{G}^1)$ such that $g \in (\mathfrak{E}^1 \cap \mathfrak{F}^1) = \mathfrak{G}^1$ for all $g \in \mathfrak{G}^1$

Combining equations (3) and (4), for all $w, t \in \hat{G}$ and $u^1 \in Q$ we have

$$\begin{aligned} T_{\Pi_1^Q(g)}(wt, u^1) &= \min \underbrace{\{T_{\Psi_1^Q(g)}(wt, u^1), T_{\mho_1^Q(g)}(wt, u^1)\}}_{(g) \in \mathfrak{E}^1 \cap \mathfrak{F}^1.} \\ &= \min \{ \underbrace{T_{\Psi_1^Q(a \odot b)}(wt, u^1), T_{\mho_1^Q(a \odot c)}(wt, u^1)}_{a \in ((\mathfrak{A}^1 \cap \mathfrak{B}^1) \cap (\mathfrak{A}^1 \cap \mathfrak{C}^1))} \end{aligned} \tag{5}$$

Considering the LHS: Let

$(\Theta_1^Q, \mathfrak{B}^1) \cap (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{D}^1)$ such that for all $w, t \in \hat{G}$ and $u^1 \in Q, we have$

$$T_{\Phi_1^Q(d)}(wt, u^1) = \min \underbrace{\{T_{\Theta_1^Q(d)}(wt, u^1), T_{\triangle_1^Q(d)}(wt, u^1)\}}_{d \in (\mathfrak{B}^1 \cap \mathfrak{C}^1)} \tag{6}$$

Also, let

$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Phi_1^Q, \mathfrak{D}^1) = (\Xi_1^Q, \mathfrak{H}^1)$ such that for all $w, t \in \hat{G}$ and $u^1 \in Q,$ we have

$$\begin{aligned} T_{\Xi_1^Q(a \odot d)}(w \odot t, u^1) = \min & \underbrace{\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Phi_1^Q(d)}(t, u^1)\}}_{\text{this implies that for all} a \in \mathfrak{A}^1, d \in \mathfrak{D}^1, a \odot d \in (\mathfrak{A}^1 \cap \mathfrak{D}^1)} \\ = \min & \underbrace{\{T_{\Lambda_1^Q(a)}(w, u^1), \min\{T_{\Theta_1^Q(d)}(t, u^1), T_{\triangle_1^Q(d)}(t, u^1)\}\}} \\ = \min & \underbrace{\{ \min\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Lambda_1^Q(a)}(w, u^1)\}, \min\{T_{\Theta_1^Q(d)}(t, u^1), T_{\triangle_1^Q(d)}(t, u^1)\}\}} \\ = \min & \underbrace{\{ \min\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Theta_1^Q(d)}(t, u^1)\}, \min\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\triangle_1^Q(d)}(t, u^1)\}\}} \\ = \min & \underbrace{\{T_{\Psi_1^Q(a \odot d)}(w \odot t, u^1), T_{\mho_1^Q(a \odot d)}(w \odot t, u^1)\}}_{(\mathfrak{A}^1 \cap \mathfrak{B}^1) \cap (\mathfrak{A}^1 \cap \mathfrak{C}^1))} \end{aligned} \tag{7}$$

Comparing (5) and (7), we have $(\Pi_1^Q, \mathfrak{G}^1) = (\Xi_1^Q, \mathfrak{H}^1)$ for the true membership degree.

Next, is to verify for indeterminate membership degree.

Let

$$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1) = (\Psi_1^Q, \mathfrak{E}^1) \text{ such that}$$
$$I_{\Psi_1^Q(a\odot b)}(w \odot t, u^1) = \max \underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\Theta_1^Q(b)}(t, u^1)\}}_{(a\odot b)=t\in \mathfrak{A}^1\cap\mathfrak{B}^1} \quad (8)$$

for all $w, t \in \hat{G}$ and $u^1 \in Q$. And, let

$$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\triangle_1^Q, \mathfrak{C}^1) = (\mho_1^Q, \mathfrak{F}^1) \text{ such that for all } w, t \in \hat{G}, \text{ and } u^1 \in Q \text{ we have}$$
$$I_{\mho_1^Q(a\odot c)}(w \odot t, u^1) = \max \underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\triangle_1^Q(c)}(t, u^1)\}}_{(a\odot c)\in \mathfrak{A}^1\cap\mathfrak{C}^1} \quad (9)$$

Let $(\Psi_1^Q, \mathfrak{E}^1) \cap (\mho_1^Q, \mathfrak{F}^1) = (\Pi_1^Q, \mathfrak{G}^1)$ such that $g \in (\mathfrak{E}^1 \cap \mathfrak{F}^1) = \mathfrak{G}^1$ for all $g \in \mathfrak{G}^1$

Combining equations (8) and (9), for all $w, t \in \hat{G}$ and $u^1 \in Q$ we have

$$\begin{aligned} I_{\Pi_1^Q(g)}(wt, u^1) &= \max \underbrace{\{I_{\Psi_1^Q(g)}(wt, u^1), I_{\mho_1^Q(g)}(wt, u^1)\}}_{(g)\in \mathfrak{E}\cap\mathfrak{F}.} \\ &= \max \{ \underbrace{I_{\Psi_1^Q(a\odot b)}(wt, u^1), I_{\mho_1^Q(a\odot c)}(wt, u^1)}_{a\in((\mathfrak{A}^1\cap\mathfrak{B}^1)\cap(\mathfrak{A}^1\cap\mathfrak{C}^1))} \end{aligned} \quad (10)$$

Considering the LHS: Let

$$(\Theta_1^Q, \mathfrak{B}^1) \cap (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{D}^1) \text{ such that for all } w, t \in \hat{G} \text{ and } u^1 \in Q, \text{we have}$$
$$I_{\Phi_1^Q(d)}(wt, u^1) = \max \underbrace{\{I_{\Theta_1^Q(d)}(wt, u^1), I_{\triangle_1^Q(d)}(wt, u^1)\}}_{d\in(\mathfrak{B}^1\cap\mathfrak{C}^1)} \quad (11)$$

Also, let

$$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Phi_1^Q, \mathfrak{D}^1) = (\Xi_1^Q, \mathfrak{H}^1) \text{ such that for all } w, t \in \hat{G} \text{ and } u^1 \in Q, \text{ we have}$$
$$\begin{aligned} I_{\Xi_1^Q(a\odot d)}(w \odot t, u^1) = \max & \underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\Phi_1^Q(d)}(t, u^1)\}}_{\text{this implies that for all} a\in\mathfrak{A}^1, d\in\mathfrak{D}^1, a\odot d\in(\mathfrak{A}^1\cap\mathfrak{D}^1)} \\ &= \max \underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), \max\{I_{\Theta_1^Q(d)}(t, u^1), I_{\triangle_1^Q(d)}(t, u^1)\}\}} \\ &= \max \underbrace{\{\max\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\Lambda_1^Q(a)}(w, u^1)\}, \max\{I_{\Theta_1^Q(d)}(t, u^1), I_{\triangle_1^Q(d)}(t, u^1)\}\}} \\ &= \max \underbrace{\{\max\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\Theta_1^Q(d)}(t, u^1)\}, \max\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\triangle_1^Q(d)}(t, u^1)\}\}} \\ &= \max \underbrace{\{I_{\Psi_1^Q(a\odot d)}(w \odot t, u^1), I_{\mho_1^Q(a\odot d)}(w \odot t, u^1)\}}_{(\mathfrak{A}^1\cap\mathfrak{B}^1)\cap(\mathfrak{A}^1\cap\mathfrak{C}^1))} \end{aligned} \quad (12)$$

Comparing (10) and (12) to get that $(\Pi_1^Q, \mathfrak{G}^1) = (\Xi_1^Q, \mathfrak{H}^1)$ for the indeterminate membership degree.

Next, verifying the falsity membership degree is similar with the result obtain for indeterminate membership degree .

(2) It has a similar argument with (1)

(3) We shall show that $(\Lambda_1^Q, \mathbb{A}) \odot \big((\Theta_1^Q, \mathfrak{B}^1) \wedge (\triangle_1^Q, \mathfrak{C}^1)\big) = \big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1)\big) \wedge \big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\triangle_1^Q, \mathfrak{C}^1)\big)$

Let

$$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1) = (\Psi_1^Q, \mathfrak{E}^1) \text{ such that}$$
$$T_{\Psi_1^Q(a\odot b)}(w \odot t, u^1) = \min \underbrace{\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Theta_1^Q(b)}(t, u^1)\}}_{(a\odot b) = t \in \mathfrak{A}^1 \cap \mathfrak{B}^1} \tag{13}$$

for all $w, t \in \hat{G}$ and $u^1 \in Q$.

And, let

$$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\triangle_1^Q, \mathfrak{C}^1) = (\mho_1^Q, \mathfrak{F}^1) \text{ such that for all } w \in \hat{G}, u^1 \in Q \text{ wt have}$$
$$T_{\mho_1^Q(a\odot c)}(w \odot t, u^1) = \min \underbrace{\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\triangle_1^Q(c)}(t, u^1)\}}_{(a\odot c) \in \mathfrak{A}^1 \cap \mathfrak{C}^1} \tag{14}$$

Now, from equations (13) and (14) we have $(\Psi_1^Q, \mathfrak{E}^1) \wedge (\mho_1^Q, \mathfrak{F}^1) = (\Pi_1^Q, \mathfrak{E}^1 \times \mathfrak{F}^1)$ $(e^*, f^*) \in (\mathfrak{E}^1 \times \mathfrak{F}^1)$ where $t^* = a \odot b$ and $f^* = a \odot c$ are parameter sets

Hence,

$$\begin{aligned} T_{\Pi_1^Q(g)}(w \odot t, u^1) &= T_{\Pi_1^Q(t^*, f^*)}((w \odot t), u^1) \\ &= \min \underbrace{\{T_{\Psi_1^Q(t)}((w \odot t), u^1), T_{\mho_1^Q(f)}((w \odot t), u^1)\}}_{(t^*, f^*) \in \mathfrak{E}^1 \times \mathfrak{F}^1.} \\ &= \min \underbrace{\{T_{\Psi_1^Q(a\odot b)}((w \odot t), u^1), T_{\mho_1^Q(a\odot c)}((w \odot t), u^1)\}}_{(t^*, f^*) \in \mathfrak{E} \times \mathfrak{F}.} \end{aligned} \tag{15}$$

Note that $t^*$ and $f^*$ are set of parameters.

Substituting (13) and (14) into (15), give

$$\begin{aligned} &\min \underbrace{\{T_{\Psi_1^Q(a\odot b)}((w \odot t), u^1), T_{\mho_1^Q(a\odot c)}((w \odot t), u^1)\}}_{(t^*, f^*) \in \mathfrak{E}^1 \times \mathfrak{F}^1.} \\ \geq \min &\left\{ \min \underbrace{\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Theta_1^Q(b)}(t, u^1)\}}_{a\odot b \in \mathfrak{A}^1 \cap \mathfrak{B}^1.}, \min \underbrace{\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\triangle_1^Q(c)}(t, u^1)\}}_{a\odot c \in \mathfrak{A}^1 \cap \mathfrak{C}^1.} \right\} \\ &= \min \underbrace{\{T_{\Psi_1^Q(a\odot b)}((w \odot t), u^1), T_{\mho_1^Q(a\odot c)}((w \odot t), u^1)\}}_{(\mathfrak{A}^1 \cap \mathfrak{B}^1) \times (\mathfrak{A}^1 \cap \mathfrak{C}^1).} \end{aligned} \tag{16}$$

Considering the LHS:

Let $(\Theta_1^Q, \mathfrak{B}^1) \wedge (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{D}^1)$ such that $(\Theta_1^Q, \mathfrak{B}^1) \wedge (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{B}^1 \times \mathfrak{C}^1)$ for all $w, t \in \hat{G}, u^1 \in Q$ and $(b, c) \in \mathfrak{B}^1 \times \mathfrak{C}^1$. Then, this follows

$$T_{\Phi_1^Q(b,c)}(wt, u^1) = \min\{T_{\Theta_1^Q(b)}(wt, u^1), T_{\triangle_1^Q(c)}(wt, u^1)\} \tag{17}$$

And, let $(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Phi_1^Q, \mathfrak{D}^1) = (\Xi_1^Q, \mathfrak{H}^1)$ such that for all $w, t \in \hat{G}, u^1 \in Q$, we have

$$T_{\Xi_1^Q(a\odot d)}(wt, u^1) = \underbrace{\min\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Phi_1^Q(d)}(t, u^1)\}}_{\text{for all } a\odot d \in \mathfrak{A}\cap\mathfrak{D} \text{ where } d=(b,c)\in\mathfrak{B}^1\times\mathfrak{C}^1} \tag{18}$$

Then, putting (17) into (18), give

$$\begin{aligned}
&T_{\Xi_1^Q(h)}(wt, u^1) = T_{\Xi_1^Q(a,d)}(wt, u^1)\\
&= \min\{ \underbrace{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Phi_1^Q(d)}(t, u^1)\}}_{\text{for all } a\odot d \in \mathfrak{A}^1\cap\mathfrak{D}^1 \text{ where } wt = t\in\hat{G}}\\
&= \min\underbrace{\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Phi_1^Q(b,c)}(wt, u^1)\}}_{(a\odot d)\in\mathfrak{A}^1\cap\mathfrak{D}^1.}\\
&= \min\Big\{ \underbrace{T_{\Lambda_1^Q(a)}(w, u^1), \min\{T_{\Theta_1^Q(b)}(t, u^1), T_{\triangle_1^Q(c)}(t, u^1)\}\Big\}}_{(a\odot d)\in\mathfrak{A}^1\cap\mathfrak{D}^1.}\\
&= \min\Big\{ \underbrace{\min\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Lambda_1^Q(a)}(w, u^1)\}, \min\{T_{\Theta_1^Q(b)}(t, u^1), T_{\triangle_1^Q(c)}(t, u^1)\}\Big\}}_{(a\odot(b,c)\in\mathfrak{A}^1\cap(\mathfrak{B}^1\times\mathfrak{C}^1).}\\
&= \min\Big\{ \underbrace{\min\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Theta_1^Q(b)}(t, u^1)\}}_{(a\odot(b))\in\mathfrak{A}^1\cap\mathfrak{B}^1.}, \underbrace{\min\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\triangle_1^Q(c)}(t, u^1)\}\Big\}}_{(a\odot c))\in\mathfrak{A}^1\cap\mathfrak{C}^1.}\\
&\min\underbrace{\{T_{\Psi_1^Q(a\odot b)}((w\odot t), u^1), T_{\mho_1^Q(a\odot c)}((w\odot t), u^1)\}}_{(\mathfrak{A}^1\cap\mathfrak{B}^1)\times(\mathfrak{A}^1\cap\mathfrak{C}^1).}
\end{aligned} \tag{19}$$

For indeterminacy membership degree.

Let

$$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1) = (\Psi_1^Q, \mathfrak{E}^1) \text{ such that}$$
$$I_{\Psi_1^Q(a\odot b)}(w\odot t, u^1) = \max\underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\Theta_1^Q(b)}(t, u^1)\}}_{(a\odot b)=t\in\mathfrak{A}^1\cap\mathfrak{B}^1} \tag{20}$$

for all $w \in \hat{G}$ and $u^1 \in Q$. And, let

$$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\triangle_1^Q, \mathfrak{C}^1) = (\mho_1^Q, \mathfrak{F}^1) \text{ such that for all } w, t\in\hat{G}, u^1\in Q \text{ we have}$$
$$I_{\mho_1^Q(a\odot c)}(w\odot t, u^1) = \max\underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\triangle_1^Q(c)}(t, u^1)\}}_{(a\odot c)\in\mathfrak{A}^1\cap\mathfrak{C}^1} \tag{21}$$

Now, from equations (20) and (21) we have that $(\Psi_1^Q, \mathfrak{E}^1) \wedge (\mho_1^Q, \mathfrak{F}^1) = (\Pi_1^Q, \mathfrak{E}^1 \times \mathfrak{F}^1)$ for all $(t^*, f^*) \in (\mathfrak{E}^1 \times \mathfrak{F}^1)$ where $t^* = a \odot b$ and $f^* = a \odot c$ are set of parameters. Then,

$$\begin{aligned} I_{\Pi_1^Q(g)}(w \odot t, u^1) &= I_{\Pi_1^Q(t,f)}((w \odot t), u^1) \\ &= \max \underbrace{\{I_{\Psi_1^Q(t)}((w \odot t), u^1), I_{\mho_1^Q(f)}((w \odot t), u^1)\}}_{(t,f) \in \mathfrak{E}^1 \times \mathfrak{F}^1.} \\ &= \max \underbrace{\{I_{\Psi_1^Q(a \odot b)}((w \odot t), u^1), I_{\mho_1^Q(a \odot c)}((w \odot t), u^1)\}}_{(t,f) \in \mathfrak{E}^1 \times \mathfrak{F}^1.} \end{aligned} \tag{22}$$

Substituting (20) and (21) into (22), give

$$\begin{aligned} &\max \underbrace{\{I_{\Psi_1^Q(a \odot b)}((w \odot t), u^1), I_{\mho_1^Q(a \odot c)}((w \odot t), u^1)\}}_{(t,f) \in \mathfrak{E}^1 \times \mathfrak{F}^1.} \\ \geq \max &\left\{ \max \underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\Theta_1^Q(b)}(t, u^1)\}}_{a \odot b \in \mathfrak{A}^1 \cap \mathfrak{B}^1.}, \max \underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\triangle_1^Q(c)}(t, u^1)\}}_{a \odot c \in \mathfrak{A}^1 \cap \mathfrak{C}^1.} \right\} \\ &= \max \underbrace{\{I_{\Psi_1^Q(a \odot b)}((w \odot t), u^1), I_{\mho_1^Q(a \odot c)}((w \odot t), u^1)\}}_{(\mathfrak{A}^1 \cap \mathfrak{B}^1) \times (\mathfrak{A}^1 \cap \mathfrak{C}^1).} \end{aligned} \tag{23}$$

Considering the LHS:

Let $(\Theta_1^Q, \mathfrak{B}^1) \wedge (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{D}^1)$ such that $(\Theta_1^Q, \mathfrak{B}^1) \wedge (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{B}^1 \times \mathfrak{C}^1)$ for all $w, t \in \hat{G}, u^1 \in Q$ and $(b, c) \in \mathfrak{B}^1 \times \mathfrak{C}^1$, It follows that

$$I_{\Phi_1^Q(b,c)}(wt, u^1) = \max\{I_{\Theta_1^Q(b)}(wt, u^1), I_{\triangle_1^Q(c)}(wt, u^1)\} \tag{24}$$

And, let $(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Phi_1^Q, \mathfrak{D}^1) = (\Xi_1^Q, \mathfrak{H}^1)$ such that for all $w, t \in \hat{G}, u^1 \in Q$,

$$I_{\Xi_1^Q(a \odot d)}(w \odot t, u^1) = \underbrace{\max\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\Phi_1^Q(d)}(t, u^1)\}}_{\text{for all } a \odot d \in \mathfrak{A}^1 \cap \mathfrak{D}^1 \text{ where } d = (b,c) \in \mathfrak{B}^1 \times \mathfrak{C}^1} \tag{25}$$

Then, putting (24) into (25), we have

$$\begin{aligned}
&I_{\Xi_1^Q(h)}(wt,u^1) = I_{\Xi_1^Q(a\odot d)}(wt,u^1)\\
&= \max\{ \underbrace{I_{\Lambda_1^Q(a)}(w,u^1), I_{\Phi_1^Q(d)}(t,u^1)\}}_{\text{for all } a\odot d\in\mathfrak{A}^1\cap\mathfrak{D}^1 \text{ where } w\odot t=t\in\hat{G}}\\
&= \max\{\underbrace{I_{\Lambda_1^Q(a)}(w,u^1), I_{\Phi_1^Q(b,c)}(wt,u^1)\}}_{(a\odot d)\in\mathfrak{A}^1\cap\mathfrak{D}^1.}\\
&= \max\Big\{\underbrace{I_{\Lambda_1^Q(a)}(w,u^1), \max\{I_{\Theta_1^Q(b)}(t,u^1), I_{\triangle_1^Q(c)}(t,u^1)\}\Big\}}_{(a\odot d)\in\mathfrak{A}^1\cap\mathfrak{D}^1.}\\
&= \max\Big\{\underbrace{\max\{I_{\Lambda_1^Q(a)}(w,u^1), I_{\Lambda_1^Q(a)}(w,u^1)\}, \max\{I_{\Theta_1^Q(b)}(t,u^1), I_{\triangle_1^Q(c)}(t,u^1)\}\Big\}}_{(a\odot(b,c))\in\mathfrak{A}^1\cap(\mathfrak{B}^1\times\mathfrak{C}^1).}\\
&= \max\Big\{\underbrace{\max\{I_{\Lambda_1^Q(a)}(w,u^1), I_{\Theta_1^Q(b)}(t,u^1)\}}_{(a\odot(b))\in\mathfrak{A}^1\cap\mathfrak{B}^1.}, \underbrace{\max\{I_{\Lambda_1^Q(a)}(w,u^1), I_{\triangle_1^Q(c)}(t,u^1)\}\Big\}}_{(a\odot c))\in\mathfrak{A}^1\cap\mathfrak{C}^1.}\\
&\max\underbrace{\{I_{\Psi_1^Q(a\odot b)}((w\odot t),u^1), I_{\mho_1^Q(a\odot c)}((w\odot t),u^1)\}}_{(\mathfrak{A}^1\cap\mathfrak{B}^1)\times(\mathfrak{A}^1\cap\mathfrak{C}^1).}
\end{aligned} \tag{26}$$

The proof of falsity membership degree has a similar argument with the proof of indeterminate membership

Therefore, we shown that $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)\big) = \big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\wedge \big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$

(4) The proof is similar with 3

(5) We shall show that $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\cup(\triangle_1^Q,\mathfrak{C}^1)\big) = \big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\cup \big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$

Let

$$\begin{aligned}
&(\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1) = (\Psi_1^Q,\mathfrak{E}^1) \text{ such that}\\
&T_{\Psi_1^Q(a\odot b)}(w\odot t,u^1) = \min\underbrace{\{T_{\Lambda_1^Q(a)}(w,u^1), T_{\Theta_1^Q(b)}(t,u^1)\}}_{(a\odot b)=s\in\mathfrak{A}^1\cap\mathfrak{B}^1}
\end{aligned} \tag{27}$$

for all $w,t\in\hat{G}$ and $u^1\in Q$. And, let

$$\begin{aligned}
&(\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1) = (\mho_1^Q,\mathfrak{F}^1) \text{ such that for all } w,t\in Q, u^1\in Q \text{ we have}\\
&T_{\mho_1^Q(a\odot c)}(w\odot t,u^1) = \max\underbrace{\{T_{\Lambda_1^Q(a)}(w,u^1), T_{\triangle_1^Q(c)}(t,u^1)\}}_{(f=a\odot c)\in\mathfrak{A}^1\cap\mathfrak{C}^1}
\end{aligned} \tag{28}$$

Now, from equations (27) and (28), we get $(\Psi_1^Q, \mathfrak{E}^1) \cup (\mho_1^Q, \mathfrak{F}^1) = (\Pi_1^Q, \mathfrak{G}^1)$. Then, for all $w, t \in \hat{G}, u^1 \in Q$, let $d \in \mathfrak{D}^1$ such that $(a \odot b) - (a \odot c) = g \in \mathfrak{G}^1 = \mathfrak{E}^1 \cup \mathfrak{F}^1$,

$$T_{\Pi_1^Q(g)}(w \odot t, u^1) = \begin{cases} T_{\Psi_1^Q(g)}(w \odot t, u^1), & \text{if } g \in \mathfrak{E}^1 - \mathfrak{F}^1 \\ T_{\mho_1^Q(g)}(w \odot t, u^1), & \text{if } g \in \mathfrak{F} - \mathfrak{E} \\ \max\{T_{\Psi_1^Q(g)}(w \odot t, u^1), T_{\mho_1^Q(g)}(w \odot t, u^1)\}, & \text{if } g \in \mathfrak{E}^1 \cap \mathfrak{F}^1 \end{cases} \tag{29}$$

$$= \begin{cases} \min\{T_{\Lambda_1^Q(g)}(w, u^1), T_{\Theta_1^Q(g)}(t, u^1)\}, & \text{if } g \in ((\mathfrak{A}^1 \cap \mathfrak{B}^1) - (\mathfrak{A}^1 \cap \mathfrak{C}^1)) \\ \min\{T_{\Lambda_1^Q(g)}(w, u^1), T_{\mho_1^Q(g)}(t, u^1)\}, & \text{if } g \in ((\mathfrak{A}^1 \cap \mathfrak{C}^1) - (\mathfrak{A}^1 \cap \mathfrak{B}^1)) \\ \max\Big\{ \min\{T_{\Lambda_1^Q(g)}(w, u^1), T_{\Theta_1^Q(g)}(t, u^1)\}, & \\ \min\{T_{\Lambda_1^Q(g)}(w, u^1), T_{\triangle_1^Q(g)}(t, u^1)\}\Big\}, & \text{if } g \in (\mathfrak{A}^1 \cap \mathfrak{B}^1) \cap (\mathfrak{A}^1 \cap \mathfrak{C}^1) \end{cases} \tag{30}$$

Considering the LHS

Let $(\Theta_1^Q, \mathfrak{B}^1) \cup (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{D}^1)$ such that for all $w, t \in \hat{G}$ and $u^1 \in Q$, we have

$$T_{\Phi_1^Q(d)}(w \odot t, u^1) = \begin{cases} T_{\Theta_1^Q(d)}(w \odot t, u^1), & \text{if } d \in \mathfrak{B}^1 - \mathfrak{C}^1 \\ T_{\triangle_1^Q(d)}(w \odot t, u^1), & \text{if } d \in \mathfrak{C}^1 - \mathfrak{B}^1 \\ \max\{T_{\Theta_1^Q(d)}(w \odot t, u^1), T_{\triangle_1^Q(d)}(w \odot t, u^1)\}, & \text{if } d \in \mathfrak{B}^1 \cap \mathfrak{C}^1 \end{cases} \tag{31}$$

Let

$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Phi_1^Q, \mathfrak{D}^1) = (\Xi_1^Q, \mathfrak{H}^1))$ such that for all $w, t \in \hat{G}, u^1 \in Q$ we have

$$T_{\Xi_1^Q(a \odot d)}(w \odot t, u^1) = \min \underbrace{\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Phi_1^Q(d)}(t, u^1)\}}_{(s = a \odot d) \in \mathfrak{A}^1 \cap \mathfrak{D}^1} \tag{32}$$

$$T_{\Xi_1^Q(t)}(w \odot t, u^1) = \begin{cases} \min\{T_{\Lambda_1^Q(s)}(w, u^1), \Theta_1^Q(t)(t, u^1)\}, & \text{if } s \in \mathfrak{A}^1 \cap (\mathfrak{B}^1 - \mathfrak{C}^1) \\ \min\{T_{\Lambda_1^Q(s)}(w, u^1), T_{\triangle_1^Q(t)}(t, u^1)\}, & \text{if } s \in \mathfrak{A}^1 \cap (\mathfrak{C}^1 - \mathfrak{B}^1) \\ \max\Big\{ \{T_{\Lambda_1^Q(s)}(w, u^1)\}, & \\ \min\{T_{\Theta_1^Q(s)}(t, u^1), T_{\triangle_1^Q(s)}(t, u^1)\}\Big\}, & \text{if } s \in \mathfrak{A}^1 \cap (\mathfrak{B}^1 \cap \mathfrak{C}^1) \end{cases} \tag{33}$$

$$= \begin{cases} \min\{T_{\Lambda_1^Q(s)}(w, u^1), \Theta_1^Q(t)(t, u^1)\}, & \text{if } s \in (\mathfrak{A}^1 \cap \mathfrak{B}^1) - (\mathfrak{A}^1 \cap \mathfrak{C}^1) \\ \min\{T_{\Lambda_1^Q(s)}(w, u^1), T_{\triangle_1^Q(s)}(t, u^1)\}, & \text{if } s \in (\mathfrak{A}^1 \cap \mathfrak{C}^1) - (\mathfrak{A}^1 \cap \mathfrak{B}^1) \\ \max\Big\{ \min\{T_{\Lambda_1^Q(s)}(w, u^1), T_{\Theta_1^Q(s)}(t, u^1)\}, & \\ \min\{T_{\Lambda_1^Q(s)}(w, u^1), T_{\triangle_1^Q(s)}(t, u^1)\}\Big\}, & \text{if } s \in (\mathfrak{A}^1 \cap \mathfrak{B}^1) \cap (\mathfrak{A}^1 \cap \mathfrak{C}^1) \end{cases} \tag{34}$$

Comparing equation (30) and (34), we shown that $(\Xi_1^Q, \mathfrak{H}^1) = (\Pi_1^Q, \mathfrak{G}^1)$

Next, the result for indeterminate membership degree is as follows

Let

$$(\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)=(\Psi_1^Q,\mathfrak{E}^1) \text{ then}$$
$$I_{\Psi_1^Q(a\odot b)}(w\odot t,u^1)=\max\underbrace{\{I_{\Lambda_1^Q(a)}(w,u^1),I_{\Theta_1^Q(b)}(t,u^1)\}}_{(a\odot b)=t\in\mathfrak{A}^1\cap\mathfrak{B}^1} \tag{35}$$

for all $w,t\in\hat{G}$ and $u^1\in Q$. And, let

$$(\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)=(\mho_1^Q,\mathfrak{F}^1) \text{ such that for all } w,t\in\hat{G},u^1\in Q \text{ we have}$$
$$I_{\mho_1^Q(a\odot c)}(w\odot t,u^1)=\max\underbrace{\{I_{\Lambda_1^Q(a)}(w,u^1),I_{\triangle_1^Q(c)}(t,u^1)\}}_{(f=a\odot c)\in\mathfrak{A}^1\cap\mathfrak{C}^1} \tag{36}$$

Using equations (35) and (36), we get $(\Psi_1^Q,\mathfrak{E}^1)\cup(\mho_1^Q,\mathfrak{F}^1)=(\Pi_1^Q,\mathfrak{G}^1)$. Then, for all $w,t\in Q,u^1\in Q$, let $d\in\mathfrak{D}^1$ such that $(a\odot b)-(a\odot c)=g\in\mathfrak{G}^1=\mathfrak{E}^1\cup\mathfrak{F}^1$, then

$$I_{\Pi_1^Q(g)}(w\odot t,u^1)=\begin{cases} I_{\Psi_1^Q(g)}(w\odot t,u^1), & \text{if } g\in\mathfrak{E}^1-\mathfrak{F}^1\\ I_{\mho_1^Q(g)}(w\odot t,u^1), & \text{if } g\in\mathfrak{F}^1-\mathfrak{E}^1\\ \min\{I_{\Psi_1^Q(g)}(w\odot t,u^1),I_{\mho_1^Q(g)}(w\odot t,u^1)\}, & \text{if } g\in\mathfrak{E}^1\cap\mathfrak{F}^1\end{cases} \tag{37}$$

$$=\begin{cases} \max\{I_{\Lambda_1^Q(g)}(w,u^1),I_{\Theta_1^Q(g)}(t,u^1)\}, & \text{if } g\in((\mathfrak{A}^1\cap\mathfrak{B}^1)-(\mathfrak{A}^1\cap\mathfrak{C}^1))\\ \max\{I_{\Lambda_1^Q(g)}(w,u^1),I_{\mho_1^Q(g)}(t,u^1)\}, & \text{if } g\in((\mathfrak{A}^1\cap\mathfrak{C}^1)-(\mathfrak{A}^1\cap\mathfrak{B}^1))\\ \min\Big\{\max\{I_{\Lambda_1^Q(g)}(w,u^1),I_{\Theta_1^Q(g)}(t,u^1)\}, & \\ \max\{I_{\Lambda_1^Q(g)}(w,u^1),I_{\triangle_1^Q(g)}(t,u^1)\}\Big\}, & \text{if } g\in(\mathfrak{A}^1\cap\mathfrak{B}^1)\cap(\mathfrak{A}^1\cap\mathfrak{C}^1)\end{cases} \tag{38}$$

Considering the LHS

Let $(\Theta_1^Q,\mathfrak{B}^1)\cup(\triangle_1^Q,\mathfrak{C}^1)=(\Phi_1^Q,\mathfrak{D}^1)$ such that for all $w,t\in\hat{G}$ and $u^1\in Q$, we have

$$I_{\Phi_1^Q(d)}(w\odot t,u^1)=\begin{cases} I_{\Theta_1^Q(d)}(w\odot t,u^1), & \text{if } d\in\mathfrak{B}^1-\mathfrak{C}^1\\ I_{\triangle_1^Q(d)}(w\odot t,u^1), & \text{if } d\in\mathfrak{C}^1-\mathfrak{B}^1\\ \min\{I_{\Theta_1^Q(d)}(w\odot t,u^1),I_{\triangle_1^Q(d)}(w\odot t,u^1)\}, & \text{if } d\in\mathfrak{B}^1\cap\mathfrak{C}^1\end{cases} \tag{39}$$

Let

$$(\Lambda_1^Q,\mathfrak{A}^1)\odot(\Phi_1^Q,\mathfrak{D}^1)=(\Xi_1^Q,\mathfrak{H}^1)) \text{ such that for all } w,t\in\hat{G},u^1\in Q \text{ we have}$$
$$I_{\Xi_1^Q(a\odot d)}(w\odot t,u^1)=\max\underbrace{\{I_{\Lambda_1^Q(a)}(w,u^1),I_{\Phi_1^Q(d)}(t,u^1)\}}_{(s=a\odot d)\in\mathfrak{A}^1\cap\mathfrak{D}^1} \tag{40}$$

$$I_{\Xi_1^Q(t)}(w \odot t, u^1) = \begin{cases} \max\{I_{\Lambda_1^Q(s)}(w,u^1), \Theta_1^Q(s)(t,u^1)\}, & \text{if } s \in \mathfrak{A}^1 \cap (\mathfrak{B}^1 - \mathfrak{C}^1) \\ \max\{I_{\Lambda_1^Q(s)}(w,u^1), I_{\triangle_1^Q(s)}(t,u^1)\}, & \text{if } s \in \mathfrak{A}^1 \cap (\mathfrak{C}^1 - \mathfrak{B}^1) \\ \min\Big\{\{I_{\Lambda_1^Q(s)}(w,u^1)\}, \\ \max\{I_{\Theta_1^Q(s)}(t,u^1), I_{\triangle_1^Q(s)}(t,u^1)\}\Big\}, & \text{if } s \in \mathfrak{A}^1 \cap (\mathfrak{B}^1 \cap \mathfrak{C}^1) \end{cases} \tag{41}$$

$$= \begin{cases} \max\{I_{\Lambda_1^Q(s)}(w,u^1), \Theta_1^Q(s)(t,u^1)\}, & \text{if } s \in (\mathfrak{A}^1 \cap \mathfrak{B}^1) - (\mathfrak{A}^1 \cap \mathfrak{C}^1) \\ \max\{I_{\Lambda_1^Q(s)}(w,u^1), I_{\triangle_1^Q(t)}(t,u^1)\}, & \text{if } s \in (\mathfrak{A}^1 \cap \mathfrak{C}^1) - (\mathfrak{A}^1 \cap \mathfrak{B}^1) \\ \min\Big\{\max\{I_{\Lambda_1^Q(s)}(w,u^1), I_{\Theta_1^Q(s)}(t,u^1)\}, \\ \max\{I_{\Lambda_1^Q(s)}(w,u^1), I_{\triangle_1^Q(s)}(s,u^1)\}\Big\}, & \text{if } s \in (\mathfrak{A}^1 \cap \mathfrak{B}^1) \cap (\mathfrak{A}^1 \cap \mathfrak{C}^1) \end{cases} \tag{42}$$

Comparing equation (38) and (42), we shown that $(\Xi_1^Q, \mathfrak{H}^1) = (\Pi_1^Q, \mathfrak{G}^1)$

Next, the result for falsity membership degree is similarly with the argument of indeterminate membership. Hence, we shown that $(\Lambda_1^Q, \mathfrak{A}^1) \odot \big((\Theta_1^Q, \mathfrak{B}^1) \cup (\triangle_1^Q, \mathfrak{C}^1)\big) = \big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1)\big) \cup \big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\triangle_1^Q, \mathfrak{C}^1)\big)$

(6) The proof is similar with (5)

(7) We shall show that $(\Lambda_1^Q, \mathfrak{A}^1) \odot \big((\Theta_1^Q, \mathfrak{B}^1) \vee (\triangle_1^Q, \mathfrak{C}^1)\big) = \big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1)\big) \vee \big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\triangle_1^Q, \mathfrak{C}^1)\big)$

Considering the RHS

Let

$$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1) = (\Psi_1^Q, \mathfrak{E}^1) \text{ such that}$$
$$T_{\Psi_1^Q(a\odot b)}(w \odot t, u^1) = \min \underbrace{\{T_{\Lambda_1^Q(a)}(w,u^1), T_{\Theta_1^Q(b)}(t,u^1)\}}_{(a\odot b) = t \in \mathfrak{A}^1 \cap \mathfrak{B}^1} \tag{43}$$

for all $t, w \in \hat{G}$ and $u^1 \in Q$. And, let

$$(\Lambda_1^Q, \mathfrak{A}^1) \odot (\triangle_1^Q, \mathfrak{C}^1) = (\mho_1^Q, \mathfrak{F}^1) \text{ such that for all } w, t \in \hat{G}, u^1 \in Q \text{ we have}$$
$$T_{\mho_1^Q(a\odot c)}(w \odot t, u^1) = \min \underbrace{\{T_{\Lambda_1^Q(a)}(w,u^1), T_{\triangle_1^Q(c)}(t,u^1)\}}_{(a\odot c) \in \mathfrak{A}^1 \cap \mathfrak{C}^1} \tag{44}$$

Now, combining equations (43) and (44) give $(\Psi_1^Q, \mathfrak{E}^1) \vee (\mho_1^Q, \mathfrak{F}^1) = (\Pi_1^Q, \mathfrak{E}^1 \times \mathfrak{F}^1)$ for all $(t^*, f^*) \in (\mathfrak{E} \times \mathfrak{F})$ where $t^* = a \odot b$ and $f^* = a \odot c$ are set of parameters. Then, equations (43) and (44) gives

$$\begin{aligned} T_{\Pi_1^Q(g)}(w \odot t, u^1) &= T_{\Pi_1^Q(t^*,f^*)}((w \odot t), u^1) \\ &= \min \underbrace{\{T_{\Psi_1^Q(t^*)}((w \odot t), u^1), T_{\mho_1^Q(f^*)}((w \odot t), u^1)\}}_{(t^*, f^*) \in \mathfrak{E}^1 \times \mathfrak{F}^1.} \\ &= \max \underbrace{\{T_{\Psi_1^Q(a\odot b)}((w \odot t), u^1), T_{\mho_1^Q(a\odot c)}((w \odot t), u^1)\}}_{(t^* = a\odot b, f^* = a \odot c) \in \mathfrak{E}^1 \times \mathfrak{F}^1.} \end{aligned} \tag{45}$$

Putting (43) and (44) into (45), we have

$$\begin{aligned} &\max \underbrace{\{T_{\Psi_1^Q(a\odot b)}((w \odot t), u^1), T_{\mho_1^Q(a\odot c)}((w \odot t), u^1)\}}_{(t, f) \in \mathfrak{E}^1 \times \mathfrak{F}^1.} \\ \geq \max &\left\{ \min \underbrace{\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Theta_1^Q(b)}(t, u^1)\}}_{a\odot b \in \mathfrak{A}^1 \cap \mathfrak{B}^1.}, \min \underbrace{\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\triangle_1^Q(c)}(t, u^1)\}}_{a \odot c \in \mathfrak{A}^1 \cap \mathfrak{C}^1.} \right\} \\ &= \max \underbrace{\{T_{\Psi_1^Q(t)}((w \odot t), u^1), T_{\mho_1^Q(f)}((w \odot t), u^1)\}}_{(\mathfrak{A}^1 \cap \mathfrak{B}^1) \times (\mathfrak{A}^1 \cap \mathfrak{C}^1).} \end{aligned} \tag{46}$$

Considering the LHS:
Let $(\Theta_1^Q, \mathfrak{B}^1) \vee (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{D}^1)$ such that $(\Theta_1^Q, \mathfrak{B}^1) \wedge (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{B}^1 \times \mathfrak{C}^1)$. For all $w, t \in \hat{G}, u^1 \in Q$ and $(b, c) \in \mathfrak{B}^1 \times \mathfrak{C}^1$ we have

$$T_{\Phi_1^Q(b,c)}(wt, u^1) = \max\{T_{\Theta_1^Q(b)}(wt, u^1), T_{\triangle_1^Q(c)}(wt, u^1)\} \tag{47}$$

Also, let $(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Phi_1^Q, \mathfrak{D}^1) = (\Xi_1^Q, \mathfrak{H}^1)$ such that for all $w, t \in \hat{G}, u^1 \in Q$, we have

$$T_{\Xi_1^Q(a\odot d)}(w \odot t, u^1) = \underbrace{\min\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Phi_1^Q(d)}(t, u^1)\}}_{\text{for all } a \odot d \in \mathfrak{A}^1 \cap \mathfrak{D}^1 \text{ where } d = (b,c) \in \mathfrak{B}^1 \times \mathfrak{C}^1} \tag{48}$$

Then, putting (47) into (48), we have

$$
\begin{aligned}
&T_{\Xi_1^Q(h)}(wt,u^1) = T_{\Xi_1^Q(a,d)}(wt,u^1)\\
&= \min\{\underbrace{T_{\Lambda_1^Q(a)}(w,u^1), T_{\Phi_1^Q(d)}(t,u^1)\}}_{\text{for all } a\odot d\in\mathfrak{A}^1\cap\mathfrak{D}^1 \text{ where } w\odot t = t\in\hat{G}}\\
&= \min\{\underbrace{T_{\Lambda_1^Q(a)}(w,u^1), T_{\Phi_1^Q(b,c)}(wt,u^1)\}}_{(a\odot d)\in\mathfrak{A}^1\cap\mathfrak{D}^1.}\\
&= \min\Big\{\underbrace{T_{\Lambda_1^Q(a)}(w,u^1), \max\{T_{\Theta_1^Q(b)}(t,u^1), T_{\triangle_1^Q(c)}(t,u^1)\}\Big\}}_{(a\odot d)\in\mathfrak{A}^1\cap\mathfrak{D}^1.}\\
&= \min\Big\{\underbrace{\max\{T_{\Lambda_1^Q(a)}(w,u^1), T_{\Lambda_1^Q(a)}(w,u^1)\}, \max\{T_{\Theta_1^Q(b)}(t,u^1), T_{\triangle_1^Q(c)}(t,u^1)\}\Big\}}_{(a\odot(b,c)\in\mathfrak{A}^1\cap(\mathfrak{B}^1\times\mathfrak{C}^1).}\\
&= \max\Big\{\underbrace{\min\{T_{\Lambda_1^Q(a)}(w,u^1), T_{\Theta_1^Q(b)}(t,u^1)\}}_{(a\odot(b))\in\mathfrak{A}^1\cap\mathfrak{B}^1.}, \underbrace{\min\{T_{\Lambda_1^Q(a)}(w,u^1), T_{\triangle_1^Q(c)}(t,u^1)\}\Big\}}_{(a\odot c))\in\mathfrak{A}^1\cap\mathfrak{C}^1.}\\
&\max\underbrace{\{T_{\Psi_1^Q(a\odot b)}((w\odot t),u^1), T_{\mho_1^Q(a\odot c)}((w\odot t),u^1)\}}_{(\mathfrak{A}^1\cap\mathfrak{B}^1)\times(\mathfrak{A}^1\cap\mathfrak{C}^1).}
\end{aligned}
\tag{49}
$$

Considering the result for indeterminate membership.

Let

$$
\begin{aligned}
&(\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1) = (\Psi_1^Q,\mathfrak{E}^1) \text{ such that}\\
&I_{\Psi_1^Q(a\odot b)}(w\odot t,u^1) = \max\underbrace{\{I_{\Lambda_1^Q(a)}(w,u^1), I_{\Theta_1^Q(b)}(t,u^1)\}}_{(a\odot b)=t\in\mathfrak{A}^1\cap\mathfrak{B}^1}
\end{aligned}
\tag{50}
$$

for all $t,w\in\hat{G}$ and $u^1\in Q$. And, let

$$
\begin{aligned}
&(\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1) = (\mho_1^Q,\mathfrak{F}^1) \text{ such that for all } t,w\in Q, u^1\in Q \text{ we get}\\
&I_{\mho_1^Q(a\odot c)}(w\odot t,u^1) = \max\underbrace{\{I_{\Lambda_1^Q(a)}(w,u^1), I_{\triangle_1^Q(c)}(t,u^1)\}}_{(a\odot c)\in\mathfrak{A}^1\cap\mathfrak{C}^1}
\end{aligned}
\tag{51}
$$

Now, from equations (50) and (51) we have $(\Psi_1^Q,\mathfrak{E}^1)\vee(\mho_1^Q,\mathfrak{F}^1) = (\Pi_1^Q,\mathfrak{E}^1\times\mathfrak{F}^1)$ for all $(t^*,f^*)\in(\mathfrak{E}^1\times\mathfrak{F}^1)$ where $t^*=a\odot b$ and $f=a\odot c$ are parameters. Then, this follows

$$\begin{aligned} I_{\Pi_1^Q(g)}(w \odot t, u^1) &= I_{\Pi_1^Q(t,f)}((w \odot t), u^1) \\ &= \min \underbrace{\{I_{\Psi_1^Q(t)}((w \odot t), u^1), I_{\mho_1^Q(f)}((w \odot t), u^1)\}}_{(t,f) \in \mathfrak{E}^1 \times \mathfrak{F}^1.} \\ &= \min \underbrace{\{I_{\Psi_1^Q(a \odot b)}((w \odot t), u^1), I_{\mho_1^Q(a \odot c)}((w \odot t), u^1)\}}_{(t = a \odot b, f = a \odot c) \in \mathfrak{E}^1 \times \mathfrak{F}^1.} \end{aligned} \tag{52}$$

Substitute (50) and (51) in (52), to get

$$\begin{aligned} &\min \underbrace{\{I_{\Psi_1^Q(a \odot b)}((w \odot t), u^1), I_{\mho_1^Q(a \odot c)}((w \odot t), u^1)\}}_{(t,f) \in \mathfrak{E} \times \mathfrak{F}.} \\ &\geq \min \left\{ \max \underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\Theta_1^Q(b)}(t, u^1)\}}_{a \odot b \in \mathfrak{A}^1 \cap \mathfrak{B}^1.}, \max \underbrace{\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\triangle_1^Q(c)}(t, u^1)\}}_{a \odot c \in \mathfrak{A} \cap \mathfrak{C}.} \right\} \\ &= \min \underbrace{\{I_{\Psi_1^Q(a \odot b)}((w \odot t), u^1), I_{\mho_1^Q(a \odot c)}((w \odot t), u^1)\}}_{(\mathfrak{A}^1 \cap \mathfrak{B}^1) \times (\mathfrak{A}^1 \cap \mathfrak{C}^1).} \end{aligned} \tag{53}$$

Considering the LHS:
Let $(\Theta_1^Q, \mathfrak{B}^1) \vee (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{D}^1)$ such that $(\Theta_1^Q, \mathfrak{B}^1) \vee (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{B}^1 \times \mathfrak{C}^1)$ for all $w, t \in \hat{G}, u^1 \in Q$ and $(b, c) \in \mathfrak{B}^1 \times \mathfrak{C}^1$. It is follows that,

$$I_{\Phi_1^Q(b,c)}(wt, u^1) = \min\{I_{\Theta_1^Q(b)}(wt, u^1), I_{\triangle_1^Q(c)}(wt, u^1)\} \tag{54}$$

Also, let $(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Phi_1^Q, \mathfrak{D}^1) = (\Xi_1^Q, \mathfrak{H}^1)$ such that for all $w, t \in \hat{G}, u^1 \in Q$,

$$I_{\Xi_1^Q(a \odot d)}(w \odot t, u^1) = \underbrace{\max\{I_{\Lambda_1^Q(a)}(w, u^1), I_{\Phi_1^Q(d)}(t, u^1)\}}_{\text{for all } a \odot d \in \mathfrak{A}^1 \cap \mathfrak{D}^1 \text{ where } d = (b,c) \in \mathfrak{B}^1 \times \mathfrak{C}^1} \tag{55}$$

are set of parameters. Then, putting (54) in (55), we get

$$\begin{aligned}
&I_{\Xi_1^Q(h)}(w\odot t,u^1) = I_{\Xi_1^Q(a\odot d)}(wt,u^1)\\
&= \max\{\underbrace{I_{\Lambda_1^Q(a)}(w,u^1), I_{\Phi_1^Q(d)}(t,u^1)\}}_{\text{for all } a\odot d\in\mathfrak{A}^1\cap\mathfrak{D}^1 \text{ where } wt=t\in\hat{G}}\\
&= \max\{\underbrace{I_{\Lambda_1^Q(a)}(w,u^1), I_{\Phi_1^Q(b,c)}(w\odot t,u^1)\}}_{(a\odot d)\in\mathfrak{A}^1\cap\mathfrak{D}^1.}\\
&= \max\{\underbrace{I_{\Lambda_1^Q(a)}(w,u^1), \min\{I_{\Theta_1^Q(b)}(t,u^1), I_{\triangle_1^Q(c)}(t,u^1)\}\}}_{(a\odot d)\in\mathfrak{A}^1\cap\mathfrak{D}^1.}\\
&= \max\{\underbrace{\min\{I_{\Lambda_1^Q(a)}(w,u^1), I_{\Lambda_1^Q(a)}(w,u^1)\}, \min\{I_{\Theta_1^Q(b)}(t,u^1), I_{\triangle_1^Q(c)}(t,u^1)\}\}}_{(a\odot(b,c)\in\mathfrak{A}^1\cap(\mathfrak{B}^1\times\mathfrak{C}^1).}\\
&= \min\{\underbrace{\max\{I_{\Lambda_1^Q(a)}(w,u^1), I_{\Theta_1^Q(b)}(t,u^1)\}}_{(a\odot(b))\in\mathfrak{A}^1\cap\mathfrak{B}^1.}, \underbrace{\max\{I_{\Lambda_1^Q(a)}(w,u^1), I_{\triangle_1^Q(c)}(t,u^1)\}\}}_{(a\odot c))\in\mathfrak{A}^1\cap\mathfrak{C}^1.}\\
&\min\{\underbrace{I_{\Psi_1^Q(a\odot b)}((w\odot t),u^1), I_{\mho_1^Q(a\odot c)}((w\odot t),u^1)\}}_{(\mathfrak{A}^1\cap\mathfrak{B}^1)\times(\mathfrak{A}^1\cap\mathfrak{C}^1).}
\end{aligned} \tag{56}$$

Next, the result for falsity membership degree is similar with the one obtained for indeterminate membership

Therefore, $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)\big) = \big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\wedge\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$

(8) Similar with the result obtained for 7

**Theorem 3.6.** *Let* $(\Lambda_1^Q,\mathfrak{A}^1),(\Theta_1^Q,\mathfrak{B}^1)$ *and* $(\triangle_1^Q,\mathfrak{C}^1)$ *be Q-neutrosophic soft quasigroups over quasigroup* $(\hat{G},\odot)$. *Then, the following holds*

(1) $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big) = \big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\odot\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$
(2) $\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\odot(\triangle_1^Q,\mathfrak{C}^1) = \big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$
(3) $(\Lambda_1^Q,\mathfrak{A}^1)/\big((\Theta_1^Q,\mathfrak{B}^1)/(\triangle_1^Q,\mathfrak{C}^1)\big) = \big((\Lambda_1^Q,\mathfrak{A}^1)/(\Theta_1^Q,\mathfrak{B}^1)\big)/\big((\Lambda_1^Q,\mathfrak{A}^1)/(\triangle_1^Q,\mathfrak{C}^1)\big)$
(4) $\big((\Lambda_1^Q,\mathfrak{A}^1)/(\Theta_1^Q,\mathfrak{B}^1)\big)/(\triangle_1^Q,\mathfrak{C}^1) = \big((\Lambda_1^Q,\mathfrak{A}^1)/(\triangle_1^Q,\mathfrak{C}^1)\big)/\big((\Theta_1^Q,\mathfrak{B}^1)/(\triangle_1^Q,\mathfrak{C}^1)\big)$
(5) $(\Lambda_1^Q,\mathfrak{A}^1)\backslash\big((\Theta_1^Q,\mathfrak{B}^1)\backslash(\triangle_1^Q,\mathfrak{C}^1)\big) = \big((\Lambda_1^Q,\mathfrak{A}^1)\backslash(\Theta_1^Q,\mathfrak{B}^1)\big)\backslash\big((\Lambda_1^Q,\mathfrak{A}^1)\backslash(\triangle_1^Q,\mathfrak{C}^1)\big)$
(6) $\big((\Lambda_1^Q,\mathfrak{A}^1)\backslash(\Theta_1^Q,\mathfrak{B}^1)\big)\backslash(\triangle_1^Q,\mathfrak{C}^1) = \big((\Lambda_1^Q,\mathfrak{A}^1)\backslash(\triangle_1^Q,\mathfrak{C}^1)\big)\backslash\big((\Theta_1^Q,\mathfrak{B}^1)\backslash(\triangle_1^Q,\mathfrak{C}^1)\big)$
(7) $\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\odot(\Theta_1^Q,\mathfrak{B}^1)\neq(\Lambda_1^Q,\mathfrak{A}^1)$
(8) $(\Theta_1^Q,\mathfrak{B}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\odot(\Lambda_1^Q,\mathfrak{A}^1)\big)\neq(\Lambda_1^Q,\mathfrak{A}^1)$

*Proof:*

(1) We want to show that $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big) = \big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\odot\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$ Considering the LHS

Let $(\Theta_1^Q, \mathfrak{B}^1) \odot (\triangle_1^Q, \mathfrak{C}^1) = (\Phi_1^Q, \mathfrak{D}^1)$ such that for all $w, t \in \hat{G}$ and $u \in Q$ we have

$$T_{\Phi_1^Q(b \odot c)}((w \odot t), u^1) = \min\{T_{\Theta_1^Q(b)}(w, u^1), T_{\triangle_1^Q(c)}(t, u^1)\} \tag{57}$$

And, let $(\Lambda_1^Q, \mathfrak{A}^1) \odot (\Phi_1^Q, \mathfrak{D}^1) = (\Xi_1^Q, \mathfrak{H}^1)$ such that for all $w, t \in \hat{G}$ and $u^1 \in Q$, we have

$$T_{\Xi_1^Q(a \odot d)}(w \odot t, u^1) = \underbrace{\max\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Phi_1^Q(d)}(t, u^1)\}}_{\text{for all } a \odot d \in \mathfrak{A}^1 \cap \mathfrak{D}^1 \text{ where } d = (b \odot c) \in \mathfrak{B}^1 \cap \mathfrak{C}^1} \tag{58}$$

.

Substituting (57) into (58), to get

$$\begin{aligned}
T_{\Xi_1^Q(a \odot d)}(w \odot t, u^1) &= T_{\Xi_1^Q(a \odot d)}(w \odot t, u^1)\\
&= \min\{ \underbrace{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Phi_1^Q(d)}(t, u^1)\}}_{\text{for all } a \odot d \in \mathfrak{A}^1 \cap \mathfrak{D}^1 \text{ and let } w \odot t = w \in \hat{G}}\\
&= \min\{\underbrace{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Phi_1^Q(b \odot c)}(w \odot t, u^1)\}}_{(a \odot d) \in \mathfrak{A}^1 \cap \mathfrak{D}^1.}\\
&= \min\Big\{\underbrace{T_{\Lambda_1^Q(a)}(w, u^1), \min\{T_{\Theta_1^Q(b)}(w, u^1), T_{\triangle_1^Q(c)}(t, u^1)\}\Big\}}_{(a \odot d) \in \mathfrak{A}^1 \cap \mathfrak{D}^1.}\\
&= \min\Big\{\underbrace{\min\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Lambda_1^Q(a)}(t, u^1)\}, \min\{T_{\Theta_1^Q(b)}(w, u^1), T_{\triangle_1^Q(c)}(t, u^1)\}\Big\}}_{(a \odot (b \odot c) \in \mathfrak{A}^1 \cap (\mathfrak{B}^1 \times \mathfrak{C}^1).}\\
&= \min\Big\{\underbrace{\min\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\Theta_1^Q(b)}(w, u^1)\}}_{(a \odot (b)) \in \mathfrak{A}^1 \cap \mathfrak{B}^1.}, \underbrace{\min\{T_{\Lambda_1^Q(a)}(w, u^1), T_{\triangle_1^Q(c)}(t, u^1)\}\Big\}}_{(a \odot c)) \in \mathfrak{A}^1 \cap \mathfrak{C}^1.}\\
&\min \underbrace{\{T_{\Psi_1^Q(a \odot b)}((w, u^1), T_{\mho_1^Q(a \odot c)}(t, u^1)\}}_{(\mathfrak{A}^1 \cap \mathfrak{B}^1) \cap (\mathfrak{A}^1 \cap \mathfrak{C}^1).} &(59)\\
&= \big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1)\big) \odot \big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\triangle_1^Q, \mathfrak{C}^1)\big) &(60)
\end{aligned}$$

Similarly, we show for indeterminate and falsity membership degrees.

(2) Follow from 1

(3) Apply Definition 3.4 along side with 1 and 2

(4) Similar with 3

(5) Similar with 4

(6) Similar with 5

(7) Proof by contradiction. Suppose that $\big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1)\big) \odot (\Theta_1^Q, \mathfrak{B}^1) = (\Lambda_1^Q, \mathfrak{A}^1)$, then we have $\big((\Lambda_1^Q, \mathfrak{A}^1) \odot (\Theta_1^Q, \mathfrak{B}^1)\big) = (\Lambda_1^Q, \mathfrak{A}^1)/(\Theta_1^Q, \mathfrak{B}^1)$.
Let $z_1 = t_1 \odot w_1, z_2 = t_2 \odot w_2$ for all $z_1, z_2 \in \hat{G}$ and $u^1 \in Q$.

Let $(\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)=(\Phi_1^Q,\mathfrak{C}^1)$ such that

$$T_{\Phi_1^Q(a\odot b)}((z_1\odot z_2),u^1)=\min\{T_{\Lambda_1^Q(a)}(z_1,u^1),T_{\Theta_1^Q(b)}(z_2,u^1)\}$$
$$=\min\left\{\min\{T_{\Lambda_1^Q(a)}(t_1,u^1),T_{\Lambda_1^Q(a)}(w_1,u^1)\},\min\{T_{\Theta_1^Q(b)}(t_2,u^1),T_{\Theta_1^Q(b)}(w_2,u^1)\}\right\}$$
$$\min\left\{\min\{T_{\Lambda_1^Q(a)}(t_1,u^1),T_{\Theta_1^Q(b)}(t_2,u^1)\},\min\{T_{\Lambda_1^Q(a)}(w_1,u^1),T_{\Theta_1^Q(b)}(w_2,u^1)\}\right\}$$
$$\min\left\{T_{\Phi_1^Q(a\odot b)}(t_1\odot t_2,u^1),T_{\Phi_1^Q(a\odot b)}(w_1\odot w_2,u^1)\right\} \tag{61}$$

Considering the RHS, let $(\Lambda_1^Q,\mathfrak{A}^1)/(\Theta_1^Q,\mathfrak{B}^1)=(\Phi_1^Q,\mathfrak{C}^1)$. Then,

$$T_{\Phi_1^Q(a/b)}((z_1/z_2),u^1)=\max\{T_{\Lambda_1^Q(a)}(z_1,u^1),T_{\Theta_1^Q(b)}(z_2,u^1)\}$$
$$=\max\left\{\min\{T_{\Lambda_1^Q(a)}(t_1,u^1),T_{\Lambda_1^Q(a)}(w_1,u^1)\},\min\{T_{\Theta_1^Q(b)}(t_2,u^1),T_{\Theta_1^Q(b)}(w_2,u^1)\}\right\}$$
$$\max\left\{\min\{T_{\Lambda_1^Q(a)}(t_1,u^1),T_{\Theta_1^Q(b)}(t_2,u^1)\},\min\{T_{\Lambda_1^Q(a)}(w_1,u^1),T_{\Theta_1^Q(b)}(w_2,u^1)\}\right\}$$
$$\max\left\{T_{\Phi_1^Q(a\odot b)}(t_1\odot t_2,u^1),T_{\Phi_1^Q(a\odot b)}(w_1\odot w_2,u^1)\right\} \tag{62}$$

Hence, $\max\left\{T_{\Phi_1^Q(a\odot b)}(t_1\odot t_2,u^1),T_{\Phi_1^Q(a\odot b)}(w_1\odot w_2,u^1)\right\}\neq\min\left\{T_{\Phi_1^Q(a\odot b)}(t_1\odot t_2,u^1),T_{\Phi_1^Q(a\odot b)}(w_1\odot w_2,u^1)\right\}$. The results for indeterminate and falsity membership degrees are similarly obtained.

(8) Similar with 7

**Theorem 3.7.** *Let $(\Lambda_1^Q,\mathfrak{A}^1),(\Theta_1^Q,\mathfrak{B}^1)$ and $(\triangle_1^Q,\mathfrak{C}^1)$ be Q-neutrosophic soft quasigroups over quasigroup $(\hat{G},\odot)$. Then, the following holds*

(1) $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)/(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)/\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$
(2) $\big((\Lambda_1^Q,\mathfrak{A}^1)/(\Theta_1^Q,\mathfrak{B}^1)\big)\odot(\triangle_1^Q,\mathfrak{C}^1)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)/\big((\Theta_1^Q,\mathfrak{B}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$
(3) $(\Lambda_1^Q,\mathfrak{A}^1)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\backslash(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\backslash\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$
(4) $\big((\Lambda_1^Q,\mathfrak{A}^1)\backslash(\Theta_1^Q,\mathfrak{B}^1)\big)\odot(\triangle_1^Q,\mathfrak{C}^1)=\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)\backslash\big((\Theta_1^Q,\mathfrak{B}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)$
(5) $(\Lambda_1^Q,\mathfrak{A}^1)/\big((\Theta_1^Q,\mathfrak{B}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)/(\Theta_1^Q,\mathfrak{B}^1)\big)\odot\big((\Lambda_1^Q,\mathfrak{A}^1)/(\triangle_1^Q,\mathfrak{C}^1)\big)$
(6) $\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)/(\triangle_1^Q,\mathfrak{C}^1)=\big((\Lambda_1^Q,\mathfrak{A}^1)/(\triangle_1^Q,\mathfrak{C}^1)\big)\odot\big((\Theta_1^Q,\mathfrak{B}^1)/(\triangle_1^Q,\mathfrak{C}^1)\big)$
(7) $(\Lambda_1^Q,\mathfrak{A}^1)\backslash\big((\Theta_1^Q,\mathfrak{B}^1)\odot(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\backslash(\Theta_1^Q,\mathfrak{B}^1)\big)\odot\big((\Lambda_1^Q,\mathfrak{A}^1)\backslash(\triangle_1^Q,\mathfrak{C}^1)\big)$
(8) $\big((\Lambda_1^Q,\mathfrak{A}^1)\odot(\Theta_1^Q,\mathfrak{B}^1)\big)\backslash(\triangle_1^Q,\mathfrak{C}^1)=\big((\Lambda_1^Q,\mathfrak{A}^1)\backslash(\triangle_1^Q,\mathfrak{C}^1)\big)\odot\big((\Theta_1^Q,\mathfrak{B}^1)\backslash(\triangle_1^Q,\mathfrak{C}^1)\big)$

*Proof:* Similar with Theorem 3.5.

**Theorem 3.8.** *Let $(\Lambda_1^Q,\mathfrak{A}^1),(\Theta_1^Q,\mathfrak{B}^1)$ and $(\triangle_1^Q,\mathfrak{C}^1)$ be Q-neutrosophic soft quasigroups over quasigroup $(\hat{G},\odot)$. Then, the following holds*

(1) $(\Lambda_1^Q,\mathfrak{A}^1)/\big((\Theta_1^Q,\mathfrak{B}^1)\cap(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)/(\Theta_1^Q,\mathfrak{B}^1)\big)\cap\big((\Lambda_1^Q,\mathfrak{A}^1)/(\triangle_1^Q,\mathfrak{C}^1)\big)$
(2) $\big((\Theta_1^Q,\mathfrak{B}^1)\cap(\triangle_1^Q,\mathfrak{C}^1)\big)/(\Lambda_1^Q,\mathfrak{A}^1)=\big((\Theta_1^Q,\mathfrak{B}^1)/(\Lambda_1^Q,\mathfrak{A}^1)\big)\cap\big((\triangle_1^Q,\mathfrak{C}^1)/(\Lambda_1^Q,\mathfrak{A}^1)\big)$

(3) $(\Lambda_1^Q,\mathfrak{A}^1)/\big((\Theta_1^Q,\mathfrak{B}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)/(\Theta_1^Q,\mathfrak{B}^1)\big)\wedge\big((\Lambda_1^Q,\mathfrak{A}^1)/(\triangle_1^Q,\mathfrak{C}^1)\big)$
(4) $\big((\Theta_1^Q,\mathfrak{B}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)\big)/(\Lambda_1^Q,\mathfrak{A}^1)=\big((\Theta_1^Q,\mathfrak{B}^1)/(\Lambda_1^Q,\mathfrak{A}^1)\big)\wedge\big((\triangle_1^Q,\mathfrak{C}^1)/(\Lambda_1^Q,\mathfrak{A}^1)\big)$
(5) $(\Lambda_1^Q,\mathfrak{A}^1)/\big((\Theta_1^Q,\mathfrak{B}^1)\cup(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)/(\Theta_1^Q,\mathfrak{B}^1)\big)\cup\big((\Lambda_1^Q,\mathfrak{A}^1)/(\triangle_1^Q,\mathfrak{C}^1)\big)$
(6) $\big((\Theta_1^Q,\mathfrak{B}^1)\cup(\triangle_1^Q,\mathfrak{C}^1)\big)/(\Lambda_1^Q,\mathfrak{A}^1)=\big((\Theta_1^Q,\mathfrak{B}^1)/(\Lambda_1^Q,\mathfrak{A}^1)\big)\cup\big((\triangle_1^Q,\mathfrak{C}^1)/(\Lambda_1^Q,\mathfrak{A}^1)\big)$
(7) $(\Lambda_1^Q,\mathfrak{A}^1)/\big((\Theta_1^Q,\mathfrak{B}^1)\vee(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)/(\Theta_1^Q,\mathfrak{B}^1)\big)\vee\big((\Lambda_1^Q,\mathfrak{A}^1)/(\triangle_1^Q,\mathfrak{C}^1)\big)$
(8) $\big((\Theta_1^Q,\mathfrak{B}^1)\vee(\triangle_1^Q,\mathfrak{C}^1)\big)/(\Lambda_1^Q,\mathfrak{A}^1)=\big((\Theta_1^Q,\mathfrak{B}^1)/(\Lambda_1^Q,\mathfrak{A}^1)\big)\vee\big((\triangle_1^Q,\mathfrak{C}^1)/(\Lambda_1^Q,\mathfrak{A}^1)\big)$

*Proof:* Similar with Theorem 3.5

**Corollary 3.9.** *Let $(\Lambda_1^Q,\mathfrak{A}^1),(\Theta_1^Q,\mathfrak{B}^1)$ and $(\triangle_1^Q,\mathfrak{C}^1)$ be Q-neutrosophic soft sets $X$. Then, the following holds*

(1) $(\Lambda_1^Q,\mathfrak{A}^1)\wedge\big((\Theta_1^Q,\mathfrak{B}^1)\vee(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\wedge(\Theta_1^Q,\mathfrak{B}^1)\big)\vee\big((\Lambda_1^Q,\mathfrak{A}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)\big)$
(2) $\big((\Lambda_1^Q,\mathfrak{A}^1\vee(\Theta_1^Q,\mathfrak{B}^1)\big)\wedge(\triangle_1^Q,\mathfrak{C}^1)=\big((\Lambda_1^Q,\mathfrak{A}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)\big)\vee\big((\Theta_1^Q,\mathfrak{B}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)\big)$
(3) $(\Lambda_1^Q,\mathfrak{A}^1)\vee\big((\Theta_1^Q,\mathfrak{B}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)\big)=\big((\Lambda_1^Q,\mathfrak{A}^1)\vee(\Theta_1^Q,\mathfrak{B}^1)\big)\wedge\big((\Lambda_1^Q,\mathfrak{A}^1)\vee(\triangle_1^Q,\mathfrak{C}^1)\big)$
(4) $\big((\Lambda_1^Q,\mathfrak{A}^1\wedge(\Theta_1^Q,\mathfrak{B}^1)\big)\vee(\triangle_1^Q,\mathfrak{C}^1)=\big((\Lambda_1^Q,\mathfrak{A}^1)\vee(\triangle_1^Q,\mathfrak{C}^1)\big)\wedge\big((\Theta_1^Q,\mathfrak{B}^1)\vee(\triangle_1^Q,\mathfrak{C}^1)\big)$

*Proof:*
Let $(\Lambda_1^Q,\mathfrak{A}^1)\wedge(\Theta_1^Q,\mathfrak{B}^1)=(\Delta_1^Q,\mathfrak{F}^1)$ and $(\Lambda_1^Q,\mathfrak{A}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)=(\Xi_1^Q,\mathfrak{G}^1)$.
Let

$$(\Theta_1^Q,\mathfrak{B}^1)\vee(\triangle_1^Q,\mathfrak{C}^1)=(\Psi_1^Q,\mathfrak{D}^1)\text{ such that}$$
$$T_{\Psi_1^Q(d)}(w,u^1)T_{\Psi_1^Q(b,c)}(w,u^1)=\max\underbrace{\{T_{\Theta_1^Q(b)}(w,u^1),T_{\triangle_1^Q(c)}(w,u^1)\}}_{(b,c)=t\in\mathfrak{B}^1\times\mathfrak{C}^1}\tag{63}$$

for all $w\in X$, $u^1\in Q$ and $(b,c)\in(\mathfrak{B}^1\times\mathfrak{C}^1)$
Let

$$(\Lambda_1^Q,\mathfrak{A}^1)\wedge(\Psi_1^Q,\mathfrak{D}^1)=(\Pi_1^Q,\mathfrak{E}^1)\text{ such that}$$
$$T_{\Pi_1^Q(t)}(w,u^1)=T_{\Pi_1^Q(a,d)}(w,u^1)=\min\underbrace{\{T_{\Lambda_1^Q(a)}(w,u^1),T_{\Psi_1^Q(d)}(w,u^1)\}}_{(a,d)=t\in\mathfrak{A}^1\times\mathfrak{D}^1}\tag{64}$$

for all $w\in X$ and $u^1\in Q$ and $(a,d)\in\mathfrak{A}^1\times\mathfrak{D}^1)$.

Substituting 63 into 64, wt have

$$T_{\Pi_1^Q(t)}(w,u^1) = \min \underbrace{\left\{ T_{\Lambda_1^Q(a)}(w,u^1), \max \underbrace{\{T_{\Theta_1^Q(b)}(w,u^1), T_{\triangle_1^Q(c)}(w,u^1)\}}_{(b,c)=t\in\mathfrak{B}^1\times\mathfrak{C}^1} \right\}}_{(a,d)=t\in\mathfrak{A}^1\times\mathfrak{D}^1}$$

$$= \min \underbrace{\left\{ \max\{T_{\Lambda_1^Q(a)}(w,u^1), T_{\Lambda_1^Q(a)}(w,u^1)\}, \max\{T_{\Theta_1^Q(b)}(w,u^1), T_{\triangle_1^Q(c)}(w,u^1)\} \right\}}_{(a,d)=\in\mathfrak{A}^1\times\mathfrak{D}^1}$$

$$\max\left\{ \min \underbrace{\{T_{\Lambda_1^Q(a)}(w,u^1), T_{\Theta_1^Q(b)}(w,u^1)\}}_{(a,b)=\in\mathfrak{A}^1\times\mathfrak{B}^1}, \min \underbrace{\{T_{\Lambda_1^Q(a)}(w,u^1), T_{\triangle_1^Q(c)}(w,u^1)\}}_{(a,c)=\in\mathfrak{A}^1\times\mathfrak{C}^1} \right\}$$

$$= \max\{T_{\Delta_1^Q(a,b)}(w,u^1), T_{\Xi_1^Q(a,c)}(w,u^1)\} \tag{65}$$

Hence, $(\Lambda_1^Q,\mathfrak{A}^1)\wedge\big((\Theta_1^Q,\mathfrak{B}^1)\vee(\triangle_1^Q,\mathfrak{C}^1)\big) = \big((\Lambda_1^Q,\mathfrak{A}^1)\wedge(\Theta_1^Q,\mathfrak{B}^1)\big)\vee\big((\Lambda_1^Q,\mathfrak{A}^1)\wedge(\triangle_1^Q,\mathfrak{C}^1)\big)$. It hold for true membership degree. Also, the proofs for indeterminate and falsity membership degrees are similar.

The results for 2 , 3 and 4 are similar to 1

## 4. **Conclusion**

In this paper, the notion of $Q-$neutrosophic soft set is extended to a non-associative algebraic structure. In particular, we focus on presenting the distributive properties of $Q-$neutrosophic soft quasigroup. Regarding the three binary operations of the quasigroup, a $Q-$neutrosophic soft set is defined under the structure of quasigroup. These three operations were used to demonstrate its characteristic in relation to the intersection, union, AND, and OR operations. A fascinating finding of the study is that the $Q-$neutrosophic soft quasigroup is self-distributive under the three binary operations and also distributive over each another. The three binary operations are distributive over intersection, union, AND and OR operations. It was further shown that $Q-$neutrosophic soft quasigroup does not adhere to left and right symmetric properties. Thus, the notion does not obey key laws. It was established that $Q-$neutrosophic soft set is a distributive lattice. In future research, Definitions 3.1, 3.2 and 3.4 will be used to examine the algebraic properties of $Q-$neutrosophic soft set under a class of qausigroup known as entropy, unipotent, and idempotent quasigroups.

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**Conflicts of Interest:** The three authors declare no conflict of interest.

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# Approximations of interval neutrosophic hyperideals in semi-hyper-rings

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**Abstract**. This paper deals with the combination of rough sets and interval neutrosophic sets. We introduce the interval neutrosophic hyper-ideals in semi-hyper-rings. Also we study the rough interval neutrosophic hyper-ideals in semi-hyper-rings.

**Keywords:** Rough sets, neutrosophic sets, interval neutrosophic sets , rough interval neutrosophic sets, semi-hyper-rings.

## 1. Introduction

In 1982, Pawlak [11] introduced the concept of rough set, as a formal tool for modeling and processing incomplete information in information systems. The basic idea of rough set is based upon the approximation of sets by a pair of sets known as the lower approximation and the upper approximation of a set. The concept of a fuzzy set, introduced by Zadeh [24] , provides a natural framework for generalizing some of the notions of classical algebraic structures. As a generalization of fuzzy sets, the intuitionistic fuzzy set was introduced by Atanassov [1] in 1986. One of the interesting generalizations of the theory of fuzzy sets and intuitionistic fuzzy sets is the theory of neutrosophic sets introduced by F. Smarandache [12]. The term neutrosophy means knowledge of neutral thought and this neutral represents the main distinction between fuzzy and intuitionistic fuzzy logic and set. It is a logic in which each proposition is estimated to have a degree of truth, a degree of indeterminacy and a degree of falsity. Unlike in intuitionistic fuzzy sets, where the incorporated uncertainty is dependent of the degree of belongingness and degree of non-belongingness, here the uncertainty present, i.e. the indeterminacy factor, is independent of truth and falsity values. Neutrosophic sets are indeed more general than Intuitionistic fuzzy set as there are no constraints between the degree of truth, degree of inde-terminacy and degree of falsity. All these degrees can individually vary within [0, 1]. The theories of neutrosophic set have achieved great success in various areas. Recently many researchers applied the notion of fuzzy neutrosophic sets to several algebraic structures. Subha et al. [17–23] studied the algebraic structures of interval rough fuzzy sets. In this paper we studied the algebraic properties of rough interval neutrosophic sets.

## 2. Preliminaries

This section we present some basic definitions related to this work.

**Definition 2.1.** [2] Let $W$ be a nonempty set, and let $P(W)$ be the set of all nonempty subsets of $W$. A hyperoperation on $W$ is a map $\circ : W \times W \leftarrow P(W)$, and the couple $(W, \circ)$ is called a hypergrupoid. If $A$ and $B$ are nonempty subsets of $W$, then we denote,

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b \; x \circ A = \{x\} \circ A,$$
$$A \circ x = A \circ \{x\}$$

**Definition 2.2.** [2] A hypergrupoid $(W, \circ)$ is called a hyper-semi-group if for all $x, y$and $z$ of $W$ we have $(x \circ y) \circ z = x \circ (y \circ z)$.
That is, $\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v$

**Definition 2.3.** [2] A is an algebraic structure $(W, +, .)$ which satisfies the following conditions.
(i) $(W, +)$ is a commutative semi-hyper-group,
(a) $(a + b) + c = a + (y + z)$ (b) $a + b = b + a$, for all $a, b, c \in W$.
(ii) $(H, .)$ is a semi-hyper-group,
(c) $(a.b).c = a.(b.c)$, for all $a, b, c \in W$.
(iii) The multiplication is distributive with respect to hyperroperation $+$,
(d) $a.(b + c) = a.b + a.c$
(e) $(a + b).c = a.c + b.c$, for all $a, b, c \in W$.

**Definition 2.4.** [2] A nonempty subset $A$ of a hyper-semi-ring $(W, +, .)$ is called sub-hyper-semi-ring if $x + y \subseteq A$ and $x.y \subseteq A$ for all $x, y \in A$.

**Definition 2.5.** [2] A left(right) hyper-ideal of a hyper-semi-ring $W$ is a nonempty subset $I$ of $W$ satisfying the following:
(i) $x + y \subseteq I$, for all $x, y \in I$.
(ii) $x.a \subseteq I(a.x \subseteq I)$, for all $a \in I$ and $x \in W$.

**Definition 2.6.** Let $R$ be a commutative semihypergroup and $\Gamma$ be a commutative group. Then $R$ is called a $\Gamma$ -semihyperring if there exists a map $R\Gamma R \to P(R)(a, \alpha, b) \to a\alpha b)$

$\forall a, b \in R, \alpha \in \Gamma$ and $P(R)$ the set of all non-empty subsets of $R$, satisfying the following conditions: (i) $(a+b)\alpha c = a\alpha c + b\alpha c$,

(ii)$a\alpha(b+c) = a\alpha b + a\alpha c$,

(iii) $a(\alpha+\beta)b = a\alpha b + a\beta b$,

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$

$\forall a, b, c \in R$ and $\forall\ \alpha, \beta \in \Gamma$

We say that $R$ is a $\Gamma$-semihyperring with zero, if there exists $0 \in R$ such that $a \in a + 0$ and $0 \in 0\alpha a, 0 \in a\alpha 0$ for all $a \in R$ and $\alpha \in \Gamma$

**Definition 2.7.** [10] Let $W$ be the universe. The neutrosophic set is an object having the form $A = \{(e, l_t(e), l_i(e), l_f(e)), e \in W\}$

where the functions $l_t, l_i, l_f : W \longrightarrow [0,1]$ define respectively the truth, the degree of indeterminacy and the degree of non-membership of the element $e \in W$ to the set $A$ with the condition $0 \leq l_t + l_i + l_q \leq 3$

## 3. Interval neutrosophic hyper-ideals(INHI) in semi-hyper-rings

In this section we studied the concept of $INLHI$ in semi-hyper-ring $W$. Also we proved nonempty intersection of $INLHI$ is also an $INLHI$. More over we discuss the pre image and image of an $INLHI$ of $W$ is also an $INLHI$. At last we proved the cartesian product of two $INLHI$ is also an $INLHI$.

**Definition 3.1.** A nonempty $IN$ subset $l$ of $W$ is said to be an $INLHI$ of $W$ if the following conditions are holds:

(C1) $\bigwedge\limits_{e \in s+q} l_t(e) \geq l_t(s) \wedge l_t(q)$

(C2) $\bigwedge\limits_{e \in s+q} l_i(e) \geq \frac{l_i(s)+l_i(q)}{2}$

(C3) $\bigvee\limits_{e \in s+q} l_f(e) \leq l_t(s) \vee l_t(q)$

(C4) $\bigwedge\limits_{e \in sq} l_t(e) \geq l_t(q)$

(C5) $\bigwedge\limits_{e \in sq} l_i(e) \geq l_i(q)$

(C6) $\bigvee\limits_{e \in sq} l_f(e) \leq l_f(q)$ for all $e, s, q \in W$

**Definition 3.2.** A nonempty $IN$ subset $l$ of $W$ is said to be an $INRHI$ of $W$ if the conditions (C1) (C2) and (C3) holds. Moreover

(C7) $\bigwedge\limits_{e \in sq} l_t(e) \geq l_t(s)$

(C8) $\bigwedge\limits_{e \in sq} l_i(e) \geq l_i(s)$

(C9) $\bigvee\limits_{e \in sq} l_f(e) \leq l_f(s)$for all $e, s, q \in W$

**Definition 3.3.** Let $l$ and $m$ be any two $IN$ subsets of $W$. Then $l \cap m$ defined by $l_t \cap m_t(e) = l_t \wedge m_t$, $l_i \cap m_i(e) = l_i \wedge m_i$ and $l_f \cap m_f(e) = l_f \vee m_f$ for all $e \in W$

**Proposition 3.4.** *A nonempty intersection of an $INLHI$ is an $INLHI$.*

**Proof :** **Assume that $\{l^k : k \in I\}$ be a family of an $INLHI$ of $W$. Let $r, s \in W$. Then**

$$\bigwedge_{e\in r+s}(\bigcap_{k\in I} l_t^k)(e) = \bigwedge_{e\in r+s}\inf_{k\in I} l_t^k(e) \geq \inf_{k\in I}\left(l_t^k(r) \wedge l_t^k(s)\right) = \inf_{k\in I} l_t^k(r) \wedge \inf_{k\in I} l_t^k(s)$$
$$= \bigcap_{k\in I} l_t^k(r) \wedge \bigcap_{k\in I} l_t^k(s)$$

**and**

$$\bigwedge_{e\in r+s}(\bigcap_{k\in I} l_i^k)(e) = \bigwedge_{e\in r+s}\inf_{k\in I} l_i^k(e) \geq \inf_{k\in I}\left[\frac{l_i^k(r)+l_i^k(s)}{2}\right] = \frac{\inf_{k\in I} l_i^k(r)+\inf_{k\in I} l_i^k(s)}{2} = \frac{\bigcap_{k\in I} l_i^k(r)+\bigcap_{k\in I} l_i^k(s)}{2}$$

**also**

$$\bigvee_{e\in r+s}(\bigcap_{k\in I} l_f^k)(e) = \bigvee_{e\in r+s}\sup_{k\in I} l_f^k(e) \leq \sup_{k\in I}\left(l_f^k(r) \vee l_f^k(s)\right) = \sup_{k\in I} l_f^k(r) \vee \sup_{k\in I} l_f^k(s)$$
$$= \bigcap_{k\in I} l_f^k(r) \vee \bigcap_{k\in I} l_f^k(s)$$

**Moreover**

$$\bigwedge_{e\in rs}(\bigcap_{k\in I} l_t^k)(e) = \bigwedge_{e\in r+s}\inf_{k\in I} l_t^k(e) \geq \inf_{k\in I} l_t^k(s) = \bigcap_{k\in I} l_t^k(s)$$

**Similarly we can prove for**

$\bigwedge_{e\in rs}(\bigcap_{k\in I} l_i^k)(e) \geq \bigcap_{k\in I} l_i^k(s)$ **and** $\bigvee_{e\in rs}(\bigcap_{k\in I} l_f^k)(e) \leq \bigcap_{k\in I} l_f^k(s)$

**Hence the theorem.**

**Definition 3.5.** Let $\sigma : F \longrightarrow E$ be a mapping from $SHR$ $W$ to $E$. Then $\sigma$ is said to be homomorphism if

(1) $\sigma(e+s) \subseteq \sigma(e) + \sigma(s)$

(2) $\sigma(es) \subseteq \sigma(e)\sigma(s)$

(3) $\sigma(0_F) = 0_E$ for all $e, s \in W$

where $0_F$ and $0_E$ are zeros of $F$ and $E$ respectively.

**Proposition 3.6.** *Let $\sigma : F \longrightarrow E$ be a homomorphism of semi-hyper-ring. If $l$ is an $INLHI$ of $W$. Then pre-image of $l$ is an $INLHI$ of $W$.*

**Proof :** **Since $\sigma : F \longrightarrow E$ be a homomorphism of $W$. Also since $l$ is an $INLHI$ of $W$ and $u, e, k \in W$.**

$$\bigwedge_{u\in e+k} \sigma^{-1}(l_t)(u) = \bigwedge_{t\in e+k} l_t(\sigma(u))$$
$$= \bigwedge_{\sigma(u)\subseteq\sigma(e)+\sigma(k)} l_t(\sigma(u))$$

$$\geq l_t(\sigma(e)) \wedge l_t(\sigma(k))$$
$$= \sigma^{-1}(l_t)(e) \wedge \sigma^{-1}(l_t)(k)$$

**Also**

$$\bigwedge_{u\in e+k} \sigma^{-1}(l_i)(u) = \bigwedge_{t\in e+k} l_i(\sigma(u))$$
$$= \bigwedge_{\sigma(u)\subseteq\sigma(e)+\sigma(k)} l_i(\sigma(u))$$
$$\geq l_i(\sigma(e)) \wedge l_i(\sigma(k))$$
$$= \sigma^{-1}(l_i)(e) \wedge \sigma^{-1}(l_i)(k)$$

**Moreover**

$$\bigvee_{u\in e+k} \sigma^{-1}(l_f)(u) = \bigvee_{t\in e+k} l_f(\sigma(u))$$
$$= \bigvee_{\sigma(u)\subseteq\sigma(e)+\sigma(k)} l_f(\sigma(u))$$
$$\leq l_f(\sigma(e)) \vee l_i(\sigma(k))$$
$$= \sigma^{-1}(l_f)(e) \vee \sigma^{-1}(l_f)(k)$$

**Again**

$$\bigwedge_{u\in ek} \sigma^{-1}(l_t)(u) = \bigwedge_{t\in ek} l_t(\sigma(u))$$
$$= \bigwedge_{\sigma(u)\subseteq\sigma(e)\sigma(k)} l_t(\sigma(u))$$
$$\geq l_t(\sigma(k)) = \sigma^{-1}(l_t)(k)$$

**Also**

$$\bigwedge_{u\in ek} \sigma^{-1}(l_i)(u) = \bigwedge_{t\in ek} l_i(\sigma(u))$$
$$= \bigwedge_{\sigma(u)\subseteq\sigma(e)\sigma(k)} l_i(\sigma(u))$$
$$\geq l_i(\sigma(k)) = \sigma^{-1}(l_i)(k)$$

**and**

$$\bigvee_{u\in ek} \sigma^{-1}(l_f)(u) = \bigvee_{t\in ek} l_f(\sigma(u))$$
$$= \bigvee_{\sigma(u)\subseteq\sigma(e)\sigma(k)} l_f(\sigma(u))$$
$$\leq l_f(\sigma(k)) = \sigma^{-1}(l_f)(k)$$

**Hence pre-image of $l$ is an $INLHI$ of $W$.**

**Proposition 3.7.** *Let $\sigma : F \longrightarrow E$ be a surjective homomorphism of semi-hyper-ring. If $l$ is an $INLHI$ of $W$. Then image of $l$ is an $INLHI$ of $W$.*

**Proof : Since $l$ is an $INLHI$ of $W$ and $u_0, e_0, k_0 \in W$. Then**

$$\bigwedge_{u_0\in e_0+k_0} \sigma(l_t)(u_0) = \bigwedge_{u_0\in e_0+k_0} \sup_{u\in\sigma^{-1}(u_0)} l_t(u)$$
$$= \bigwedge_{u_0\in e_0+k_0} \sup_{e\in\sigma^{-1}(e_0), k\in\sigma^{-1}(k_0)} l_t(u)$$
$$\geq \sup_{e\in\sigma^{-1}(e_0), k\in\sigma^{-1}(k_0)} \{l_t(u) \vee l_t(k)\}$$
$$= \sup_{e\in\sigma^{-1}(e_0)} l_t(u) \wedge \sup_{k\in\sigma^{-1}(k_0)} l_t(k)$$

$$= \sigma(l_t)(e_0) \wedge \sigma(l_t)(k_0)$$

**Also**

$$\begin{aligned}
\bigwedge_{u_0 \in e_0 + k_0} \sigma(l_i)(u_0) &= \bigwedge_{u_0 \in e_0 + k_0} \sup_{u \in \sigma^{-1}(u_0)} l_i(\sigma(u)) \\
&= \bigwedge_{u_0 \in e_0 + k_0} \sup_{e \in \sigma^{-1}(e_0), k \in \sigma^{-1}(k_0)} l_i(\sigma(u)) \\
&\geq \sup_{e \in \sigma^{-1}(e_0), k \in \sigma^{-1}(k_0)} \frac{l_i(e) + l_i(k)}{2} \\
&= 1/2 \left[ \sup_{e \in \sigma^{-1}(e_0)} l_i(u) + \sup_{k \in \sigma^{-1}(k_0)} l_i(k) \right] \\
&= 1/2 \left[ \sigma(l_i)(e_0) + \sigma(l_i)(k_0) \right]
\end{aligned}$$

$$\begin{aligned}
\bigvee_{u_0 \in e_0 + k_0} \sigma(l_f)(u_0) &= \bigvee_{u_0 \in e_0 + k_0} \inf_{u \in \sigma^{-1}(u_0)} l_f(u) \\
&= \bigvee_{u_0 \in e_0 + k_0} \inf_{e \in \sigma^{-1}(e_0), k \in \sigma^{-1}(k_0)} l_f(u) \\
&\leq \inf_{e \in \sigma^{-1}(e_0), k \in \sigma^{-1}(k_0)} \{ l_t(e) \vee l_t(k) \} \\
&= \inf_{e \in \sigma^{-1}(e_0)} l_t(e) \vee \inf_{k \in \sigma^{-1}(k_0)} l_t(k) \\
&= \sigma(l_t)(e_0) \vee \sigma(l_t)(k_0)
\end{aligned}$$

**Moreover**

$$\begin{aligned}
\bigwedge_{u_0 \in e_0 k_0} \sigma(l_t)(u_0) &= \bigwedge_{u_0 \in e_0 k_0} \sup_{u \in \sigma^{-1}(u_0)} l_t(u) \\
&= \bigwedge_{u_0 \in e_0 + k_0} \sup_{e \in \sigma^{-1}(e_0), k \in \sigma^{-1}(k_0)} l_t(u) \\
&\geq \sup_{k \in \sigma^{-1}(k_0)} l_t(k) \\
&= \sigma(l_t)(k_0)
\end{aligned}$$

$$\begin{aligned}
\bigwedge_{u_0 \in e_0 k_0} \sigma(l_i)(u_0) &= \bigwedge_{u_0 \in e_0 k_0} \sup_{u \in \sigma^{-1}(u_0)} l_i(u) \\
&= \bigwedge_{u_0 \in e_0 + k_0} \sup_{e \in \sigma^{-1}(e_0), k \in \sigma^{-1}(k_0)} l_i(u) \\
&\geq \sup_{k \in \sigma^{-1}(k_0)} l_i(k) \\
&= \sigma(l_i)(k_0)
\end{aligned}$$

**Also**

$$\begin{aligned}
\bigvee_{u_0 \in e_0 k_0} \sigma(l_f)(u_0) &= \bigvee_{u_0 \in e_0 k_0} \inf_{u \in \sigma^{-1}(u_0)} l_f(u) \\
&= \bigvee_{u_0 \in e_0 + k_0} \inf_{e \in \sigma^{-1}(e_0), k \in \sigma^{-1}(k_0)} l_f(u) \\
&\leq \inf_{k \in \sigma^{-1}(k_0)} l_f(k) \\
&= \sigma(l_f)(k_0)
\end{aligned}$$

**Definition 3.8.** Cartesian product of two $IN$ subsets $l$ and $m$ of $W$ is defined by,

$(l_t \times m_t)(e, k) = l_t \wedge m_t$

$(l_i \times m_i)(e,k) = \frac{l_i+m_i}{2}$

$(l_f \times m_f)(e,k) = l_f \vee m_f$ for all $e,k \in W$

**Theorem 3.9.** *Cartesian product of two $INLHI$ is also an $INLHI$.*

**Proof: Let $l$ and $m$ be two $INLHI$ of $W$. Let $(e_1,e_2),(k_1,k_2),(u_1,u_2) \in W \times W$. Then**

$$\begin{aligned}\bigwedge_{(e_1,e_2)\in(k_1,k_2)+(u_1,u_2)} (l_t \times m_t)(e_1,e_2) &= \bigwedge_{e_1\in(k_1+u_1),e_2\in(k_2+u_2)} (l_t \times m_t)(e_1,e_2)\\ &= \bigwedge_{e_1\in(k_1+u_1),e_2\in(k_2+u_2)} (l_t(e_1) \wedge m_t(e_2))\\ &\geq min\left\{(l_t(k_1) \wedge l_t(u_1)), (m_t(k_1) \wedge m_t(u_1))\right\}\\ &= min\left\{(l_t(k_1) \wedge l_t(k_2)), (m_t(u_1) \wedge m_t(u_2))\right\}\\ &= min\left\{(l_t \times m_t)(k_1,k_2), (l_t \times m_t)(u_1,u_2)\right\}\end{aligned}$$

**Also**

$$\begin{aligned}\bigwedge_{(e_1,e_2)\in(k_1,k_2)+(u_1,u_2)} (l_i \times m_i)(e_1,e_2) &= \bigwedge_{e_1\in(k_1+u_1),e_2\in(k_2+u_2)} (l_i \times m_i)(e_1,e_2)\\ &= \bigwedge_{e_1\in(k_1+u_1),e_2\in(k_2+u_2)} \frac{l_i(e_1)+m_i(e_2)}{2}\\ &\geq 1/2\left[\frac{l_i(k_1)+m_i(u_1)}{2} + \frac{l_i(k_2)+m_i(u_2)}{2}\right]\\ &= 1/2\left[\frac{l_i(k_1)+m_i(k_2)}{2} + \frac{l_i(u_1)+m_i(u_2)}{2}\right]\\ &= 1/2\left[(l_i \times m_i)(k_1,k_2) + (l_i \times m_i)(u_1,u_2)\right]\end{aligned}$$

**and**

$$\begin{aligned}\bigvee_{(e_1,e_2)\in(k_1,k_2)+(u_1,u_2)} (l_f \times m_f)(e_1,e_2) &= \bigvee_{e_1\in(k_1+u_1),e_2\in(k_2+u_2)} (l_f \times m_f)(e_1,e_2)\\ &= \bigvee_{e_1\in(k_1+u_1),e_2\in(k_2+u_2)} (l_t(e_1) \vee m_t(e_2))\\ &\leq max\left\{(l_t(k_1) \vee l_t(u_1)), (m_t(k_1) \vee m_t(u_1))\right\}\\ &= max\left\{(l_t(k_1) \vee l_t(k_2)), (m_t(u_1) \vee m_t(u_2))\right\}\\ &= max\left\{(l_t \times m_t)(k_1,k_2), (l_t \times m_t)(u_1,u_2)\right\}\end{aligned}$$

**In similar manner we prove**

$$\begin{aligned}\bigwedge_{(e_1,e_2)\in(k_1,k_2)(u_1,u_2)} (l_t \times m_t)(e_1,e_2) &= \bigwedge_{e_1\in(k_1u_1),e_2\in(k_2u_2)} (l_t \times m_t)(e_1,e_2)\\ &= \bigwedge_{e_1\in(k_1u_1),e_2\in(k_2u_2)} (l_t(e_1) \wedge m_t(e_2))\\ &\geq min\left\{l_t(u_1) \wedge m_t(u_2)\right\}\\ &= min\left\{(l_t \times m_t)(u_1,u_2)\right\}\end{aligned}$$

**also**

$$\begin{aligned}\bigwedge_{(e_1,e_2)\in(k_1,k_2)(u_1,u_2)} (l_i \times m_i)(e_1,e_2) &= \bigwedge_{e_1\in(k_1u_1),e_2\in(k_2u_2)} (l_i \times m_i)(e_1,e_2)\\ &= \bigwedge_{e_1\in(k_1u_1),e_2\in(k_2u_2)} \frac{l_i(e_1)+m_i(e_2)}{2}\\ &\geq \frac{l_i(u_1)+m_i(u_2)}{2} = (l_i \times m_i)(u_1,u_2)\end{aligned}$$

**Moreover**

$$\begin{aligned}\bigvee_{(e_1,e_2)\in(k_1,k_2)(u_1,u_2)}(l_f\times m_f)(e_1,e_2) &= \bigvee_{e_1\in(k_1u_1),e_2\in(k_2u_2)}(l_f\times m_f)(e_1,e_2)\\ &= \bigvee_{e_1\in(k_1+u_1),e_2\in(k_2+u_2)}(l_t(e_1)\vee m_t(e_2))\\ &\leq l_f(u_1)\vee m_f(u_2)=(l_f\times m_f)(u_1,u_2)\end{aligned}$$

## 4. Rough interval neutrosophic hyper-ideal ($RINHI$) in semihyperrings

**This section deals with the new concept $RINHI$ of semihyperrings. Let $\phi$ be a congruence relation on $W$.**
**$\phi$ is an equivalence relation on $W$ such that $(e,s)\in\phi \implies (ew,sw)\in\phi$ and $(we,ws)\in\phi$ for every $w\in W$.**

**Definition 4.1.** An $INHI$ is called an $\phi$-lower(upper)$INHI$ of $W$ if its lower(upper) approximation is also an $INHI$.

**Definition 4.2.** An $INHI$ is said to be an $RINHI$ if it is both $\phi$-lower and $\phi$-upper $INHI$ of $W$.

**Theorem 4.3.** *Let $l$ be an $INHI$ of $W$. Then $l$ is an $RINHI$.*

**Proof: Since $l$ is an $INHI$ of $W$. Let $e,s,q\in W$ then**

$$\begin{aligned}\bigwedge_{e\in s+q}\overline{\phi}(l_t)(e) &= \bigwedge_{e\in s+q}\bigvee_{r\in[s+q]_\phi} l_t(r)\\ &\geq \bigwedge_{e\in s+q}\bigvee_{r\in[s]_\phi+[q]_\phi} l_t(r)\\ &\geq \bigwedge_{r\in i+j}\bigvee_{i+j\subseteq[s]_\phi+[q]_\phi} l_t(r)\\ &= \bigvee_{i\in[s]_\phi,j\in[q]_\phi}\bigwedge_{r\in i+j} l_t(r)\\ &\geq \bigvee_{i\in[s]_\phi,j\in[q]_\phi}\{l_t(i)\wedge l_t(j)\}\\ &= \bigvee_{i\in[s]_\phi} l_t(s)\wedge\bigvee_{j\in[q]_\phi} l_t(q)\\ &= \overline{\phi}(l_t)(s)\wedge\overline{\phi}(l_t)(q)\end{aligned}$$

**and**

$$\begin{aligned}\bigwedge_{e\in s+q}\overline{\phi}(l_i)(e) &= \bigwedge_{e\in s+q}\bigwedge_{r\in[s+q]_\phi} l_i(r)\\ &\geq \bigwedge_{e\in s+q}\bigwedge_{r\in[s]_\phi+[q]_\phi} l_i(r)\\ &\geq \bigwedge_{r\in i+j}\bigwedge_{i+j\subseteq[s]_\phi+[q]_\phi} l_i(r)\\ &= \bigwedge_{i\in[s]_\phi,j\in[q]_\phi}\bigwedge_{r\in i+j} l_i(r)\\ &\geq \bigwedge_{i\in[s]_\phi,j\in[q]_\phi}\left[\frac{l_i(i)+l_i(j)}{2}\right]\\ &= \frac{1}{2}\left[\bigwedge_{i\in[s]_\phi} l_i(i)+\bigwedge_{j\in[q]_\phi} l_i(j)\right]\end{aligned}$$

$$= \frac{1}{2}\left[\overline{\phi}(l_i)(s) + \overline{\phi}(l_i)(q)\right]$$

**also**

$$\begin{aligned}
\bigvee_{e\in s+q} \overline{\phi}(l_f)(e) &= \bigvee_{e\in s+q} \bigvee_{r\in[s+q]_\phi} l_f(r) \\
&\leq \bigvee_{e\in s+q} \bigvee_{r\in[s]_\phi+[q]_\phi} l_f(r) \\
&\leq \bigvee_{r\in i+j} \bigvee_{i+j\subseteq[s]_\phi+[q]_\phi} l_f(r) \\
&= \bigvee_{i\in[s]_\phi, j\in[q]_\phi} \bigvee_{r\in i+j} l_f(r) \\
&\leq \bigvee_{i\in[s]_\phi, j\in[q]_\phi} \{l_f(i) \vee l_f(j)\} \\
&= \bigvee_{i\in[s]_\phi} l_f(s) \vee \bigvee_{j\in[q]_\phi} l_f(q) \\
&= \overline{\phi}(l_f)(s) \vee \overline{\phi}(l_f)(q)
\end{aligned}$$

**Moreover**

$$\begin{aligned}
\bigwedge_{e\in sq} \overline{\phi}(l_t)(e) &= \bigwedge_{e\in sq} \bigvee_{r\in[sq]_\phi} l_t(r) \\
&= \bigwedge_{e\in sq} \bigvee_{r\in[s]_\phi[q]_\phi} l_t(r) \\
&= \bigwedge_{r\in ij} \bigvee_{ij\subseteq[s]_\phi[q]_\phi} l_t(r) \\
&= \bigvee_{i\in[s]_\phi j\in[q]_\phi} \bigwedge_{r\in ij} l_t(r) \\
&\geq \bigvee_{i\in[s]_\phi j\in[q]_\phi} l_t(j) \\
&\geq \bigvee_{j\in[q]_\phi} l_t(j) \\
&= \overline{\phi}(l_t)(q)
\end{aligned}$$

**Similarly we can prove for**

$\bigwedge_{e\in sq} \overline{\phi}(l_f)(e) \geq \overline{\phi}(l_f)(q)$ **and** $\bigwedge_{e\in sq} \overline{\phi}(l_i)(e) \leq \overline{\phi}(l_i)(q)$

**Consequently we can prove for lower approximation**

**ie.,**

$$\bigwedge_{e\in s+q} \underline{\phi}(l_t)(e) \geq \underline{\phi}(l_t)(s) \wedge \underline{\phi}(l_t)(q)$$

$$\bigwedge_{e\in s+q} \underline{\phi}(l_i)(e) \geq \frac{1}{2}\left[\underline{\phi}(l_i)(s) + \underline{\phi}(l_i)(q)\right]$$

$$\bigwedge_{e\in s+q} \underline{\phi}(l_f)(e) \leq \underline{\phi}(l_f)(s) \wedge \underline{\phi}(l_f)(q)$$

**and**

$$\bigwedge_{e\in sq} \underline{\phi}(l_t)(e) \geq \underline{\phi}(l_t)(q)$$

$$\bigwedge_{e\in sq} \underline{\phi}(l_f)(e) \geq \underline{\phi}(l_f)(q)$$

$$\bigwedge_{e\in sq} \underline{\phi}(l_i)(e) \leq \underline{\phi}(l_i)(q)$$

**Hence $l$ is a $RINLHI$ of $W$.**

## 5. Conclusions

**In this paper we introduce the notion of rough interval neutrosophic hyperideals in semihyperrings. Some basic properties of this ideals are studied. We apply rough interval neutrosophic set to some more algebraic structures. Moreover in future we apply rough interval neutrosophic sets to some applications like multi criteria decision making, medical analysis, decision making, gray analysis etc.,**

**Conflicts of Interest: The authors declare that there is no conflict of interest regarding the publication of the paper.**

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# An Intelligent Decision-Making Model to Analysis and Assess the Strategies of International Business Administrations Under Neutrosophic Environment

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**Abstract**: Global company management relies heavily on the techniques developed in the field of International Business Administration. This paper summarizes the most important strategies used by companies to deal with the challenges of doing business on a global scale. It stresses the necessity for localization and adaptation to accommodate varied market preferences and underscores the relevance of market entrance techniques including exporting, licensing, and foreign direct investment. Strategies for obtaining economies of scale and uniformity across markets are examined, along with the role of standardization and global integration in doing so. The need to develop one's cross-cultural competency as a means of appreciating and valuing one another's cultural backgrounds is emphasized. Approaches to logistics, procurement, and distribution management at the global level are explored. The authors argue that strategic partnerships and joint ventures might be used to tap into local expertise and infrastructure. This paper used the neutrosophic set integrated with the multi-criteria decision-making (MCDM) tools to analyze and evaluate the various strategists. The TOPSIS and VIKOR methods are used to rank the alternatives and obtain the most suitable strategy. This paper used ten factors and seven strategies to be evaluated and analyzed. These elements are applied in experiments to use the best strategy in the organization.

**Keywords**: Neutrosophic Set, TOPSIS, VIKOR, MCDM, International Business Administration.

## 1. Introduction

International company administrations in today's linked global economy confront difficult problems and possibilities as they negotiate the varied business environments of various nations and regions. To survive and prosper in today's fast-paced world, businesses need to create and execute strategies that give them a leg up in the global marketplace. Assessing the efficacy of these approaches, pinpointing improvement opportunities, and propelling organizational success are all dependent on careful analysis and assessment. The purpose of this study is to investigate and assess the approaches used by multinational company administrations, illuminating their viability, difficulties, and possibilities for gaining a lasting edge in the marketplace[1], [2].

Global business administrations compete in a dynamic and ever-evolving environment. Their chances of survival and growth are profoundly affected by the methods they use to expand into new markets, handle cross-border activities, accommodate cultural differences, and take advantage of international possibilities. Organizations may learn a lot about their strengths, shortcomings, and competitive standing by analyzing and assessing these tactics. Strategic decisions may then be made using the knowledge gained from this kind of analysis, which also aids in capitalizing on possibilities and reducing the risks involved with doing business in a globalized world[3], [4].

The fundamental purpose of this study is to assess and compare the approaches used by multinational business administrators in different sectors. The study's overarching goal is to learn about various approaches and their components, drivers, and consequences. The study also aims to evaluate how well these tactics help businesses accomplish their stated goals of increasing their reach in the market, their bottom line, the visibility of their brand, and the efficacy of their operations. This study's overarching goal is to help businesses improve their international strategy by analyzing current practices and offering suggestions for how they might do better in the future[5], [6].

Both qualitative and quantitative approaches will be used to complete the study. The basis of understanding will be established by conducting a comprehensive literature study of relevant academic research, case studies, and industry reports. Further, qualitative interviews and questionnaires will be administered to professionals and executives in the worldwide business community to glean insights from the actual world. Research techniques such as these will shed light on the tactics used by multinational business administrations and allow for an assessment of how successful these approaches are[7], [8].

A proposition in Boolean logic is either true or false; in classical set theory, an item either belongs to a set or it does not; and in optimization, an approach is either viable or it is not. However, in practice, almost everything is a question of degree and cannot be precisely characterized by the usual reasoning[9], [10]. Zadeh developed the fuzzy sets theory to cope with this sort of ambiguity. Since its inception in 1965, various variations of fuzzy sets have been developed. Zadeh created the type-n fuzzy set to address the ambiguity of the membership function in fuzzy set theory. Atanassov introduced intuitionistic fuzzy sets (IFSs) to clarify how membership functions, their degrees of membership and non-membership, and the uncertainty of decision-makers are defined[11]. Torra first introduced hesitant fuzzy sets (HFSs), which are an extension of conventional fuzzy sets in which many values are allowed for the membership of one component. However, the opinions of decision-makers may not be well reflected in the enlarged description of membership duties. As a result, we still need to add some additional extensions[12], [13].

Smarandache extended intuitionistic fuzzy sets with neutrosophic logic and neutrosophic sets (NSs) to address this shortcoming[14]. The neutrosophic set is the set in which every conceivable thing in the cosmos has a truthiness, indeterminacy, or falsity between zero and one. While levels of belongingness non-belongingness and indeterminacy value were factored in as equivalence or absoluteness, in the neutrosophic sets, ambiguity is expressed as truth and falsity numbers. In addition to coping with systemic ambiguity, this designation for neutrosophic sets helps eliminate the paralysis caused by conflicting data[9], [15]. Thus, the truth number, the falsehood value, and the indeterminacy value may be thought of as the membership level, the non-membership level, and the hesitant degree, respectively[16]. This paper used the neutrosophic set with the multi-criteria decision-making (MCDM) methods like TOPSIS and VIKOR method to evaluate the strategies of international business administration[15], [17], [18].

In this study, we shall dissect and assess several approaches used by global company administrations. Market entry strategies, international supply chain management, cultural adaptation and localization, strategic alliances and partnerships, international marketing and branding, technological innovation and digital transformation, risk management, and compliance; are just some of the areas that fall under the remit of this study. Key elements, success indicators, difficulties, and best practices for each approach will be investigated, providing a formal framework for the study. Organizations will be able to obtain insights and make well-informed choices about their international commercial operations thanks to the paper's framework, which will allow for a full review of these tactics[19], [20].

Organizations that want to succeed in the global market must devote significant resources to the study and assessment of international business strategy. Organizations may improve their decision-making processes, increase their competitive edge, and achieve sustainable development if they have a thorough awareness of the pros and cons of various strategies. The purpose of this study is to add to the current body of literature by analyzing and evaluating in depth the tactics used by multinational company administrations. This study's results and suggestions will help businesses of all sizes create competitive advantages, adapt to new markets, and expand internationally.

The rest of this paper is organized as follows: Strategies of international business administration are presented in section 2. The materials and methods are presented in section 3. The application and experiment of methodologies are presented in section 4. The conclusions of this study are presented in section 5.

## 2. Strategies of International Business Administration

Examining the methods and strategies used by companies competing on a worldwide scale is an important part of any analysis and evaluation of international business administrations. We may learn about the efficacy, obstacles, and possibilities that multinational firms confront by analyzing these tactics. Here, we'll examine the effectiveness of some of the most often-used methods among multinational corporations' top management[21], [22].

Exporting, licensing, franchising, joint ventures, and foreign direct investment (FDI) are just a few examples of market entrance techniques used by international enterprises. Companies may enter new markets, get access to resources, and increase their consumer base with the help of these techniques. Market dynamics, legal frameworks, cultural variables, and competitive landscapes all have a role in determining the success of these approaches. Market entrance tactics may be judged on their ability to break into the market, win over customers, turn a profit, and fuel sustainable expansion[23], [24].

Management of the global supply chain is essential for international companies to guarantee successful product sourcing, manufacturing, and distribution. Supplier selection, logistical optimization, inventory management, and risk avoidance are all examples of supply chain management strategies. Cost-effectiveness, speed of response, dependability, sustainability, and resistance to interruptions are only a few of the criteria that must be considered when evaluating the efficacy of supply chain methods[25], [26].

Successful global companies know the value of adopting localized strategies that account for cultural differences. Customer acceptability and market penetration may be increased by tailoring goods, services, marketing messaging, and business practices to local tastes and cultural norms. The success of these tactics may be gauged by looking at metrics like increased sales and word-of-mouth advertising.

Global companies often develop strategic alliances and collaborations with regional competitors to better understand target markets, expand distribution channels, pool resources, and spread potential dangers. Joint ventures, collaborations, and strategic partnerships are all examples of types of strategic alliances. Synergy, knowledge transfer, resource sharing, competitive advantage, and financial performance are just a few of the indicators that may be used to gauge the success of these methods[27], [28].

Marketing and branding on a worldwide scale: How multinational corporations raise their profiles, strengthen their brands, and carve out a niche for themselves. These methods entail communicating effectively across cultural boundaries and tailoring messages to specific target audiences. Brand awareness, consumer participation, increased market share, and a positive return on ad spend are just a few of the metrics that must be considered when assessing the success of a worldwide marketing campaign.

International corporations now use technology and digital platforms to speed up innovation, boost productivity, and enrich their customers' interactions. Data analytics, AI, e-commerce, digital marketing, and process automation are just a few of the areas that may be addressed by technological innovation and digital transformation strategies. Considerations including technological uptake, competitive advantage, satisfied customers, and increased profits are all part of an effective evaluation of these tactics[29], [30].

Political, legal, financial, operational, and reputational risks are just some of the threats that international firms confront. Risk assessment, risk mitigation, contingency planning, and adherence to local legislation are all essential components of an effective risk management strategy. Considerations including risk exposure, risk mitigation efficiency, legal compliance, and organizational resilience are essential for assessing these plans[3], [31].

It is crucial to take into account the industry, competitive dynamics, macroeconomic conditions, and organizational goals while assessing the strategies of international company administrations. Financial performance indicators, market share statistics, customer feedback, and industry benchmarking are all examples of quantitative measurements that might be included in the research. Organizations may enhance their performance, build on their successes, and adjust to the dynamic nature of the global business environment by conducting in-depth analyses of these tactics[32], [33].

## 3. Materials and Methods

Here, we'll describe how the distance measure and score function we just described are employed in the neutrosophic TOPSIS and neutrosophic VIKOR methods' calculation procedures[34], [35]. Figure 1 shows the steps of the proposed model.

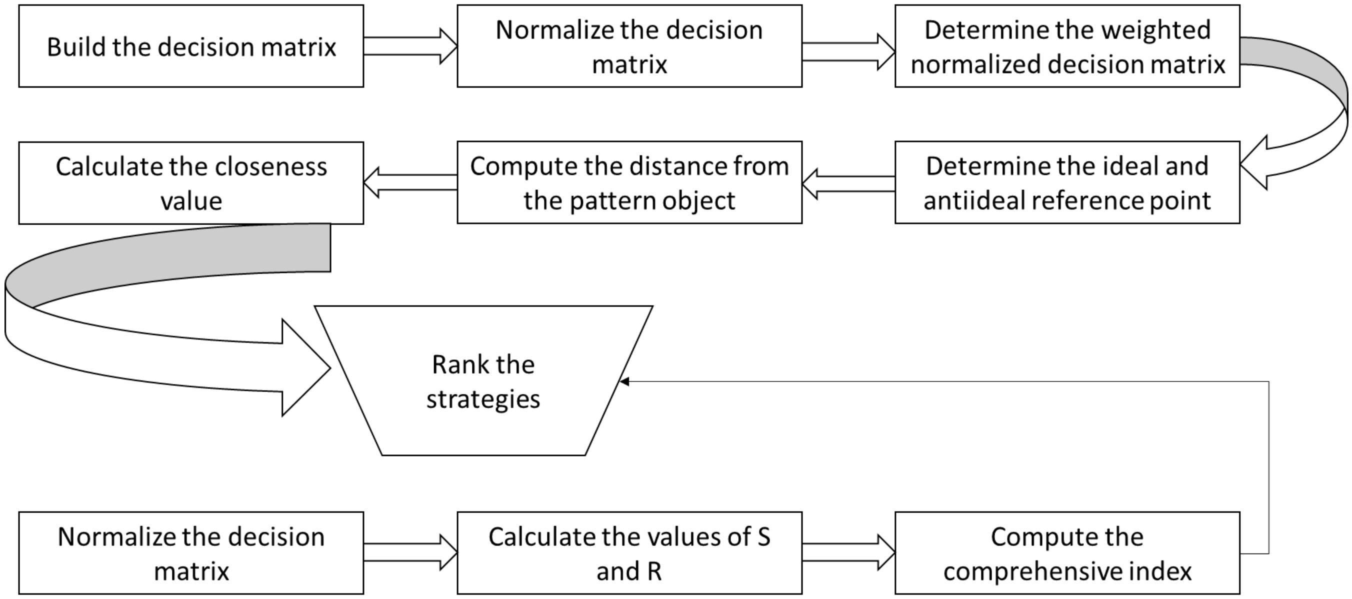


**Figure** 1. The framework of this study.

We used the interval-valued neutrosophic set. So, we can define some mathematical models of interval-valued neutrosophic sets as:

$$T_{IJ} = \frac{a_{ij}^{pos}}{a_{ij}^{pos}+a_{ij}^{neu}+a_{ij}^{neg}} \quad 1.1$$

$$I_{IJ} = \frac{a_{ij}^{neu}}{a_{ij}^{pos}+a_{ij}^{neu}+a_{ij}^{neg}} \quad 1.2$$

$$F_{IJ} = \frac{a_{ij}^{neg}}{a_{ij}^{pos}+a_{ij}^{neu}+a_{ij}^{neg}} \quad 1.3$$

We can define the interval-valued neutrosophic number as:

$$T_{ij}^{L} = T_{ij} - z_{\theta/2} \times \sqrt{\frac{T_{ij}(1-T_{ij})}{a_{ij}^{pos}+a_{ij}^{neu}+a_{ij}^{neg}}} \quad 1.4$$

$$T_{ij}^{U} = T_{ij} + z_{\theta/2} \times \sqrt{\frac{T_{ij}(1-T_{ij})}{a_{ij}^{pos}+a_{ij}^{neu}+a_{ij}^{neg}}} \quad 1.5$$

$$I_{ij}^{L} = I_{ij} - z_{\theta/2} \times \sqrt{\frac{I_{ij}(1-I_{ij})}{a_{ij}^{pos}+a_{ij}^{neu}+a_{ij}^{neg}}} \quad 1.6$$

$$I_{ij}^{U} = I_{ij} + z_{\theta/2} \times \sqrt{\frac{I_{ij}(1-I_{ij})}{a_{ij}^{pos}+a_{ij}^{neu}+a_{ij}^{neg}}} \quad 1.7$$

$$F_{ij}^{L} = F_{ij} - z_{\theta/2} \times \sqrt{\frac{F_{ij}(1-F_{ij})}{a_{ij}^{pos}+a_{ij}^{neu}+a_{ij}^{neg}}} \quad 1.8$$

$$F_{ij}^{U} = F_{ij} + z_{\theta/2} \times \sqrt{\frac{F_{ij}(1-F_{ij})}{a_{ij}^{pos}+a_{ij}^{neu}+a_{ij}^{neg}}} \quad 1.9$$

Where $z_{\theta/2}$ refers to the critical value

### 3.1 Neutrosophic TOPSIS Method

Decision alternatives are ranked (ordered) according to how closely they resemble the preferred pattern using the TOPIS approach. The ideal substitute achieves this by having the smallest possible distance to the pattern, whereas the anti-ideal reference substitute has the largest possible distance to the anti-pattern. The distances between each design and the ideal and anti-ideal patterns are determined. That's what makes the final tally possible.

3.1.1. Build the decision matrix.

This step used the opinions of experts to build the decision matrix.

3.1.2. Normalize the decision matrix.

$$a_i^* = \frac{a_i}{\sum_{k=1}^{n} a_j} \quad (1)$$

Where $i = 1,2,3, \dots . m; j = 1,2,3 \dots . n$

3.1.3. Determine the weighted normalized decision matrix.

$$t_i = a_i^* * w_i \quad (2)$$

3.1.4. Determine the ideal and anti-ideal reference point

$$t_i^+ = \begin{cases} \max\limits_j t_i & for\ positive\ criteria \\ \min\limits_j t_i & for\ cost\ criteria \end{cases} \quad (3.1)$$

$$t_i^- = \begin{cases} \min\limits_j t_i & for\ positive\ criteria \\ \max\limits_j t_i & for\ cost\ criteria \end{cases} \quad (3.2)$$

3.1.5. Compute the distance from the pattern object

$$e_j^+ = \sqrt{\sum_{i=1}^{m} \left| t_i - t_i^+ \right|^p} \quad (4)$$

$$e_j^- = \sqrt{\sum_{i=1}^{m} \left| t_i - t_i^- \right|^p} \quad (5)$$

3.1.6. Calculate the closeness value.

$$C_j = \frac{e_j^-}{e_j^- + e_j^+} \tag{6}$$

3.1.7 Rank the strategies.

### 3.2 Neutrosophic VIKOR Method

Using the VICOR technique, we may identify alternative courses of action and choose a middle ground that considers competing assessment criteria. Distances from the ideal and anti-ideal points are used to rank all the options.

3.2.1. Normalize the decision matrix.

$$z_i = \frac{a_i - \min x_i}{\max x_i - \min x_i} \tag{7}$$

3.2.2. Calculate the values of S and R

$$S_j = \sum_{i=1}^{m} w_i * z_i \tag{8}$$

$$R_j = \max_i (w_i z_i) \tag{9}$$

3.2.3 Compute the comprehensive index.

$$Q_i = u \frac{S_j - \min_j S_j}{\max_j S_j - \min_j S_j} + (1-u) \frac{R_j - \min_j R_j}{\max_j R_j - \min_j R_j} \tag{10}$$

3.2.4 Rank the strategies.

## 4. Application

Many organizations and firms tend to select the best strategy for their work. So, this section introduces various strategies and their factors to analyze and evaluate them. We used ten factors and seven strategies in this paper. The ten factors are used in this paper organized as: Several important elements must be considered when assessing the success of a global marketing strategy. These metrics are useful for gauging how far a campaign has traveled, how many people it has reached, and whether it has accomplished its goals. Some crucial elements are as follows:

Assess how well the company has been able to break into new markets thanks to its worldwide marketing efforts. Think about things like increasing your market share, attracting more customers, and opening in new areas.

Measure the success of your worldwide advertising campaign by asking consumers how well they know and understand your brand. Find out how your brand is doing in terms of recognition, affiliations, and overall brand image in various regions.

Examine how well marketing tactics strike a balance between being consistent internationally and tailoring to specific local markets. Evaluate how effectively the company has kept its brand image and messages similar throughout markets while catering to the needs of local consumers.

Determine how involved and responsive your customers are with global marketing campaigns. Measure the success of your marketing efforts by looking at indicators like website visits, social media shares, email opens, email clicks, and customer satisfaction.

Evaluate the company's proficiency in determining and appealing to desirable subsets of its target market. Assess how well the company's worldwide marketing efforts cater to the wants, requirements, and actions of its target consumers throughout the world.

Examine the marketing return on investment (ROI) for all of your international campaigns. Consider customer acquisition expenses, conversion rates, sales income, and overall profitability when you assess the value of marketing initiatives.

Examine the company's worldwide marketing strategy to see how well they set them out from the competition. Evaluate how well marketing has been in portraying the company as cutting-edge and ahead of the competition.

Ability to adjust global marketing tactics quickly and effectively in response to new opportunities and changing customer preferences. Evaluate how quickly and well you adapt to changes in the market and customer needs.

Assess how well your marketing material and messaging have been localized for various markets. Evaluate how successful localized marketing was at reaching the intended population, overcoming any language and cultural hurdles, and increasing consumer participation.

Analyzing and measuring the success of global marketing initiatives requires the use of marketing analytics and metrics. Analyze how well KPIs, marketing analytics tools, and data-driven insights are used to monitor progress and guide strategy.

Organizations may learn a lot about the success of their international marketing campaigns by keeping these things in mind. The results of this analysis may be used to better focus marketing efforts and adapt tactics to the dynamic nature of global marketplaces.

We used the experts and decision-makers to evaluate the factors and strategies. We build the decision makers by the interval-valued neutrosophic numbers. Then we applied the TOPSIS steps. We build the decision matrix by using interval-valued neutrosophic numbers as shown in Table 1. We use Eq. (1) to normalize the decision matrix. Then we construct the weighted normalized matrix by using Eq. (2) as shown in Table 2. Then compute the weights of factors. Then compute the ideal and anti-ideal reference point by using Eqs. (3.1 and 3.2). All factors are positive factors. Then compute the distance from the pattern object using Eqs. (4 and 5). Then compute the closeness value as shown in Figure 2. The results show strategy 6 is the best and strategy 1 is the worst.

Table 1. The interval-valued neutrosophic numbers between factors and strategies.

| | $INTC_1$ | $INTC_2$ | $INTC_3$ | $INTC_4$ | $INTC_5$ | $INTC_6$ | $INTC_7$ | $INTC_8$ | $INTC_9$ | $INTC_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $INTS_1$ | ([0.610, 0.713] [0.133, 0.215] [0.124, 0.205]) | ([0.581, 0.686] [0.119, 0.198] [0.130, 0.212]) | ([0.581, 0.686] [0.119, 0.198] [0.130, 0.212]) | ([0.829, 0.904] [0.017, 0.058] [0.021, 0.066]) | ([0.610, 0.713] [0.133, 0.215] [0.124, 0.205]) | ([0.581, 0.686] [0.119, 0.198] [0.130, 0.212]) | ([0.829, 0.904] [0.017, 0.058] [0.021, 0.066]) | ([0.610, 0.713] [0.133, 0.215] [0.124, 0.205]) | ([0.819, 0.895] [0.046, 0.103] [0.002, 0.029]) | ([0.610, 0.713] [0.133, 0.215] [0.124, 0.205]) |
| $INTS_2$ | ([0.711, 0.805] [0.061, 0.125] [0.094, 0.167]) | ([0.610, 0.713] [0.133, 0.215] [0.124, 0.205]) | ([0.829, 0.904] [0.017, 0.058] [0.021, 0.066]) | ([0.610, 0.713] [0.133, 0.215] [0.124, 0.205]) | ([0.819, 0.895] [0.046, 0.103] [0.002, 0.029]) | ([0.711, 0.805] [0.061, 0.125] [0.094, 0.167]) | ([0.819, 0.895] [0.046, 0.103] [0.002, 0.029]) | ([0.711, 0.805] [0.061, 0.125] [0.094, 0.167]) | ([0.819, 0.895] [0.046, 0.103] [0.002, 0.029]) | ([0.711, 0.805] [0.061, 0.125] [0.094, 0.167]) |
| $INTS_3$ | ([0.610, 0.713] [0.133, 0.215] [0.124, 0.205]) | ([0.819, 0.895] [0.046, 0.103] [0.002, 0.029]) | ([0.819, 0.895] [0.046, 0.103] [0.002, 0.029]) | ([0.819, 0.895] [0.046, 0.103] [0.002, 0.029]) | ([0.829, 0.904] [0.017, 0.058] [0.021, 0.066]) | ([0.819, 0.895] [0.046, 0.103] [0.002, 0.029]) | ([0.819, 0.895] [0.046, 0.103] [0.002, 0.029]) | ([0.819, 0.895] [0.046, 0.103] [0.002, 0.029]) | ([0.829, 0.904] [0.017, 0.058] [0.021, 0.066]) | ([0.610, 0.713] [0.133, 0.215] [0.124, 0.205]) |
| $INTS_4$ | ([0.711, 0.805] [0.061, 0.125] [0.094, 0.167]) | ([0.819, 0.895] [0.046, 0.103] [0.002, 0.029]) | ([0.829, 0.904] [0.017, 0.058] [0.021, 0.066]) | ([0.711, 0.805] [0.061, 0.125] [0.094, 0.167]) | ([0.819, 0.895] [0.046, 0.103] [0.002, 0.029]) | ([0.819, 0.895] [0.046, 0.103] [0.002, 0.029]) | ([0.829, 0.904] [0.017, 0.058] [0.021, 0.066]) | ([0.711, 0.805] [0.061, 0.125] [0.094, 0.167]) | ([0.819, 0.895] [0.046, 0.103] [0.002, 0.029]) | ([0.711, 0.805] [0.061, 0.125] [0.094, 0.167]) |
| $INTS_5$ | ([0.610, 0.713] [0.133, 0.215] [0.124, 0.205]) | ([0.819, 0.895] [0.046, 0.103] [0.002, 0.029]) | ([0.581, 0.686] [0.119, 0.198] [0.130, 0.212]) | ([0.819, 0.895] [0.046, 0.103] [0.002, 0.029]) | ([0.581, 0.686] [0.119, 0.198] [0.130, 0.212]) | ([0.610, 0.713] [0.133, 0.215] [0.124, 0.205]) | ([0.819, 0.895] [0.046, 0.103] [0.002, 0.029]) | ([0.581, 0.686] [0.119, 0.198] [0.130, 0.212]) | ([0.819, 0.895] [0.046, 0.103] [0.002, 0.029]) | ([0.581, 0.686] [0.119, 0.198] [0.130, 0.212]) |
| $INTS_6$ | ([0.711, 0.805] [0.061, 0.125] [0.094, 0.167]) | ([0.711, 0.805] [0.061, 0.125] [0.094, 0.167]) | ([0.819, 0.895] [0.046, 0.103] [0.002, 0.029]) | ([0.829, 0.904] [0.017, 0.058] [0.021, 0.066]) | ([0.829, 0.904] [0.017, 0.058] [0.021, 0.066]) | ([0.711, 0.805] [0.061, 0.125] [0.094, 0.167]) | ([0.829, 0.904] [0.017, 0.058] [0.021, 0.066]) | ([0.829, 0.904] [0.017, 0.058] [0.021, 0.066]) | ([0.711, 0.805] [0.061, 0.125] [0.094, 0.167]) | ([0.610, 0.713] [0.133, 0.215] [0.124, 0.205]) |
| $INTS_7$ | ([0.610, 0.713] [0.133, 0.215] [0.124, 0.205]) | ([0.711, 0.805] [0.061, 0.125] [0.094, 0.167]) | ([0.829, 0.904] [0.017, 0.058] [0.021, 0.066]) | ([0.610, 0.713] [0.133, 0.215] [0.124, 0.205]) | ([0.711, 0.805] [0.061, 0.125] [0.094, 0.167]) | ([0.610, 0.713] [0.133, 0.215] [0.124, 0.205]) | ([0.829, 0.904] [0.017, 0.058] [0.021, 0.066]) | ([0.610, 0.713] [0.133, 0.215] [0.124, 0.205]) | ([0.829, 0.904] [0.017, 0.058] [0.021, 0.066]) | ([0.610, 0.713] [0.133, 0.215] [0.124, 0.205]) |

Table 2. The weighted normalized matrix.

| | $INTC_1$ | $INTC_2$ | $INTC_3$ | $INTC_4$ | $INTC_5$ | $INTC_6$ | $INTC_7$ | $INTC_8$ | $INTC_9$ | $INTC_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $INTS_1$ | 0.031167 | 0.027723 | 0.029414 | 0.043114 | 0.031864 | 0.030356 | 0.04299 | 0.032232 | 0.043384 | 0.032076 |
| $INTS_2$ | 0.036328 | 0.029107 | 0.041969 | 0.031725 | 0.042781 | 0.037148 | 0.042471 | 0.037569 | 0.043384 | 0.037387 |
| $INTS_3$ | 0.031167 | 0.03908 | 0.041462 | 0.042594 | 0.043303 | 0.042791 | 0.042471 | 0.043275 | 0.043914 | 0.032076 |
| $INTS_4$ | 0.036328 | 0.03908 | 0.041969 | 0.036977 | 0.042781 | 0.042791 | 0.04299 | 0.037569 | 0.043384 | 0.037387 |
| $INTS_5$ | 0.031167 | 0.03908 | 0.029414 | 0.042594 | 0.030349 | 0.031871 | 0.042471 | 0.0307 | 0.043384 | 0.030551 |
| $INTS_6$ | 0.036328 | 0.033927 | 0.041462 | 0.043114 | 0.043303 | 0.037148 | 0.04299 | 0.043804 | 0.037663 | 0.032076 |
| $INTS_7$ | 0.031167 | 0.033927 | 0.041969 | 0.031725 | 0.037139 | 0.031871 | 0.04299 | 0.032232 | 0.043914 | 0.032076 |

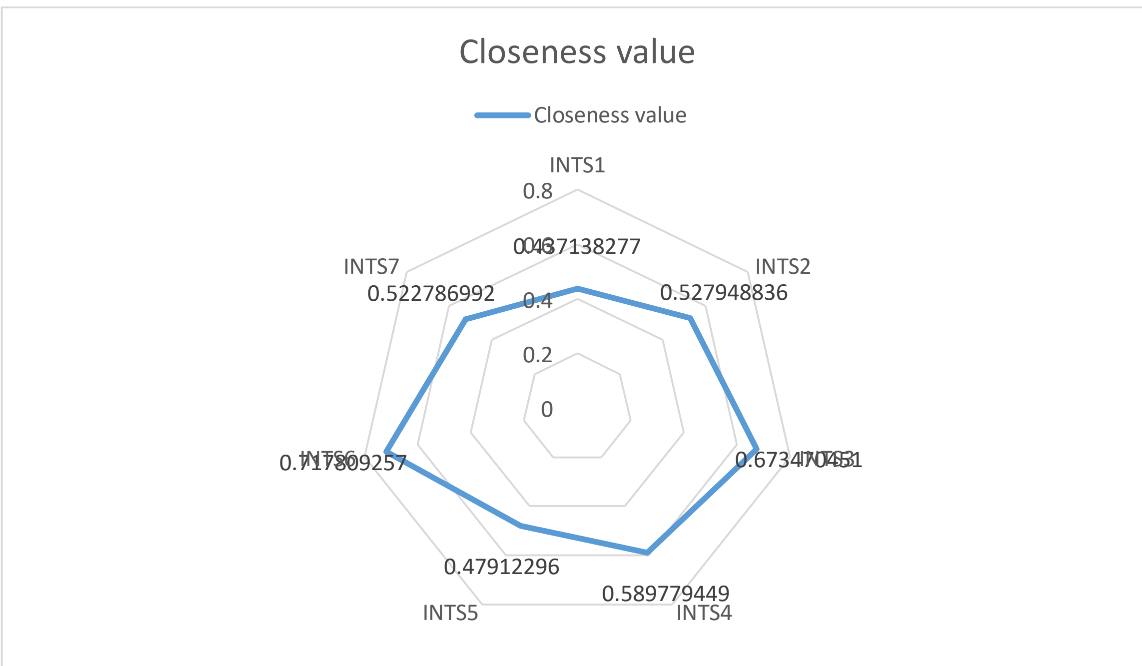


**Figure** 2. The value of closeness by the TOPSIS method.

Then apply the steps of the interval-valued neutrosophic VIKOR method to analyze and evaluate the strategies in international business administration. We normalize the decision matrix by using Eq. (7) as shown in Table 3. Then we compute the values of S and R by using Eqs. (8 and 9). Then we obtain the index of comprehensive by using Eq. (10). We used the u=0.5 to compute the value as shown in Figure 3. Figure 4 shows the rank of strategies by the interval-valued neutrosophic TOPSIS and interval-valued neutrosophic VIKOR method. The results show that strategy 4 is the best and strategy 5 is the worst. We obtained that Technological Innovation and Digital Transformation is the best strategy to be applied in organization and firm.

Table 3. The normalized decision matrix by the VIKOR method.

| | $INTC_1$ | $INTC_2$ | $INTC_3$ | $INTC_4$ | $INTC_5$ | $INTC_6$ | $INTC_7$ | $INTC_8$ | $INTC_9$ | $INTC_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $INTS_1$ | 0.08857 | 0.092193 | 0.102238 | 0 | 0.091458 | 0 | 0 | 0.086678 | 0.009589 | 0.019758 |
| $INTS_2$ | 0 | 0.08096 | 0 | 0.103568 | 0.004176 | 0.052888 | 0.113155 | 0.046703 | 0.009589 | 0.08857 |
| $INTS_3$ | 0.08857 | 0 | 0.004123 | 0.004729 | 0 | 0.096826 | 0.113155 | 0.003958 | 0 | 0.019758 |
| $INTS_4$ | 0 | 0 | 0 | 0.055804 | 0.004176 | 0.096826 | 0 | 0.046703 | 0.009589 | 0.08857 |
| $INTS_5$ | 0.08857 | 0 | 0.102238 | 0.004729 | 0.103568 | 0.011798 | 0.113155 | 0.098156 | 0.009589 | 0 |
| $INTS_6$ | 0 | 0.041836 | 0.004123 | 0 | 0 | 0.052888 | 0 | 0 | 0.113155 | 0.019758 |
| $INTS_7$ | 0.08857 | 0.041836 | 0 | 0.103568 | 0.049279 | 0.011798 | 0 | 0.086678 | 0 | 0.019758 |

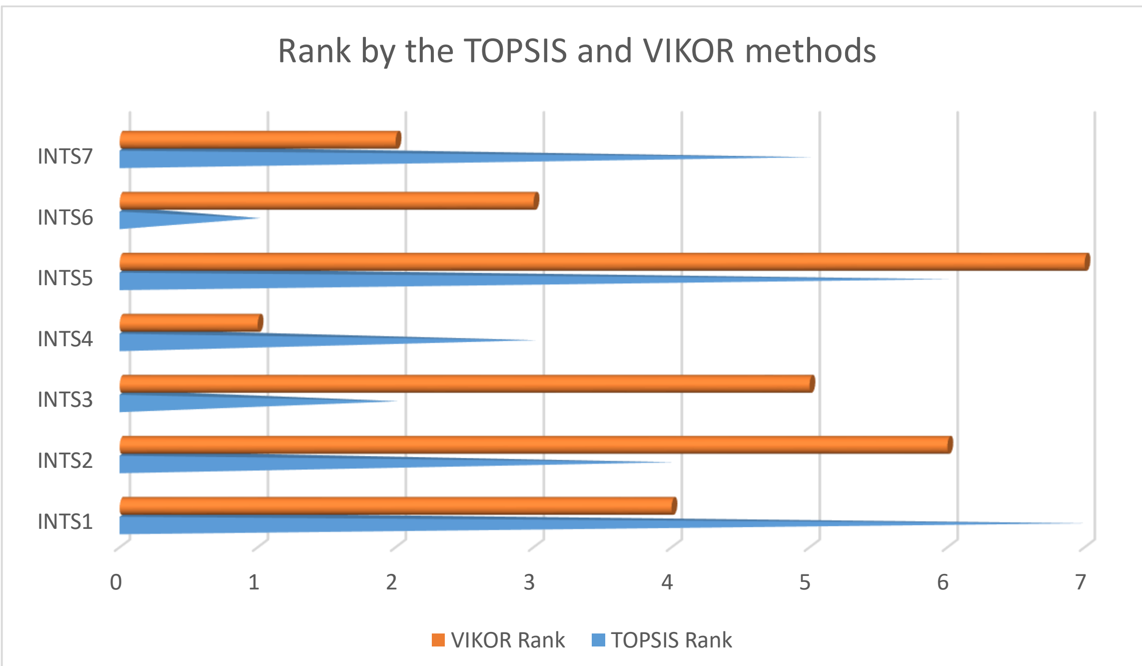


Figure 3. The rank of strategies by the interval-valued neutrosophic TOPSIS and VIKOR methods.

We change the value of u in the VIKOR method to show the rank of strategies. We change this value between 0.1 and 1. The rank of strategies after this change is shown in Figure 4. All values show that strategy 4 is the best and strategy 5 is the worst.

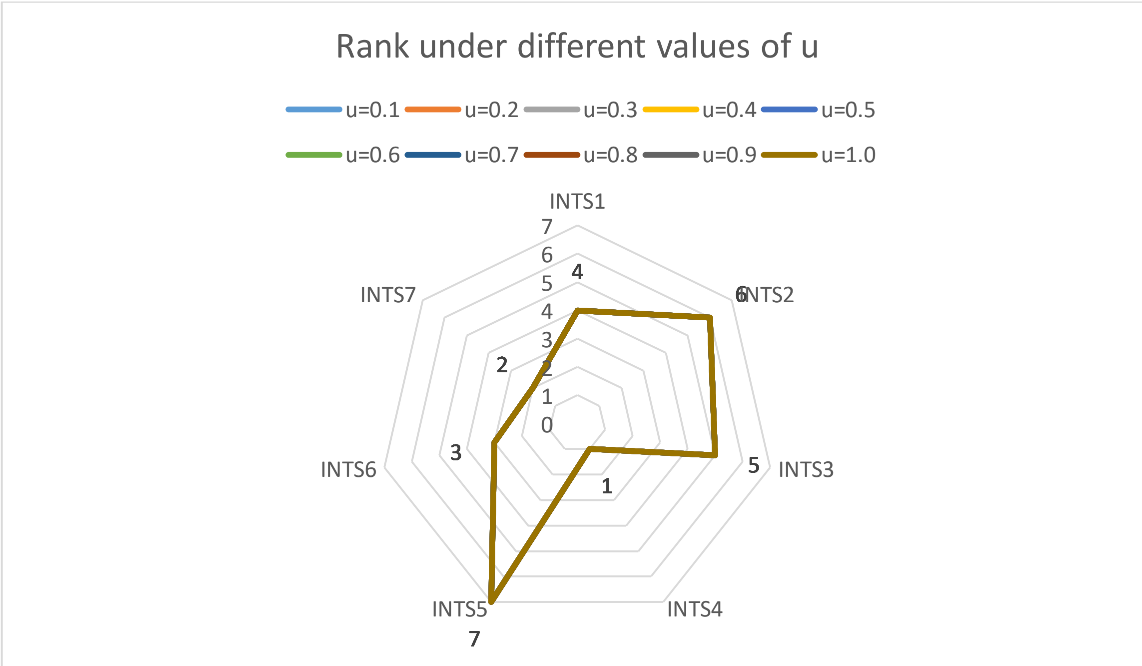


Figure 4. The rank of strategies when changing the value of u between 0.1 and 1.

### 4.1 Sensitivity Analysis

In this part, we change the weights of the criteria to check the rank of alternatives. The rank of alternatives is evaluated by the two MCDM methods. We change the weights of the criteria by eleven cases. In the first case, we put all criteria equal weights. In the second and rest of the cases, we put the one criterion with 0.5 weight and all criteria take the sum of 0.5 weight as shown in Figure 5. Then we apply the TOPSIS and VIKOR method in eleven cases. We got the eleven ranks in TOPSIS and VIKOR methods as shown in Figures 6 and 7.

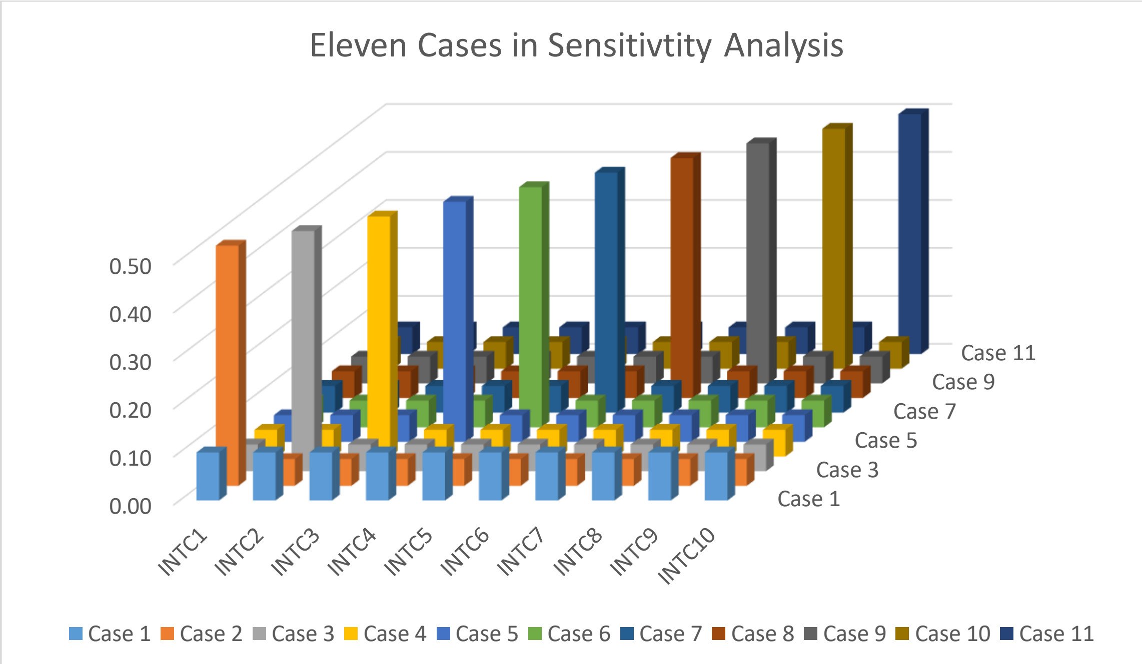


Figure 5. The eleven cases in changing the weights of criteria.

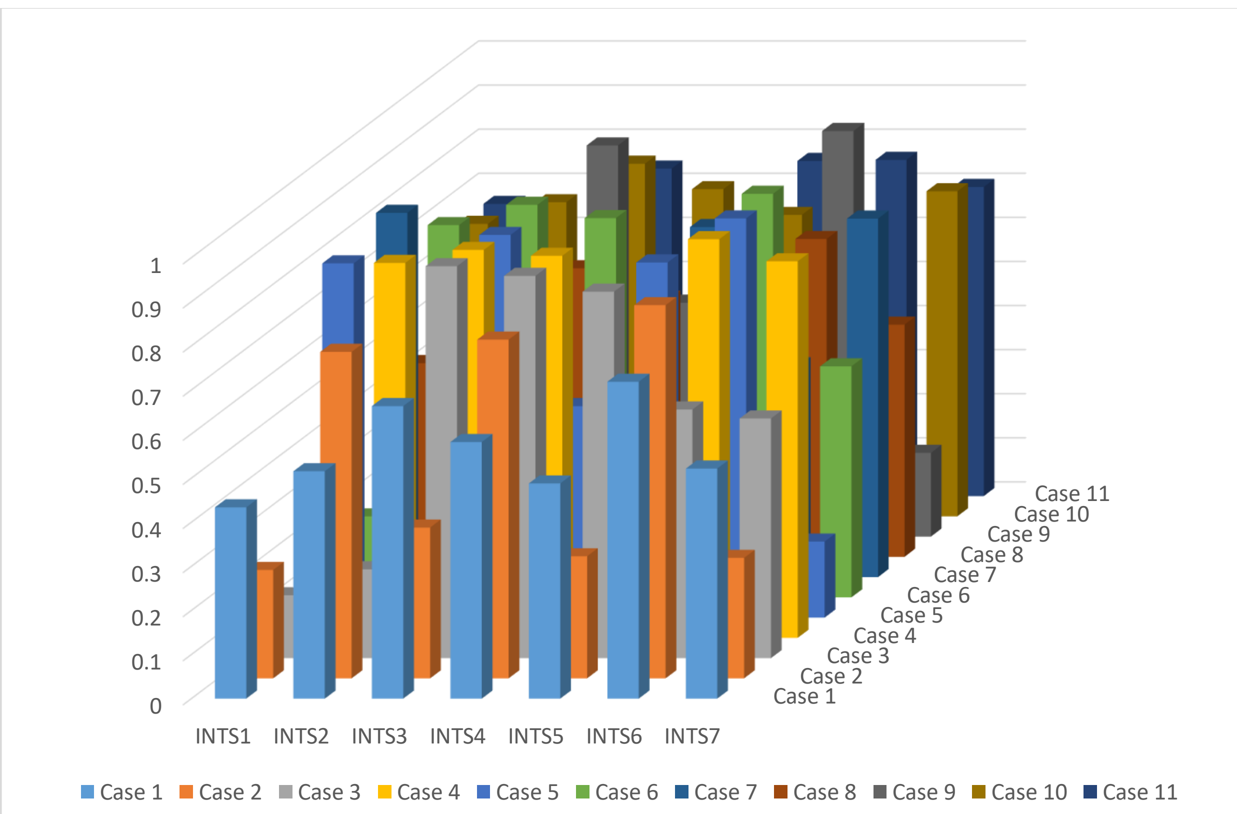


Figure 6. The rank of alternatives by the TOPSIS method under eleven cases.

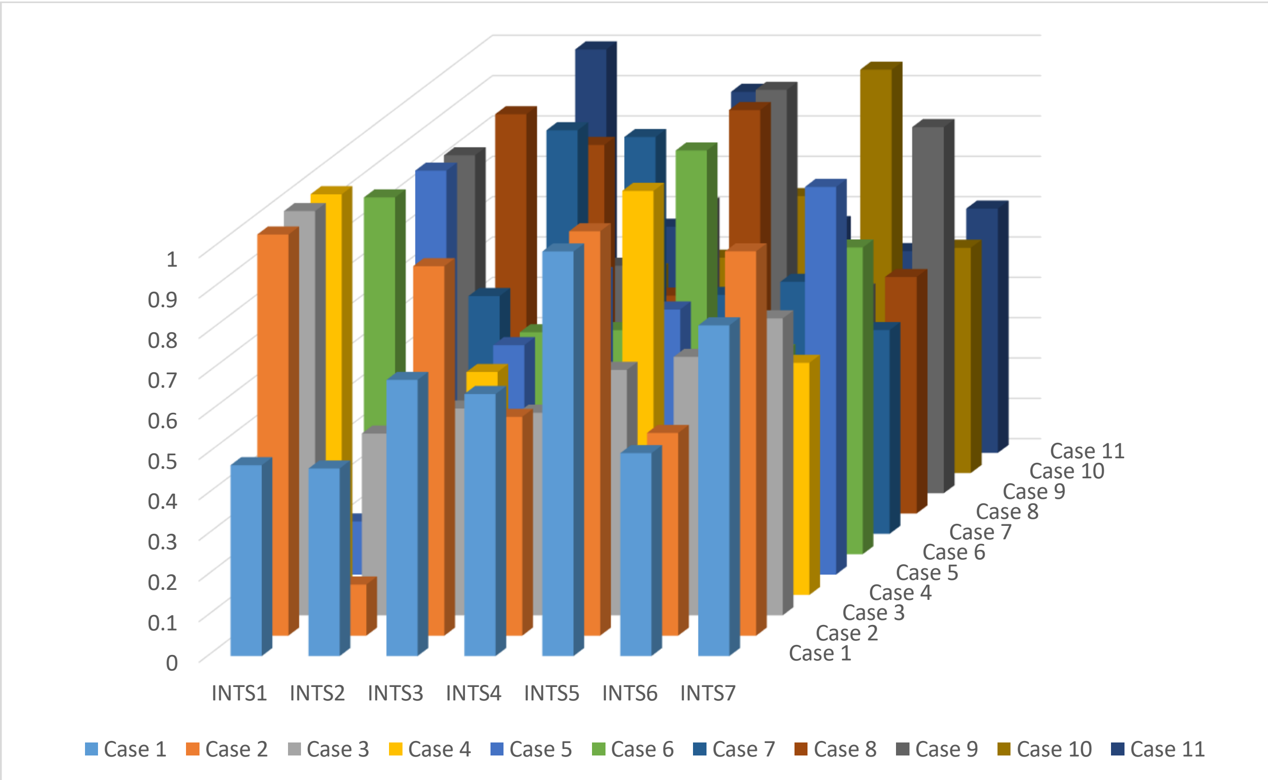


Figure 7. The rank of alternatives by the VIKOR method under eleven cases.

## 4.2 Comparative Analysis

In this part, we compare the rank of alternatives by other MCDM methods like MABAC, CODAS, and MULTIMOORA. We applied these methods to the weights of criteria by the proposed method. This part aims to conclude the correlation between the proposed method and the other MCDM methods. Figure 8 shows the comparative study. We conclude the proposed model is robust compared with other MCDM methods.

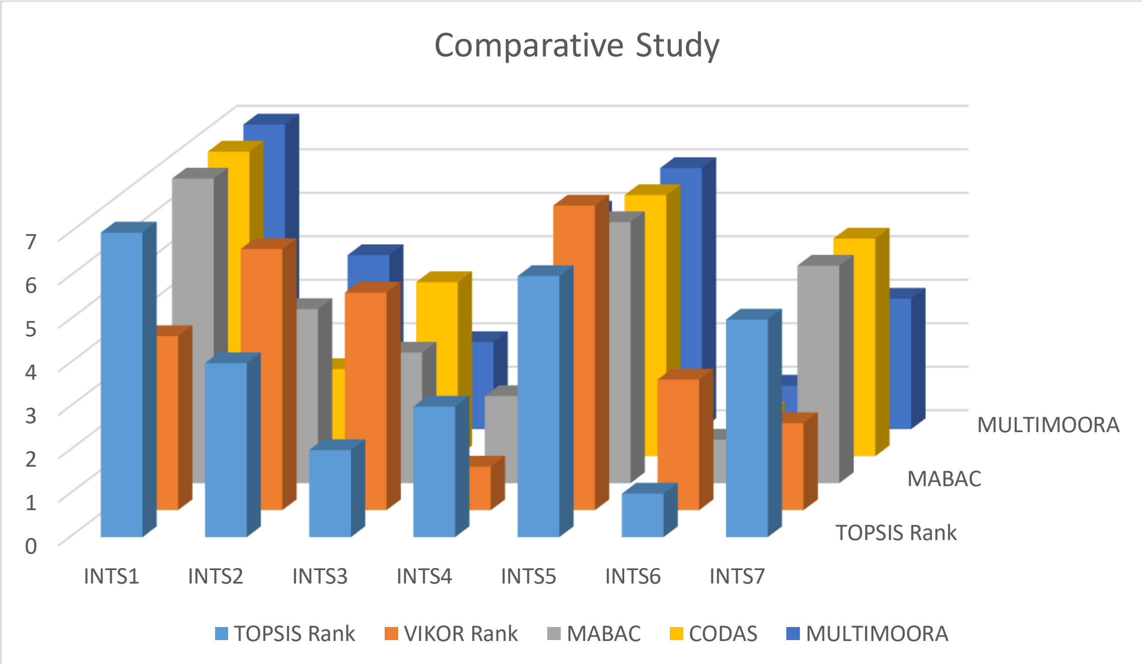


Figure 8. The Comparative study between the proposed method and other MCDM methods.

### 4.3 Managerial Implications

Several management considerations arise when businesses adopt foreign business strategies. To successfully implement international company plans and compete in global marketplaces, managers must consider these ramifications. Managers should be aware of the following consequences of adopting various approaches to doing business abroad:

International Plan for Standardization:

Managers should centralize decision-making to guarantee market-wide uniformity in product development, manufacturing, and sales and distribution.

To achieve economies of scale and cost savings, management should create standardized procedures and systems.

Managers in charge of global branding must create and maintain a unified visual identity for their products or services across all markets.

Approach to Localization:

Managers need to put in the time and effort to learn about the local market to meet the needs of local customers and comply with local regulations.

Managers should delegate authority to divisions so that employees on the ground may make choices and adjust tactics in response to changing market circumstances.

Managers need to set up adaptable production and supply chain systems to account for regional differences in product demand and availability.

*Cross-Border Tactics:*

Managers need to balance the priorities of globalization and localization by considering the unique characteristics of both global and local markets.

Managers should encourage employees at all levels to share their expertise and work together across regions and subsidiaries to make the most of available global resources and skills.

To successfully manage multicultural teams and deal with the inevitable cultural frictions that arise, managers need to cultivate their cross-cultural competence.

*Trade Policy:*

Managers should do extensive market research to determine markets of interest, learn about client wants and requirements, and plot market entrance tactics.

Managers are responsible for ensuring that all export documents and export laws are met to facilitate efficient international commerce.

Managers should set up reliable distribution and logistics networks to back up export efforts and guarantee on-time delivery to consumers.

*Alliances and Joint Ventures:*

Managers are responsible for selecting and overseeing all relationships, taking into account variables including compatibility, shared vision, and mutual advantages. It's important to establish and nurture healthy connections with collaborators.

Managers must enable cultural integration amongst collaborating organizations to harmonize objectives, values, and methods of operation.

Managers should anticipate and address potential risks and disputes in strategic partnerships and joint ventures.

*Investments made from outside the country:*

To maintain seamless operations and reduce legal risks, managers must be familiar with and abide by the host country's laws, regulations, and business practices.

To help FDI operations and close cultural and skill gaps, managers need to do a good job of attracting, developing, and retaining local talent.

Managers need to find a happy medium between globalizing all operations and localizing certain of them to maximize efficiency and success.

*Franchising:*

Managers are responsible for selecting franchisees who share the company's values and have the skills to successfully implement the business plan.

Managers should provide extensive training programs to franchisees so that all customers get the same high-quality goods and services.

Managers have a responsibility to ensure that the brand's values are consistently upheld and enforced throughout all franchised businesses.

*Acquisitions and Mergers (A&M):*

When two or more organizations merge, it may be difficult to integrate their own cultures, values, and work styles.

The key to realizing cost savings, operational efficiency, and growth potential from a merger is for managers to discover and capitalize on synergies between the merging firms.

To keep disruptions to a minimum and integration to a minimum, managers must effectively communicate and manage change throughout the M&A process.

To successfully negotiate the management implications of international company strategy, managers need to be flexible, culturally aware, and equipped with excellent leadership and communication abilities. Also, they need to encourage a global perspective and learn everything they can about the many marketplaces in which they compete.

## 5. Conclusion

The efficacy, difficulties, and prospects for success in the global marketplace may be gained useful insights via the examination and assessment of techniques implemented by multinational corporate administrations. Market entrance, supply chain management, cultural adaptation, strategic partnerships, global marketing, technical innovation, and risk management are just some of the international business strategy topics we've covered. Organizations may better understand their own capabilities, limitations, and development prospects by analyzing these approaches. Market share, brand recognition, customer loyalty, competitive advantage, flexibility, and return on investment are just a few of the metrics that must be taken into account when assessing an international company strategy. These aspects provide a complete picture of the strategies' efficacy and influence in realizing the organization's goals. The study process guarantees a vital analysis that includes real-world insights and experiences by combining a literature review, case studies, interviews, and surveys. This study's conclusions stress the value of planning and executing successful strategies in international companies. Organizations may expand into new markets and take advantage of global possibilities with the help of well-thought-out strategies that allow them to establish powerful brands, navigate cultural differences, form strategic partnerships, adopt new technologies, and mitigate risks. Companies may enhance their worldwide operations and achieve long-term growth goals by analyzing and readjusting their global strategy. This study used the interval-valued neutrosophic set to analyze and evaluate the strategies of international business administration. This study integrated the interval-valued neutrosophic set with the MCDM methods such as TOPSIS and VIKOR methods. These methods are used to rank and select the best strategy. This study used ten factors to be evaluated. Then these factors are used in the analysis of the seven strategies. We obtained that Technological Innovation and Digital Transformation is the best strategy to be applied in organizations and firm.

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# On $S_\lambda$-summability in neutrosophic soft normed linear spaces

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**Abstract**. In present paper, we aim to define $\lambda$-statistical convergence, $\lambda$-statistical Cauchy and $\lambda$-statistical completeness of sequences in neutrosophic soft normed linear spaces (briefly called $NSNLS$). We study certain properties of these notions and provide example to show that $\lambda$-statistical convergence is a more general method of summability in these spaces.

**Keywords:** $\lambda$-statistical convergence, $\lambda$-statistical Cauchy, soft sets, soft normed linear spaces.

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## 1. Introduction

For any non-decreasing sequence $\lambda = (\lambda_n)$ of positive reals with $\lambda_n \to \infty$, $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, the notion of the $\lambda$-statistical convergence was explored by Mursaleen [17] as a generalization of statistical convergence that was initially introduced by Fast [9] and Schoenberg [10] independently.

If we denote $I_n = [n - \lambda_n + 1, n]$, then the $\lambda$-density of any subset $K$ of $\mathbb{N}$ is defined as follows.

"For $K \subseteq \mathbb{N}$, the $\lambda$-density of $K$ is denoted by $\delta_\lambda(K)$ and is defined by

$$\delta_\lambda(K) = \lim_n \frac{1}{\lambda_n} |\{k \in I_n : k \in K\}|$$

provided the limit exists, where the vertical bars denote the cardinality of the enclosed set.

A sequence $x = (x_k)$ is said to be $\lambda$-statistical convergent to $x_0$ if for each $\epsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : |x_k - x_0| \geq \epsilon\}| = 0,$$

i.e., $\delta_\lambda(K_\epsilon) = 0$, where $K_\epsilon = \{k \in I_n : |x_k - x_0| \geq \epsilon\}$. We write, in this case $S_\lambda - \lim_n x_k = x_0$."

Subsequently, statistical convergence and its generalizations have been developed by numerous authors including Hazarika et al.[5], Maddox[11], Fridy[12], Connor[13], Šalát[28], Kumar et al.[32] and many others.

On the other side, many problems of the real world are so complicated due to the uncertainty of data. Therefore, it is very difficult to model these problems mathematically via crisp set theory. So far we have many approaches including, the theory of probability, theory of rough sets[35], theory of fuzzy sets[16], theory of intuitionistic fuzzy sets[15], and theory of neutrosophic sets[7,8] to deal with such situations. In the present study, we are interested in the latter one i.e., the neutrosophic sets, which were initially introduced by Smarandache[7,8] as a generalization of fuzzy sets and intuitionistic fuzzy sets. He used the idea of indeterminacy function along with membership and non-membership functions to define a neutrosophic set. These sets have been further developed by numerous authors in [1], [14], [21], [22], [23], etc.

Kirişçi and Şimşek[18] used neutrosophic logic to define a new kind of norm, called neutrosophic norm and studied statistical convergence in neutrosophic normed linear spaces. Their pioneer work attracted many researchers to work in this direction and nowadays many interesting methods of summability theory have been extended in neutrosophic normed linear spaces. For a wide view in this direction, we refer to the reader [2], [3], [31].

Many approaches discussed above to minimize the uncertainty have their own drawbacks due to the inadequacy of the parametrization. In view of this, Molodtsov[6] proposed a new approach, called soft set theory to reduce the uncertainty during mathematical modelling. These sets turn out very useful tools in many areas of engineering and medical sciences. For instance: Maji et al.[20] applied the theory of soft sets in decision-making problems. Kong et al.[36] presented a heuristic algorithm of normal parameter reduction of soft sets. Zou and Xiao[33] presented a data analysis approach of soft sets under incomplete information. Recently, Yuksel et al.[24] applied soft set theory to diagnose the prostate cancer risk in human beings whereas Çelik and Yamak[34] applied fuzzy soft set theory for medical diagnosis using fuzzy arithmetic operations. Shabir and Naz[19] used soft sets to define soft topological spaces and studied some of their properties. However, Das et al.[25] defined soft normed linear spaces and investigated some of their properties. Recently, Bera and Mahapatra [29] united the concepts of softness and neutrosophic logic to define a generalized norm and called it as neutrosophic soft norm. They also studied some properties of $NSNLS$ and developed fundamental concepts of sequences in these spaces. In present study, we continue to define a more generalized convergence which we called $S_\lambda$-convergence in $NSNLS$. We also introduce the concepts of $S_\lambda$-Cauchy sequence, $S_\lambda$-completeness and develop some of their properties.

## 2. Preliminaries

This section starts with a brief information on soft sets, soft vector spaces and neutrosophic soft normed spaces. We begin with the following notations and definitions.

Throughout this work, $\mathbb{N}$ will denote the set of positive integers, $\mathbb{R}$ the set of reals and $\mathbb{R}^+$ the set of positive real numbers.

**Definition 2.1** [4] A binary operation $\circ : [0,1] \times [0,1] \to [0,1]$ is continuous $t$-norm if $\circ$ satisfies the following conditions:

(i) $x \circ y = y \circ x$ and $x \circ (y \circ z) = (x \circ y) \circ z$.

(ii) $\circ$ is continuous.

(iii) $x \circ 1 = 1 \circ x = x$ for all $x \in [0,1]$.

(iv) $w \circ x \leq y \circ z$ if $w \leq y, x \leq z$ with $w, x, y, z \in [0,1]$.

**Definition 2.2** [4] A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is continuous $t$-conorm($s$-norm) if $\diamond$ satisfies the following conditions:

(i) $x \diamond y = y \diamond x$ and $x \diamond (y \diamond z) = (x \diamond y) \diamond z$.

(ii) $\diamond$ is continuous.

(iii) $x \diamond 0 = 0 \diamond x = x$ for all $x \in [0,1]$.

(iv) $w \diamond x \leq y \diamond z$ if $w \leq y, x \leq z$ with $w, x, y, z \in [0,1]$.

For any universe set $U$ and the set $E$ of the parameters, the soft set is defined as follows:

**Definition 2.3** [6] A pair $(H, E)$ is called a soft set over $U$ if and only if $H$ is a mapping from $E$ into the set of all subsets of the set $U$. i.e., the soft set is a parametrized family of subsets of the set $U$.

Moreover, every set $H(\epsilon), \epsilon \in E$, from this family may be considered as the set of $\epsilon$-elements of the soft set $(H, E)$, or as the set of $\epsilon$-approximate elements of the set.

**Definition 2.4** [6] A soft set $(H, E)$ over $U$ is said to be absolute soft set if for all $\epsilon \in E$, $H(\epsilon) = U$. We will denote it by $\widetilde{U}$.

**Definition 2.5** [26] Let $\mathbb{R}$ be the set of real numbers, $B(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and $E$ taken as a set of parameters. Then a mapping $F : E \to B(\mathbb{R})$ is called a soft real set. If a soft real set is a singleton soft set, then it is called a soft real number and denoted by $\widetilde{r}, \widetilde{s}, \widetilde{t}$, etc. $\widetilde{0}, \widetilde{1}$ are the soft real numbers where $\widetilde{0}\ (e) = 0, \widetilde{1}\ (e) = 1$ for all $e \in E$ respectively.

Let $\mathbb{R}(E)$ and $\mathbb{R}^+(E)$ respectively denote the sets of all soft real numbers and all positive soft real numbers.

**Definition 2.6** [27] Let $(H, E)$ be a soft set over $U$. The set $(H, E)$ is said to be a soft point, denoted by $H_e^u$ if there is exactly one $e \in E$ s.t $H(e) = \{u\}$ for some $u \in U$ and $H(e^{'}) = \phi$ for all $e^{'} \in E - \{e\}$.

Two soft points $H_e^u, H_{e'}^w$ are said to be equal if $e = e'$ and $u = w$. Let $\Delta_{\widetilde{U}}$ denotes the set of all soft points on $\widetilde{U}$.

In case $U$ is a vector space over $\mathbb{R}$ and the parameter set $E = \mathbb{R}$, the soft point is called a soft vector.

Soft vector spaces are used to define soft norm as follows:

**Definition 2.7** [30] Let $\widetilde{U}$ be a absolute soft vector space. Then a mapping $\|\cdot\| : \widetilde{U} \to \mathbb{R}^+(E)$ is said to be a soft norm on $\widetilde{U}$, if $\|\cdot\|$ satisfies the following conditions:

(i) $\|u_e\| \geq \widetilde{0}$ for all $u_e \in \widetilde{U}$ and $\|u_e\| = \widetilde{0} \Leftrightarrow u_e = \widetilde{\theta}_0$ where $\widetilde{\theta}_0$ denotes the zero element of $\widetilde{U}$.

(ii) $\|\widetilde{\alpha}\, u_e\| = |\widetilde{\alpha}|\|u_e\|$ for all $u_e \in \widetilde{U}$ and for every soft scalar $\widetilde{\alpha}$.

(iii) $\|u_e + v_{e'}\| \leq \|u_e\| + \|v_{e'}\|$ for all $u_e, v_{e'} \in \widetilde{U}$.

(iv) $\|u_e \cdot v_{e'}\| = \|u_e\|\, \|v_{e'}\|, \forall\, u_e, v_{e'} \in \widetilde{U}$.

The soft vector space $\widetilde{U}$ with a soft norm $\|\cdot\|$ on $\widetilde{U}$ is said to be a soft normed linear space and is denoted by $(\widetilde{U}, \|\cdot\|)$.

We now recall the definition of neutrosophic soft normed linear spaces and the convergence structure in these spaces.

**Definition 2.8** [29] Let $\widetilde{U}$ be a soft linear space over the field $F$ and $\mathbb{R}(E), \Delta_{\widetilde{U}}$ denote respectively, the set of all soft real numbers and the set of all soft points on $\widetilde{U}$. Then a neutrosophic subset $N$ over $\Delta_{\widetilde{U}} \times \mathbb{R}(E)$ is called a neutrosophic soft norm on $\widetilde{U}$ if for $u_e, v_{e'} \in \widetilde{U}$ and $\widetilde{\alpha} \in F$ ($\widetilde{\alpha}$ being soft scalar), the following conditions hold.

(i) $0 \leq G_N(u_e, \widetilde{\eta_1}), B_N(u_e, \widetilde{\eta_1}), Y_N(u_e, \widetilde{\eta_1}) \leq 1, \forall\, \widetilde{\eta_1} \in \mathbb{R}(E)$.

(ii) $0 \leq G_N(u_e, \widetilde{\eta_1}) + B_N(u_e, \widetilde{\eta_1}) + Y_N(u_e, \widetilde{\eta_1}) \leq 3, \forall\, \widetilde{\eta_1} \in \mathbb{R}(E)$.

(iii) $G_N(u_e, \widetilde{\eta_1}) = 0$ with $\widetilde{\eta_1} \leq \widetilde{0}$ .

(iv) $G_N(u_e, \widetilde{\eta_1}) = 1,$ with $\widetilde{\eta_1} > \widetilde{0}$ if and only if $u_e = \widetilde{\theta}$, the null soft vector.

(v) $G_N(\widetilde{\alpha}\, u_e, \widetilde{\eta_1}) = G_N\left(u_e, \frac{\widetilde{\eta_1}}{|\widetilde{\alpha}|}\right), \forall\, \widetilde{\alpha}\, (\neq \widetilde{0}), \widetilde{\eta_1} > \widetilde{0}$.

(vi) $G_N(u_e, \widetilde{\eta_1}) \circ G_N(v_{e'}, \widetilde{\eta_2}) \leq G_N(u_e \oplus v_{e'}, \widetilde{\eta_1} \oplus \widetilde{\eta_2}), \forall\, \widetilde{\eta_1}, \widetilde{\eta_2} \in \mathbb{R}(E)$

(vii) $G_N(u_e, \cdot)$ is continuous non-decreasing function for $\widetilde{\eta_1} > \widetilde{0}$ and $\lim_{\widetilde{\eta_1} \to \infty} G_N(u_e, \widetilde{\eta_1}) = 1$.

(viii) $B_N(u_e, \widetilde{\eta_1}) = 1$ with $\widetilde{\eta_1} \leq \widetilde{0}$ .

(ix) $B_N(u_e, \widetilde{\eta_1}) = 0,$ with $\widetilde{\eta_1} > \widetilde{0}$ if and only if $u_e = \widetilde{\theta}$, the null soft vector.

(x) $B_N(\widetilde{\alpha}\, u_e, \widetilde{\eta_1}) = B_N\left(u_e, \frac{\widetilde{\eta_1}}{|\widetilde{\alpha}|}\right), \forall\, \widetilde{\alpha}\, (\neq \widetilde{0}), \widetilde{\eta_1} > \widetilde{0}$ .

(xi) $B_N(u_e, \widetilde{\eta_1}) \diamond B_N(v_{e'}, \widetilde{\eta_2}) \geq B_N(u_e \oplus v_{e'}, \widetilde{\eta_1} \oplus \widetilde{\eta_2})\ \forall\, \widetilde{\eta_1}, \widetilde{\eta_2} \in \mathbb{R}(E)$.

(xii) $B_N(u_e, \cdot)$ is continuous non-increasing function for $\widetilde{\eta_1} > \widetilde{0}$ and $\lim_{\widetilde{\eta_1} \to \infty} B_N(u_e, \widetilde{\eta_1}) = 0$.

(xiii) $Y_N(u_e, \widetilde{\eta_1}) = 0$ with $\widetilde{\eta_1} \leq \widetilde{0}$ .

(xiv)$Y_N(u_e, \widetilde{\eta_1}) = 0,$ with $\widetilde{\eta_1} > \widetilde{0}$ if and only if $u_e = \widetilde{\theta}$, the null soft vector.

(xv) $Y_N(\widetilde{\alpha}\, u_e, \widetilde{\eta_1}) = Y_N\left(u_e, \frac{\widetilde{\eta_1}}{|\widetilde{\alpha}|}\right), \forall\, \widetilde{\alpha}\, (\neq \widetilde{0}), \widetilde{\eta_1} > \widetilde{0}$ .

(xvi) $Y_N(u_e, \widetilde{\eta_1}) \diamond Y_N(v_{e'}, \widetilde{\eta_2}) \geq Y_N(u_e \oplus v_{e'}, \widetilde{\eta_1} \oplus \widetilde{\eta_2})\ \forall\ \widetilde{\eta_1}, \widetilde{\eta_2} \in \mathbb{R}(E)$.
(xvii) $Y_N(u_e, \cdot)$ is continuous non-increasing function for $\widetilde{\eta_1} > \widetilde{0}$ and $\lim\limits_{\widetilde{\eta_1}\to\infty} B_N(u_e, \widetilde{\eta_1}) = 0$.

In this case $\mathcal{N} = (G_N, B_N, Y_N)$ is called the neutrosophic soft norm and $(\widetilde{U}(F), G_N, B_N, Y_N, \circ, \diamond)$ is an neutrosophic soft normed linear space ($NSNLS$ briefly).

Let $(\widetilde{U}, \|\cdot\|)$ be a soft normed space. Take the operations $\circ$ and $\diamond$ as $x \circ y = xy$; $x \diamond y = x + y - xy$. For $\widetilde{\eta} > \widetilde{0}$, define

$$G_N(u_e, \widetilde{\eta}) = \begin{cases} \frac{\widetilde{\eta}}{\widetilde{\eta}+\|u_e\|} \text{ if } \widetilde{\eta} > \|u_e\| \\ 0 \text{ otherwise} \end{cases}$$

$$B_N(u_e, \widetilde{\eta}) = \begin{cases} \frac{\|u_e\|}{\widetilde{\eta}+\|u_e\|} \text{ if } \widetilde{\eta} > \|u_e\| \\ 0 \text{ otherwise} \end{cases}$$

$$Y_N(u_e, \widetilde{\eta}) = \begin{cases} \frac{\|u_e\|}{\widetilde{\eta}} \text{ if } \widetilde{\eta} > \|u_e\| \\ 0 \text{ otherwise,} \end{cases}$$

then $(\widetilde{U}(F), G_N, B_N, Y_N, \circ, \diamond)$ is an $NSNLS$. From now onwards, unless otherwise stated by $\widetilde{V}$ we shall denote the $NSNLS$ $(\widetilde{U}(F), G_N, B_N, Y_N, \circ, \diamond)$.

A sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$ is said to be convergent to a soft point $v_e \in \widetilde{V}$ if for $0 < \epsilon < 1$ and $\widetilde{\eta} > \widetilde{0}\ \exists\ n_0 \in \mathbb{N}$ s.t $G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1 - \epsilon,\ B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon,\ Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon$. In this case, we write $\lim\limits_{k\to\infty} v^k_{e_k} = v_e$. A sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$ is said to be Cauchy sequence if for $0 < \epsilon < 1$ and $\widetilde{\eta} > \widetilde{0}\ \exists\ n_0 \in \mathbb{N}$ s.t for all $k, p \geq n_0$ $G_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) > 1 - \epsilon,\ B_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) < \epsilon,\ Y_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) < \epsilon$.

Throughout this paper, $\oplus$ and $\ominus$ denote the sum and difference of soft points respectively.

## 3. $\lambda$-Statistical convergence in NSNLS

In this section, we define $\lambda$-statistical convergence in neutrosophic soft normed linear spaces and develop some of its properties.

**Definition 3.1** A sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$ is said to be $\lambda$-statistical convergent or $S_\lambda$-convergent to a soft point $v_e$ in $\widetilde{V}$ if for each $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left| \left\{ k \in I_n : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon \right\} \right| = 0,$$

i.e., $\delta_\lambda(K) = 0$ where

$$K = \{k \in I_n : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\}.$$

In this case, we write $S_\lambda - \lim_{k\to\infty} v^k_{e_k} = v_e$.

Let, $S_\lambda(G_N, B_N, Y_N)$ denotes the set of all sequences of soft points in $\widetilde{V}$ which are $S_\lambda$-convergent with respect to the neutrosophic soft norm $(G_N, B_N, Y_N)$.

**Remark 3.1** Since the $\lambda$-density of a finite set is zero, therefore, a convergent sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$ is $S_\lambda$-statistical convergent to the same limit. However, the converse may not be true in general.

**Remark 3.2** For particular choice $\lambda_n = n$, $S_\lambda$-convergence coincides with statistical convergence in neutrosophic soft normed linear space.

**Exapmle 3.1** Let $(\widetilde{\mathbb{R}}, \|\cdot\|)$ be a soft normed linear space. For $v_e$ in $\widetilde{\mathbb{R}}$ and $\widetilde{\eta} > \widetilde{0}$, if we define

$$G_N(v_e, \widetilde{\eta}) = \frac{\widetilde{\eta}}{\widetilde{\eta} \oplus \|v_e\|},\ B_N(v_e, \widetilde{\eta}) = \frac{\|v_e\|}{\widetilde{\eta} \oplus \|v_e\|},\ Y_N(v_e, \widetilde{\eta}) = \frac{\|v_e\|}{\widetilde{\eta}},$$

$x \circ y = xy$ and $x \diamond y = \min\{x + y, 1\}$, then it is easy to see that $\widetilde{V} = (\widetilde{\mathbb{R}}, G_N, B_N, Y_N, \circ, \diamond)$ is a neutrosophic soft normed linear space.

Now define a sequence $v = (v^k_{e_k})$ in $\widetilde{V}$ by

$$v^k_{e_k} = \begin{cases} \widetilde{k} & \text{if } n - [\sqrt{\lambda_n}] + 1 \le k \le n, \\ \widetilde{0} & \text{otherwise.} \end{cases}$$

Now, for each $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$, let

$$\begin{aligned} A(\epsilon, \widetilde{\eta}) &= \left\{ k \in I_n : G_N(v^k_{e_k}, \widetilde{\eta}) \le 1 - \epsilon \text{ or } B_N(v^k_{e_k}, \widetilde{\eta}) \ge \epsilon, Y_N(v^k_{e_k}, \widetilde{\eta}) \ge \epsilon \right\} \\ &= \left\{ k \in I_n : \frac{\widetilde{\eta}}{\widetilde{\eta} \oplus \|v^k_{e_k}\|} \le 1 - \epsilon \text{ or } \frac{\|v^k_{e_k}\|}{\widetilde{\eta} \oplus \|v^k_{e_k}\|} \ge \epsilon, \frac{\|v^k_{e_k}\|}{\widetilde{\eta}} \ge \epsilon \right\} \\ &= \left\{ k \in I_n : \|v^k_{e_k}\| \ge \frac{\widetilde{\eta}\,\epsilon}{1-\epsilon} \text{ or } \|v^k_{e_k}\| \ge \widetilde{\eta}\,\epsilon \right\} \\ &= \left\{ k \in I_n : v^k_{e_k} = \widetilde{k} \right\} \\ &= \left\{ k \in I_n : n - [\sqrt{\lambda_n}] + 1 \le k \le n \right\} \end{aligned}$$

and so we get

$$\frac{1}{\lambda_n}|A(\epsilon, \widetilde{\eta})| = \frac{1}{\lambda_n}|\{k \in I_n : n - [\sqrt{\lambda_n}] + 1 \le k \le n\}| \le \frac{\sqrt{\lambda_n}}{\lambda_n}.$$

Taking $n \to \infty$,

$\lim_{n\to\infty} \frac{1}{\lambda_n}|A(\epsilon, \widetilde{\eta})| \le \lim_{n\to\infty} \frac{\sqrt{\lambda_n}}{\lambda_n} = 0$, i.e., $\delta_\lambda(A(\epsilon, \widetilde{\eta})) = 0$.

This shows that, $v = (v^k_{e_k})$ is $\lambda$-statistically convergent to $\widetilde{0}$. But by the structure of the sequence, $v = (v^k_{e_k})$ is not $(G_N, B_N, Y_N)$-convergent to $\widetilde{0}$.

**Lemma 3.1** For any sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$, the following statements are equivalent:

(i) $S_\lambda - \lim_{k\to\infty} v^k_{e_k} = v_e$;

(ii) $\delta_\lambda\{k \in I_n : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1-\epsilon\} = \delta_\lambda\{k \in I_n : B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\} = \delta_\lambda\{k \in I_n : Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\} = 0$;

(iii) $\delta_\lambda\{k \in I_n : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1-\epsilon$ and $B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon\} = 1$;

(iv)$\delta_\lambda\{k \in I_n : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1-\epsilon\} = \delta_\lambda\{k \in I_n : B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon\} = \delta_\lambda\{k \in I_n : Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon\} = 1$;

(v) $S_\lambda - \lim_{k\to\infty} G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) = 1$ and $S_\lambda - \lim_{k\to\infty} B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) = 0$ , $S_\lambda - \lim_{k\to\infty} Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) = 0$.

**Proof**. Omitted.□

**Theorem 3.2** For any sequence $v = (v^k_{e_k})$ in $\widetilde{V}$, if $S_\lambda - \lim_{k\to\infty} v^k_{e_k}$ exists, then it is unique.

**Proof**. Suppose that $S_\lambda - \lim_{n\to\infty} v^k_{e_k} = v_{e_1}$ and $S_\lambda - \lim_{n\to\infty} v^k_{e_k} = v'_{e_2}$, where $v_{e_1} \neq v'_{e_2}$. Let $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$. Choose $\epsilon_1 > 0$ s.t.

$$(1-\epsilon_1) \circ (1-\epsilon_1) > 1-\epsilon \text{ and } \epsilon_1 \Diamond \epsilon_1 < \epsilon \tag{1}$$

Define the following sets:

$$A_{G_N,1}(\epsilon_1, \widetilde{\eta}) = \left\{k \in I_n : G_N\left(v^k_{e_k} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \leq 1-\epsilon_1\right\}.$$

$$A_{G_N,2}(\epsilon_1, \widetilde{\eta}) = \left\{k \in I_n : G_N\left(v^k_{e_k} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \leq 1-\epsilon_1\right\}.$$

$$A_{B_N,1}(\epsilon_1, \widetilde{\eta}) = \left\{k \in I_n : B_N\left(v^k_{e_k} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \geq \epsilon_1\right\}.$$

$$A_{B_N,2}(\epsilon_1, \widetilde{\eta}) = \left\{k \in I_n : B_N\left(v^k_{e_k} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \geq \epsilon_1\right\}.$$

$$A_{Y_N,1}(\epsilon_1, \widetilde{\eta}) = \left\{k \in I_n : Y_N\left(v^k_{e_k} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \geq \epsilon_1\right\}.$$

$$A_{Y_N,2}(\epsilon_1, \widetilde{\eta}) = \left\{k \in I_n : Y_N\left(v^k_{e_k} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \geq \epsilon_1\right\}.$$

Since $S_\lambda - \lim_{k\to\infty} v^k_{e_k} = v_{e_1}$, so
$\delta_\lambda\{A_{G_N,1}(\epsilon_1, \widetilde{\eta})\} = \delta_\lambda\{A_{B_N,1}(\epsilon_1, \widetilde{\eta})\} = \delta_\lambda\{A_{Y_N,1}(\epsilon_1, \widetilde{\eta})\} = 0$ and therefore $\delta_\lambda\{A^C_{G_N,1}(\epsilon_1, \widetilde{\eta})\} = \delta_\lambda\{A^C_{B_N,1}(\epsilon_1, \widetilde{\eta})\} = \delta_\lambda\{A^C_{Y_N,1}(\epsilon_1, \widetilde{\eta})\} = 1$.

Further, $S_\lambda - \lim_{k\to\infty} v^k_{e_k} = v'_{e_2}$, so
$\delta_\lambda\{A_{G_N,2}(\epsilon_1, \widetilde{\eta})\} = \delta_\lambda\{A_{B_N,2}(\epsilon_1, \widetilde{\eta})\} = \delta_\lambda\{A_{Y_N,2}(\epsilon_1, \widetilde{\eta})\} = 0$ and therefore $\delta_\lambda\{A^C_{G_N,2}(\epsilon_1, \widetilde{\eta})\} = \delta_\lambda\{A^C_{B_N,2}(\epsilon_1, \widetilde{\eta})\} = \delta_\lambda\{A^C_{Y_N,2}(\epsilon_1, \widetilde{\eta})\} = 1$ for all $\widetilde{\eta} > \widetilde{0}$. Define

$$\begin{aligned} K_{G_N,B_N,Y_N}(\epsilon, \widetilde{\eta}) = &\{A_{G_N,1}(\epsilon_1, \widetilde{\eta}) \cup A_{G_N,2}(\epsilon_1, \widetilde{\eta})\} \\ &\cap\{A_{B_N,1}(\epsilon_1, \widetilde{\eta}) \cup A_{B_N,2}(\epsilon_1, \widetilde{\eta})\} \cap \{A_{Y_N,1}(\epsilon_1, \widetilde{\eta}) \cup A_{Y_N,2}(\epsilon_1, \widetilde{\eta})\}, \end{aligned}$$

then $\delta_\lambda\{K_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta})\} = 0$ and therefore, $\delta_\lambda\{K^C_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta})\} = 1$. Let $m \in K^C_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta})$, then we have following possibilities.

1. $m \in \left\{A_{G_N,1}(\epsilon_1,\widetilde{\eta}) \cup A_{G_N,2}(\epsilon_1,\widetilde{\eta})\right\}^C$;
2. $m \in \left\{A_{B_N,1}(\epsilon_1,\widetilde{\eta}) \cup A_{B_N,2}(\epsilon_1,\widetilde{\eta})\right\}^C$;
3. $m \in \left\{A_{Y_N,1}(\epsilon_1,\widetilde{\eta}) \cup A_{Y_N,2}(\epsilon_1,\widetilde{\eta})\right\}^C$.

Case 1: Let $m \in \left\{A_{G_N,1}(\epsilon_1,\widetilde{\eta}) \cup A_{G_N,2}(\epsilon_1,\widetilde{\eta})\right\}^C$, then $m \in A^C_{G_N,1}(\epsilon_1,\widetilde{\eta})$ and $m \in A^C_{G_N,2}(\epsilon_1,\widetilde{\eta})$ and therefore,

$$G_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) > 1-\epsilon_1 \text{ and } G_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) > 1-\epsilon_1. \tag{2}$$

Now

$$\begin{aligned} G_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) &= G_N\left(v_{e_1} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &= G_N\left(v^m_{e_m} \ominus v^m_{e_m} \oplus v_{e_1} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\geq G_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \circ G_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \\ &> (1-\epsilon_1) \circ (1-\epsilon_1) \qquad \text{by (2)} \\ &> 1-\epsilon. \qquad \text{by (1)} \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, so we have $G_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) = 1$ for all $\widetilde{\eta} > \widetilde{0}$, which gives $v_{e_1} \ominus v'_{e_2} = \widetilde{\theta}$, i.e., $v_{e_1} = v'_{e_2}$.

Case 2: Let $m \in \left\{A_{B_N,1}(\epsilon_1,\widetilde{\eta}) \cup A_{B_N,2}(\epsilon_1,\widetilde{\eta})\right\}^C$, then $m \in A^C_{B_N,1}(\epsilon_1,\widetilde{\eta}))$ and $m \in A^C_{B_N,2}(\epsilon_1,\widetilde{\eta})$ and therefore,

$$B_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) < \epsilon_1 \text{ and } B_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) < \epsilon_1. \tag{3}$$

Now

$$\begin{aligned} B_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) &= B_N\left(v_{e_1} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &= B_N\left(v^m_{e_m} \ominus v^m_{e_m} \oplus v_{e_1} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\leq B_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \diamond B_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \\ &< \epsilon_1 \diamond \epsilon_1 \qquad \text{by (3)} \\ &< \epsilon. \qquad \text{by (1)} \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, so we have $B_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) = 0$ for all $\widetilde{\eta} > \widetilde{0}$ , which gives $v_{e_1} \ominus v'_{e_2} = \widetilde{\theta}$, i.e., $v_{e_1} = v'_{e_2}$.

Case 3: Let $m \in \left\{A_{Y_N,1}(\epsilon_1, \widetilde{\eta}) \cup A_{Y_N,2}(\epsilon_1, \widetilde{\eta})\right\}^C$ , then $m \in A^C_{Y_N,1}(\epsilon_1, \widetilde{\eta})$ and $m \in A^C_{Y_N,2}(\epsilon_1, \widetilde{\eta})$ and therefore,

$$Y_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) < \epsilon_1 \text{ and } Y_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) < \epsilon_1. \tag{4}$$

Now

$$\begin{aligned} Y_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) &= Y_N\left(v_{e_1} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &= Y_N\left(v^m_{e_m} \ominus v^m_{e_m} \oplus v_{e_1} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\le Y_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \diamond Y_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \\ &< \epsilon_1 \diamond \epsilon_1 \qquad \text{by (4)} \\ &< \epsilon. \qquad \text{by (1)} \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, so we have $Y_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) = 0$ for all $\widetilde{\eta} > \widetilde{0}$ , which gives $v_{e_1} \ominus v'_{e_2} = \widetilde{\theta}$, i.e., $v_{e_1} = v'_{e_2}$.

Hence, in all cases we have $v_{e_1} = v'_{e_2}$, i.e., the $\lambda$-statistical limit of $(v^k_{e_k})$ is unique.□

**Theorem 3.3** Let $u = (u^k_{e_k})$ and $v = (v^k_{e_k})$ be any two sequences in $\widetilde{V}$ s.t $S_\lambda - \lim_{k\to\infty} (u^k_{e_k}) = u_{e_1}$ and $S_\lambda - \lim_{k\to\infty} (v^k_{e_k}) = v_{e_2}$. Then

(i)$S_\lambda - \lim_{k\to\infty} (u^k_{e_k} \oplus v^k_{e_k}) = u_{e_1} \oplus v_{e_2}$

(ii) $S_\lambda - \lim_{k\to\infty} (\widetilde{\alpha}\, u^k_{e_k}) = \widetilde{\alpha}\, u_{e_1}$, where $\widetilde{0} \neq \widetilde{\alpha} \in F$.

**Proof**. Omitted.□

**Theorem 3.4** A sequence $v = (v^k_{e_k})$ in $\widetilde{V}$ is $\lambda$-statistically convergent, if and only if $\exists$ a subset $K = \{k_1, k_2, k_3, ...\}$ of $\mathbb{N}$ s.t $\delta_\lambda(K) = 1$ and $(G_N, B_N, Y_N) - \lim_{\substack{k\in K \\ k\to\infty}} v^k_{e_k} = v_e$.

**Proof**. First suppose that $S_\lambda - \lim_{k\to\infty} v^k_{e_k} = v_e$. For $\widetilde{\eta} > \widetilde{0}$ and $p \in \mathbb{N}$, define the set

$$\begin{aligned} K_{G_N,B_N,Y_N}(p, \widetilde{\eta}) = \Big\{k \in I_n : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1 - \frac{1}{p} \text{ and} \\ B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \frac{1}{p}, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \frac{1}{p}\Big\} \text{ and} \\ K^C_{G_N,B_N,Y_N}(p, \widetilde{\eta}) = \Big\{k \in I_n : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \le 1 - \frac{1}{p} \text{ or} \\ B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \ge \frac{1}{p}, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \ge \frac{1}{p}\Big\}. \end{aligned}$$

Since $S_\lambda - \lim_{k\to\infty} v^k_{e_k} = v_e$, it follows that $\delta_\lambda(K^C_{G_N,B_N,Y_N}(p,\widetilde{\eta})) = 0$. Furthermore, for $\widetilde{\eta} > \widetilde{0}$ and $p \in \mathbb{N}$, we observe $K_{G_N,B_N,Y_N}(p,\widetilde{\eta}) \supset K_{G_N,B_N,Y_N}(p+1,\widetilde{\eta})$ and

$$\delta_\lambda(K_{G_N,B_N,Y_N}(p,\widetilde{\eta})) = 1. \tag{5}$$

Now, we have to show that, for $k \in K_{G_N,B_N,Y_N}(p,\widetilde{\eta})$, $(G_N,B_N,Y_N) - \lim_{\substack{k\in K\\ k\to\infty}} v^k_{e_k} = v_e$. Suppose for $k \in K_{G_N,B_N,Y_N}(p,\widetilde{\eta})$, $(v^k_{e_k})$ is not convergent to $v_e$ w.r.t $(G_N,B_N,Y_N)$. Then $\exists$ some $q > 0$ s.t $\{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) \leq 1-q$ or $B_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) \geq q, Y_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) \geq q\}$ for infinitely many terms of the sequence $v = (v^k_{e_k})$. If we take

$$K_{G_N,B_N,Y_N}(q,\widetilde{\eta}) = \Big\{k \in I_n : G_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) > 1-q \text{ and } B_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) < q, Y_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) < q\Big\}$$

and choose $q > \frac{1}{p}$ where $p \in \mathbb{N}$, then we have $\delta_\lambda(K_{G_N,B_N,Y_N}(q,\widetilde{\eta})) = 0$. Further, $K_{G_N,B_N,Y_N}(p,\widetilde{\eta}) \subset K_{G_N,B_N,Y_N}(q,\widetilde{\eta})$ implies that $\delta_\lambda(K_{G_N,B_N,Y_N}(p,\widetilde{\eta})) = 0$. In this way we obtained a contradiction to (5) as $\delta_\lambda(K_{G_N,B_N,Y_N}(p,\widetilde{\eta})) = 1$. Hence, $(G_N,B_N,Y_N) - \lim_{\substack{k\in K\\ k\to\infty}} v^k_{e_k} = v_e$.
Conversely, Suppose $\exists$ a subset $K = \{k_1,k_2,...,k_j,...\}$ of $\mathbb{N}$ with $\delta_\lambda(K) = 1$ and $(G_N,B_N,Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_e$ over K i.e., $(G_N,B_N,Y_N) - \lim_{\substack{k\in K\\ k\to\infty}} v^k_{e_k} = v_e$. Let $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$, $\exists$ $k_{j_0} \in \mathbb{N}$ s.t for all $k_j \geq k_{j_0}$, $G_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) > 1-\epsilon$ and $B_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) < \epsilon, Y_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) < \epsilon$. So if we consider the set

$$T_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta}) = \Big\{k \in I_n : G_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) \leq 1-\epsilon \text{ or } B_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) \geq \epsilon\Big\},$$

then it is easy to see that $T_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta}) \subset \mathbb{N} - \{k_{j_0},k_{j_0+1},k_{j_0+2},...\}$. This immediately implies that $\delta_\lambda\Big(T_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta})\Big) \leq \delta_\lambda(\mathbb{N}) - \delta_\lambda(\{k_{j_0},k_{j_0+1},k_{j_0+2},...\}) = 1-1 = 0$ and therefore $\delta_\lambda\Big(T_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta})\Big) = 0$ as $\delta_\lambda\Big(T_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta})\Big)$ can not be negative. This shows that $v = (v^k_{e_k})$ is $\lambda$-statistical convergent to $v_e$ i.e., $S_\lambda - \lim_{n\to\infty} v^k_{e_k} = v_e$.$\square$

## 4. $\lambda$-Statistical Cauchy sequence in NSNLS

**Definition 4.1** A sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$ is said to be $\lambda$-statistically Cauchy if for each $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$, $\exists$ $n_0 \in \mathbb{N}$ s.t for all $k,p \geq n_0$

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\left\{k\in I_n : G_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\leq 1-\epsilon \text{ or } B_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon, Y_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon\right\}\right|=0,$$

or equivalently, the $\lambda$-density of the set $K$ is zero, i.e., $\delta_\lambda(K)=0$ where

$$K=\{k\in I_n : G_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\leq 1-\epsilon \text{ or } B_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon, Y_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon\}.$$

**Theorem 4.1** For any sequence $v=(v^k_{e_k})$ in $\widetilde{V}$, the following are equivalent:
(i) $v=(v^k_{e_k})$ is $\lambda$-statistically Cauchy.
(ii) $\exists$ a subset $K=\{k_1,k_2,...,k_j,...\}$ of $\mathbb{N}$ with $\delta_\lambda(K)=1$ and $(v^{k_j}_{e_{k_j}})$ is Cauchy sequence over $K$.

**Proof**. Omitted.□

**Theorem 4.2** Every $\lambda$-statistically convergent sequence of soft points in $\widetilde{V}$ is $\lambda$-statistically Cauchy.

**Proof**. Let $v=(v^k_{e_k})$ be any $\lambda$-statistically convergent sequence with $S_\lambda-\lim\limits_{k\to\infty}v^k_{e_k}=v_e$. Let $\epsilon>0$ and $\widetilde{\eta}>\widetilde{0}$. Choose $\epsilon_1>0$ s.t (1) is satisfied. Define a set,

$$M(\epsilon_1,\widetilde{\eta})=\left\{k\in I_n : G_N(v^k_{e_k}\ominus v_e,\frac{\widetilde{\eta}}{2})\leq 1-\epsilon_1 \text{ or } B_N(v^k_{e_k}\ominus v_e,\frac{\widetilde{\eta}}{2})\geq\epsilon_1, Y_N(v^k_{e_k}\ominus v_e,\frac{\widetilde{\eta}}{2})\geq\epsilon_1\right\},$$

then

$$M^C(\epsilon_1,\widetilde{\eta})=\left\{k\in I_n : G_N(v^k_{e_k}\ominus v_e,\frac{\widetilde{\eta}}{2})> 1-\epsilon_1 \text{ and } B_N(v^k_{e_k}\ominus v_e,\frac{\widetilde{\eta}}{2})<\epsilon_1, Y_N(v^k_{e_k}\ominus v_e,\frac{\widetilde{\eta}}{2})<\epsilon_1\right\}.$$

Since $S_\lambda-\lim\limits_{n\to\infty}v^k_{e_k}=v_e$, so $\delta_\lambda(M(\epsilon_1,\widetilde{\eta}))=0$ and $\delta_\lambda(M^C(\epsilon_1,\widetilde{\eta}))=1$. Let $p\in M^C(\epsilon_1,\widetilde{\eta})$, then

$$G_N\left(v^p_{e_p}\ominus v_e,\frac{\widetilde{\eta}}{2}\right)>1-\epsilon_1 \text{ and } B_N\left(v^p_{e_p}\ominus v_e,\frac{\widetilde{\eta}}{2}\right)<\epsilon_1, Y_N\left(v^p_{e_p}\ominus v_e,\frac{\widetilde{\eta}}{2}\right)<\epsilon_1. \tag{6}$$

Now, let $T(\epsilon,\widetilde{\eta})=\{k\in I_n : G_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\leq 1-\epsilon$ or $B_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon, Y_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon\}$, then we show that $T(\epsilon,\widetilde{\eta})\subseteq M(\epsilon_1,\widetilde{\eta})$. Let $m\in T(\epsilon,\widetilde{\eta})$, then

$$G_N(v^m_{e_m}\ominus v^p_{e_p},\widetilde{\eta})\leq 1-\epsilon \text{ or } B_N(v^m_{e_m}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon, Y_N(v^m_{e_m}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon. \tag{7}$$

Case 1: If $G_N(v^m_{e_m} \ominus v^p_{e_p}, \widetilde{\eta}) \leq 1-\epsilon$, then $G_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \leq 1-\epsilon_1$ and therefore $m \in M(\epsilon_1, \widetilde{\eta})$. As otherwise i.e., if $G_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) > 1-\epsilon_1$, then by (1), (6) and (7) we get

$$\begin{aligned} 1-\epsilon \geq G_N(v^m_{e_m} \ominus v^p_{e_p}, \widetilde{\eta}) &= G_N\left(v^m_{e_m} \ominus v_e \oplus \ v_e \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\geq G_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \circ G_N\left(v^p_{e_p} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ &> (1-\epsilon_1) \circ (1-\epsilon_1) > 1-\epsilon, \end{aligned}$$

which is not possible. Thus, $T(\epsilon, \widetilde{\eta}) \subseteq M(\epsilon_1, \widetilde{\eta})$.
Case 2: If $B_N(v^m_{e_m} \ominus v^p_{e_p}, \widetilde{\eta}) \geq \epsilon$, then $B_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \geq \epsilon_1$ and therefore $m \in M(\epsilon_1, \widetilde{\eta})$. As otherwise i.e., if $B_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) < \epsilon_1$, then by (1), (6) and (7) we get

$$\begin{aligned} \epsilon \leq B_N(v^m_{e_m} \ominus v^p_{e_p}, \widetilde{\eta}) &= B_N\left(v^m_{e_m} \ominus v_e \oplus \ v_e \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\leq B_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \diamond B_N\left(v^p_{e_p} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ &< \epsilon_1 \diamond \epsilon_1 < \epsilon, \end{aligned}$$

which is not possible.
Also, If $Y_N(v^m_{e_m} \ominus v^p_{e_p}, \widetilde{\eta}) \geq \epsilon$, then $Y_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \geq \epsilon_1$ and therefore $m \in M(\epsilon_1, \widetilde{\eta})$. As otherwise i.e., if $Y_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) < \epsilon_1$, then by (1), (6) and (7) we get

$$\begin{aligned} \epsilon \leq Y_N(v^m_{e_m} \ominus v^p_{e_p}, \widetilde{\eta}) &= Y_N\left(v^m_{e_m} \ominus v_e \oplus \ v_e \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\leq Y_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \diamond Y_N\left(v^p_{e_p} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ &< \epsilon_1 \diamond \epsilon_1 < \epsilon, \end{aligned}$$

which is not possible. Thus, $T(\epsilon, \widetilde{\eta}) \subseteq M(\epsilon_1, \widetilde{\eta})$.
Hence in all cases, $T(\epsilon, \widetilde{\eta}) \subseteq M(\epsilon_1, \widetilde{\eta})$. Since $\delta_\lambda(M(\epsilon_1, \widetilde{\eta})) = 0$, so $\delta_\lambda(T(\epsilon, \widetilde{\eta})) = 0$, and therefore $v = (v^k_{e_k})$ is $\lambda$-statistically Cauchy.□

**Example 4.1** Let $R_1 = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $\| \cdot \| = |\cdot|$ i.e., the usual norm on $R_1$, then $(R_1, |\cdot|)$ is a normed linear space. For $\widetilde{\eta} > \widetilde{0}$, if we define $G_N(v_e, \widetilde{\eta}) = \frac{\widetilde{\eta}}{\widetilde{\eta} \oplus \|v_e\|}$; $B_N(v_e, \widetilde{\eta}) = \frac{\|v_e\|}{\widetilde{\eta} \oplus \|v_e\|}$; $Y_N(v_e, \widetilde{\eta}) = \frac{\|v_e\|}{\widetilde{\eta}}$; $x \circ y = xy$ and $x \diamond y = x + y - xy$, then it is easy to see that $(\widetilde{R_1}\ (\mathbb{R}), G_N, B_N, Y_N, \circ, \diamond)$ is a neutrosophic soft normed linear space.
If we define a sequence of soft points $v = (v^k_{e_k})$ by $v^k_{e_k} = \widetilde{\frac{1}{k}}$ and select $\lambda_n = n$ then $(v^k_{e_k})$ is

$\lambda$-statistical Cauchy and $S_\lambda - \lim_{k\to\infty} v_{e_k}^k = \widetilde{0}$ but $\widetilde{0}$ is not a member of the space.

**Theorem 4.3** If $(v_{e_k}^k)$ , $(w_{e_k}^k)$ are $\lambda$-statistical Cauchy sequences of soft vectors and $(\widetilde{\alpha}_k)$ is a $\lambda$-statistical Cauchy sequence of soft scalars in $\widetilde{V}$, then $(v_{e_k}^k \oplus\ w_{e_k}^k)$ and $(\widetilde{\alpha}_k w_{e_k}^k)$ are also $\lambda$-statistical Cauchy in $\widetilde{V}$.

**Proof**. Omitted.□

**Definition 4.2** A $NSNLS$ $\widetilde{V}$ is said to be $\lambda$-statistically complete if every $\lambda$-statistical Cauchy sequence in $\widetilde{V}$ is $\lambda$-statistical convergent w.r.t $(G_N, B_N, Y_N)$.

**Theorem 4.4** If every $\lambda$-statistical Cauchy sequence of soft points in $\widetilde{V}$ has a $\lambda$-statistical convergent subsequence then $\widetilde{V}$ is $\lambda$-statistically complete.

**Proof**. Let $v = (v_{e_k}^k)$ be any $\lambda$-statistically Cauchy sequence of soft points in $\widetilde{V}$ which has a $\lambda$-statistical convergent subsequence $(v_{e_{k(j)}}^{k(j)})$ i.e., $S_\lambda - \lim_{j\to\infty} v_{e_{k(j)}}^{k(j)} = v_e$ for some $v_e$ in $\widetilde{V}$. Let $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$. Choose $\epsilon_1 > 0$ s.t (1) is satisfied. Since $v = (v_{e_k}^k)$ is $\lambda$-statistically Cauchy, so $\exists\ n_0 \in \mathbb{N}$ s.t $\forall\ k, p \geq n_0$, $\delta_\lambda(A) = 0$ where

$$A = \left\{k \in I_n : G_N\left(v_{e_k}^k \ominus v_{e_p}^p, \frac{\widetilde{\eta}}{2}\right) \leq 1 - \epsilon_1 \text{ or } B_N\left(v_{e_k}^k \ominus v_{e_p}^p, \frac{\widetilde{\eta}}{2}\right) \geq \epsilon_1, Y_N\left(v_{e_k}^k \ominus v_{e_p}^p, \frac{\widetilde{\eta}}{2}\right) \geq \epsilon_1\right\}.$$

Again since $S_\lambda - \lim_{j\to\infty} v_{e_{k(j)}}^{k(j)} = v_e$. So we have $\delta_\lambda(B) = 0$, where

$$B = \left\{k(j) \in I_n : G_N\left(v_{e_{k(j)}}^{k(j)} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \leq 1 - \epsilon_1 \text{ or } B_N\left(v_{e_{k(j)}}^{k(j)} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \geq \epsilon_1, Y_N\left(v_{e_{k(j)}}^{k(j)} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \geq \epsilon_1\right\}.$$

Now define

$$D = \{k \in I_n : G_N(v_{e_k}^k \ominus v_e, \widetilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v_{e_k}^k \ominus v_e, \widetilde{\eta}) \geq \epsilon, Y_N(v_{e_k}^k \ominus v_e, \widetilde{\eta}) \geq \epsilon\}.$$

We now show that $A^C \cap B^C \subseteq D^C$. Let $m \in A^C \cap B^C$. As $m \in A^C$, so

$$G_N\left(v_{e_m}^m \ominus\ v_{e_p}^p, \frac{\widetilde{\eta}}{2}\right) > 1 - \epsilon_1 \text{ and } B_N\left(v_{e_m}^m \ominus\ v_{e_p}^p, \frac{\widetilde{\eta}}{2}\right) < \epsilon_1, Y_N\left(v_{e_m}^m \ominus\ v_{e_p}^p, \frac{\widetilde{\eta}}{2}\right) < \epsilon_1, \tag{8}$$

and since $m \in B^C$, so $m = k(j_0)$ for $j_0 \in \mathbb{N}$ and

$$G_N\left(v^{k(j_0)}_{e_{k(j_0)}} \ominus\ v_e, \frac{\widetilde{\eta}}{2}\right) > 1 - \epsilon_1 \text{ and}$$
$$B_N\left(v^{k(j_0)}_{e_{k(j_0)}} \ominus\ v_e, \frac{\widetilde{\eta}}{2}\right) < \epsilon_1, Y_N\left(v^{k(j_0)}_{e_{k(j_0)}} \ominus\ v_e, \frac{\widetilde{\eta}}{2}\right) < \epsilon_1. \tag{9}$$

Now

$$\begin{aligned} G_N(v^m_{e_m} \ominus v_e, \widetilde{\eta}) &= G_N\left(v^m_{e_m} \ominus v^{k(j_0)}_{e_{k(j_0)}} \oplus v^{k(j_0)}_{e_{k(j_0)}} \ominus v_e, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\geq G_N\left(v^m_{e_m} \ominus v^{k(j_0)}_{e_{k(j_0)}}, \frac{\widetilde{\eta}}{2}\right) \circ G_N\left(v^{k(j_0)}_{e_{k(j_0)}} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ &> (1 - \epsilon_1) \circ (1 - \epsilon_1) \quad \text{for } p = k(j_0) \\ &> 1 - \epsilon \end{aligned}$$

and

$$\begin{aligned} B_N(v^m_{e_m} \ominus v_e, \widetilde{\eta}) &= B_N\left(v^m_{e_m} \ominus v^{k(j_0)}_{e_{k(j_0)}} \oplus v^{k(j_0)}_{e_{k(j_0)}} \ominus v_e, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\leq B_N\left(v^m_{e_m} \ominus v^{k(j_0)}_{e_{k(j_0)}}, \frac{\widetilde{\eta}}{2}\right) \diamond B_N\left(v^{k(j_0)}_{e_{k(j_0)}} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ &< \epsilon_1 \diamond \epsilon_1 \quad \text{for } p = k(j_0) \\ &< \epsilon, \end{aligned}$$

$$\begin{aligned} Y_N(v^m_{e_m} \ominus v_e, \widetilde{\eta}) &= Y_N\left(v^m_{e_m} \ominus v^{k(j_0)}_{e_{k(j_0)}} \oplus v^{k(j_0)}_{e_{k(j_0)}} \ominus v_e, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\leq Y_N\left(v^m_{e_m} \ominus v^{k(j_0)}_{e_{k(j_0)}}, \frac{\widetilde{\eta}}{2}\right) \diamond Y_N\left(v^{k(j_0)}_{e_{k(j_0)}} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ &< \epsilon_1 \diamond \epsilon_1 \quad \text{for } p = k(j_0) \\ &< \epsilon, \qquad \text{by (1), (8) and (9)} \end{aligned}$$

which implies that $m \in D^C$, so $A^C \cap B^C \subseteq D^C$ or $D \subseteq A \cup B$. Therefore, $\delta_\lambda(D) \leq \delta_\lambda(A \cup B) = 0$. This shows that $v = (v^k_{e_k})$ is $\lambda$-statistically convergent and therefore, $\widetilde{V}$ is $\lambda$-statistically complete.□

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# Fermatean Neutrosophic Matrices and Their Basic Operations

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**Abstract:** This paper aims to define a special case of neutrosophic matrices referred to as Fermatean neutrosophic matrices (FNMs). FNMs were introduced as generalization of Fermatean fuzzy matrices, intuitionistic fuzzy matrices, and Pythagorean neutrosophic matrices. In FNMs, some properties were discussed in connection with the well-known standard operations (⊕, ⊗, ∧ and ∨). In addition, the scalar multiplication (nA) and exponentiation ($A^n$) of a Fermatean neutrosophic matrix A were proposed, and their basic properties were investigated. Lastly, a new operation denoted by @ was defined on Fermatean neutrosophic matrices, and some of the properties of these matrices were examined.

**Keywords:** Fermatean Neutrosophic Matrices; Pythagorean neutrosophic sets; Fermatean Neutrosophic sets; Pythagorean Neutrosophic Matrices; Scalar multiplication; Exponentiation operations.

## 1. Introduction

The Fuzzy set theory [1] was introduced by Zadeh in 1965. This theory provides a way to deal with vague concepts by assigning a degree of membership to each element of a set. Fuzzy set theory was extensively examined and put together by academics and technology professionals, with broadened usage in fuzzy logic, fuzzy topology, fuzzy control systems, etc. Additionally, theories like fuzzy probability, soft set concepts, and rough set concepts are employed to tackle similar issues. Atanassov's intuitionistic fuzzy sets (IFS) described in [2] are suitable in this circumstance. Only imperfect information taking into account both the truth-membership and falsity-membership values may be handled by intuitionistic fuzzy sets. The ambiguous and contradictory information that is present in belief systems aren't dealt with by it. In 1995, neutrosophic sets are a mathematical notion established by Smarandache [3] that can be used to solve issues involving imperfect, ambiguous, and inconsistent data. Numerous scholars have looked at the applications of Neutrosophic sets and their extensions to ambiguous real-world situations. In the paper authored by Senapati et al. [4], the concept of Fermatean fuzzy sets is introduced, which is characterized by a restriction on the sum of the cubes of membership and non-membership degrees that is not to exceed a value of 1. This characteristic gives FFS a wider range of applicability compared to both IFSs and PFSs. They then produce certain operations for Fermatean Fuzzy Sets. Several studies on Fermatean fuzzy sets were applied in different fields later on. Ganie [5] developed distance and knowledge measurements with Fermatean fuzzy sets. Xu and Shen[6] suggested a technique for Fermatean fuzzy sets to identify patterns that utilizes similarity metrics. Zhou[7] demonstrated a Fermatean Fuzzy ELECTRE Technique for MCDM. Yang[8] illustrates the calculus of Continuities, Derivatives, and Differentials in Fermatean Fuzzy Processes. Barraza [9] included a Fermatean fuzzy matrix which is utilized while collaborating on municipal construction projects. Aydemir [10] and Mishra[11] proposed fermatean fuzzy TOPSIS and WASPAS in MCDM.

By extending Fermatean fuzzy sets, Sweety and R. Jansi [12] presented the concept of Fermatean neutrosophic sets. Fermatean neutrosophic sets are specific types of neutrosophic sets that are used to model uncertainty, indeterminacy, and incomplete information in decision-making processes. These sets have found real applications under uncertainty in decision-making [13-14] and graph theory [15-17, 18].

The theory of fuzzy matrix proposed by Thomason in his paper [19] has been extended and generalized in many ways. To overcome the limitations of fuzzy matrices .The idea of intuitionistic fuzzy matrix (IFM) was proposed by Khan et al. [20, 21] and Im et al. [22] model more complex decision-making problems than fuzzy matrix. In such a situation, IFM fails to produce a reasonable solution. To address this situation, in 2018, Silambarasan and Sriram [23] designed Pythagorean fuzzy matrices (PFMs) and studied their algebraic operations. Recently, major contributions on the extension of IFMs have been published ( see Table 1).

**Table 1.** Review on the Extensions of Intuitionistic fuzzy matrices.

| References | Extensions of Intuitionistic fuzzy matrices | Year |
|---|---|---|
| [24] | Single valued neutrosophic matrices | 2018 |
| [25] | Spherical fuzzy matrices | 2020 |
| [26] | Picture fuzzy matrices | 2020 |
| [27] | T-Spherical Fuzzy Matrices | 2022 |
| [28] | Multi-valued neutrosophic fuzzy matrix (MVNFM) | 2022 |
| [29] | Fermatean fuzzy matrices | 2022 |
| proposed | Fermatean neutrosophic matrices | 2023 |

Based on literature review that reflects no research has been carried out on Fermatean neutrosophic matrix and to merge this gap, we established a new special class of neutrosophic matrices.

These following concepts constitute the components of this paper. Basic concepts of existing work have been given in introduction. Some fundamental definitions of IFMs and PFMs are presented in the Preliminaries section. In third section, Fermatean neutrosophic matrices and their fundamental operations are described. Also a new operation (@) on Fermatean neutrosophic matrices is presented and some algebraic characteristics are discussed. In the final segment, the work is concluded.

## 2. Preliminaries

This section of the article presents some fundamental ideas concerning the Pythagorean fuzzy matrix (PyFM), intuitionistic fuzzy matrix (IFM), Fermatean fuzzy matrices (FFM), Pythagorean neutrosophic sets and Fermatean neutrosophic sets.

**Definition 2.1 [20]**

The definition of an intuitionistic fuzzy matrix (IFM) $\mathcal{R}$ with dimensions $m \times n$ is given by.

$$\mathcal{R} = \left[X_{ij}, \langle T_{ijp} \quad , F_{ijp} \rangle\right]_{m\times n}$$

Where $T_{ijp}, F_{ijp} \in [0,1]$ are referred to as truth, and falsity of in $\mathcal{R}$, which maintaining the condition $0 \leq T_{ijp} + F_{ijp} \leq 1$, to simplify matters, we express it as R=$\left[X_{ij}, \mathcal{R}_{ij}\right]_{m\times n}$ or $\left[\mathcal{R}_{ij}\right]_{m\times n}$ where

$$\mathcal{R}_{ij}=\langle T_{ijp}, F_{ijp}\rangle$$

**Example 2.1.** Let $\mathcal{R}$ be a $2\times 2$ IFM.

$\mathcal{R}=\begin{bmatrix}(0.2,\ 0.3) & (0.5,\ 0.1)\\(0.3, 0.5) & (0.6,\ 0.2)\end{bmatrix}$ is not a FM, but $\mathcal{R}$ is a IFM

**Definition 2.2 [23]**

The definition of Pythagorean fuzzy matrix (PyFM) $\mathcal{R}$ with dimensions $m\times n$ is given by.

$$\mathcal{R}=[X_{ij}, \langle T_{ij\mathcal{R}}, F_{ij\mathcal{R}}\rangle]_{m\times n}$$

Where $T_{ij\mathcal{R}}, F_{ij\mathcal{R}}\in[0,1]$ are referred to as the degrees of the truth and the falsity of in $\mathcal{R}$, which maintaining the condition $0\le T_{ij\mathcal{R}}{}^2+\ F_{ij\mathcal{R}}{}^2\le 2$, to simplify matters, we express it as R=$[X_{ij}, \mathcal{R}_{ij}]_{m\times n}$ or $[\mathcal{R}_{ij}]_{m\times n}$ where

$$\mathcal{R}_{ij}=\langle T_{ij\mathcal{R}},, F_{ij\mathcal{R}}\rangle$$

**Example 2.1**. Let $\mathcal{R}$ be a $2\times 2$ PyFM

$\mathcal{R}=\begin{bmatrix}(0.5, 0.3) & (0.3,\ 0.6)\\(0.7, 0.3) & [(0.5,\ 0.2)\end{bmatrix}$ is not a IFM, but $\mathcal{R}$ is a PyFM.

**Definition 2.3 [29]**

The definition of Fermatean fuzzy matrix (FFM) $\mathcal{R}$ with dimensions $m\times n$ is given by.

$$\mathcal{R}=[X_{ij}, \langle T_{ij\mathcal{R}}\ \ , F_{ij\mathcal{R}}\rangle]_{m\times n}$$

Where $T_{ij\mathcal{R}}, F_{ij\mathcal{R}}\in[0,1]$ are referred to as truth, and falsity of in $\mathcal{R}$, which maintaining the condition $0\le T_{ij\mathcal{R}}{}^3\ +\ F_{ij\mathcal{R}}{}^3\le 2$, to simplify matters, we express it as R=$[X_{ij}, \mathcal{R}_{ij}]_{m\times n}$ or $[\mathcal{R}_{ij}]_{m\times n}$ where

$$\mathcal{R}_{ij}=\langle T_{ij\mathcal{R}}, F_{ij\mathcal{R}}\rangle$$

**Example 2.3**. Let $\mathcal{R}$ be a $2\times 2$ FFM.

$\mathcal{R}=\begin{bmatrix}(0.7,\ 0.7) & (0.3,\ 0.1)\\(0.7,\ 0.8) & (0.5,\ 0.2)\end{bmatrix}$ is not a IFM and not a PyFM , but $\mathcal{R}$ is a FFM

**Definition 2.4 [30]**

The concept of Pythagorean neutrosophic sets (PyN sets) $\tilde{P}$ on $\mathfrak{U}$ is an object that can be expressed as

$\tilde{P}=\{(\ \mathfrak{r},\ \ T_{\tilde{P}}(\mathfrak{r}),\ I_{\tilde{P}}(\mathfrak{r}),\ F_{\tilde{P}}(\mathfrak{r})):\ \mathfrak{r}\in\mathfrak{U}\}$, where $ó_{\tilde{P}}\ (\mathfrak{r})\in[0,1]$ represents the membership degree of $\mathfrak{r}$ in $\mathfrak{U}$, $I_{\tilde{P}}(\mathfrak{r})\in[0,1]$ is the indeterminacy degree of $\mathfrak{r}$ in $\mathfrak{U}$ and $F_{\tilde{P}}(\mathfrak{r})\in[0,1]$ denotes the non-membership degree of $\mathfrak{r}$ in $\mathfrak{U}$, and these three are satisfying the relation;

$$0\le (T_{\tilde{P}}(\mathfrak{r}))\ ^2+\ (I_{\tilde{P}}(\mathfrak{r}))\ ^2+\ (F_{\tilde{P}}(\mathfrak{r}))\ ^2\ \le 2$$

Here $T_{\tilde{P}}(\mathfrak{r})$ and $I_{\tilde{P}}(\mathfrak{r})$ are dependent component and $F_{\tilde{P}}(\mathfrak{r})$ is an independent component.

**Definition 2.5 [12]**

The concept of Fermatean neutrosophic sets (FN sets) $\tilde{N}$ on $\mathfrak{U}$ is an object that can be represented mathematically by

$\tilde{N}$ = {( $\mathfrak{r}$, $T_{\tilde{N}}(\mathfrak{r})$, $I_{\tilde{N}}(\mathfrak{r})$, $F_{\tilde{N}}(\mathfrak{r})$): $\mathfrak{r} \in \mathfrak{U}$}, where $T_{\tilde{N}}(\mathfrak{r}) \in [0,1]$ represents the membership degree of $\mathfrak{r}$ in $\mathfrak{U}$, $I_{\tilde{N}}(\mathfrak{r}) \in [0,1]$ is the indeterminacy degree of $\mathfrak{r}$ in $\mathfrak{U}$ and $F_{\tilde{N}}(\mathfrak{r}) \in [0,1]$ denotes the non-membership degree of $\mathfrak{r}$ in $\mathfrak{U}$, and these three are satisfying the relation;

$$0 \leq (T(\mathfrak{r}))^3 + (F_{\tilde{N}}(\mathfrak{r}))^3 \leq 1 \text{ and } 0 \leq (I_{\tilde{N}}(\mathfrak{r}))^3 \leq 1;$$

Then, $$0 \leq (T(\mathfrak{r}))^3 + (I_{\tilde{N}}(\mathfrak{r}))^3 + (F_{\tilde{N}}(\mathfrak{r}))^3 \leq 2$$

Here $T_{\tilde{N}}(\mathfrak{r})$ and $F_{\tilde{N}}(\mathfrak{r})$ are dependent component and $I_{\tilde{N}}(\mathfrak{r})$ is an independent component.

**Definition 2.6 [12]**

Let's suppose that the universe of discourse is termed by $\mathfrak{A}$ and the unit interval [0,1]. Let K and L two Fermatean neutrosophic sets expressed mathematically by

$K$ = {( $\mathfrak{r}$, $T_{\tilde{P}}(\mathfrak{r})$, $I_{\tilde{P}}(\mathfrak{r})$, $F_{\tilde{P}}(\mathfrak{r})$)/ $\mathfrak{r} \in \mathfrak{A}$ } and L = {( $\mathfrak{r}$, $T_{\tilde{N}}(\mathfrak{r})$, $I_{\tilde{N}}(\mathfrak{r})$, $F_{\tilde{N}}(\mathfrak{r})$/$\mathfrak{r} \in \mathfrak{A}$ }. Then

a. $K^c$ = {( $\mathfrak{r}$, $F_{\tilde{P}}(\mathfrak{r})$, $1 - I_{\tilde{P}}(\mathfrak{r})$, $T_{\tilde{P}}(\mathfrak{r})$): $\mathfrak{r} \in \mathfrak{A}$ }

b. K∪ L ={( $\mathfrak{r}$, max($T_{\tilde{P}}(\mathfrak{r})$, $T_{\tilde{N}}(\mathfrak{r})$), min($I_{\tilde{P}}(\mathfrak{r})$, $I_{\tilde{N}}(\mathfrak{r})$), min ($F_{\tilde{P}}(\mathfrak{r})$, $F_{\tilde{N}}(\mathfrak{r})$): $\mathfrak{r} \in \mathfrak{A}$ }

c. K ∩ L ={( $\mathfrak{r}$, min($T_{\tilde{P}}(\mathfrak{r})$, $T_{\tilde{N}}(\mathfrak{r})$), max($I_{\tilde{P}}(\mathfrak{r})$, $I_{\tilde{N}}(\mathfrak{r})$), max($F_{\tilde{P}}(\mathfrak{r})$, $F_{\tilde{N}}(\mathfrak{r})$): $\mathfrak{r} \in \mathfrak{A}$ }

## 3. Fermatean Neutrosophic Matrices

Before introducing the concept of Fermatean neutrosophic matrices, we briefly presented the definition of Pythagorean neutrosophic matrix (PyNM) which will be used in this section

**Definition 3.1: Pythagorean neutrosophic matrix**

The definition of Pythagorean neutrosophic matrix (PyNM) $\mathcal{R}$ with dimensions $m \times n$ is given by.

$$\mathcal{R} = \left[X_{ij}, \langle T_{ij\mathcal{R}}, I_{ij\mathcal{R}}, F_{ij\mathcal{R}} \rangle\right]_{m\times n}$$

Where $T_{ij\mathcal{R}}, I_{ij\mathcal{R}}, F_{ij\mathcal{R}} \in [0,1]$ are referred to as the degrees of the truth, the indeterminacy, and the falsity of in $\mathcal{R}$, which maintaining the condition $0 \leq {T_{ij\mathcal{R}}}^2 + {I_{ij\mathcal{R}}}^2 + {F_{ij\mathcal{R}}}^2 \leq 2$, to simplify matters, we express it as R=$\left[X_{ij}, \mathcal{R}_{ij}\right]_{m\times n}$ or $\left[\mathcal{R}_{ij}\right]_{m\times n}$ where

$$\mathcal{R}_{ij} = \langle T_{ij\mathcal{R}}, I_{ij\mathcal{R}}, F_{ij\mathcal{R}} \rangle$$

**Example 3.1**. Let $\mathcal{R}$ be a $2 \times 2$ PyNM

$\mathcal{R} = \begin{bmatrix} (0.5, 0.7, 0.3) & (0.3, 0.4, 0.6) \\ (0.7, 0.4, 0.3) & (0.5, 0.4, 0.2) \end{bmatrix}$ is not a IFM, but $\mathcal{R}$ is a PyNM.

Each element in an PyNM is expressed by an ordered pair $T_{\tilde{P}}$ (ɼ), $I_{\tilde{P}}$ (ɼ), $F_{\tilde{P}}$ (ɼ) with $T_{\tilde{P}}$ (ɼ), $I_{\tilde{P}}$ (ɼ) and $F_{\tilde{P}}$ (ɼ)∈ [0, 1] and 0 ≤ $T_{\tilde{P}}$ (ɼ)$^2$+ $I_{\tilde{P}}$ (ɼ)$^2$+ $F_{\tilde{P}}$ (ɼ)$^2$≤ 2 and 0 ≤ $T_{\tilde{P}}$ (ɼ)$^2$+ $F_{\tilde{P}}$ (ɼ)$^2$≤ 1. It became obvious to observe this $(0.7)^2 + (0.7)^2 + (0.8)^2$= = 1.62 ≤ 2 and $(0.7)^2 + (0.8)^2$ =1.13 > 1, and therefore, Its description was beyond the scope of PyNM. In this study, the Fermatean neutrosophic matrix (FNM) and related algebraic operations to characterize an assessment of this sort are provided. A pair of ordered elements can be utilized to represent each element in a FNM, $T_{\tilde{P}}$ (ɼ), $I_{\tilde{P}}$ (ɼ), $F_{\tilde{P}}$ (ɼ) with $T_{\tilde{P}}$ (ɼ), $I_{\tilde{P}}$ (ɼ), $F_{\tilde{P}}$ (ɼ) ∈ [0, 1] and 0 ≤ $T_{\tilde{P}}$ (ɼ)$^3$+ $I_{\tilde{P}}$ (ɼ)$^3$+ $F_{\tilde{P}}$ (ɼ)$^3$≤ 2 and 0 ≤ $T_{\tilde{P}}$ (ɼ)$^3$+ $F_{\tilde{P}}$ (ɼ)$^3$≤ 1.Also, we can get $(0.7)^3 + (0.8)^3$ =0.855 < 1 and $(0.7)^3 + (0.7)^3 + (0.8)^3$= 1.198 ≤ 2, which enables the FNM to be utilized for managing it.

By limiting the measure of truth , indeterminacy and falsity membership but preserving their total in the range [0, $\sqrt[2]{2}$], Fermatean neutrosophic matrices algebraic operations are provided in.

**Definition 3.2:**

The definition of Fermatean neutrosophic matrix (FNM) $\mathcal{R}$ with dimensions $m \times n$ is given by.

$$\mathcal{R} = \left[X_{ij}, \langle T_{ij\mathcal{R}}, I_{ij\mathcal{R}}, F_{ij\mathcal{R}} \rangle\right]_{m\times n}$$

Where $T_{ij\mathcal{R}}, I_{ij\mathcal{R}}, F_{ij\mathcal{R}} \in [0,1]$ are referred to as the degrees of the truth, the indeterminacy, and the falsity of in $\mathcal{R}$, which maintaining the condition 0 ≤ $T_{ij\mathcal{R}}3$ + $F_{ij\mathcal{R}}3$ + $F_{ij\mathcal{R}}3$ ≤ 2, to simplify matters, we express it as R=$\left[X_{ij}, \mathcal{R}_{ij}\right]_{m\times n}$ or $\left[\mathcal{R}_{ij}\right]_{m\times n}$ where

$$\mathcal{R}_{ij} = \langle T_{ij\mathcal{R}}, I_{ij\mathcal{R}}, F_{ij\mathcal{R}} \rangle$$

**Example 3.2.** Let $\mathcal{R}$ be a $2 \times 2$ FNM.

$\mathcal{R} = \begin{bmatrix} (0.7, 0.7, 0.7) & (0.3, 0.4, 0.1) \\ (0.3, 0.4, 0.2) & [(0.5, 0.4, 0.2) \end{bmatrix}$ is not a FFM and not , but $\mathcal{R}$ is a FNM.

In Fermatean neutrosophic matrix FNMs the cube sum of the triplet of membership, non-membership, and indeterminacy of the Fermatean neutrosophic element is is less than or equal to 2 , while in PyNMs the square sum of these triplet is bounded by 2

On the other hand, if we drop the indeterminacy degree from each triplet of a Fermatean neutrosophic matrix in example 3.2 then FNM, is reduced to Fermatean fuzzy matrix

**Definition 3.3.**

Suppose $\mathbb{N}$ and $\wp$ are two Fermatean neutrosophic matrices, then

- $\mathbb{N} < \wp$ iff $\forall$ i,j$\in \mathfrak{U}$, $\tilde{T}_{a_{ij}} \leq \tilde{T}_{b_{ij}}$, $\tilde{I}_{a_{ij}} \leq \tilde{I}_{b_{ij}}$ or $\tilde{I}_{a_{ij}} \geq \tilde{I}_{b_{ij}}$, $\tilde{F}_{a_{ij}} \geq \tilde{F}_{b_{ij}}$:
- $\mathbb{N}^C = \left(\langle \tilde{F}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{T}_{a_{ij}} \rangle\right)_{mxn}$
- $\mathbb{N} \vee \wp = \left(\langle \max(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}), \min(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}) \ , \min \ (\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}) \rangle\right)_{mxn}$
- $\mathbb{N} \wedge \wp) = \left(\langle \min(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}), \max(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \max(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}) \rangle\right)_{mxn}$
- $\mathbb{N} \oplus \wp = \left(\langle \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}, \tilde{I}_{a_{ij}} \ \tilde{I}_{b_{ij}} \ , \tilde{F}_{a_{ij}} \ \tilde{F}_{b_{ij}} \rangle\right)_{mxn}$
- $\mathbb{N} \otimes \wp = \left(\langle \tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}, \sqrt[3]{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{b_{ij}} - \tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\tilde{F}^3_{a_{ij}} + \tilde{F}^3_{b_{ij}} - \tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}} \rangle\right)_{mxn}$

**Definition 3.4**

A scalar multiplication operation on FNM, n $\mathbb{N}$ and is defined as follow:

$$n \ \mathbb{N} = \langle \left( \sqrt[3]{1 - [1 - \tilde{T}^3_{a_{ij}}]^n}, \left[\tilde{I}_{a_{ij}}\right]^n, \left[\tilde{F}_{a_{ij}}\right]^n \right) \rangle_{mxn}$$

**Definition 3.5**

An exponentiation operation on FNM, $\mathbb{N}$ and is defined as follow:

$$\mathbb{N}^n = \langle \left( \left[\tilde{T}_{a_{ij}}\right]^n, \sqrt[3]{1 - [1 - \tilde{I}^3_{a_{ij}}]^n}, \sqrt[3]{1 - [1 - \tilde{F}^3_{a_{ij}}]^n} \right) \rangle_{mxn}$$

Let $FN_{mxn}$ be the set of all the Fermatean neutrosophic matrices.

The following theorem relation between algebraic sum, and algebraic product of FNMs.

**Theorem 3.1**.If $\mathbb{N}$, $\wp \in FN_{mxn}$, then $\mathbb{N} \otimes \wp \leq \mathbb{N} \oplus \wp$.

Proof

Let $\mathbb{N} \oplus \wp = \left(\langle \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}, \tilde{I}_{a_{ij}} \ \tilde{I}_{b_{ij}} \ , \tilde{F}_{a_{ij}} \ \tilde{F}_{b_{ij}} \rangle\right)_{mxn}$

$$\mathbb{N} \otimes \wp = \left(\langle \tilde{T}_{a_{ij}} \ \tilde{T}_{b_{ij}} \ , \sqrt[3]{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{b_{ij}} - \tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\tilde{F}^3_{a_{ij}} + \tilde{F}^3_{b_{ij}} - \tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}} \rangle\right)_{mxn}$$

Suppose that

$\tilde{T}_{a_{ij}} \ \tilde{T}_{b_{ij}} \leq \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}$

Cubing both sides

$\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}} \leq \tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}$

$\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}} - \tilde{T}^3_{b_{ij}} + \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}} \leq 0$

$\tilde{T}^3_{a_{ij}}\left(\tilde{T}^3_{b_{ij}} - 1\right) - \tilde{T}^3_{b_{ij}}\left(1 - \tilde{T}^3_{a_{ij}}\right) \leq 0$

$-\tilde{T}^3_{a_{ij}}\left(1 - \tilde{T}^3_{b_{ij}}\right) - \tilde{T}^3_{b_{ij}}\left(1 - \tilde{T}^3_{a_{ij}}\right) \leq 0$

$\tilde{T}^3_{a_{ij}}\left(1 - \tilde{T}^3_{b_{ij}}\right) + \tilde{T}^3_{b_{ij}}\left(1 - \tilde{T}^3_{a_{ij}}\right) \geq 0$

It is clear that $0 \leq \tilde{T}^3_{a_{ij}} \leq 1$ and $0 \leq \tilde{T}^3_{b_{ij}} \leq 1$

And

$\tilde{I}_{a_{ij}} \ \tilde{I}_{b_{ij}} \leq \sqrt[3]{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{b_{ij}} - \tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}$

Ie, $\tilde{I}_{a_{ij}} \ \tilde{I}_{b_{ij}} - \sqrt[3]{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{b_{ij}} - \tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}} \geq 0$

Ie,$\tilde{I}^3_{a_{ij}}\left(1 - \tilde{I}^3_{b_{ij}}\right) + \tilde{I}^3_{b_{ij}}(1 - \tilde{I}^3_{a_{ij}}) \geq 0$

It is clear that $0 \leq \tilde{I}^3_{a_{ij}} \leq 1$ and $0 \leq \tilde{I}^3_{b_{ij}} \leq 1$

And

$\tilde{F}_{a_{ij}} \ \tilde{F}_{b_{ij}} \leq \sqrt[3]{\tilde{F}^3_{a_{ij}} + \tilde{F}^3_{b_{ij}} - \tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}}$

Ie, $\tilde{F}_{a_{ij}} \ \tilde{F}_{b_{ij}} - \sqrt[3]{\tilde{F}^3_{a_{ij}} + \tilde{F}^3_{b_{ij}} - \tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}} \geq 0$

Ie,$\tilde{F}^3_{a_{ij}}\left(1 - \tilde{F}^3_{b_{ij}}\right) + \tilde{F}^3_{b_{ij}}(1 - \tilde{F}^3_{a_{ij}}) \geq 0$

It is clear that $0 \leq \tilde{F}^3_{a_{ij}} \leq 1$ and $0 \leq \tilde{F}^3_{b_{ij}} \leq 1$

Hence, $\mathbb{N} \otimes \wp \leq \mathbb{N} \oplus \wp$.

**Theorem 3.2**. For any Fermatean neutrosophic matrix $\mathbb{N}$,

(i) $\mathbb{N} \oplus \mathbb{N} \geq \mathbb{N}$

(ii) $\mathbb{N} \otimes \mathbb{N} \leq \mathbb{N}$.

Proof.

(i) Let $\mathbb{N} \oplus \mathbb{N} = (\langle \tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}} \rangle) \oplus (\langle \tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}} \rangle)$

$$\mathbb{N} \oplus \mathbb{N} = \left(\langle \sqrt[3]{2\ \tilde{T}^3_{a_{ij}} - \left(\tilde{T}^3_{a_{ij}}\right)^2}, \left(\tilde{I}_{a_{ij}}\right)^2, \left(\tilde{F}_{a_{ij}}\right)^2 \rangle\right)_{mxn}$$

$$\sqrt[3]{2\ \tilde{T}^3_{a_{ij}} - \left(\tilde{T}^3_{a_{ij}}\right)^2} = \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{a_{ij}}\left(1 - \tilde{T}^3_{a_{ij}}\right)} \geq \tilde{T}^3_{a_{ij}}, \forall\ i,j \in \mathfrak{A}$$

And $\left(\tilde{I}_{a_{ij}}\right)^2 \leq \tilde{I}_{a_{ij}},$

And $\left(\tilde{F}_{a_{ij}}\right)^2 \leq \tilde{F}_{a_{ij}},$

Hence $\mathbb{N} \oplus \mathbb{N} \geq \mathbb{N}$

(ii) Let $\mathbb{N} \otimes \mathbb{N} = (\langle \tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}} \rangle) \otimes (\langle \tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}} \rangle)$

$$\mathbb{N} \otimes \mathbb{N} = \left(\langle \left(\tilde{T}_{a_{ij}}\right)^2, \sqrt[3]{2\ \tilde{I}^3_{a_{ij}} - \left(\tilde{I}^3_{a_{ij}}\right)^2}, \sqrt[3]{2\ \tilde{F}^3_{a_{ij}} - \left(\tilde{F}^3_{a_{ij}}\right)^2} \rangle\right)_{mxn}$$

$$\sqrt[3]{2\ \tilde{I}^3_{a_{ij}} - \left(\tilde{I}^3_{a_{ij}}\right)^2} = \sqrt[3]{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{a_{ij}}\left(1 - \tilde{I}^3_{a_{ij}}\right)} \leq \tilde{I}^3_{a_{ij}}\ , \forall\ \mathfrak{r} \in \mathfrak{A}$$

$$\sqrt[3]{2\ \tilde{F}^3_{a_{ij}} - \left(\tilde{F}^3_{a_{ij}}\right)^2} = \sqrt[3]{\tilde{F}^3_{a_{ij}} + \tilde{F}^3_{a_{ij}}\left(1 - \tilde{F}^3_{a_{ij}}\right)} \leq \tilde{F}^3_{a_{ij}}\ , \forall\ \mathfrak{r} \in \mathfrak{A}$$

And $\left(\tilde{T}_{a_{ij}}\right)^2 \geq \tilde{T}_{a_{ij}},$

Hence $\mathbb{N} \otimes \mathbb{N} \leq \mathbb{N}$.

**Theorem 3.3**. If $\mathbb{N}$, $\wp$, M $\in FN_{mxn}$, then

(i) $\mathbb{N} \oplus \wp = \wp \oplus \mathbb{N}$,

(ii) $\mathbb{N} \otimes \wp = \wp \otimes \mathbb{N}$,

(iii) $(\mathbb{N} \oplus \wp) \oplus M = \mathbb{N} \oplus (\wp \oplus M)$,

(iv) $(\mathbb{N} \otimes \wp) \otimes M = \mathbb{N} \otimes (\wp \otimes M)$.

Proof. (i) Assume $\mathbb{N} \oplus \wp = \left(\langle \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}, \tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}} \rangle\right)_{mxn}$

$$= \left(\langle \sqrt[3]{\tilde{T}^3_{b_{ij}} + \tilde{T}^3_{a_{ij}} - \tilde{T}^3_{b_{ij}}\tilde{T}^3_{a_{ij}}}, \tilde{I}_{b_{ij}}\tilde{I}_{a_{ij}}, \tilde{F}_{b_{ij}}\tilde{F}_{a_{ij}} \rangle\right)_{mxn}$$

$= \wp \oplus \mathbb{N}$

(iii) $\mathbb{N} \otimes \wp = \left(\langle \tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}, \sqrt[3]{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{b_{ij}} - \tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\tilde{F}^3_{a_{ij}} + \tilde{F}^3_{b_{ij}} - \tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}} \rangle\right)_{mxn}$

$$= \left(\langle \tilde{T}_{b_{ij}}\tilde{T}_{a_{ij}}, \sqrt[3]{\tilde{I}^3_{b_{ij}} + \tilde{I}^3_{a_{ij}} - \tilde{I}^3_{b_{ij}}\tilde{I}^3_{a_{ij}}}, \sqrt[3]{\tilde{F}^3_{b_{ij}} + \tilde{F}^3_{a_{ij}} - \tilde{F}^3_{b_{ij}}\tilde{F}^3_{a_{ij}}} \rangle\right)_{mxn}$$

$= \wp \otimes \mathbb{N}$

(v) $(\mathbb{N} \oplus \wp) \oplus M = \mathbb{N} \oplus (\wp \oplus M)$, Let (ℕ ⊕ ℘) ⊕ M

$$=\left(\langle(\sqrt[3]{\tilde{T}_{a_{ij}}^3+\tilde{T}_{b_{ij}}^3-\tilde{T}_{a_{ij}}^3\tilde{T}_{b_{ij}}^3}, \tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}) \oplus (\tilde{T}_{c_{ij}}, \tilde{I}_{c_{ij}}, \tilde{F}_{c_{ij}})\rangle\right)_{mxn}$$

$$=\begin{bmatrix}\sqrt[3]{\left(\sqrt[3]{\tilde{T}_{a_{ij}}^3+\tilde{T}_{b_{ij}}^3-\tilde{T}_{a_{ij}}^3\tilde{T}_{b_{ij}}^3}\right)^{2}+\tilde{T}_{c_{ij}}^3-\left(\sqrt[3]{\tilde{T}_{a_{ij}}^3+\tilde{T}_{b_{ij}}^3-\tilde{T}_{a_{ij}}^3\tilde{T}_{b_{ij}}^3}\right)^2\tilde{T}_{c_{ij}}^3}, \\ \tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\tilde{I}_{c_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\tilde{F}_{c_{ij}}\end{bmatrix}_{mxn}$$

$$=\begin{bmatrix}\sqrt[3]{\tilde{T}_{a_{ij}}^3+\tilde{T}_{b_{ij}}^3+\tilde{T}_{c_{ij}}^3-\tilde{T}_{a_{ij}}^3\tilde{T}_{b_{ij}}^3-\tilde{T}_{a_{ij}}^3\tilde{T}_{c_{ij}}^3-\tilde{T}_{b_{ij}}^3\tilde{T}_{c_{ij}}^3+\tilde{T}_{a_{ij}}^3\tilde{T}_{b_{ij}}^3\tilde{T}_{c_{ij}}^3} \\ \tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\tilde{I}_{c_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\tilde{F}_{c_{ij}}\end{bmatrix}_{mxn}$$

Let ℕ ⊕ (℘ ⊕ M) =

$$=\left((\tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}}) \oplus \langle\sqrt[3]{\tilde{T}_{b_{ij}}^3+\tilde{T}_{c_{ij}}^3-\tilde{T}_{b_{ij}}^3\tilde{T}_{c_{ij}}^3}, \tilde{I}_{b_{ij}}\tilde{I}_{c_{ij}}, \tilde{F}_{b_{ij}}\tilde{F}_{c_{ij}})\rangle\right)$$

$$=\begin{bmatrix}\sqrt[3]{\tilde{T}_{a_{ij}}^3+\left(\sqrt[3]{\tilde{T}_{b_{ij}}^3+\tilde{T}_{c_{ij}}^3-\tilde{T}_{b_{ij}}^3\tilde{T}_{c_{ij}}^3}\right)^{2}-\tilde{T}_{a_{ij}}^3\left(\sqrt[3]{\tilde{T}_{b_{ij}}^3+\tilde{T}_{c_{ij}}^3-\tilde{T}_{b_{ij}}^3\tilde{T}_{c_{ij}}^3}\right)^2}, \\ \tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\tilde{I}_{c_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\tilde{F}_{c_{ij}}\end{bmatrix}_{mxn}$$

$$=\sqrt[3]{\tilde{T}_{a_{ij}}^3+\tilde{T}_{b_{ij}}^3+\tilde{T}_{c_{ij}}^3-\tilde{T}_{a_{ij}}^3\tilde{T}_{b_{ij}}^3-\tilde{T}_{a_{ij}}^3\tilde{T}_{c_{ij}}^3-\tilde{T}_{b_{ij}}^3\tilde{T}_{c_{ij}}^3+\tilde{T}_{a_{ij}}^3\tilde{T}_{b_{ij}}^3\tilde{T}_{c_{ij}}^3}$$
$$\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\tilde{I}_{c_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\tilde{F}_{c_{ij}}$$

Hence (ℕ ⊕ ℘) ⊕ M = ℕ ⊕ (℘ ⊕ M),

iv) (ℕ ⊗ ℘) ⊗ M = ℕ ⊗ (℘ ⊗ M)

(ℕ ⊗ ℘)

$$=\left(\langle\tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}, \sqrt[3]{\tilde{I}_{a_{ij}}^3+\tilde{I}_{b_{ij}}^3-\tilde{I}_{a_{ij}}^3\tilde{I}_{b_{ij}}^3}, \sqrt[3]{\tilde{F}_{a_{ij}}^3+\tilde{F}_{b_{ij}}^3-\tilde{F}_{a_{ij}}^3\tilde{F}_{b_{ij}}^3}\rangle\right)_{mxn}$$

(ℕ ⊗ ℘) ⊗ M

$$=\left(\langle\tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}, \sqrt[3]{\tilde{I}_{a_{ij}}^3+\tilde{I}_{b_{ij}}^3-\tilde{I}_{a_{ij}}^3\tilde{I}_{b_{ij}}^3}, \sqrt[3]{\tilde{F}_{a_{ij}}^3+\tilde{F}_{b_{ij}}^3-\tilde{F}_{a_{ij}}^3\tilde{F}_{b_{ij}}^3}\rangle \otimes \langle\tilde{F}_{a_{ij}}^3\tilde{F}_{b_{ij}}^3\tilde{F}_{c_{ij}}^3\rangle\right)_{mxn}$$

$$=\left(\begin{matrix} \tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}\tilde{T}_{c_{ij}} \\ \langle \sqrt[3]{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}+\tilde{I}^3_{c_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{c_{ij}}-\tilde{I}^3_{b_{ij}}\tilde{I}^3_{c_{ij}}+\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}\tilde{I}^3_{c_{ij}}}, \rangle \\ \sqrt[3]{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}+\tilde{F}^3_{c_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{c_{ij}}-\tilde{F}^3_{b_{ij}}\tilde{F}^3_{c_{ij}}+\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}\tilde{F}^3_{c_{ij}}} \end{matrix}\right)_{mxn}$$

Let $\mathbb{N} \otimes (\wp \otimes \mathrm{M})$

$(\wp \otimes \mathrm{M})$

$$= \left(\langle \tilde{T}_{a_{ij}}\tilde{T}_{c_{ij}}, \sqrt[3]{\tilde{I}^3_{b_{ij}}+\tilde{I}^3_{c_{ij}}-\tilde{I}^3_{b_{ij}}\tilde{I}^3_{c_{ij}}}, \sqrt[3]{\tilde{F}^3_{b_{ij}}+\tilde{F}^3_{c_{ij}}-\tilde{F}^3_{b_{ij}}\tilde{F}^3_{c_{ij}}}\rangle\right)_{\boldsymbol{mxn}}$$

$\mathbb{N} \otimes (\wp \otimes \mathrm{M}) =$

$$\left(\begin{matrix} \tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}\tilde{T}_{c_{ij}}, \\ \langle \sqrt[3]{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}+\tilde{I}^3_{c_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{c_{ij}}-\tilde{I}^3_{b_{ij}}\tilde{I}^3_{c_{ij}}+\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}\tilde{I}^3_{c_{ij}}}, \rangle \\ \sqrt[3]{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}+\tilde{F}^3_{c_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{c_{ij}}-\tilde{F}^3_{b_{ij}}\tilde{F}^3_{c_{ij}}+\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}\tilde{F}^3_{c_{ij}}} \end{matrix}\right)_{mxn}$$

Hence $(\mathbb{N} \otimes \wp) \otimes \mathrm{M} = \mathbb{N} \otimes (\wp \otimes \mathrm{M})$

**Theorem 3.4.** If $\mathbb{N}$, $\wp \in FN_{mxn}$, then

(i) $\mathbb{N} \oplus (\mathbb{N} \otimes \wp) \geq \mathbb{N}$,

(ii) $\mathbb{N} \otimes (\mathbb{N} \oplus \wp) \leq \mathbb{N}$.

Proof. (i) Let $\mathbb{N} \oplus (\mathbb{N} \otimes \wp) =$

$$(\mathbb{N} \otimes \wp) = \left(\langle \tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}, \sqrt[3]{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}}\rangle\right)_{\boldsymbol{mxn}}$$

$\mathbb{N} \oplus (\mathbb{N} \otimes \wp) =$

$$= \left(\langle \tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}}\rangle\right) \oplus$$

$$\left(\langle \tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}, \sqrt[3]{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}}\rangle\right)$$

$$=\left[\begin{matrix} \sqrt[3]{\tilde{T}^3_{a_{ij}}+\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}-\tilde{T}^3_{a_{ij}}\left[\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}\right]}, \tilde{I}_{a_{ij}}\left[\sqrt[3]{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \right] \\ \tilde{F}_{a_{ij}}\left[\sqrt[3]{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}}\right] \end{matrix}\right]_{mxn}$$

$$=$$

$$\left[\begin{matrix} \sqrt[3]{\tilde{T}^3_{a_{ij}}+\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}-\tilde{T}^3_{a_{ij}}\left[\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}\right]}, \tilde{I}_{a_{ij}}\sqrt[3]{1-\left[1-\tilde{I}^3_{a_{ij}}\right]\left[1-\tilde{I}^3_{b_{ij}}\right]} \\ \tilde{F}_{a_{ij}}\left[\sqrt[3]{1-\left[1-\tilde{F}^3_{a_{ij}}\right]\left[1-\tilde{F}^3_{b_{ij}}\right]}\right] \end{matrix}\right]_{mxn}$$

$\geq \mathbb{N}$

Hence
$\mathbb{N} \oplus (\mathbb{N} \otimes \wp) \geq \mathbb{N}$.

ii)$\mathbb{N} \quad \oplus \wp = \left( \langle \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}} \tilde{T}^3_{b_{ij}}}, \tilde{I}_{a_{ij}} \tilde{I}_{b_{ij}}, \tilde{F}_{a_{ij}} \tilde{F}_{b_{ij}} \rangle \right)_{mxn}$

$$\mathbb{N} \otimes (\mathbb{N} \oplus \wp) = \left( \langle \tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}} \rangle \right) \otimes \left( \langle \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}} \tilde{T}^3_{b_{ij}}}, \tilde{I}_{a_{ij}} \tilde{I}_{b_{ij}}, \tilde{F}_{a_{ij}} \tilde{F}_{b_{ij}} \rangle \right)$$

$$= \left[ \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{a_{ij}} \tilde{T}^3_{b_{ij}} \left[1 - \tilde{T}^3_{a_{ij}}\right]}, \tilde{I}_{a_{ij}} \left( \sqrt[3]{1 - \left[1 - \tilde{I}^3_{a_{ij}}\right]\left[1 - \tilde{I}^3_{b_{ij}}\right]} \right) \right], \quad \tilde{F}_{a_{ij}} \left[ \sqrt[3]{1 - \left[1 - \tilde{F}^3_{a_{ij}}\right]\left[1 - \tilde{F}^3_{b_{ij}}\right]} \right]$$

$$= \begin{bmatrix} \tilde{T}_{a_{ij}} \sqrt[3]{1 - \left[1 - \tilde{T}^3_{a_{ij}}\right]\left[1 - \tilde{T}^3_{b_{ij}}\right]}, \sqrt[3]{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{a_{ij}} \tilde{I}^3_{b_{ij}} \left[1 - \tilde{I}^3_{a_{ij}}\right]}, \\ \sqrt[3]{\tilde{F}^3_{a_{ij}} + \tilde{F}^3_{a_{ij}} \tilde{F}^3_{b_{ij}} \left[1 - \tilde{F}^3_{a_{ij}}\right]}, \end{bmatrix}_{mxn}$$

$\leq \mathbb{N}$.

Hence $\mathbb{N} \otimes (\mathbb{N} \oplus \wp) \leq \mathbb{N}$

**Theorem 3.5.** If $\mathbb{A}$, $\wp \in FN_{mxn}$, Then

(i) $\mathbb{N} \vee \wp = \mathbb{N} \vee \wp$

(ii) $\mathbb{N} \wedge \wp = \mathbb{N} \wedge \wp$

(iii) $\mathbb{N} \vee \wp = \wp \vee \mathbb{N}$

$$\mathbb{N} \vee \wp = \left( \langle \max(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}), \min(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \min(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}) \rangle \right)_{mxn}$$

$$\wp \vee \mathbb{N} = \left( \langle \max(\tilde{T}_{b_{ij}}, \tilde{T}_{a_{ij}}), \min(\tilde{I}_{b_{ij}}, \tilde{I}_{a_{ij}}), \min(\tilde{F}_{b_{ij}}, \tilde{F}_{a_{ij}}) \rangle \right)_{mxn}$$

$$= \left( \langle \max(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}), \min(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \min(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}) \rangle \right)_{mxn}$$

$= \mathbb{N} \vee \wp$

(i) $\mathbb{N} \wedge \wp = \wp \wedge \mathbb{N}$

$$\mathbb{N} \wedge \wp = \left( \langle \min\left(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}\right), \max(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \max(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}) \rangle \right)_{mxn}$$

$$\wp \wedge \mathbb{N} = \left( \langle \min(\tilde{T}_{b_{ij}}, \tilde{T}_{a_{ij}}), \max(\tilde{I}_{b_{ij}}, \tilde{I}_{a_{ij}}), \max(\tilde{F}_{b_{ij}}, \tilde{F}_{a_{ij}}) \rangle \right)_{mxn}$$

$$=\left(\langle \min(\tilde{T}_{a_{ij}},\tilde{T}_{b_{ij}}),\ \max(\tilde{I}_{a_{ij}},\tilde{I}_{b_{ij}}),\ \max(\tilde{F}_{a_{ij}},\tilde{F}_{b_{ij}})\ \rangle\right)_{mxn}$$

$= \ \wp \vee \mathbb{N}$

**Theorem 3.6.** Let $\mathbb{N},\wp\ ,\mathbb{C} \in FN_{mxn}$, then

(i). $\mathbb{N} \oplus (\wp \vee \mathbb{C}) = (\mathbb{N} \oplus \wp) \vee (\mathbb{N} \oplus \mathbb{C})$,

(ii). $\mathbb{N} \otimes (\wp \vee \mathbb{C}) = (\mathbb{N} \otimes \wp) \vee (\mathbb{N} \otimes \mathbb{C})$,

(iii). $\mathbb{N} \oplus (\wp \wedge \mathbb{C}) = (\mathbb{N} \oplus \wp) \wedge (\mathbb{N} \oplus \mathbb{C})$,

(iv). $\mathbb{N} \otimes (\wp \wedge \mathbb{C}) = (\mathbb{N} \otimes \wp) \wedge (\mathbb{N} \otimes \mathbb{C})$,

Proof.

(i) Let $\mathbb{N} \oplus (\wp \vee \mathbb{C})$

$$= \left( \left\langle \begin{matrix} \sqrt[3]{\tilde{T}^3_{a_{ij}} + \max\left(\tilde{T}^3_{b_{ij}}\tilde{T}^3_{c_{ij}}\right) - \tilde{T}^3_{a_{ij}}.\max\left(\tilde{T}^3_{b_{ij}}\tilde{T}^3_{c_{ij}}\right)}, \\ \tilde{I}_{a_{ij}}, \max\left(\tilde{I}^3_{b_{ij}}\tilde{I}^3_{c_{ij}}\right) \\ \tilde{F}_{a_{ij}}, \max\left(\tilde{F}^3_{b_{ij}}\tilde{F}^3_{c_{ij}}\right) \end{matrix} \right\rangle \right)_{mxn}$$

$$= \left( \left\langle \begin{matrix} \sqrt[3]{\max\left(\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}}, \tilde{T}^3_{a_{ij}} + \tilde{T}^3_{c_{ij}}\right) - \max\left(\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}, \tilde{T}^3_{a_{ij}}\tilde{T}^3_{c_{ij}}\right)} \\ , \min\left(\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}, \tilde{I}_{a_{ij}}\tilde{I}_{c_{ij}}\right) \\ , \min\left(\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{c_{ij}}\right) \end{matrix} \right\rangle \right)_{mxn}$$

$$= \left( \left\langle \begin{matrix} \sqrt[3]{\max\left(\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}, \tilde{T}^3_{a_{ij}} + \tilde{T}^3_{c_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{c_{ij}}\right)} \\ , \min\left(\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}, \tilde{I}_{a_{ij}}\tilde{I}_{c_{ij}}\right) \\ , \min\left(\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{c_{ij}}\right) \end{matrix} \right\rangle \right)_{mxn}$$

$= (\mathbb{N} \oplus \wp) \vee (\mathbb{N} \oplus \mathbb{C})$

**Theorem 3.7**. Let $\mathbb{N},\wp \in FN_{mxn}$, then

(i) $(\mathbb{N} \wedge \wp) \oplus (\mathbb{N} \vee \wp) = \mathbb{N} \oplus \wp$,

(ii) $(\mathbb{N} \wedge \wp) \otimes (\mathbb{N} \vee \wp) = \mathbb{N} \otimes \wp$,

(iii) $(\mathbb{N} \oplus \wp) \wedge (\mathbb{N} \otimes \wp) = \mathbb{N} \otimes \wp$,

(iv) $(\mathbb{N} \oplus \wp) \vee (\mathbb{N} \otimes \wp) = \mathbb{N} \oplus \wp$.

**Proof.** In the following, we shall prove (i), and (ii) – (iv) can be proved similarly.

(v) $(\mathbb{N} \wedge \wp) \oplus (\mathbb{N} \vee \wp) = \mathbb{N} \oplus \wp$,

Let $(\mathbb{N} \wedge \wp) \oplus (\mathbb{N} \vee \wp)$
=

$$\begin{bmatrix} \sqrt[3]{\min\left(\tilde{T}^3_{a_{ij}}, \tilde{T}^3_{b_{ij}}\right) + \max\left(\tilde{T}^3_{a_{ij}}, \tilde{T}^3_{b_{ij}}\right) - min\left(\tilde{T}^3_{a_{ij}}, \tilde{T}^3_{b_{ij}}\right).max\left(\tilde{T}^3_{a_{ij}}, \tilde{T}^3_{b_{ij}}\right)}, \\ \max\left(\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\right), \min\left(\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\right), \\ \max\left(\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\right), \min\left(\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\right) \end{bmatrix}_{mxn}$$

$$= \left(\langle \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}, \tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}} \rangle\right)_{mxn}$$

$= \mathbb{N} \oplus \wp$

**Theorem 3.8.** If $\mathbb{N}, \wp \in FN_{mxn}$, then

(i) $(\mathbb{N} \oplus \wp)^C = \mathbb{N}^C \otimes \wp^C$,
(ii) $(\mathbb{N} \otimes \wp)^C = \mathbb{N}^C \oplus \wp^C$,
(iii) $(\mathbb{N} \oplus \wp)^C \leq \mathbb{N}^C \oplus \wp^C$,
(iv) $(\mathbb{N} \otimes \wp)^C \geq \mathbb{N}^C \otimes \wp^C$.

**Proof**.

(i) $(\mathbb{N} \oplus \wp)^C = \mathbb{N}^C \otimes \wp^C$

$(\mathbb{N} \oplus \wp)^C$

$$=\left(\langle \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}, \sqrt[3]{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{b_{ij}} - \tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}} \rangle\right)_{mxn}$$

$$=(\langle \tilde{F}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{T}_{a_{ij}} \rangle \otimes (\langle \tilde{F}_{b_{ij}}, \tilde{I}_{b_{ij}}, \tilde{T}_{b_{ij}} \rangle$$

$= \mathbb{N}^C \otimes \wp^C$

(ii) $(\mathbb{N} \otimes \wp)^C = \mathbb{N}^C \oplus \wp^C$,

$$= \left(\langle \tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}, \sqrt[3]{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{b_{ij}} - \tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\tilde{F}^3_{a_{ij}} + \tilde{F}^3_{b_{ij}} - \tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}} \rangle\right)^c$$

$$=\left(\langle \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}, \sqrt[3]{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{b_{ij}} - \tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}} \rangle\right)_{mxn}$$

$$=(\langle \tilde{F}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{T}_{a_{ij}} \rangle \oplus \left(\langle \tilde{F}_{b_{ij}}, \tilde{I}_{b_{ij}}, \tilde{T}_{b_{ij}} \rangle\right)$$

$= \mathbb{N}^C \oplus \wp^C$

(iii) $(\mathbb{N} \oplus \wp)^C \leq \mathbb{N}^C \oplus \wp^C$,

$(\mathbb{N} \oplus \wp)^C$

$$= \left(\langle \sqrt[3]{\tilde{T}_{a_{ij}}^3 + \tilde{T}_{b_{ij}}^3 - \tilde{T}_{a_{ij}}^3 \tilde{T}_{b_{ij}}^3}, \tilde{I}_{a_{ij}} \tilde{I}_{b_{ij}}, \tilde{F}_{a_{ij}} \tilde{F}_{b_{ij}} \rangle\right)_{mxn}^{c}$$

$$= \left(\langle \tilde{F}_{a_{ij}} \tilde{F}_{b_{ij}}, \sqrt[3]{\tilde{I}_{a_{ij}}^3 + \tilde{I}_{b_{ij}}^3 - \tilde{I}_{a_{ij}}^3 \tilde{I}_{b_{ij}}^3}, \sqrt[3]{\tilde{T}_{a_{ij}}^3 + \tilde{T}_{b_{ij}}^3 - \tilde{T}_{a_{ij}}^3 \tilde{T}_{b_{ij}}^3} \rangle\right)_{mxn}$$

$\mathbb{N}^C \oplus \wp^C$

$$= \left(\tilde{F}_{a_{ij}} \tilde{F}_{b_{ij}}, \sqrt[3]{\tilde{I}_{a_{ij}}^3 + \tilde{I}_{b_{ij}}^3 - \tilde{I}_{a_{ij}}^3 \tilde{I}_{b_{ij}}^3}, \sqrt[3]{\tilde{T}_{a_{ij}}^3 + \tilde{T}_{b_{ij}}^3 - \tilde{T}_{a_{ij}}^3 \tilde{T}_{b_{ij}}^3}\right)_{mxn}$$

$$= \left(\sqrt[3]{\tilde{F}_{a_{ij}}^3 + \tilde{F}_{b_{ij}}^3 - \tilde{F}_{a_{ij}}^3 \tilde{F}_{b_{ij}}^3}, \quad \tilde{I}_{a_{ij}} \tilde{I}_{b_{ij}} \;, \tilde{T}_{a_{ij}} \tilde{T}_{b_{ij}}\right)_{mxn}$$

$$\tilde{F}_{a_{ij}} \tilde{F}_{b_{ij}} \leq \sqrt[3]{\tilde{F}_{a_{ij}}^3 + \tilde{F}_{b_{ij}}^3 - \tilde{F}_{a_{ij}}^3 \tilde{F}_{b_{ij}}^3},$$

$$\sqrt[3]{\tilde{I}_{a_{ij}}^3 + \tilde{I}_{b_{ij}}^3 - \tilde{I}_{a_{ij}}^3 \tilde{I}_{b_{ij}}^3} \geq \tilde{I}_{a_{ij}} \tilde{I}_{b_{ij}}$$

$$\sqrt[3]{\tilde{T}_{a_{ij}}^3 + \tilde{T}_{b_{ij}}^3 - \tilde{T}_{a_{ij}}^3 \tilde{T}_{b_{ij}}^3} \geq \tilde{T}_{a_{ij}} \tilde{T}_{b_{ij}}$$

Hence $(\mathbb{N} \oplus \wp)^C \leq \mathbb{N}^C \oplus \wp^C$,

(ii) $(\mathbb{N} \otimes \wp)^C \geq \mathbb{N}^C \otimes \wp^C$.

$$\mathbb{N} \otimes \wp = \left(\langle \tilde{T}_{a_{ij}} \tilde{T}_{b_{ij}}, \sqrt[3]{\tilde{I}_{a_{ij}}^3 + \tilde{I}_{b_{ij}}^3 - \tilde{I}_{a_{ij}}^3 \tilde{I}_{b_{ij}}^3}, \sqrt[3]{\tilde{F}_{a_{ij}}^3 + \tilde{F}_{b_{ij}}^3 - \tilde{F}_{a_{ij}}^3 \tilde{F}_{b_{ij}}^3} \rangle\right)_{mxn}$$

$$(\mathbb{N} \otimes \wp)^C = \left(\langle \tilde{T}_{a_{ij}} \tilde{T}_{b_{ij}}, \sqrt[3]{\tilde{I}_{a_{ij}}^3 + \tilde{I}_{b_{ij}}^3 - \tilde{I}_{a_{ij}}^3 \tilde{I}_{b_{ij}}^3}, \sqrt[3]{\tilde{F}_{a_{ij}}^3 + \tilde{F}_{b_{ij}}^3 - \tilde{F}_{a_{ij}}^3 \tilde{F}_{b_{ij}}^3} \rangle\right)_{mxn}^{c}$$

$$= \left(\langle \sqrt[3]{\tilde{F}_{a_{ij}}^3 + \tilde{F}_{b_{ij}}^3 - \tilde{F}_{a_{ij}}^3 \tilde{F}_{b_{ij}}^3}, \tilde{I}_{a_{ij}} \tilde{I}_{b_{ij}}, \tilde{T}_{a_{ij}} \tilde{T}_{b_{ij}} \rangle\right)_{mxn}$$

$$\mathbb{N}^C \otimes \wp^C = (\langle \tilde{F}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{T}_{a_{ij}} \rangle \otimes \left(\langle \tilde{F}_{b_{ij}}, \tilde{I}_{b_{ij}}, \tilde{T}_{b_{ij}} \rangle\right)$$

Since

$$\sqrt[3]{\tilde{F}_{a_{ij}}^3 + \tilde{F}_{b_{ij}}^3 - \tilde{F}_{a_{ij}}^3 \tilde{F}_{b_{ij}}^3} \geq \tilde{F}_{a_{ij}} \tilde{F}_{b_{ij}}$$

$$\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}} \le \sqrt[3]{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{b_{ij}} - \tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}$$

$$\tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}} \le \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}$$

Hence $(\mathbb{N} \otimes \wp)^C \ge \mathbb{N}^C \otimes \wp^C$

**Theorem 3.9** . Let $\mathbb{N}, \wp \in FN_{mxn}$, then

(i). $(\mathbb{N}^C)^C = \mathbb{N}$,

(ii). $(\mathbb{N} \vee \wp)^C = \mathbb{N}^C \wedge \wp^C$,

(iii). $(\mathbb{N} \wedge \wp)^C = \mathbb{N}^C \vee \wp^C$.

Proof.

(i) $((\mathbb{N}^C)^C = \mathbb{N}$,

$\mathbb{N} = (\langle \tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}} \rangle$

$\mathbb{N}^C = (\langle \tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}} \rangle)^c = \langle \tilde{F}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{T}_{a_{ij}} \rangle$

$(\mathbb{N}^C)^C = (\langle \tilde{F}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{T}_{a_{ij}} \rangle^c = \langle \tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}} \rangle$

(ii) $(\mathbb{N} \vee \wp)^C = \mathbb{N}^C \wedge \wp^C$,

$$\mathbb{N} \vee \wp = \left(\langle \max(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}), \min(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \min(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}) \rangle\right)_{mxn}$$

$$(\mathbb{N} \vee \wp)^C = \left(\langle \max(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}), \min(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \min(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}) \rangle\right)_{mxn}^c$$

$$= \left(\langle \min(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}), \max(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \max(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}) \rangle\right)_{mxn}$$

$\mathbb{N}^C = \langle \tilde{F}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{T}_{a_{ij}} \rangle$

$\wp^C = \langle \tilde{F}_{b_{ij}}, \tilde{I}_{b_{ij}}, \tilde{T}_{b_{ij}} \rangle$

$$\mathbb{N}^C \wedge \wp^C = \left(\langle \min(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}), \max(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \max(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}) \rangle\right)_{mxn}$$

Hence $(\mathbb{N} \vee \wp)^C = \mathbb{N}^C \wedge \wp^C$.

(iii) $(\mathbb{N} \wedge \wp)^C = \mathbb{N}^C \vee \wp^C$.

$$\mathbb{N} \wedge \wp = \left(\langle \min(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}), \max(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \max(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}) \rangle\right)_{mxn}$$

$$(\mathbb{N} \wedge \wp)^C = \left(\langle \min(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}), \max(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \max(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}) \rangle\right)_{mxn}^c$$

$$= \left(\langle \max(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}), \min(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \min(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}) \rangle\right)_{mxn}$$

$\mathbb{N}^C = \langle \tilde{F}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{T}_{a_{ij}} \rangle$

$\wp^C = \langle \tilde{F}_{b_{ij}}, \tilde{I}_{b_{ij}}, \tilde{T}_{b_{ij}} \rangle$

$$\mathbb{N}^C \vee \wp^C = \left( \langle \max(\tilde{F}_{a_{ij}}, \tilde{F}_{b_{ij}}), \min(\tilde{I}_{a_{ij}}, \tilde{I}_{b_{ij}}), \min(\tilde{T}_{a_{ij}}, \tilde{T}_{b_{ij}}) \rangle \right)_{mxn}$$

$$= (\mathbb{N} \wedge \wp)^C$$

Within this section, we will show that the operations of scalar multiplication and exponentiation presented in definitions 3.3, 3.4, and 3.5 are well-defined for Fermatean neutrosophic matrices.

**Theorem 3.10** For $\mathbb{N}$, $\wp \in FN_{mxn}$ and n, n1, n2 > 0, we have

(i) n($\mathbb{N} \oplus \wp$) = n $\mathbb{N} \oplus$ n$\wp$,

(ii) n1 $\mathbb{N} \oplus$ n2 $\mathbb{N}$ = (n1 + n2) $\mathbb{N}$,

(iii) $(\mathbb{N} \otimes \wp)^n = \mathbb{N}^n \otimes \wp^n$ ,

(iv) $\mathbb{N}^{n_1} \otimes \mathbb{N}^{n_2} = \mathbb{N}^{n_1+n_2}$

Proof. We will consider two Fermatean neutrosophic matrices $\mathbb{N}$ and $\wp$ and positive real numbers n, $n_1$, and $n_2$ . As stated in the definition, the following can be obtained.

(i) n($\mathbb{N} \oplus \wp$) = n $\mathbb{N} \oplus$ n$\wp$,

$$= n \left( \langle \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}, \tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}} \rangle \right)_{mxn}$$

$$= \left( \langle \sqrt[3]{1 - \left[1 - \tilde{T}^3_{a_{ij}}][1 - \tilde{T}^3_{b_{ij}}\right]^n}, \left[\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\right]^n, \left[\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\right]^n \rangle \right)_{mxn}$$

$$= \left( \langle \sqrt[3]{1 - \left[1 - \tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}\right]^n}, \left[\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\right]^n, \left[\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\right]^n \rangle \right)_{mxn}$$

n $\mathbb{N} \oplus$ n$\wp$

$$= \left( \langle \left( \sqrt[3]{1 - [1 - \tilde{T}^3_{a_{ij}}]^n}, \left[\tilde{I}_{a_{ij}}\right]^n, \left[\tilde{F}_{a_{ij}}\right]^n \right) \oplus \left( \sqrt[3]{1 - [1 - \tilde{T}^3_{b_{ij}}]^n}, \left[\tilde{I}_{b_{ij}}\right]^n, \left[\tilde{F}_{b_{ij}}\right]^n \right) \rangle \right)_{mxn}$$

$$= \begin{bmatrix} \sqrt[3]{\left(1 - \left[1 - \tilde{T}^3_{a_{ij}}\right]^n + 1 - \left[1 - \tilde{T}^3_{b_{ij}}\right]^n\right) - \left(1 - \left[1 - \tilde{T}^3_{a_{ij}}\right]^n\right)\left(1 - \left[1 - \tilde{T}^3_{b_{ij}}\right]^n\right)} \\ \left[\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\right]^n, \left[\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\right]^n \end{bmatrix}_{mxn}$$

$$= \left( \langle \sqrt[3]{1 - \left[1 - \tilde{T}^3_{a_{ij}}][1 - \tilde{T}^3_{b_{ij}}\right]^n}, \left[\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\right]^n, \left[\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\right]^n \rangle \right)_{mxn}$$

$$= \left( \langle \sqrt[3]{1 - \left[1 - \tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}\right]^n}, \left[\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}\right]^n, \left[\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\right]^n \rangle \right)_{mxn}$$

= n($\mathbb{N} \oplus \wp$)

(ii) Let n1 $\mathbb{N} \oplus$ n2 $\mathbb{N}$ = (n1 + n2) $\mathbb{N}$,

n1 $\mathbb{N} \oplus$ n2 $\mathbb{N}$

$$= \left( \langle \left( \sqrt[3]{1 - [1 - \tilde{T}^3_{a_{ij}}]^{n_1}}, \left[\tilde{I}_{a_{ij}}\right]^{n_1}, \left[\tilde{F}_{a_{ij}}\right]^{n_1} \right) \oplus \left( \sqrt[3]{1 - [1 - \tilde{T}^3_{a_{ij}}]^{n_2}}, \left[\tilde{I}_{a_{ij}}\right]^{n_2}, \left[\tilde{F}_{a_{ij}}\right]^{n_2} \right) \rangle \right)_{mxn}$$

$$= \left[ \begin{array}{c} \sqrt[3]{\left(1-\left[1-\tilde{T}_{a_{ij}}^{3}\right]^{n_1}+1-\left[1-\tilde{T}_{a_{ij}}^{3}\right]^{n_2}\right)-\left(1-\left[1-\tilde{T}_{a_{ij}}^{3}\right]^{n_1}\right)\left(1-\left[1-\tilde{T}_{a_{ij}}^{3}\right]^{n_2}\right)} \\ \left[\tilde{I}_{a_{ij}}\right]^{n_1}\left[\tilde{I}_{a_{ij}}\right]^{n_2}, \left[\tilde{F}_{a_{ij}}\right]^{n_1}\left[\tilde{F}_{a_{ij}}\right]^{n_2} \end{array} \right]_{mxn}$$

$$= \left( \langle \sqrt[3]{1-[1-\tilde{T}_{a_{ij}}^{3}]^{n_1+n_2}}, \left[\tilde{I}_{a_{ij}}^{3}\right]^{n_1+n_2}, \left[\tilde{F}_{a_{ij}}\right]^{n_1+n_2} \rangle \right)_{mxn}$$

(iii)

$$(\mathbb{N} \otimes \wp)^n = \left( \langle \begin{array}{c} \left(\tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}\right)^n, \sqrt[3]{1-\left[1-\tilde{I}_{a_{ij}}^{3}+\tilde{I}_{b_{ij}}^{3}-\tilde{I}_{a_{ij}}^{3}\tilde{I}_{b_{ij}}^{3}\right]^n}, \\ \sqrt[3]{1-\left[1-\tilde{F}_{a_{ij}}^{3}+\tilde{F}_{b_{ij}}^{3}-\tilde{F}_{a_{ij}}^{3}\tilde{F}_{b_{ij}}^{3}\right]^n} \end{array} \rangle \right)_{\boldsymbol{mxn}}$$

$$= \left( \langle \begin{array}{c} \left(\tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}\right)^n, \sqrt[3]{1-\left[1-\tilde{I}_{a_{ij}}^{3}\right]^n\left[1-\tilde{I}_{b_{ij}}^{3}\right]^n}, \\ \sqrt[3]{1-\left[1-\tilde{F}_{a_{ij}}^{3}\right]^n\left[1-\tilde{F}_{b_{ij}}^{3}\right]^n} \end{array} \rangle \right)_{\boldsymbol{mxn}}$$

$\mathbb{N}^n \otimes \wp^n$

$$= \left[ \begin{array}{c} \left(\tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}\right)^n, \sqrt[3]{1-\left[1-\tilde{I}_{a_{ij}}^{3}\right]^n+1-\left[1-\tilde{I}_{b_{ij}}^{3}\right]^n-\left(1-\left[1-\tilde{I}_{a_{ij}}^{3}\right]^n\right)\left(1-\left[1-\tilde{I}_{b_{ij}}^{3}\right]^n\right)} \\ \sqrt[3]{1-\left[1-\tilde{F}_{a_{ij}}^{3}\right]^n+1-\left[1-\tilde{F}_{b_{ij}}^{3}\right]^n-\left(1-\left[1-\tilde{F}_{a_{ij}}^{3}\right]^n\right)\left(1-\left[1-\tilde{F}_{b_{ij}}^{3}\right]^n\right)} \end{array} \right]_{mxn}$$

$$= \left( \langle \begin{array}{c} \left(\tilde{T}_{a_{ij}}\tilde{T}_{b_{ij}}\right)^n, \sqrt[3]{1-\left[1-\tilde{I}_{a_{ij}}^{3}\right]^n\left[1-\tilde{I}_{b_{ij}}^{3}\right]^n}, \\ \sqrt[3]{1-\left[1-\tilde{F}_{a_{ij}}^{3}\right]^n\left[1-\tilde{F}_{b_{ij}}^{3}\right]^n} \end{array} \rangle \right)_{mxn}$$

$= (\mathbb{N} \otimes \wp)^n$

iv) $\mathbb{N}^{n_1} \otimes \mathbb{N}^{n_2} = \mathbb{N}^{(n_1+n_2)}$

$\mathbb{N}^{n_1} \otimes \mathbb{N}^{n_2} =$

$$\left[ \begin{array}{c} \left(\tilde{T}_{a_{ij}}\right)^{n_1+n_2}, \sqrt[3]{1-\left[1-\tilde{I}_{a_{ij}}^{3}\right]^{n_1}+1-\left[1-\tilde{I}_{a_{ij}}^{3}\right]^{n_2}-\left(1-\left[1-\tilde{I}_{a_{ij}}^{3}\right]^{n_1}\right)\left(1-\left[1-\tilde{I}_{a_{ij}}^{3}\right]^{n_2}\right)} \\ \sqrt[3]{1-\left[1-\tilde{F}_{a_{ij}}^{3}\right]^{n_1}+1-\left[1-\tilde{F}_{a_{ij}}^{3}\right]^{n_2}-\left(1-\left[1-\tilde{F}_{a_{ij}}^{3}\right]^{n_1}\right)\left(1-\left[1-\tilde{F}_{a_{ij}}^{3}\right]^{n_2}\right)} \end{array} \right]_{mxn}$$

$$=\left(\langle \left(\tilde{T}_{a_{ij}}\right)^{n_1+n_2}, \sqrt[3]{1-\left[1-\tilde{I}^3_{a_{ij}}\right]^{n_1+n_2}}, \sqrt[3]{1-\left[1-\tilde{F}^3_{a_{ij}}\right]^{n_1+n_2}} \rangle\right) = \mathbb{N}^{(n_1+n_2)}$$

As a result, it is proven.

**Theorem 3.11.** Given that $\mathbb{N}$ and $\wp$ are matrices of size m×n and n>0, the following holds.

(i) $n\mathbb{N} \le n\wp$,

(ii) $\mathbb{N}^n \le \wp^n$

Proof:

Suppose $\mathbb{N} \le \wp$

$\Rightarrow \tilde{T}_{a_{ij}} \le \tilde{T}_{b_{ij}}$ and $\tilde{I}_{a_{ij}} \ge \tilde{I}_{b_{ij}}$ and $\tilde{F}_{a_{ij}} \ge \tilde{F}_{b_{ij}}$ for all i,j.

$$\Rightarrow \sqrt[3]{1-[1-\tilde{T}^3_{a_{ij}}]^n} \le \sqrt[3]{1-[1-\tilde{T}^3_{b_{ij}}]^n}$$

$\left[\tilde{I}^3_{a_{ij}}\right]^n \ge \left[\tilde{I}^3_{b_{ij}}\right]^n$ and $\left[\tilde{F}^3_{a_{ij}}\right]^n \ge \left[\tilde{F}^3_{a_{ij}}\right]^n$ ,for all i,j.

ii) also $\left[\tilde{F}^3_{a_{ij}}\right]^n \ge \left[\tilde{F}^3_{b_{ij}}\right]^n$,for all i,j.

$\sqrt[3]{1-[1-\tilde{I}^3_{a_{ij}}]^n} \le \sqrt[3]{1-[1-\tilde{I}^3_{b_{ij}}]^n}$,

$\sqrt[3]{1-[1-\tilde{F}^3_{a_{ij}}]^n} \le \sqrt[3]{1-[1-\tilde{F}^3_{b_{ij}}]^n}$ for all i,j.

**Theorem 3.12.** Given that $\mathbb{N}$ and $\wp$ are matrices of size m×n and n>0, the following holds.

(i) $n(\mathbb{N} \wedge \wp) = n\ \mathbb{N} \wedge n\wp$,

(ii) $n(\mathbb{N} \vee \wp) = n\ \mathbb{N} \vee n\wp$

Proof

(i) $n(\mathbb{N} \wedge \wp)$

$$= \left(\langle \sqrt[3]{1-\left[1-\min\left(\tilde{T}^3_{a_{ij}}, \tilde{T}^3_{b_{ij}}\right)\right]^n}, \max\left(\left|\tilde{I}_{a_{ij}}\right|^n, \left|\tilde{I}_{b_{ij}}\right|^n\right), \max\left(\left|\tilde{F}_{a_{ij}}\right|^n, \left|\tilde{F}_{b_{ij}}\right|^n\right)\rangle\right)_{mxn}$$

$$= \left[\sqrt[3]{1-\left[\max\left(1-\tilde{T}^3_{a_{ij}}, 1-\tilde{T}^3_{b_{ij}}\right)\right]^n}, \max\left(\left|\tilde{I}_{a_{ij}}\right|^n, \left|\tilde{I}_{b_{ij}}\right|^n\right), \max\left(\left|\tilde{F}_{a_{ij}}\right|^n, \left|\tilde{F}_{b_{ij}}\right|^n\right)\right]_{mxn}$$

$$= \left[\sqrt[3]{1-(\max\left(\left[1-\tilde{T}^3_{a_{ij}}\right]^n, \left[1-\tilde{T}^3_{b_{ij}}\right]^n\right)}, \max\left(\left|\tilde{I}_{a_{ij}}\right|^n, \left|\tilde{I}_{b_{ij}}\right|^n\right), \max\left(\left|\tilde{F}_{a_{ij}}\right|^n, \left|\tilde{F}_{b_{ij}}\right|^n\right)\right]_{mxn}$$

$$= \left[\max\left(\sqrt[3]{1-\left[1-\tilde{T}^3_{a_{ij}}\right]^n}, \sqrt[3]{1-\left[1-\tilde{T}^3_{b_{ij}}\right]^n}\right), \max\left(\left|\tilde{I}_{a_{ij}}\right|^n, \left|\tilde{I}_{b_{ij}}\right|^n\right), \max\left(\left|\tilde{F}_{a_{ij}}\right|^n, \left|\tilde{F}_{b_{ij}}\right|^n\right)\right]_{mxn}$$

= n ℕ ∧ n℘

Hence, n(ℕ ∧ ℘) = n ℕ ∧ n℘,

,

Similarly we can prove n(ℕ ∨ ℘) = nℕ ∨ n℘

**Theorem 3.13**. For ℕ, ℘ ∈ $FN_{mxn}$ and n > 0, the following holds

(i) $(\mathbb{N} \wedge \wp)^n = \mathbb{N}^n \wedge \wp^n$,

(ii) $(\mathbb{N} \vee \wp)^n = \mathbb{N}^n \vee \wp^n$

Let $(\mathbb{N} \wedge \wp)^n$

$$= \begin{bmatrix} \min\left(\left|\tilde{T}_{a_{ij}}\right|^n, \left|\tilde{T}_{b_{ij}}\right|^n\right), \sqrt[3]{1-\max\left(\left[1-\tilde{I}^3_{a_{ij}}\right]^{\mathrm{n}}, \left[1-\tilde{I}^3_{b_{ij}}\right]^{\mathrm{n}}\right)}, \\ \sqrt[3]{1-\max\left(\left[1-\tilde{F}^3_{a_{ij}}\right]^{\mathrm{n}}, \left[1-\tilde{F}^3_{b_{ij}}\right]^{\mathrm{n}}\right)} \end{bmatrix}_{mxn}$$

$$= \begin{bmatrix} \min\left(\left|\tilde{T}_{a_{ij}}\right|^n, \left|\tilde{T}_{b_{ij}}\right|^n\right), \sqrt[3]{1-\left[1-\min\left(\left[1-\tilde{I}^3_{a_{ij}}\right]^{\mathrm{n}}, \left[1-\tilde{I}^3_{b_{ij}}\right]^{\mathrm{n}}\right)\right]}, \\ \sqrt[3]{1-\left[1-\min\left(\left[1-\tilde{F}^3_{a_{ij}}\right]^{\mathrm{n}}, \left[1-\tilde{F}^3_{b_{ij}}\right]^{\mathrm{n}}\right)\right]} \end{bmatrix}_{mxn}$$

$$= \begin{bmatrix} \min\left(\left|\tilde{T}_{a_{ij}}\right|^n, \left|\tilde{T}_{b_{ij}}\right|^n\right), \\ max\left(\sqrt[3]{1-\left[1-\tilde{I}^3_{a_{ij}}\right]^{\mathrm{n}}}, \sqrt[3]{1-\left[1-\tilde{I}^3_{b_{ij}}\right]^{\mathrm{n}}}\right) \\ \max\left(\sqrt[3]{1-\left[1-\tilde{F}^3_{a_{ij}}\right]^{\mathrm{n}}}, \sqrt[3]{1-\left[1-\tilde{F}^3_{b_{ij}}\right]^{\mathrm{n}}}\right) \end{bmatrix}_{mxn}$$

$$\mathbb{N}^n \wedge \wp^n = \left[\left(\left|\tilde{T}_{a_{ij}}\right|^n, \sqrt[3]{1-\left[1-\tilde{I}^3_{a_{ij}}\right]^{\mathrm{n}}}, \sqrt[3]{1-\left[1-\tilde{I}^3_{b_{ij}}\right]^{\mathrm{n}}}\right) \wedge \left(\left|\tilde{T}_{b_{ij}}\right|^n, \sqrt[3]{1-\left[1-\tilde{F}^3_{a_{ij}}\right]^{\mathrm{n}}}, \sqrt[3]{1-\left[1-\tilde{F}^3_{b_{ij}}\right]^{\mathrm{n}}}\right)\right]$$

$$= \begin{bmatrix} \min\left(\left|\tilde{T}_{a_{ij}}\right|^n, \left|\tilde{T}_{b_{ij}}\right|^n\right), \\ max\left(\sqrt[3]{1-\left[1-\tilde{I}^3_{a_{ij}}\right]^{\mathrm{n}}}, \sqrt[3]{1-\left[1-\tilde{I}^3_{b_{ij}}\right]^{\mathrm{n}}}\right) \\ \max\left(\sqrt[3]{1-\left[1-\tilde{F}^3_{a_{ij}}\right]^{\mathrm{n}}}, \sqrt[3]{1-\left[1-\tilde{F}^3_{b_{ij}}\right]^{\mathrm{n}}}\right) \end{bmatrix}_{mxn}$$

$= (\mathbb{N} \wedge \wp)^n$

Hence $(\mathbb{N} \wedge \wp)^n = \mathbb{N}^n \wedge \wp^n$

Similarly we can prove $(\mathbb{N} \vee \wp)^n = \mathbb{N}^n \vee \wp^n$

**Theorem 3.14.**

For ℕ, ℘ ∈ $FN_{mxn}$ and n > 0, we have (ℕ ⊕ ℘)$^n$ ≠ ℕ$^n$ ⊕ ℘$^n$

Proof: Let (ℕ ⊕ ℘)$^n$

$$\mathbb{N} \oplus \wp = \left(\langle \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}, \tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}} \rangle\right)_{mxn}$$

$$(\mathbb{N} \oplus \wp)^n = \left(\langle \left(\sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}\right)^n, \sqrt[3]{1-\left[1-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}\right]^n}, \sqrt[3]{1-\left[1-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}\right]^n} \rangle\right)_{mxn}$$

$$\mathbb{N}^n = \left(\langle \left|\tilde{T}_{a_{ij}}\right|^n, \sqrt[3]{1-\left[1-\tilde{I}^3_{a_{ij}}\right]^n}, \sqrt[3]{1-\left[1-\tilde{F}^3_{a_{ij}}\right]^n} \rangle\right)_{mxn}$$

$$\wp^n = \left(\langle \left|\tilde{T}^3_{b_{ij}}\right|^n, \sqrt[3]{1-\left[1-\tilde{I}^3_{b_{ij}}\right]^n}, \sqrt[3]{1-\left[1-\tilde{F}^3_{b_{ij}}\right]^n} \rangle\right)_{mxn}$$

$$\mathbb{N}^n \oplus \wp^n = \begin{bmatrix} \sqrt[3]{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}} - \tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}, \left(\sqrt[3]{1-\left[1-\tilde{I}^3_{a_{ij}}\right]^n}\right)^{n} \left(\sqrt[3]{1-\left[1-\tilde{I}^3_{b_{ij}}\right]^n}\right)^{n}, \\ \left(\sqrt[3]{1-\left[1-\tilde{F}^3_{a_{ij}}\right]^n}\right)^{n} \left(\sqrt[3]{1-\left[1-\tilde{F}^3_{b_{ij}}\right]^n}\right)^{n} \end{bmatrix}_{mxn}$$

Hence (ℕ ⊕ ℘)$^n$ ≠ ℕ$^n$ ⊕ ℘$^n$

## 4. A new operation (@) on Fermatean neutrosophic matrices

Motivated by the existing operations presented in [29].A novel operation on Fermatean neutrosophic matrices, symbolized by (@), is presented along with the demonstration of its favorable characteristics.

**Definition 4.1** If ℕ = $\langle \tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}} \rangle$ and ℘=$\langle \tilde{T}_{b_{ij}}, \tilde{I}_{b_{ij}}, \tilde{F}_{b_{ij}} \rangle$are two Fermatean Neutrosophic Matrices ,then the new operation of FNM is defined by

$$\mathbb{N} \,@\wp = \left[\langle \sqrt[3]{\frac{\tilde{T}^3_{a_{ij}}+\tilde{T}^3_{b_{ij}}}{2}}, \sqrt[3]{\frac{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}}{2}}, \sqrt[3]{\frac{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}}{2}} \rangle\right]_{mxn}$$

**Note :**

It is obvious that for any two Fermatean neutrosophic matrices ℕ and ℘, the matrix ℕ @℘ is also a Fermatean neutrosophic matrix.

i.e., $0 \le \frac{\tilde{T}^3_{a_{ij}}+\tilde{T}^3_{b_{ij}}}{2} + \frac{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}}{2} + \frac{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}}{2}$

$$\le \frac{\tilde{T}^3_{a_{ij}} + \tilde{I}^3_{a_{ij}} + \tilde{F}^3_{a_{ij}}}{2} + \frac{\tilde{T}^3_{b_{ij}} + \tilde{I}^3_{b_{ij}} + \tilde{F}^3_{b_{ij}}}{2} \le \frac{1}{2} + \frac{1}{2} = 1$$

**Theorem 4.1.** If ℕ is a Fermatean Neutrosophic Matrix, then ℕ @ ℕ = ℕ.

Proof: Let ℕ @ ℕ = $\left(\langle\sqrt[3]{\frac{\tilde{T}_{a_{ij}}^3+\tilde{T}_{a_{ij}}^3}{2}},\ \sqrt[3]{\frac{\tilde{I}_{a_{ij}}^3+\tilde{I}_{a_{ij}}^3}{2}},\ \sqrt[3]{\frac{\tilde{F}_{a_{ij}}^3+\tilde{F}_{a_{ij}}^3}{2}}\ \rangle\right)$

$$= \left(\langle\left(\sqrt[3]{\frac{\tilde{T}_{a_{ij}}^3+\tilde{T}_{a_{ij}}^3}{2}}\right)^2, \left(\sqrt[3]{\frac{\tilde{I}_{a_{ij}}^3+\tilde{I}_{a_{ij}}^3}{2}}\right)^2, \left(\sqrt[3]{\frac{\tilde{F}_{a_{ij}}^3+\tilde{F}_{a_{ij}}^3}{2}}\right)^2 \rangle\right)_{mxn}$$

$$=\left(\langle\left(\sqrt[3]{\frac{2\tilde{T}_{a_{ij}}^3}{2}}\right)^2, \left(\sqrt[3]{\frac{2\tilde{I}_{a_{ij}}^3}{2}}\right)^2, \left(\sqrt[3]{\frac{2\tilde{F}_{a_{ij}}^3}{2}}\right)^2 \rangle\right) = \left(\langle\tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}})\rangle\right)_{mxn}$$

Since $\tilde{T}_{a_{ij}}^3 \le \tilde{T}_{a_{ij}}, \tilde{I}_{a_{ij}}^3 \le \tilde{I}_{a_{ij}}, \tilde{F}_{a_{ij}}^3 \le \tilde{F}_{a_{ij}}$.

Note 4.2 . if x, y ∈ [0, 1], then $xy \le \frac{x\ +\ y}{2}$ , $\frac{x\ +y}{2} \le x + y - xy$.

**Theorem 4.2.** Let ℕ, ℘ $\in FN_{mxn}$, then

(i) (ℕ ⊕ ℘) ∨ (ℕ @ ℘) = ℕ ⊕ ℘,

(ii) (ℕ ⊗ ℘) ∧ (ℕ @ ℘) = ℕ ⊗ ℘,

(iii) (ℕ ⊕ ℘) ∧ (ℕ @ ℘) = ℕ @ ℘,

(iv) (ℕ ⊗ ℘) ∨ (ℕ @ ℘) = ℕ @ ℘

Proof:

(i) (ℕ ⊕ ℘) ∨ (ℕ @ ℘) = ℕ ⊕ ℘

Let (ℕ ⊕ ℘) ∨ (ℕ @ ℘)

$$= \begin{bmatrix} max\left(\sqrt[3]{\tilde{T}_{a_{ij}}^3 + \tilde{T}_{b_{ij}}^3 - \tilde{T}_{a_{ij}}^3\tilde{T}_{b_{ij}}^3}, \sqrt[3]{\frac{\tilde{T}_{a_{ij}}^3+\tilde{T}_{b_{ij}}^3}{2}}\right), \\ min\left(\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}, \sqrt[3]{\frac{\tilde{I}_{a_{ij}}^3+\tilde{I}_{b_{ij}}^3}{2}}\right), min\left(\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}, \sqrt[3]{\frac{\tilde{F}_{a_{ij}}^3+\tilde{F}_{b_{ij}}^3}{2}}\right) \end{bmatrix}_{mxn}$$

$$=\left[\sqrt[3]{\tilde{T}_{a_{ij}}^3 + \tilde{T}_{b_{ij}}^3 - \tilde{T}_{a_{ij}}^3\tilde{T}_{b_{ij}}^3}, \tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}, \tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}\right]_{mxn}$$

= ℕ ⊕ ℘

(ii) (ℕ ⊗ ℘) ∧ (ℕ @ ℘) = ℕ ⊗℘

Let (ℕ ⊗ ℘) ∧ (ℕ @ ℘)

=

$$=\left[\left\langle \begin{matrix} min\left(\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}, \sqrt[3]{\frac{\tilde{T}^3_{a_{ij}}+\tilde{T}^3_{b_{ij}}}{2}}\right), max\left(\sqrt[3]{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\frac{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}}{2}}\right) \\ max\left(\sqrt[3]{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}}, \sqrt[3]{\frac{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}}{2}}\right) \end{matrix} \right\rangle\right]_{mxn}$$

$$= \left[\langle\left(\langle\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}, \sqrt[3]{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}}\rangle\right)\rangle\right]_{mxn}$$

$= \mathbb{N} \otimes \wp$

(iii) $(\mathbb{N} \oplus \wp) \wedge (\mathbb{N} \ @\ \wp) = \mathbb{N} \ @\ \wp$

Let $(\mathbb{N} \oplus \wp) \wedge (\mathbb{N} \ @\ \wp)$

$$=\left[\begin{matrix} min\left(\sqrt[3]{\tilde{T}^3_{a_{ij}}+\tilde{T}^3_{b_{ij}}-\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}}, \sqrt[3]{\frac{\tilde{T}^3_{a_{ij}}+\tilde{T}^3_{b_{ij}}}{2}}\right), \\ max\left(\tilde{I}_{a_{ij}}\tilde{I}_{b_{ij}}, \sqrt[3]{\frac{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}}{2}}\right), \\ max\left(\tilde{F}_{a_{ij}}\tilde{F}_{b_{ij}}, \sqrt[3]{\frac{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}}{2}}\right) \end{matrix}\right]_{mxn}$$

$$= \left(\langle\sqrt[3]{\frac{\tilde{T}^3_{a_{ij}}+\tilde{T}^3_{b_{ij}}}{2}}, \sqrt[3]{\frac{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}}{2}}, \sqrt[3]{\frac{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}}{2}}\rangle\right)_{mxn} = (\mathbb{N} \ @\ \wp)$$

(iv) $(\mathbb{N} \otimes \wp) \vee (\mathbb{N} \ @\ \wp) = \mathbb{N} \ @\ \wp$

Let $(\mathbb{N} \otimes \wp) \vee (\mathbb{N} \ @\ \wp)$

$$=\left[\left\langle \begin{matrix} max\left(\tilde{T}^3_{a_{ij}}\tilde{T}^3_{b_{ij}}, \sqrt[3]{\frac{\tilde{T}^3_{a_{ij}}+\tilde{T}^3_{b_{ij}}}{2}}\right), min\left(\sqrt[3]{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}-\tilde{I}^3_{a_{ij}}\tilde{I}^3_{b_{ij}}}, \sqrt[3]{\frac{\tilde{I}^3_{a_{ij}}+\tilde{I}^3_{b_{ij}}}{2}}\right) \\ min\left(\sqrt[3]{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}-\tilde{F}^3_{a_{ij}}\tilde{F}^3_{b_{ij}}}, \sqrt[3]{\frac{\tilde{F}^3_{a_{ij}}+\tilde{F}^3_{b_{ij}}}{2}}\right) \end{matrix} \right\rangle\right]_{mxn}$$

$$= \left( \langle \sqrt[3]{\frac{\tilde{T}^3_{a_{ij}} + \tilde{T}^3_{b_{ij}}}{2}}, \sqrt[3]{\frac{\tilde{I}^3_{a_{ij}} + \tilde{I}^3_{b_{ij}}}{2}}, \sqrt[3]{\frac{\tilde{F}^3_{a_{ij}} + \tilde{F}^3_{b_{ij}}}{2}} \rangle \right)_{mxn} = (\mathbb{N} \; @ \; \wp)$$

Hence, $(\mathbb{N} \otimes \wp) \vee (\mathbb{N} \; @ \; \wp) = \mathbb{N} \; @ \; \wp$

## 5. Conclusion

This research aims to enrich the domain of neutrosophic matrix theory and neutrosophic logic by investigating Fermatean neutrosophic matrices (FNMs), a novel approach to handling uncertainty. The paper study some fundamental algebraic operations on Fermatean neutrosophic matrices. The Fermatean neutrosophic matrices considered as a generalization of Fermatean fuzzy matrices, intuitionistic fuzzy matrices, Pythagorean fuzzy matrices, and Pythagorean neutrosophic matrices. The paper shows that the properties of Fermatean neutrosophic matrices are consistent with the properties of the standard operations. To conclude, a novel operation (@) on Fermatean neutrosophic matrices is defined and distributive rules are examined. In the upcoming, it will be significant to probe how the suggested aggregating operators of FNMs might be used in decision-making problem, Information fusion, and Operations research under neutrosophic environment.

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# Plithogenic functions value

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**Abstract:** The goal of this paper is to find a formula through which we find all the values of the plithogenic functions. we found this formula and proved it, in addition to providing a definition of the exponential, logarithmic, trigonometric plithogenic function. Also, the absolute value of the plithogenic number was defined.

**Keywords:** plithogenic functions, plithogenic number, plithogenic exponential.

## 1. Introduction

The genesis, origination, formation, development, and evolution of new entities through dynamics of contradictory and/or neutral and/or noncontradictory multiple old entities is known as plithogenic. Plithogeny advocates for the integration of theories from several fields.

We use numerous "knowledges" from domains like soft sciences, hard sciences, arts and literature theories, etc. as "entities" in this study, this is what Smarandache introduced, as he presented a study on Plithogeny, Plithogenic Set, Logic, Probability, and Statistics [2], in addition to presenting Introduction to the Symbolic Plithogenic Algebraic Structures (revisited), through which he discussed several ideas, including mathematical operations on Plithogenic numbers [1]. Also, An Overview of Plithogenic Set and Symbolic Plithogenic Algebraic Structures was discussed by him [3]. It is thought that the symbolic n-plithogenic sets are a good place to start when developing algebraic extensions for other classical structures including rings, vector spaces, modules, and equations [4-5-6-7].

Paper is divided into four parts. provides an introduction in the first portion, which includes a review of Plithogenic science. A few definitions of a Plithogenic and operations with plithogenic numbers are covered in the second section. the third section defined plithogenic functions. The paper's conclusion is provided in the fourth section.

## 2. Preliminaries

### 2.1. Definition of Plithogenic Numbers (PN) [1]

The numbers of the form $PN = a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n$ defined as above are called Plithogenic Numbers, where $a_nP_n$ is called the leading (strongest) term.

### 2.2 Operations with Plithogenic Numbers [1]

Let's consider two plithogenic numbers:

$$PN_1 = a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n$$

$$PN_2 = b_0 + b_1P_1 + b_2P_2 + \cdots + b_nP_n$$

*2.2.1. Addition of Plithogenic Numbers*

$$PN_1 + PN_2 = (a_0 + b_0) + \sum_{i=1}^{n} (a_i + b_i) P_i$$

*2.2.2. Subtraction of Plithogenic Numbers*

$$PN_1 - PN_2 = (a_0 - b_0) + \sum_{i=1}^{n} (a_i - b_i) P_i$$

(SPS, +) is a Symbolic Plithogenic Commutative Group

*2.2.3. Scalar Multiplication of Plithogenic Numbers*

$$c.PN_1 = c.(a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n) = c.a_0 + c.a_1P_1 + c.a_2P_2 + \cdots + c.a_nP_n$$

*2.2.4. Multiplication of and Ppower of Plithogenic Numbers*

$$PN_1.PN_2 = (a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n).(b_0 + b_1P_1 + b_2P_2 + \cdots + b_nP_n)$$

and then one multiplies them, term by term $(a_iP_i).(a_jP_j) = a_i.a_j.P_{max\{i,j\}}$ ,where · is the classical multiplication, as in classical algebra, using the above multiplication of symbolic plithogenic components.

*2.2.5. Division of Symbolic Plithogenic Components*

$$\frac{P_i}{P_j} = \begin{cases} x_0 + x_1P_1 + x_2P_2 + \cdots + x_jP_j + P_i & x_0 + x_1 + x_2 + \cdots + x_j = 0 \quad i > j \\ x_0 + x_1P_1 + x_2P_2 + \cdots + x_iP_i & x_0 + x_1 + x_2 + \cdots + x_i = 1 \quad i = j \\ \emptyset & i < j \end{cases}$$

where all coefficients $x_0, x_1, x_2, \dots, x_i, \dots \in$ SPS.

*2.2.5. Division of Symbolic Plithogenic Numbers*

Let consider two symbolic plithogenic numbers as below:

$$PN_r = a_0 + a_1P_1 + a_2P_2 + \cdots + a_rP_r$$

$$PN_s = b_0 + b_1P_1 + b_2P_2 + \cdots + b_sP_s$$

$$\frac{PN_r}{PN_s} = \begin{cases} none, one \;\; many & r \geq s \\ \emptyset & r < s \end{cases}$$

## 3. Plithogenic functions value

**Definition 1**

Let $a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n$ *and* $b_0 + b_1P_1 + b_2P_2 + \cdots + b_nP_n$ are numbers of plithogenic, then we say that:

$a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n \leq b_0 + b_1P_1 + b_2P_2 + \cdots + b_nP_n$ if and only if:

$a_0 \le b_0$, $a_0 + a_1 \le b_0 + b_1$, $a_0 + a_1 + a_2 \le b_0 + b_1 + b_2$, ... , and $a_0 + a_1 + a_2 + \cdots + a_n \le b_0 + b_1 + b_2 + \cdots + b_n$

**Definition 2**

We say that the plithogenic number $a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n$ is positive if the following conditions is met:

$a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n \ge 0 + 0P_1 + 0P_2 + \cdots + 0P_n$, then:

$$a_0 \ge 0 \;\; , a_0 + a_1 \ge 0 \;\; , \ldots . , and \;\; a_0 + a_1 + a_2 + \cdots + a_n \ge 0$$

**Definition 3**

Let $a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n$ is number of plithogenic, and $f$ is a plithogenic function, then:

$$f(a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n) = f(a_0) + P_1[f(a_0 + a_1) - f(a_0)] + P_2[f(a_0 + a_1 + a_2) - f(a_0 + a_1)] + \cdots + P_n[f(a_0 + a_1 + a_2 + \cdots + a_n) - f(a_0 + a_1 + \cdots + a_{n-1})]$$

Proof:

**First: we will proof it for $n = 1$**

$$f(a_0 + a_1P_1) = f(a_0) + P_1[f(a_0 + a_1) - f(a_0)]$$

$$(a_0 + a_1P_1)^{\acute{n}} = a_0^{\acute{n}} + P_1\left[(a_0 + a_1)^{\acute{n}} - a_0^{\acute{n}}\right]$$

for $\acute{n} = 1$, we get:

$(a_0 + a_1P_1) = a_0 + P_1[a_0 + a_1 - a_0] = a_0 + a_1P_1$ (The formula is correct for $\acute{n} = 1$)

Assume The formula is correct for $\acute{n} = k$

$$(a_0 + a_1P_1)^k = a_0^k + P_1[(a_0 + a_1)^k - a_0^k]$$

for $\acute{n} = 1$, we get:

Let's check it for $\acute{n} = k + 1$

$$(a_0 + a_1P_1)^{k+1} = a_0^{k+1} + P_1[(a_0 + a_1)^{k+1} - a_0^{k+1}]$$

$$(a_0 + a_1P_1)^{k+1} = (a_0 + a_1P_1)^k(a_0 + a_1P_1) = (a_0^k + P_1[(a_0 + a_1)^k - a_0^k])(a_0 + a_1P_1)$$

$$= a_0^{k+1} + a_0^k a_1P_1 + P_1[a_0(a_0 + a_1)^k - a_0^{k+1}] + P_1[a_1P_1(a_0 + a_1)^k - a_0^k a_1P_1]$$

$$= a_0^{k+1} + a_0^k a_1P_1 + a_0(a_0 + a_1)^k P_1 - a_0^{k+1}P_1 + a_1(a_0 + a_1)^k P_1 - a_0^k a_1P_1$$

$$= a_0^{k+1} + (a_0 + a_1)^k(a_0 + a_1)P_1 - a_0^{k+1}P_1$$

$$= a_0^{k+1} + P_1[(a_0 + a_1)^{k+1} - a_0^{k+1}] \text{ (The formula is correct for } \acute{n} = k + 1)$$

then: $(a_0 + a_1P_1)^{\acute{n}} = a_0^{\acute{n}} + P_1\left[(a_0 + a_1)^{\acute{n}} - a_0^{\acute{n}}\right]$ is true for any $\acute{n}$

Let's proof it for:

$$e^{a_0 + a_1P_1} = e^{a_0} + P_1[e^{a_0 + a_1} - e^{a_0}]$$

$$e^{a_0+a_1P_1} = \sum_{\acute{n}=0}^{\infty} \frac{1}{\acute{n}!}(a_0 + a_1P_1)^{\acute{n}}$$

$$= \sum_{\acute{n}=0}^{\infty} \frac{1}{\acute{n}!}\left(a_0{}^{\acute{n}} + P_1\left[(a_0 + a_1)^{\acute{n}} - a_0{}^{\acute{n}}\right]\right)$$

$$= \sum_{\acute{n}=0}^{\infty} \frac{a_0{}^{\acute{n}}}{\acute{n}!} + P_1\left[\sum_{\acute{n}=0}^{\infty} \frac{(a_0 + a_1)^{\acute{n}}}{\acute{n}!} + \sum_{\acute{n}=0}^{\infty} \frac{a_0{}^{\acute{n}}}{\acute{n}!}\right]$$

$$= e^{a_0} + P_1[e^{a_0+a_1} - e^{a_0}]$$

**Second: we will proof it for $n = 2$**

$$f(a_0 + a_1P_1 + a_2P_2) = f(a_0) + P_1[f(a_0 + a_1 + a_2) - f(a_0 + a_1)]$$

$$(a_0 + a_1P_1 + a_2P_2)^{\acute{n}} = a_0{}^{\acute{n}} + P_1\left[(a_0 + a_1)^{\acute{n}} - a_0{}^{\acute{n}}\right] + P_2\left[(a_0 + a_1 + a_2)^{\acute{n}} - (a_0 + a_1)^{\acute{n}}\right]$$

for $\acute{n} = 1$, we get:

$(a_0 + a_1P_1 + a_2P_2) = a_0 + P_1[a_0 + a_1 - a_0] + P_2[a_0 + a_1 + a_2 - (a_0 + a_1)] = a_0 + a_1P_1 + a_2P_2$

(The formula is correct for $\acute{n} = 1$)

Assume The formula is correct for $\acute{n} = k$

$$(a_0 + a_1P_1 + a_2P_2)^k = a_0{}^k + P_1[(a_0 + a_1)^k - a_0{}^k] + P_2[(a_0 + a_1 + a_2)^k - (a_0 + a_1)^k]$$

for $n = 1$, we get:

Let's check it for $\acute{n} = k + 1$

$$(a_0 + a_1P_1 + a_2P_2)^{k+1} = a_0{}^{k+1} + P_1[(a_0 + a_1)^{k+1} - a_0{}^{k+1}] + P_2[(a_0 + a_1 + a_2)^{k+1} - (a_0 + a_1)^{k+1}]$$

$$(a_0 + a_1P_1 + a_2P_2)^{k+1} = (a_0 + a_1P_1 + a_2P_2)^k(a_0 + a_1P_1 + a_2P_2)$$

$$= (a_0{}^k + P_1[(a_0 + a_1)^k - a_0{}^k] + P_2[(a_0 + a_1 + a_2)^k - (a_0 + a_1)^k])(a_0 + a_1P_1 + a_2P_2)$$

$$\begin{aligned} &= a_0a_0{}^k + P_1[a_0(a_0 + a_1)^k - a_0a_0{}^k] + P_2[a_0(a_0 + a_1 + a_2)^k - a_0(a_0 + a_1)^k] + a_0{}^ka_1P_1 \\ &\quad + a_1P_1P_1[(a_0 + a_1)^k - a_0{}^k] + a_1P_1P_2[(a_0 + a_1 + a_2)^k - (a_0 + a_1)^k] + a_0{}^ka_2P_2 \\ &\quad + a_2P_2P_1[(a_0 + a_1)^k - a_0{}^k] + a_2P_2P_2[(a_0 + a_1 + a_2)^k - (a_0 + a_1)^k] \\ &= a_0a_0{}^k + P_1[a_0(a_0 + a_1)^k - a_0a_0{}^k] + P_2[a_0(a_0 + a_1 + a_2)^k - a_0(a_0 + a_1)^k] + a_0{}^ka_1P_1 \\ &\quad + P_1[a_1(a_0 + a_1)^k - a_1a_0{}^k] + P_2[a_1(a_0 + a_1 + a_2)^k - a_1(a_0 + a_1)^k] + a_0{}^ka_2P_2 \\ &\quad + P_2[a_2(a_0 + a_1)^k - a_2a_0{}^k] + P_2[a_2(a_0 + a_1 + a_2)^k - a_2(a_0 + a_1)^k] \end{aligned}$$

$$\begin{aligned} &= a_0{}^{k+1} + P_1[a_0(a_0 + a_1)^k + a_1(a_0 + a_1)^k - a_0{}^{k+1}] \\ &\quad + P_2[a_0(a_0 + a_1 + a_2)^k + a_1(a_0 + a_1 + a_2)^k + a_2(a_0 + a_1 + a_2)^k - a_0(a_0 + a_1)^k \\ &\quad - a_1(a_0 + a_1)^k] \end{aligned}$$

$$= a_0{}^{k+1} + P_1[(a_0 + a_1)(a_0 + a_1)^k - a_0{}^{k+1}] + P_2[(a_0 + a_1 + a_2)(a_0 + a_1 + a_2)^k - (a_0 + a_1)(a_0 + a_1)^k]$$

$$= {a_0}^{k+1} + P_1[(a_0 + a_1)^{k+1} - {a_0}^{k+1}] + P_2[(a_0 + a_1 + a_2)^{k+1} - (a_0 + a_1)^{k+1}]$$

(The formula is correct for $ń = k + 1$)

then: $(a_0 + a_1P_1 + a_2P_2)^{ń} = {a_0}^{ń} + P_1\left[(a_0 + a_1)^{ń} - {a_0}^{ń}\right] + P_2\left[(a_0 + a_1 + a_2)^{ń} - (a_0 + a_1)^{ń}\right]$ is true for any $ń$

Let's proof it for:

$$e^{a_0+a_1P_1+a_2P_2} = e^{a_0} + P_1[e^{a_0+a_1} - e^{a_0}] + P_2[e^{a_0+a_1+a_2} - e^{a_0+a_1}]$$

$$e^{a_0+a_1P_1+a_2P_2} = \sum_{ń=0}^{\infty} \frac{1}{ń!}(a_0 + a_1P_1 + a_2P_2)^{ń}$$

$$= \sum_{ń=0}^{\infty} \frac{1}{ń!}\left({a_0}^{ń} + P_1\left[(a_0 + a_1)^{ń} - {a_0}^{ń}\right] + P_2\left[(a_0 + a_1 + a_2)^{ń} - (a_0 + a_1)^{ń}\right]\right)$$

$$= \sum_{ń=0}^{\infty} \frac{{a_0}^{ń}}{ń!} + P_1\left[\sum_{ń=0}^{\infty} \frac{(a_0 + a_1)^{ń}}{ń!} + \sum_{ń=0}^{\infty} \frac{{a_0}^{ń}}{ń!}\right] + P_2\left[\sum_{ń=0}^{\infty} \frac{(a_0 + a_1 + a_2)^{ń}}{ń!} + \sum_{ń=0}^{\infty} \frac{(a_0 + a_1)^{ń}}{ń!}\right]$$

$$= e^{a_0} + P_1[e^{a_0+a_1} - e^{a_0}] + P_2[e^{a_0+a_1+a_2} - e^{a_0+a_1}]$$

Hence, we can apply:

$$f(a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n)$$
$$= f(a_0) + P_1[f(a_0 + a_1) - f(a_0)] + P_2[f(a_0 + a_1 + a_2) - f(a_0 + a_1)] + \cdots$$
$$+ P_n[f(a_0 + a_1 + a_2 + \cdots + a_n) - f(a_0 + a_1 + \cdots + a_{n-1})]$$

for any $n$.

Example1

$$e^{6+4P_1-P_2+2P_3+7P_4} = e^6 + P_1[e^{10} - e^6] + P_2[e^9 - e^{10}] + P_3[e^{11} - e^9] + P_2[e^{18} - e^{11}]$$

**Remark 1**

Absolute value of the plithogenic number defined as follows:

$$|a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n|$$
$$= |a_0| + P_1[|a_0 + a_1| - |a_0|] + P_2[|a_0 + a_1 + a_2| - |a_0 + a_1|] + \cdots$$
$$+ P_n[|a_0 + a_1 + a_2 + \cdots + a_n| - |a_0 + a_1 + \cdots + a_{n-1}|] \geq 0$$

Example 2

- $|9 - 4P_1 + 2P_2 - P_3| = 9 + P_1[|9 - 4| - |9|] + P_2[|9 - 4 + 2| - |9 - 4|] + P_3[|9 - 4 + 2 - 1| - |9 - 4 + 2|]$ $= 9 - 4P_1 + 2P_2 - P_3 > 0$

- $|5 - 7P_1 - 8P_2| = 5 + P_1[|5 - 7| - |5|] + P_2[|5 - 7 - 8| - |5 - 7|]$

  $= 5 - 3P_1 + 8P_2 > 0$

clear the Absolute value of the plithogenic number according to the definition 2.

**Definition 4**

1) Plithogenic exponential functions is defined as formula:

$$e^{(a_0+a_1P_1+a_2P_2+\cdots+a_nP_n)x} = e^{a_0x} + P_1\left[e^{(a_0+a_1)x} - e^{a_0x}\right] + P_2\left[e^{(a_0+a_1+a_2)x} - e^{(a_0+a_1)x}\right] + \cdots + P_n\left[e^{(a_0+a_1+a_2+\cdots+a_n)x} - e^{(a_0+a_1+a_2+\cdots+a_{n-1})}\right]$$

2) Plithogenic logarithmic functions is defined as formula:

$$ln\big((a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n)x\big) = ln(a_0x) + P_1\big[ln\big((a_0 + a_1)x\big) - ln(a_0x)\big] + P_2\big[ln\big((a_0 + a_1 + a_2)x\big) - ln\big((a_0 + a_1)x\big)\big] + \cdots + P_n\big[ln\big((a_0 + a_1 + a_2 + \cdots + a_{n-1})x\big) - ln\big((a_0 + a_1 + a_2 + \cdots + a_{n-1})x\big)\big]$$

where: $(a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n)x > 0$

**Definition 5**

Plithogenic trigonometric functions is defined as formulas:

1) $sin\big((a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n)x\big) = sin(a_0x) + P_1\big[sin\big((a_0 + a_1)x\big) - sin(a_0x)\big] + P_2\big[sin\big((a_0 + a_1 + a_2)x\big) - sin\big((a_0 + a_1)x\big)\big] + \cdots + P_n\big[sin\big((a_0 + a_1 + a_2 + \cdots + a_{n-1})x\big) - sin\big((a_0 + a_1 + a_2 + \cdots + a_{n-1})x\big)\big]$
2) $cos\big((a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n)x\big) = cos(a_0x) + P_1\big[cos\big((a_0 + a_1)x\big) - cos(a_0x)\big] + P_2\big[cos\big((a_0 + a_1 + a_2)x\big) - cos\big((a_0 + a_1)x\big)\big] + \cdots + P_n\big[cos\big((a_0 + a_1 + a_2 + \cdots + a_{n-1})x\big) - cos\big((a_0 + a_1 + a_2 + \cdots + a_{n-1})x\big)\big]$
3) $tan\big((a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n)x\big) = tan(a_0x) + P_1\big[tan\big((a_0 + a_1)x\big) - tan(a_0x)\big] + P_2\big[tan\big((a_0 + a_1 + a_2)x\big) - tan\big((a_0 + a_1)x\big)\big] + \cdots + P_n\big[tan\big((a_0 + a_1 + a_2 + \cdots + a_{n-1})x\big) - tan\big((a_0 + a_1 + a_2 + \cdots + a_{n-1})x\big)\big]$
4) $cot\big((a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n)x\big) = cot(a_0x) + P_1\big[cot\big((a_0 + a_1)x\big) - cot(a_0x)\big] + P_2\big[cot\big((a_0 + a_1 + a_2)x\big) - cot\big((a_0 + a_1)x\big)\big] + \cdots + P_n\big[cot\big((a_0 + a_1 + a_2 + \cdots + a_{n-1})x\big) - cot\big((a_0 + a_1 + a_2 + \cdots + a_{n-1})x\big)\big]$

## 4. Conclusions

This paper is considered important in the field of plithogenic, as it presented the concept of the plithogenic function, and how to calculate its values by finding a formula through which we find the values of the plithogenic functions.

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# Neutrosophic Price Indices with an Applied Study on Pesticide Prices

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**Abstract:** An index is a tool for comparing a phenomenon or several phenomena dating back to different periods to know the amount of change in the phenomena or the difference between them. For example, we compare the price of a commodity in a particular year with its price in a given year, compare the prices of a group of commodities in one year with their prices in another year, or compare the price of a commodity in two different places. For example, we compare the price of a commodity in one city with its price in another city. This paper presents a study of these Indices from a neutrosophic point of view. Which allows us to study Indices with uncertain or non-well-defined price data. In addition, access to results that determine the extent of the change in prices as accurately as possible, and thus the optimal planning for the next stage and the development of appropriate solutions. An applied example was presented to study the change in the prices of pesticides used to combat the "vine stem borer" insect using the neutrosophic Indices.

**Keywords:** Indices, Neutrosophic Logic, Base Year, Studying Year, Comparison Year, Target Year.

## 1. Introduction

Neutrosophic means the study of ideas and concepts that are neither right nor wrong, but between that, and this means (neutrality, indeterminacy, ambiguity, contradiction, hesitancy), in which every field of knowledge and experience has its neutrosophic part which contains indeterminacy. The first scientist who established the neutrosophic theory was the American philosopher and mathematician F. Smarandache, he presented neutrosophic logic in 1995 as a generalization of fuzzy logic [1,2]. As an extension of this, Ahmed Salama presented the theory of classical neutrosophic sets as a generalization of the theory of classical sets [3,4]. The neutrosophic has grown significantly in recent years and many researchers have worked with it around the world because it formed a real revolution in science through its application in many disciplines and scientific and practical fields [5-12]. In this research, we highlight the study of the Indices from a neutrosophic point of view. Which allows us to study Indices with uncertain or non-well-defined price data. This matter is not available for study in the classical logic that does not recognize the existence of uncertain cases.

The neutrosophic indices are numerous because of the different ways they are created or because of the different phenomena that he compares. According to the phenomena that he compares, we find a large number of indices, for example, the neutrosophic index for wages and the neutrosophic index for agriculture...etc.

When creating the index for prices, we compare the prices of a year, which we will call the "studying year" or the "target year" (comparison year), and symbolize its prices with the symbol $Np_I$, with the prices of another year, which is the "base year", and symbolize its prices with the symbol $Np_0$. This comparison is carried out in different ways, depending on the establishment of the index of prices.

Before we start creating and applying the neutrosophic index, we must take the following three requirements into consideration:

1- Determine the base year. (It is preferable that this year be free from any abnormal circumstances).
2- Choosing the commodities that will be included in the calculation of the Neutrosophic Index of Prices. (It is preferable that these commodities are stable in quality and representative of people's habits).
3- Determining prices. (It is preferable to take them from official authorities). Here, we can take the prices in a neutrosophic way (that is, taking into account the change in the price of the commodity in the "base year" and the "studying year").

The aim of the study of the neutrosophic indices is to determine the extent of the change in prices and thus to study the economic or living ratio in the country for the sake of optimal planning for the future and for public spending. In addition to developing appropriate solutions for all problems to serve this goal.

## 2. The Theoretical Part of the Case Study is Supported by Applied Examples

### 2.1 Neutrosophic Index (NSN)

It is a tool for comparing a phenomenon or several phenomena dating back to different periods to know the amount of change in the phenomena, or between two different places, according to the neutrosophic logic which takes into account non-specific changes that occur and affect the study.

### 2.2 Simple Neutrosophic Ratio Indices:

It is either the neutrosophic ratio of one phenomenon value in the "comparison year" to the "base year", or the neutrosophic ratio of one phenomenon value in the "comparison place" to the "base place". Note that, the authors will be interested here in the general ways to create the neutrosophic index of prices. This Index is calculated from the following mathematical formula:

$$\text{NSN} = \frac{Np_1}{Np_0} * 100$$

Where the price in the "base year" is $Np_0 = p_{0L} + p_{0U}I$. Here, $p_{0L}$ is denoted to the specified part of the price in the base year, while $p_{0U}I$ is denoted the undefined part of the price in the base year, the price in the "year of study" is $Np_1 = p_{1L} + p_{1U}I$. Where $p_{1L}$ is denoted to the specified part of the price in the study year, and $p_{1U}I$ is the undefined part of the price in the study year.

#### 2.2.2 Example

This study is dedicated to evaluating the change in the prices of one of the pesticides (aluminium phosphide) used to Combat the “vine borer” insect, as its price in 2020 was equal to 20 $ (with an unspecified amount [0,2]). while its price in 2022 was $30 (with an unspecified amount [0,5]), it is worth mentioning that the unspecified prices are appearing due to price instability in these years. So, the neutrosophic index for the price in 2022 compared to the year 2020:

$$NSN = \frac{Np_1}{Np_0} * 100$$

$Np_0 = p_{0L} + p_{0U}I = 20 + [0.2] = [20.22]$ $Np_1 = p_{1L} + p_{1U}I = 30 + [0.5] = [30.35]$ $NSN = \frac{[30.35]}{[20.22]} * 100 = [150.159]\%$

That is, the price of the pesticide increased by [50.59] % compared to its price in 2020.

**Explanation**:

To analyze the values of the neutrosophic index $NSN = [NSN_L, NSN_U]$, we have the following cases:

1- If $NSN = [NSN_L, NSN_U] = [100.100]$, that is, the value of the phenomenon has not changed compared to the base year or place of the f base.

2- If $NSN_L > 100$ *or* $NSN_U > 100$, that is, the value of the phenomenon has increased compared to the base year or place of base, and the percentage of rise is the difference between the index and 100.

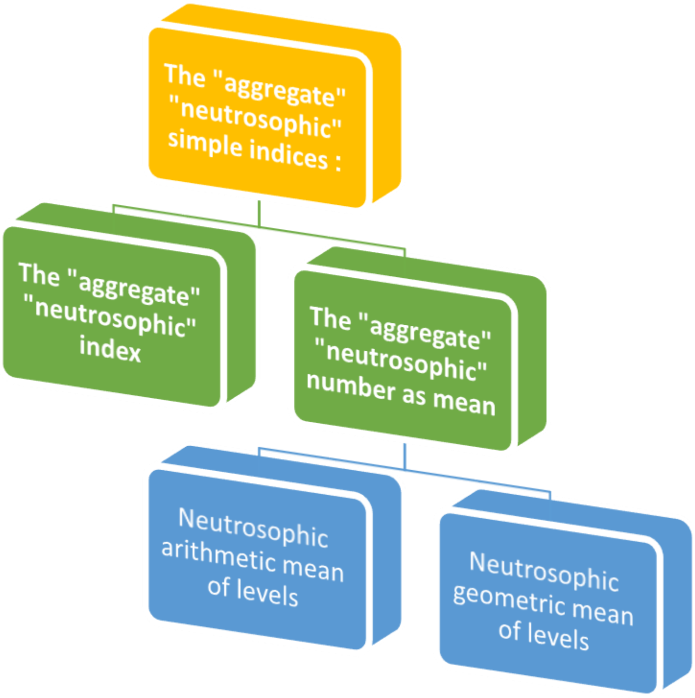
3- If $NSN_L < 100$ *or* $NSN_U < 100$, that is, the value of the phenomenon decreased compared to the base year or the place of base, and the percentage of decline is the difference between the index and 100.

4- The same interpretation is done in the case if $NSN_U > 100$ and $NSN_L < 100$. Or vice versa (but here we explain each side separately).

For example if we had $NSN = [75.159]\%$, the result explanation is that, at the beginning of the year, the prices were down by 25% compared to 2020, then until the end of the year the prices had increased by 59% compared to 2020 and so on.

## 2.3 The Aggregate Neutrosophic Simple indices:

The below flow chart illustrates the types of the aggregate neutrosophic simple indices which are divided into two types, the aggregate neutrosophic index, and the aggregate neutrosophic mean number, which is also can be divided into two categories the neutrosophic arithmetic mean of the levels, and the neutrosophic geometric mean of the levels.




### 2.2. 1. The Simple and Assembly Neutrosophic Number:

We express here the neutrosophic index of prices as the ratio of the sum of the neutrosophic prices of commodities in the comparison year to the sum of the neutrosophic prices of the commodities in the base year, multiplied by 100. Note that we use this number when the required is to compare more than one commodity at the same time

$$NSN_s = \frac{\sum Np_1}{\sum Np_0} * 100$$

Here, $\sum \mathrm{Np}_1 = \sum[\ \mathrm{p}_{1L} + \mathrm{p}_{1U}\mathrm{I}]$ represents the sum of the neutrosophic prices for commodities in the comparison year. While, $\sum Np_0 = \sum[\ p_{0L} + p_{0U}I]$ represents the sum of the neutrosophic prices for commodities in the base year.

**Example**

In this example, we recorded the prices in USD for three pesticides used to combat the "vine stem borer" insect. They are (chlorpyrifos - diflubenzuron - aluminium phosphide) in both the "base year" and the "comparison year". Where each of these commodities had a fixed price $p_L$ with an unspecified amount that expresses the price change within one year $p_U I$, with assuming that the base year is 2010, and the comparison year is 2022, then the recorded price table is as follow:

| **Pesticides** | **The price in the base year** $Np_0$ | **The price in the study year** $Np_1$ |
|---|---|---|
| Chlorpyrifos | 1.2 + [0, 0.8] | 24.2 + [0, 0.8] |
| Diflubenzuron | 1.6 + [0, 0.5] | 28.6 + [0, 0.4] |
| Aluminum phosphide | 0.8 + [0, 0.1] | 30 + [ 0, 1.5] |
| $sum$ | [3.6, 5] | [82.8, 85.5] |

$$NSN_s = \frac{\sum Np_1}{\sum Np_0} * 100$$

$$NSN_s = \frac{[82.8\ ,\ 85.5]}{[3.6,\ 5]} * 100 = [2300, 1710]\%$$

To analyze the above result, it is clear that the prices in the comparison year 2022 increased by a very large percentage compared to the base year 2010, and this explains and shows the significant inflation that occurred in prices between the comparison year and the base year. Again to make another comparison the prices of these pesticides between 2020 and 2022:

| **pesticides** | **The price in the base year** $Np_0$ | **Price in the study year** $Np_I$ |
|---|---|---|
| Chlorpyrifos | 14.2 + [0, 0.8] | 24.2 + [0, 0.8] |
| Diflubenzuron | 18.8 + [0, 1.3] | 28.6 + [0, 0.4] |
| Aluminum phosphide | 20.8 + [0, 2.1] | 30 + [0, 1.5] |
| $sum$ | [53.8, 58] | [82.8, 85.5] |

$$NSN_s = \frac{\sum Np_1}{\sum Np_0} * 100$$

$$NSN_s = \frac{[82.8,\ 85.5]}{[53.8,\ 58]} * 100 = [147, 154\ ]\ \%$$

That is, pesticide prices in 2022 increased by $[47, 54]$ % over their prices in 2020.

## 2.2.2 The Simple Neutrosophic Average of Ratios:

As seen from the previous section, the assembly method to compare neutrosophic prices has a defect when we use it to compare a large number of commodities. This section is dedicated to overcoming this defect, where the average of a large number of ratios of commodities is used, either by using the neutrosophic arithmetic mean or by using the neutrosophic geometric mean.

i. **Neutrosophic Arithmetic Mean of ratios:**

The following mathematical formula is precisely derived for the above-mentioned purpose

$$M(NSN) = \frac{\sum \frac{Np_{1i}}{Np_{0i}}}{n} * 100$$

**ii. Neutrosophic Geometric Mean of Ratios:**

It is clear that the bellow formula service the same aim of this section as we previously stated:

$$M(NSN) = \sqrt[n]{\frac{Np_{11}}{Np_{01}} \cdot \frac{Np_{12}}{Np_{02}} \dots \frac{Np_{1n}}{Np_{0n}}} . 100$$

Again, the mathematical statement $Np_0 = p_{0L} + p_{0U}I$ represents the price in the base year. While the formula $Np_1 = p_{1L} + p_{1U}I$ represents the price in the studying year. The symbol $n$ denotes to the number of commodities.

**Example**

The below data table contains the price of pesticides A and B in both the base year and comparison year. The neutrosophic index, as an arithmetic mean and as a geometric mean, have been calculated in the same table, Given that pesticide A represents the diflubenzuron pesticide, and pesticide B is the aluminium phosphide pesticide. Also, suppose that the base year is 2020, and the comparison year is 2022.

| **Pesticides** | **the price** | | $\frac{Np_1}{Np_0}$ |
|---|---|---|---|
| | $Np_1$ | $Np_0$ | |
| A | 28.6 + [0,1.4] | 18.8 + [0,0.2] | [1.52,1.57] |
| B | 29 + [0,2.2] | 20 + [0,1.2] | [1.45,1.47] |
| The sum | [57.6, 61.2] | [38.8,40.2] | [2.97,3.04] |
| Neutrosophic arithmetic mean of ratios | | M(NSN)=[148.5, 152]% | |
| Neutrosophic geometric mean of ratios | | M(NSN)=[148.45, 151.9]% | |

**i. Neutrosophic Arithmetic Mean of Ratios:**

$$M(NSN) = \frac{\sum \frac{Np_{1i}}{Np_{0i}}}{n}.100 = \frac{[2.97, 3.04]}{2}.100 = [148.5, 152]\%$$

The neutrosophic arithmetic mean of the ratios indicates that the prices of pesticides increased between $[48.5, 52]\%$ between the base year and the comparison year.

ii. **Neutrosophic geometric mean of ratios**:

Here we have $n = 2$

$$M(NSN) = \sqrt[n]{\frac{Np_{11}}{Np_{01}}.\frac{Np_{12}}{Np_{02}}}.100 = \sqrt[2]{\ [1.52, 1.57].[1.45, 1.47]}\ .100 =$$

$$\sqrt[2]{[2.204, 2.308]}\ .100 = [148.45, 151.9]\%.$$

We notice that the neutrosophic geometric mean of the ratios indicates that the prices of pesticides increased by a range between $[48.45, 51.9]\%$ between the base year and the comparison year. However, the neutrosophic arithmetic mean of the ratios and the neutrosophic geometric mean of the ratios are very close and give almost the same amount of increase in prices that occurred in the prices of pesticides, and this confirms the reliability of this amount as an increase. Especially since we know that the arithmetic mean is easier to use, but the geometric mean is the best for this purpose.

## 3. Conclusion:

We conclude from this study that the generalization of classical indices to neutrosophic indices provides a more general and clear view in uncertain environments. In addition, it gave us better results in terms of determining the extent of the change in prices more precisely. Thus, the cost of the insect control process changes on the farms. This helps us in optimal planning for the next stage and developing appropriate solutions that serve this goal.

In the near future, we look forward to studying the neutrosophic indices for the weighted ratios, which gives each commodity its importance in terms of the number of units sold**.**

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# A novel computational method for neutrosophic uncertainty related quadratic fractional programming problems

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**Abstract:** This study introduces a novel method for addressing the pentagonal quadratic fractional programming problem (PQFPP). We employ pentagonal neutrosophic numbers for the objective function's cost, resources, and technological coefficients. The paper transforms the PQFPP into a standard quadratic fractional programming (QFP) problem via the score function. By leveraging the Taylor series approach, the modified QFP is simplified to a single-objective linear programming (LP) task, amenable to resolution through conventional LP algorithms or software tools. A numerical example serves to demonstrate the efficacy of the suggested approach. Moreover, comparative analyses and benefits reveal that the newly developed techniques outperform existing solutions in current scholarly works.

**Keywords:** Quadratic fractional programming; Score function; Taylor series; Linear programming; Decision making; Optimal solution.

## 1. Introduction

The issue of fractional programming (FP) comes into play when the goal is to optimize the relationship between variables and constraints in decision-making scenarios. In problems of numerical optimization, FP can be considered an extension of linear fractional programming (LFP). In FP, the objective function is composed of a ratio between two generally nonlinear functions.

Fractional programming finds applications in diverse areas of decision-making, including but not limited to traffic management (cited from Dantzig et al., 1966), network flow optimization (referenced from Arisawa and Elmaghraby, 1972), and strategic games (based on Isbell and Marlow, 1956). Schaible (1976 and 1982) gives a review of various applications. Enormous approaches are introduced to solve LFP problems (Gupta and Chakraborty, 1998; Tantawy, 2007; 2008; Pandey and Punnen, 2007; Pop and Stancu, 2008; Kim and Mehrotra, 2021; Bennani et al., 2021; Park and Lim, 2021; Das and Mandal, 2017; Das, 2019; 2021; Farnam and Darehmiraki, 2021; Mekawy, 2022; Edalatpanah, 2023; Jiao and Shang, 2023).

In the domain of operations research, quadratic fractional programming (QFP) issues are widely applicable. These problems can be categorized by the uniformity of the constraints and the divisibility of the objective function, as outlined by Sharma and Singh in 2013. Ibaraki et al. (1976) introduced some models for solving QFP. Gupta and Puri's 1994 work focused on a specialized QFP scenario, aiming to minimize a certain quadratic fractional function under generalized constraints, further narrowed down to an extreme point of a convex polytope. Benson, in 2006, explored fractional programming problems that maximize a particular ratio of two convex functions, with at least one being a quadratic form, and detailed their mathematical attributes. In 2006, Mishra and Ghosh introduced an interactive fuzzy programming technique to find satisfactory solutions for a two-tier QFP problem involving dual decision-makers. Zhang and Hayashi, in 2011, dealt with a fractional programming problem that minimizes the ratio of two indefinite quadratic functions under dual quadratic constraints, converting the original problem into a univariate nonlinear equation.

Khurana and Arora, in 2011, put forth a methodology for tackling QFP problems incorporating uniform constraints. Sharma and Singh, in 2013, devised an iterative approach based on simplex techniques for solving QFP problems with factorization. Suleiman and Nawkhass, in 2013, advocated that a revised simplex method outperforms in addressing QFP issues and applied Wolfe's method to solve them. Lur and colleagues, in 2014, proposed a new continuous-time QFP (CQFP) model using the parametric and discretization methods, leading to an approximation algorithm with any desired accuracy.

Continuing, Singh and Haldar, in 2015, innovated a method for bi-level quadratic linear fractional programming issues by transforming the original problem. Youness et al., in 2016, introduced a parametric approach to solve nonlinear fractional optimization problems, relying on a two-dimensional algorithm. Sharma et al., in 2017, employed the ϵϵ-scalarization technique coupled with an integer feasible solution ranking to identify all non-dominated points for bi-objective quadratic fractional integer programming issues. Jain and colleagues, in 2018, offered an algorithmic solution for quadratic fractional integer programming problems involving bounded variables, using complete ranking and scanning.

Kassa and Tsegay, in 2018, presented an algorithm for a tri-level programming issue involving quadratic fractional objectives at each tier, using a fuzzy goal programming strategy. Sivri and colleagues, in 2018, proposed a computational technique that simplifies QFP into a linear programming task. Lara, in 2019, established optimality conditions for general QFP issues using a generalized asymptotic function for dealing with quasi-convexity. Gharanjik et al., in 2019, introduced a novel optimization schema for signal design problems involving max-min FQP issues, simplifying the original problem using a penalized version.

Lastly, Consolini et al., in 2020, rephrased an FQP issue into a Celis–Dennis–Tapia (CDT) problem, which served to outline a local search algorithm. Lachhwani, in 2020, recommended a holistic method for solving multi-level QFP problems based on fuzzy goal programming. For other recent research on the topic, refer to works by Taghi-Nezhad and Taleshian (2018), Badrloo and Husseinzadeh Kashan (2019), Yang and Xia (2020), Kausar et al. (2021), Jafari and Sheykhan (2021), Rani et al. (2021), Xiao et al. (2022), Ju et al. (2022), Zhou et al. (2022), and Berahas et al. (2023).

Real-world data is inaccurate and very difficult to be determined exactly. Therefore, a mathematical model of a problem does not generally have accurate output to fulfill sufficient efficiency. As a result, in optimization problems, an appropriate tool is required by which the uncertainty of data is overcome. Fuzzy set theory serves as a pivotal research methodology for addressing issues associated with vagueness and uncertainty, and it has found applications across diverse academic disciplines. Initially introduced by Zadeh in 1965 (Zadeh, 1965), fuzzy numbers are constrained to a single membership function. In practical scenarios, the attributes of data such as certainty, accuracy, and reliability are typically elusive. Given that fuzzy numbers' optimal solutions are bound by a restricted set of constraints, a new theoretical framework known as 'Neutrosophic sets' was introduced. This approach was first conceptualized by Smarandache in 1995 (Smarandache, 1998). After that this logic was developed and studied by several scholars (Rivieccio, 2008; Guo and Cheng, 2009; Ye, 2014; Smarandache, 2020; Edalatpanah, 2018; Garg, 2020; Abdel-Basset et al., 2020; Debnath, 2021; Mohanta and Toragay, 2023; Edalatpanah et al., 2023; Bhat, 2023, etc.). For researchers, it's crucial to extend traditional linear programming issues into their neutrosophic counterparts, incorporating three distinct membership functions: truth, indeterminacy, and falsity. This capacity to manage ambiguous and nebulous data can significantly enhance the adoption and utility of linear programming. Unlike fuzzy linear programming, which relies solely on a single membership function, neutrosophic linear programming offers a more nuanced approach by employing three types of membership functions. For an in-depth understanding, refer to works by Ye (2018), Abdel-Basset et al. (2019), Khatter (2020), Basumatary and Broumi (2020), Das and Dash (2020), Das et al. (2020a,b,c), Das and Edalatpanah (2022), Kumar et al. (2021), and Abdelfattah (2021).

For the best of our mind, it has been observed from the literature study that there was no study in a neutrosophic quadratic fractional programming problems. Taking this opportunity, we introduced a new method for solving PQFPP.

**Contribution:** One of the key strengths of the neutrosophic set lies in its ability to aid decision-makers through its incorporation of degrees of truth, falsity, and indeterminacy. In this context, the degree of indeterminacy is often viewed as an autonomous variable with a crucial role in decision processes. Given the inherent uncertainties in real-world scenarios, utilizing pentagonal neutrosophic linear fractional programming problems (PNQLFPP) offers a more realistic approach than traditional PQFPP. In this study, we introduce a PNQLFPP model, wherein all coefficients are treated as pentagonal neutrosophic numbers. We present a novel algorithm that leverages a recently-developed ranking function along with the Taylor series method to solve PNQLFPPs. As far as we are aware, this is the inaugural methodology for addressing PNQLFPPs using a ranking function. Consequently, a direct comparison with existing techniques is not applicable for validating our approach. We illustrate the utility and efficacy of our method through a diet planning example, thereby showcasing its real-world applicability.

**Motivation:**

Neutrosophic sets serve as a cornerstone in modeling uncertainty, a key element in the creation of applied mathematical models in science, engineering structures, and medical diagnostic problems. Given the absence of existing studies that tackle PNQLFPP, our work pioneers a new approach that employs a ranking function along with the Taylor series method for resolving issues related to PNQLFP. Queries such as the feasibility of incorporating this into operations research grounded in linear programming, or its real-world applicability, have yet to be answered. It is with this backdrop that we seek to advance the discourse through this paper.

**Novelties:**

In recent years, scholarly focus has shifted toward the enhancement and refinement of theories within the realm of neutrosophic studies, with ongoing efforts to broaden their utility across diverse neutrosophic subfields. Against this backdrop, our core objective in relation to PNQLFP problem theory is to validate the conceptual framework through several pivotal aspects:

- ✓ Unveiling an efficient ranking function.
- ✓ Incorporating the Taylor series method and elucidating its applications.
- ✓ Real-world applications of the PNQLFP problem.
- ✓ Benchmarking our findings against earlier established outcomes.

In addressing this research void, our paper debuts the concept of pentagonal neutrosophic quadratic fractional programming problems. We then transform this into a more straightforward problem via the scoring function associated with pentagonal neutrosophic numbers, ultimately reducing it to a linear programming (LP) issue through the application of the Taylor Series. The rest of the paper unfolds as follows: Section 2 provides essential background information. The formulation of the pentagonal neutrosophic quadratic fractional programming issue is detailed in Section 3. A methodology to arrive at an optimal solution is laid out in Section 4. Section 5 brings in a numerical example to elucidate the concept. Insights into the results and merits of our approach are discussed in Section 6, and Section 7 closes with concluding observations.

## 2. Foundational Concepts

**3.**

In this part, we outline fundamental ideas and findings concerning fuzzy numbers, pentagonal fuzzy numbers, the neutrosophic set, pentagonal neutrosophic numbers, and the arithmetic operations associated with them.

**Definition 1.** (Cited from Zadeh, 1965). A set A is termed as a fuzzy set within the realm of real numbers R if the range of its membership function falls between [0, 1].

**Definition 2.** (As per Abbasbandy and Hajjari, 2009). A number $\widetilde{A}_P = (r, s, t, u, v), r \leq s \leq t \leq u \leq v$, in the set of real numbers $\Re$, is identified as a pentagonal fuzzy number when its membership function is expressed as:

$$\mu_{\tilde{A}_P} = \begin{cases} 0, \quad x < r, \\ w_1\left(\frac{x-r}{s-r}\right), \text{for } r \leq x \leq s, \\ 1-(1-w_1)\left(\frac{x-s}{t-s}\right), \text{for } s \leq x \leq t \\ 1, \text{for } x = t, \\ 1-(1-w_2)\left(\frac{u-x}{u-t}\right), \text{for } t \leq x \leq u, \\ w_2\left(\frac{v-x}{v-u}\right), \text{for } u \leq x \leq v, \\ 0, \text{for } x > v. \end{cases}$$

**Definition 3.** (Based on Smarandache, 1998). A neutrosophic set, denoted as $\widetilde{B}^N$, within a non-empty set $X$ is characterized as:

$$\widetilde{B}^N = \{\langle x;\ I_{\widetilde{B}^N}(x),\ J_{\widetilde{B}^N}(x), V_{\widetilde{B}^N}(x)\rangle : x \in X, I_{\widetilde{B}^N}(x),\ J_{\widetilde{B}^N}(x), V_{\widetilde{B}^N}(x) \in ]0_-, 1^+[\},$$

here, $I_{\widetilde{B}^N}(x), J_{\widetilde{B}^N}(x)$, and $V_{\widetilde{B}^N}(x)$ stand for the truth, indeterminacy, and falsity membership functions, respectively. The sum of them is unrestricted, falling within the range $0^- \leq \text{Sup}\{I_{\widetilde{B}^N}(x)\} + \text{Sup}\{J_{\widetilde{B}^N}(x)\} + \text{Sup}\{V_{\widetilde{B}^N}(x)\} \leq 3^+$. Additionally, $]0^-, 1^+[$ represents a nonstandard unit interval.

**Definition 4.** Referring to the terms outlined in Definition 3, if these membership functions are confined to the interval [0,1] and their aggregate sum lies in the range [0,3], such a set is termed a Single-Valued Neutrosophic set.

**Definition 5.** Assume that $\tau_{\tilde{p}}, \varphi_{\tilde{p}}, \omega_{\tilde{p}} \in [0, 1]$, and $r, s, t, u, v \in \mathbb{R}$ satisfying $r \leq s \leq t \leq\ u \leq v$. A Single-Valued Pentagonal Neutrosophic Set (SVPN), denoted as $\tilde{p}^{PN} = \langle (r, s, t, u, v) : \tau_{\tilde{p}}, \varphi_{\tilde{p}}, \omega_{\tilde{p}} \rangle$, is a specialized neutrosophic set on $\Re$. In this set, the truth-membership, hesitant-membership, and falsity-membership functions are represented by:

$$\mu_{\tilde{p}^{PN}}(x)=\begin{cases} 0, & x<r;\\ \tau_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(s-r)^2}(x-r)^2\right), & r\le x\le s;\\ \tau_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(t-s)^2}(x-t)^2+1\right), & s\le x\le t;\\ \tau_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(u-t)^2}(x-t)^2+1\right), & t\le x\le u;\\ \tau_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(v-u)^2}(x-v)^2\right), & u\le x\le v;\\ 0, & x>v. \end{cases}$$

$$\rho_{\tilde{p}^{PN}}(x)=\begin{cases} 0, & x<r;\\ \varphi_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(s-r)^2}(x-r)^2\right), & r\le x\le s;\\ \varphi_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(t-s)^2}(x-t)^2+1\right), & s\le x\le t;\\ \varphi_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(u-t)^2}(x-t)^2+1\right), & t\le x\le u;\\ \varphi_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(v-u)^2}(x-v)^2\right), & u\le x\le v;\\ 0, & x>v. \end{cases}$$

$$\sigma_{\tilde{p}^{PN}}(x)=\begin{cases} 0, & x<r;\\ \omega_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(s-r)^2}(x-r)^2\right), & r\le x\le s;\\ \omega_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(t-s)^2}(x-t)^2+1\right), & s\le x\le t;\\ \omega_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(u-t)^2}(x-t)^2+1\right), & t\le x\le u;\\ \omega_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(v-u)^2}(x-v)^2\right), & u\le x\le v;\\ 0, & x>v. \end{cases}$$

Here, $\tau_{\tilde{p}^{PN}}, \varphi_{\tilde{p}^{PN}}$, and $\omega_{\tilde{p}^{PN}}$ indicate the peak truth, nadir-hesitant, and nadir-falsity membership degrees, correspondingly. The SVPN $\tilde{p}^{PN} = \langle (r, s, t, u, v): \tau_{\tilde{p}^{PN}}, \varphi_{\tilde{p}^{PN}}, \omega_{\tilde{p}^{PN}} \rangle$ can depict a vaguely defined value approximating $[s, u]$.

**Definition 6.** Suppose $\tilde{p}^{PN} = \langle (r, s, t, u, v): \tau_{\tilde{p}^{PN}}, \varphi_{\tilde{p}^{PN}}, \omega_{\tilde{p}^{PN}} \rangle$ and $\tilde{q}^{PN} = \langle (r', s', t', u', v'): \tau_{\tilde{q}^{PN}}, \varphi_{\tilde{q}^{PN}}, \omega_{\tilde{q}^{PN}} \rangle$ are two distinct single-valued PQFNs. The following describes the arithmetic procedures that apply to them:

1. $\tilde{p}^{PN} \oplus \tilde{q}^{PN} = \langle (r + r', s + s', t - t', u + u', v + v');\ \tau_{\tilde{p}^{PN}} \wedge \tau_{\tilde{q}^{PN}}, \varphi_{\tilde{p}^{PN}} \vee \varphi_{\tilde{q}^{PN}}, \omega_{\tilde{p}^{PN}} \vee \omega_{\tilde{q}^{PN}} \rangle$

2. $\tilde{p}^{PN} \ominus \tilde{q}^{PN} = \langle \left( r - v', s - u', t + t', u - s', v - r' \right);\ \tau_{\tilde{p}^{PN}} \wedge \tau_{\tilde{q}^{PN}}, \varphi_{\tilde{p}^{PN}} \vee \varphi_{\tilde{q}^{PN}}, \omega_{\tilde{p}^{PN}} \vee \omega_{\tilde{q}^{PN}} \rangle,$

3. $\tilde{p}^{PN} \otimes \tilde{q}^{PN} = \frac{1}{5}\gamma_q \langle (r, s, t, u, v);\ \tau_{\tilde{p}^{PN}} \wedge \tau_{\tilde{q}^{PN}}, \varphi_{\tilde{p}^{PN}} \vee \varphi_{\tilde{q}^{PN}}, \omega_{\tilde{p}^{PN}} \vee \omega_{\tilde{q}^{PN}} \rangle, \gamma_q = \frac{1}{3}(r' + s' + t' + u' + v')\left(2 + \tau_{\tilde{q}^{PN}} - \varphi_{\tilde{q}^{PN}}\right) \neq 0,$

4. $\tilde{p}^{N} \oslash \tilde{q}^{N} = \frac{5}{\gamma_q} \langle (r, s, t, u, v);\ \tau_{\tilde{p}^{PN}} \wedge \tau_{\tilde{q}^{PN}}, \varphi_{\tilde{p}^{PN}} \vee \varphi_{\tilde{q}^{PN}}, \omega_{\tilde{p}^{PN}} \vee \omega_{\tilde{q}^{PN}} \rangle, \gamma_q \neq 0,$

5. $k\,\tilde{p}^{PN} = f(x) = \begin{cases} \langle (kr, ks, kt, ku, kv): \tau_{\tilde{p}^{PN}}, \varphi_{\tilde{p}^{PN}}, \omega_{\tilde{p}^{PN}} \rangle, k > 0, \\ \langle (kv, ku, kt, ks, kr);\ \tau_{\tau_{\tilde{p}^{PN}}},\ \varphi_{\tau_{\tilde{p}^{PN}}}, \omega_{\tau_{\tilde{p}^{PN}}} \rangle, k < 0, \end{cases}$

6. $\tilde{p}^{PN^{-1}} = \langle \left( \frac{1}{v}, \frac{1}{u}, \frac{1}{t}, \frac{1}{s}, \frac{1}{r} \right);\ \tau_{\tau_{\tilde{p}^{PN}}},\ \varphi_{\tau_{\tilde{p}^{PN}}}, \omega_{\tau_{\tilde{p}^{PN}}} \rangle, \tilde{p}^{PN} \neq 0.$

**Definition 7.** Assume $\tilde{p}^{PN} = \langle (r, s, t, u, v): \tau_{\tilde{p}^{PN}}, \varphi_{\tilde{p}^{PN}}, \omega_{\tilde{p}^{PN}} \rangle$ is a single-valued pentagonal neutrosophic number. In this context, the Accuracy and Score functions are delineated in the following manner:

$$AC(\tilde{p}^{PN}) = \left(\frac{1}{15}\right)(r + s + t + u + v) * [2 + \tau_{\tilde{p}^{PN}} - \varphi_{\tilde{p}^{PN}}].$$

$$SC(\tilde{p}^{PN}) = \left(\frac{1}{15}\right)(r + s + t + u + v) * \left[2 + \tau_{\tilde{p}^{PN}} - \varphi_{\tilde{p}^{PN}} - \omega_{\tilde{p}^{PN}}\right].$$

**Definition 8.** (As per Thamariselvi and Santhi, 2016). Referring to the terms outlined in Definition 7, the ordinal relationships between A and B, predicated on their Accuracy and Score functions, are specified as follows:

1. If $SC(\tilde{p}^{PN}) < SC(\tilde{q}^{PN})$, then $p < q$,
2. If $SC(\tilde{p}^{PN}) = SC(\tilde{q}^{PN})$, then $p = q$,
3. If $AC(\tilde{p}^{PN}) < AC(\tilde{q}^{PN})$, then $p < q$
4. If $AC(\tilde{p}^{PN}) > AC(\tilde{q}^{PN})$, then $p > q$,
5. If $AC(\tilde{p}^{PN}) = AC(\tilde{q}^{PN})$, then$p = q$.

**Definition 9.** (Based on Sivri et al., 2018). The initial pair of terms in the Taylor series, stemming from $f(X_1, X_2, \dots, X_2)$, when evaluated at a given point $Q = (q_1, q_2, \dots, q_n)$, are characterized as follows:

$$f(Q) + \nabla_{X_1} f(Q)(X_1 - q_1) + \nabla_{X_2} f(Q)(X_2 - q_2) + \cdots + \nabla_{X_n} f(Q)(X_n - q_n) = 0.$$

**3. Problem statement**

Quadratic fractional programming with pentagonal neutrosophic parameters can be formulated as

$$\text{(PQFP)} \quad \max(\text{or} \min)\, \tilde{Z}^{PN}(X) = \frac{f(\tilde{C}^{PN}, \tilde{E}^{PN})}{g(\tilde{D}^{PN}, \tilde{G}^{PN})} = \frac{(\tilde{C}^T)^{PN} X + \frac{1}{2} X^T \tilde{E}^{PN} X}{(\tilde{D}^T)^{PN} X + \frac{1}{2} X^T \tilde{G}^{PN} X}$$

Subject to

$$X \in \tilde{M}^{PN} = \left\{X \in \Re^n : \tilde{A}^{PN} \otimes X (\lesssim, \backsimeq, \gtrsim) \tilde{b}^{PN}; X \geq 0\right\}$$

Where, $\tilde{b}^{PN} = (\tilde{b}_1^{PN}, \tilde{b}_2^{PN}, \ldots, \tilde{b}_m^{PN})$, $\tilde{c}^{PN} = (\tilde{c}_1^{PN}, \tilde{c}_2^{PN}, \ldots, \tilde{c}_n^{PN})$, $\tilde{d}^{PN} = (\tilde{d}_1^{PN}, \tilde{d}_2^{PN}, \ldots, \tilde{d}_n^{PN})$, and are neutrosophic cost vector and neutrosophic right- hand side vector. $X = (X_1, X_2, \ldots, X_n)$ is a vector of decision variables, and $\tilde{E}^{PN} = [\tilde{e}_{ij}^{PN}]_{n\times n}$, $\tilde{G}^{PN} = [\tilde{g}_{ij}^{PN}]_{n\times n}$ is a matrix of quadratic form which is symmetric and positive semi-definite, and $\tilde{A}^{PN} = [\tilde{a}_{ij}^{PN}]_{m\times n}$. It is assumed that all of $\tilde{A}^{PN}, \tilde{b}^{PN}, \tilde{C}^{PN}, \tilde{D}^{PN}, \tilde{E}^{PN} and\ \tilde{G}^{PN} \in F(R)$, where $F(R)$ denotes the set of all pentagonal neutrosophic parameters.

**Definition 10.** In the context of the PQFP problem, a feasible fuzzy solution, denoted as $\overline{X}$, is termed an optimal fuzzy solution if $\tilde{Z}^{PN}(X) \binom{\gtrsim}{\lesssim} \tilde{Z}^{PN}(\overline{X})$ for every individual $x$.

Utilizing the score function associated with the Pentagonal fuzzy number, the PQFP problem is reformulated as follows:

$$\text{(QFP)} \qquad \max(\text{or } \min)\ Z(X) = \frac{C^T X + \frac{1}{2} X^T E X}{D^T X + \frac{1}{2} X^T G X}$$

Subject to

$$x \in M = \{X \in \mathfrak{R}^n : A\,X(\leq, =, \geq)b; X \geq 0\}.$$

It assumed that $Z(X)$ is a function of class $C^{(1)}$.

**4. Solution method**

The steps of the solution procedure are:

**Step 1:** Consider the PQFP problem.

**Step 2:** Convert the PQFP into the QFP based on the score function.

**Step 3:** Choose an arbitrary initial non-zero feasible point, say $x^*$.

**Step 4:** Expand the Taylor series at $x^*$ (Definition 9) to linearize the objective function.

**Step 5:** Solve the linear programming

$$\text{(LP)} \qquad \max(\text{or } \min)\ Z^{Linear}(X) = H^T X$$

Subject to $x \in M$.

Assume $X^{\circ}$ represents the optimal solution for the Linear Programming (LP) problem.

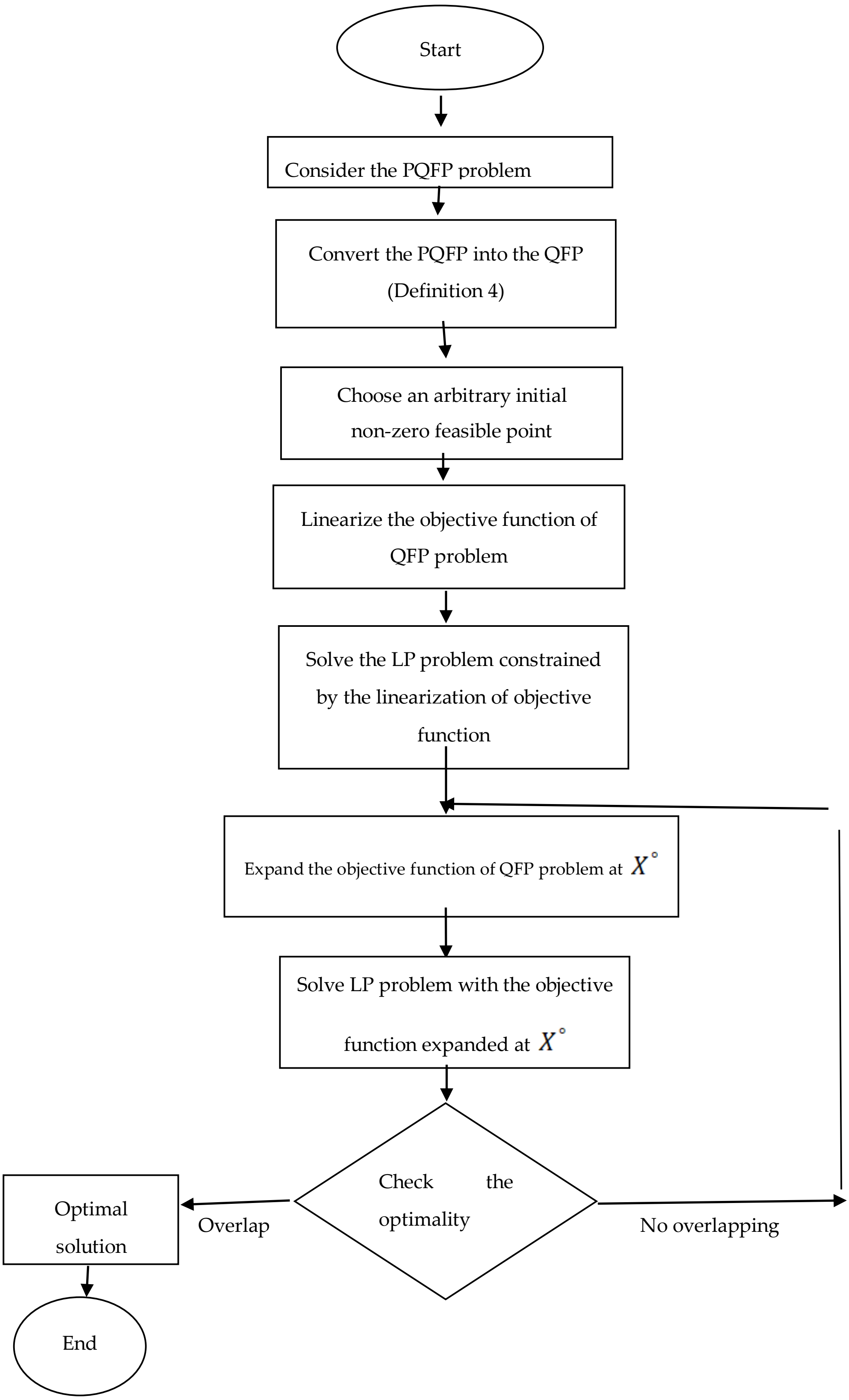

**Fig.1. Solution method flow chart**

**Step 6:** Expand the objective function of the QFP problem (Definition 9) at $X^{\circ}$.

**Step 7:** Solve LP problem with expanded objective function resulting from step 6 as constrained. Let $\widehat{X}$ be the solution.

**Step 8:** Check the optimality

If the solutions $X^{\circ}$ and $\widehat{X}$ overlap stop with the final optimal solution. Otherwise, assign $X^{\circ}$ to $\widehat{X}$ and return to step 6.

Fig. 1 depicts the flowchart outlining the steps of the solution method.

## 5. Illustrative Example

This section is dedicated to the application of our proposed method. We examine the following PQFP issue:

$$\max \tilde{Z}^{PN}(x) = \frac{(x_1 x_2)\begin{pmatrix} 0 & \tilde{e}_{12}^{PN} \\ 0 & \tilde{e}_{22}^{PN} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} (+)\tilde{c}_1^{PN}x_1(+)\tilde{c}_2^{PN}x_2(+)\tilde{c}_0^{PN}}{(x_1 x_2)\begin{pmatrix} \tilde{g}_{11}^{PN} & 0 \\ \tilde{g}_{21}^{PN} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} (+)\tilde{d}_1^{PN}x_1(+)\tilde{d}_2^{PN}x_2(+)\tilde{d}_0^{PN}}$$

Subject to (1)

$$\tilde{a}_{11}^{PN}x_1(+)\tilde{a}_{12}^{PN}x_2 \lesssim \tilde{b}_1^{PN},$$

$$\tilde{a}_{21}^{PN}x_1(+)\tilde{a}_{2}^{PN}x_2 \lesssim \tilde{b}_2^{PN},$$

$$x_1,\ x_2 \geq 0.$$

Where,

$$\tilde{e}_{12}^{PN} = \tilde{e}_{22}^{PN} = \tilde{g}_{11}^{PN} = \tilde{g}_{21}^{PN} = \langle(1,1,1,1,1);1,0,0\rangle;\ \tilde{c}_1^{PN} = \langle(1,1,1,1,1);1,0,0\rangle;$$

$$\tilde{c}_2^{PN} = \langle(2,3,4,5,6);1,0,0\rangle; \tilde{c}_0^{PN} = \langle(1,2,3,4,5);1,0,0\rangle = \tilde{d}_2^{PN}$$

$$\tilde{d}_1^{PN} = \langle(1,5,6,7,11);1,0,0\rangle; \tilde{d}_0^{PN} = \langle(2,6,8,10,14);1,0,0\rangle;$$

$$\tilde{a}_{11}^{PN} = \tilde{a}_{12}^{PN} = \tilde{a}_{21}^{PN}x_1 = \tilde{b}_1^{PN} = \langle(1,1,1,1,1);1,0,0\rangle;$$

$\tilde{a}_2^{PN} = \langle(0,1,2,3,4);1,0,0\rangle;\ \tilde{b}_2^{PN} = \langle(1,4,5,9,16);1,0,0\rangle$

Let us apply the steps of the solution method as

Step 2: Based on the ranking function of the pentagonal neutrosophic numbers, Problem (1) converts into

$$\max Z(x) = \frac{(x_1 x_2)\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + x_1 + 4x_2 + 3}{(x_1 x_2)\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + 6x_1 + 4x_2 + 8}$$

Subject to (2)

$-x_1 + x_2 \leq 1,$

$x_1 + 2x_2 \leq 7,$

$x_1,\ x_2 \geq 0.$

Or equivalently,

$$\max Z(x) = \frac{x_2^2 + x_1x_2 + x_1 + 4x_2 + 3}{x_1^2 + x_1x_2 + 6x_1 + 4x_2 + 8}$$

Subject to (3)

$-x_1 + x_2 \leq 1,$

$x_1 + 2x_2 \leq 7,$

$x_1,\ x_2 \geq 0.$

Steps 3 and 4: Select $x^* = (1,1)$ to be a random optimal point that is not zero. Utilize the Taylor series centered at this point to approximate the objective function in a linear form as follows:

$$\nabla_{x_1} Z = \frac{(x_1^2 + x_1x_2 + 6x_1 + 4x_2 + 8)(x_2 + 1) - (x_2^2 + x_1x_2 + x_1 + 4x_2 + 3)(x_2 + 4)}{(x_1^2 + x_1x_2 + 6x_1 + 4x_2 + 8)^2}$$

$$\nabla_{x_2} Z = \frac{(x_1^2 + x_1x_2 + 6x_1 + 4x_2 + 8)(2x_2 + x_1 + 4) - (x_2^2 + x_1x_2 + x_1 + 4x_2 + 3)(2x_1 + x_2 + 6)}{(x_1^2 + x_1x_2 + 6x_1 + 4x_2 + 8)^2}$$

Step 5: Construct the LP problem as

$$\max Z = -\frac{1}{8}x_1 + \frac{9}{40}x_2$$

Subject to (4)

$$-x_1 + x_2 \le 1,$$

$$x_1 + 2x_2 \le 7,$$

$$x_1,\ x_2 \ge 0.$$

The optimal solution is $X^{\circ} = (1.67, 2.67)$.

Steps 6 and 7: Elaborate on the objective function corresponding to the QFP issue (as per Definition 6) around the point $X^{\circ}$. Subsequently, construct and resolve the ensuing LP problem as:

$$\max Z = -0.148x_1 + 0.188x_2$$

Subject to (5)

$$-x_1 + x_2 \le 1,$$

$$x_1 + 2x_2 \le 7,$$

$$x_1,\ x_2 \ge 0.$$

The optimal solution is $X^{\circ} = (1.67, 2.67) = \hat{X}$, with the optimum value $Z = 0.75$, and

$$\tilde{Z}^{NP} = \langle (1.61216, 1.91406, 2.21596, 2.51786, 2.8198); 1, 0, 0 \rangle.$$

## 6. Result Analysis

In this section, we analyze the efficacy of our proposed approach in comparison to the existing technique by Sivri et al. (2018) for addressing the PNQFP issue. Utilizing ranking functions and order definitions, we establish that our method yields more effective outcomes, as evidenced by the comparison:

$$Z_{\text{Proposed method}} = 0.75 > Z_{\text{Sivri et al. method (2018)}} = 0.74$$

Additionally, it's crucial to note that the literature currently lacks a method for addressing the PNQFP issue. Consequently, we juxtaposed our novel approach with prevalent techniques for solving the C-QFP problem.

Our findings indicate that the objective value yielded by our method surpasses those of existing approaches. This leads us to conclude that our method is notably more effective. Furthermore, the objective value generated by our method resides within the realm of neutrosophic values.

Benefits of the proposed approach are as follow:

i) The outcomes generated by our model outperform those of Sivri's. As evidenced in the results section, our objective function value stands at 0.75, compared to Sivri's 0.74. Given that the problem aims for maximization, our solution effectively achieves this goal.

ii) Crucially, in real-world scenarios, managers often grapple with options of agreement, uncertainty, and disagreement. Sivri's model restricts them to parameters set by decision-makers, a limitation we've addressed by incorporating a neutrosophic model in our approach.

iii) Our model is versatile enough to be applicable to both real-world and large-scale issues.

Summing up the discussion, our newly proposed algorithm presents an innovative avenue for tackling both uncertainty and indeterminacy in real-world situations.

## 7. Conclusions

In real-world settings, dealing with ambiguous, unclear, or incomplete information often necessitates the use of neutrosophic sets. This study focuses on a neutrosophic linear fractional programming issue involving pentagonal neutrosophic numbers and converts it into a QFP issue through a ranking function. Utilizing the Taylor series method, we further simplify the QFP issue into a linear programming (LP) problem solvable via standard LP algorithms or software. Scholars in this domain could find our approach useful for addressing both intricate and straightforward challenges. An example is included to validate the efficiency of our methodology. This new framework not only augments the realm of uncertain linear fractional programming but also introduces a novel, effective strategy for managing indeterminate optimization issues. Comparative evaluations with existing methodologies underscore the merits of our ranking approach. Future extensions could involve incorporating other specialized neutrosophic sets like pentagonal neutrosophic sets, neutrosophic rough sets, interval-valued neutrosophic sets, and plithogenic contexts.

**Conflicts of Interest**

Authors do not have any conflicts of interest.

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# Studying Transport Models with the Shortest Time According to the Neutrosophic Logic

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**Abstract:**
Time factor plays an important role in many issues, and the most important of these issues is the issue of transportation, when we need to transport perishable materials such as milk, medicines ,blood,….. etc. or develop war plans to secure the requirements of battle of ammunition - food - and soldiers …..etc., at maximum speed, we need a careful scientific study that enables us to avoid losses, so the researchers studied transport models in the shortest possible time using the values of classic logic and the best solution for such models is a specific value subject to increase or decrease because there is nothing certain in the real reality, all the results of the studies are related to the surrounding conditions of the system under study, due to the sensitivity of these issues had to be reformulated according to a science that takes into account all the cases that the system can go through so that we can take all possible precautions that help us reduce losses and secure the required in the shortest possible time, we have in This research formulates transport models with the shortest time and we presented a special way to solve such models using neutrosophic values, based on the concept of the neutrosophic linear mathematical model, a model that has something of non-determination (indeterminacy) and we reached transport models with the shortest time that are considered a generalization of the existing models because they give us optimal neutrosophic values for time, which are unspecified values $Nt^* = t^* \pm \varepsilon$ where $\varepsilon$ is indeterminism and we will take it in the form then become the matrix of times $Nt^* = [t^* \pm \varepsilon]$, where $Nt^*$ is any neighborhood of $t^*$, it should be noted that in this research when viewing the example we took some transfer times neutrosophic values of the form $Nt \in [t_1, t_2]$ to be able to clarify the main goal of the research.

**Key words:** Linear programming - Simplex - the problem of transportation at the lowest cost - the issue of transportation in the shortest time - Ways to solve the problem of transport .

**Introduction:**

Linear programming is one of the most methods of operations research that has been used to address many realistic issues and provided optimal solutions to them that helped reduce losses, and since the optimal solution to such issues depends on the data provided by those in charge of the work and this data lack stability , we are unable to study these issues and develop accurate future action plans It was necessary for a new science that provides us with a margin of freedom and helps us reduce losses, so it was a neutrosophic science, which was laid by the American scientist and mathematical philosopher Florentin Samarandche and came as a generalization of the fuzzy logic presented by Lutfizadeh [1], this science has received great attention by researchers as many researches and studies have been published in various fields of science using the basic concepts developed by the founder of this science [2-8], and since operations research is considered the applied aspect of mathematics, it was necessary to reformulate its topics using the basic concepts of neutrosophic science, where we published the feast of research on various topics such as static stock models, simulation, decision-making, dynamic programming, and in the topics of linear programming, and transport problems at the lowest cost. Waiting queues, Machine Learning...etc. [9-26], a complement to our research in which we present a study of transportation models in the shortest possible time.

**Discussion:**

Through our study of some topics of linear programming using neutrosophic science and due to the interest that these researches have received from researchers and those interested in the development of science, we present this research as a complementary study to what we have done in two previous researches on the issue of transport [18,19] It is dedicated to transport models with the shortest time using neutrosophic values that take into account all changes that may occur during the functioning of the systems under study and provide We have optimal transit time.

**Model of transporting materials in the shortest possible time using neutrosophic values:**

**Based on the study contained in the research [19], we can formulate the issue as follows:**

**The text of the issue in general form:**

Suppose we want to move a material from production centers $A_i$ where $i = 1,2,\ldots,m$ ,whose production capacities are respectively $Na_1, Na_2, \ldots, Na_m$, to consumption centers $B_j$ where $j = 1,2,\ldots,n$ whose needs is $Nb_1, Nb_2, \ldots, Nb_n$ It is in order, and let the matrix of times necessary to transfer the appropriate quantity from the center $i$ to the center $j$ be known and equal $NT = [Nt_{ij}]$, it is required to formulate the appropriate mathematical model to transfer all the quantities ,available in the production centers and meet the needs of all consumption centers in the shortest time. In order to build the appropriate mathematical model, we denote $Nx_{ij}$ for the quantity transferred from the production center $i$ where $i = 1,2,3,\ldots,m$ to the consumption center j where $= 1,2,3,\ldots,n$ , then we can put the problem unknowns in matrix form. $NX = [Nx_{ij}]$ , and we put the information in the question in a table as follows:

| consumption<br>production | $B_1$ | $B_2$ | $B_3$ | … | $B_n$ | quantities |
|---|---|---|---|---|---|---|
| $A_1$ | $Nt_{11}$<br>$Nx_{11}$ | $Nt_{12}$<br>$Nx_{12}$ | $Nt_{13}$<br>$Nx_{13}$ | … | $Nt_{1n}$<br>$Nx_{1n}$ | $Na_1$ |
| $A_2$ | $Nt_{21}$<br>$Nx_{21}$ | $Nt_{22}$<br>$Nx_{22}$ | $Nt_{23}$<br>$Nx_{23}$ | … | $Nt_{2n}$<br>$Nx_{2n}$ | $Na_2$ |
| $A_3$ | $Nt_{31}$<br>$Nx_{31}$ | $Nt_{32}$<br>$Nx_{32}$ | $Nt_{33}$<br>$Nx_{33}$ | … | $Nt_{3n}$<br>$Nx_{3n}$ | $Na_3$ |
| … | … | … | … | … | … | … |
| $A_m$ | $Nt_{m1}$<br>$Nx_{m1}$ | $Nt_{m2}$<br>$Nx_{m2}$ | $Nt_{m3}$<br>$Nx_{m3}$ | … | $Nt_{mn}$<br>$Nx_{mn}$ | $Na_m$ |
| Required quant | $Nb_1$ | $Nb_2$ | $Nb_3$ | … | $Nb_n$ | |

**Table No. (1) Data of the issue of transport in the shortest time**

To build the appropriate mathematical model, we distinguish two cases:

First case: the model is balanced:
The model is balanced if:

$$\sum_{i=1}^{m} Na_i = \sum_{j=1}^{n} Nb_j$$

Second Case: Unbalanced Model:
The model is unbalanced if:

$$\sum_{i=1}^{m} Na_i \neq \sum_{j=1}^{n} Nb_j$$

From this case, two cases result:
Overproduction:

$$\sum_{i=1}^{m} Na_i \geq \sum_{j=1}^{n} Nb_j$$

This model is returned to a balanced model by adding an imaginary consumer center that needs it:

$$Nb_{n+1} = \sum_{i=1}^{m} Na_i - \sum_{j=1}^{n} Nb_j$$

Deficit in production:
This model is returned to a balanced model by adding a fictitious production center with a production capacity:

$$Na_{m+1} = \sum_{j=1}^{n} Nb_i - \sum_{i=1}^{m} Na_i$$

In both cases($b$ & $a$), a case of surplus production and a deficit in production, we get a balanced model.
Formulation of the mathematical model of transport models in the shortest possible time:
Our symbol for the quantity transferred from the center $i$ to the center $j$ with the symbol $Nx_{ij}$ then these variables must meet the following conditions:

$$\sum_{j=1}^{n} Nx_{ij} = Na_i \qquad (i = 1,2,3, \dots, m)$$

$$\sum_{i=1}^{m} Nx_{ij} = Nb_j \qquad (j = 1,2,3, \dots, n)$$

$$Nx_{ij} \geq 0 \quad (i = 1,2,3, \dots, m), (j = 1,2,3, \dots, n)$$

In these models, the objective function cannot be formulate d with a mathematical follower, so we extract its most important qualities and properties through the following discussion:
To find the optimal solution for any transport model, we must find the values of unknowns: $Nx_{ij}$ ; $(i = 1,2,3, \dots, m)$, $(j = 1,2,3, \dots, n)$. In a previous research [18] we have presented ways to find the preliminary solution to the problem of neutrosophic transport at the lowest cost, taking advantage of the study contained in the research we find a primary basic solution using one of the methods, we know that any optimal solution that includes n+m-1 basic solution that are not equal to zero , and against this solution there is a set of times that we will symbolize as $[Nt_{ij}]_X$.

It represents the time required to transport all materials available in all production centers and meet the needs of all consumption centers.
The time required to finish the transfer process, which we will symbolize as corresponding to the largest element of the matrix $[Nt_{ij}]_X$ must achieve the following relationship:

$$Nt_x = Max_{i,j}[Nt_{ij}]_X$$

Since we have a large number of acceptable solutions, the optimal solution is given by the following relationship:

$$Nt_x^* = MinNt_x = Min(Max_{i,j}[Nt_{ij}]_X)$$

This means that we solve the model without a target function we get a base solution and then we determine the number set from the matrix $[Nt_{ij}]_X$ corresponding to this base solution.

**Note 1:**

If the issue is unbalanced and when adding an imaginary production center or an imaginary consumer center, we determine the time according to the following:
Since the time required to finish the transfer process achieves the following relationship:

$$Nt_x = Max_{i,j}[Nt_{ij}]_X$$

So we take the time required to transfer the quantities available in this imaginary production center to all consumption centers equal to zero.
And we take the time required to transport quantities from all production centers to the imaginary consumer center is equal to zero.

**Note 2:**

In order for the transport model to be a neutrosophic transport model, at least one of the data in Table 1 must be a neutrosophic value.
The general method used to obtain the smallest transfer time is to move from one neutrosophic base solution to another base solution using the simplex method to solve the neutrosophic linear models described in the research [12] ,and the goal is to make the largest elements $Nt_x$ in the matrix $NT = [Nt_{ij}]_X$ as small as possible.
In this research, we will use a special method to solve neutrosophic transport models according to time, which we explain through the following example:

**Example:**

Four pharmaceutical plants distribute their production of one type to three pharmacies the available quantities, the quantities required and the times required to transport them are shown in the following table:

| Consumption center / Production centers | $B_1$ | $B_2$ | $B_3$ | Required quantities |
|---|---|---|---|---|
| $A_1$ | $[1, 1.5]$ $Nx_{11}$ | $[2, 2.4]$ $Nx_{12}$ | $6$ $Nx_{13}$ | 11 |
| $A_2$ | $[3, 3.2]$ $Nx_{21}$ | $8$ $Nx_{22}$ | $[1, 1.5]$ $Nx_{23}$ | 9 |
| $A_3$ | $[7, 7.5]$ $Nx_{31}$ | $10$ $Nx_{32}$ | $[4, 4.6]$ $Nx_{33}$ | 13 |
| $A_4$ | $12$ $Nx_{41}$ | $8$ $Nx_{42}$ | $[5, 5.1]$ $Nx_{43}$ | 17 |

| | | | | |
|---|---|---|---|---|
| Available quantity | $18$ | $10$ | $12$ | $40$<br>$40$ |

**Table No. (2) Example data**

We look for the smallest time in the rooms $(i, j)$, we find that it is found in two rooms $(1, 1)$ and $(3, 2)$ :

$$Min(Nt_{ij)} = Nt_{11} = Nt_{32} \in [1,1.5]$$

We denote b.i $\Omega_1$ for all table stone except the two rooms $(1, 1)$ and $(2, 3)$

We saturate the two chambers opposite them $(1, 1)$ $and$ $(2, 3)$ then we put in the other stone $(*)$ we get the following table:

| consumption<br>production | | | | Required quantities |
|---|---|---|---|---|
| $A_1$ | $11$ | $*$ | $*$ | $11$ |
| $A_2$ | $*$ | $*$ | $9$ | $9$ |
| $A_3$ | $*$ | $*$ | $*$ | $13$ |
| $A_4$ | $*$ | $*$ | $*$ | $7$ |
| Available quantities | $18$ | $10$ | $12$ | $40$<br>$40$ |

**Table No. (3) First Step**

The first solution is $x_{11}^{(1)} = 11$ , $x_{23}^{(1)} = 9$ which expresses the total quantities transferred and is equal to $x_{ij} = 11 + 9 = 20$ it crosses a quantity less than the quantity required for that time solution from which we started and the author of $Nt_{11} = Nt_{23} \in [1,1.5]$ is an imperfect solution to the problem at hand.

From the elements of the sat $\Omega_1$, where $\Omega_1$ equal :

$$\Omega_1 = \{ [3 ,3.2], [7 ,7.5], 12, [2 ,2.4], 8,10,8,6, [4 ,4.6], [5 ,5.1] \}$$

We look for the smallest time we find:

$$(Nt_{ij}) = Min\{[3 ,3.2], [7 ,7.5], 12, [2 ,2.4], 8,10,8,6, [4 ,4.6], [5 ,5.1]\} \in [2 ,2.4]$$

Any smallest time is this $Nt_{12} \in [2 ,2.4]$ means that we must transfer the quantities available in the production center $A_1$ to the consumption center $B_2$ and this is not possible because the center $A_1$ no longer contains any quantity and therefore this step is not useful.

We form the sat $\Omega_2 = \{ [3 ,3.2], [7 ,7.5], 12,8,10,8,6, [4 ,4.6], [5 ,5.1] \}$ is the resulting $\Omega_1$ after deleting $Nt_{12} \in [2 ,2.4]$ and we choose from $\Omega_2$ the smallest time we find:

$$(Nt_{ij}) = Min\{[3 ,3.2], [7 ,7.5], 12,8,10,8,6, [4 ,4.6], [5 ,5.1]\} \in [3 ,3.2]$$

Any smaller time is this $Nt_{21} \in [3 ,3.2]$ means that we have to transfer the quantities available in the production center $A_2$ to the consumption center $B_1$ and this is not possible because the center $A_2$ no longer contains any quantity and therefore this step is also not useful

We form $\Omega_3 = \{[7 ,7.5], 12,8,10,8,6, [4 ,4.6], [5 ,5.1] \}$the sat which is the resulting sat $\Omega_2$ after deleting $Nt_{21} \in [3 ,3.2]$ and choosing from the $\Omega_3$ smallest time we find :

$$(Nt_{ij}) = Min\{[7 ,7.5], 12,8,10,8,6, [4 ,4.6], [5 ,5.1]\} \in [4 ,4.6]$$

Any smallest time is $Nt_{33} \in [4 ,4.6]$ this means that we must transfer the quantities available in the production center $A_3$ to the consumption center $B_3$ i.e. we put $x_{33} = 12$ and therefore we must shift the amount in the room $(2,3)$ with the same time $[1 ,1.5]$ to a room with a time immediately followed and located in the same line i.e. to the room $(2,1)$ with the same time $[3 ,3.2]$ and we put $x_{21} = 9$ ,then we make a balance process for the quantities that have been distributed and here we must reduce the quantity in the room $(1,1)$ becomes $x_{11} = 9$ and we put the quantity that has been reduced in the room with the lowest time immediately following the time in that room and in the same row, i.e in the room $(1,2)$ we get $x_{12} = 2$ from this step we get the following distribution:

$x_{11} = 9$ ، $x_{12} = 2 , x_{21} = 9 , x_{33} = 12$

But this solution is not ideal because the sum of the quantities that were distributed is not equal to the 40 total quantities available, that is, the time $Nt_{33} \in [4 ,4.6]$ Not the shortest time, we proceed in the same way in the eighth time to reach the distribution shown in the following table:

| consumption<br>production | $B_1$ | $B_2$ | $B_3$ | Required amounts |
|---|---|---|---|---|
| $A_1$ | 1 | 10 | $*$ | 11 |
| $A_2$ | 9 | $*$ | $*$ | 9 |
| $A_3$ | 8 | $*$ | 5 | 13 |
| $A_4$ | $*$ | $*$ | 7 | 7 |
| Available quantities | 18 | 10 | 12 | 40<br>40 |

**Table No. (4) Optimal Solution**

In return for $Min(Nt_{ij}) = Nt_{31} \in [7 ,7.5]$ this time, the entire quantities available in the production centers have been transferred and the needs of all consumer centers have been met, the ideal solution is

$$x_{11} = 1 ,\ x_{21} = 9 ,\ x_{31} = 8 ,\ x_{12} = 10 ,\ x_{33} = 5 , x_{43} = 7$$

The rest of the variables is equal to zero, and the shortest time is $Nt^* = Nt_{31} \in [7 ,7.5]$

It should be noted that the same example was presented and solved according to classical logic in reference [20], and the data of the problem were as in the following table:

The available quantities, the quantities required and the times required for their transportation are shown in the following table:

| consumption<br>production | $B_1$ | $B_2$ | $B_3$ | Required amounts |
|---|---|---|---|---|
| $A_1$ | 1<br>$Nx_{11}$ | 2<br>$Nx_{12}$ | 6<br>$Nx_{13}$ | 11 |
| $A_2$ | 3<br>$Nx_{21}$ | 8<br>$Nx_{22}$ | 1<br>$Nx_{23}$ | 9 |
| $A_3$ | 7<br>$Nx_{31}$ | 10<br>$Nx_{32}$ | 4<br>$Nx_{33}$ | 13 |
| $A_4$ | 12<br>$Nx_{41}$ | 8<br>$Nx_{42}$ | 5<br>$Nx_{43}$ | 7 |

| | | | | |
|---|---|---|---|---|
| Available amounts | 18 | 10 | 12 | 40<br>40 |

**Table No. (5) Example data according to classical logic**

The optimal solution was as follows:
Time $Min(t_{ij}) = t_{31} = 7$ and in exchange for this time, the entire quantities available in the production centers have been transferred and the needs of all consumer centers have been met, the best solution is

$$x_{11} = 1 \ , x_{21} = 9 \ , \ x_{31} = 8 \ , \ x_{12} = 10 \ , x_{33} = 5 \ , x_{43} = 7$$

The rest of the variables is equal to zero, and the shortest time is $t^* = t_{31} = 7$

**Conclusion and results:**

The solution using neutrosophic values is an undefined neutrosophic value, which can be any value belonging to the domain $Nt^* \in [7 \ ,7.5]$, and this value is not specified shows its impact according to one time used and according to the nature of the material transported and the need of consumption centers for it in the end The difference between it and the value $t^* = 7$ obtained when solving this problem using classical values and its impact is determined by those responsible for the work, in addition to that the method used to find the shortest time is an iterative method and in this example we repeated it eight times until we reached the required despite the number of production centers and consumption centers a small number and in Real reality is more numerous, so we recommend using computers and new technologies used in programming when applying this method to real systems from reality,

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# Ambiguous Set is a subclass of the Double Refined Indeterminacy Neutrosophic Set, and of the Refined Neutrosophic Set in general

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**Abstract**: In this short note we show that the so-called Ambiguous Set (2019) is a subclass of the Double Refined Indeterminacy Neutrosophic Set (2017) and is a particular case of the Refined Neutrosophic Set (2013). Also, the Ambiguous Set is similar to the Quadripartitioned Neutrosophic Set (2016), and Belnap's Four-Valued Logic (1975).

**Keywords:** Double Refined Indeterminacy Neutrosophic Set (DRINS); Refined Neutrosophic Set (RNS); Ambiguous Set (AS); Quadripartitioned Neutrosophic Set (QNS); Belnap's Four-Valued Logic (BFVL).

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## 1. Introduction

We provide the definitions of the previous five types of sets, and we prove that the Ambiguous Set is a particular case of the Refined Neutrosophic Set (RNS), Quadripartitioned Neutrosophic Set (QNS), and Belnap's Four-Valued Logic (BFVL), and mostly that the Ambiguous Set coincides with the Double Refined Indeterminacy Neutrosophic Set with the distinction that the sum of quadruple components is ≤ 2 for the AS, which makes it a subclass of the DRINS where the sum is any number between 0 and 4.

## 2. Ambiguous Set

The definition of the Ambiguous Set (AS) according to [1, 2] is given as follows:

Let U = {g} be the universe for any event *g*, which is fixed. An AS Ś for g ∈ U is defined by:

$$Ś = \{g, \Pi t(g), \Pi f(g), \Pi ta(g), \Pi fa(g) \mid g \in U\}$$

where, $\Pi t(g): U \to [0,1]$, $\Pi f(g): U \to [0,1]$, $\Pi ta(g): U \to [0,1]$, and $\Pi fa(g): U \to [0,1]$ are called the true membership degree (TMD), false membership degree (FMD), true-ambiguous membership degree (TAMD), and false-ambiguous membership degree (FAMD), respectively.

Where Πt(g), Πf(g), Πta(g) and Πfa(g) must satisfy the following condition as:

$$0 \leq \Pi t(g) + \Pi f(g) + \Pi ta(g) + \Pi fa(g) \leq 2$$

## 3. Double Refined Indeterminacy Neutrosophic Set (DRINS)

The definition of Double Refined Indeterminacy Neutrosophic Set is given in [3] as follows:

Let X be a space of points (objects) with generic elements in X denoted by x.

A Double Refined Indeterminacy Neutrosophic Set (DRINS) A in X is characterized by four components:

truth membership function $T_A(x)$, indeterminacy leaning towards truth membership function $I_{TA}(x)$,

indeterminacy leaning towards falsity membership function $I_{FA}(x)$, and falsity membership function $F_A(x)$.

For each generic element x ∈ X, there are $T_A(x)$, $I_{TA}(x)$, $I_{FA}(x)$, $F_A(x) \in [0, 1]$,

and $0 \leq T_A(x)+I_{TA}(x)+I_{FA}(x)+ F_A(x) \leq 4$.

Therefore, a DRINS A can be represented by

$A = \{\langle x, T_A(x), I_{TA}(x), I_{FA}(x), F_A(x)\rangle \mid x \in X\}$.

## 4. Ambiguous Set vs. Double Refined Indeterminacy Neutrosophic Set

Let's compare the two definitions.

The definition of Ambiguous Set, as presented by Singh, Huang, & Lee [1, 2] in 2019 and in 2023, coincides with that of Double Refined Indeterminacy Neutrosophic Set introduced by Ilanthenral & Smarandache [3] in 2017, ahead of them.

They only renamed:

the indeterminacy leaning towards truth membership function $I_{TA}(x)$, as true-ambiguous membership degree (TAMD),

and the indeterminacy leaning towards falsehood membership function $I_{TA}(x)$, as false-ambiguous membership degree (FAMD).

The only distinction between AS and DRINS is that:

the sum of AS quadruple components is restricted to be ≤ 2, while the sum of DRINS quadruple components is ≤ 4 (no restriction), which means that one can take any number between 0 and 4, in the particular case they took the number 2, whence AS is a subclass of the DRINS.

## 5. Refined Neutrosophic Set

The Definition of Refined Neutrosophic Set is the following.

Let X be a space of points (objects) with generic elements in X denoted by x.

A Refined Neutrosophic Set (RNS) A in X is characterized by *n* sub-components:

sub-truth membership functions $T_{1A}(x)$, $T_{2A}(x)$, … , $T_{pA}(x)$;

sub-indeterminacy membership functions $I_{1A}(x)$, $I_{2A}(x)$, …, $I_{rA}(x)$;

and sub-falsehood membership functions $F_{1A}(x)$, $F_{2A}(x)$, … , $F_{sA}(x)$;

where p, r, s ≥ 0 are integers, and p + r + s = n ≥ 2, such that at least one of p, r, s is ≥ 2 for assuring the refinement of at least one neutrosophic component amongst T, I, or F.

For each generic element x ∈ X, the functions

$T_{1A}(x), T_{2A}(x), \dots, T_{pA}(x), I_{1A}(x), I_{2A}(x), \dots, I_{rA}(x), F_{1A}(x), F_{2A}(x), \dots, F_{sA}(x) \in [0, 1]$,

with their sum

$$0 \le T_{1A}(x) + T_{2A}(x) + \dots + T_{pA}(x) + I_{1A}(x) + I_{2A}(x) + \dots + I_{rA}(x) + F_{1A}(x) + F_{2A}(x) + \dots + F_{sA}(x) \le n$$

Therefore, a RNS A can be represented by

$A_{RNS} = \{\langle x, T_{1A}(x), T_{2A}(x), \dots, T_{pA}(x), I_{1A}(x), I_{2A}(x), \dots, I_{rA}(x), F_{1A}(x), F_{2A}(x), \dots, F_{sA}(x)\rangle, \mid x \in X\}$.

The Ambiguous Set is a particular case of the Refined Neutrosophic Set, since one takes

p = 1 (only one true membership);

r = 2 (two types of indeterminacy memberships,

$I_1$ = true-ambiguous membership degree (TAMD),

and

$I_2$ = false-ambiguous membership degree (FAMD);

s = 1 (only one false membership).

Therefore, the Ambiguous Set is a particular case of the Refined Neutrosophic Set.

In the same way it is proven that the Double Refined Indeterminacy Neutrosophic Set is a particular of the Refined Neutrosophic Set.

## 6. Ambigous Set vs. Refined Neutrosophic Set

Both, the so-called Ambiguous Set and the Double Refined Indeterminacy Neutrosophic Set are particular cases of the Refined Neutrosophic Set [4] introduced by Smarandache in 2013.

## 7. Quadripartitioned Neutrosophic Set

The Definition of single-valued Quadripartitioned Neutrosophic Set [5]

Let X be a non-empty set. The Quadripartitioned single-valued Neutrosophic Set (QNS) A over X characterizes each element x in X by a truth-membership function $T_A$ , a contradiction membership function $C_A$ , an ignorance–membership function $U_A$ and a falsity membership function $F_A$ such that:

for each $x \in X$ one has $T_A(x), C_A(x), U_A(x), F_A(x) \in [0,1]$ and

$0 \le T_A(x) + C_A(x) + U_A(x) + F_A(x) \in [0,1] \le 4$.

When X is discrete, A is represented as

$$A = \sum_{i=1}^{n} < T_A(x_i), C_A(x_i), U_A(x_i), F_A(x_i) > / x_i, x_i \in X.$$

However, when X is continuous, A is represented as:

$$\int_X < T_A(x), C_A(x), U_A(x), F_A(x) > / x, x \in X.$$

It is clear the Quadripartitioned Neutrosophic Set (no matter if it is single-valued, interval-valued, or set-valued in general) is a particular case of the Refined Neutrosophic Set, of the form T, F, and indeterminacy I is split into two parts: $I_1$ = C (contradiction-membership) and $I_2$ = U (ignorance-membership).

While the Ambiguous Set is similar with the Quadripartitioned Neutrosophic Set, where the two types of sub-indeterminacies $I_1$ and $I_2$ are named differently: true-ambiguous membership and respectively false-ambiguous membership.

Surely, one can rename the sub-indeterminacies $I_1$ and $I_2$ in many ways, since there are many types of indeterminacies / uncertainties / vagueness / conflicting informations etc.

## 8. **Belnap's Four-Valued Logic**

In 1975 Belnap has considered a logic of four values: true, false, both (true and false), and neither (neither true, nor false). We can denote them by T (true), F (false), C (true and false = contradiction), U (neither true not false = ignorance) respectively and we see that the Ambiguous Set and Quadripartitioned Neutrosophic Set are similar to Belnap's Logic

Further on, the *Belnap's 4-valued Logic* is a particular case of the *Refined Neutrosophic n-valued Logic* that has types of truths $T_1$, $T_2$, …, $T_p$, types of indeterminacies $I_1$, $I_2$, …, $I_r$, and types of falsehoods: $F_1$, $F_2$, …, $F_s$.

## 9. Conclusion

We proved that the so-called Ambiguous Set coincides with the Double Refined Indeterminacy Neutrosophic Set with respect their quadruple structures, while, with respect to the sum of components, AS is a subclass of the DRINS.

Also, AS is similar with the Quadripartitioned Neutrosophic Set and Belnap Four-Valued Logic as well.

Further on, we proved that the AS, DRINS, QNS and BFVL are particular cases of the Refined Neutrosophic Set / Logic respectively.

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# Uncertainty-infused Representation Learning Using Neutrosophic-based Transformer Network

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**Abstract** Representation learning and interactive modeling of visual content are critical aspects for advancing visual interpretation in different computer vision tasks. However, visual data's inherent uncertainty and ambiguity remain a critical challenge facing representation learning algorithms. In response to this challenge, this study presents a simple but effective model, namely, the neutrosophic-based transformer network (NTN), which integrate the theory of neutrosophic logic and transformer architecture to offer unprecedented challenge in managing uncertainties. The design of NTN includes three primary building blocks: neutrosophic encoding, multipath network, and fusion and decision modules. Motivated by the success of neutrosophic in interpreting indeterminacy involved in visual data, we introduce a neutrosophic encoding module that applies a convolving window to map image data into the neutrosophic domain (truth, indeterminacy, and falsehood). This helps the NTN mitigate spatial and intensity uncertainties present in image patches, thereby enhancing boundary and uniformity retention while minimizing discontinuities. Then, multipath networks are built with visual transformer encoding blocks (composed of multi-head self-attention, feed-forward network, and residual link) to take the responsibility of learning rich representations from the generated neutrosophic image. By the end of NTN, the multiplicative fusion module is presented to fuse diverse knowledge from different network paths to obtain insightful representation that can assist in making informed decisions about the input. A set of proof-of-concept experiments are conducted to evaluate the proposed NTN against cutting-edge approaches using two image recognition datasets (namely Fashion-MNIST and CIFAR-10) with different uncertainty settings, and the findings demonstrate the potential of NTN in maintaining high representation power through efficient modeling of uncertainty information within visual recognition tasks.

**Keywords:** Uncertainty; Neutrosophic set; Representation learning; Vision Transformer; Machine learning; Image Recognition.

## 1. Introduction

Representational learning has been recognized as a core concept in machine learning (ML) that usually refers to the ability of an algorithm/model to extract complex and meaningful features from the training or inference data. This motivates research communities to develop a wide variety of ML techniques that can have effective representational power for different tasks, including image identification, natural language processing, and audio analysis. With this representation power, the ML systems can decode intricate patterns and make informed decisions [1]. The quality of data being passed to the model contributes significantly to the ability of machines to understand and engage with the real world more efficiently. Nevertheless, the real-world data is not usually clean but unpredictable and ambiguous limiting the representation power. This highlights the need for a new method to augment the representational learning capabilities with uncertainty handling capabilities to adequately handle the intricate and diverse nature of uncertainties seen in real data [2].

Vision data obtained from dynamic and unexpected surroundings inherently contain some level of uncertainty caused by a variety of factors, including occlusion, fluctuations in lighting conditions, changes in viewpoint, and deformations of the object [3]. In traditional ML methods, like probabilistic modeling, the range and likelihood of results are estimated to put measurable bounds on uncertainty in the data. However, the problem with these methods lies in their inability to handle situations encountering both ambiguities and contradictions simultaneously. In another way, the fuzzy logic was presented by Zadeh, to handle uncertainty by reconsidering system parameters as fuzzy numbers (instead of crisp values), a membership function characterized each. However, it may be insufficient to address more complex uncertainty categories such as epistemic or probabilistic uncertainty [4]. These constraints bring a critical need to explore other methodologies that can address the above limitation and thereby provide a more inclusive handling of uncertainties.

In 1995, Florentin Smarandache introduced the concept of Neutrosophic logic, in which each proposition is projected to have a level of truthfulness, a level of indeterminacy, and a level of falseness. With its triadic nature, the neutrosophic logic offers a great opportunity to deal with situations encountering uncertainties and contradictions, which substantially correlate with the complexities of visual recognition problems [5]. With the introduction of the idea of "neutrosophic membership", it could be easy to get a detailed measurement of the levels of truth, indeterminacy, and falsehood in a statement, offering a comprehensive depiction of different types of uncertainty [6]. The literature on computer vision has witnessed many breakthroughs in recent years due to the continuous evolution of deep image recognition models. Among these models, Vision transformers (ViTs) have been achieving remarkable success in capturing hierarchical patterns within images. ViTs were developed as customization of Transformer architecture, which was initially established for language modeling tasks, but with some key edits to suit image processing. One of the main distinctions about ViT lies in representing input images as a sequence of patches of equal size. These patches are passed to self-attention layers to enable the model to have long-range dependencies between them, as it grants the model to learn how the representations of distinctive fragments of an image contribute to a final decision. However, the ViT is not designed to deal with the inherent ambiguity and uncertainties in the input images, which limit their representation power and lead to poor classification performance [7]. This significant research gap highlighted the need for a novel solution that helps keep the representational power of ViTs under serious and multidimensional uncertainty prevalent in visual data [8].

This paper presents a hybrid framework, called a neutrosophic-based transformer network (NTN), that integrates neutrosophic logic into the visual learning of ViTs, aiming to provide a consistent mechanism for modeling uncertainties inherent in visual recognition tasks. The proposed NTN is designed to introduce a neutrosophic encoding module to transform the noisy and ambiguous images into a neutrosophic domain before being fed into a representational layer within the attention layers of our model. Proof-of-concept experimentations are performed on two image recognition datasets (namely Fashion-MNIST and CIFAR-10) under different uncertainty settings, and the results demonstrated the effectiveness of our NTN in maintaining high representation power through effective modeling of diverse uncertainty patterns in visual recognition tasks.

## 2. Literature Review

In this section, we provide an overview of background literature and related works that emphasize the representation of learning and uncertainty modelling, which shape the foundations of our search.

In their book, Smarandache and Said [6] laid the groundwork for incorporating neutrosophic theories into different application domains (e.g., product acceptance determination). They also studied the concept of neutrosophic graphs by investigating their association with different machine-learning algorithms. They also discussed the use of neutrosophic representation to solve the systems of linear equations. Pamucar et al. [7] introduced an approach for evaluating and selecting suppliers based on fuzzy neutrosophic decision-making, in which a Dombi aggregator was used as a weight aggregator that applies pairwise comparison based on trapezoidal neutrosophic linguistic variables. Their approach used MABAC (multiattribute border approximation area comparison) tool to analyze the suppliers in a resilient supply chain management (RSCM) system with an uncertain environment and numerous factors; meanwhile, sensitivity analysis tests were applied to evaluate the model examined. This study proposes a novel fuzzy-neutrosophic-based approach for resilient supplier selection. Besides, Haq et al. [8] integrated entropy–MultiAtributive Ideal-Real Comparative Analysis (MAIRCA) Interval-Valued Neutrosophic Sets (IVNSs) in a unified framework to concurrently handle the subjective measures with vague or uncertain data and objective measures with crisp inputs. They used a wing-spar of a Human-Powered Aircraft (HPA) to prove the applicability of their framework, in which a committee of three subordinate experts makes decisions on the substitutes or criteria lingually via IVNSs. Then, the level of skill of an expert implicated in the decision-making process is evaluated utilizing a weighting methodology with verbal information. Finally, they used the MAIRCA method to assess materials for the HPA spar using the entropy weighting method.

Abdelhafeez et al. [9] studied the clustering of breast cancer through the application of a broadly adopted c-means algorithm and its improved versions: fuzzy c-means algorithm and neutrosophic c-means algorithm. They conducted an in-depth comparative study between these algorithms to analyze and interpret both the qualitative and quantitative efficiency metrics. In addition, Essameldin et al. [10] introduced a neutrosophic-based approach for opinion mining on Twitter, aiming to tackle perspectivism, its effects, and indeterminateness. For the first aim, they used Graphistry to conduct social network analysis (SNA), whereby a neural network was applied to impact weighting based on SNA-generated output and the public's responses to analyzed texts. Banerjee et al. [11] explored the diagnosis of melanoma lesions using a hybrid approach that integratess the triangular neutrosophic concept into the deep convolutional network, aiming to handle the uncertainty in dermoscopic and digital pictures and providing improved and reliable classification decision. Moreover, Karam and Ali [12] introduced a structural method for evaluating the performance of functioning onshore wind resources, respecting the three scopes of sustainability. Om their method, the energy manufacture, ecological validity, and practicability of onshore wind accommodations are all conditional on precise evaluations. They also discussed many criteria, including site accessibility, transmission, environmental impact, wind resources, permitting and regulatory requirements, turbine technology, and others. Lo et al. [14] developed an inclusive approach for the selection of strategic alliance partners, in which the neutrosophic ITARA technique was presented to create a group of criteria objective weights, and the neutrosophic TOPSIS method was used to regulate the performance and precedence of strategic alliance partners. They also used neutrosophic fuzzy logic to replicate the uncertainty in complicated problems inherent in realistic data from a multinational reactive factor manufacturing company. Furthermore, Singh [15] introduced a three-way n-valued neutrosophic concept lattice, illustrating the integration of neutrosophic frameworks into formal concept analysis. The author also introduced a mathematical mechanism to autonomously portray the n-valued neutrosophic according to its n-valued indeterminacy, n-valued falsity n-valued truth. His method presented a granular-centered computing model to find some n-valued neutrosophic theories, which offer many ways to transform the given n-valued neutrosophic context into binary context at the user's essential degree of granulation. Thanikachalam et al. [16] integrated the Interval Neutrosophic Set into a deep-learning model to assist in distinguishing the contaminated regions in the fundus image. Their model included three feature extractors: texture features, histogram features, and wavelet features, where the Optimal Deep Belief Network (ODBN) and Shuffled Shepherd Optimization (SSO) algorithm were used to make classification and optimize hyperparameters, respectively. From the above discussions, it can be noted that there is growing interest in the applicability of neutrosophic methods to handle uncertainty in different applications. Motivated by that, this work seeks to advance the understanding of uncertainty-infused representation learning through the novel integration of neutrosophic logic into ViTs.

## 3. Methodology

This section explains the methodology of designing the proposed NTN to give insight into the role of neutrosophic logic in empowering the representational learning capabilities with an effective ability to handle uncertainty during the image recognition process. This is achieved by taking advantage of ViTs, with the unique capacities of neutrosophic logic, to build a novel representation learning solution that explicitly accounts for uncertainties. In the following, the details of the structural design of the NTN are presented along with its main components, including the integrated neutrosophic logic, for learning informative representations from data.

### *3.1. Preliminaries*

Derived from the concept of neutrosophic logic, the neutrosophic set was introduced as an extension of classical sets by enabling representation and dealing with indeterminacy exhibited in real-world data. In other words, the Neutrosophic Set provided a step toward representing incomplete and uncertain information, and it's composed of three essential components, namely truth membership $\omega_{\hat{A}_{\mathcal{N}}}(x)$, indeterminacy membership $\sigma_{\hat{A}_{\mathcal{N}}}(x)$, and falsity membership $\eta_{\hat{A}_{\mathcal{N}}}(x)$. mathematically speaking, given a neutrosophic set $\hat{A}_{\mathcal{N}} = \{(x; [\omega_{\hat{A}_{\mathcal{N}}}(x), \sigma_{\hat{A}_{\mathcal{N}}}(x), \eta_{\hat{A}_{\mathcal{N}}}(x)]) \,:\, x \in X\}$, the corresponding three components $\omega_{\hat{A}_{\mathcal{N}}}(x): X \to ]0^-, 1^+[$, $\sigma_{\hat{A}_{\mathcal{N}}}(x): X \to ]0^-, 1^+[$, $\eta_{\hat{A}_{\mathcal{N}}}(x): X \to ]0^-, 1^+[$ represents the degree to which an element $x$ belongs to the set; the degree to which an element $x$ is indeterminate in its membership status; and the degree to which an element $x$ does not belong to the set, respectively [11-14]. In this context, the definition of superior sum of the above components is given as follows:

$$n_{sup} = sup\left(\omega_{\hat{A}_{\mathcal{N}}}(x)\right) + sup\left(\sigma_{\hat{A}_{\mathcal{N}}}(x)\right) + sup\left(\eta_{\hat{A}_{\mathcal{N}}}(x)\right) \in ]^{-}0, 3^{+}[\,, \quad (1)$$

In the above expression, the term $Sup$ symbolizes the supremum, or, in other words, the least upper bound. The definition of the superior sum of the Neutrosophic components is given as follows:

$$n_{inf} = inf\left(\omega_{\hat{A}_{\mathcal{N}}}(x)\right) + inf\left(\sigma_{\hat{A}_{\mathcal{N}}}(x)\right) + inf\left(\eta_{\hat{A}_{\mathcal{N}}}(x)\right) \in ]^{-}0, 3^{+}[\,, \quad (2)$$

According to the above two expressions, the set is called an intuitionistic set in the case of $n_{sup} < 1$, and is referred to as a paraconsistent set in case of $n_{sup} > 1$.

The literature contains popular operations that can be applied to neutrosophic sets including union, intersection, and complement, which is defined using the following expressions for two neutrosophic sets $\hat{A}_{\mathcal{N}}$ and $\hat{B}_{\mathcal{N}}$.

$$\hat{A}_{\mathcal{N}} \cup \hat{B}_{\mathcal{N}} = \left\{x \mid \max\left(\omega_{\hat{A}_{\mathcal{N}}}(x), \omega_{\hat{B}_{\mathcal{N}}}(y)\right), \max\left(\sigma_{\hat{A}_{\mathcal{N}}}(x), \sigma_{\hat{B}_{\mathcal{N}}}(y)\right), \max\left(\eta_{\hat{A}_{\mathcal{N}}}(x), \eta_{\hat{B}_{\mathcal{N}}}(y)\right)\right\}. \quad (3)$$

$$\hat{A}_{\mathcal{N}} \cap \hat{B}_{\mathcal{N}} = \left\{x \mid \min\left(\omega_{\hat{A}_{\mathcal{N}}}(x), \omega_{\hat{B}_{\mathcal{N}}}(y)\right), \min\left(\sigma_{\hat{A}_{\mathcal{N}}}(x), \sigma_{\hat{B}_{\mathcal{N}}}(y)\right), \min\left(\eta_{\hat{A}_{\mathcal{N}}}(x), \eta_{\hat{B}_{\mathcal{N}}}(y)\right)\right\}. \quad (4)$$

$$\neg\hat{A}_{\mathcal{N}} = \{x \mid 1 - \omega_{\hat{A}_{\mathcal{N}}}(x), 1 - \sigma_{\hat{A}_{\mathcal{N}}}(x), 1 - \eta_{\hat{A}_{\mathcal{N}}}(x)\}. \quad (5)$$

Generally, neutrosophic sets can be leveraged in image processing tasks to represent composite and uncertain pixel information that classical set theory might not sufficiently tackle.

When the three components of neuromorphic sets satisfy the following $\omega_{\hat{A}_{\mathcal{N}}} \in [0,1]$, $\sigma_{\hat{A}_{\mathcal{N}}}(x) \in [0,1]$, $\eta_{\hat{A}_{\mathcal{N}}}(x) \in [0,1]$, then $\hat{A}_{\mathcal{N}}$ is known as single-valued neutrosophic (SVNS), which obeys the following constraints.

$$0 < \omega_{\hat{A}_{\mathcal{N}}}(x) + \sigma_{\hat{A}_{\mathcal{N}}}(x) + \eta_{\hat{A}_{\mathcal{N}}}(x) < 3\,, \tag{6}$$

Assume that there is SVNS denoted as $\hat{A}_{\mathcal{N}}$ in $X$, with triplet $\langle \omega_{\hat{A}_{\mathcal{N}}}(x), \sigma_{\hat{A}_{\mathcal{N}}}(x), \eta_{\hat{A}_{\mathcal{N}}}(x)\rangle$, then each element $x \in X$ if this set is called a singled-valued neutrosophic number (SVNN). To make things simpler, an SVNN $c$ or $e$ in SVNS can be written as $\langle \omega_c, \sigma_c, \eta_c \rangle$ or $\langle \omega_e, \sigma_e, \eta_e \rangle$, where each element has a unique value for its truth, indeterminacy, and falsity membership components [15-18]. The SVNNs are subject to many operations to manipulate their membership values, and these operations include Addition, Subtraction, Multiplication, Division, Power, aggregation, and variance, which are defined as follows:

$$c + e = \langle \omega_c + \omega_e, \sigma_c + \sigma_e, \eta_c + \eta_e \rangle, \tag{7}$$

$$c - e = \langle \omega_c - \omega_e, \sigma_c - \sigma_e, \eta_c - \eta_e \rangle. \tag{8}$$

$$c \cdot e = \langle \omega_c \cdot \omega_e, \sigma_c \cdot \sigma_e, \eta_c \cdot \eta_e \rangle. \tag{9}$$

$$c/e = \langle \omega_c/\omega_e, \sigma_c/\sigma_e, \eta_c/\eta_e \rangle \tag{10}$$

$$c^n = \langle \omega_c^n, \sigma_c^n\ , \eta_c^n \rangle \tag{11}$$

To aggregate SVNNs, we can define a weighted sum as follows:

$$X_{\text{aggregated}} = \sum_{i-1}^{n} w_i \cdot x_i = \left\langle \sum_{i-1}^{n} w_i \cdot \omega_{\hat{A}_{\mathcal{N}}}(x_i), \sum_{i-1}^{n} w_i \cdot \sigma_{\hat{A}_{\mathcal{N}}}(x_i), \sum_{i-1}^{n} w_i \cdot \eta_{\hat{A}_{\mathcal{N}}}(x_i) \right\rangle, \tag{12}$$

In addition, to quantify the distance between two SVNNs, we can use normalized hamming distance or Hausdorff distance, which can be expressed as follows:

$$d_{hamm}(c,e) = \frac{1}{3}\{|\omega_c - \omega_e| \ + |\sigma_c - \sigma_e| + |\eta_c - \eta_e|\}, \tag{13}$$

$$d_{haus}(c,e) = \max\{|\omega_c - \omega_e|, |\sigma_c - \sigma_e|, |\eta_c - \eta_e|\}, \tag{14}$$

### *3.2. Proposed Network*

Herein, we debate and discuss the detailed architecture of the proposed NTN, which is composed of three main building modules: neutrosophic encoding, multipath network, and fusion and decision modules. These modules jointly enable the NTN to handle ambiguities and uncertainties throughout representation learning effectively. In the following subsection, we dive into details of the design of each module, its contribution to the representation learning process, and its relation to other modules in the NTN.

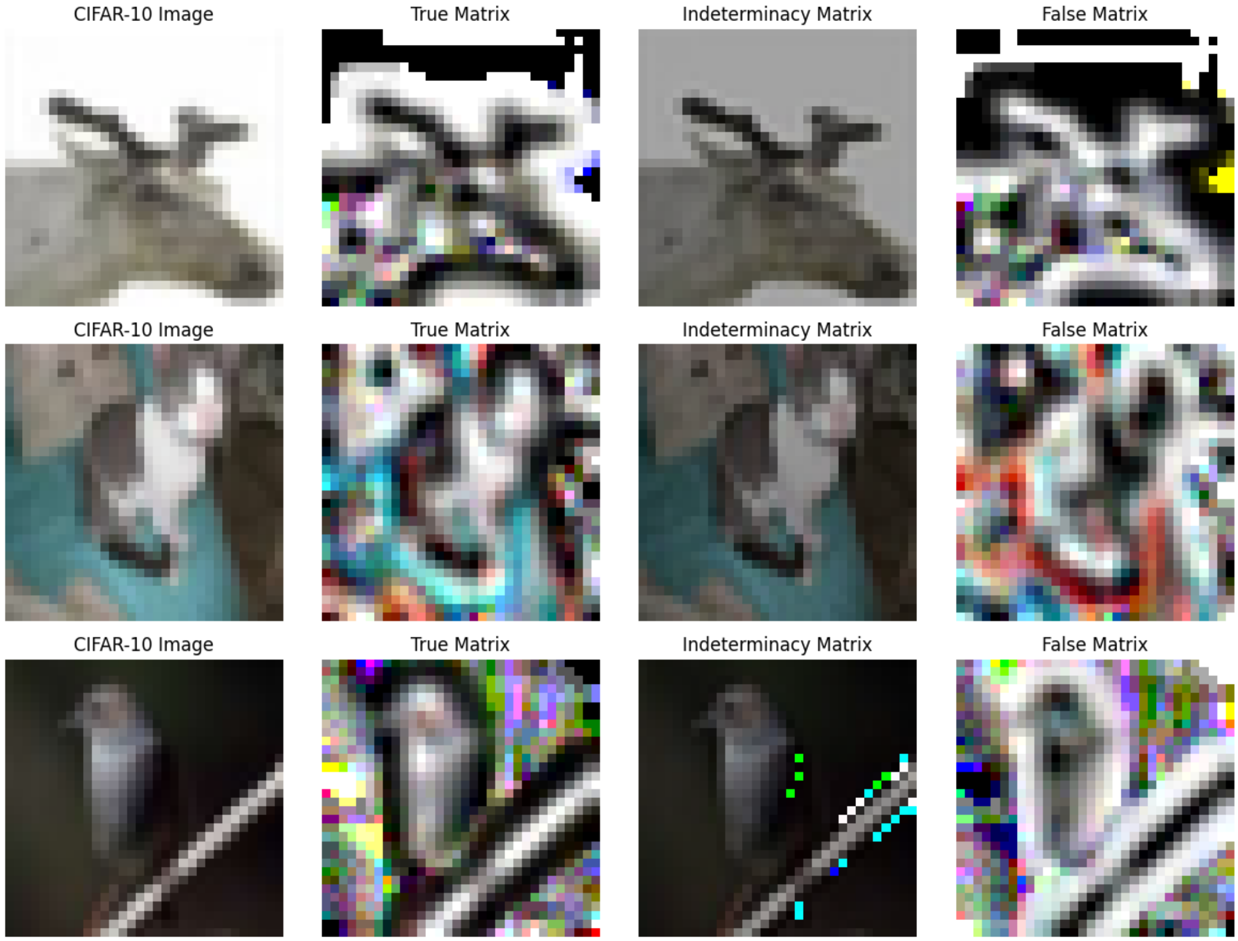



**Figure 1**. Visualization of the samples of CIFAR-10 data in the original and neutrosophic domains.

### *3.3. Neutrosophic Encoding Module*

In the early phase of the proposed NTN framework, the input image is received and encoded be passed to subsequent feature extraction layers. Unlike transition ViT, in which the input images get patched and linearly projected before being passed to the network, the proposed encoding module takes the responsibility of transforming original pixel values into neutrosophic numbers. This encoding process is made before the inputs are projected to the feature extraction layer, leading to three types of maps corresponding to the components of neutrosophic. This mapping process, which is integral to encoding images in the neutrosophic domain, is tailored to the specific image processing application at hand. In mathematical terms, the pixel in location $(i, j)$ of the input image can be denoted as $P(i, j) = \left(\omega_{\hat{A}_{\mathcal{N}}}(i, j), \sigma_{\hat{A}_{\mathcal{N}}}(i, j), \eta_{\hat{A}_{\mathcal{N}}}(i, j)\right)$, or for simplicity $P\left(\omega_{\hat{A}_{\mathcal{N}}}, \sigma_{\hat{A}_{\mathcal{N}}}, \eta_{\hat{A}_{\mathcal{N}}}\right)$. This representation conveys valuable information about the pixel's composition, specifying the percentage of its truth membership (white), indeterminacy membership (noise), and falsity membership (black). The calculation of these membership values $\omega_{\hat{A}_{\mathcal{N}}}$, $\sigma_{\hat{A}_{\mathcal{N}}}$, and $\eta_{\hat{A}_{\mathcal{N}}}$ are a serious characteristic of encoding images before they get patched. Motivated by the existing literature [4,5], the proposed encoding module computes the $\omega_{\hat{A}_{\mathcal{N}}}$ as the likelihood of a pixel being white, $\sigma_{\hat{A}_{\mathcal{N}}}$ as the degree of uncertainty or noise, and $\eta_{\hat{A}_{\mathcal{N}}}$ as the likelihood of a pixel being black.

$$\omega_{\hat{A}_{\mathcal{N}}}(i,j) = \frac{\overline{g}_{(i,j)} - \overline{g}_{min}}{\overline{g}_{max} - \overline{g}_{min}} \quad (15)$$

$$\sigma_{\hat{A}_{\mathcal{N}}}(i,j) = \frac{\partial_{(i,j)} - \partial_{min}}{\partial_{max} - \partial_{min}} \quad (16)$$

$$\eta_{\hat{A}_{\mathcal{N}}}(i,j) = 1 - \omega_{\hat{A}_{\mathcal{N}}}(i,j) = \frac{\overline{g}_{max} - \overline{g}_{(i,j)}}{\overline{g}_{max} - \overline{g}_{min}} \quad (17)$$

In the above formula, $\overline{g}_{(i,j)}$ denotes the average concentration, calculated as follows:

$$\overline{g}_{(i,j)} = \frac{1}{p*p}\sum_{m=i-p/2}^{m=i+p/2} \sum_{n=j-p/2}^{n=j+p/2} g(m,n), \quad (18)$$

where $p$ denotes the spatial dimension of the squared subwindow representing the influence of the adjacent points.

$$\overline{g}_{max} = Max(\overline{g}_{m(i,j)}), \overline{g}_{min} = Min(\overline{g}_{m(i,j)}) \quad (19)$$

$$\partial = |g_{(i,j)} - \overline{g}_{(i,j)}| \quad (20)$$

$$\partial_{max} = Max(\partial_{(i,j)}) \text{and}\, \partial_{min} = Min(\partial_{(i,j)}) \quad (21)$$

These calculated values of $\omega_{\hat{A}_{\mathcal{N}}}$, $\sigma_{\hat{A}_{\mathcal{N}}}$, and $\eta_{\hat{A}_{\mathcal{N}}}$ provide a nuanced description of each pixel's composition and offer insights into the extent of white, noise, and black contents within the pixel. Figure 1 illustrates the results of applying neutrosophic encoding to samples of CIFAR-10 data. This visualization provides an insightful view of how the encoding process translates the typical color images into the neutrosophic domain, allowing us to analyze representations in terms of truth, indeterminacy, and falsity. In addition to capturing the pixel values, the neutrosophic domain representation also helps model the inherent ambiguities and uncertainties in the images. This graphic depiction of the encoding output makes it easier to understand the encoding process and its potential advantages for managing uncertainties in visual data.

### *3.4. Multipath Network Module*

Following the encoding process, the multipath network is presented as the next building module in the proposed NTN. This multipath structure is designated with transformer encoding blocks to take the responsibility of feature extraction and representation learning from the neutrosophic encoded inputs. To recap, the architecture of the transformer encoder is primarily composed of multihead self-attention (MHSA) mechanisms, feedforward neural networks (FNNs), and residual connections that jointly process the received patches of input to learn intricate data patterns, including those related to uncertainties. The MHSA was designed to learn global interdependencies between received sequences of patches [19]. Given neutrosophic encoded input sequence $X = (x_1, x_2, \ldots, x_n)$, where $n$ is the sequence length, the linear transformations are applied to encode sequence $X$ into three matrices, namely the query matrix $(Q)$, the key matrix $(K)$, and the value matrix $(V)$:

$$Q = X \cdot W_Q, K = X \cdot W_K, V = X \cdot W_V, \quad (22)$$

where $W_Q$, $W_K$, and $W_V$ are learnable weight matrices. Then, a matrix of attention scores ($A$) is calculated based on the dot product of the query matrix with the transposed key matrix, followed by scaling, as shown as follows:

$$A = \text{softmax}\left(\frac{oK^T}{d_k}\right), \tag{23}$$

where $d_k$ is the dimension of the key vectors [20]. The matrix $A$ is later adopted to weight the value matrix, leading to self-attention output given as follows:

$$SA(Q, K, V) = A \cdot V. \tag{24}$$

The above computations are performed concurrently across different multiple heads to allow the TNT to focus on different parts of the neutrosophic encoded input sequence:

$$MHSA_{\text{multihead}} = \text{Concat}\,(SA_1, SA_2, \dots, SA_h) \cdot W_o, \tag{25}$$

where $h$ is the number of attention heads. $W_O$ is an additional learnable weight matrix for output transformation. Then, the output of different attention heads is passed to FNN with two linear transformations separated by a nonlinear activation function:

$$FFN_1 = MHSA_{\text{multihead}} \cdot W_1 + b_1. \tag{26}$$

Then, the activation function, typically rectified Linear Units ($ReLU$), is applied element-wise, as shown as follows:

$$FFN_2 = \text{ReLU}\,(FFN_1). \tag{27}$$

The result is further transformed with another set of weight matrices:

$$FFN_{\text{output}} = FFN_2 \cdot W_2 + b_2, \tag{28}$$

where $W_1$, $W_2$, $b_1$, and $b_2$ are learnable weight and bias matrices. The building of each network path in TNT is composed of a stack of three multiple transformer blocks, in which each block takes the output of the previous block as input, tolerating the network to capture progressively composite patterns and dependencies in the neutrosophic-encoded data.

### *3.5. Fusion and Decision Modules*

By the end of NTN architecture, the fusion module is introduced as a pivotal building block for consolidating the various knowledge obtained from different paths in the previous network. We apply multiplicative fusion to combine the outcomes of the various representational learning paths, which aim to inform the model's final decision with uncertainty-infused representation. The choice of multiplicative fusion can be attributed to its ability to capture non-linear and complex interactions among features, empowering the model to emphasize relevant information by defeating inputs according to the learned weights. In mathematical terms, given the outputs $O_1, O_2, \dots, O_n$ from the respective paths, the multiplicative fusion can be described as follows:

$$F_{\text{multiplicative}} = \prod_{i=1}^{n} O_i, \tag{29}$$

where $O_i$ represents the output of the $i$th path. Finally, the classification decision is made based on the cross-entropy given below:

$$L_{CCE} = -\frac{1}{N}\sum_{i=1}^{N}\sum_{c=1}^{C}\left(y_{actual}^{i,c}\right)\log\left(y_{model}^{i,c}\right). \tag{30}$$

In the above formula, $N$, $C$ symbolize the number of samples and number of classes, respectively.

## 4. Experimental Setup

This section debates the empirical preparation and related setups for the proof-of-concept experiments conducted in this work. In other words, our discussion here covers data description, evaluation metrics, training hyperparameters, execution settings, and.

First, we used popular datasets (Fashion-MNIST, CIFAR-10) to train and evaluate NTN and competing methods. The former dataset encompasses 70,000 images ($size = 28 \times 28$) fitting to distinct classes, where 7,000 images exist per class. Among them, 60,000 images are used as a training set, while 10,000 images are used for testing. The second dataset contains 60,000 images ($size = 32 \times 32 \times 3$) distributed across 10 classes, where the training set is selected to contain only 50,000 images, and the test set contains 10,000 images. During the evaluation phase, the model performance is measured by displaying a confusion matrix, from which the following metrics are calculated:

$$\text{Accuracy} = \frac{\text{TP+TN}}{\text{TP+TN+FP+FN}}, \quad (31)$$

$$\text{Precision} = \frac{\text{TP}}{\text{TP+FP}}, \quad (32)$$

$$\text{Recall} = \frac{\text{TP}}{\text{TP+FN}}, \quad (33)$$

$$\text{F1} - \text{measure} = 2 * \frac{\text{Recall} \times \text{Precision}}{\text{Recall+Precision}}. \quad (34)$$

As an essential step for the reproducibility of our work, we chart out the implementation setup associated with our experiment. Also, the training hyper-parameters are presented to facilitate interpreting the different configurations taken into account throughout our experiments. Table 1 introduces a comprehensive detail of our implementations in terms of parameters and corresponding values.

**Table** 1. Summary of implementation setup for our experiments

| **Parameter** | **Value** |
|---|---|
| Device | Dell Workstation |
| CPU | Intel(R) Core (TM) i5-3317U CPU @ 1.70–1.70 GHz |
| GPU | NVIDIA RTX 2060 |
| RAM | 32 GB |
| OS | Windows 10 |
| Frameworks | TensorFlow 2.8.0, Sk-learn, Sci-Py, MatPlot |
| Batch Size | 64 |
| Learning Rate | 0.001 |
| Training Epochs | 60 |

Given that we study uncertainty modeling, we apply Gaussian noise to the datasets in our experiments. This approach can be achieved in different scenarios according to the level of noise injected into the data. Table 2 summarizes the different scenarios adopted in our experiments, in which the "Mean (μ)" column specifies the mean of the Gaussian noise, and the "Standard Deviation (σ)" column indicates the standard deviation of the noise.

**Table 2**: Noise setting summary

| Scenario | Noise Level | Mean (μ) | Standard Deviation (σ) |
|---|---|---|---|
| **Sc-0** | Clean (No Noise) | 0.00 | 0.00 |
| Sc-1 | Low Noise | 0.10 | 0.05 |
| Sc-2 | Medium Noise | 0.30 | 0.10 |
| Sc-3 | High Noise | 0.50 | 0.15 |
| Sc-4 | Very High Noise | 0.70 | 0.20 |

## 5. Results and Analysis

In this section, we provide a detailed explanation of the experimental results, offering an inclusive evaluation of the performance of the proposed model across various datasets. The results act as a lens by which we can obtain valuable insights about the interaction between the visual deep network and neutrosophic logic when modeling the representational patterns under high uncertainty settings.

To interpret the performance ability of the proposed model in modeling uncertainties in visual representational learning tasks, we conduct fair experimental comparisons to quantitively evaluate the performance of our model against the cutting-edge baselines. In these experiments, the data from scenarios Sc-3, Sc-4, and Sc-5 are used. Table 3 summarizes the numerical classification results on Fashion-MNIST data under different scenarios. Our model achieves remarkable improvements over competing models under different noise levels. Table 4 summarizes the numerical classification results on CIFAR-10 data under different scenarios. Remarkably, the classification performance of our model outperforms the competing models under different noise levels. As observed, the performance improvement becomes more significant in the case of high noise scenarios, further validating the advantage of our model for modeling uncertainties in noisy data.

**Table 3**. Comparison of quantitative results of different classifiers on Fashion-MNIST dataset.

| | **Sc-2** | | | **Sc-3** | | | **Sc-4** | | |
|---|---|---|---|---|---|---|---|---|---|
| **Model** | Accuracy | F1-score | AUC | Accuracy | F1-score | AUC | Accuracy | F1-score | AUC |
| **SVM** | 94.19±2.49 | 94.15±2.42 | 97.11±0.05 | 91.28±5.52 | 92.88±5.82 | 95.46±0.1 | 88.47±5.29 | 87.34±2.64 | 93.1±4.77 |
| **LeNet** | 95.21±3.07 | 94.87±2.88 | 97.76±5.43 | 92.65±1.36 | 92.43±5.43 | 97.08±5.51 | 90.06±5.78 | 89.67±5.2 | 95.88±1.13 |
| **ResNet-18** | 96.88±3.44 | 96.74±3.24 | 98.99±2.54 | 93.20±1.17 | 95.29±3.00 | 98.03±1.38 | 90.78±3.71 | 90.12±1.8 | 94.69±0.36 |
| **ViT** | 97.02±3.54 | 96.64±5.13 | 98.26±1.23 | 93.61±2.83 | 93.56±3.25 | 96.97±3.79 | 90.93±5.29 | 90.54±1.05 | 94.84±1.9 |
| **CCT** | 97.31±2.1 | 96.87±2.45 | 98.74±3.37 | 95.13±1.48 | 94.89±3.14 | 97.53±1.34 | 91.87±1.56 | 91.45±2.37 | 96.92±2.13 |
| **Proposed** | 98.28±1.4 | 97.99±1.91 | 99.31±0.94 | 97.17±1.89 | 97.01±2.67 | 98.33±1.02 | 94.29±1.91 | 94.22±2.33 | 97.33±1.11 |

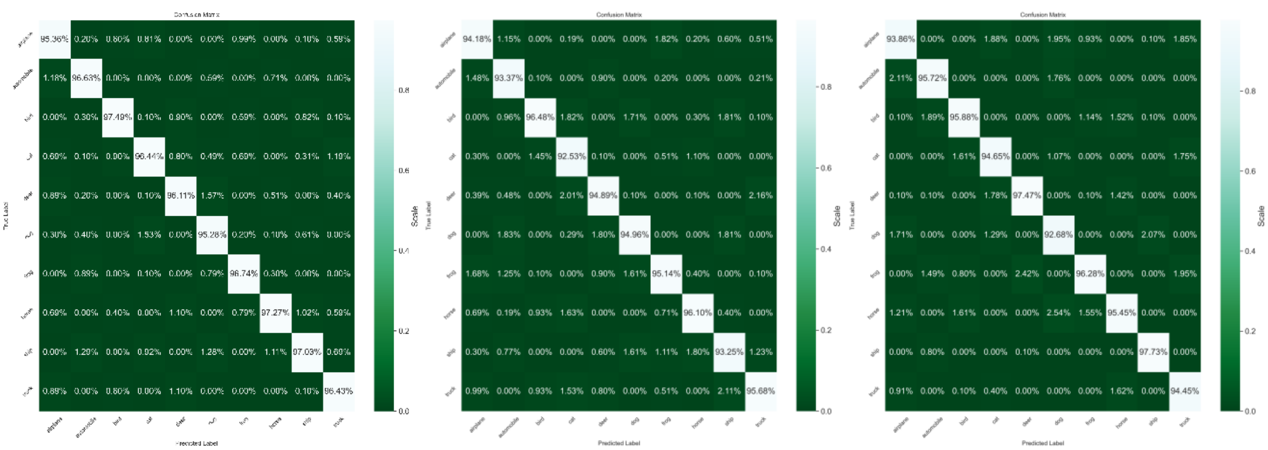
**Table 4.** Comparison of quantitative results of different classifiers on Fashion-MNIST dataset.

| | **Sc-2** | | | **Sc-3** | | | **Sc-4** | | |
|---|---|---|---|---|---|---|---|---|---|
| **Model** | Accuracy | F1-score | AUC | Accuracy | F1-score | AUC | Accuracy | F1-score | AUC |
| **SVM** | 90.12±0.03 | 88.69±5.59 | 96.13±4.48 | 85.97±0.7 | 84.49±1.76 | 95.31±4.95 | 83.04±0.35 | 82.38±0.23 | 91.1±2.11 |
| **LeNet** | 92.01±5.16 | 91.55±5.51 | 97.08±0.94 | 87.02±5.52 | 85.49±4.08 | 94.18±4.15 | 82.32±5.81 | 81.25±1.66 | 90.04±3.97 |
| **ResNet-18** | 92.22±4.99 | 91.06±3.92 | 97.9±1.88 | 87.6±2.8 | 87.14±3.91 | 96.25±2.97 | 83.48±5.4 | 82.43±3.15 | 90.43±4.78 |
| **ViT** | 94.95±5.94 | 94.4±1.95 | 98.05±5.54 | 90.6±1.68 | 89.26±0.78 | 95.93±0.92 | 86.78±3.71 | 85.92±4.9 | 91.33±2.99 |
| **CCT** | 95.16±3.95 | 94.55±3.08 | 98.07±4.48 | 91.89±5.35 | 90.8±5.93 | 95.83±1.33 | 87.26±5.19 | 85.77±1.64 | 90.12±4.48 |
| **Proposed** | 96.41±1.22 | 96.33±2.22 | 98.44±1.87 | 94.64±2.27 | 94.21±1.69 | 97.66±2.16 | 95.41±2.42 | 95.39±2.21 | 97.18±1.02 |

Beyond the numerical improvements achieved by the proposed NTN in the above comparisons, the t-test is used to provide a comprehensive statistical significance analysis to ensure that the achieved improvements are not occurring by chance. This experimental analysis computes p-value statistics (refer to Table 5), which are compared with a suitable threshold (e.g., 0.05), that is typically determined according to a predefined confidence level (e.g., 95%). As notable in Table 5, the obtained p-values are below the significance threshold, which leads to drawing strong conclusions regarding the statistical significance of improvements achieved by the proposed NTN, demonstrating its competitive advantage.

**Table 5**: Statistical significance analysis results.

| | Fashion-MNIST | | | CIFAR-10 | | |
|---|---|---|---|---|---|---|
| **Model** | **Sc-2** | **Sc-3** | **Sc-4** | **Sc-2** | **Sc-3** | **Sc-4** |
| **Proposed vs SVM** | 8.86E-03 | 8.71E-03 | 1.25E-02 | 1.33E-02 | 5.27E-03 | 3.31E-04 |
| **Proposed vs LeNet** | 8.94E-03 | 1.34E-03 | 1.34E-04 | 1.57E-02 | 1.58E-03 | 1.33E-05 |
| **Proposed vs ResNet-18** | 1.68E-05 | 1.39E-04 | 8.17E-03 | 3.18E-03 | 1.37E-02 | 1.11E-06 |
| **Proposed vs ViT** | 7.01E-03 | 6.42E-06 | 6.66E-05 | 1.21E-04 | 7.17E-09 | 1.49E-08 |
| **Proposed vs CCT** | 1.12E-07 | 1.44E-08 | 1.03E-07 | 1.19E-08 | 1.50E-06 | 6.10E-04 |



**Sc-2** **Sc-3** **Sc-4**

**Figure** 2. Visual illustration of confusion matrices for NTN under different uncertainty settings on Fashion-MNIST data.

In the previous experiments, the focus was given to the global view of model performance, but, to get a more detailed view of the class-level performance of the proposed NTN, we display its confusion matrix in both Figures 2-3. Figure 2 displays three confusion matrices corresponding to different uncertainty scenarios in the Fashion-MNIST dataset. It is worth noting that the average recognition accuracy of each class is almost similar with only 1%-2% variations, which reflect the representational power of NTN, demonstrating their consistent detection performance across various uncertainty levels. Moreover, in Figure 3, we display three confusion matrices corresponding to different uncertainty scenarios in the CIFAR-10 dataset. By observing the class-level precision, we can further drive critical insights on the consistent discrimination ability of NTN under RGB settings, which conform to our findings in grayscale scenario.

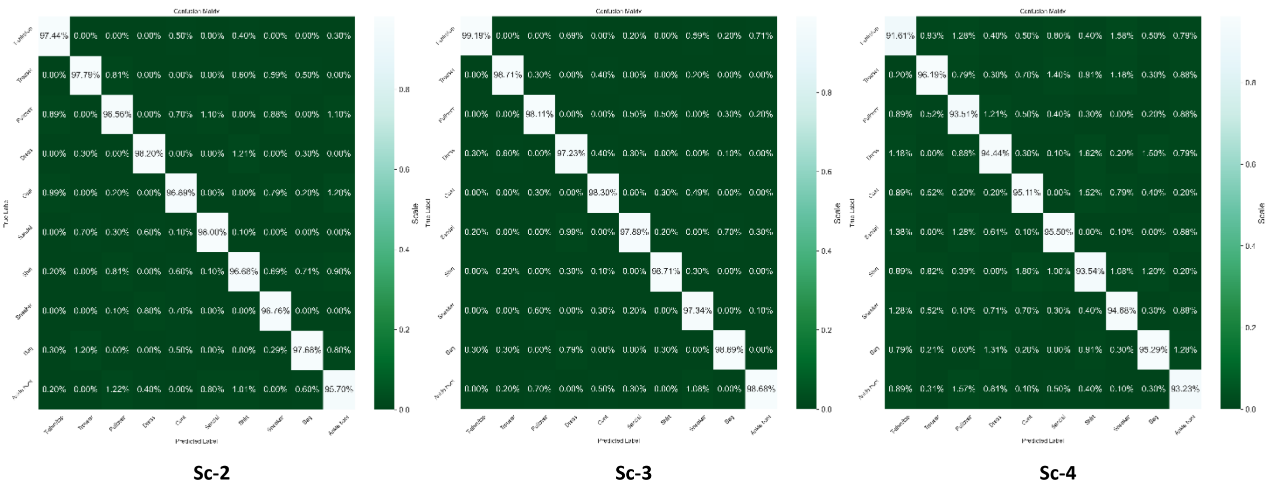



**Figure 3**. Visual illustration of confusion matrices for NTN under different uncertainty settings on CIFAR-10 data.

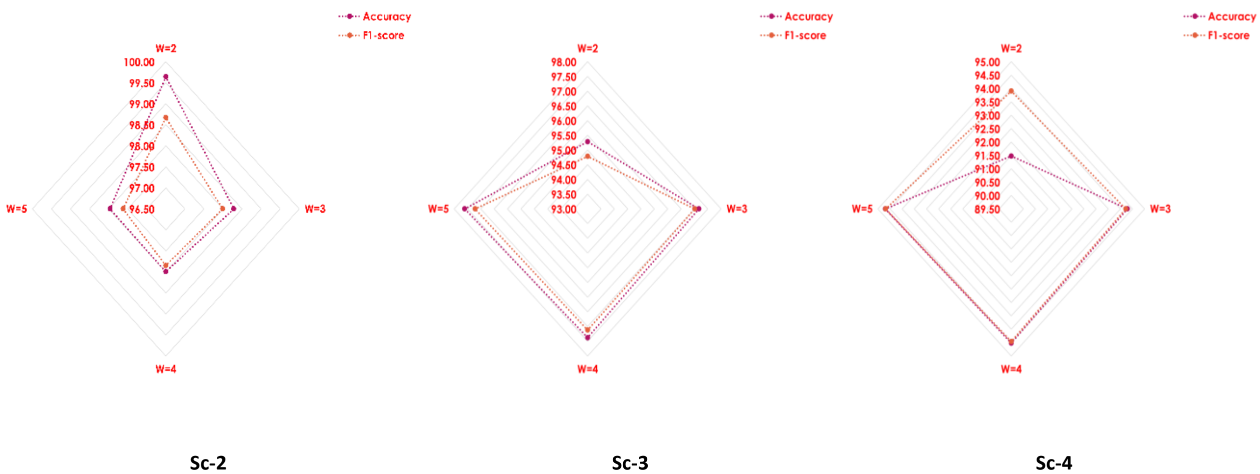



**Figure** 4. Ablation analysis for NTN under different uncertainty settings on Fashion-MNIST data.

Moreover, to capture the effectiveness of window size in the neutrosophic encoding module, a set of ablation experiments is performed to provide an in-depth evaluation of this hyperparameter. Different window sizes are independently used to assess their influences on the model's performance, where the corresponding numerical results on Fashion-MNIST are presented in Figure 4. As displayed, we can draw the conclusion that smaller window sizes work better on Sc-1, whereas larger window sizes are preferable in noisy scenarios (Sc-3 and Sc-4). This can be attributed to the fact that when the window size grows, more contextual information around each pixel is contemplated in the neutrosophic domain, making the image more tolerant to artifacts produced by noisy pixels in the near locality. Conversely, in Sc-2 data, we require only nearby details, and this effect may not exist, alleviating the need for such background locality knowledge.

## 6. Conclusion

This study presents an exploration of the impact of uncertainty on representational learning through developing a multi-path framework for modeling uncertainty in visual inputs during the process of feature extraction. A neutrosophic encoding is applied to map the image patches into triplet neutrosophic components, while visual transformer encoding is applied to extract insightful representation across different paths. The experimental comparisons on the Fashion-MNIST and CIFAR-10 datasets demonstrated the efficiency of the proposed NTN over the state-of-the-art methods even under scenarios with increased levels of uncertainty, showcasing its compliance and flexibility. The experimental findings demonstrate the promise of neutrosophic logic in revolutionizing the architectural design of the existing ML systems to be robust against ambiguity in different types of data, including time series, text, and graphs. Furthermore, the implications of this study can extend to support the explainability of ML decisions in complex data scenarios.

**Conflict of interest**

The authors declare that there is no conflict of interest in the research.

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**Ethical approval**

This article does not contain any studies with human participants or animals performed by any of the authors.

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