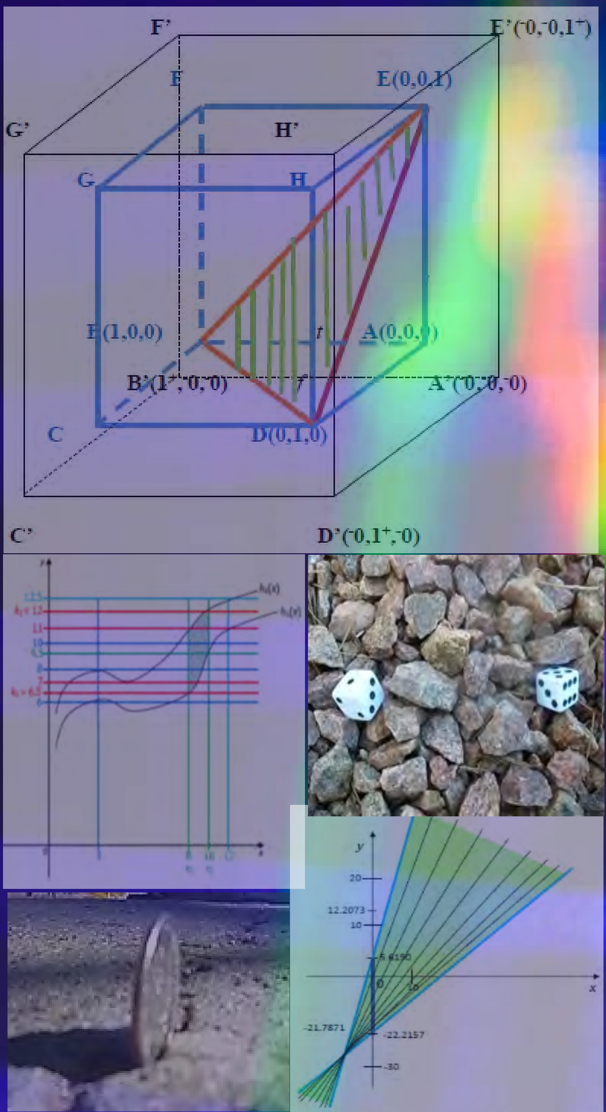
Volume 59,2023

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

Special Issue in: Symbolic Plithogenic Algebraic Structures



Florentin Smarandache . Mohamed Abdel-Basset . Said Broumi
Editors-in-Chief

Necati.Olgun, Memet Şahin, Ahmed Hatip, Mohammad Abobala
Guest Editors

ISSN 2331-6055 (Print)
ISSN 2331-608X (Online)

Neutrosophic Science International Association (NSIA)

ISSN 2331-6055 (print) ISSN 2331-608X (online)

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

NSS

University of New Mexico

ISSN 2331-6055 (print) *ISSN 2331-608X (online)*

University of New Mexico

NSS

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

Information for Authors and Subscribers

*"*Neutrosophic Sets and Systems*"* has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

All submissions should be designed in MS Word format using our template file:

http://fs.unm.edu/NSS/NSS-paper-template.doc.

A variety of scientific books in many languages can be downloaded freely from the Digital Library of Science:

http://fs.unm.edu/ScienceLibrary.htm.

To submit a paper, mail the file to the Editor-in-Chief. To order printed issues, contact the Editor-in-Chief. This journal is non-commercial, academic edition. It is printed from private donations.

Information about the neutrosophics you get from the UNM website:

http://fs.unm.edu/neutrosophy.htm. The

home page of the journal is accessed on

http://fs.unm.edu/NSS.

ISSN 2331-6055 (print) ISSN 2331-608X (online)

University of New Mexico

NSS

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

**** NSS has been accepted by SCOPUS. Starting with Vol. 19, 2018, the NSS articles are indexed in Scopus.**

NSS ABSTRACTED/INDEXED IN

SCOPUS,
Google Scholar,
Google Plus,
Google Books,
EBSCO,
Cengage Thompson Gale (USA),
Cengage Learning (USA),
ProQuest (USA),
Amazon Kindle (USA),
University Grants Commission (UGC) - India,
DOAJ (Sweden),
International Society for Research Activity (ISRA),
Scientific Index Services (SIS),
Academic Research Index (ResearchBib),
Index Copernicus (European Union),
CNKI (Tongfang Knowledge Network Technology Co.,
Beijing, China),
Baidu Scholar (China),
Redalyc - Universidad Autonoma del Estado de Mexico (IberoAmerica),
Publons,
Scimago, etc.

Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

University of New Mexico

.

Contents

Generating Pythagoras Quadruples in Symbolic 2-Plithogenic Commutative Rings

Yaser Ahmad Alhasan[1,*], Abuobida Mohammed A. Alfahal[2], Raja Abdullah Abdulfatah[3]

[1]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia
[2]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia
[3]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia

Emails:[1.] y.alhasan@psau.edu.sa; [2.]a.alfahal@psau.edu.sa;[3.] r.abdulfatah@psau.edu.sa ;
Corresponding author: y.alhasan@psau.edu.sa

Abstract:

This paper is dedicated to find a general algorithm for generating different solutions for Pythagoras non-linear Diophantine equation in four variables $x^2 + y^2 + z^2 = t^2$ in symbolic 2-plithogenic rings, which are known as Pythagoras quadruples.

Also, we present some examples about those quadruples in some finite symbolic 2-plithogenic rings.

Keywords: symbolic 2-plithogenic ring, Pythagoras quadruples, Diophantine equations

Introduction and Preliminaries

Symbolic n-plithogenic algebraic structures are a new generalization of classical algebraic structures, as they have serious algebraic properties to study.

In the previous literature, we can clearly note several algebraic studies that were interested in discovering the properties of these algebraic structures, for example we can find some applications of plithogenic structures in probability, ring theory, linear spaces, matrices, and equations [1-10].

zenodo.10031125/10.5281

Researchers have studied Pythagorean quadruples in the ring of ordinary algebraic numbers [11-14].

Several efficient algorithms for calculating these quadruples have been presented, as solutions to the corresponding Diophantine equation.

This has motivated us to study Pythagoras quadruples in the symbolic 2-plithogenic commutative case, where we find a general algorithm for generating different solutions for Pythagoras non-linear Diophantine equation in four variables $\boldsymbol{x^2 + y^2 + z^2 = t^2}$ in symbolic 2-plithogenic rings.

Definition.

The symbolic 2-plithogenic ring of real numbers is defined as follows:

$$2 - SP_R = \{t_0 + t_1P_1 + t_2P_2; t_i \in R, P_1 \times P_2 = P_2 \times P_1 = P_2, {P_1}^2 = {P_2}^2 = P_2\}$$

The addition operation on $2 - SP_R$ is defined as follows:

$$(t_0 + t_1P_1 + t_2P_2) + (t_0' + t_1'P_1 + t_2'P_2) = (t_0 + t_0') + (t_1 + t_1')P_1 + (t_2 + t_2')P_2$$

The multiplication on $2 - SP_R$ is defined as follows:

$$(t_0 + t_1P_1 + t_2P_2)(t_0' + t_1'P_1 + t_2'P_2)$$
$$= t_0t_0' + (t_0t_1' + t_1t_0' + t_1t_1')P_1 + (t_0t_2' + t_1t_2' + t_2t_2' + t_2t_0' + t_2t_1')P_2$$

Main Discussion

Definition.

Let $T = t_0 + t_1P_1 + t_2P_2, S = s_0 + s_1P_1 + s_2P_2, K = k_0 + k_1P_1 + k_2P_2, L = l_0 + l_1P_1 + l_2P_2$ be four symbolic 2-plithogenic elements of a symbolic 2-plithogenic commutative ring

$2 - SP_R$, then (T, S, K, L) is called a symbolic 2-plithogenic Pythagoras quadruple if and only if $T^2 + S^2 + K^2 = L^2$.

Theorem.

Let $T = t_0 + t_1P_1 + t_2P_2, S = s_0 + s_1P_1 + s_2P_2, K = k_0 + k_1P_1 + k_2P_2, L = l_0 + l_1P_1 + l_2P_2 \in 2 - SP_R$, then (T, S, K, L) is a symbolic 2-plithogenic Pythagoras quadruple if and only if:

$(t_0, s_0, k_0, l_0), (t_0 + t_1, s_0 + s_1, k_0 + k_1, l_0 + l_1), (t_0 + t_1 + t_2, s_0 + s_1 + s_2, k_0 + k_1 + k_2, l_0 + l_1 + l_2)$ are three Pythagoras quadruples in R.

Proof.

We have:

$T^2 = {t_0}^2 + [(t_0 + t_1)^2 - {t_0}^2]P_1 + [(t_0 + t_1 + t_2)^2 - (t_0 + t_1)^2]P_2,$

$S^2 = {s_0}^2 + [(s_0 + s_1)^2 - {s_0}^2]P_1 + [(s_0 + s_1 + s_2)^2 - (s_0 + s_1)^2]P_2,$

$K^2 = {k_0}^2 + [(k_0 + k_1)^2 - {k_0}^2]P_1 + [(k_0 + k_1 + k_2)^2 - (k_0 + k_1)^2]P_2,$

$L^2 = {l_0}^2 + [(l_0 + l_1)^2 - {l_0}^2]P_1 + [(l_0 + l_1 + l_2)^2 - (l_0 + l_1)^2]P_2,$

The equation $T^2 + S^2 + K^2 = L^2$ is equivalent to:

${t_0}^2 + {s_0}^2 + {k_0}^2 = {l_0}^2$ (1)

$(t_0 + t_1)^2 + (s_0 + s_1)^2 + (k_0 + k_1)^2 = (l_0 + l_1)^2$ (2)

$(t_0 + t_1 + t_2)^2 + (s_0 + s_1 + s_2)^2 + (k_0 + k_1 + k_2)^2 = (l_0 + l_1 + l_2)^2$ (3)

Thus, the proof holds.

Theorem.

Let $(t_0, s_0, k_0, l_0), (t_1, s_1, k_1, l_1), (t_2, s_2, k_2, l_2)$ be three Pythagoras quadruples in R, then the corresponding Pythagoras quadruple in $2 - SP_R$ is (T, S, K, L), where:

$T = t_0 + [t_1 - t_0]P_1 + [t_2 - t_1]P_2,$

$S = s_0 + [s_1 - s_0]P_1 + [s_2 - s_1]P_2,$

$K = k_0 + [k_1 - k_0]P_1 + [k_2 - k_1]P_2,$

$L = l_0 + [l_1 - l_0]P_1 + [l_2 - l_1]P_2.$

Proof.

We must compute $T^2 + S^2 + K^2$,

$T^2 + S^2 + K^2 = {t_0}^2 + ({t_1}^2 - {t_0}^2)P_1 + ({t_2}^2 - {t_1}^2)P_2 + {s_0}^2 + ({s_1}^2 - {s_0}^2)P_1 + ({s_2}^2 - {s_1}^2)P_2 + {k_0}^2 + ({k_1}^2 - {k_0}^2)P_1 + ({k_2}^2 - {k_1}^2)P_2 = ({t_0}^2 + {s_0}^2 + {k_0}^2) + ({t_1}^2 + {s_1}^2 + {k_1}^2 - {t_0}^2 - {s_0}^2 - {k_0}^2)P_1 + ({t_2}^2 + {s_2}^2 + {k_2}^2 - {t_1}^2 - {s_1}^2 - {k_1}^2)P_2 = {l_0}^2 + ({l_1}^2 - {l_0}^2)P_1 + ({l_2}^2 - {l_1}^2)P_2 = L^2.$

So that, the proof is complete.

Example.

We have $L_1 = (1, -1, i, 1), L_2 = (i, 1, -1, -1), L_3 = (-i, -1,1, -1)$ are three Pythagoras quadruples in C.

The corresponding 2-plithogenic Pythagoras quadruple is (T, S, K, L), where:

$T = 1 + (-1 + i)P_1 - 2iP_2$

$S = -1 + 2P_1 - 2P_2$

$K = i + (-1 - i)P_1 + 2P_2$

$L = 1 - 2P_1 + 2P_2$

On the other hand, we have:

$T^2 = 1 - 2P_1,$

$S^2 = 1,$

$K^2 = -1 + 2P_1,$

$L^2 = 1 = T^2 + S^2 + K^2.$

Example.

Consider the following three Pythagoras quadruples in Z_2:

$L_1 = (0,0,0,0), L_2 = (1,1,1,1), L_3 = (1,1,0,0)$

For every triple $(L_i, L_j, L_s); 1 \leq i, j, s \leq 3$, we can get a symbolic 2-plithogenic pythagoras quadruple.

We will find some symbolic 2-plithogenic Pythagoras quadruple in $2 - SP_{Z_2}$.

Let us write the following quadruples:

$$\begin{cases} Y_1 = 0 \\ \acute{Y}_1 = 0 \\ {Y_1}'' = 0 \\ {Y_1}''' = 0 \end{cases}$$

$$\begin{cases} Y_2 = P_2 \\ \acute{Y}_2 = P_2 \\ {Y_2}'' = P_2 \\ {Y_2}''' = P_2 \end{cases}$$

$$\begin{cases} Y_3 = 0 \\ \acute{Y}_3 = 0 \\ {Y_3}'' = P_2 \\ {Y_3}''' = P_2 \end{cases}$$

$$\begin{cases} Y_4 = P_1 + P_2 \\ \acute{Y}_4 = P_1 + P_2 \\ Y_4'' = P_1 + P_2 \\ Y_4''' = P_1 + P_2 \end{cases}$$

$$\begin{cases} Y_5 = P_1 + P_2 \\ \acute{Y}_5 = P_1 + P_2 \\ Y_5'' = 0 \\ Y_5''' = 0 \end{cases}$$

$$\begin{cases} Y_6 = 0 \\ \acute{Y}_6 = 0 \\ Y_6'' = P_1 + P_2 \\ Y_6''' = P_1 + P_2 \end{cases}$$

$$\begin{cases} Y_7 = 1 + P_1 \\ \acute{Y}_7 = 1 + P_1 \\ Y_7'' = 1 + P_1 \\ Y_7''' = 1 + P_1 \end{cases}$$

$$\begin{cases} Y_8 = 1 + P_1 \\ \acute{Y}_8 = 1 + P_1 \\ Y_8'' = 0 \\ Y_8''' = 0 \end{cases}$$

$$\begin{cases} Y_9 = 0 \\ \acute{Y}_9 = 0 \\ Y_9'' = 1 + P_1 \\ Y_9''' = 1 + P_1 \end{cases}$$

$$\begin{cases} Y_{10} = 1 \\ \acute{Y}_{10} = 1 \\ Y_{10}'' = 1 \\ Y_{10}''' = 1 \end{cases}$$

$$\begin{cases} Y_{11} = 1 + P_2 \\ \acute{Y}_{11} = 1 + P_2 \\ Y_{11}'' = 1 + P_2 \\ Y_{11}''' = 1 + P_2 \end{cases}$$

$$\begin{cases} Y_{12} = 1 + P_2 \\ \acute{Y}_{12} = 1 + P_2 \\ Y_{12}'' = 1 \\ Y_{12}''' = 1 \end{cases}$$

$$\begin{cases} Y_{13} = 1 + P_1 + P_2 \\ \acute{Y}_{13} = 1 + P_1 + P_2 \\ Y_{13}'' = 1 + P_1 + P_2 \\ Y_{13}''' = 1 + P_1 + P_2 \end{cases}$$

$$\begin{cases} Y_{14} = 1 \\ \acute{Y}_{14} = 1 \\ Y_{14}'' = 1 + P_1 + P_2 \\ Y_{14}''' = 1 + P_1 + P_2 \end{cases}$$

$$\begin{cases} Y_{15} = 1 + P_1 + P_2 \\ \acute{Y}_{15} = 1 + P_1 + P_2 \\ Y_{15}'' = 1 \\ Y_{15}''' = 1 \end{cases}$$

$$\begin{cases} Y_{16} = P_1 \\ \acute{Y}_{16} = P_1 \\ Y_{16}'' = P_1 \\ Y_{16}''' = P_1 \end{cases}$$

$$\begin{cases} Y_{17} = 1 \\ \acute{Y}_{17} = 1 \\ Y_{17}'' = P_1 \\ Y_{17}''' = P_1 \end{cases}$$

$$\begin{cases} Y_{26} = P_1 \\ \acute{Y}_{26} = P_1 \\ Y_{26}'' = 1 \\ Y_{26}''' = 1 \end{cases}$$

$$\begin{cases} Y_{27} = 1 \\ \acute{Y}_{27} = 1 \\ Y_{27}'' = 0 \\ Y_{27}''' = 0 \end{cases}$$

$$\begin{cases} Y_{32} = 1 \\ \acute{Y}_{32} = 1 \\ Y_{32}'' = P_2 \\ Y_{32}''' = P_2 \end{cases}$$

$$\begin{cases} Y_{33} = 1 + P_2 \\ \acute{Y}_{33} = 1 + P_2 \\ Y_{33}'' = P_2 \\ Y_{33}''' = P_2 \end{cases}$$

$$\begin{cases} Y_{35} = 1 \\ \acute{Y}_{35} = 1 \\ Y_{35}'' = P_1 + P_2 \\ Y_{35}''' = P_1 + P_2 \end{cases}$$

$$\begin{cases} Y_{36} = 1 + P_1 + P_2 \\ \acute{Y}_{36} = 1 + P_1 + P_2 \\ Y_{36}'' = P_1 + P_2 \\ Y_{36}''' = P_1 + P_2 \end{cases}$$

$$\begin{cases} Y_{37} = P_1 \\ \acute{Y}_{37} = P_1 \\ Y_{37}'' = 0 \\ Y_{37}''' = 0 \end{cases}$$

$$\begin{cases} Y_{38} = 1 \\ \acute{Y}_{38} = 1 \\ Y_{38}'' = 1 + P_1 \\ Y_{38}''' = 1 + P_1 \end{cases}$$

$$\begin{cases} Y_{39} = P_1 \\ \acute{Y}_{39} = P_1 \\ Y_{39}'' = 1 + P_1 \\ Y_{39}''' = 1 + P_1 \end{cases}$$

$$\begin{cases} Y_{40} = 0 \\ \acute{Y}_{40} = 0 \\ Y_{40}'' = 1 \\ Y_{40}''' = 1 \end{cases}$$

$$\begin{cases} Y_{44} = 0 \\ \acute{Y}_{44} = 0 \\ Y_{44}'' = 1 + P_2 \\ Y_{44}''' = 1 + P_2 \end{cases}$$

$$\begin{cases} Y_{45} = P_2 \\ \acute{Y}_{45} = P_2 \\ Y_{45}'' = 1 \\ Y_{45}''' = 1 \end{cases}$$

$$\begin{cases} Y_{46} = P_2 \\ \acute{Y}_{46} = P_2 \\ Y_{46}'' = 1 + P_2 \\ Y_{46}''' = 1 + P_2 \end{cases}$$

$$\begin{cases} Y_{47} = 0 \\ \acute{Y}_{47} = 0 \\ Y_{47}'' = 1 + P_1 + P_2 \\ Y_{47}''' = 1 + P_1 + P_2 \end{cases}$$

$$\begin{cases} Y_{48} = P_1 + P_2 \\ \acute{Y}_{48} = P_1 + P_2 \\ Y_{48}'' = 1 \\ Y_{48}''' = 1 \end{cases}$$

$$\begin{cases} Y_{49} = P_1 + P_2 \\ \acute{Y}_{49} = P_1 + P_2 \\ Y_{49}'' = 1 + P_1 + P_2 \\ Y_{49}''' = 1 + P_1 + P_2 \end{cases}$$

$$\begin{cases} Y_{50} = 0 \\ \acute{Y}_{50} = 0 \\ {Y_{50}}'' = P_1 \\ {Y_{50}}''' = P_1 \end{cases}$$

$$\begin{cases} Y_{51} = 1 + P_1 \\ \acute{Y}_{51} = 1 + P_1 \\ {Y_{51}}'' = 1 \\ {Y_{51}}''' = 1 \end{cases}$$

$$\begin{cases} Y_{52} = 1 + P_1 \\ \acute{Y}_{52} = 1 + P_1 \\ {Y_{52}}'' = P_1 \\ {Y_{52}}''' = P_1 \end{cases}$$

$$\begin{cases} Y_{53} = P_1 \\ \acute{Y}_{53} = P_1 \\ {Y_{53}}'' = P_1 + P_2 \\ {Y_{53}}''' = P_1 + P_2 \end{cases}$$

$$\begin{cases} Y_{54} = P_1 + P_2 \\ \acute{Y}_{54} = P_1 + P_2 \\ {Y_{54}}'' = P_1 + P_2 \\ {Y_{54}}''' = P_1 + P_2 \end{cases}$$

$$\begin{cases} Y_{55} = P_1 \\ \acute{Y}_{55} = P_1 \\ {Y_{55}}'' = P_2 \\ {Y_{55}}''' = P_2 \end{cases}$$

$$\begin{cases} Y_{56} = P_1 + P_2 \\ \acute{Y}_{56} = P_1 + P_2 \\ {Y_{56}}'' = P_2 \\ {Y_{56}}''' = P_2 \end{cases}$$

$$\begin{cases} Y_{57} = P_2 \\ \acute{Y}_{57} = P_2 \\ {Y_{57}}'' = P_1 \\ {Y_{57}}''' = P_1 \end{cases}$$

$$\begin{cases} Y_{58} = P_2 \\ \acute{Y}_{58} = P_2 \\ {Y_{58}}'' = P_1 + P_2 \\ {Y_{58}}''' = P_1 + P_2 \end{cases}$$

$$\begin{cases} Y_{59} = 1 + P_1 + P_2 \\ \acute{Y}_{59} = 1 + P_1 + P_2 \\ {Y_{59}}'' = 1 + P_1 \\ {Y_{59}}''' = 1 + P_1 \end{cases}$$

$$\begin{cases} Y_{60} = 1 + P_1 \\ Y_{60}' = 1 + P_1 \\ Y_{60}'' = P_2 \\ Y_{60}''' = P_2 \end{cases}$$

$$\begin{cases} Y_{61} = 1 + P_1 + P_2 \\ Y_{61}' = 1 + P_1 + P_2 \\ Y_{61}'' = P_2 \\ Y_{61}''' = P_2 \end{cases}$$

$$\begin{cases} Y_{62} = 1 + P_1 \\ Y_{62}' = 1 + P_1 \\ Y_{62}'' = P_2 \\ Y_{62}''' = P_2 \end{cases}$$

$$\begin{cases} Y_{63} = P_2 \\ Y_{63}' = P_2 \\ Y_{63}'' = 1 + P_2 \\ Y_{63}''' = 1 + P_2 \end{cases}$$

$$\begin{cases} Y_{64} = P_2 \\ Y_{64}' = P_2 \\ Y_{64}'' = 1 \\ Y_{64}''' = 1 \end{cases}$$

$$\begin{cases} Y_{65} = 1 + P_2 \\ Y_{65}' = 1 + P_2 \\ Y_{65}'' = 1 \\ Y_{65}''' = 1 \end{cases}$$

$$\begin{cases} Y_{66} = 1 + P_2 \\ Y_{66}' = 1 + P_2 \\ Y_{66}'' = 1 + P_1 + P_2 \\ Y_{66}''' = 1 + P_1 + P_2 \end{cases}$$

$$\begin{cases} Y_{67} = 1 + P_2 \\ Y_{67}' = 1 + P_2 \\ Y_{67}'' = P_1 + P_2 \\ Y_{67}''' = P_1 + P_2 \end{cases}$$

$$\begin{cases} Y_{68} = 1 + P_2 \\ Y_{68}' = 1 + P_2 \\ Y_{68}'' = 1 \\ Y_{68}''' = 1 \end{cases}$$

$$\begin{cases} Y_{69} = 1 + P_1 \\ Y_{69}' = 1 + P_1 \\ Y_{69}'' = P_1 \\ Y_{69}''' = P_1 \end{cases}$$

$$\begin{cases} Y_{70} = 1 + P_1 + P_2 \\ \acute{Y}_{70} = 1 + P_1 + P_2 \\ Y_{70}'' = P_1 \\ Y_{70}''' = P_1 \end{cases}$$

$$\begin{cases} Y_{71} = P_1 + P_2 \\ \acute{Y}_{71} = P_1 + P_2 \\ Y_{71}'' = 1 \\ Y_{71}''' = 1 \end{cases}$$

$$\begin{cases} Y_{72} = P_1 \\ \acute{Y}_{72} = P_1 \\ Y_{72}'' = 1 + P_1 + P_2 \\ Y_{72}''' = 1 + P_1 + P_2 \end{cases}$$

$$\begin{cases} Y_{73} = P_1 + P_2 \\ \acute{Y}_{73} = P_1 + P_2 \\ Y_{73}'' = 1 + P_2 \\ Y_{73}''' = 1 + P_2 \end{cases}$$

$$\begin{cases} Y_{74} = P_1 \\ \acute{Y}_{74} = P_1 \\ Y_{74}'' = 1 + P_2 \\ Y_{74}''' = 1 + P_2 \end{cases}$$

$$\begin{cases} Y_{75} = 1 + P_1 \\ \acute{Y}_{75} = 1 + P_1 \\ Y_{75}'' = 1 + P_2 \\ Y_{75}''' = 1 + P_2 \end{cases}$$

$$\begin{cases} Y_{76} = 1 + P_1 + P_2 \\ \acute{Y}_{76} = 1 + P_1 + P_2 \\ Y_{76}'' = 1 + P_2 \\ Y_{76}''' = 1 + P_2 \end{cases}$$

$$\begin{cases} Y_{77} = 1 \\ \acute{Y}_{77} = 1 \\ Y_{77}'' = 1 + P_2 \\ Y_{77}''' = 1 + P_2 \end{cases}$$

Conclusion.

In this paper, we have studied Pythagoras quadruples in symbolic 2-plithogenic commutative rings, where necessary and sufficient conditions for a symbolic 2-plithogenic quadruple (x, y, z, t) to be a Pythagoras quadruple.

Also, we have presented some related examples that explain how to find 2-plithogenic quadruples from classical quadruples.

Acknowledgments " This study is supported via funding from Prince sattam bin Abdulaziz University project number (PSAU/2023/R/1445) ".

References

[1] M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, “Introduction to Symbolic 2-Plithogenic Probability Theory,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.

[2] Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.

[3] Nader Mahmoud Taffach , Ahmed Hatip., "A Review on Symbolic 2-Plithogenic Algebraic Structures " Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.

[4] Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.

[5] Ben Othman, K., "On Some Algorithms For Solving Symbolic 3-Plithogenic Equations", Neoma Journal Of Mathematics and Computer Science, 2023.

[6] Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Concept Of Symbolic 7-Plithogenic Real Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

[7] Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Symbolic 8-Plithogenic Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

[8]Mohamed, M., "Modified Approach for Optimization of Real-Life Transportation Environment: Suggested Modifications", American Journal of Business and Operations research, 2021.

[9] Nabeeh, N., Alshaimaa, A., and Tantawy, A., "A Neutrosophic Proposed Model For Evaluation Blockchain Technology in Secure Enterprise Distributed Applications", Journal of Cypersecurity and Information Management, 2023.

[10]Abualkishik, A., Almajed, R., Thompson, W., "Improving The Performance of Fog-assisted Internet of Things Networks Using Bipolar Trapezoidal Neutrosophic Sets", Journal of Wireless and Ad Hoc Communication, 2023.

[11] R. A. Trivedi, S. A. Bhanotar, Pythagorean Triplets - Views, Analysis and Classification, IIOSR J. Math., 11 (2015).

[12] J. Rukavicka, Dickson's method for generating Pythagorean triples revisited, Eur. J. Pure Appl.Math., 6 (2013).

[13] T.Roy and F J. Sonia, A Direct Method To Generate Pythagorean Triples And Its Generalization To Pythagorean Quadruples And n-tuples, , Jadavpur University, India (2010)

[14] Paul Oliverio, " Self-Generating Pythagorean Quadruples And TV-Tuples", Los Angeles, (1993)

Received:1/5/2023, Accepted: 5/10/2023

On The Symbolic 2-plithogenic Fermat's Non-Linear Diophantine Equation

[1]Heba Alrawashdeh, [2]Othman Al-Basheer, [3]Arwa Hajjari

[1]Faculty of Computer Information Systems, The University of Jordan, Aqaba, Jordan,

[2]Sudan University of Science and Technology, Faculty of Science, Khartoum, Sudan

[3]Cairo University, Department of Mathematics, Cairo, Egypt

Co- hsrawashdeh@yahoo.com

Abstract:

This paper is dedicated to find all symbolic 2-plithogenic integer solutions for the symbolic 2-plithogenic Fermat's Diophantine equation $X^n + Y^n = Z^n$ for $n \geq 3$.

We prove that it has exactly 27 solutions, and we find all possible solutions.

Keywords: symbolic 2-plithogenic integer, Fermat's Diophantine equation, symbolic 2-plithogenic ring.

Introduction and basic definitions.

The theory of Diophantine equations is considered as an important and central theory in commutative algebra.

In our days, many developments of algebraic structures have helped us with general cases of Diophantine equations, for example, neutrosophic rings and their generalizations [1-4] have led to many new related Diophantine equations such as neutrosophic Pell's equation [5], refined neutrosophic Diophantine equation [6] and n-refined equations [7]. The main application of generalized versions of number theory is cryptography algorithms, see [15-19].

The concept of symbolic 2-plithogenic rings was defined in [8], then it was studied and generalized by many authors, see [9-14].

zenodo.10031141/10.5281

The Fermat's triple is defined as a solution of the Diophantine non-linear equation $X^n + Y^n = Z^n$ in the ring R, with $n \geq 3$.

We refer to that for the special case of $n = 2$, the Fermat's triple is called a Pythagoras triple.

It is useful for the reader to ensure that neutrosophic and plithogenic number theory is useful in cryptography [16-20].

In this work, we find all solutions of $X^n + Y^n = Z^n$ in the symbolic ring of integers $2 - SP_Z$.

Definition.

Let Z be the ring of integers, the corresponding symbolic 2-plithogenicc ring of integers is defined as follow:

$2 - SP_Z = \{x + yP_1 + zP_2; x, y, z \in Z, {P_i}^2 = P_i, P_1 \times P_2 = P_2 \times P_1 = P_2\}$.

Theorem.

For $X = l_0 + l_1P_1 + l_2P_2 \in 2 - SP_Z$, then;

$X^n = {l_0}^n + P_1[(l_0 + l_1)^n - {l_0}^n] + P_2[(l_0 + l_1 + l_2)^n - (l_0 + l_1)^n]$.

Remark.

In Z, we have three Fermat's triples:

$(0,1,1), (1,0,1), (0,0,0)$ for all $n \geq 3$.

Main results

Theorem.

Let $2 - SP_Z$ be the symbolic 2-plithogenic ring of integers, then it has exactly 27 Fermat's triples.

Proof.

Let (T, S, K) be a Fermat's triple of $2 - SP_Z$

with $T = t_0 + t_1P_1 + t_2P_2, S = s_0 + s_1P_1 + s_2P_2, K = k_0 + k_1P_1 + k_2P_2$,

the equation $T^n + S^n = K^n$ is equivalent to:

$$\begin{cases} {t_0}^n + {s_0}^n = {k_0}^n \\ (t_0 + t_1)^n + (s_0 + s_1)^n = (k_0 + k_1)^n \\ (t_0 + t_1 + t_2)^n + (s_0 + s_1 + s_2)^n = (k_0 + k_1 + k_2)^n \end{cases}$$

Thus $A_1 = (t_0, s_0, k_0), A_2 = (t_0 + t_1, s_0 + s_1, k_0 + k_1), A_3 = (t_0 + t_1 + t_2, s_0 + s_1 + s_2, k_0 + k_1 + k_2)$ are three triples in Z.

So that, $A_1, A_2, A_3 \in \{(0,1,1), (1,0,1), (0,0,0)\}$, thus three exists 27 solutions of the symbolic 2-plithogenic Fermat's Diophantine equation.

Now, we discus all possible cases:

Case1.

$$\begin{cases} t_0 = s_0 = k_0 = 0 \\ t_0 + t_1 = s_0 + s_1 = k_0 + k_1 = 0 \\ t_0 + t_1 + t_2 = s_0 + s_1 + s_2 = k_0 + k_1 + k_2 = 0 \end{cases}$$

Thus $F_1 = (0,0,0)$

Case2.

$$\begin{cases} t_0 = s_0 = k_0 = 0 \\ t_0 + t_1 = s_0 + s_1 = k_0 + k_1 = 0 \\ t_0 + t_1 + t_2 = 0, s_0 + s_1 + s_2 = k_0 + k_1 + k_2 = 1 \end{cases}$$

Thus $F_2 = (0, P_2, P_2)$

Case3.

$$\begin{cases} t_0 = s_0 = k_0 = 0 \\ t_0 + t_1 = s_0 + {s_1}^n = k_0 + k_1 = 0 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = 1, s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_3 = (P_2, 0, P_2)$

Case4.

$$\begin{cases} t_0 = s_0 = k_0 = 0 \\ t_0 + t_1 = k_0 + k_1 = 1, s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_4 = (P_1 - P_2, 0, P_1 - P_2)$

Case5.

$$\begin{cases} t_0 = s_0 = k_0 = 0 \\ t_0 + t_1 = k_0 + k_1 = 1, s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = 1, s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_5 = (P_1, 0, P_1)$

Case6.

$$\begin{cases} t_0 = s_0 = k_0 = 0 \\ t_0 + t_1 = k_0 + k_1 = 1, s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = 0, k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 1 \end{cases}$$

Thus $F_6 = (P_1 - P_2, P_2, P_1)$

Case7.

$$\begin{cases} t_0 = s_0 = k_0 = 0 \\ t_0 + t_1 = 0, k_0 + k_1 = s_0 + s_1 = 1 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_7 = (0, P_1 - P_2, P_1 - P_2)$

Case8.

$$\begin{cases} t_0 = s_0 = k_0 = 0 \\ t_0 + t_1 = 0, k_0 + k_1 = s_0 + s_1 = 1 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = 1, s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_8 = (P_2, P_1 - P_2, P_1)$

Case9.

$$\begin{cases} t_0 = s_0 = k_0 = 0 \\ t_0 + t_1 = 0, k_0 + k_1 = s_0 + s_1 = 1 \\ t_0 + t_1 + t_2 = 0, k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 1 \end{cases}$$

Thus $F_9 = (0, P_1, P_1)$

Case10.

$$\begin{cases} t_0 = k_0 = 1, s_0 = 0 \\ t_0 + t_1 = k_0 + k_1 = s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{10} = (1 - P_1, 0, 1 - P_1)$

Case11.

$$\begin{cases} t_0 = k_0 = 1, s_0 = 0 \\ t_0 + t_1 = k_0 + k_1 = s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = 1, s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{11} = (1 - P_1, 0, 1 - P_1 + P_2)$

Case12.

$$\begin{cases} t_0 = k_0 = 1, s_0 = 0 \\ t_0 + t_1 = k_0 + k_1 = s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = 0, k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 1 \end{cases}$$

Thus $F_{12} = (1 - P_1, P_2, 1 - P_1 + P_2)$

Case13.

$$\begin{cases} t_0 = k_0 = 1, s_0 = 0 \\ t_0 + t_1 = k_0 + k_1 = 1, s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{13} = (1 - P_1, 0,1 - P_2)$

Case14.

$$\begin{cases} t_0 = k_0 = 1, s_0 = 0 \\ t_0 + t_1 = k_0 + k_1 = 1, s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = 1, s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{14} = (1,0,1)$

Case15.

$$\begin{cases} t_0 = k_0 = 1, s_0 = 0 \\ t_0 + t_1 = k_0 + k_1 = 1, s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = 0, k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{15} = (1 - P_1, P_2, 1)$

Case16.

$$\begin{cases} t_0 = k_0 = 1, s_0 = 0 \\ t_0 + t_1 = 0, k_0 + k_1 = s_0 + s_1 = 1 \\ t_0 + t_1 + t_2 = 0, k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{16} = (1 - P_1, P_1 - P_2, 1 - P_1)$

Case17.

$$\begin{cases} t_0 = k_0 = 1, s_0 = 0 \\ t_0 + t_1 = 0, k_0 + k_1 = s_0 + s_1 = 1 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = 1, s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{17} = (1 - P_1 + P_2, P_1 - P_2, 1)$

Case18.

$$\begin{cases} t_0 = k_0 = 1, s_0 = 0 \\ t_0 + t_1 = 0, k_0 + k_1 = s_0 + s_1 = 1 \\ t_0 + t_1 + t_2 = 0, k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 1 \end{cases}$$

Thus $F_{18} = (1 - P_1, P_1, 1)$

Case19.

$$\begin{cases} t_0 = 0, k_0 = s_0 = 1 \\ t_0 + t_1 = k_0 + k_1 = s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{19} = (0,1 - P_1, 1 - P_1)$

Case20.

$$\begin{cases} t_0 = 0, k_0 = s_0 = 1 \\ t_0 + t_1 = k_0 + k_1 = s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = 1, s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{20} = (P_2, 1 - P_1, 1 - P_1 + P_2)$

Case21.

$$\begin{cases} t_0 = 0, k_0 = s_0 = 1 \\ t_0 + t_1 = k_0 + k_1 = s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = 0, k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{21} = (0,1 - P_1 + P_2, 1 - P_1 + P_2)$

Case22.

$$\begin{cases} t_0 = 0, k_0 = s_0 = 1 \\ t_0 + t_1 = k_0 + k_1 = 1, s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = 0, k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{22} = (P_1 - P_2, 1 - P_1, 1 - P_2)$

Case23.

$$\begin{cases} t_0 = 0, k_0 = s_0 = 1 \\ t_0 + t_1 = k_0 + k_1 = 1, s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = 1, s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{23} = (P_1, 1 - P_1, 1)$

Case24.

$$\begin{cases} t_0 = 0, k_0 = s_0 = 1 \\ t_0 + t_1 = k_0 + k_1 = 1, s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = 0, k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 1 \end{cases}$$

Thus $F_{22} = (P_1 - P_2, 1 - P_1 + P_2, 1)$

Case25.

$$\begin{cases} t_0 = 0, k_0 = s_0 = 1 \\ t_0 + t_1 = 0, k_0 + k_1 = s_0 + s_1 = 1 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{25} = (0,1 - P_2, 1 - P_2)$

Case26.

$$\begin{cases} t_0 = 0, k_0 = s_0 = 1 \\ t_0 + t_1 = k_0 + k_1 = 1, s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = k_0 + k_1 + k_2 = 1, s_0 + s_1 + s_2 = 0 \end{cases}$$

Thus $F_{26} = (P_2, 1 - P_2, 1)$

Case27.

$$\begin{cases} t_0 = 0, k_0 = s_0 = 1 \\ t_0 + t_1 = k_0 + k_1 = 1, s_0 + s_1 = 0 \\ t_0 + t_1 + t_2 = 0, k_0 + k_1 + k_2 = s_0 + s_1 + s_2 = 1 \end{cases}$$

Thus $F_{27} = (0,1,1)$.

Conclusion

In this paper, we have studied the solutions of symbolic 2-plithogenic Fermat's non-linear Diophantine equation, where we have proved that it has exactly 27 solutions, and we presented the all-27 possible solutions.

References

[1] Kandasamy, V.W.B., and Smarandache, F., "Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures", Hexis, Phonex, Arizona, 2006.

[2] Abobala, M, "n-Cyclic Refined Neutrosophic Algebraic Systems Of Sub-Indeterminacies, An Application To Rings and Modules", International Journal of Neutrosophic Science, Vol. 12, pp. 81-95 . 2020.

[3] Ahmad, K., Bal, M., and Aswad, M.," A Short Note on The Solution Of Fermat's Diophantine Equation In Some Neutrosophic Rings", Journal of Neutrosophic and Fuzzy Systems, Vol. 1, 2022.

[4] Ceven, Y., and Tekin, S., " Some Properties of Neutrosophic Integers", Kırklareli University Journal of Engineering and Science, Vol. 6, pp.50-59, 2020.

[5] Sankari, H., and Abobala, M., "Neutrosophic Linear Diophantine Equations With two Variables", Neutrosophic Sets and Systems, Vol. 38, pp. 22-30, 2020.

[6] Ibrahim, M., and Abobala, M., "An Introduction To Refined Neutrosophic Number Theory", Neutrosophic Sets and Systems, Vol. 45, 2021.

[7] Abobala, M., "On Some Algebraic Properties of n-Refined Neutrosophic Elements and n-Refined Neutrosophic Linear Equations", Mathematical Problems in Engineering, Hindawi, 2021.

[8] Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.

[9] Nader Mahmoud Taffach , Ahmed Hatip., "A Review on Symbolic 2-Plithogenic Algebraic Structures " Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.

[10] Nader Mahmoud Taffach , Ahmed Hatip.," A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations ", Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.

[11] M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, "Introduction to Symbolic 2-Plithogenic Probability Theory," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.

[12] Ali, R., and Hasan, Z., "An Introduction To The Symbolic 3-Plithogenic Vector Spaces", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.

[13] Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.

[14] Ben Othman, K., "On Some Algorithms For Solving Symbolic 3-Plithogenic Equations", Neoma Journal Of Mathematics and Computer Science, 2023.

[15] Merkepci, M., Abobala, M., and Allouf, A., " The Applications of Fusion Neutrosophic Number Theory in Public Key Cryptography and the Improvement of RSA Algorithm ", Fusion: Practice and Applications, 2023.

[16] Merkepci, M,; Sarkis, M. An Application of Pythagorean Circles in Cryptography and Some Ideas for Future Non Classical Systems. Galoitica Journal of Mathematical Structures and Applications **2022**.

[17] Merkepci, M., and Abobala, M., " Security Model for Encrypting Uncertain Rational Data Units Based on Refined Neutrosophic Integers Fusion and El Gamal Algorithm ", Fusion: Practice and Applications, 2023.

[18] Alhasan, Y., Alfahal, A., Abdulfatah, R., Ali, R., and Aljibawi, M., " On A Novel Security Algorithm For The Encryption Of 3×3 Fuzzy Matrices With Rational

Entries Based On The Symbolic 2-Plithogenic Integers And El-Gamal Algorithm", International Journal of Neutrosophic Science, 2023.

[19] Abobala, M., and Allouf, A., " On A Novel Security Scheme for The Encryption and Decryption Of 2×2 Fuzzy Matrices with Rational Entries Based on The Algebra of Neutrosophic Integers and El-Gamal Crypto-System", Neutrosophic Sets and Systems, vol.54, 2023.

[20] Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Concept Of Symbolic 7-Plithogenic Real Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

[21] Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Symbolic 8-Plithogenic Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

[22] Nabeeh, N., Alshaimaa, A., and Tantawy, A., "A Neutrosophic Proposed Model For Evaluation Blockchain Technology in Secure Enterprise Distributed Applications", Journal of Cypersecurity and Information Management, 2023.

[23] Ben Othman, K., Albasheer, O., Nadweh, R., Von Shtawzen, O., and Ali, R., "On The Symbolic 6-Plithogenic and 7-Plithogenic Rings", Galoitica Journal of Mathematical Structures and Applications, Vol.8, 2023.

Received 1/5/2023, Accepted 3/9/2023

Neutrosophic Sets and Systems, Vol. 59, 2023
University of New Mexico

On The Algebraic Homomorphisms Between Symbolic 2-plithogenic Rings And 2-cyclic Refined Rings

[1.]Hasan Sankari, [2.] Mohammad Abobala

Tishreen University, Department of Mathematics, Latakia, Syria

Co-mohammadabobala777@gmail.com

Abstract:

The main goal of this research paper is to find an algebraic ring homomorphism between symbolic 2-plithogenic ring and the corresponding 2-cyclic refined ring.

This work presents some applications of the defined homomorphism to explain some algebraic relationships between symbolic 2-plithogenic algebraic structures and 2-cyclic refined structures.

Keywords: 2-cyclic refined ring, symbolic 2-plithogenic ring, symbolic 2-plithogenic matrix, 2-cyclic refined vector spaces.

Introduction and preliminaries

Algebraic homomorphisms play a central role in the classification of rings, where they are considered a very rich material to find the algebraic relationships between different rings.

The symbolic 2-plithogenic rings were defined in [3], they have many interesting properties, since they are a good extension of classical rings, see [1-2, 7-10]. Symbolic 2-plithogenic rings are examples about symbolic n-plithogenic sets and structures founded by Smarandache [4, 11-12].

zenodo.10031149/10.5281

On the other hand, another extension of rings was defined and handled by many authors, where n-cyclic refined rings are neutrosophic structures with an algebraic structure similar to the cyclic ring of integers [5-6].

This work is dedicated to find an algebraic relation by using homomorphisms between symbolic 2-plithogenic rings and 2-cyclic refined rings, where these homomorphisms can be used between the matrices defined over these rings, and vectors defined over them.

Many examples will be presented as a sign of the validity of our work.

For the definitions of algebraic relations between symbolic 2-plithogenic elements see [3]. For the definitions of algebraic relations between n—cyclic refiend elements see [6].

Main discussion

Theorem.

Let $R_2(I)$ be the 2-cyclic refined ring, ideals of the ring R, $2-SP_R$ be the symbolic 2-plithogenic ring refined over the ring R, then there exists a ring homomorphism $f: R_2(I) \to 2-SP_R$.

Proof.

We define $f: R_2(I) \to 2-SP_R$ such that:

$f(V_0+V_1I_1+V_2I_2) = V_0+(V_1+V_2)P_1-2V_1P_2$

f is well defined:

Assume that $V_0+V_1I_1+V_2I_2 = w_0+w_1I_1+w_2I_2$, then $V_i = w_i$ for all $0 \le i \le 2$, thus:

$V_0+(V_1+V_2)P_1-2V_1P_2 = w_0+(w_1+w_2)P_1-2w_1P_2$,

hence $f(V_0+V_1I_1+V_2I_2) = f(w_0+w_1I_1+w_2I_2)$.

f preserves addition:

For $V = V_0+V_1I_1+V_2W = I_2, w_0+w_1I_1+w_2I_2$, we have:

$V + W = (V_0 + w_0) + (V_1 + w_1)I_1 + (V_2 + w_2)I_2$

$f(V + W) = (V_0 + w_0) + (V_1 + w_1 + V_2 + w_2)P_1 - 2(V_1 + w_1)P_2 =$

$[V_0 + (V_1 + V_2)P_1 - 2V_1P_2] + [w_0 + (w_1 + w_2)P_1 - 2w_1P_2] = f(V) + f(W).$

f preserves multiplication:

$V.W = V_0.w_0 + (V_0w_1 + V_1w_0 + V_1w_2 + V_2w_1)I_1 + (V_0w_2 + V_2w_0 + V_2w_2 + V_1w_1)I_2$

$$f(V.W) = V_0.w_0 + (V_0w_1 + V_1w_0 + V_1w_2 + V_2w_1 + V_0w_2 + V_2w_0 + V_2w_2 + V_1w_1)P_1 - 2(V_0w_1 + V_1w_0 + V_1w_2 + V_2w_1)P_2$$

On the other hand, we have:

$f(V).f(W) = [V_0 + (V_1 + V_2)P_1 - 2V_1P_2].[w_0 + (w_1 + w_2)P_1 - 2w_1P_2] = V_0.w_0 + (V_0w_1 + V_1w_0 + V_1w_2 + V_2w_1 + V_0w_2 + V_2w_0 + V_2w_2 + V_1w_1)P_1 - 2(V_0w_1 + V_1w_0 + V_1w_2 + V_2w_1)P_2 = f(V.W).$

So that, f is a ring homomorphism.

Theorem.

Let f be the previous homomorphism defined with $f: R_2(I) \to 2 - SP_R$, then:

1. $ker(f) = \{yI_1 - yI_2; 2y = 0, y \in R\}$
2. $ker(f)$ is a zero ring.

Proof.

1. $ker(f) = \{x + yI_1 + zI_2; f(x + yI_1 + zI_2) = 0\}$, hence $x + (y + z)P_1 - 2yP_2 = 0$,

   thus $x = 0, z = -y, 2y = 0$ which implies that:

$ker(f) = \{yI_1 - yI_2; 2y = 0, y \in R\}$.

2. Let $M = mI_1 - mI_2, N = nI_1 - nI_2 \in ker(f)$, then $2m - 2n = 0$, thus:

$M.N = (mI_1 - mI_2)(nI_1 - nI_2) = mnI_2 - mnI_1 - mnI_1 + mnI_2 = -2mnI_1 + 2mnI_2 = 0 - 0 = 0$, which means that $ker(f)$ is a zero ring.

Theorem.

Let R be a field, then $R_2(I) \cong 2 - SP_R$

Proof.

According to the previous theorem, we have $2y = 0$ implies that $y = 0$, thus $ker(f) = \{O\}$ and f is injective.

To prove that the homomorphism f is surjective, we take an arbitrary element

$V_0 + V_1P_1 + V_2P_2 \in 2 - SP_R$, then there exists

$W = V_0 + \left(\frac{-V_2}{2}\right)I_1 + \left(V_1 + \frac{V_2}{2}\right)I_2 \in R_2(I)$.

Such that:

$f(W) = V_0 + V_1P_1 + V_2P_2$, so that f is an isomorphism.

Applications to Vector Spaces.

By using the previous relationship between symbolic 2-plithogenic ring and 2-cyclic refined rings, we will be able to show the algebraic relations between symbolic 2-plithogenic vector spaces and 2-cyclic refined vector spaces.

Theorem.

Let F be an algebraic field, T be a vector space over F. Assume that $2 - SP_T = \{t_0 + t_1P_1 + t_2P_2; t_i \in T\}$ is the corresponding symbolic 2-plithogenic vector space over $2 - SP_F$.

$T_2(I) = \{t_0 + t_1P_1 + t_2P_2; t_i \in T\}$ is the corresponding 2-cyclic refined vector space over $F_2(I)$, then there exists a semi module homomorphism between $T_2(I)$ and $2 - SP_T$.

Proof.

According to the previous theorems, there exists a ring homomorphism $f: F_2(I) \to 2 - SP_F$ such that

$f(V_0 + V_1I_1 + V_2I_2) = V_0 + (V_1 + V_2)P_1 - 2V_1P_2$

We define $g: T_2(I) \to 2 - SP_T$ such that:

$g(t_0 + t_1I_1 + t_2I_2) = t_0 + (t_1 + t_2)P_1 - 2t_1P_2$

g is well defined:

If $t_0 + t_1I_1 + t_2I_2 = \acute{t}_0 + \acute{t}_1I_1 + \acute{t}_2I_2$, then $t_i = \acute{t}_i; 0 \le i \le 2$ and

$t_0 + (t_1 + t_2)P_1 - 2t_1P_2 = \acute{t}_0 + (\acute{t}_1 + \acute{t}_2)P_1 - 2\acute{t}_1P_2$,

which means that

$g(t_0 + t_1I_1 + t_2I_2) = g(t́_0 + t́_1I_1 + t́_2I_2)$.

g preserves addition:

For $M = t_0 + t_1I_1 + t_2I_2, N = t́_0 + t́_1I_1 + t́_2I_2 \in R_2(I)$.

$M + N = (t_0 + t́_0) + (t_1 + t́_1)I_1 + (t_2 + t́_2)I_2$

$g(M + N) = (t_0 + t́_0) + (t_1 + t́_1 + t_2 + t́_2)P_1 - 2(t_1 + t́_1)P_2 = [t_0 + (t_1 + t_2)P_1 - 2t_1P_2] + [t́_0 + (t́_1 + t́_2)P_1 - 2t́_1P_2] = g(M) + g(N)$.

To complete the proof, we must prove that:

$g(qM) = f(q).g(M); q = q_0 + q_1I_1 + q_2I_2 \in F_2(I)$ and $M = t_0 + t_1I_1 + t_2I_2 \in T_2(I)$.

First, we have:

$qM = q_0.t_0 + (q_0t_1 + q_1t_0 + q_1t_2 + q_2t_1)I_1 + (q_0t_2 + q_2t_0 + q_2t_2 + q_1t_1)I_2$

$g(qM) = q_0.t_0 + (q_0t_1 + q_1t_0 + q_1t_2 + q_2t_1 + q_0t_2 + q_2t_0 + q_2t_2 + q_1t_1)P_1 - 2(q_0t_1 + q_1t_0 + q_1t_2 + q_2t_1)P_2$

$f(q) = q_0 + (q_1 + q_2)P_1 - 2q_1P_2$

$g(M) = t_0 + (t_1 + t_2)P_1 - 2t_1P_2$

$f(q).g(M) = q_0.t_0 + (q_0t_1 + q_1t_0 + q_1t_2 + q_2t_1 + q_0t_2 + q_2t_0 + q_2t_2 + q_1t_1)P_1 - 2(q_0t_1 + q_1t_0 + q_1t_2 + q_2t_1)P_2 = g(qM)$.

This implies that f is a semi-module homomorphism, and $T_2(I)$ is semi homomorphic to $2 - SP_T$.

Remark.

Consider that $H = \{h_0 + h_1I_1 + h_2I_2;\ h_i \in H́, H́\ is\ a\ subspace\ of\ T\}$, then H is a submodule of $T_2(I)$, let us find it direct image according to the semi-homomorphism g.

$Im(H) = g(H) = \{h_0 + (h_1 + h_2)P_1 - 2h_1P_2;\ h_i \in H́\}$.

We prove that $Im(H)$ is a submodule of $2 - SP_T$.

Let $X = x_0 + (x_1 + x_2)P_1 - 2x_1P_2, Y = y_0 + (y_1 + y_2)P_1 - 2y_1P_2 \in Im(H)$, where $x_i, y_i \in H́$

$X + Y = (x_0 + y_0) + (x_1 + x_2 + y_1 + y_2)P_1 - 2(x_1 + y_1)P_2 \in Im(H)$

Let $q = q_0 + q_1I_1 + q_2I_2 \in 2 - SP_F$, then:

$qX = q_0.x_0 + (q_0x_1 + q_0x_2 + q_1x_0 + q_1x_2 + q_1x_1)P_1 + (-2q_0x_1 + q_0x_2 - 2q_1x_1 + q_2x_1 + q_2x_2)P_2$

$\acute{q} = q_0 + \left(\frac{-q_2}{2}\right) I_1 + \left(q_1 + \frac{q_2}{2}\right) I_2 \in F_2(I)$, then:

$f(\acute{q}) = q$, which implies that:

$qX = f(\acute{q})g(\acute{X})$, $\acute{X} = x_0 + \left(\frac{-x_2}{2}\right) I_1 + \left(x_1 + \frac{x_2}{2}\right) I_2$, hence $qX = g(\acute{q}\acute{X}) \in Im(H)$ is a submodule of $2 - SP_T$.

Remark.

Let us find the kernel of the semi-homomorphism g.

$ker(g) = \{M = t_0 + t_1 I_1 + t_2 I_2 \in T_2(I), g(M) = 0\}$, so that:

$$\begin{cases} t_0 = 0 \\ t_1 + t_2 = 0 \Rightarrow t_2 = 0 \\ -2t_1 = 0 \Rightarrow t_1 = 0 \end{cases}$$

Hence, $ker(g) = \{O\}$.

Result:

Let $M = t_0 + t_1 I_1 + t_2 I_2 \in T_2(I)$, assume that $E = \{e_0, \dots, e_k\}$ is a basis of $T_2(I)$ over $F_2(I)$, then

$M = n_0 e_0 + n_1 e_1 + \cdots + n_k e_k$; $n_i \in F_2(I)$.

By taking the direct image of M, we can find $g(M) = g\left(\sum_{i=0}^{k} n_i e_i\right) = \sum_{i=0}^{k} f(n_i) g(e_i)$.

This means that the elements of $Im(T_2(I))$ can be written as a linear combination with respect to the elements of the basis E.

On the other hand, the set $g(E) = \{g(e_0), \dots, g(e_k)\}$ is linearly independent, that is because $\sum_{i=0}^{k} f(n_i) g(e_i) = 0 \Rightarrow \sum_{i=0}^{k} g(n_i e_i) = 0 \Rightarrow g\left(\sum_{i=0}^{k} n_i e_i\right) = 0 \Rightarrow \sum_{i=0}^{k} n_i e_i = 0 \Rightarrow n_i = 0; 0 \leq i \leq k$.

Applications to matrices.

Let $M = (m_{ij})_{k \times k}$ be a square matrix with 2-cyclic refined entries, then $M = M_0 + M_1 I_1 + M_2 I_2$, where M_0, M_1, M_2 are there $k \times k$ classical matrices.

If we take the direct image of M by the ring homomorphism

$f(M) = M_0 + (M_1 + M_2) P_1 - 2 M_1 P_2$, we get a symbolic 2-plithogenic matrix.

For a 2-cyclic refined real number $q = q_0 + q_1 I_1 + q_2 I_2$, we see that qM is a 2-cyclic refined square real matrix.

We can use the semi-homomorphism g to write:

$g(qM) = f(q)g(M); g(M) = M_0 + (M_1 + M_2)P_1 - 2M_1P_2$

Example.

Consider the following 2-cyclic refined real square matrix:

$$M = \begin{pmatrix} 2 + I_1 - I_2 & 1 + 3I_1 + I_2 \\ I_1 - I_2 & I_1 + I_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} I_1 + \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} I_2$$

$$= M_0 + M_1I_1 + M_2I_2$$

The corresponding symbolic 2-plithogeni matrix $g(M)$ is equal to:

$$g(M) = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ 0 & 2 \end{pmatrix} P_1 + \begin{pmatrix} -2 & -6 \\ -2 & -2 \end{pmatrix} P_2 = \begin{pmatrix} 2 - 2P_2 & 1 + 4P_1 - 6P_2 \\ -2P_2 & 2P_1 - 2P_2 \end{pmatrix}$$

Remark.

Let us study the relation between $\det M$ and $det(g(M))$.

$\det M = \det M_0 + I_1[det(M_0 + M_1 + M_2) - det(M_0 - M_1 + M_2)] + \frac{1}{2}I_1[det(M_0 + M_1 + M_2) - det(M_0 - M_1 + M_2) - 2\det M_0 +]$ see [1].

$det(g(M)) = det[M_0 + (M_1 + M_2)P_1 - 2M_1P_2] = \det M_0 + P_1[det(M_0 + M_1 + M_2) - \det M_0] + P_2[det(M_0 - M_1 + M_2) - det(M_0 + M_1 + M_2)]$ see [2].

Consider the ring homomorphism:

$f: R_2(I) \to 2 - SP_R$; $f(a_0 + a_1I_1 + a_2I_2) = a_0 + (a_1 + a_2)P_1 - 2a_1P_2$, then:

$f(detM) = \det M_0 + P_1[det(M_0 + M_1 + M_2) - \det M_0] + P_2[det(M_0 - M_1 + M_2) - det(M_0 + M_1 + M_2)]$

Applications to modules.

If R is a ring, K_R be a module over R.

Let $2 - SP_R$, $R_2(I)$ be the corresponding symbolic 2-plithogenic ring and 2-cyclic refined respectively.

Let $2 - SP_M$ be the corresponding symbolic 2-plithogenic module over $2 - SP_R$ and $M_2(I)$ be the corresponding 2-cyclic refined module over $R_2(I)$, then by a similar discussion of the case of vector spaces, we can write:

1. $g: M_2(I) \to 2 - SP_M$ such that:

$g(m_0 + m_1I_1 + m_2I_2) = m_0 + (m_0 + m_1)P_1 - 2m_1P_2$ is a semi module homomorphism.

2. If S is a submodule of $M_2(I)$, then $Im(S) = g(S)$ is a submodule of $2 - SP_M$.
3. For $q = q_0 + q_1 I_1 + q_2 I_2 \in R_2(I)$ and $m = m_0 + m_1 I_1 + m_2 I_2 \in M_2(I)$, then $g(q) = 0$ does not imply that $m = 0$, because:

$g(qm) = 0 \Rightarrow f(q)g(m) = 0$, since R is not a field, then it may has zero divisors, which means that $f(q) = 0$ without $q = 0$.

This is a big difference between the case of 2-cyclic refined vector spaces and 2-cyclic refined modules.

Conclusion

In this paper, we have found an algebraic homomorphism between 2-cyclic refined rings and symbolic 2-plithogenic rings, and we have used this homomorphism to study some algebraic relations between 2-cyclic refined matrices and symbolic 2-plithogenic matrices.

In the future, we aim to find the algebraic relations between other kinds of neutrosophic rings and symbolic n-plithogenic rings.

References

1. Nader Mahmoud Taffach , Ahmed Hatip., "A Review on Symbolic 2-Plithogenic Algebraic Structures " Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.
2. Nader Mahmoud Taffach , Ahmed Hatip.," A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations ", Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.
3. Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.
4. Smarandache, F., " Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)", Neutrosophic Sets and Systems, vol. 53, 2023.

5. Von Shtawzen, O., " Conjectures For Invertible Diophantine Equations Of 3-Cyclic and 4-Cyclic Refined Integers", Journal Of Neutrosophic And Fuzzy Systems, Vol.3, 2022.

6. Basheer, A., Ahmad, K., and Ali, R., " A Short Contribution To Von Shtawzen's Abelian Group In n-Cyclic Refined Neutrosophic Rings", Journal Of Neutrosophic And Fuzzy Systems,, 2022.
7. Ali, R., and Hasan, Z., " An Introduction To The Symbolic 3-Plithogenic Modules ", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.
8. Ali, R., and Hasan, Z., "An Introduction To The Symbolic 3-Plithogenic Vector Spaces", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.
9. Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.
10. Ben Othman, K., "On Some Algorithms For Solving Symbolic 3-Plithogenic Equations", Neoma Journal Of Mathematics and Computer Science, 2023.
11. Florentin Smarandache, Plithogenic Algebraic Structures. Chapter in "Nidus idearum Scilogs, V: joining the dots" (third version), Pons Publishing Brussels, pp. 123-125, 2019.
12. Florentin Smarandache: Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets – Revisited, *Neutrosophic Sets and Systems*, vol. 21, 2018, pp. 153-166.
13. Ben Othman, K., Albasheer, O., Nadweh, R., Von Shtawzen, O., and Ali, R., "On The Symbolic 6-Plithogenic and 7-Plithogenic Rings", Galoitica Journal of Mathematical Structures and Applications, Vol.8, 2023.

Received 10/4/2023, Accepted 17/9/2023

Neutrosophic Sets and Systems, Vol. 59, 2023
University of New Mexico

An Introduction to The Split-Complex Symbolic 2-Plithogenic Numbers

1.Rama Asad Nadweh, 2.Oliver Von Shtawzen, 3.Ahmad Khaldi

Islamic Online University, Department Of Science and Information Technology, Doha, Qatar

ramaanadwehh@gmail.com

3. University of Nizwa, Sultanate of Oman, Vonshtawzen1970abc@gmail.com

3 Mutah University, Faculty of Science, Jordan, khaldiahmad1221@gmail.com

Abstract:

The objective of this paper is to combine split-complex numbers with symbolic 2-plithogenic numbers in one algebraic structure called split-complex symbolic 2-plithogenic real numbers.

Also, many elementary properties of the suggested system will be handled such as Invertibility and idempotency by many related theorems and examples.

Keywords: Split-complex, symbolic 2-plithogenic number, invertible, idempotent.

Introduction and basic concepts

Split-complex numbers are considered as a generalization of real numbers, where they are defined as follows:

$S = \{a + bJ;\ J^2 = 1, a, b \in R\}$ [1].

Split-complex numbers together make a commutative ring with many interesting properties [2-4, 24].

Addition on S is defined as follows:

$(t_0 + t_1 J) + (k_0 + k_1 J) = (t_0 + k_0) + J(t_1 + k_1)$.

Multiplication on S is defined as follows:

zenodo.10031152/10.5281

$(t_0 + t_1 J) \times (k_0 + k_1 J) = (t_0 k_0 + t_1 k_1) + J(t_0 k_1 + t_1 k_0)$.

In [5], Smarandache presented symbolic n-plithogenic sets, then they were used in generalizing many famous algebraic structures such as ring, matrices, and other structures [6-9,15-18].

We refer to many similar numerical systems that generalize real numbers, such as neutrosophic, refined neutrosophic numbers and weak fuzzy numbers [10-14,19-23].

Through this paper, we use symbolic 2-plithogenic real numbers with split-complex numbers to build a new generalization of real numbers, and we present some of its elementary algebraic properties.

Definition.

The symbolic 2-plithogenic ring of real numbers is defined as follows:

$2 - SP_R = \{t_0 + t_1 P_1 + t_2 P_2; t_i \in R, P_1 \times P_2 = P_2 \times P_1 = P_2, {P_1}^2 = {P_2}^2 = P_2\}$.

The addition operation on $2 - SP_R$ is defined as follows:

$(t_0 + t_1 P_1 + t_2 P_2) + (t'_0 + t'_1 P_1 + t'_2 P_2) = (t_0 + t'_0) + (t_1 + t'_1)P_1 + (t_2 + t'_2)P_2$

The multiplication on $2 - SP_R$ is defined as follows:

$(t_0 + t_1 P_1 + t_2 P_2)(t'_0 + t'_1 P_1 + t'_2 P_2) = t_0 t'_0 + (t_0 t'_1 + t_1 t'_0 + t_1 t'_1)P_1 + (t_0 t'_2 + t_1 t'_2 + t_2 t'_2 + t_2 t'_0 + t_2 t'_1)P_2$.

Main concepts.

Definition.

The set of split-complex symbolic 2-plithogenic numbers is defined as follows:

$2 - SP_S = \{(x_0 + x_1 P_1 + x_2 P_2) + J(y_0 + y_1 P_1 + y_2 P_2); x_i, y_i \in R\ J^2 = 1\}$.

Definition.

Addition on $2 - SP_S$ is defined as follows:

$[(x_0 + x_1 P_1 + x_2 P_2) + J(y_0 + y_1 P_1 + y_2 P_2)] + [(z_0 + z_1 P_1 + z_2 P_2) + J(t_0 + t_1 P_1 + t_2 P_2)] = [(x_0 + z_0) + (x_1 + z_1)P_1 + (x_2 + z_2)P_2] + J[(y_0 + t_0) + (y_1 + t_1)P_1 + (y_2 + t_2)P_2]$.

Multiplication on $2 - SP_S$ is defined as follows

For $X=(x_0+x_1P_1+x_2P_2)+J(\acute{x}_0+\acute{x}_1P_1+\acute{x}_2P_2), Y=(y_0+y_1P_1+y_2P_2)+J(\acute{y}_0+\acute{y}_1P_1+\acute{y}_2P_2)$,

then

$X.Y=(x_0+x_1P_1+x_2P_2)(y_0+y_1P_1+y_2P_2)+(\acute{x}_0+\acute{x}_1P_1+\acute{x}_2P_2)(\acute{y}_0+\acute{y}_1P_1+\acute{y}_2P_2)+J[(x_0+x_1P_1+x_2P_2)(\acute{y}_0+\acute{y}_1P_1+\acute{y}_2P_2)+(\acute{x}_0+\acute{x}_1P_1+\acute{x}_2P_2)(y_0+y_1P_1+y_2P_2)]$.

Example.

Consider $X=(P_1+P_2)+J(1+3P_2), Y=(1-P_2)+J(2-P_1)$, then:

$X+Y=(1+P_1)+J(3-P_1+3P_2)$

$X.Y=(P_1+P_2)(1-P_2)+(1+3P_2)(2-P_1)+J[(P_1+P_2)(2-P_1)+(1+3P_2)(1-P_2)]=P_1-P_2+P_2-P_2+2-P_1+6P_2-3P_2+J[2P_1-P_1+2P_2-P_2+1-P_2+3P_2-3P_2]=(2+2P_2)+J(1+P_1)$.

Remark.

$(2-SP_S,+,.)$ Is a commutative ring.

Invertibility.

Theorem.

Let $X=(m_0+m_1P_1+m_2P_2)+J(n_0+n_1P_1+n_2P_2)\in 2-SP_S$, then X is invertible if and only if:

$$\begin{cases}(m_0-n_0)+(m_1-n_1)P_1+(m_2-n_2)P_2\\(m_0+n_0)+(m_1+n_1)P_1+(m_2+n_2)P_2\end{cases}$$

Are invertible in $2-SP_R$.

On the other hand:

$$X^{-1}=\frac{1}{X}=\frac{m_0}{{m_0}^2-{n_0}^2}+P_1\left[\frac{m_0+m_1}{(m_0+m_1)^2-(n_0+n_1)^2}-\frac{m_0}{{m_0}^2-{n_0}^2}\right]+P_2\left[\frac{m_0+m_1+m_2}{(m_0+m_1+m_2)^2-(n_0+n_1+n_2)^2}-\frac{m_0+m_1}{(m_0+m_1)^2-(n_0+n_1)^2}\right]-J\left[\frac{n_0}{{m_0}^2-{n_0}^2}+P_1\left[\frac{n_0+n_1}{(m_0+m_1)^2-(n_0+n_1)^2}-\frac{n_0}{{m_0}^2-{n_0}^2}\right]+P_2\left[\frac{n_0+n_1+n_2}{(m_0+m_1+m_2)^2-(n_0+n_1+n_2)^2}-\frac{n_0+n_1}{(m_0+m_1)^2-(n_0+n_1)^2}\right]\right]$$

Proof.

Put $X = M + NJ; M = m_0 + m_1P_1 + m_2P_2, N = n_0 + n_1P_1 + n_2P_2$.

According to [2], X is invertible if and only if $M + N, M - N$ are invertible, hence:

$$\begin{cases}(m_0 - n_0) + (m_1 - n_1)P_1 + (m_2 - n_2)P_2 \\ (m_0 + n_0) + (m_1 + n_1)P_1 + (m_2 + n_2)P_2\end{cases}$$

Are invertible in $2 - SP_R$.

Also, $X^{-1} = \frac{1}{X} = \frac{M-NJ}{M^2-N^2} = \frac{M}{M^2-N^2} - J\frac{N}{M^2-N^2}$.

According to [ ], we can write:

$$\frac{M}{M^2 - N^2} = M.\frac{1}{M^2 - N^2}$$
$$= (m_0 + m_1P_1 + m_2P_2)\left[\frac{1}{{m_0}^2 - {n_0}^2}\right.$$
$$+ P_1\left[\frac{1}{(m_0 + m_1)^2 - (n_0 + n_1)^2} - \frac{1}{{m_0}^2 - {n_0}^2}\right]$$
$$\left.+ P_2\left[\frac{1}{(m_0 + m_1 + m_2)^2 - (n_0 + n_1 + n_2)^2} - \frac{1}{(m_0 + m_1)^2 - (n_0 + n_1)^2}\right]\right]$$
$$= \frac{m_0}{{m_0}^2 - {n_0}^2} + P_1\left[\frac{m_0 + m_1}{(m_0 + m_1)^2 - (n_0 + n_1)^2} - \frac{m_0}{{m_0}^2 - {n_0}^2}\right]$$
$$+ P_2\left[\frac{m_0 + m_1 + m_2}{(m_0 + m_1 + m_2)^2 - (n_0 + n_1 + n_2)^2} - \frac{m_0 + m_1}{(m_0 + m_1)^2 - (n_0 + n_1)^2}\right]$$

By a similar argument, we get:

$$\frac{N}{M^2 - N^2} = \frac{n_0}{{m_0}^2 - {n_0}^2} + P_1\left[\frac{n_0 + n_1}{(m_0 + m_1)^2 - (n_0 + n_1)^2} - \frac{n_0}{{m_0}^2 - {n_0}^2}\right]$$
$$+ P_2\left[\frac{n_0 + n_1 + n_2}{(m_0 + m_1 + m_2)^2 - (n_0 + n_1 + n_2)^2} - \frac{n_0 + n_1}{(m_0 + m_1)^2 - (n_0 + n_1)^2}\right]$$

Thus, the proof holds.

Example.

Take $X = (1 + P_1) + J(2 - P_1 + 2P_2) \in 2 - SP_S$, then:

$M = 1 + P_1, N = 2 - P_1 + 2P_2, M + N = 3 + P_2, M - N = -1 + 2P_1 - 2P_2$

$M + N, M - N$ are invertible in $2 - SP_R$.

$$X^{-1} = \frac{1}{-3} + P_1\left(\frac{2}{3} - \frac{1}{-3}\right) + P_2\left(\frac{2}{-5} - \frac{2}{3}\right) - J\left[\frac{2}{-3} + P_1\left(\frac{1}{3} - \frac{1}{-3}\right) + P_2\left(\frac{3}{-5} - \frac{1}{3}\right)\right]$$
$$= \frac{-1}{3} + P_1 + P_2\left(-\frac{16}{15}\right) + J\left(\frac{2}{3} - \frac{2}{3}P_1 + \frac{4}{15}P_2\right)$$

Idempotenty.

Let $X = M + NJ; M = m_0 + m_1P_1 + m_2P_2, N = n_0 + n_1P_1 + n_2P_2 \in 2 - SP_R$.

then X is called idempotent if and only if $X^2 = X$.

First, we compute X^2:

$X^2 = M^2 + N^2 + 2MNJ == {m_0}^2 + {n_0}^2 + P_1[(m_0 + m_1)^2 + (n_0 + n_1)^2 - {m_0}^2 - {n_0}^2] + P_2[(m_0 + m_1 + m_2)^2 + (n_0 + n_1 + n_2)^2 - (m_0 + m_1)^2 - (n_0 + n_1)^2]$.

The equation $X^2 = X$ is equivalent to:

$$\begin{cases} {m_0}^2 + {n_0}^2 = m_0 \quad (1) \\ (m_0 + m_1)^2 + (n_0 + n_1)^2 = m_0 + m_1 \quad (2) \\ (m_0 + m_1 + m_2)^2 + (n_0 + n_1 + n_2)^2 = m_0 + m_1 + m_2 \quad (3) \\ N(2M - 1) = 0 \quad (4) \end{cases}$$

Equation (4) is equivalent to:

$$\begin{cases} n_0(2m_0 - 1) = 0 \quad (5) \\ (n_0 + n_1)(2m_0 + 2m_1 - 1) = 0 \quad (6) \\ (n_0 + n_1 + n_2)(2m_0 + 2m_1 + 2m_2 - 1) = 0 \quad (7) \end{cases}$$

Equation (5) implies that $n_0 = 0$ or $m_0 = \frac{1}{2}$

If $n_0 \neq 0, m_0 \neq \frac{1}{2}$, then form (1), we get $m_1 = 0$ or $m_0 = 1$.

If $n_0 \neq 0, m_0 = \frac{1}{2}$, then form (1), we get $n_0 = \frac{1}{2}$ or $n_0 = -\frac{1}{2}$.

If $n_0 = 0, m_0 = \frac{1}{2}$, we get a contradiction:

Equation (6) implies that $n_0 + n_1 = 0$ or $m_0 + m_1 = \frac{1}{2}$

If $n_0 + n_1 \neq 0, m_0 + m_1 \neq \frac{1}{2}$, then $m_0 + m_1 = 0$ or $m_0 + m_1 = 1$.

If $n_0 + n_1 \neq 0, m_0 + m_1 = \frac{1}{2}$, then $n_0 + n_1 = \frac{1}{2}$ or $n_0 + n_1 = -\frac{1}{2}$.

If $n_0 + n_1 = 0, m_0 + m_1 = \frac{1}{2}$, we get a contradiction:

Equation (7) implies that $n_0 + n_1 + n_2 = 0$ or $m_0 + m_1 + m_2 = \frac{1}{2}$

If $n_0 + n_1 + n_2 = 0, m_0 + m_1 + m_2 \neq \frac{1}{2}$, then $m_0 + m_1 + m_2 = 0$ or $m_0 + m_1 + m_2 = 1$.

If $n_0 + n_1 + n_2 \neq 0, m_0 + m_1 + m_2 = \frac{1}{2}$, then $n_0 + n_1 + n_2 = \frac{1}{2}$ or $n_0 + n_1 + n_2 = -\frac{1}{2}$.

If $n_0 + n_1 + n_2 = 0, m_0 + m_1 + m_2 = \frac{1}{2}$, we get a contradiction:

The possible cases are:

Case 1.

$n_0 = 0, m_0 = 0, n_0 + n_1 = 0, m_0 + m_1 = 0, n_0 + n_1 + n_2 = 0, m_0 + m_1 + m_2 = 0$, then $X = 0$.

Case 2.

$n_0 = 0, m_0 = 0, n_0 + n_1 = 0, m_0 + m_1 = 1, n_0 + n_1 + n_2 = 0, m_0 + m_1 + m_2 = 0$, then $X = P_1 - P_2$.

Case 3.

$n_0 = 0, m_0 = 0, n_0 + n_1 = 0, m_0 + m_1 = 1, n_0 + n_1 + n_2 = 0, m_0 + m_1 + m_2 = 1$, then $X = P_1$.

Case 4.

$n_0 = 0, m_0 = 0, n_0 + n_1 = 0, m_0 + m_1 = 0, n_0 + n_1 + n_2 = 0, m_0 + m_1 + m_2 = 1$, then $X = P_2$.

Case 5.

$n_0 = 0, m_0 = 1, n_0 + n_1 = 0, m_0 + m_1 = 0, n_0 + n_1 + n_2 = 0, m_0 + m_1 + m_2 = 0$, then $X = 1 - P_1$.

Case 6.

$n_0 = 0, m_0 = 1, n_0 + n_1 = 0, m_0 + m_1 = 0, n_0 + n_1 + n_2 = 0, m_0 + m_1 + m_2 = 1$, then $X = 1 - P_1 + P_2$.

Case 7.

$n_0 = 0, m_0 = 1, n_0 + n_1 = 0, m_0 + m_1 = 1, n_0 + n_1 + n_2 = 0, m_0 + m_1 + m_2 = 0$, then $X = 1 - P_2$.

Case 8.

$n_0 = 0, m_0 = 1, n_0 + n_1 = 0, m_0 + m_1 = 1, n_0 + n_1 + n_2 = 0, m_0 + m_1 + m_2 = 1$, then $X = 1$

Case 9.

$n_0 = 0, m_0 = 0, n_0 + n_1 = \frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = 0, m_0 + m_1 + m_2 = 0$, then $X = \left(\frac{1}{2}P_1 - \frac{1}{2}P_2\right) + J\left(\frac{1}{2}P_1 - \frac{1}{2}P_2\right)$.

Case 10.

$n_0=0, m_0=0, n_0+n_1=-\frac{1}{2}, m_0+m_1=\frac{1}{2}, n_0+n_1+n_2=0, m_0+m_1+m_2=0$, then $X=\left(\frac{1}{2}P_1-\frac{1}{2}P_2\right)+J\left(-\frac{1}{2}P_1+\frac{1}{2}P_2\right)$.

Case 11.

$n_0=0, m_0=0, n_0+n_1=\frac{1}{2}, m_0+m_1=\frac{1}{2}, n_0+n_1+n_2=0, m_0+m_1+m_2=1$, then $X=\left(\frac{1}{2}P_1+\frac{1}{2}P_2\right)+J\left(\frac{1}{2}P_1-\frac{1}{2}P_2\right)$.

Case 12.

$n_0=0, m_0=0, n_0+n_1=-\frac{1}{2}, m_0+m_1=\frac{1}{2}, n_0+n_1+n_2=0, m_0+m_1+m_2=1$, then $X=\left(\frac{1}{2}P_1+\frac{1}{2}P_2\right)+J\left(-\frac{1}{2}P_1+\frac{1}{2}P_2\right)$.

Case 13.

$n_0=0, m_0=1, n_0+n_1=\frac{1}{2}, m_0+m_1=\frac{1}{2}, n_0+n_1+n_2=0, m_0+m_1+m_2=0$, then $X=\left(1-\frac{1}{2}P_1-\frac{1}{2}P_2\right)+J\left(\frac{1}{2}P_1-\frac{1}{2}P_2\right)$.

Case 14.

$n_0=0, m_0=1, n_0+n_1=-\frac{1}{2}, m_0+m_1=\frac{1}{2}, n_0+n_1+n_2=0, m_0+m_1+m_2=1$, then $X=\left(1-\frac{1}{2}P_1+\frac{1}{2}P_2\right)+J\left(-\frac{1}{2}P_1+\frac{1}{2}P_2\right)$.

Case 15.

$n_0=0, m_0=1, n_0+n_1=\frac{1}{2}, m_0+m_1=\frac{1}{2}, n_0+n_1+n_2=0, m_0+m_1+m_2=1$, then $X=\left(1-\frac{1}{2}P_1+\frac{1}{2}P_2\right)+J\left(\frac{1}{2}P_1-\frac{1}{2}P_2\right)$.

Case 16.

$n_0=0, m_0=1, n_0+n_1=-\frac{1}{2}, m_0+m_1=\frac{1}{2}, n_0+n_1+n_2=0, m_0+m_1+m_2=0$, then $X=\left(1-\frac{1}{2}P_1-\frac{1}{2}P_2\right)+J\left(-\frac{1}{2}P_1+\frac{1}{2}P_2\right)$.

Case 17.

$n_0=0, m_0=0, n_0+n_1=0, m_0+m_1=0, n_0+n_1+n_2=\frac{1}{2}, m_0+m_1+m_2=\frac{1}{2}$, then $X=\frac{1}{2}P_2+J\left(\frac{1}{2}P_2\right)$.

Case 18.

$n_0 = 0, m_0 = 0, n_0 + n_1 = 0, m_0 + m_1 = 0, n_0 + n_1 + n_2 = -\frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2}$, then $X = \frac{1}{2}P_2 - \frac{1}{2}P_2 J$.

Case 19.

$n_0 = 0, m_0 = 0, n_0 + n_1 = 0, m_0 + m_1 = 1, n_0 + n_1 + n_2 = \frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2}$, then $X = P_1 - \frac{1}{2}P_2 + \frac{1}{2}P_2 J$.

Case 20.

$n_0 = 0, m_0 = 0, n_0 + n_1 = 0, m_0 + m_1 = 1, n_0 + n_1 + n_2 = -\frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2}$, then $X = P_1 - \frac{1}{2}P_2 - \frac{1}{2}P_2 J$.

Case 21.

$n_0 = 0, m_0 = 1, n_0 + n_1 = 0, m_0 + m_1 = 0, n_0 + n_1 + n_2 = \frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2}$, then $X = \left(1 - P_1 + \frac{1}{2}P_2\right) + \frac{1}{2}P_2 J$.

Case 22.

$n_0 = 0, m_0 = 1, n_0 + n_1 = 0, m_0 + m_1 = 0, n_0 + n_1 + n_2 = -\frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2}$, then $X = \left(1 - P_1 + \frac{1}{2}P_2\right) - \frac{1}{2}P_2 J$.

Case 23.

$n_0 = 0, m_0 = 1, n_0 + n_1 = 0, m_0 + m_1 = 1, n_0 + n_1 + n_2 = \frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2}$, then $X = \left(1 - \frac{1}{2}P_2\right) + \frac{1}{2}P_2 J$.

Case 24.

$n_0 = 0, m_0 = 1, n_0 + n_1 = 0, m_0 + m_1 = 1, n_0 + n_1 + n_2 = -\frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2}$, then $X = \left(1 - \frac{1}{2}P_2\right) - \frac{1}{2}P_2 J$.

Case 25.

$n_0 = 0, m_0 = 1, n_0 + n_1 = \frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = \frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2}$, then $X = \left(\frac{1}{2}P_1\right) + \frac{1}{2}P_1 J$.

Case 26.

$n_0 = 0, m_0 = 1, n_0 + n_1 = \frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = -\frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2}$, then $X = \left(\frac{1}{2}P_1\right) + \left(\frac{1}{2}P_1 - P_2\right)J$.

Case 27.

$n_0 = 0, m_0 = 1, n_0 + n_1 = -\frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = \frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2}$, then $X = \left(\frac{1}{2}P_1\right) + \left(-\frac{1}{2}P_1 + P_2\right)J$.

Case 28.

$n_0 = 0, m_0 = 1, n_0 + n_1 = -\frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = -\frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2}$, then $X = \left(\frac{1}{2}P_1\right) + \left(-\frac{1}{2}P_1\right)J$.

Case 29.

$n_0 = 0, m_0 = 1, n_0 + n_1 = \frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = \frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2}$, then $X = \left(1 - \frac{1}{2}P_1\right) + \left(\frac{1}{2}P_1\right)J$.

Case 30.

$n_0 = 0, m_0 = 1, n_0 + n_1 = \frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = -\frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2}$, then $X = \left(1 - \frac{1}{2}P_1\right) + \left(\frac{1}{2}P_1 - P_2\right)J$.

Case 31.

$n_0 = 0, m_0 = 1, n_0 + n_1 = -\frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = \frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2}$, then $X = \left(1 - \frac{1}{2}P_1\right) + \left(-\frac{1}{2}P_1 + P_2\right)J$.

Case 32.

$n_0 = 0, m_0 = 1, n_0 + n_1 = -\frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = -\frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2}$, then $X = \left(1 - \frac{1}{2}P_1\right) + \left(-\frac{1}{2}P_1\right)J$.

Case 33.

$n_0 = \frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = 0, m_0 + m_1 = 0, n_0 + n_1 + n_2 = 0, m_0 + m_1 + m_2 = 0$, then $X = \left(\frac{1}{2} - \frac{1}{2}P_1\right) + \left(\frac{1}{2} - \frac{1}{2}P_1\right)J$.

Case 34.

$n_0=\frac{1}{2}, m_0=\frac{1}{2}, n_0+n_1=0, m_0+m_1=0, n_0+n_1+n_2=0, m_0+m_1+m_2=1$, then

$X=\left(\frac{1}{2}-\frac{1}{2}P_1+P_2\right)+\left(\frac{1}{2}-\frac{1}{2}P_1\right)J$.

Case 35.

$n_0=\frac{1}{2}, m_0=\frac{1}{2}, n_0+n_1=0, m_0+m_1=1, n_0+n_1+n_2=0, m_0+m_1+m_2=0$, then

$X=\left(\frac{1}{2}+\frac{1}{2}P_1-P_2\right)+\left(\frac{1}{2}-\frac{1}{2}P_1\right)J$.

Case 36.

$n_0=\frac{1}{2}, m_0=\frac{1}{2}, n_0+n_1=0, m_0+m_1=1, n_0+n_1+n_2=0, m_0+m_1+m_2=0$, then

$X=\left(\frac{1}{2}+\frac{1}{2}P_1-P_2\right)+\left(\frac{1}{2}-\frac{1}{2}P_1\right)J$.

Case 37.

$n_0=-\frac{1}{2}, m_0=\frac{1}{2}, n_0+n_1=0, m_0+m_1=0, n_0+n_1+n_2=0, m_0+m_1+m_2=0$,

then $X=\left(\frac{1}{2}-\frac{1}{2}P_1\right)+\left(-\frac{1}{2}+\frac{1}{2}P_1\right)J$.

Case 38.

$n_0=-\frac{1}{2}, m_0=\frac{1}{2}, n_0+n_1=0, m_0+m_1=0, n_0+n_1+n_2=0, m_0+m_1+m_2=1$,

then $X=\left(\frac{1}{2}-\frac{1}{2}P_1+P_2\right)+\left(-\frac{1}{2}+\frac{1}{2}P_1\right)J$.

Case 39.

$n_0=-\frac{1}{2}, m_0=\frac{1}{2}, n_0+n_1=0, m_0+m_1=1, n_0+n_1+n_2=0, m_0+m_1+m_2=0$,

then $X=\left(\frac{1}{2}+\frac{1}{2}P_1-P_2\right)+\left(-\frac{1}{2}+\frac{1}{2}P_1\right)J$.

Case 40.

$n_0=-\frac{1}{2}, m_0=\frac{1}{2}, n_0+n_1=0, m_0+m_1=1, n_0+n_1+n_2=0, m_0+m_1+m_2=1$,

then $X=\left(\frac{1}{2}-\frac{1}{2}P_1\right)+\left(-\frac{1}{2}+\frac{1}{2}P_1\right)J$.

Case 41.

$n_0=\frac{1}{2}, m_0=\frac{1}{2}, n_0+n_1=0, m_0+m_1=0, n_0+n_1+n_2=\frac{1}{2}, m_0+m_1+m_2=\frac{1}{2}$, then

$X=\left(\frac{1}{2}-\frac{1}{2}P_1+\frac{1}{2}P_2\right)+\left(\frac{1}{2}-\frac{1}{2}P_1+\frac{1}{2}P_2\right)J$.

Case 42.

$n_0 = \frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = 0, m_0 + m_1 = 0, n_0 + n_1 + n_2 = -\frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2},$

then $X = \left(\frac{1}{2} - \frac{1}{2}P_1 + \frac{1}{2}P_2\right) + \left(\frac{1}{2} - \frac{1}{2}P_1 - \frac{1}{2}P_2\right)J.$

Case 43.

$n_0 = -\frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = 0, m_0 + m_1 = 0, n_0 + n_1 + n_2 = \frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2},$

then $X = \left(\frac{1}{2} - \frac{1}{2}P_1 + \frac{1}{2}P_2\right) + \left(-\frac{1}{2} + \frac{1}{2}P_1 + \frac{1}{2}P_2\right)J.$

Case 44.

$n_0 = -\frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = 0, m_0 + m_1 = 0, n_0 + n_1 + n_2 = -\frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2},$

then $X = \left(\frac{1}{2} - \frac{1}{2}P_1 + \frac{1}{2}P_2\right) + \left(-\frac{1}{2} + \frac{1}{2}P_1 - \frac{1}{2}P_2\right)J.$

Case 45.

$n_0 = \frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = 0, m_0 + m_1 = 0, n_0 + n_1 + n_2 = \frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2}$, then $X = \left(\frac{1}{2} + \frac{1}{2}P_1 - \frac{1}{2}P_2\right) + \left(\frac{1}{2} - \frac{1}{2}P_1 + \frac{1}{2}P_2\right)J.$

Case 46.

$n_0 = \frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = 0, m_0 + m_1 = 1, n_0 + n_1 + n_2 = -\frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2},$

then $X = \left(\frac{1}{2} + \frac{1}{2}P_1 - \frac{1}{2}P_2\right) + \left(\frac{1}{2} - \frac{1}{2}P_1 - \frac{1}{2}P_2\right)J.$

Case 47.

$n_0 = -\frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = 0, m_0 + m_1 = 1, n_0 + n_1 + n_2 = \frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2},$

then $X = \left(\frac{1}{2} + \frac{1}{2}P_1 - \frac{1}{2}P_2\right) + \left(-\frac{1}{2} + \frac{1}{2}P_1 + \frac{1}{2}P_2\right)J.$

Case 48.

$n_0 = -\frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = 0, m_0 + m_1 = 1, n_0 + n_1 + n_2 = -\frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2},$

then $X = \left(\frac{1}{2} + \frac{1}{2}P_1 - \frac{1}{2}P_2\right) + \left(-\frac{1}{2} + \frac{1}{2}P_1 - \frac{1}{2}P_2\right)J.$

Case 49.

$n_0 = \frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = \frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = 0, m_0 + m_1 + m_2 = 0$, then $X = \left(\frac{1}{2} - \frac{1}{2}P_2\right) + \left(\frac{1}{2} - \frac{1}{2}P_2\right)J.$

Case 50.

$n_0 = \frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = \frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = 0, m_0 + m_1 + m_2 = 1$, then $X = \left(\frac{1}{2} + \frac{1}{2}P_2\right) + \left(\frac{1}{2} - \frac{1}{2}P_2\right)J$.

Case 51.

$n_0 = -\frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = \frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = 0, m_0 + m_1 + m_2 = 0$, then $X = \left(\frac{1}{2} - \frac{1}{2}P_2\right) + \left(-\frac{1}{2} + P_1 - \frac{1}{2}P_2\right)J$.

Case 52.

$n_0 = -\frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = \frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = 0, m_0 + m_1 + m_2 = 1$, then $X = \left(\frac{1}{2} + \frac{1}{2}P_2\right) + \left(-\frac{1}{2} + P_1 - \frac{1}{2}P_2\right)J$.

Case 53.

$n_0 = \frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = -\frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = 0, m_0 + m_1 + m_2 = 0$, then $X = \left(\frac{1}{2} - \frac{1}{2}P_2\right) + \left(\frac{1}{2} - P_1 + \frac{1}{2}P_2\right)J$.

Case 54.

$n_0 = \frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = -\frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = 0, m_0 + m_1 + m_2 = 1$, then $X = \left(\frac{1}{2} + \frac{1}{2}P_2\right) + \left(\frac{1}{2} - P_1 + \frac{1}{2}P_2\right)J$.

Case 55.

$n_0 = -\frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = -\frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = 0, m_0 + m_1 + m_2 = 0$, then $X = \left(\frac{1}{2} - \frac{1}{2}P_2\right) + \left(-\frac{1}{2} + \frac{1}{2}P_2\right)J$.

Case 56.

$n_0 = -\frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = -\frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = 0, m_0 + m_1 + m_2 = 1$, then $X = \left(\frac{1}{2} + \frac{1}{2}P_2\right) + \left(-\frac{1}{2} + \frac{1}{2}P_2\right)J$.

Case 57.

$n_0 = \frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = \frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = \frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2}$, then $X = \frac{1}{2} + \frac{1}{2}J$.

Case 58.

$n_0 = \frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = \frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = -\frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2},$

then $X = \frac{1}{2} + \left(\frac{1}{2} - P_2\right)J.$

Case 59.

$n_0 = \frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = -\frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = \frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2},$

then $X = \frac{1}{2} + \left(\frac{1}{2} - P_1 + P_2\right)J.$

Case 60.

$n_0 = \frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = -\frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = -\frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2},$

then $X = \frac{1}{2} + \left(\frac{1}{2} - P_1\right)J.$

Case 61.

$n_0 = -\frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = -\frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = \frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2},$

then $X = \frac{1}{2} + \left(-\frac{1}{2} + P_1\right)J.$

Case 62.

$n_0 = -\frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = \frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = -\frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2},$

then $X = \frac{1}{2} + \left(-\frac{1}{2} + P_1 - P_2\right)J.$

Case 63.

$n_0 = -\frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = -\frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = \frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2},$

then $X = \frac{1}{2} + \left(-\frac{1}{2} + P_2\right)J.$

Case 64.

$n_0 = -\frac{1}{2}, m_0 = \frac{1}{2}, n_0 + n_1 = -\frac{1}{2}, m_0 + m_1 = \frac{1}{2}, n_0 + n_1 + n_2 = -\frac{1}{2}, m_0 + m_1 + m_2 = \frac{1}{2}$, then $X = \frac{1}{2} - \frac{1}{2}J.$

Conclusion

In this paper, we have defined for the first time the ring of split-complex symbolic 2-plithogenic numbers by combining split-complex numbers with symbolic 2-plithogenic numbers.

We have studied many elementary properties of the novel generalized system, where necessary and sufficient conditions for Invertibility and idempotency of symbolic 2-plithogenic split-complex numbers were handled by many related theorems and valid examples.

In the future, we aim to study matrix systems and functional systems generated by the novel algebraic structure.

References

1. Deckelman, S.; Robson, B. Split-Complex Numbers and Dirac Bra-kets. Communications In Information and Systems **2014**, 14, 135-159.
2. Merkepci, M., and Abobala, M., " On Some Novel Results About Split-Complex Numbers, The Diagonalization Problem And Applications To Public Key Asymmetric Cryptography", Journal of Mathematics, Hindawi, 2023.
3. Khaldi, A., " A Study On Split-Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.
4. Ahmad, K., " On Some Split-Complex Diophantine Equations", Neoma Journal Of Mathematics and Computer Science, 2023.
5. Smarandache, F., " Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)", Neutrosophic Sets and Systems, vol. 53, 2023.
6. Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.
7. Nader Mahmoud Taffach , Ahmed Hatip.," A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations ", Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.
8. M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, “Introduction to Symbolic 2-Plithogenic Probability Theory,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.
9. Merkepci, H., and Rawashdeh, A., " On The Symbolic 2-Plithogenic Number Theory and Integers ", Neutrosophic Sets and Systems, Vol 54, 2023.
10. Abobala, M., "On The Characterization of Maximal and Minimal Ideals In Several Neutrosophic Rings", Neutrosophic sets and systems, Vol. 45, 2021.

11. Abobala, M., "A Study Of Nil Ideals and Kothe's Conjecture In Neutrosophic Rings", International Journal of Mathematics and Mathematical Sciences, hindawi, 2021.
12. Abobala, M., "On Some Algebraic Properties of n-Refined Neutrosophic Elements and n-Refined Neutrosophic Linear Equations", Mathematical Problems in Engineering, Hindawi, 2021
13. Sarkis, M., " On The Solutions Of Fermat's Diophantine Equation In 3-refined Neutrosophic Ring of Integers", Neoma Journal of Mathematics and Computer Science, 2023.
14. Abobala, M., On Refined Neutrosophic Matrices and Their Applications In Refined Neutrosophic Algebraic Equations, Journal Of Mathematics, Hindawi, 2021
15. Taffach, N., " An Introduction to Symbolic 2-Plithogenic Vector Spaces Generated from The Fusion of Symbolic Plithogenic Sets and Vector Spaces", Neutrosophic Sets and Systems, Vol 54, 2023.
16. Ben Othman, K., Albasheer, O., Nadweh, R., Von Shtawzen, O., and Ali, R., "On The Symbolic 6-Plithogenic and 7-Plithogenic Rings", Galoitica Journal of Mathematical Structures and Applications, Vol.8, 2023.
17. M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, “Introduction to Symbolic 2-Plithogenic Probability Theory,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.
18. Ali, R., and Hasan, Z., " An Introduction To The Symbolic 3-Plithogenic Modules ", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.
19. Ali, R., " On The Weak Fuzzy Complex Inner Products On Weak Fuzzy Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.

20. Hatip, A., "An Introduction To Weak Fuzzy Complex Numbers ", Galoitica Journal Of Mathematical Structures and Applications, Vol.3, 2023.
21. Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.
22. Ben Othman, K., "On Some Algorithms For Solving Symbolic 3-Plithogenic Equations", Neoma Journal Of Mathematics and Computer Science, 2023.
23. Abobala, M., and Zeina, M.B., " A Study Of Neutrosophic Real Analysis By Using One Dimensional Geometric AH-Isometry", Galoitica Journal Of Mathematical Structures And Applications, Vol.3, 2023.
24. Abobala, M., " A Short Contribution to Split-Complex Linear Diophantine Equations in Two Variables", Galoitica Journal of Mathematical Structures and Applications, Vol.6, 2023.
25. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Concept Of Symbolic 7-Plithogenic Real Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.
26. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Symbolic 8-Plithogenic Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

Received 5/7/2023, Accepted 27/9/2023

An Introduction to The Dual Symbolic 3-Plithogenic And 4-Plithogenic Numbers

[1.]Khadija Ben Othman, [2.]Maretta Sarkis, [3.] Djamal Lhiani

[1..] Umm Al-Qura Univerity, Mekka, Saudi Arabia

[2.]Abu Dhabi University, Abu Dhabi, United Arab Emirates

[3.]University of Blida 1, Department of Mathematics, Algeria
Co-khagijabenothman33@gmail.com

Abstract:

The objective of this paper is to use dual numbers with symbolic 3-plithogenic and 4-plithogenic numbers in one numerical system called dual symbolic 3-plithogenic/4-plithogenic numbers.

Also, the elementary algebraic properties of the suggested systems will be discussed in terms of theorems and related examples that explain the validity of these algebraic number systems.

Keywords: Symbolic 3-plithogenic number, dual number, dual symbolic 3-plithogenic number, Symbolic 4-plithogenic number, dual symbolic 4-plithogenic number.

Introduction and preliminaries.

Dual numbers are considered as a generalization of real numbers, where they are defined as follows:

$D = \{a + bt;\ t^2 = 0, a, b \in R\}$[1]. Dual numbers make together a commutative ring with many interesting properties.

Addition on D is defined as follows:

zenodo.10031156/10.5281

$(a_0 + b_0 t) + (a_1 + b_1 t) = (a_0 + a_1) + (b_0 + b_1)t$

Multiplication on D is defined as follows:

$(a_0 + b_0 t).(a_1 + b_1 t) = (a_0 a_1) + (a_0 b_1 + b_0 a_1)t$

In [2-4], smarandache presented symbolic n-plithogenic sets, then they were used in generalizing many famous algebraic structures such as rings, matrices, and other structures [6-11].

We refer to many similar numerical systems that generalize real number, such as neutrosophic numbers, split-complex number, and weak fuzzy numbers [12-18]. These generalized numbers were applicable in cryptography and matrix theory[19-24].

Through this paper, we use symbolic 3-plithogenic real numbers and symbolic 4-plithogenic real numbers to build a new generalization of real numbers, and we present some of its elementary algebraic properties.

Main concepts.

Definition.

The set of symbolic 3-plithogenic dual numbers is defined as follows:

$3 - SP_D = \{(x_0 + x_1 t) + (y_0 + y_1 t)P_1 + (z_0 + z_1 t)P_2 + (s_0 + s_1 t)P_3; x_i, y_i, z_i, s_i \in R, t^2 = 0\}$.

Definition.

Addition of $3 - SP_D$ is defined:

$[(m_0 + m_1 t) + (k_0 + k_1 t)P_1 + (s_0 + s_1 t)P_2 + (r_0 + r_1 t)P_3] + [(n_0 + n_1 t) + (l_0 + l_1 t)P_1 + (q_0 + q_1 t)P_2 + (g_0 + g_1 t)P_3] = (m_0 + n_0) + (m_1 + n_1)t + [(k_0 + l_0) + (k_1 + l_1)t]P_1 + [(s_0 + q_0) + (s_1 + q_1)t]P_2 + [(r_0 + g_0) + (r_1 + g_1)t]P_3$.

$(3 - SP_D, +)$ is an abelian group.

Remark.

A symbolic 3-plithogenic dual number $X = (x_0 + x_1 t) + (y_0 + y_1 t)P_1 + (z_0 + z_1 t)P_2 + (s_0 + s_1 t)P_3$

can be written:

$X = (x_0 + y_0 P_1 + z_0 P_2 + s_0 P_3) + t(x_1 + y_1 P_1 + z_1 P_2 + s_1 P_3)$.

Definition.

Let

$X = (x_0 + x_1P_1 + x_2P_2 + x_3P_3) + t(\acute{x}_0 + \acute{x}_1P_1 + \acute{x}_2P_2 + \acute{x}_3P_3) = M_1 + M_2t,$

$Y = (y_0 + y_1P_1 + y_2P_2 + y_3P_3) + t(\acute{y}_0 + \acute{y}_1P_1 + \acute{y}_2P_2 + \acute{y}_3P_3) = N_1 + N_2t \in 3 - SP_D,$

then:

Multiplication on $3 - SP_D$ is defined as follows:

$X.Y = M_1N_1 + t(M_1N_2 + N_1M_2)$

Example.

Consider $X = (1 + P_1 + P_2 + P_3) + t(2 - P_3), Y = P_1 + t(1 - P_3)$, we have:

$X + Y = (1 + 2P_1 + P_2 + P_3) + t(3 - 2P_3)$

$X.Y = (1 + P_1 + P_2 + P_3)P_1 + t[(1 + P_1 + P_2 + P_3)(1 - P_3) + (2 - P_3)P_1] =$

$(2P_1 + P_2 + P_3) + t[(1 - P_3 + P_2 - P_3 + P_3 - P_3 + P_1 - P_3) + 2P_1 - P_3] =$

$(2P_1 + P_2 + P_3) + t(1 + P_1 + P_2 - 3P_3).$

Remark.

$(3 - SP_D, +, .)$ Is a commutative ring.

Invertibility:

Theorem.

Let $X = (m_0 + m_1P_1 + m_2P_2 + m_3P_3) + t(n_0 + n_1P_1 + n_2P_2 + n_3P_3) \in 3 - SP_D$, then X is invertible if and only if $m_0 \neq 0, m_0 + m_1 \neq 0, m_0 + m_1 + m_2 \neq 0, m_0 + m_1 + m_2 + m_3 \neq 0$ and:

$$X^{-1} = \frac{1}{X} = \left[\frac{1}{m_0} + \left(\frac{1}{m_0 + m_1} - \frac{1}{m_0}\right)P_1 + \left(\frac{1}{m_0 + m_1 + m_2} - \frac{1}{m_0 + m_1}\right)P_2 + \left(\frac{1}{m_0 + m_1 + m_2 + m_3} - \frac{1}{m_0 + m_1 + m_2}\right)P_3\right] - t\left[\frac{n_0}{(m_0)^2} + \left(\frac{n_0 + n_1}{(m_0 + m_1)^2} - \frac{n_0}{(m_0)^2}\right)P_1 + \left(\frac{n_0 + n_1 + n_2}{(m_0 + m_1 + m_2)^2} - \frac{n_0 + n_1}{(m_0 + m_1)^2}\right)P_2 + \left(\frac{n_0 + n_1 + n_2 + n_3}{(m_0 + m_1 + m_2 + m_3)^2} - \frac{n_0 + n_1 + n_2}{(m_0 + m_1 + m_2)^2}\right)P_3\right]$$

Proof.

X is invertible if and only if $\frac{1}{X}$ is defined as follows:

$$\frac{1}{X}=\frac{1}{(m_0+m_1P_1+m_2P_2+m_3P_3)+t(n_0+n_1P_1+n_2P_2+n_3P_3)}$$
$$=\frac{(m_0+m_1P_1+m_2P_2+m_3P_3)-t(n_0+n_1P_1+n_2P_2+n_3P_3)}{[(m_0+m_1P_1+m_2P_2+m_3P_3)+t(n_0+n_1P_1+n_2P_2+n_3P_3)][(m_0+m_1P_1+m_2P_2+m_3P_3)-t(n_0+n_1P_1+n_2P_2+n_3P_3)]}$$
$$=\frac{(m_0+m_1P_1+m_2P_2+m_3P_3)-t(n_0+n_1P_1+n_2P_2+n_3P_3)}{(m_0+m_1P_1+m_2P_2+m_3P_3)^2}$$

So that $m_0+m_1P_1+m_2P_2+m_3P_3$ is invertible in $3-SP_R$.

This is equivalent to $m_0\neq 0, m_0+m_1\neq 0, m_0+m_1+m_2\neq 0, m_0+m_1+m_2+m_3\neq 0$.

On the other hand, $\frac{1}{X}=\frac{1}{m_0+m_1P_1+m_2P_2+m_3P_3}-t\frac{(n_0+n_1P_1+n_2P_2+n_3P_3)^2}{(m_0+m_1P_1+m_2P_2+m_3P_3)^2}$

Put $Y=\left[\frac{1}{m_0}+\left(\frac{1}{m_0+m_1}-\frac{1}{m_0}\right)P_1+\left(\frac{1}{m_0+m_1+m_2}-\frac{1}{m_0+m_1}\right)P_2+\left(\frac{1}{m_0+m_1+m_2+m_3}-\frac{1}{m_0+m_1+m_2}\right)P_3\right]-t\left[\frac{n_0}{(m_0)^2}+\left(\frac{n_0+n_1}{(m_0+m_1)^2}-\frac{n_0}{(m_0)^2}\right)P_1+\left(\frac{n_0+n_1+n_2}{(m_0+m_1+m_2)^2}-\frac{n_0+n_1}{(m_0+m_1)^2}\right)P_2+\left(\frac{n_0+n_1+n_2+n_3}{(m_0+m_1+m_2+m_3)^2}-\frac{n_0+n_1+n_2}{(m_0+m_1+m_2)^2}\right)P_3\right]$.

Compute the result of XY to get:

$XY=1$

So that, $X^{-1}=\frac{1}{X}=Y$

Example.

Take $X=(1+P_3)+t(2+P_3)\in 3-SP_D$:

$X^{-1}=\frac{1}{1}+\left(\frac{1}{2}-\frac{1}{1}\right)P_2-t\left[\frac{2}{1}+\left(\frac{2}{1}-\frac{2}{1}\right)P_1+\left(\frac{2}{1}-\frac{2}{1}\right)P_2+\left(\frac{3}{4}-\frac{2}{1}\right)P_2\right]=1-\frac{1}{2}P_3-t\left(2-\frac{5}{4}P_3\right)$.

Natural power.

Theorem.

Let $X=(m_0+m_1P_1+m_2P_2+m_3P_3)+t(n_0+n_1P_1+n_2P_2+n_3P_3)\in 3-SP_D$, then:

$X^n=(m_0)^n+((m_0+m_1)^n-(m_0)^n)P_1+((m_0+m_1+m_2)^n-(m_0+m_1)^n)P_2+((m_0+m_1+m_2+m_3)^n-(m_0+m_1+m_2)^n)P_3+n(n_0+n_1P_1+n_2P_2+n_3P_3)[(m_0)^{n-1}+((m_0+m_1)^{n-1}-(m_0)^{n-1})P_1+((m_0+m_1+m_2)^{n-1}-(m_0+m_1)^{n-1})P_2((m_0+m_1+m_2+m_3)^{n-1}-(m_0+m_1+m_2)^{n-1})P_3]$ for $n\in N$.

Proof.

Let $X=A+Bt; A,B\in 3-SP_D$, then:

$A^n = A^n + nA^{n-1}Bt$, we get:

$A^n = (m_0)^n + ((m_0 + m_1)^n - (m_0)^n)P_1 + ((m_0 + m_1 + m_2)^n - (m_0 + m_1)^n)P_2 + ((m_0 + m_1 + m_2 + m_3)^n - (m_0 + m_1 + m_2)^n)P_3$, then the proof holds.

Example.

Take $X = (1 + P_3) + t(2 - P_3) \in 3 - SP_D$

$X^3 = 1 + (1 - 1)P_1 + (1 - 1)P_2 + (8 - 1)P_3 + 3t(2 - P_3)[1 + (1 - 1)P_1 + (1 - 1)P_2 + (4 - 1)P_3] = 1 + 7P_3 + 3t[(2 - P_3)(1 + 3P_3)] = 1 + 7P_3 + t(6 + 6P_3)$.

Idempotency.

Definition.

Let $X \in 3 - SP_D$, then X is called idempotent if and only if $X^2 = X$.

Theorem.

Let $X = (m_0 + m_1P_1 + m_2P_2 + m_3P_3) + t(n_0 + n_1P_1 + n_2P_2 + n_3P_3) \in 3 - SP_D$, then X is called idempotent if and only if:

1. $m_0 + m_1P_1 + m_2P_2 + m_3P_3$ is idempotent.
2. $(n_0 + n_1P_1 + n_2P_2 + n_3P_3)[2m_0 - 1 + 2m_1P_1 + 2m_2P_2 + 2m_3P_3] = 0$

Proof.

$X = M + Nt$ is idempotent if and only if:

$$X^2 = X \Longrightarrow \begin{cases} M^2 = M \\ 2MN = N \Longrightarrow N(2M - 1) = 0 \end{cases}$$

For $M = m_0 + m_1P_1 + m_2P_2 + m_3P_3, N = n_0 + n_1P_1 + n_2P_2 + n_3P_3 \in 3 - SP_R$.

This implies the proof.

Definition.

The set of symbolic 4-plithogenic dual numbers is defined as follows:

$4 - SP_D = \{(x_0 + x_1t) + (y_0 + y_1t)P_1 + (z_0 + z_1t)P_2 + (s_0 + s_1t)P_3 + (l_0 + l_1t)P_4; x_i, y_i, z_i, s_i, l_i \in R, t^2 = 0\}$.

Definition.

Addition of $4 - SP_D$ is defined:

$[(m_0 + m_1t) + (k_0 + k_1t)P_1 + (s_0 + s_1t)P_2 + (r_0 + r_1t)P_3 + (d_0 + d_1t)P_4] + [(n_0 + n_1t) + (l_0 + l_1t)P_1 + (q_0 + q_1t)P_2 + (g_0 + g_1t)P_3 + (c_0 + c_1t)P_4] =$

$(m_0+n_0)+(m_1+n_1)t+[(k_0+l_0)+(k_1+l_1)t]P_1+[(s_0+q_0)+(s_1+q_1)t]P_2+[(r_0+g_0)+(r_1+g_1)t]P_3+[(d_0+c_0)+(d_1+c_1)t]P_4$.

$(4-SP_D,+)$ is an abelian group.

Remark.

A symbolic 4-plithogenic dual number $X=(x_0+x_1t)+(y_0+y_1t)P_1+(z_0+z_1t)P_2+(s_0+s_1t)P_3+(d_0+d_1t)P_4$

can be written:

$X=(x_0+y_0P_1+z_0P_2+s_0P_3+d_0P_4)+t(x_1+y_1P_1+z_1P_2+s_1P_3+d_1P_4)$.

Definition.

Let

$X=(x_0+x_1P_1+x_2P_2+x_3P_3+x_4P_4)+t(\acute{x}_0+\acute{x}_1P_1+\acute{x}_2P_2+\acute{x}_3P_3+\acute{x}_4P_4)=M_1+M_2t$,

$Y=(y_0+y_1P_1+y_2P_2+y_3P_3+y_4P_4)+t(\acute{y}_0+\acute{y}_1P_1+\acute{y}_2P_2+\acute{y}_3P_3+\acute{y}_4P_4)=N_1+N_2t\in 4-SP_D$,

then:

Multiplication on $4-SP_D$ is defined as follows:

$X.Y=M_1N_1+t(M_1N_2+N_1M_2)$

Example.

Consider $X=(1+P_4)+t(2-P_3), Y=P_1+t(1-P_4)$, we have:

$X+Y=(1+P_1+P_4)+t(3-P_3-P_4)$

$X.Y=(1+P_4)P_1+t[(1+P_4)(1-P_4)+(2-P_3)P_1]=(P_1+P_4)+t[(1-P_4)+2P_1-P_3]=(P_1+P_4)+t(1+2P_1-P_3-P_4)$.

Remark.

$(4-SP_D,+,.)$ Is a commutative ring.

Invertibility:

Theorem.

Let

$X=(m_0+m_1P_1+m_2P_2+m_3P_3+m_4P_4)+t(n_0+n_1P_1+n_2P_2+n_3P_3+n_4P_4)\in 4-SP_D$, then X is invertible if and only if $m_0 \neq 0, m_0 + m_1 \neq 0, m_0 + m_1 + m_2 \neq 0, m_0 + m_1 + m_2 + m_3 \neq 0, m_0 + m_1 + m_2 + m_3 + m_4 \neq 0$ and:

$$X^{-1} = \frac{1}{X} = \left[\frac{1}{m_0} + \left(\frac{1}{m_0+m_1} - \frac{1}{m_0}\right)P_1 + \left(\frac{1}{m_0+m_1+m_2} - \frac{1}{m_0+m_1}\right)P_2 + \left(\frac{1}{m_0+m_1+m_2+m_3} - \frac{1}{m_0+m_1+m_2}\right)P_3 + \left(\frac{1}{m_0+m_1+m_2+m_3+m_4} - \frac{1}{m_0+m_1+m_2+m_3}\right)P_4\right] - t\left[\frac{n_0}{(m_0)^2} + \left(\frac{n_0+n_1}{(m_0+m_1)^2} - \frac{n_0}{(m_0)^2}\right)P_1 + \left(\frac{n_0+n_1+n_2}{(m_0+m_1+m_2)^2} - \frac{n_0+n_1}{(m_0+m_1)^2}\right)P_2 + \left(\frac{n_0+n_1+n_2+n_3}{(m_0+m_1+m_2+m_3)^2} - \frac{n_0+n_1+n_2}{(m_0+m_1+m_2)^2}\right)P_3 + \left(\frac{n_0+n_1+n_2+n_3+n_4}{(m_0+m_1+m_2+m_3+m_4)^2} - \frac{n_0+n_1+n_2+n_3}{(m_0+m_1+m_2+m_3)^2}\right)P_4\right].$$

Proof.

X is invertible if and only if $\frac{1}{X}$ is defined as follows:

$$\frac{1}{X} = \frac{1}{(m_0+m_1P_1+m_2P_2+m_3P_3+m_4P_4)+t(n_0+n_1P_1+n_2P_2+n_3P_3+n_4P_4)} = \frac{(m_0 + m_1P_1 + m_2P_2 + m_3P_3 + m_4P_4) - t(n_0 + n_1P_1 + n_2P_2 + n_3P_3 + n_4P_4)}{[(m_0 + m_1P_1 + m_2P_2 + m_3P_3 + m_4P_4) + t(n_0 + n_1P_1 + n_2P_2 + n_3P_3 + n_4P_4)][(m_0 + m_1P_1 + m_2P_2 + m_3P_3 + m_4P_4) - t(n_0 + n_1P_1 + n_2P_2 + n_3P_3 + n_4P_4)]} = \frac{(m_0+m_1P_1+m_2P_2+m_3P_3+m_4P_4)-t(n_0+n_1P_1+n_2P_2+n_3P_3+n_4P_4)}{(m_0+m_1P_1+m_2P_2+m_3P_3+m_4P_4)^2},$$

So that $m_0 + m_1P_1 + m_2P_2 + m_3P_3 + m_4P_4$ is invertible in $4 - SP_R$.

This is equivalent to $m_0 \neq 0, m_0 + m_1 \neq 0, m_0 + m_1 + m_2 \neq 0, m_0 + m_1 + m_2 + m_3 \neq 0, m_0 + m_1 + m_2 + m_3 + m_4 \neq 0$.

On the other hand,

$$\frac{1}{X} = \frac{1}{m_0+m_1P_1+m_2P_2+m_3P_3+m_4P_4} - t\,\frac{(n_0+n_1P_1+n_2P_2+n_3P_3+n_4P_4)^2}{(m_0+m_1P_1+m_2P_2+m_3P_3+m_4P_4)^2}$$

Put

$$Y = \left[\frac{1}{m_0} + \left(\frac{1}{m_0+m_1} - \frac{1}{m_0}\right)P_1 + \left(\frac{1}{m_0+m_1+m_2} - \frac{1}{m_0+m_1}\right)P_2 + \left(\frac{1}{m_0+m_1+m_2+m_3} - \frac{1}{m_0+m_1+m_2}\right)P_3 + \left(\frac{1}{m_0+m_1+m_2+m_3+m_4} - \frac{1}{m_0+m_1+m_2+m_3}\right)P_4\right] - t\left[\frac{n_0}{(m_0)^2} + \left(\frac{n_0+n_1}{(m_0+m_1)^2} - \frac{n_0}{(m_0)^2}\right)P_1 + \left(\frac{n_0+n_1+n_2}{(m_0+m_1+m_2)^2} - \frac{n_0+n_1}{(m_0+m_1)^2}\right)P_2 + \left(\frac{n_0+n_1+n_2+n_3}{(m_0+m_1+m_2+m_3)^2} - \frac{n_0+n_1+n_2}{(m_0+m_1+m_2)^2}\right)P_3 + \left(\frac{n_0+n_1+n_2+n_3+n_4}{(m_0+m_1+m_2+m_3+m_4)^2} - \frac{n_0+n_1+n_2+n_3}{(m_0+m_1+m_2+m_3)^2}\right)P_4\right].$$

Compute the result of XY to get:

$XY = 1$

So that, $X^{-1} = \frac{1}{X} = Y$

Natural power.

Theorem.

Let

$X = (m_0 + m_1P_1 + m_2P_2 + m_3P_3 + m_4P_4) + t(n_0 + n_1P_1 + n_2P_2 + n_3P_3 + n_4P_4) \in 4 - SP_D$,

then:

$X^n = (m_0)^n + ((m_0 + m_1)^n - (m_0)^n)P_1 + ((m_0 + m_1 + m_2)^n - (m_0 + m_1)^n)P_2 + ((m_0 + m_1 + m_2 + m_3)^n - (m_0 + m_1 + m_2)^n)P_3 + ((m_0 + m_1 + m_2 + m_3 + m_4)^n - (m_0 + m_1 + m_2 + m_3)^n)P_4 + n(n_0 + n_1P_1 + n_2P_2 + n_3P_3 + n_4P_4)[(m_0)^{n-1} + ((m_0 + m_1)^{n-1} - (m_0)^{n-1})P_1 + ((m_0 + m_1 + m_2)^{n-1} - (m_0 + m_1)^{n-1})P_2((m_0 + m_1 + m_2 + m_3)^{n-1} - (m_0 + m_1 + m_2)^{n-1})P_3 + ((m_0 + m_1 + m_2 + m_3 + m_4)^{n-1} - (m_0 + m_1 + m_2 + m_3)^{n-1})P_4]$ for $n \in N$.

Proof.

Let $X = A + Bt; A, B \in 4 - SP_D$, then:

$A^n = A^n + nA^{n-1}Bt$, we get:

$A^n = (m_0)^n + ((m_0 + m_1)^n - (m_0)^n)P_1 + ((m_0 + m_1 + m_2)^n - (m_0 + m_1)^n)P_2 + ((m_0 + m_1 + m_2 + m_3)^n - (m_0 + m_1 + m_2)^n)P_3 + ((m_0 + m_1 + m_2 + m_3 + m_4)^n - (m_0 + m_1 + m_2 + m_3)^n)P_4$, then the proof holds.

Idempotency.

Definition.

Let $X \in 4 - SP_D$, then X is called idempotent if and only if $X^2 = X$.

Theorem.

Let $X = (m_0 + m_1P_1 + m_2P_2 + m_3P_3 + m_4P_4) + t(n_0 + n_1P_1 + n_2P_2 + n_3P_3 + m_4P_4) \in 4 - SP_D$, then X is called idempotent if and only if:

1. $m_0 + m_1P_1 + m_2P_2 + m_3P_3 + m_4P_4$ is idempotent.
2. $(n_0 + n_1P_1 + n_2P_2 + n_3P_3 + n_4P_4)[2m_0 - 1 + 2m_1P_1 + 2m_2P_2 + 2m_3P_3 + 2m_4P_4] = 0$

Proof.

$X = M + Nt$ is idempotent if and only if:

$$X^2 = X \Longrightarrow \begin{cases} M^2 = M \\ 2MN = N \Longrightarrow N(2M - 1) = 0 \end{cases}$$

For $M = m_0 + m_1P_1 + m_2P_2 + m_3P_3 + m_4P_4, N = n_0 + n_1P_1 + n_2P_2 + n_3P_3 + n_4P_4 \in 4 - SP_R$.

This implies the proof.

Conclusion

In this paper, we have studied for the first time the combination of symbolic 3-plithogenic numbers and 4-plithogenic numbers with dual numbers. The novel algebraic structures generated by them are called dual symbolic 3-plithogenic numbers and dual symbolic 4-plithogenic numbers.

We have determined the invertibility condition and the formula of the inverse for dual symbolic 3-plithogenic and 4-plithogenic numbers.

References

1. Akar, M.; Yuce, S.; Sahin, S. On The Dual Hyperbolic Numbers and The Complex Hyperbolic Numbers. Jcscm **2018,** 8, DOI: 10.20967/jcscm.2018.01.001.
2. Smarandache, F., " Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)", Neutrosophic Sets and Systems, vol. 53, 2023.
3. Florentin Smarandache, Plithogenic Algebraic Structures. Chapter in "Nidus idearum Scilogs, V: joining the dots" (third version), Pons Publishing Brussels, pp. 123-125, 2019.
4. Florentin Smarandache: Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets – Revisited, *Neutrosophic Sets and Systems*, vol. 21, 2018, pp. 153-166.
5. Ali, R., and Hasan, Z., "An Introduction To The Symbolic 3-Plithogenic Vector Spaces", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.
6. Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.

7. Ben Othman, K., "On Some Algorithms For Solving Symbolic 3-Plithogenic Equations", Neoma Journal Of Mathematics and Computer Science, 2023.
8. Ali, R., and Hasan, Z., " An Introduction To The Symbolic 3-Plithogenic Modules ", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.
9. M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, “Introduction to Symbolic 2-Plithogenic Probability Theory,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.
10. Nader Mahmoud Taffach , Ahmed Hatip.," A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations ", Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.
11. Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.
12. Sarkis, M., " On The Solutions Of Fermat's Diophantine Equation In 3-refined Neutrosophic Ring of Integers", Neoma Journal of Mathematics and Computer Science, 2023.
13. Abobala, M., "On Some Algebraic Properties of n-Refined Neutrosophic Elements and n-Refined Neutrosophic Linear Equations", Mathematical Problems in Engineering, Hindawi, 2021
14. Abobala, M., "A Study Of Nil Ideals and Kothe's Conjecture In Neutrosophic Rings", International Journal of Mathematics and Mathematical Sciences, hindawi, 2021
15. Khaldi, A**.,** " A Study On Split-Complex Vector Spaces"**,** Neoma Journal Of Mathematics and Computer Science, 2023.
16. Ahmad, K., " On Some Split-Complex Diophantine Equations", Neoma Journal Of Mathematics and Computer Science, 2023.

17. Ali, R., " On The Weak Fuzzy Complex Inner Products On Weak Fuzzy Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.

18. Hatip, A., "An Introduction To Weak Fuzzy Complex Numbers ", Galoitica Journal Of Mathematical Structures and Applications, Vol.3, 2023.

19. Merkepci, M., Abobala, M., and Allouf, A., " The Applications of Fusion Neutrosophic Number Theory in Public Key Cryptography and the Improvement of RSA Algorithm ", Fusion: Practice and Applications, 2023.

20. Merkepci, M,; Sarkis, M. An Application of Pythagorean Circles in Cryptography and Some Ideas for Future Non Classical Systems. Galoitica Journal of Mathematical Structures and Applications **2022**.

21. Merkepci, M., and Abobala, M., " Security Model for Encrypting Uncertain Rational Data Units Based on Refined Neutrosophic Integers Fusion and El Gamal Algorithm ", Fusion: Practice and Applications, 2023.

22. Alhasan, Y., Alfahal, A., Abdulfatah, R., Ali, R., and Aljibawi, M., " On A Novel Security Algorithm For The Encryption Of 3×3 Fuzzy Matrices With Rational Entries Based On The Symbolic 2-Plithogenic Integers And El-Gamal Algorithm", International Journal of Neutrosophic Science, 2023.

23. Alfahal, A., Alhasan, Y., Abdulfatah, R., Mehmood, A., and Kadhim, T., " On Symbolic 2-Plithogenic Real Matrices And Their Algebraic Properties ", International Journal of Neutrosophic Science, 2023.

24. Abobala, M., and Allouf, A., " On A Novel Security Scheme for The Encryption and Decryption Of 2×2 Fuzzy Matrices with Rational Entries Based on The Algebra of Neutrosophic Integers and El-Gamal Crypto-System", Neutrosophic Sets and Systems, vol.54, 2023.

25. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Concept Of Symbolic 7-Plithogenic Real Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.
26. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Symbolic 8-Plithogenic Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.
27. Ben Othman, K., Albasheer, O., Nadweh, R., Von Shtawzen, O., and Ali, R., "On The Symbolic 6-Plithogenic and 7-Plithogenic Rings", Galoitica Journal of Mathematical Structures and Applications, Vol.8, 2023.

Received 18/6/2023, Accepted 10/10/2023

Neutrosophic Sets and Systems, Vol. 59, 2023
University of New Mexico

Foundation of 2-Symbolic Plithogenic Maximum a Posteriori Estimation

Nizar Altounji[1], Moustafa Mazhar Ranneh[2] and Mohamed Bisher Zeina[3]

[1] Faculty of Science, Dept. of Mathematical Statistics, University of Aleppo, Aleppo, Syria; e-mail: nizar.altounji.94@hotmail.com
[2] Faculty of Science, Dept. of Mathematical Statistics, University of Aleppo, Aleppo, Syria; e-mail: mazhar.ranneh@gmail.com
[3] Faculty of Science, Dept. of Mathematical Statistics, University of Aleppo, Aleppo, Syria; e-mail: bisher.zeina@gmail.com

Abstract: In this paper, we introduce and define for the first time the plithogenic loss function, plithogenic risk function, plithogenic maximum likelihood function, and plithogenic posterior risk function, which form a base to easily define the plithogenic maximum a posteriori estimator (Plithogenic MAP) and its conditions, algebraic isomorphism was used through equations, and finally, we worked on an example of a plithogenic random variables sample exponentially distributed with a gamma prior for the parameter distribution and used the quadratic loss function, we found the posterior distribution of the parameter which is also a plithogenic gamma distribution and taken the posterior mean as an estimate of the parameter, such results are similar to the classical case of MAP taking in consideration the plithogenic parts which represent generalized indeterminacy.

Keywords: Plithogenic; Loss Function; Risk Function; Plithogenic Probability Density Function; Maximum Likelihood Function; Posterior Risk Function; Maximum a Posteriori Estimator.

1. Introduction

As Uncertainty and Ambiguity are more observed in real life applications, traditional probability theory methods of studying such applications became less

zenodo.10031161/10.5281

effective with capturing all information needed, which led us to define and work with a new extension of probability theory that deals with the indeterminacy that we face in life applications to better understand its complexity, this new extension helps in many real life fields such as psychology, economics, mathematics, data analysis, artificial intelligence, etc.

Neutrosophic probability theory was first introduced in 1995, which deals with the probability as a triplet values, which represents the degree of truth, false, and indeterminacy, F. Smarandache presented neutrosophic sets and its applications in [1]–[7], M. Abobala and A. Hatip built the concept of Euclidean neutrosophic geometry which opens the world of many mathematical concepts such as real analysis and probability theory represented by using an algebraic structure depending on the indeterminacy element I that satisfies $I^2 = I$ [8]–[20].

Plithogenic probability theory is also an extension of classical probability theory that studies the indeterminacy related to the occurrence or non-occurrence of an event, it is a more generalized than neutrosophic since it deals with indeterminacy as two parts, F. Smarandache et al also presented symbolic plithogenic algebraic structures and plithogenic probability and statistics. Also, N. M. Taffach and A. Hatip gave a review on symbolic 2-plithogenic algebraic structures, as there are lots of papers related to plithogenic probability in many fields [21]–[37].

This paper deals with symbolic plithogenic numbers that take the form $a_P = a_0 + a_1P_1 + a_2P_2$; ${P_1}^2 = P_1$, ${P_2}^2 = P_2$, $P_1.P_2 = P_2.P_1 = P_2$, we will introduce important classical definitions in terms of symbolic plithogenic and focus on the definition of the plithogenic maximum a posteriori estimator (Plithogenic MAP), which can be considered as a generalization of our work in [38].

2. Preliminaries:

Definition 2.1

Let $R(P_1, P_2) = \{a + bP_1 + cP_2; a, b, c \in R, {P_1}^2 = P_1,\ {P_2}^2 = P_2,\ P_1.P_2 = P_2.P_1 = P_2\}$ be the plithogenic field of reals. The one-dimensional AH-Isometry and its inverse are defined as following:

$$T: R(P_1, P_2) \longrightarrow R^3\ ; T(a + bP_1 + cP_2) = \ (a, a + b, a + b + c)$$
$$T^{-1}: R^3 \longrightarrow R(P_1, P_2)\ ; T^{-1}(a, b, c) = \ a + (b - a)P_1 + (c - b)P_2$$

Definition 2.2

Let $f: R(P_1, P_2) \longrightarrow R(P_1, P_2); f = f(x_P), x_P \in R(P_1, P_2),\ f$ is called a plithogenic real function with one plithogenic variable.

Definition 2.3

Plithogenic random variable X_P is defined as follows:

$$X_P: \Omega_P \longrightarrow R(P_1, P_2);\ \Omega_P = \Omega_0 \times \Omega_1(P_1) \times \Omega_2(P_2)$$
$$X_P = X_0 + X_1P_1 + X_2P_2;\ {P_1}^2 = P_1, {P_2}^2 = P_2, P_1.P_2 = P_2.P_1 = P_2$$

Where X_0, X_1, X_2 are classical random variables defined on $\Omega_0, \Omega_1, \Omega_2$ respectively.

Definition 2.4

Let $a_P = a_0 + a_1P_1 + a_2P_2,\ b_P = b_0 + b_1P_1 + b_2P_2 \in R(P_1, P_2)$, we say that $a_P \geq_P b_P$ if:

$$a_0 \geq b_0, a_0 + a_1 \geq b_0 + b_1, a_0 + a_1 + a_2 \geq b_0 + b_1 + b_2$$

3. Plithogenic Density, Plithogenic Conditional Density, Plithogenic Conditional Expectation, Plithogenic Loss and Risk Functions:

Theorem 3.1

Let X_P be a plithogenic random variable that has a probability density function $f(x_P; \Theta_P)$ with $\Theta_P = (\theta_{1P}, \dots, \theta_{kP}); \theta_{iP} = \theta_{i0} + \theta_{i1}P_1 + \theta_{i2}P_2\ ; i = 1,2, \dots, k$ a vector of parameters, then $f(x_P; \Theta_P)$ is written in its formal plithogenic form as the following:

$$f(x_P;\Theta_P) = f(x_0;\Theta_0) + [f(x_0+x_1;\Theta_0+\Theta_1)\text{-}f(x_0;\Theta_0)]\ P_1\text{+}[f(x_0+x_1+x_2;\Theta_0+\Theta_1+\Theta_2)\text{-}\ f(x_0+x_1;\Theta_0+\Theta_1)]P_2 \quad (1)$$

Where: $\Theta_0 = (\theta_{10},\dots,\theta_{k0})$, $\Theta_0+\Theta_1 = (\theta_{10}+\theta_{11},\dots,\theta_{k0}+\theta_{k1})$, $\Theta_0+\Theta_1+\Theta_2 = (\theta_{10}+\theta_{11}+\theta_{12},\dots,\theta_{k0}+\theta_{k1}+\theta_{k2})$

Proof:

See [37].

Theorem 3.2

Let X_P, Y_P be two plithogenic random variables, and let $f(x_P|y_P;\Theta_P)$ be the conditional probability density function of X_P given Y_P with $\Theta_P = (\theta_{1P},\dots,\theta_{kP}); \theta_{iP} = \theta_{i0}+\theta_{i1}P_1+\theta_{i2}P_2\ ; i = 1,2,\dots,k$ a vector of parameters, then:

$$\begin{aligned} f(x_P|y_P;\Theta_P) = f(x_0|y_0;\Theta_0) &+ [f(x_0+x_1|y_0+y_1;\Theta_0+\Theta_1) - f(x_0|y_0;\Theta_0)]P_1 \\ &+ [f(x_0+x_1+x_2|y_0+y_1+y_2;\Theta_0+\Theta_1+\Theta_2) \\ &- f(x_0+y_1|x_0+y_1;\Theta_0+\Theta_1)\ ]P_2 \end{aligned} \quad (2)$$

Proof:

Straightforward using theorem 3.1.

Theorem 3.3

Let X_P, Y_P be two plithogenic random variables, and let $f(x_P|y_P)$ be the conditional probability density function of X_P given Y_P , then plithogenic conditional expectation is:

$$\begin{aligned} E(X_P|Y_P) = E(X_0|Y_0) &+ [E(X_0+X_1|Y_0+Y_1) - E(X_0|Y_0)]P_1 \\ &+ [E(X_0+X_1+X_2|Y_0+Y_1+Y_2) - E(X_0+X_1|Y_0+Y_1)\ ]P_2 \end{aligned} \quad (3)$$

Proof:

$$E(X_P|Y_P) = \int x_P f(x_P|y_P).\,dx_P$$

$$= \int (x_0 + x_1P_1 + x_2P_2) f(x_0 + x_1P_1 + x_2P_2|y_0 + y_1P_1 + y_2P_2).\,d(x_0 + x_1P_1 + x_2P_2)$$

Let's take the one-dimensional AH-Isometry:

$$T(E(X_P|Y_P)) = T\left(\int (x_0 + x_1P_1 + x_2P_2) f(x_0 + x_1P_1 + x_2P_2|y_0 + y_1P_1 + y_2P_2).\,d(x_0 + x_1P_1 + x_2P_2)\right)$$

$$= (\int x_0 f(x_0|y_0).\,dx_0\,, \int (x_0 + x_1) f(x_0 + x_1|y_0 + y_1).\,d(x_0 + x_1)\,, \int (x_0 + x_1 + x_2) f(x_0 + x_1 + x_2|y_0 + y_1 + y_2).\,d(x_0 + x_1 + x_2))$$

$$= (E(X_0|Y_0), E(X_0 + X_1|Y_0 + Y_1), E(X_0 + X_1 + X_2|Y_0 + Y_1 + Y_2))$$

Now we take T^{-1}:

$$\Longrightarrow E(X_P|Y_P) = E(X_0|Y_0) + [E(X_0 + X_1|Y_0 + Y_1) - E(X_0|Y_0)]P_1 + [E(X_0 + X_1 + X_2|Y_0 + Y_1 + Y_2) - E(X_0 + X_1|Y_0 + Y_1)\,]P_2$$

Theorem 3.4

Let $\theta_P = \theta_0 + \theta_1P_1 + \theta_2P_2$ be a plithogenic parameter of a probability distribution, and let $\hat{\theta}_P = \hat{\theta}_0 + \hat{\theta}_1P_1 + \hat{\theta}_2P_2$ be an estimation of θ_P, we can prove that the loss function of θ_P is:

$$Loss(\theta_P, \hat{\theta}_P) = Loss(\theta_0, \hat{\theta}_0) + \left[Loss(\theta_0 + \theta_1, \hat{\theta}_0 + \hat{\theta}_1) - Loss(\theta_0, \hat{\theta}_0)\right]P_1 + [Loss(\theta_0 + \theta_1 + \theta_2, \hat{\theta}_0 + \hat{\theta}_1 + \hat{\theta}_2) - Loss(\theta_0 + \theta_1, \hat{\theta}_0 + \hat{\theta}_1)]P_2 \quad (4)$$

Proof

Straightforward.

Remark

Classical loss could take the form: $Loss(a,\hat{a}) = |a - \hat{a}|$ or $Loss(a,\hat{a}) = (a - \hat{a})^2$, or other loss functions.

Theorem 3.5

Let $\theta_P = \theta_0 + \theta_1 P_1 + \theta_2 P_2$ be a plithogenic parameter of a probability distribution, and let $\hat{\theta}_P = \hat{\theta}_0 + \hat{\theta}_1 P_1 + \hat{\theta}_2 P_2$ be an estimation of θ_P, the risk function of θ_P is:

$$R(\theta_P,\hat{\theta}_P) = E(Loss(\theta_P,\hat{\theta}_P))$$

$$= R(\theta_0,\hat{\theta}_0) + \left[R(\theta_0+\theta_1,\hat{\theta}_0+\hat{\theta}_1) - R(\theta_0,\hat{\theta}_0)\right]P_1 + [R(\theta_0+\theta_1+\theta_2,\hat{\theta}_0+\hat{\theta}_1 + \hat{\theta}_2) - R(\theta_0+\theta_1,\hat{\theta}_0+\hat{\theta}_1)]P_2 \quad (5)$$

Proof

Straightforward.

Theorem 3.6

Let $\theta_P = \theta_0 + \theta_1 P_1 + \theta_2 P_2$ be a plithogenic parameter of a probability distribution, and let $\hat{\theta}_P = \hat{\theta}_0 + \hat{\theta}_1 P_1 + \hat{\theta}_2 P_2$ be an estimation of θ_P, the posterior risk function of θ_P given X_P is:

$$R(\theta_P,\hat{\theta}_P|X_P) = E(Loss(\theta_P,\hat{\theta}_P)|X_P)$$

$$= R(\theta_0,\hat{\theta}_0|X_0) + \left[R(\theta_0+\theta_1,\hat{\theta}_0+\hat{\theta}_1|X_0+X_1) - R(\theta_0,\hat{\theta}_0|X_0)\right]P_1 + [R(\theta_0 + \theta_1+\theta_2,\hat{\theta}_0+\hat{\theta}_1+\hat{\theta}_2|X_0+X_1+X_2) - R(\theta_0+\theta_1,\hat{\theta}_0 + \hat{\theta}_1|X_0+X_1)]P_2 \quad (6)$$

Proof

Straightforward.

Theorem 3.7

Let $\mathbb{X}_P = (X_{1P},\dots,X_{nP})$ be a sample of independent and identically distributed plithogenic random variables and $\Theta_P = (\theta_{1P},\dots,\theta_{kP}); \theta_{iP} = \theta_{i0} + \theta_{i1}P_1 + \theta_{i2}P_2\,; i = 1,2,\dots,k$ a vector of parameters. The plithogenic maximum likelihood function is given by:

$$L_P(\Theta_P) = f(\mathbb{x}_P;\Theta_P) = \prod_{i=1}^{n} f(x_{iP};\Theta_P)$$
$$L_P(\Theta_P) = L(\Theta_0) + [L(\Theta_0+\Theta_1) - L(\Theta_0)]P_1 + [L(\Theta_0+\Theta_1+\Theta_2) - L(\Theta_0+\Theta_1)]P_2 \tag{7}$$

Where:

$$L(\Theta_0) = f(\mathbb{x}_0;\Theta_0)$$
$$L(\Theta_0+\Theta_1) = f(\mathbb{x}_0+\mathbb{x}_1;\Theta_0+\Theta_1)$$
$$L(\Theta_0+\Theta_1+\Theta_2) = f(\mathbb{x}_0+\mathbb{x}_1+\mathbb{x}_2;\Theta_0+\Theta_1+\Theta_2)$$

And:

$$\Theta_0 = (\theta_{10},\dots,\theta_{k0}),\ \Theta_0 +\ \Theta_1 = (\theta_{10}+\theta_{11},\dots,\theta_{k0}+\theta_{k1}),\ \Theta_0 +\ \Theta_1+\Theta_2 = (\theta_{10}+\theta_{11}+\theta_{12},\dots,\theta_{k0}+\theta_{k1}+\theta_{k2})$$
$$\mathbb{X}_0 = (\mathbb{X}_{10},\dots,\mathbb{X}_{n0}),\ \mathbb{X}_0 +\ \mathbb{X}_1 = (\mathbb{X}_{10}+\mathbb{X}_{11},\dots,\mathbb{X}_{n0}+\mathbb{X}_{n1}),\ \mathbb{X}_0 +\ \mathbb{X}_1+\mathbb{X}_2 = (\mathbb{X}_{10}+\mathbb{X}_{11}+\mathbb{X}_{12},\dots,\mathbb{X}_{n0}+\mathbb{X}_{n1}+\mathbb{X}_{n2})$$

Proof

Taking one-dim AH-Isometry:

$$T(L_P(\Theta_P)) = T\left(\prod_{i=1}^{n} f(x_{i0}+x_{i1}P_1+x_{i2}P_2;\Theta_0+\Theta_1P_1+\Theta_2P_2)\right)$$
$$= \prod_{i=1}^{n} f((x_{i0};\Theta_0, x_{i0}+x_{i1};\Theta_0+\Theta_1, x_{i0}+x_{i1}+x_{i2};\Theta_0+\Theta_1+\Theta_2))$$
$$= (\prod_{i=1}^{n} f(x_{i0};\Theta_0), \prod_{i=1}^{n} f(x_{i0}+x_{i1};\Theta_0+\Theta_1), \prod_{i=1}^{n} f(x_{i0}+x_{i1}+x_{i2};\Theta_0+\Theta_1+\Theta_2))$$
$$= (L(\Theta_0), L(\Theta_0+\Theta_1), L(\Theta_0+\Theta_1+\Theta_2))$$

Again, taking T^{-1} yields:

$$L_P(\Theta_P) = L(\Theta_0) + [L(\Theta_0+\Theta_1) - L(\Theta_0)]P_1 + [L(\Theta_0+\Theta_1+\Theta_2) - L(\Theta_0+\Theta_1)]P_2$$

4. Plithogenic Maximum a Posteriori Estimation:

Theorem 4.1

Let $\mathbb{X}_P = (X_{1P}, \dots, X_{nP})$ be a sample of independent and identically distributed plithogenic random variables and $\Theta_P = (\theta_{1P}, \dots, \theta_{kP}); \theta_{iP} = \theta_{i0} + \theta_{i1}P_1 + \theta_{i2}P_2 \,; i = 1,2, \dots, k$ a vector of parameters, suppose Θ_P is a random variable follows a distribution that has a probability density function of $g(\Theta_P)$, which we call it a prior distribution, and suppose $L_P(\Theta_P)$ is the plithogenic maximum likelihood function of the sample, hence the posterior density function of Θ_P is given as follows:

$$f(\Theta_P|\mathbb{x}_P) \sim L(\Theta_0).\, g(\Theta_0) + [L(\Theta_0 + \Theta_1).\, g(\Theta_0 + \Theta_1) - L(\Theta_0).\, g(\Theta_0)]P_1 + [L(\Theta_0 + \Theta_1 + \Theta_2).\, g(\Theta_0 + \Theta_1 + \Theta_2) - L(\Theta_0 + \Theta_1).\, g(\Theta_0 + \Theta_1)]P_2 \quad (9)$$

Proof

We write $f(\Theta_P|\mathbb{x}_P)$ by using equation (2) as the following:

$$f(\Theta_P|\mathbb{x}_P) = f(\Theta_0|\mathbb{x}_0) + [f(\Theta_0 + \Theta_1|\mathbb{x}_0 + \mathbb{x}_1) - f(\Theta_0|\mathbb{x}_0)]P_1 + [f(\Theta_0 + \Theta_1 + \Theta_2|\mathbb{x}_0 + \mathbb{x}_1 + \mathbb{x}_2) - f(\Theta_0 + \Theta_1|\mathbb{x}_0 + \mathbb{x}_1)\,]P_2$$

$$= \frac{f(\mathbb{x}_0|\Theta_0)g(\Theta_0)}{f(\mathbb{x}_0)} + \left[\frac{f(\mathbb{x}_0 + \mathbb{x}_1|\Theta_0 + \Theta_1)g(\Theta_0 + \Theta_1)}{f(\mathbb{x}_0 + \mathbb{x}_1)} - \frac{f(\mathbb{x}_0|\Theta_0)g(\Theta_0)}{f(\mathbb{x}_0)}\right]P_1 + \left[\frac{f(\mathbb{x}_0 + \mathbb{x}_1 + \mathbb{x}_2|\Theta_0 + \Theta_1 + \Theta_2)g(\Theta_0 + \Theta_1 + \Theta_2)}{f(\mathbb{x}_0 + \mathbb{x}_1 + \mathbb{x}_2)} - \frac{f(\mathbb{x}_0 + \mathbb{x}_1|\Theta_0 + \Theta_1)g(\Theta_0 + \Theta_1)}{f(\mathbb{x}_0 + \mathbb{x}_1)}\right]P_2$$

By taking one-dim AH-Isometry and excluding the denominators due to they don't affect the final shape of the distribution then retaking the inverse isometry we get:

$$f(\Theta_P|\mathbb{x}_P) \sim f(\mathbb{x}_0|\Theta_0)g(\Theta_0) + [f(\mathbb{x}_0 + \mathbb{x}_1|\Theta_0 + \Theta_1)g(\Theta_0 + \Theta_1) - f(\mathbb{x}_0|\Theta_0)g(\Theta_0)]P_1 + [f(\mathbb{x}_0 + \mathbb{x}_1 + \mathbb{x}_2|\Theta_0 + \Theta_1 + \Theta_2)g(\Theta_0 + \Theta_1 + \Theta_2) - f(\mathbb{x}_0 + \mathbb{x}_1|\Theta_0 + \Theta_1)g(\Theta_0 + \Theta_1)\,]P_2$$

$$\Rightarrow f(\Theta_P|\mathbb{x}_P)\sim L(\Theta_0).g(\Theta_0)+[L(\Theta_0+\Theta_1).g(\Theta_0+\Theta_1)-L(\Theta_0).g(\Theta_0)]P_1$$
$$+[L(\Theta_0+\Theta_1+\Theta_2).g(\Theta_0+\Theta_1+\Theta_2)-L(\Theta_0+\Theta_1).g(\Theta_0+\Theta_1)]P_2$$

Theorem 4.2

If $\widehat{\Theta}_P$ is the posterior mean of Θ_P for a plithogenic quadratic loss function, or $\widehat{\Theta}_P$ the posterior median for a plithogenic absolute loss function, then $\widehat{\Theta}_P$ is the estimator that minimizes the plithogenic posterior risk function.

Proof

The minimization of $R(\Theta_P,\widehat{\Theta}_P|\mathbb{X}_P)$ occurs when $\frac{d}{\mathrm{d}\Theta_P}R(\Theta_P,\widehat{\Theta}_P|\mathbb{X}_P)=0$, and by the equation (6) we see that this happens when:

$$\frac{d}{\mathrm{d}\Theta_0}R(\Theta_0,\widehat{\Theta}_0|\mathbb{X}_0)=0$$
$$\frac{d}{\mathrm{d}(\Theta_0+\Theta_1)}(\Theta_0+\Theta_1,\widehat{\Theta}_0+\widehat{\Theta}_1|\mathbb{X}_0+\mathbb{X}_1)=0$$
$$\frac{d}{\mathrm{d}(\Theta_0+\Theta_1+\Theta_2)}(\Theta_0+\Theta_1+\Theta_2,\widehat{\Theta}_0+\widehat{\Theta}_1+\widehat{\Theta}_2|\mathbb{X}_0+\mathbb{X}_1+\mathbb{X}_2)=0$$

We deal with these three conditions as we do in the classical case, i.e., for a quadratic loss function, $\widehat{\Theta}_P$ must equal:

$$\widehat{\Theta}_P=E(\Theta_P|\mathbb{X}_P)\Leftrightarrow\begin{cases}\widehat{\Theta}_0=E(\Theta_0|\mathbb{X}_0)\\ \widehat{\Theta}_0+\widehat{\Theta}_1=E(\Theta_0+\Theta_1|\mathbb{X}_0+\mathbb{X}_1)\\ \widehat{\Theta}_0+\widehat{\Theta}_1+\widehat{\Theta}_2=E(\Theta_0+\Theta_1+\Theta_2|\mathbb{X}_0+\mathbb{X}_1+\mathbb{X}_2)\end{cases}\quad(10)$$

Which is the posterior mean.

And for the absolute loss function, we take the posterior median.

Example

Let $X_{1P},\dots,X_{nP}\sim Exp(\theta_P)$, and let θ_P be an unknown plithogenic random variable that we want to estimate which is plithogenically gamma distributed with two known parameters r_P,λ_P ; $r_P=r_0+r_1P_1+r_2P_2,\lambda_P=\lambda_0+\lambda_1P_1+\lambda_2P_2$, then:

$$g(\theta_P)=\frac{{\lambda_P}^{r_P}}{(r_P-1)!}{\theta_P}^{r_P-1}e^{-\lambda_P\theta_P}$$

We write $g(\theta_P)$ using equation (1) as following:

$$\begin{aligned}g(\theta_P)=&\frac{{\lambda_0}^{r_0}}{(r_0-1)!}{\theta_0}^{r_0-1}e^{-\lambda_0\theta_0}\\&+\left[\frac{(\lambda_0+\lambda_1)^{r_0+r_1}}{(r_0+r_1-1)!}(\theta_0+\theta_1)^{r_0+r_1-1}e^{-(\lambda_0+\lambda_1)(\theta_0+\theta_1)}\right.\\&\left.-\frac{{\lambda_0}^{r_0}}{(r_0-1)!}{\theta_0}^{r_0-1}e^{-\lambda_0\theta_0}\right]P_1\\&+[\frac{(\lambda_0+\lambda_1+\lambda_2)^{r_0+r_1+r_2}}{(r_0+r_1+r_2-1)!}(\theta_0+\theta_1+\theta_2)^{r_0+r_1+r_2-1}e^{-(\lambda_0+\lambda_1+\lambda_2)(\theta_0+\theta_1+\theta_2)}\\&-\frac{(\lambda_0+\lambda_1)^{r_0+r_1}}{(r_0+r_1-1)!}(\theta_0+\theta_1)^{r_0+r_1-1}e^{-(\lambda_0+\lambda_1)(\theta_0+\theta_1)}]P_2\end{aligned}$$

Also:

$$L(\theta_P)=\prod_{i=1}^{n}\theta_P e^{-\theta_P\, x_{iP}}$$

Which can be written by (7) as:

$$\begin{aligned}L(\theta_P)=&\prod_{i=1}^{n}\theta_0 e^{-\theta_0\, x_{i0}}+\left[\prod_{i=1}^{n}(\theta_0+\theta_1)e^{-(\theta_0+\theta_1)(\,x_{i0}+x_{i1})}-\prod_{i=1}^{n}\theta_0 e^{-\theta_0\, x_{i0}}\right]P_1\\&+[\prod_{i=1}^{n}(\theta_0+\theta_1+\theta_2)e^{-(\theta_0+\theta_1+\theta_2)(\,x_{i0}+x_{i1}+x_{i2})}\\&-\prod_{i=1}^{n}(\theta_0+\theta_1)e^{-(\theta_0+\theta_1)(\,x_{i0}+x_{i1})}]P_2\end{aligned}$$

Hence by (9):

$$f(\theta_P|\mathbb{x}_P)\sim{\theta_0}^{n}e^{-\theta_0\sum x_{i0}}\frac{{\lambda_0}^{r_0}}{(r_0-1)!}{\theta_0}^{r_0-1}e^{-\lambda_0\theta_0}+$$

$$[(\theta_0+\theta_1)^n e^{-(\theta_0+\theta_1)\sum(x_{i0}+x_{i1})}\frac{(\lambda_0+\lambda_1)^{r_0+r_1}}{(r_0+r_1-1)!}(\theta_0+\theta_1)^{r_0+r_1-1}e^{-(\lambda_0+\lambda_1)(\theta_0+\theta_1)}$$
$$-\theta_0{}^n e^{-\theta_0\sum x_{i0}}\frac{\lambda_0{}^{r_0}}{(r_0-1)!}\theta_0{}^{r_0-1}e^{-\lambda_0\theta_0}]P_1+$$
$$[(\theta_0+\theta_1+\theta_2)^n e^{-(\theta_0+\theta_1+\theta_2)\sum(x_{i0}+x_{i1}+x_{i2})}\frac{(\lambda_0+\lambda_1+\lambda_2)^{r_0+r_1+r_2}}{(r_0+r_1+r_2-1)!}(\theta_0+\theta_1$$
$$+\theta_2)^{r_0+r_1+r_2-1}e^{-(\lambda_0+\lambda_1+\lambda_2)(\theta_0+\theta_1+\theta_2)}$$
$$-(\theta_0+\theta_1)^n e^{-(\theta_0+\theta_1)\sum(x_{i0}+x_{i1})}\frac{(\lambda_0+\lambda_1)^{r_0+r_1}}{(r_0+r_1-1)!}(\theta_0+\theta_1)^{r_0+r_1-1}e^{-(\lambda_0+\lambda_1)(\theta_0+\theta_1)}]P_2$$

$$\Rightarrow=\frac{\lambda_0{}^{r_0}}{(r_0-1)!}\theta_0{}^{n+r_0-1}e^{-\theta_0(\lambda_0+\sum x_{i0})}$$
$$+[\frac{(\lambda_0+\lambda_1)^{r_0+r_1}}{(r_0+r_1-1)!}(\theta_0+\theta_1)^{n+r_0+r_1-1}e^{-(\theta_0+\theta_1)(\lambda_0+\lambda_1+\sum(x_{i0}+x_{i1}))}$$
$$-\frac{\lambda_0{}^{r_0}}{(r_0-1)!}\theta_0{}^{n+r_0-1}e^{-\theta_0(\lambda_0+\sum x_{i0})}]P_1$$
$$+[\frac{(\lambda_0+\lambda_1+\lambda_2)^{r_0+r_1+r_2}}{(r_0+r_1+r_2-1)!}(\theta_0+\theta_1$$
$$+\theta_2)^{n+r_0+r_1+r_2-1}e^{-(\theta_0+\theta_1+\theta_2)(\lambda_0+\lambda_1+\lambda_2+\sum(x_{i0}+x_{i1}+x_{i2}))}$$
$$-\frac{(\lambda_0+\lambda_1)^{r_0+r_1}}{(r_0+r_1-1)!}(\theta_0+\theta_1)^{n+r_0+r_1-1}e^{-(\theta_0+\theta_1)(\lambda_0+\lambda_1+\sum(x_{i0}+x_{i1}))}]P_2$$

Excluding constants after taking suitable isomorphism and then taking its inverse yields to:

$$\Rightarrow f(\theta_P|\mathbb{x}_P)\sim\theta_0{}^{n+r_0-1}e^{-\theta_0(\lambda_0+\sum x_{i0})}$$
$$+\left[(\theta_0+\theta_1)^{n+r_0+r_1-1}e^{-(\theta_0+\theta_1)(\lambda_0+\lambda_1+\sum(x_{i0}+x_{i1}))}\right.$$
$$\left.-\theta_0{}^{n+r_0-1}e^{-\theta_0(\lambda_0+\sum x_{i0})}\right]P_1$$
$$+[(\theta_0+\theta_1+\theta_2)^{n+r_0+r_1+r_2-1}e^{-(\theta_0+\theta_1+\theta_2)(\lambda_0+\lambda_1+\lambda_2+\sum(x_{i0}+x_{i1}+x_{i2}))}$$
$$-(\theta_0+\theta_1)^{n+r_0+r_1-1}e^{-(\theta_0+\theta_1)(\lambda_0+\lambda_1+\sum(x_{i0}+x_{i1}))}]P_2$$

Which yields that $\theta_P\sim Gamma(n+r_P,\lambda_P+\sum x_{iP})$

If we use the plithogenic quadratic loss function, then:

$$\widehat{\Theta}_P = \frac{n + r_P}{\lambda_P + \sum x_{iP}}$$

$$T(\widehat{\Theta}_P) = (\widehat{\Theta}_0, \widehat{\Theta}_0 + \widehat{\Theta}_1, \widehat{\Theta}_0 + \widehat{\Theta}_1 + \widehat{\Theta}_2) = T\left(\frac{n + r_P}{\lambda_P + \sum x_{iP}}\right) = \frac{T(n + r_P)}{T(\lambda_P + \sum x_{iP})}$$

$$= \frac{(n + r_0, n + r_0 + r_1, n + r_0 + r_1 + r_2)}{(\lambda_0 + \sum x_{i0}\,, \lambda_0 + \lambda_1 + \sum(x_{i0} + x_{i1})\,, \lambda_0 + \lambda_1 + \lambda_2 + \sum(x_{i0} + x_{i1} + x_{i2}))}$$

$$\Rightarrow \begin{cases} \widehat{\Theta}_0 = \dfrac{n + r_0}{\lambda_0 + \sum x_{i0}} \\ \widehat{\Theta}_0 + \widehat{\Theta}_1 = \dfrac{n + r_0 + r_1}{\lambda_0 + \lambda_1 + \sum(x_{i0} + x_{i1})} \\ \widehat{\Theta}_0 + \widehat{\Theta}_1 + \widehat{\Theta}_2 = \dfrac{n + r_0 + r_1 + r_2}{\lambda_0 + \lambda_1 + \lambda_2 + \sum(x_{i0} + x_{i1} + x_{i2})} \end{cases}$$

$$\Rightarrow \widehat{\Theta}_P = \frac{n + r_0}{\lambda_0 + \sum x_{i0}} + \left[\frac{n + r_0 + r_1}{\lambda_0 + \lambda_1 + \sum(x_{i0} + x_{i1})} - \frac{n + r_0}{\lambda_0 + \sum x_{i0}}\right] P_1 + \left[\frac{n + r_0 + r_1 + r_2}{\lambda_0 + \lambda_1 + \lambda_2 + \sum(x_{i0} + x_{i1} + x_{i2})} - \frac{n + r_0 + r_1}{\lambda_0 + \lambda_1 + \sum(x_{i0} + x_{i1})}\right] P_2$$

5. Conclusions and future research directions:

We found the formal definitions of the plithogenic maximum a posteriori estimator, posterior density function, and posterior risk function, which we used to find the estimation of parameter that has a gamma prior with an exponentially distributed plithogenic random variables, the results were similar to the classical case but takes into consideration the plithogeny, we also defined the plithogenic maximum likelihood function and other definitions related to our work, future researches will focus on studying more cases of conjugate priors and non-informative priors, also on finding formal definitions that deal with neutrosophic sample and plithogenic prior or vice versa.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

Acknowledgement: Authors are thankful for the reviewers because of their important comments and edits that powered the work.

References

[1] F. Smarandache, "Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability," *ArXiv,* 2013.

[2] F. Smarandache, "Symbolic Neutrosophic Theory," *ArXiv,* 2015, doi: 10.5281/ZENODO.32078.

[3] F. Smarandache, "Introduction to Neutrosophic Statistics," *Branch Mathematics and Statistics Faculty and Staff Publications,* Jan. 2014, Accessed: Feb. 21, 2023. [Online]. Available: https://digitalrepository.unm.edu/math_fsp/33

[4] F. Smarandache *et al.*, "Introduction to neutrosophy and neutrosophic environment," *Neutrosophic Set in Medical Image Analysis,* pp. 3–29, 2019, doi: 10.1016/B978-0-12-818148-5.00001-1.

[5] F. Smarandache, "Neutrosophic theory and its applications," *Brussels,* vol. I, 2014.

[6] F. Smarandache, "Indeterminacy in neutrosophic theories and their applications," *International Journal of Neutrosophic Science,* vol. 15, no. 2, 2021, doi: 10.5281/zenodo.5295819.

[7] F. Smarandache, "(T, I, F)-Neutrosophic Structures," *Applied Mechanics and Materials*, vol. 811, 2015, doi: 10.4028/www.scientific.net/amm.811.104.

[8] M. B. Zeina and A. Hatip, "Neutrosophic Random Variables," *Neutrosophic Sets and Systems*, vol. 39, 2021, doi: 10.5281/zenodo.4444987.

[9] M. Abobala, M. B. Ziena, R. I. Doewes, and Z. Hussein, "The Representation of Refined Neutrosophic Matrices By Refined Neutrosophic Linear Functions," *International Journal of Neutrosophic Science,* vol. 19, no. 1, 2022, doi: 10.54216/IJNS.190131.

[10] M. Abobala, “On the Representation of Neutrosophic Matrices by Neutrosophic Linear Transformations,” *Journal of Mathematics*, vol. 2021, 2021, doi: 10.1155/2021/5591576.

[11] M. B. Zeina and Y. Karmouta, “Introduction to Neutrosophic Stochastic Processes,” *Neutrosophic Sets and Systems*, vol. 54, 2023.

[12] C. Granados and J. Sanabria, “On Independence Neutrosophic Random Variables,” *Neutrosophic Sets and Systems*, vol. 47, 2021.

[13] M. Bisher Zeina and M. Abobala, “On The Refined Neutrosophic Real Analysis Based on Refined Neutrosophic Algebraic AH-Isometry,” *Neutrosophic Sets and Systems*, vol. 54, 2023.

[14] C. Granados, “New Notions On Neutrosophic Random Variables,” *Neutrosophic Sets and Systems*, vol. 47, 2021.

[15] A. Astambli, M. B. Zeina, and Y. Karmouta, “Algebraic Approach to Neutrosophic Confidence Intervals,” *Journal of Neutrosophic and Fuzzy Systems*, vol. 5, no. 2, pp. 08–22, 2023, doi: 10.54216/JNFS.050201.

[16] M. Abobala and A. Hatip, “An Algebraic Approach to Neutrosophic Euclidean Geometry,” *Neutrosophic Sets and Systems*, vol. 43, 2021.

[17] M. B. Zeina, M. Abobala, A. Hatip, S. Broumi, and S. Jalal Mosa, “Algebraic Approach to Literal Neutrosophic Kumaraswamy Probability Distribution,” *Neutrosophic Sets and Systems*, vol. 54, pp. 124–138, 2023.

[18] A. Astambli, M. B. Zeina, and Y. Karmouta, “On Some Estimation Methods of Neutrosophic Continuous Probability Distributions Using One-Dimensional AH-Isometry,” *Neutrosophic Sets and Systems*, vol. 53, 2023.

[19] M. B. Zeina and M. Abobala, “A novel approach of neutrosophic continuous probability distributions using AH-isometry with applications in medicine,” *Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics*, pp. 267–286, Jan. 2023, doi: 10.1016/B978-0-323-99456-9.00014-3.

[20] M. Abobala and M. B. Zeina, “A Study of Neutrosophic Real Analysis by Using the One-Dimensional Geometric AH-Isometry,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. 3, no. 1, pp. 18–24, 2023, doi: 10.54216/GJMSA.030103).

[21] P. K. Singh, “Complex Plithogenic Set,” *International Journal of Neutrosophic Science*, vol. 18, no. 1, 2022, doi: 10.54216/IJNS.180106.

[22] S. Alkhazaleh, “Plithogenic Soft Set,” *Neutrosophic Sets and Systems*, 2020.

[23] N. Martin and F. Smarandache, “Introduction to Combined Plithogenic Hypersoft Sets,” *Neutrosophic Sets and Systems*, vol. 35, 2020, doi: 10.5281/zenodo.3951708.

[24] F. Smarandache, *Plithogeny, Plithogenic Set, Logic, Probability, and Statistics*. Belgium: Pons, 2018. Accessed: Feb. 23, 2023. [Online]. Available: http://arxiv.org/abs/1808.03948

[25] F. Smarandache, “Introduction to the Symbolic Plithogenic Algebraic Structures (revisited),” *Neutrosophic Sets and Systems* , vol. 53, Jan. 2023, doi: 10.5281/ZENODO.7536105.

[26] N. M. Taffach and A. Hatip, “A Review on Symbolic 2-Plithogenic Algebraic Structures,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. 5, no. 1, pp. 08–16, 2023, doi: 10.54216/GJMSA.050101.

[27] R. Ali and Z. Hasan, “An Introduction To The Symbolic 3-Plithogenic Modules,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. 6, no. 1, pp. 13–17, 2023, doi: 10.54216/GJMSA.060102.

[28] F. Smarandache, “Introduction to Plithogenic Logic as generalization of MultiVariate Logic,” *Neutrosophic Sets and Systems*, vol. 45, 2021.

[29] R. Ali and Z. Hasan, “An Introduction to The Symbolic 3-Plithogenic Vector Spaces,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. 6, no. 1, pp. 08–12, 2023, doi: 10.54216/GJMSA.060101.

[30] F. Smarandache, “Plithogenic Probability & Statistics are generalizations of MultiVariate Probability & Statistics,” *Neutrosophic Sets and Systems*, vol. 43, 2021.

[31] F. Smarandache, “Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets,” *Neutrosophic Sets and Systems*, vol. 21, pp. 153–166, 2018.

[32] M. Abdel-Basset and R. Mohamed, “A novel plithogenic TOPSIS- CRITIC model for sustainable supply chain risk management,” *J Clean Prod*, vol. 247, Feb. 2020, doi: 10.1016/J.JCLEPRO.2019.119586.

[33] M. Abdel-Basset, M. El-hoseny, A. Gamal, and F. Smarandache, “A novel model for evaluation Hospital medical care systems based on plithogenic sets,” *Artif Intell Med*, vol. 100, Sep. 2019, doi: 10.1016/J.ARTMED.2019.101710.

[34] N. M. Taffach and A. Hatip, “A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. 5, no. 1, pp. 36–44, 2023, doi: 10.54216/GJMSA.050103.

[35] F. Sultana *et al.*, “A study of plithogenic graphs: applications in spreading coronavirus disease (COVID-19) globally,” *J Ambient Intell Humaniz Comput*, p. 1, 2022, doi: 10.1007/S12652-022-03772-6.

[36] M. Abdel-Basset, R. Mohamed, A. E. N. H. Zaied, and F. Smarandache, “A hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics,” *Symmetry (Basel)*, vol. 11, no. 7, Jul. 2019, doi: 10.3390/SYM11070903.

[37] M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, “Introduction to Symbolic 2-Plithogenic Probability Theory,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 2, 2023.

[38] N. Altounji, M. B. Zeina, and M. M. Ranneh, “Introduction to Neutrosophic Bayes Estimation Theory,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. Volume 7, no. 1, pp. 43–50, Sep. 2023, doi: 10.54216/GJMSA.070105.

Received 12/8/2023, Accepted 10/10/2023

Neutrosophic Sets and Systems, Vol. 59, 2023
University of New Mexico

On The Computing of Symbolic 2-Plithogenic And 3-Plithogenic Complex Roots of Unity

Nabil Khuder Salman

College Of Pharmacy, AL-Farahidi University, Baghdad, Iraq

nsalman@uoalfarahidi.edu.iq

Abstract:

The concept of unity roots plays a central role in the theory of field extensions and polynomials roots' computing.

The objective of this paper is to find the algebraic formula for computing the symbolic 2-plithogenic and 3-plithogenic complex roots of unity, where a general formula will be provided with many related examples up to the exponent 3.

Keywords: symbolic 2-plithogenic complex number, symbolic 3-plithogenic complex number, symbolic n-plithogenic roots of unity.

Introduction and preliminaries.

The symbolic n-plithogenic set was supposed by Smarandache in [1-3]. Symbolic n-plithogenic sets were very helpful in algebra, where this concept has helped with developing algebraic structures, where we can see easily that for any value of n, we get a bigger structure.

Symbolic 2-plithogenic structures and 3-plithogenic were defined and handled by many authors around the globe.

For example, by now we have symbolic 2-plithogenic spaces, modules, matrix [4-7], and same thing for 3-plithogenic structures, see [8-11].

zenodo.10031165/10.5281

In this paper, we are trying to close an important research gap by answering the following question.

How can we find all the roots of unity in the symbolic 2-plithogenic complex ring, and in the 3-plithogenic complex ring?

The symbolic 2-plithogenic or 3-plithogenic root of unity is a symbolic plithogenic number $\boldsymbol{x}$ with the following algebraic property $\boldsymbol{x^n = 1}$.

Definition.

Let C be the complex field, we have:

1). $2-SP_C = \{v_0 + v_1P_1 + v_2P_2;\ v_i \in C\}$ is called the symbolic 2-plithogenic complex ring.

2). $3-SP_C = \{v_0 + v_1P_1 + v_2P_2 + v_3P_3;\ v_i \in C\}$ is called the symbolic 3-plithogenic complex ring.

Algebraic operations on $2-SP_C, 3-SP_C$ are defined as follows:

$(+): 2-SP_C * 2-SP_C \to 2-SP_C$ such that:

$(v_0 + v_1P_1 + v_2P_2) + (u_0 + u_1P_1 + u_2P_2) = (v_0 + u_0) + (v_1 + u_1)P_1 + (v_2 + u_2)P_2.$

$(+): 3-SP_C * 3-SP_C \to 3-SP_C$ such that:

$(v_0 + v_1P_1 + v_2P_2 + v_3P_3) + (u_0 + u_1P_1 + u_2P_2 + u_3P_3) = (v_0 + u_0) + (v_1 + u_1)P_1 + (v_2 + u_2)P_2 + (v_3 + u_3)P_3.$

$(.): 2-SP_C * 2-SP_C \to 2-SP_C$ such that:

$(v_0 + v_1P_1 + v_2P_2)(u_0 + u_1P_1 + u_2P_2) = v_0u_0 + (v_0u_1 + v_1u_0 + v_1u_1)P_1 + (v_0u_2 + v_2u_1 + v_2u_2 + v_2u_0 + v_1u_2)P_2.$

$(.): 3-SP_C * 3-SP_C \to 3-SP_C$ such that:

$(v_0 + v_1P_1 + v_2P_2 + v_3P_3).(u_0 + u_1P_1 + u_2P_2 + u_3P_3) = v_0u_0 + (v_0u_1 + v_1u_0 + v_1u_1)P_1 + (v_0u_2 + v_2u_1 + v_2u_2 + v_2y_0 + v_1u_2)P_2 + (v_0u_3 + v_1u_3 + v_2u_3 + v_3u_3 + v_3u_0 + v_3u_1 + v_3u_2)P_3.$

Multiplication is defined with the following property:

$P_i \times P_j = P_{\max(i,j)}, P_i \times P_i = P_i; 1 \le i \le 3, 1 \le j \le 3$

Main discussion.

Definition.

Let $2-SP_C$ be the symbolic 2-plithogenic complex ring,

then $v=v_0+v_1P_1+v_2P_2 \in 2-SP_C$ is called n-th root of unity if and only if $v^n = 1$.

Definition.

Let $3-SP_C$ be the symbolic 2-plithogenic complex ring,

then $v=v_0+v_1P_1+v_2P_2+v_3P_3 \in 3-SP_C$ is called n-th root of unity if and only if $v^n = 1$.

Definition.

Let $v=v_0+v_1P_1+v_2P_2 \in 2-SP_C, u=u_0+u_1P_1+u_2P_2+u_3P_3 \in 3-SP_C$, then we define:

$\bar{v}=\overline{v_0}+\overline{v_1}P_1+\overline{v_2}P_2, \bar{u}=\overline{u_0}+\overline{u_1}P_1+\overline{u_2}P_2+\overline{u_3}P_3$.

Remark.

For $v=v_0+v_1P_1+v_2P_2 \in 2-SP_C$, we have:

$v^n={v_0}^n+[(v_0+v_1)^n-{v_0}^n]P_1+[(v_0+v_1+v_2)^n-(v_0+v_1)^n]P_2; n \in N.$

For $v=v_0+v_1P_1+v_2P_2+v_3P_3 \in 3-SP_C$, we have:

$v^n={v_0}^n+[(v_0+v_1)^n-{v_0}^n]P_1+[(v_0+v_1+v_2)^n-(v_0+v_1)^n]P_2+[(v_0+v_1+v_2+v_3)^n-(v_0+v_1+v_2)^n]P_3; n \in N.$

Theorem.

Let $v=v_0+v_1P_1+v_2P_2 \in 2-SP_C, u=u_0+u_1P_1+u_2P_2+u_3P_3 \in 3-SP_C$, then:

1. $\|v\|=|v_0|+[|(v_0+v_1)|-|v_0|]P_1+[|(v_0+v_1+v_2)|-|(v_0+v_1)|]P_2$
2. $\|u\|=|u_0|+[|(u_0+u_1)|-|u_0|]P_1+[|(u_0+u_1+u_2)|-|(u_0+u_1)|]P_2+[|(u_0+u_1+u_2+u_3)|-|(u_0+u_1+u_2)|]P_3$

Proof.

1. $\|v\|^2=v.\bar{v}=(v_0+v_1P_1+v_2P_2)(\overline{v_0}+\overline{v_1}P_1+\overline{v_2}P_2)=v_0\overline{v_0}+(v_0\overline{v_1}+v_1\overline{v_0}+v_1\overline{v_1})P_1+(v_0\overline{v_2}+v_2\overline{v_1}+v_2\overline{v_2}+v_2\overline{v_0}+v_1\overline{v_2})P_2=|v_0|^2+\left((v_0+v_1)(\overline{v_0}+\overline{v_1})-v_0\overline{v_0}\right)P_1+\left((v_0+v_1+v_2)(\overline{v_0}+\overline{v_1}+\overline{v_2})-(v_0+v_1)(\overline{v_0}+\overline{v_1})\right)P_2=|v_0|^2+[|(v_0+v_1)|^2-|v_0|^2]P_1+[|(v_0+v_1+v_2)|^2-|(v_0+v_1)|^2]P_2$

Now, we put

$R = |v_0| + [|(v_0 + v_1)| - |v_0|]P_1 + [|(v_0 + v_1 + v_2)| - |(v_0 + v_1)|]P_2.$

We get

$$R^2 = |v_0|^2 + [|(v_0 + v_1)|^2 - |v_0|^2]P_1 + [|(v_0 + v_1 + v_2)|^2 - |(v_0 + v_1)|^2]P_2$$
$$+ 2|v_0|P_1[|(v_0 + v_1)| - |v_0|] + 2|v_0|[|(v_0 + v_1 + v_2)| - |(v_0 + v_1)|]P_2$$
$$+ 2(|(v_0 + v_1)| - |v_0|)[|(v_0 + v_1 + v_2)| - |(v_0 + v_1)|]P_1P_2$$
$$= |v_0|^2$$
$$+ [|(v_0 + v_1)|^2 - |v_0|^2 - 2|v_0||v_0 + v_1| + 2|v_0||v_0 + v_1| - 2|v_0|^2]P_1$$
$$+ [|(v_0 + v_1 + v_2)|^2 - |(v_0 + v_1)|^2 + 2|v_0 + v_1||(v_0 + v_1 + v_2)|$$
$$+ 2|v_0||(v_0 + v_1 + v_2)| - 2|v_0||v_0 + v_1| + 2|v_0 + v_1||(v_0 + v_1 + v_2)|$$
$$- |v_0 + v_1|^2 - 2|v_0||(v_0 + v_1 + v_2)| + 2|v_0||v_0 + v_1|]P_2$$
$$= |v_0|^2 + [|(v_0 + v_1)|^2 - |v_0|^2]P_1 + [|(v_0 + v_1 + v_2)|^2 - |(v_0 + v_1)|^2]P_2$$

This implies that

$\|v\| = R = |v_0| + [|(v_0 + v_1)| - |v_0|]P_1 + [|(v_0 + v_1 + v_2)| - |(v_0 + v_1)|]P_2.$

2. $\|u\|^2 = u.\bar{u} = (u_0 + u_1P_1 + u_2P_2 + u_3P_3)(\overline{u_0} + \overline{u_1}P_1 + \overline{u_2}P_2 + \overline{u_3}P_3) = u_0\overline{u_0} + (u_0\overline{u_1} + u_1\overline{u_0} + u_1\overline{u_1})P_1 + (u_0\overline{u_2} + u_2\overline{u_1} + u_2\overline{u_2} + u_2\overline{u_0} + u_1\overline{u_2})P_2 + (u_0\overline{u_3} + u_1\overline{u_3} + u_2\overline{u_3} + u_3\overline{u_0} + u_3\overline{u_1} + u_3\overline{u_2} + u_3\overline{u_3})P_3 = |u_0|^2 + \big((u_0 + u_1)(\overline{u_0} + \overline{u_1}) - u_0\overline{u_0}\big)P_1 + \big((u_0 + u_1 + u_2)(\overline{u_0} + \overline{u_1} + \overline{u_2}) - (u_0 + u_1)(\overline{u_0} + \overline{u_1})\big)P_2 + \big((u_0 + u_1 + u_2 + u_3)(\overline{u_0} + \overline{u_1} + \overline{u_2} + \overline{u_3}) - (u_0 + u_1 + u_2)(\overline{u_0} + \overline{u_1} + \overline{u_2})\big)P_3 = |u_0|^2 + [|(u_0 + u_1)|^2 - |u_0|^2]P_1 + [|(u_0 + u_1 + u_2)|^2 - |(u_0 + u_1)|^2]P_2 + [|(u_0 + u_1 + u_2 + u_3)|^2 - |(u_0 + u_1 + u_2)|^2]P_3$

We put

$R = |u_0| + [|(u_0 + u_1)| - |u_0|]P_1 + [|(u_0 + u_1 + u_2)| - |(u_0 + u_1)|]P_2 + [|(u_0 + u_1 + u_2 + u_3)| - |(u_0 + u_1 + u_2)|]P_3$

by an easy computing, we get $R^2 = |u|^2$, thus $|u| = R$.

Example.

Take $v = (2 + i) + (1 - i)P_1 + 2iP_2 \in 2 - SP_C$, we have:

$v_0 = 2 + i, v_1 = 1 - i, v_2 = 2i.$

$\overline{v_0} = 2 - i, \overline{v_1} = 1 + i, \overline{v_2} = -2i$

$\bar{v} = (2 - i) + (1 + i)P_1 - 2iP_2$

Also, $\begin{cases} |v_0| = \sqrt{5}, |v_0 + v_1| = |3| = 3, |v_0 + v_1 + v_2| = |3 + 2i| = \sqrt{13} \\ \|v\| = \sqrt{5} + \big(3 - \sqrt{5}\big)P_1 + \big(\sqrt{13} - 3\big)P_2 \end{cases}$

Example.

Take $v=(1+i)+(1-3i)P_1+(5+i)P_2+(4+3i)P_3$, we have:

$v_0=1+i, v_1=1-3i, v_2=5+i, v_3=4+3i.$

$\overline{v_0}=1-i, \overline{v_1}=1+3i, \overline{v_2}=5-i, \overline{v_3}=4-3i$

$\bar{v}=1-i+(1+3i)P_1+(5-i)P_2+(4-3i)P_3$

Also,

$$\begin{cases}|v_0|=\sqrt{2}, |v_0+v_1|=|2-2i|=\sqrt{8}, |v_0+v_1+v_2|=|7-i|=\sqrt{50}, |v_0+v_1+v_2+v_3|=|11+2i| \\ \|v\|=\sqrt{2}+\left(\sqrt{8}-\sqrt{2}\right)P_1+\left(\sqrt{50}-\sqrt{8}\right)P_2+\left(\sqrt{125}-\sqrt{50}\right)P_3\end{cases}$$

Theorem.

Let $v=v_0+v_1P_1+v_2P_2\in 2-SP_C$, then v is a symbolic 2-plithogenic n-th root of unity if and only if $v_0, v_0+v_1, v_0+v_1+v_2$ are classical n-th roots of unity in the field C.

Proof.

It is know $v^n=1$, which is equivalent to:

$${v_0}^n+[(v_0+v_1)^n-{v_0}^n]P_1+[(v_0+v_1+v_2)^n-(v_0+v_1)^n]P_2=1$$

$$\begin{cases}{v_0}^n=1 \\ (v_0+v_1)^n-{v_0}^n=0 \Longrightarrow (v_0+v_1)^n={v_0}^n=1 \\ (v_0+v_1+v_2)^n-(v_0+v_1)^n=0 \Longrightarrow (v_0+v_1+v_2)^n=(v_0+v_1)^n=1\end{cases}$$

So that, $v_0, v_0+v_1, v_0+v_1+v_2$ are n-th roots of unity.

Example.

Let us find all a symbolic 2-plithogenic roots of unity order 2.

The classical set of the roots of unity of order 2 is $E_1=\{-1,1\}$.

The corresponding 2-symbolic plithogenic roots of unity of order 2 are:

1). $v_0=v_0+v_1=v_0+v_1+v_2=1 \Longrightarrow R_1=1$

2). $v_0=v_0+v_1=1, v_0+v_1+v_2=-1 \Longrightarrow R_2=1-2P_2$

3). $v_0=v_0+v_1+v_2=1, v_0+v_1=-1 \Longrightarrow R_3=1-2P_1+2P_2$

4). $v_0=1, v_0+v_1=v_0+v_1+v_2=-1 \Longrightarrow R_4=1-2P_1$

5). $v_0=v_0+v_1+v_2=1=v_0+v_1=-1 \Longrightarrow R_5=-1$

6). $v_0=v_0+v_1=-1, v_0+v_1+v_2s=1 \Longrightarrow R_6=-1+2P_2$

7). $v_0=v_0+v_1+v_2=-1, v_0+v_1=-1 \Longrightarrow R_7=-1+2P_1-2P_2$

8). $v_0 = -1, v_0 + v_1 + v_2 = v_0 + v_1 = 1 \Longrightarrow R_8 = -1 + 2P_1$

Example.

Let us find all a symbolic 2-plithogenic roots of unity order 3.

The classical set of the roots of unity of order 3 is $E_2 = \left\{-1, e^{2\frac{\pi}{3}i}, e^{4\frac{\pi}{3}i}\right\}$.

The corresponding 2-symbolic plithogenic roots of unity of order 3 are:

1). $v_0 = v_0 + v_1 = v_0 + v_1 + v_2 = 1 \Longrightarrow R_1 = 1$

2). $v_0 = v_0 + v_1 = 1, v_0 + v_1 + v_2 = e^{2\frac{\pi}{3}i} \Longrightarrow R_2 = 1 + \left(e^{2\frac{\pi}{3}i} - 1\right) P_2$

3). $v_0 = v_0 + v_1 + v_2 = 1, v_0 + v_1 = e^{4\frac{\pi}{3}i} \Longrightarrow R_3 = 1 + \left(e^{4\frac{\pi}{3}i} - 1\right) P_2$

4). $v_0 = v_0 + v_1 + v_2 = 1, v_0 + v_1 = e^{2\frac{\pi}{3}i} \Longrightarrow R_4 = 1 + \left(e^{2\frac{\pi}{3}i} - 1\right) P_1 + \left(e^{2\frac{\pi}{3}i} - 1\right) P_2$

5). $v_0 = v_0 + v_1 + v_2 = 1 = v_0 + v_1 = e^{4\frac{\pi}{3}i} \Longrightarrow R_5 = 1 + \left(e^{4\frac{\pi}{3}i} - 1\right) P_1 + \left(1 - e^{4\frac{\pi}{3}i}\right) P_2$

6). $v_0 = 1, v_0 + v_1 = v_0 + v_1 + v_2 = e^{2\frac{\pi}{3}i} \Longrightarrow R_6 = 1 + \left(e^{4\frac{\pi}{3}i} - 1\right) P_1$

7). $v_0 = 1, v_0 + v_1 + v_2 = v_0 + v_1 = e^{4\frac{\pi}{3}i} \Longrightarrow R_7 = 1 + \left(e^{4\frac{\pi}{3}i} - 1\right) P_2$

8). $v_0 = 1, v_0 + v_1 + v_2 = e^{4\frac{\pi}{3}i}, v_0 + v_1 = e^{2\frac{\pi}{3}i} \Longrightarrow R_8 = 1 + \left(e^{2\frac{\pi}{3}i} - 1\right) P_1 + \left(e^{4\frac{\pi}{3}i} - e^{2\frac{\pi}{3}i}\right) P_2$

9). $v_0 = 1, v_0 + v_1 + v_2 = e^{2\frac{\pi}{3}i}, v_0 + v_1 = e^{4\frac{\pi}{3}i} \Longrightarrow R_9 = 1 + \left(e^{4\frac{\pi}{3}i} - 1\right) P_1 + \left(e^{2\frac{\pi}{3}i} - e^{4\frac{\pi}{3}i}\right) P_2$

10). $v_0 = e^{2\frac{\pi}{3}i}, v_0 + v_1 + v_2 = v_0 + v_1 = 1 \Longrightarrow R_{10} = e^{2\frac{\pi}{3}i} + \left(1 - e^{2\frac{\pi}{3}i}\right) P_1$

11). $v_0 = e^{2\frac{\pi}{3}i}, v_0 + v_1 + v_2 = v_0 + v_1 = e^{2\frac{\pi}{3}i} \Longrightarrow R_{11} = e^{2\frac{\pi}{3}i}$

12). $v_0 = e^{2\frac{\pi}{3}i}, v_0 + v_1 + v_2 = v_0 + v_1 = e^{4\frac{\pi}{3}i} \Longrightarrow R_{12} = e^{2\frac{\pi}{3}i} + \left(e^{4\frac{\pi}{3}i} - e^{2\frac{\pi}{3}i}\right) P_1$

13). $v_0 = e^{2\frac{\pi}{3}i}, v_0 + v_1 + v_2 = e^{2\frac{\pi}{3}i}, v_0 + v_1 = 1 \Longrightarrow R_{13} = e^{2\frac{\pi}{3}i} + \left(1 - e^{2\frac{\pi}{3}i}\right) P_1 + \left(e^{2\frac{\pi}{3}i} - 1\right) P_2$

14). $v_0 = e^{2\frac{\pi}{3}i}, v_0 + v_1 + v_2 = e^{4\frac{\pi}{3}i}, v_0 + v_1 = 1 \Longrightarrow R_{14} = e^{2\frac{\pi}{3}i} + \left(1 - e^{2\frac{\pi}{3}i}\right) P_1 + \left(e^{4\frac{\pi}{3}i} - 1\right) P_2$

15). $v_0 = e^{2\frac{\pi}{3}i}, v_0 + v_1 + v_2 = 1, v_0 + v_1 = e^{2\frac{\pi}{3}i} \Longrightarrow R_{15} = e^{2\frac{\pi}{3}i} + \left(1 - e^{2\frac{\pi}{3}i}\right) P_2$

16). $v_0 = e^{2\frac{\pi}{3}i}, v_0 + v_1 + v_2 = 1, v_0 + v_1 = e^{4\frac{\pi}{3}i} \Longrightarrow R_{16} = e^{2\frac{\pi}{3}i} + \left(e^{4\frac{\pi}{3}i} - e^{2\frac{\pi}{3}i}\right) P_1 + \left(1 - e^{4\frac{\pi}{3}i}\right) P_2$

17). $v_0 = e^{2\frac{\pi}{3}i}, v_0 + v_1 + v_2 = e^{4\frac{\pi}{3}i}, v_0 + v_1 = e^{2\frac{\pi}{3}i} \Longrightarrow R_{17} = e^{2\frac{\pi}{3}i} + \left(e^{4\frac{\pi}{3}i} - e^{2\frac{\pi}{3}i}\right) P_2$

18). $v_0 = e^{2\frac{\pi}{3}i}, v_0 + v_1 + v_2 = e^{2\frac{\pi}{3}i}, v_0 + v_1 = e^{4\frac{\pi}{3}i} \Longrightarrow R_{18} = e^{2\frac{\pi}{3}i} + \left(e^{4\frac{\pi}{3}i} - e^{2\frac{\pi}{3}i}\right) P_1 + \left(e^{2\frac{\pi}{3}i} - e^{4\frac{\pi}{3}i}\right) P_2$

19). $v_0 = e^{4\frac{\pi}{3}i}, v_0 + v_1 + v_2 = v_0 + v_1 = 1 \Longrightarrow R_{19} = e^{4\frac{\pi}{3}i} + \left(1 - e^{4\frac{\pi}{3}i}\right) P_1$

20). $v_0 = e^{4\frac{\pi}{3}i}, v_0 + v_1 + v_2 = v_0 + v_1 = e^{2\frac{\pi}{3}i} \Longrightarrow R_{20} = e^{4\frac{\pi}{3}i} + \left(e^{2\frac{\pi}{3}i} - e^{4\frac{\pi}{3}i}\right) P_1$

21). $v_0 = e^{4\frac{\pi}{3}i}, v_0 + v_1 + v_2 = v_0 + v_1 = e^{4\frac{\pi}{3}i} \Longrightarrow R_{21} = e^{4\frac{\pi}{3}i}$

22). $v_0 = e^{2\frac{\pi}{3}i}, v_0 + v_1 + v_2 = e^{2\frac{\pi}{3}i}, v_0 + v_1 = 1 \Longrightarrow R_{22} = e^{4\frac{\pi}{3}i} + \left(1 - e^{4\frac{\pi}{3}i}\right) P_1 + \left(e^{2\frac{\pi}{3}i} - 1\right) P_2$

23). $v_0 = e^{4\frac{\pi}{3}i}, v_0 + v_1 + v_2 = e^{4\frac{\pi}{3}i}, v_0 + v_1 = 1 \Longrightarrow R_{23} = e^{4\frac{\pi}{3}i} + \left(1 - e^{4\frac{\pi}{3}i}\right) P_1 + \left(e^{4\frac{\pi}{3}i} - 1\right) P_2$

24). $v_0 = e^{4\frac{\pi}{3}i}, v_0 + v_1 + v_2 = 1, v_0 + v_1 = e^{2\frac{\pi}{3}i} \Longrightarrow R_{24} = e^{4\frac{\pi}{3}i} + \left(e^{2\frac{\pi}{3}i} - e^{4\frac{\pi}{3}i}\right) P_1 + \left(1 - e^{2\frac{\pi}{3}i}\right) P_2$

25). $v_0 = e^{4\frac{\pi}{3}i}, v_0 + v_1 + v_2 = 1, v_0 + v_1 = e^{4\frac{\pi}{3}i} \Longrightarrow R_{25} = e^{4\frac{\pi}{3}i} + \left(1 - e^{4\frac{\pi}{3}i}\right) P_2$

26). $v_0 = e^{4\frac{\pi}{3}i}, v_0 + v_1 + v_2 = e^{2\frac{\pi}{3}i}, v_0 + v_1 = e^{2\frac{\pi}{3}i} \Longrightarrow R_{26} = e^{4\frac{\pi}{3}i} + \left(e^{2\frac{\pi}{3}i} - e^{4\frac{\pi}{3}i}\right) P_1 + \left(e^{4\frac{\pi}{3}i} - e^{2\frac{\pi}{3}i}\right) P_2$

27). $v_0 = e^{4\frac{\pi}{3}i}, v_0 + v_1 + v_2 = e^{2\frac{\pi}{3}i}, v_0 + v_1 = e^{4\frac{\pi}{3}i} \Longrightarrow R_{27} = e^{4\frac{\pi}{3}i} + \left(e^{2\frac{\pi}{3}i} - e^{4\frac{\pi}{3}i}\right) P_2$

Theorem.

Let $v = v_0 + v_1 P_1 + v_2 P_2 + v_3 P_3 \in 3 - SP_C$, then v is an n-th root of unity if and only if $v_0, v_0 + v_1, v_0 + v_1 + v_2$ are n-th root of unity in C.

Proof.

It is know that $v^n = 1$, which is equivalent to:

$$v_0{}^n + [(v_0 + v_1)^n - v_0{}^n]P_1 + [(v_0 + v_1 + v_2)^n - (v_0 + v_1)^n]P_2 + [(v_0 + v_1 + v_2 + v_3)^n - (v_0 + v_1 + v_2)^n]P_3 = 1$$

$$\begin{cases} {v_0}^n = 1 \\ (v_0 + v_1)^n - {v_0}^n = 0 \Rightarrow (v_0 + v_1)^n = {v_0}^n = 1 \\ (v_0 + v_1 + v_2)^n - (v_0 + v_1)^n = 0 \Rightarrow (v_0 + v_1 + v_2)^n = (v_0 + v_1)^n = 1 \\ (v_0 + v_1 + v_2 + v_3)^n - (v_0 + v_1 + v_2)^n = 0 \Rightarrow (v_0 + v_1 + v_2 + v_3)^n = (v_0 + v_1 + v_2)^n = 1 \end{cases}$$

Thus the proof holds.

Example.

Let us find all 0f 2-nd roots of unity in $3 - SP_C$.

1). $v_0 = v_0 + v_1 = v_0 + v_1 + v_2 = v_0 + v_1 + v_2 + v_3 = 1 \Rightarrow R_1 = 1$

2). $v_0 = v_0 + v_1 = v_0 + v_1 + v_2 = 1, v_0 + v_1 + v_2 + v_3 = -1 \Rightarrow R_2 = 1 - 2P_3$

3). $v_0 = v_0 + v_1 = v_0 + v_1 + v_2 + v_3 = 1, v_0 + v_1 + v_2 = -1 \Rightarrow R_3 = 1 - 2P_2 + 2P_3$

4). $v_0 = v_0 + v_1 + v_2 = v_0 + v_1 + v_2 + v_3 = 1, v_0 + v_1 = -1 \Rightarrow R_4 = 1 - 2P_1 + 2P_2$

5). $v_0 = v_0 + v_1 = 1, v_0 + v_1 + v_2 = v_0 + v_1 + v_2 + v_3 = -1 \Rightarrow R_5 = 1 - 2P_2$

6). $v_0 = v_0 + v_1 + v_2 = 1, v_0 + v_1 = v_0 + v_1 + v_2 + v_3 = -1 \Rightarrow R_6 = 1 + 2P_2 + 2P_3$

7). $v_0 = v_0 + v_1 + v_2 + v_3 = 1, v_0 + v_1 = v_0 + v_1 + v_2 = -1 \Rightarrow R_7 = 1 - 2P_1 + 2P_3$

8). $v_0 = 1, v_0 + v_1 + v_2 + v_3 == v_0 + v_1 = v_0 + v_1 + v_2 = -1 \Rightarrow R_8 = 1 - 2P_1$

9). $R_9 = -1 = -R_1$

10). $R_{10} = -R_2 = -1 + 2P_3$

11). $R_{11} = -R_3 = -1 + 2P_2 - 2P_3$

12). $R_{12} = -R_4 = -1 + 2P_1 - 2P_2$

13). $R_{13} = -R_5 = -1 + 2P_2$

14). $R_{14} = -R_6 = -1 - 2P_2 - 2P_3$

15). $R_{15} = -R_7 = -1 + 2P_1 - 2P_3$

16). $R_{16} = -R_8 = -1 + 2P_1$.

The group of unity roots classification

It is known that the set of all n-th roots of unity forms a subgroup of C^* denoted by U_C with respect to the multiplication operation and this group is isomorphic to the additive group Z_n (integers modulo n).

By a similar approach, we can see easily that the set of all symbolic 2-plithogenic complex n-th roots of unity forms a group with respect to multiplication operation, and the set of all symbolic 3-plithogenic complex n-th roots of unity forms a group with respect to multiplication operation.

The following theorem classifies the symbolic 2-plithogenic and 3-plithogenic groups of n-th roots of unity.

Theorem:

Let U_{2-SPC} be the group of n-th unity roots of symbolic 2-plithogenic complex numbers, and U_{3-SPC} be the group of n-th unity roots of symbolic 3-plithogenic complex numbers, then:

1-) $U_{2-SPC} \cong Z_n \times Z_n \times Z_n$.

2-) $U_{3-SPC} \cong Z_n \times Z_n \times Z_n \times Z_n$.

Proof:

1-) Define the mapping $f: U_{2-SPC} \to U_C \times U_C \times U_C$ such that:

$f(e_0 + e_1P_1 + e_2P_2) = (e_0, e_0 + e_1, e_0 + e_1 + e_2)$.

The mapping f is well defined:

For $M = m_0 + m_1P_1 + m_2P_2 = N = n_0 + n_1P_1 + n_2P_2$, we get:

$m_0 = n_0, m_0 + m_1 = n_0 + n_1, m_0 + m_1 + m_2 = n_0 + n_1 + n_2$,

Thus $f(M) = f(N)$.

The mapping f preserves multiplication:

For $M = m_0 + m_1P_1 + m_2P_2, N = n_0 + n_1P_1 + n_2P_2$, we get:

$f(MN) = f(m_0n_0 + [m_0n_1 + m_1n_0 + m_1n_1]P_1 + [m_0n_2 + m_1n_2 + m_2n_0 + m_2n_1 + m_2n_2]P_2) = (m_0n_0, m_0n_0 + m_0n_1 + m_1n_0 + m_1n_1, m_0n_0 + m_0n_1 + m_1n_0 + m_1n_1 + m_0n_2 + m_1n_2 + m_2n_0 + m_2n_1 + m_2n_2) = (m_0, m_0 + m_1, m_0 + m_1 + m_2).(n_0, n_0 + n_1, n_0 + n_1 + n_2) = f(M)f(N)$.

The mapping f is injective:

$Ker(f) = \{M = m_0 + m_1P_1 + m_2P_2; f(M) = (1,1,1)\}$,

So that, $m_0 = 1, m_1 = m_2 = 0, thus\ Ker(f) = \{1\}$.

The mapping f is surjective:

$Im(f) = U_C \times U_C \times U_C$.

Thus, the mapping f is a group isomorphism, which means that $U_{2-SPC} \cong U_C \times U_C \times U_C$. Since $U_C \cong Z_n$, we get $U_{2-SPC} \cong Z_n \times Z_n \times Z_n$.

2-) Define the mapping $f: U_{3-SPC} \to U_C \times U_C \times U_C \times U_C$ such that:

$f(e_0 + e_1P_1 + e_2P_2) = (e_0, e_0 + e_1, e_0 + e_1 + e_2, e_0 + e_1 + e_2 + e_3)$.

The mapping f is well defined:

For $M = m_0 + m_1P_1 + m_2P_2 + m_3P_3 = N = n_0 + n_1P_1 + n_2P_2 + n_3P_3$, we get:

$m_0 = n_0, m_0 + m_1 = n_0 + n_1, m_0 + m_1 + m_2 = n_0 + n_1 + n_2, m_0 + m_1 + m_2 + m_3 = n_0 + n_1 + n_2 + n_3$,

Thus $f(M) = f(N)$.

The mapping f preserves multiplication:

For $M = m_0 + m_1P_1 + m_2P_2 + m_3P_3, N = n_0 + n_1P_1 + n_2P_2 + n_3P_3$, we get:

$f(MN) = f(m_0n_0 + [m_0n_1 + m_1n_0 + m_1n_1]P_1 + [m_0n_2 + m_1n_2 + m_2n_0 + m_2n_1 + m_2n_2]P_2 + [m_0n_3 + m_1n_3 + m_2n_3 + m_3n_0 + m_3n_1 + m_3n_2 + m_3n_3]P_3) = (m_0n_0, m_0n_0 + m_0n_1 + m_1n_0 + m_1n_1, m_0n_0 + m_0n_1 + m_1n_0 + m_1n_1 + m_0n_2 + m_1n_2 + m_2n_0 + m_2n_1 + m_2n_2, m_0n_0 + m_0n_1 + m_1n_0 + m_1n_1 + m_0n_2 + m_1n_2 + m_2n_0 + m_2n_1 + m_2n_2 + m_0n_3 + m_1n_3 + m_2n_3 + m_3n_0 + m_3n_1 + m_3n_2 + m_3n_3) = (m_0, m_0 + m_1, m_0 + m_1 + m_2, m_0 + m_1 + m_2 + m_3).(n_0, n_0 + n_1, n_0 + n_1 + n_2, n_0 + n_1 + n_2 + n_3) = f(M)f(N)$.

The mapping f is injective:

$Ker(f) = \{M = m_0 + m_1P_1 + m_2P_2 + m_3P_3; f(M) = (1,1,1)\}$,

So that, $m_0 = 1, m_1 = m_2 = m_3 = 0, thus\ Ker(f) = \{1\}$.

The mapping f is surjective:

$Im(f) = U_C \times U_C \times U_C \times U_C$.

Thus, the mapping f is a group isomorphism, which means that $U_{3-SPC} \cong U_C \times U_C \times U_C \times U_C$. Since $U_C \cong Z_n$, we get $U_{3-SPC} \cong Z_n \times Z_n \times Z_n \times Z_n$.

Conclusion.

In this paper, we presented an algebraic algorithm to compute n-th roots of unity in symbolic 2-plithogenic/3-plithogenic complex ring respectively.

Also, we have illustrated some examples to clarify the flow of our algorithm.

In the future, we aim to find n-th roots of unity in symbolic m-plithogenic complex ring for any value of m.

References:

1. Smarandache, F., " Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)", Neutrosophic Sets and Systems, vol. 53, 2023.
2. Florentin Smarandache, Plithogenic Algebraic Structures. Chapter in “Nidus idearum Scilogs, V: joining the dots” (third version), Pons Publishing Brussels, pp. 123-125, 2019.
3. Florentin Smarandache: Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets – Revisited, *Neutrosophic Sets and Systems*, vol. 21, 2018, pp. 153-166.
4. Alfahal, A., Alhasan, Y., Abdulfatah, R., Mehmood, A., and Kadhim,M., " On Symbolic 2-Plithogenic Real Matrices and Their Algebraic Properties", International Journal of Neutrosophic Science, Vol 21, 2023.
5. Nader Mahmoud Taffach , Ahmed Hatip., "A Review on Symbolic 2-Plithogenic Algebraic Structures " Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.
6. M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, “Introduction to Symbolic 2-Plithogenic Probability Theory,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.
7. Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.
8. Ali, R., and Hasan, Z., " An Introduction To The Symbolic 3-Plithogenic Modules ", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.
9. Ali, R., and Hasan, Z., "An Introduction To The Symbolic 3-Plithogenic Vector Spaces", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.

10. Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.
11. Ben Othman, K., "On Some Algorithms For Solving Symbolic 3-Plithogenic Equations", Neoma Journal Of Mathematics and Computer Science, 2023.
12. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Concept Of Symbolic 7-Plithogenic Real Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.
13. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Symbolic 8-Plithogenic Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.
14. Ben Othman, K., Albasheer, O., Nadweh, R., Von Shtawzen, O., and Ali, R., "On The Symbolic 6-Plithogenic and 7-Plithogenic Rings", Galoitica Journal of Mathematical Structures and Applications, Vol.8, 2023.

Received 19/7/2023, Accepted 5/9/2023

On The Symbolic 2-Plithogenic Real Analysis by Using Algebraic Functions

Nabil Khuder Salman

College Of Pharmacy, AL-Farahidi University, Baghdad, Iraq

nsalman@uoalfarahidi.edu.iq

Abstract:

Symbolic 2-plithogenic sets as a generalization of classical concept of sets were applicable to algebraic structures. The symbolic 2-plithogenic rings and fields are good generalizations of classical corresponding systems.

In this paper, we study the symbolic 2-plithogenic real functions with one variable by using a special algebraic function called AH-isometry. In addition, we discuss the symbolic 2-plithogenic simple differential equations and conic sections by using this isometry. Also, many examples will be presented to explain the novelty of this work.

Keywords: symbolic 2-plithogenic set, symbolic 2-plithogenic real function, symbolic 2-plithogenic circle, symbolic 2-plithogenic ellipse.

Introduction and basic definitions

Symbolic n-plithogenic algebraic structures are considered as new generalizations of classical algebraic structures [1-3], such as symbolic 2-plithogenic integers, modules, and vector spaces [4-8].

zenodo.10031172/10.5281

Symbolic 2-plithogenic structures have a similar structure to the refined neutrosophic structures and many non classical algebraic structures defined by many authors in [9-17,20-23, 24-30].

In the literature, many mathematical approaches were carried out on neutrosophic and refined neutrosophic structures, were a special function called AH-isometry was used to study the analytical properties and conic sections [11-12, 18-19], and that occurs by taking the direct image of neutrosophic elements to the classical Cartesian product of the real field with itself.

In this work, we follow the previous efforts, and we define for the first time a special AH-isometry on the symbolic 2-plithogenic field of reals, and we use this isometry to obtain many formulas and properties about the symbolic 2-plithogenic analytical concepts such as differentiability, continuity, and integrability. Also, symbolic 2-plithogenic conic sections will find a place in our study.

Definition.

The symbolic 2-plithogenic ring of real numbers is defined as follows:

$$2-SP_R=\{t_0+t_1P_1+t_2P_2; t_i\in R, P_1\times P_2=P_2\times P_1=P_2, {P_1}^2={P_2}^2=P_2\}$$

The addition operation on $2-SP_R$ is defined as follows:

$$(t_0+t_1P_1+t_2P_2)+(t_0'+t_1'P_1+t_2'P_2)=(t_0+t_0')+(t_1+t_1')P_1+(t_2+t_2')P_2$$

The multiplication on $2-SP_R$ is defined as follows:

$$(t_0+t_1P_1+t_2P_2)(t_0'+t_1'P_1+t_2'P_2)$$
$$=t_0t_0'+(t_0t_1'+t_1t_0'+t_1t_1')P_1+(t_0t_2'+t_1t_2'+t_2t_2'+t_2t_0'+t_2t_1')P_2$$

Remark.

If $T=t_0+t_1P_1+t_2P_2\in 2-SP_R$, then:

$T^{-1}=\frac{1}{T}=\frac{1}{t_0}+\left[\frac{1}{t_0+t_1}-\frac{1}{t_0}\right]P_1+\left[\frac{1}{t_0+t_1+t_2}-\frac{1}{t_0+t_1}\right]P_2$, with $t_0\neq 0, t_0+t_1\neq 0, t_0+t_1+t_2\neq 0$.

Main Results

Definition.

Let $2-SP_R=\{a+bP_1+cP_2; a,b,c\in R\}$ be the 2-plithogenic field of real numbers, a function $f=f(X): 2-SP_R\to 2-SP_R$ is called one variable symbolic 2-plithogenic real function, with

$X=x_0+x_1P_1+x_2P_2\in 2-SP_R$.

Definition.

Let $2-SP_R$ be the symbolic 2-plithogenic field of reals, we define its AH-isometry as follows:

$I: 2-SP_R\to R\times R\times R$ such that:

$I(x+yP_1+zP_2)=(x,x+y,x+y+z)$.

It is easy to see that I is a ring isomorphism with the inverse:

$I^{-1}: R\times R\times R\to 2-SP_R$ such that:

$I^{-1}(x,y,z)=x+(y-x)P_1+(z-y)P_2$

Definition.

Let $f: 2-SP_R\to 2-SP_R$ be a symbolic 2-plithogenic real function with one variable, we define the canonical formula as follows:

$$I^{-1}\circ I(f): 2-SP_R\to 2-SP_R$$

Example.

Consider$f(X)=X^2+2-P_1+P_2$, its canonical formula is:

$I(f(X))=[I(X)]^2+I(2-P_1+P_2)=(x_0,x_0+x_1,x_0+x_1+x_2)^2+(2,1,2)$

$=({x_0}^2+2,(x_0+x_1)^2+1,(x_0+x_1+x_2)^2+2)$

$I^{-1}\circ I(f(X))=$

${x_0}^2+2+P_1[(x_0+x_1)^2-{x_0}^2-1]+P_2[(x_0+x_1+x_2)^2-(x_0+x_1)^2+1]$

For example:

$f(1+P_1)=(1+P_1)^2+2-P_1+P_2=1+P_1+2P_1+2-P_1+P_2=3+2P_1+P_2$.

If we put values $x_0=1, x_1=1, x_2=0$,

in the canonical formula, then we get:

$I^{-1}\circ I(f(X))=(1)^2+2+P_1[(2)^2-(1)^2-1]+P_2[(2)^2-(2)^2+1]=3+2P_1+P_2$.

The canonical formulas of famous functions:

1. The exponent function:

$e^X; X = x_0 + x_1P_1 + x_2P_2, I^{-1} \circ I(e^X) =$

$I^{-1}(e^{x_0}, e^{x_0+x_1}, e^{x_0+x_1+x_2}) =$

$e^{x_0} + P_1[e^{x_0+x_1} - e^{x_0}] + P_2[e^{x_0+x_1+x_2} - e^{x_0+x_1}]$

2. The logarithmic function:

$\ln(X)\,; X = x_0 + x_1P_1 + x_2P_2,$

$$I^{-1} \circ I(\ln(X)) = I^{-1}(\ln(x_0)\,, \ln(x_0 + x_1)\,, \ln(x_0 + x_1 + x_2)) =$$

$\ln(x_0) + P_1[\ln(x_0 + x_1) - \ln(x_0)] + P_2[\ln(x_0 + x_1 + x_2) - \ln(x_0 + x_1)].$

3. Famous trigonometric functions:

$\sin(X) = \sin(x_0) + P_1[\sin(x_0 + x_1) - \sin(x_0)] + P_2[\sin(x_0 + x_1 + x_2) - \sin(x_0 + x_1)]$

$\cos(X) = \cos(x_0) + P_1[\cos(x_0 + x_1) - \cos(x_0)]$
$\quad + P_2[\cos(x_0 + x_1 + x_2) - \cos(x_0 + x_1)]$

$\tan(X) = \tan(x_0) + P_1[\text{taan}(x_0 + x_1) - \tan(x_0)]$
$\quad + P_2[\tan(x_0 + x_1 + x_2) - \tan(x_0 + x_1)]$

And so no.

Definition.

A symbolic 2-plithogenic real number $T = t_0 + t_1P_1 + t_2P_2$ is called positive if and only if

$t_0 \geq 0, t_0 + t_1 \geq 0, t_0 + t_1 + t_2 \geq 0.$

For example $3 + 2P_1 - P_2 > 0$, that is because, $3 > 0$, $5 > 0, 4 > 0$.

Definition.

Let $f: 2 - SP_R \to 2 - SP_R$ be a symbolic 2-plithogenic real function with one variable $X = x_0 + x_1P_1 + x_2P_2$, then:

a. f is differentiable if and only if $I(f(X))$ is differentiable.
b. f is continuous if and only if $I(f(X))$ is continuous.
c. f is integrable if and only if $I(f(X))$ is integrable.

Example.

Find the derivation of $f(X) = X^2 + X + P_1$ in two different ways.

Solution.

The regular way is $\acute{f}(X) = 2X + 1$.

The canonical way is:

$I^{-1} \circ I\big(f(X)\big) =$

$I^{-1}({x_0}^2, (x_0+x_1)^2, (x_0+x_1+x_2)^2) + I^{-1}(x_0, x_0+x_1, x_0+x_1+x_2) + I^{-1}(0,1,1)$

$$= {x_0}^2 + x_0 + P_1[(x_0+x_1)^2 - {x_0}^2 + (x_0+x_1) - x_0 + 1]$$
$$+ P_2[(x_0+x_1+x_2)^2 - (x_0+x_1)^2 + (x_0+x_1+x_2) - (x_0+x_1) + 0] =$$
$$= {x_0}^2 + x_0 + P_1[(x_0+x_1)^2 - {x_0}^2 + x_1 + 1]$$
$$+ P_2[(x_0+x_1+x_2)^2 - (x_0+x_1)^2 + x_1]$$

First, we have: $({x_0}^2 + x_0)'_{x_0} = 2x_0 + 1.$

$[(x_0+x_1)^2 - {x_0}^2 + x_1 + 1 + {x_0}^2 + x_0]'_{x_0+x_1} =$

$[(x_0+x_1)^2 + (x_0+x_1) + 1]'_{x_0+x_1} = 2(x_0+x_1) + 1$

$[(x_0+x_1+x_2)^2 - (x_0+x_1)^2 + x_2 + (x_0+x_1)^2 - {x_0}^2 + x_1 + 1]'_{x_0+x_1+x_2}$

$$= [(x_0+x_1+x_2)^2 + (x_0+x_1+x_2) + 1]'_{x_0+x_1+x_2} = 2(x_0+x_1+x_2) + 1$$

Thus, $\acute{f}(X) = 2x_0 + 1 + P_1[2(x_0+x_1) + 1 - 2x_0 - 1] + P_2[2(x_0+x_1+x_2) + 1 - 2(x_0+x_1) - 1] = (2x_0+1) + 2x_1P_1 + 2x_2P_2 = 2X + 1$

Example.

Find the value of $\int_0^{1+P_1+P_2} e^X dX$ in two different ways.

Solution;

The regular way: $\int_0^{1+P_1+P_2} e^X dX = [e^X]_0^{1+P_1+P_2} = e^{1+P_1+P_2} - e^0 = e^{1+P_1+P_2} - 1.$

The canonical formula way:

$I^{-1} \circ I\big(f(X)\big) = e^{x_0} + P_1[e^{x_0+x_1} - e^{x_0}] + P_2[e^{x_0+x_1+x_2} - e^{x_0+x_1}]$

We have:

$$\int_0^1 e^{x_0} dx_0 = e - 1, \int_0^2 e^{e^{x_0+x_1}} d(x_0+x_1) = e^2 - 1, \int_0^3 e^{x_0+x_1+x_2} d(x_0+x_1+x_2) = e^3 - 1$$

Thus, $\int_0^{1+P_1+P_2} e^X dX = e - 1 + P_1[e^2 - 1 - e + 1] + P_2[e^3 - 1 - e^2 + 1] = e - 1 + P_1[e^2 - e] + P_2[e^3 - e^2] = e^{1+P_1+P_2} - 1$

Applications to differential equations.

Example.

Solve the equation $\acute{Y} = C; Y = y_0 + y_1P_1 + y_2P_2$ is a function, and $C = c_0 + c_1P_1 + c_2P_2$ is a constant.

We have $y_0 = f_0, y_1 = f_1, y_2 = f_2 : R \to R.$

$$\acute{Y} = (y_0)'_{x_0} + P_1\left[(y_0 + y_1)'_{x_0+x_1} - (y_0)'_{x_0}\right] +$$

$$P_2\left[(y_0 + y_1 + y_2)'_{x_0+x_1+x_2} - (y_0 + y_1)'_{x_0+x_1}\right] = c_0 + c_1P_1 + c_2P_2,$$

so that:

$$\begin{cases} (y_0)'_{x_0} = c_0 \\ (y_0 + y_1)'_{x_0+x_1} = c_1 \\ (y_0 + y_1 + y_2)'_{x_0+x_1+x_2} = c_2 \end{cases} \Longrightarrow \begin{cases} y_0 = c_0x_0 + m_0 \\ y_0 + y_1 = c_1(x_0 + x_1) + m_1 \\ y_0 + y_1 + y_2 = c_2(x_0 + x_1 + x_2) + m_2 \end{cases}$$

This implies that:

$$Y = (c_0x_0 + m_0) + P_1[c_1(x_0 + x_1) + m_1 - (c_0x_0 + m_0)] +$$

$P_2[c_2(x_0 + x_1 + x_2) + m_2 - c_1(x_0 + x_1) - m_1]$; x_i are real variables, m_i are real constants.

Example.

Solve the differential equation $\acute{Y} = CY$, where $C = c_0 + c_1P_1 + c_2P_2, Y = y_0 + y_1P_1 + y_2P_2$.

Solution.

$\acute{Y} = CY$ equivalents:

$$\begin{cases} (y_0)'_{x_0} = c_0y_0 \\ (y_0 + y_1)'_{x_0+x_1} = (c_0 + c_1)(y_0 + y_1) \\ (y_0 + y_1 + y_2)'_{x_0+x_1+x_2} = (c_0 + c_1 + c_2)(y_0 + y_1 + y_2) \end{cases}$$

So that:

$$\begin{cases} y_0 = k_0e^{c_0x_0} \\ y_0 + y_1 = k_1e^{(c_0+c_1)(x_0+x_1)} \\ y_0 + y_1 + y_2 = k_2e^{(c_0+c_1+c_2)(x_0+x_1+x_2)} \end{cases}$$

Thus: $Y = k_0e^{c_0x_0} + P_1\left[k_1e^{(c_0+c_1)(x_0+x_1)} - k_0e^{c_0x_0}\right] + P_2\left[k_2e^{(c_0+c_1+c_2)(x_0+x_1+x_2)} - k_1e^{(c_0+c_1)(x_0+x_1)}\right]$.

Example.

Solve the differential equation $Y'' = C$, where $C = c_0 + c_1P_1 + c_2P_2, Y = y_0 + y_1P_1 + y_2P_2$.

Solution.

$Y'' = C$ equivalents:

$$\begin{cases} (Y_{x_0})'' = c_0 \\ (y_0+y_1)_{x_0+x_1}'' = c_1 \\ (y_0+y_1+y_2)_{x_0+x_1+x_2}'' = c_2 \end{cases}$$

So that:

$$\begin{cases} y_0 = \frac{c_0}{2}{x_0}^2 + m_0x_0 + n_0; m_0, n_0 \in R \\ y_0 + y_1 = \frac{c_1}{2}(x_0+x_1)^2 + m_1(x_0+x_1) + n_1; m_1, n_1 \in R \\ y_0+y_1+y_2 = \frac{c_2}{2}(x_0+x_1+x_2)^2 + m_2(x_0+x_1+x_2) + n_2; m_2, n_2 \in R \end{cases}$$

And

$$Y = \left(\frac{c_0}{2}{x_0}^2 + m_0x_0 + n_0\right) +$$

$$P_1\left[\frac{c_1}{2}(x_0+x_1)^2 + m_1(x_0+x_1) + n_1 - \frac{c_0}{2}{x_0}^2 - m_0x_0 - n_0\right] +$$

$$P_2\left[\frac{c_2}{2}(x_0+x_1+x_2)^2 + m_2(x_0+x_1+x_2) + n_2 - \frac{c_1}{2}(x_0+x_1)^2 - m_1(x_0+x_1) - n_1\right].$$

Applications to geometric shapes:

Definition.

1). We define the symbolic 2-plithogenic circle as follows:

$(X-A)^2 + (Y-B)^2 = R^2; A = a_0 + a_1P_1 + a_2P_2, B = b_0 + b_1P_1 + b_2P_2, R = r_0 + r_1P_1 + r_2P_2, Y = y_0 + y_1P_1 + y_2P_2, x = x_0 + x_1P_1 + x_2P_2,$ with $a_i, b_i, r_i, x_i, y_i \in R$

2). We define the symbolic 2-plithogenic sphere as follows:

$(X-A)^2 + (Y-B)^2 + (Z-C)^2 = R^2; X, A, B, C, R, Y, Z \in 2-SP_R$

3). We define the symbolic 2-plithogenic ellipse as follows:

$\frac{(X-A)^2}{T^2} + \frac{(Y-B)^2}{S^2} = 1; X, A, B, T, S, Y \in 2-SP_R$ and T, S invertible.

4). We define the symbolic 2-plithogenic hyperbola as follows:

$\frac{(X-A)^2}{T^2} - \frac{(Y-B)^2}{S^2} = 1; X, A, B, T, S, Y \in 2-SP_R$ and T, S invertible.

Example.

1). $(X-1-P_1+P_2)^2 + (Y-3+2P_1-P_2)^2 = (1+P_2)^2$ is a 2-plithogenic circle.

2). $(X-10+P_1)^2 + (Y+P_2)^2 + (Z-P_1+P_2)^2 = (1+P_1+13P_2)^2$ is a 2-plithogenic sphere.

3). $\frac{(X-P_1)^2}{(1+P_1+P_2)^2} + \frac{(Y+P_1-P_2)^2}{(2-P_1+5P_2)^2} = 1$ is a 2-plithogenic ellipse.

4). $\frac{(X-2P_2)^2}{(1+P_1+3P_2)^2}-\frac{(Y+1+4P_2)^2}{(13-2P_1+P_2)^2}=1$ is a 2-plithogenic hyperbola.

Theorem.

1. Any symbolic 2-plithogenic circle is equivalent to three classical circles.
2. Any symbolic 2-plithogenic sphare is equivalent to three classical spheres.
3. Any symbolic 2-plithogenic ellipse is equivalent to three classical ellipses.
4. Any symbolic 2-plithogenic hyperbola is equivalent to three classical hyperbolas.

Proof.

1. Consider the symbolic 2-plithogenic circle:

$(X-A)^2+(Y-B)^2=R^2$, then by using the isomprphism defined before, we get:

$$I[(X-A)^2]=[I(X)-I(A)]^2$$
$$=\left(x_0-a_0,(x_0+x_1)-(a_0+a_1),(x_0+x_1+x_2)-(a_0+a_1+a_2)\right)^2$$
$$=\left((x_0-a_0)^2,\left((x_0+x_1)-(a_0+a_1)\right)^2,\left((x_0+x_1+x_2)-(a_0+a_1+a_2)\right)^2\right)$$

$$I[(Y-B)^2]=[I(Y)-I(B)]^2$$
$$=\left((y_0-b_0)^2,\left((y_0+y_1)-(b_0+b_1)\right)^2,\left((y_0+y_1+y_2)-(b_0+b_1+b_2)\right)^2\right)$$

$$I(R^2)=[I(R)]^2=({r_0}^2,(r_0+r_1)^2,(r_0+r_1+r_2)^2)$$

Thus, it is equivalent to:

$$\begin{cases}(x_0-a_0)^2+(y_0-b_0)^2={r_0}^2\\ \left((x_0+x_1)-(a_0+a_1)\right)^2+\left((y_0+y_1)-(b_0+b_1)\right)^2=(r_0+r_1)^2\\ \left((x_0+x_1+x_2)-(a_0+a_1+a_2)\right)^2+\left((y_0+y_1+y_2)-(b_0+b_1+b_2)\right)^2=(r_0+r_1+r_2)^2\end{cases}$$

2. Consider the sphere $(X-A)^2+(Y-B)^2+(Z-C)^2=R^2$, we use the isomorphism I, to get:

$$I[(X-A)^2]=\left((x_0-a_0)^2,\left((x_0+x_1)-(a_0+a_1)\right)^2,\left((x_0+x_1+x_2)-(a_0+a_1+a_2)\right)^2\right)$$

$$I[(Y-B)^2]=\left((y_0-b_0)^2,\left((y_0+y_1)-(b_0+b_1)\right)^2,\left((y_0+y_1+y_2)-(b_0+b_1+b_2)\right)^2\right)$$

$$I[(Z-C)^2]=\left((z_0-c_0)^2,\left((z_0+z_1)-(c_0+c_1)\right)^2,\left((z_0+z_1+z_2)\right.\right.$$
$$\left.\left.-(c_0+c_1+c_2)\right)^2\right)$$

$I(R^2)=[I(R)]^2=({r_0}^2,(r_0+r_1)^2,(r_0+r_1+r_2)^2)$, hence we get:

$$\begin{cases} (x_0-a_0)^2+(y_0-b_0)^2+(z_0-c_0)^2={r_0}^2 \\ \left((x_0+x_1)-(a_0+a_1)\right)^2+\left((y_0+y_1)-(b_0+b_1)\right)^2+\left((z_0+z_1)-(c_0+c_1)\right)^2=(\\ \left((x_0+x_1+x_2)-(a_0+a_1+a_2)\right)^2+\left((y_0+y_1+y_2)-(b_0+b_1+b_2)\right)^2+\left((z_0+z_1+z_2)-(c_0+c \right. \end{cases}$$

3. Consider the ellipse $\frac{(X-A)^2}{T^2}+\frac{(Y-B)^2}{S^2}=1$, we use the isomorphism I to get:

$$I\left[\frac{(X-A)^2}{T^2}\right]$$
$$=\left(\frac{(x_0-a_0)^2}{{t_0}^2},\frac{\left((x_0+x_1)-(a_0+a_1)\right)^2}{(t_0+t_1)^2},\frac{\left((x_0+x_1+x_2)-(a_0+a_1+a_2)\right)^2}{(t_0+t_1+t_2)^2}\right)$$
$$I\left[\frac{(Y-B)^2}{S^2}\right]$$
$$=\left(\frac{(y_0-b_0)^2}{{s_0}^2},\frac{\left((y_0+y_1)-(b_0+b_1)\right)^2}{(s_0+s_1)^2},\frac{\left((y_0+y_1+y_2)-(b_0+b_1+b_2)\right)^2}{(s_0+s_1+s_2)^2}\right)$$

$I(1)=(1,1,1)$, thus:

$$\begin{cases} \frac{(x_0-a_0)^2}{{t_0}^2}+\frac{(y_0-b_0)^2}{{s_0}^2}=1 \\ \frac{\left((x_0+x_1)-(a_0+a_1)\right)^2}{(t_0+t_1)^2}+\frac{\left((y_0+y_1)-(b_0+b_1)\right)^2}{(s_0+s_1)^2}=1 \\ \frac{\left((x_0+x_1+x_2)-(a_0+a_1+a_2)\right)^2}{(t_0+t_1+t_2)^2}+\frac{\left((y_0+y_1+y_2)-(b_0+b_1+b_2)\right)^2}{(s_0+s_1+s_2)^2}=1 \end{cases}$$

4. Consider the hyperbola $\frac{(X-A)^2}{T^2}-\frac{(Y-B)^2}{S^2}=1$, by a similar discussion, we get

$$\begin{cases} \frac{(x_0-a_0)^2}{{t_0}^2}-\frac{(y_0-b_0)^2}{{s_0}^2}=1 \\ \frac{\left((x_0+x_1)-(a_0+a_1)\right)^2}{(t_0+t_1)^2}-\frac{\left((y_0+y_1)-(b_0+b_1)\right)^2}{(s_0+s_1)^2}=1 \\ \frac{\left((x_0+x_1+x_2)-(a_0+a_1+a_2)\right)^2}{(t_0+t_1+t_2)^2}-\frac{\left((y_0+y_1+y_2)-(b_0+b_1+b_2)\right)^2}{(s_0+s_1+s_2)^2}=1 \end{cases}$$

Example.

Consider the symbolic 2-plithogenic circle:

$(X-1+P_1)^2+(Y-3+2P_1+2P_2)^2=(1+4P_1+4P_2)^2$, it is equivalent to:

$$\begin{cases} (x_0-1)^2+(y_0-3)^2=1 \\ \left((x_0+x_1)\right)^2+\left((y_0+y_1)-1\right)^2=25 \\ \left((x_0+x_1+x_2)\right)^2+\left((y_0+y_1+y_2)\right)^2=81 \end{cases}$$

The first circle has $(1,3)$ as entre and radius 1.

The second circle has $(0,1)$ as entre and radius 5.

The third circle has $(0,-1)$ as entre and radius 9.

Example:

Consider the symbolic 2-plithogenic sphere:

$(X-1+P_1+3P_2)^2+(Y-P_2)^2+(Z-4+P_2)^2=(3-P_1+P_2)^2$, it is equivalent to:

$$\begin{cases} (x_0-2)^2+(y_0)^2+(z_0-4)^2=9 \\ \left((x_0+x_1)-1\right)^2+\left((y_0+y_1)\right)^2+\left((z_0+z_1)-1\right)^2=4 \\ \left((x_0+x_1+x_2)+4\right)^2+\left((y_0+y_1+y_2)-1\right)^2+\left((z_0+z_1+z_2)-3\right)^2=9 \end{cases}$$

The first sphere has $(2,0,4)$ as entre and radius 3.

The second sphere has $(1,0,4)$ as entre and radius 2.

The third sphere has $(-4,1,3)$ as entre and radius 3.

Example:

Consider the symbolic 2-plithogenic ellipse:

$\frac{(X-1+P_1+3P_2)^2}{(3+P_1+P_2)^2}+\frac{(Y-1-P_2)^2}{(4+2P_1+3P_2)^2}=1$, it is equivalent to:

$$\begin{cases} \frac{(x_0)^2}{3^2}+\frac{(y_0-1)^2}{4^2}=1 \\ \frac{\left((x_0+x_1)-4\right)^2}{4^2}+\frac{\left((y_0+y_1)-1\right)^2}{6^2}=1 \\ \frac{\left((x_0+x_1+x_2)-3\right)^2}{5^2}+\frac{\left((y_0+y_1+y_2)-2\right)^2}{9^2}=1 \end{cases}$$

Example:

Consider the symbolic 2-plithogenic hyperbola:

$\frac{(X-1)^2}{(5+2P_1)^2}-\frac{(Y-P_1)^2}{(1+P_1+P_2)^2}=1$, it is equivalent to:

$$\begin{cases} \frac{(x_0-1)^2}{5^2}-\frac{(y_0)^2}{1^2}=1 \\ \frac{\left((x_0+x_1)-1\right)^2}{7^2}-\frac{\left((y_0+y_1)-1\right)^2}{2^2}=1 \\ \frac{\left((x_0+x_1+x_2)-1\right)^2}{7^2}-\frac{\left((y_0+y_1+y_2)-1\right)^2}{3^2}=1 \end{cases}$$

Example.

Let us the parametric representation of the symbolic 2-plithogenic ellipse:

$$\frac{(X-1-P_1-P_2)^2}{(2+P_1)^2}+\frac{(Y-2-3P_1)^2}{(1+2P_1-P_2)^2}=1$$

The previous ellipse is equivalent to:

$$\begin{cases} \frac{(x_0-1)^2}{2^2}+\frac{(y_0-2)^2}{1^2}=1 \ldots (1) \\ \frac{\left((x_0+x_1)-2\right)^2}{3^2}+\frac{\left((y_0+y_1)-5\right)^2}{3^2}=1 \ldots (2) \\ \frac{\left((x_0+x_1+x_2)-3\right)^2}{3^2}+\frac{\left((y_0+y_1+y_2)-5\right)^2}{2^2}=1 \ldots (3) \end{cases}$$

Equation (1) implies $\frac{x_0-1}{2}=cos\theta_0$, $\frac{y_0-2}{1}=sin\theta_0$, hence $x_0=2cos\theta_0+1, y_0=sin\theta_0+2$.

Equation (2) implies $\frac{(x_0+x_1)-2}{3}=cos\theta_1, \frac{(y_0+y_1)-5}{2}=sin\theta_1$, hence $x_1=3cos\theta_1+2-x_0=3cos\theta_1-2cos\theta_0+1, y_1=2sin\theta_1+5-y_0=2sin\theta_1-sin\theta_0+3$

Equation (3) implies $\frac{(x_0+x_1+x_2)-3}{3}=cos\theta_2$, $\frac{(y_0+y_1+y_2)-5}{2}=sin\theta_2$, hence $x_2=3cos\theta_2-(x_0+x_1)+3=3cos\theta_2-3cos\theta_1+1, y_2=2sin\theta_2-(y_0+y_1)+5=2sin\theta_2-2sin\theta_1$

This means that:

$X=(2cos\theta_0+1)+P_1[3cos\theta_1-2cos\theta_0+1]+P_2[3cos\theta_2-3cos\theta_1+1]$

$Y=(sin\theta_0+2)+P_1[2sin\theta_1-sin\theta_0+3]+P_2[2sin\theta_2-2sin\theta_1]$.

Example.

Consider the symbolic 2-plithogenic ellipse:

$$\frac{(X-2-4P_1+3P_2)^2}{(1+5P_1+7P_2)^2}+\frac{(Y+1+P_2)^2}{(3-P_1-P_2)^2}=1$$

it is equivalent to:

$$\begin{cases} \frac{(x_0-2)^2}{1}+\frac{(y_0+1)^2}{3^2}=1 \\ \frac{[(x_0+x_1)-6]^2}{6^2}+\frac{[(y_0+y_1)+1]^2}{2^2}=1 \\ \frac{[(x_0+x_1+x_2)-3]^2}{13^2}+\frac{[(y_0+y_1+y_2)+2]^2}{1}=1 \end{cases}$$

Example.

Consider the symbolic 2-plithogenic circle:

$(X-2-4P_1+3P_2)^2+(Y+1+P_2)^2=1$

It is equivalent to:

$$\begin{cases} (x_0-2)^2+(y_0+1)^2=1 \\ [(x_0+x_1)-6]^2+[(y_0+y_1)+1]^2=1 \\ [(x_0+x_1+x_2)-3]^2+[(y_0+y_1+y_2)+2]^2=1 \end{cases}$$

Example.

Consider the symbolic 2-plithogenic hyperbola:

$$\frac{(X-2-4P_1+3P_2)^2}{(1+5P_1+7P_2)^2}-\frac{(Y+1+P_2)^2}{(3-P_1-P_2)^2}=1$$

it is equivalent to:

$$\begin{cases} \frac{(x_0-2)^2}{1}-\frac{(y_0+1)^2}{3^2}=1 \\ \frac{[(x_0+x_1)-6]^2}{6^2}-\frac{[(y_0+y_1)+1]^2}{2^2}=1 \\ \frac{[(x_0+x_1+x_2)-3]^2}{13^2}-\frac{[(y_0+y_1+y_2)+2]^2}{1}=1 \end{cases}$$

Conclusion

In this paper, we defined for the first time a special AH-isometry on the symbolic 2-plithogenic fields of reals, and we used this isometry to obtain many formulas and properties about the symbolic 2-plithogenic analytical concepts such as differentiability, continuity, and integrability. Also, symbolic 2-plithogenic conic sections were handled by using the mentioned isometry. In addition, many related examples were presented to clarify the novelty of our work.

References

1. Nader Mahmoud Taffach , Ahmed Hatip., "A Review on Symbolic 2-Plithogenic Algebraic Structures " Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.
2. Nader Mahmoud Taffach , Ahmed Hatip.," A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations ", Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.
3. Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.
4. Smarandache, F., " Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)", Neutrosophic Sets and Systems, vol. 53, 2023.
5. Taffach, N., " An Introduction to Symbolic 2-Plithogenic Vector Spaces Generated from The Fusion of Symbolic Plithogenic Sets and Vector Spaces", Neutrosophic Sets and Systems, Vol 54, 2023.
6. Taffach, N., and Ben Othman, K., " An Introduction to Symbolic 2-Plithogenic Modules Over Symbolic 2-Plithogenic Rings", Neutrosophic Sets and Systems, Vol 54, 2023.
7. Merkepci, H., and Rawashdeh, A., " On The Symbolic 2-Plithogenic Number Theory and Integers ", Neutrosophic Sets and Systems, Vol 54, 2023.
8. Albasheer, O., Hajjari., A., and Dalla., R., " On The Symbolic 3-Plithogenic Rings and TheirAlgebraic Properties", Neutrosophic Sets and Systems, Vol 54, 2023.
9. Abobala, M., On Refined Neutrosophic Matrices and Their Applications In Refined Neutrosophic Algebraic Equations, Journal Of Mathematics, Hindawi, 2021
10. Olgun, N., Hatip, A., Bal, M., and Abobala, M., " A Novel Approach To Necessary and Sufficient Conditions For The Diagonalization of Refined

Neutrosophic Matrices", International Journal of neutrosophic Science, Vol. 16, pp. 72-79, 2021.

11. Merkepci, H., and Abobala, M., " The Application of AH-isometry In The Study Of Neutrosophic Conic Sections", Galoitica Journal Of Mathematical Structures And Applications, Vol.2, 2022.
12. Abobala, M., and Zeina, M.B., " A Study Of Neutrosophic Real Analysis By Using One Dimensional Geometric AH-Isometry", Galoitica Journal Of Mathematical Structures And Applications, Vol.3, 2023.
13. Khaldi, A**.,** " A Study On Split-Complex Vector Spaces"**,** Neoma Journal Of Mathematics and Computer Science, 2023.
14. Ahmad, K., " On Some Split-Complex Diophantine Equations", Neoma Journal Of Mathematics and Computer Science, 2023.
15. Ali, R., " On The Weak Fuzzy Complex Inner Products On Weak Fuzzy Complex Vector Spaces"**,** Neoma Journal Of Mathematics and Computer Science, 2023.
16. Von Shtawzen, O., " Conjectures For Invertible Diophantine Equations Of 3-Cyclic and 4-Cyclic Refined Integers", Journal Of Neutrosophic And Fuzzy Systems, Vol.3, 2022.
17. Ben Othman, K., Albasheer, O., Nadweh, R., Von Shtawzen, O., and Ali, R., "On The Symbolic 6-Plithogenic and 7-Plithogenic Rings", Galoitica Journal of Mathematical Structures and Applications, Vol.8, 2023.
18. Bisher Ziena, M., and Abobala, M., " On The Refined Neutrosophic Real Analysis Based on Refined Neutrosophic Algebraic AH-Isometry", Neutrosophic Sets and Systems, vol. 54, 2023.
19. M. B. Zeina and M. Abobala, "A Novel Approach of Neutrosophic Continuous Probability Distributions using AH-Isometry with Applications

in Medicine," in *Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics*, Elsevier, 2023.

20. M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, "Introduction to Symbolic 2-Plithogenic Probability Theory," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.
21. Ali, R., and Hasan, Z., "An Introduction To The Symbolic 3-Plithogenic Vector Spaces", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.
22. Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.
23. Ben Othman, K., "On Some Algorithms For Solving Symbolic 3-Plithogenic Equations", Neoma Journal Of Mathematics and Computer Science, 2023.
24. Florentin Smarandache, Plithogenic Algebraic Structures. Chapter in "Nidus idearum Scilogs, V: joining the dots" (third version), Pons Publishing Brussels, pp. 123-125, 2019.
25. Florentin Smarandache: Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets – Revisited, *Neutrosophic Sets and Systems*, vol. 21, 2018, pp. 153-166.
26. Sarkis, M., " On The Solutions Of Fermat's Diophantine Equation In 3-refined Neutrosophic Ring of Integers", Neoma Journal of Mathematics and Computer Science, 2023.
27. Abobala, M., "On Some Algebraic Properties of n-Refined Neutrosophic Elements and n-Refined Neutrosophic Linear Equations", Mathematical Problems in Engineering, Hindawi, 2021
28. Abobala, M., "A Study Of Nil Ideals and Kothe's Conjecture In Neutrosophic Rings", International Journal of Mathematics and Mathematical Sciences, hindawi, 2021

29. Abobala, M., "On The Characterization of Maximal and Minimal Ideals In Several Neutrosophic Rings", Neutrosophic sets and systems, Vol. 45, 2021.

30. Abobala, M., Bal, M., Aswad, M., "A Short Note On Some Novel Applications of Semi Module Homomorphisms", International journal of neutrosophic science, 2022.

31. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Concept Of Symbolic 7-Plithogenic Real Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

32. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Symbolic 8-Plithogenic Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

33. Abualkishik, A., Almajed, R., Thompson, W., "Improving The Performance of Fog-assisted Internet of Things Networks Using Bipolar Trapezoidal Neutrosophic Sets", Journal of Wireless and Ad Hoc Communication, 2023.

Received 10/4/2023, Accepted 7/10/2023

Neutrosophic Sets and Systems, Vol. 59, 2023
University of New Mexico

On The Orthogonality in Real Symbolic 2-Plithogenic and 3-Plithogenic Vector Spaces

[1.]Ahmed Hatip, [2.] Mohammad Alsheikh, [3.] Iyad Alhamadeh

[1.]Department of Mathematics, Gaziantep University, Gaziantep, Turkey

[2.]Faculty of Science, Department of Mathematics, Idlib university, Syria

[3.]Department of Mathematics and Science Education, Faculty of Education in Afrin, Gaziantep university, Turkey

Co: kollnaar5@gmail.com

Abstract:

This paper is dedicated to study the real inner product defined over symbolic 2-plithogenic vector spaces, where we discuss the concept of inner (scalar) products over the symbolic 2-plithogenic vector spaces by using the corresponding Euclidean scalar products to get theorems that describe the conditions of orthogonality in this class of spaces. Also, we give many examples to explain the ortho-normed symbolic 2-plithogenic spaces.

Key words: Symbolic 2-plithogenic vector space, inner product, orthogonal basis, ortho-normed basis.

Introduction

Symbolic 2-plithogenic vector spaces and modules were defined in [1-5] as a new generalization of classical vector spaces, and with a similar algebraic structure of refined neutrosophic vector spaces .

Many algebraic properties of these spaces such as basis, and semi homomorphic images were studied on a wide range.

zenodo.10031182/10.5281

Also, we can see symbolic 3-plithogenic vector spaces/modules defined over 3-plithogenic rings. Symbolic plithogenic algebraic structures are generally very rich in their concepts and meta-properties, see [5-8 ,23-28].

In this work, we will study the concept of real inner product over symbolic 2-plithogenic vector spaces, where we use these inner products to study the orthogonality between symbolic 2-plithogenic vectors and ortho-normed basis.

This work is motivated by the previous published works in [9-22] that study neutrosophic vector spaces, matrices and their refined neutrosophic extensions.

For basic definitions about symbolic 2-plithogenic vector spaces, check [1].

Main Discussion

Definition:

Let V be a vector space over the field R.

Let $2-SP_R=\{l_0+l_1P_1+l_2P_2; l_i\in R\}$ be the corresponding symbolic 2-plithogenic field, $2-SP_V=\{q_0+q_1P_1+q_2P_2; q_i\in V\}$ the corresponding symbolic 2-plithogenic vector space, then:

$\varphi: 2-SP_V\times 2-SP_V\to 2-SP_R$ is called a symbolic 2-plithogenic real inner product if and only if:

1). $\varphi(w,n)=\varphi(n,w)$ for all $n,w\in 2-SP_V$.

2). $\varphi(w,w)\geq 0$ for all $w\in 2-SP_V$ (with respect to the corresponding partial order relation defined on $2-SP_R$).

3). $\varphi(u+v,w)=\varphi(u,w)+\varphi(v,w)$.

4). $\varphi(a.w,v)=a\varphi(w,v); w,v\in 2-SP_V, a\in 2-SP_R$.

Theorem.

If there exists a classical inner product $f: V\times V\to R$ such that:

For $w=w_0+w_1P_1+w_2P_2, n=n_0+n_1P_1+n_2P_2\in 2-SP_V$, we define:

$\varphi(w,n)=f(w_0,n_0)+P_1[f(w_0+w_1,n_0+n_1)-f(w_0,n_0)]+P_2[f(w_0+w_1+w_2,n_0+n_1+n_2)-f(w_0+w_1,n_0+n_1)]$.

Then, $\varphi: 2-SP_V \times 2-SP_V \to 2-SP_R$ is a symbolic 2-plithogenic real inner product.

Proof.

Assume that $f: V \times V \to R$ is a real inner product defined on V.

We put $\varphi: 2-SP_V \times 2-SP_V \to 2-SP_R$, where:

$\varphi(w,n) = f(w_0,n_0) + P_1[f(w_0+w_1,n_0+n_1) - f(w_0,n_0)] + P_2[f(w_0+w_1+w_2,n_0+n_1+n_2) - f(w_0+w_1,n_0+n_1)]$.

We must prove that φ is a symbolic 2-plithogenic real inner product on $2-SP_V$.

$\varphi(w,n) = f(w_0,n_0) + P_1[f(w_0+w_1,n_0+n_1) - f(w_0,n_0)] + P_2[f(w_0+w_1+w_2,n_0+n_1+n_2) - f(w_0+w_1,n_0+n_1)] = f(n_0,w_0) + P_1[f(n_0+n_1,w_0+w_1) - f(n_0,w_0)] + P_2[f(n_0+n_1+n_2,w_0+w_1+w_2) - f(n_0+n_1,w_0+w_1)] = \varphi(n,w)$.

$\varphi(w,w) = f(w_0,w_0) + P_1[f(w_0+w_1,w_0+w_1) - f(w_0,w_0)] + P_2[f(w_0+w_1+w_2,w_0+w_1+w_2) - f(w_0+w_1,w_0+w_1)] = \|w_0\|^2 + P_1[\|w_0+w_1\|^2 - \|w_0\|^2] + P_2[\|w_0+w_1+w_2\|^2 - \|w_0+w_1\|^2]$.

And that is because:

$$\begin{cases} \|w_0\|^2 \geq 0 \\ \|w_0\|^2 + [\|w_0+w_1\|^2 - \|w_0\|^2] = \|w_0+w_1\|^2 \geq 0 \\ \|w_0\|^2 + [\|w_0+w_1\|^2 - \|w_0\|^2] + [\|w_0+w_1+w_2\|^2 - \|w_0+w_1\|^2] = \|w_0+w_1+w_2\|^2 \geq 0 \end{cases}$$

Now, we let $u = u_0 + u_1P_1 + u_2P_2 \in 2-SP_V$, we have:

$$\begin{aligned} \varphi(u+w,n) = {} & f(u_0+w_0,n_0) + P_1[f(u_0+u_1+w_0+w_1,n_0+n_1) - f(u_0+w_0,n_0)] \\ & + P_2[f(u_0+u_1+u_2+w_0+w_1+w_2,n_0+n_1+n_2) \\ & - f(u_0+u_1+w_0+w_1,n_0+n_1)] = \varphi(u,n) + \varphi(w,n) \end{aligned}$$

Let $a = a_0 + a_1P_1 + a_2P_2 \in 2-SP_R$, then:

$\varphi(a.w,n) = f(a_0w_0,n_0) + P_1\big[f\big((a_0+a_1+a_2)(u_0+u_1+w_0+w_1),n_0+n_1\big) - f(a_0w_0,n_0)\big] + P_2\big[f\big((a_0+a_1)(u_0+u_1+u_2+w_0+w_1+w_2),n_0+n_1+n_2\big) - f\big((a_0+a_1+a_2)(u_0+u_1+w_0+w_1),n_0+n_1\big)\big] = (a_0+a_1P_1+a_2P_2)\varphi(w,n) = a.\varphi(w,n)$.

So that, φ is a symbolic 2-plithogenic inner product.

Remark:

Suppose that $\varphi: 2-SP_V \times 2-SP_V \to 2-SP_R$ is a symbolic 2-plithogenic inner product.

Put $f: V \times V \to R$, with $f(w_0, n_0) = \varphi(w_0 + 0.P_1 + 0.P_2, n_0 + 0.P_1 + 0.P_2)$, where $w_0, n_0 \in V$.

f is a classical real inner product, that is because:

$f(w_0, n_0) = \varphi(w_0, n_0) = \varphi(n_0, w_0) = f(n_0, w_0)$

$f(w_0, w_0) = \varphi(w_0, w_0) = \|w_0\|^2 \geq 0$

$f(u_0 + w_0, n_0) = \varphi(u_0 + w_0, n_0) = \varphi(u_0, n_0) + \varphi(w_0, n_0) = f(u_0, n_0) + f(w_0, n_0)$

$f(a_0 w_0, n_0) = \varphi(a_0 w_0, n_0) = a_0 \varphi(w_0, n_0) = a_0 f(w_0, n_0); w_0 \in V, a_0 \in R$

Example.

Consider the Euclidean real inner product defined on $V = R^2$ with:

$f(X_1, X_2) = f[(x_1', x_1''), (x_2', x_2'')] = x_1' x_2' + x_1'' x_2''$.

The corresponding symbolic 2-plithogenic inner product defined on $2-SP_V$ is:

$\varphi: 2-SP_V \times 2-SP_V \to 2-SP_R$;

$$\begin{aligned}
&\varphi[(x_0, y_0) + (x_1, y_1)P_1 + (x_2, y_2)P_2, (z_0, t_0) + (z_1, t_1)P_1 + (z_2, t_2)P_2] \\
&= f\big((x_0, y_0), (z_0, t_0)\big) \\
&+ P_1\big[f\big((x_0 + x_1, y_0 + y_1), (z_0 + z_1, t_0 + t_1)\big) - f\big((x_0, y_0), (z_0, t_0)\big)\big] \\
&+ P_2\big[f\big((x_0 + x_1 + x_2, y_0 + y_1 + y_2), (z_0 + z_1 + z_2, t_0 + t_1 + t_2)\big) \\
&- f\big((x_0 + x_1, y_0 + y_1), (z_0 + z_1, t_0 + t_1)\big)\big] \\
&= x_0 z_0 + y_0 t_0 \\
&+ P_1[(x_0 + x_1)(z_0 + z_1) + (y_0 + y_1)(t_0 + t_1) - x_0 z_0 - y_0 t_0] \\
&+ P_2[(x_0 + x_1 + x_2)(z_0 + z_1 + z_2) + (y_0 + y_1 + y_2)(t_0 + t_1 + t_2) \\
&- (x_0 + x_1)(z_0 + z_1) - (y_0 + y_1)(t_0 + t_1)]
\end{aligned}$$

For example:

Consider $X = (1,1) + (1,2)P_1 + (0,1)P_2, Y = (1,0) + (-1,0)P_1 + (1,1)P_2$, we have:

$$\begin{cases} (x_0, y_0) = (1,1), (z_0, t_0) = (1,0) \\ (x_0 + x_1, y_0 + y_1) = (2,3), (z_0 + z_1, t_0 + t_1) = (0,0) \\ (x_0 + x_1 + x_2, y_0 + y_1 + y_2) = (2,4), (z_0 + z_1 + z_2, t_0 + t_1 + t_2) = (1,1) \end{cases}$$

And:

$$\begin{cases} f\big((x_0,y_0),(z_0,t_0)\big)=1 \\ f\big((x_0+x_1,y_0+y_1),(z_0+z_1,t_0+t_1)\big)=0 \\ f\big((x_0+x_1+x_2,y_0+y_1+y_2),(z_0+z_1+z_2,t_0+t_1+t_2)\big)=6 \end{cases}$$

This implies that:

$\varphi(X,Y)=1+P_1[0-1]+P_2[6-0]=1-P_1+6P_2$

Remark.

The norm of $w=w_0+w_1P_1+w_2P_2$ is defined with respect to φ as follows:

$\|w\|=\sqrt{\varphi(w,w)}=\|w_0\|+P_1[\|w_0+w_1\|-\|w_0\|]+P_2[\|w_0+w_1+w_2\|-\|w_0+w_1\|]$.

In the previous example, we can see:

$\|X\|=\|(1,1)\|+P_1[\|(2,3)\|-\|(1,1)\|]+P_2[\|(2,4)\|-\|(2,3)\|]=\sqrt{2}+P_1\big[\sqrt{13}-\sqrt{2}\big]+P_2\big[\sqrt{20}-\sqrt{13}\big]$.

Theorem.

Let φ be a symbolic 2-plithogenic inner product on $2-SP_V$, then:

1). $\|w\|\geq 0; w\in 2-SP_V$

2). $\|a.w\|=|a|.\|w\|; a\in 2-SP_R$

3). $\|w+n\|\leq\|w\|+\|n\|; n\in 2-SP_V$

4). $|\varphi(w,n)|\leq\|w\|.\|n\|$

5). $w\perp n$ if and only if $w_0\perp n_0, w_0+w_1\perp n_0+n_1, w_0+w_1+w_2\perp n_0+n_1+n_2$

6). If $w\perp n$, then $\|w+n\|^2=\|w\|^2+\|n\|^2$

Proof.

1). It holds directly from the definition of norm, and from the partial order relation defined on $2-SP_R$.

2). Let $a=a_0+a_1P_1+a_2P_2\in 2-SP_R, w=w_0+w_1P_1+w_2P_2\in 2-SP_V$, we have:

$\|a.w\|^2=\varphi(a.w,a.w)=a^2\varphi(w,w)$, thus $\|a.w\|=|a|.\|w\|$.

3). $\|w+n\|=\|w_0+n_0\|+P_1[\|w_0+w_1+n_0+n_1\|-\|w_0+n_0\|]+P_2[\|w_0+w_1+w_2+n_0+n_1+n_2\|-\|w_0+w_1+n_0+n_1\|]$, we have:

$$\begin{cases} \|w_0+n_0\|\leq\|w_0\|+\|n_0\| \\ \|w_0+w_1+n_0+n_1\|\leq\|w_0+w_1\|+\|n_0+n_1\| \\ \|w_0+w_1+w_2+n_0+n_1+n_2\|\leq\|w_0+w_1+w_2\|+\|n_0+n_1+n_2\| \end{cases}$$

So that:

$\|w+n\| \le \|w_0\| + \|n_0\| + P_1[\|w_0+w_1\| + \|n_0+n_1\| - \|w_0\| - \|n_0\|] + P_2[\|w_0 + w_1 + w_2\| + \|n_0+n_1+n_2\| - \|w_0+w_1\| - \|n_0+n_1\|] \le \|w\| + \|n\|$.

4). We have:

$|\varphi(w,n)| = |\varphi(w_0,n_0)| + P_1[|\varphi(w_0+w_1, n_0+n_1)| - |\varphi(w_0,n_0)|] + P_2[|\varphi(w_0+w_1+w_2, n_0+n_1+n_2)| - |\varphi(w_0+w_1, n_0+n_1)|]$.

According to Cauchy-Shwartz inequality, we can write:

$$\begin{cases} |f(w_0,n_0)| \le \|w_0\| + \|n_0\| \\ |f(w_0+w_1, n_0+n_1)| \le \|w_0+w_1\| + \|n_0+n_1\| \\ |f(w_0+w_1+w_2, n_0+n_1+n_2)| \le \|w_0+w_1+w_2\| + \|n_0+n_1+n_2\| \end{cases}$$

So that,

$$\begin{aligned} |\varphi(w,n)| \le \|w_0\|.\|n_0\| + P_1[\|w_0+w_1\|.\|n_0+n_1\| - \|w_0\|.\|n_0\|] \\ + P_2[\|w_0+w_1+w_2\|.\|n_0+n_1+n_2\| - \|w_0+w_1\|.\|n_0+n_1\|] \\ = \|w\|.\|n\| \end{aligned}$$

5). $w \perp n$ if and only if $\varphi(w,n) = 0$, which is equivalent to:

$$\begin{cases} f(w_0,n_0) = 0 \Longrightarrow w_0 \perp n_0 \\ f(w_0+w_1, n_0+n_1) \Longrightarrow w_0+w_1 \perp n_0+n_1 \\ f(w_0+w_1+w_2, n_0+n_1+n_2) \Longrightarrow w_0+w_1+w_2 \perp n_0+n_1+n_2 \end{cases}$$

6). Assume that $w \perp n$, then $\|w+n\|^2 = \varphi(w+n, w+n) = \varphi(w,w) + \varphi(n,n) + 2\varphi(w,n) = \|w\|^2 + \|n\|^2$.

Example.

Consider the Euclidean real inner symbolic 2-plithogenic product defined previously on $2 - SP_R$, we have:

$w = (1,0) + (0,1)P_1, u = w = (2,2) + (1,3)P_1 - (1,1)P_2, s = (0,1) + (1,-2)P_1$, we can see:

$\varphi(w,s) = 0, \|w\| = 1 + (\sqrt{2}-1)P_1, \|s\| = 1 + (\sqrt{2}-1)P_1, w+s = (1,1) + (1,-1)P_1$

$\|w+s\|^2 = 2 + 2P_1 = \|w\|^2 + \|s\|^2$

$\varphi(w,u) = 2 + (8-2)P_1 + (6-8)P_2 = 2 + 6P_1 - 2P_2$

$|\varphi(w,u)| = |2| + [|8| - |2|]P_1 + [|6| - |8|]P_2 = 2 + 6P_1 - 2P_2$

$\|u\| = \sqrt{8} + (\sqrt{34} - \sqrt{8})P_1 + (\sqrt{20} - \sqrt{34})P_2$

$$\|w\|.\|u\| = 2\sqrt{8} + \left(2\sqrt{34} - 2\sqrt{8} + 4\sqrt{8} + 2\sqrt{34} - 2\sqrt{8}\right)P_1 + \left(2\sqrt{20} - 2\sqrt{34} + 2\sqrt{20} - 2\sqrt{34}\right)P_2 = 2\sqrt{8} + 4\sqrt{34}P_1 + \left(4\sqrt{20} - 4\sqrt{34}\right)P_2$$

We have $2 \leq 2\sqrt{8}, 2 + 6 = 8 \leq 2\sqrt{8} + 4\sqrt{34}, 2 + 6 - 2 = 8 \leq 2\sqrt{8} + 4\sqrt{34} + 4\sqrt{20} - 4\sqrt{34} = 2\sqrt{8} + 4\sqrt{20}$

Hence $|\varphi(w,u)| \leq \|w\|.\|u\|$.

Symbolic 2-plithogenic orthogonal basis.

Let $N = \{V_1, \dots, V_n\}$ be a basis of symbolic 2-plithogenic vector space $2 - SP_V$, where n is the number of elements in the basis of V.

We say that N is the orthogonal if and only if:

$\varphi(V_i, V_j) = 0; i \neq j, 1 \leq i, j \leq 3n$

It is called ortho-normed if and only if:

$$\begin{cases} \varphi(V_i, V_j) = 0 \\ \varphi(V_i, V_i) = 1 \end{cases}; i \neq j, 1 \leq i, j \leq 3n$$

We will answer the following question:

How can we build a symbolic 2-plithogenic ortho-normed basis of $2 - SP_V$?

Theorem.

Let $\Delta = \{q_1, q_2, \dots, q_n\}$ be an ortho-normed basis of V with respect to the inner product $f: V \times V \to R$.

The set $\Delta_P = \{m_i + (n_j - m_i)P_1 + (s_k - n_j)P_2; m_i, n_j, s_k \in \Delta, 1 \leq i, j, k \leq n\}$ is an ortho-normed basis of $2 - SP_V$.

Proof.

According to [4], the set Δ_P is a basis of $2 - SP_V$.

Let

$M_1 = m_i + (n_j - m_i)P_1 + (s_k - n_j)P_2 \in \Delta, M_2 = \acute{m}_i + (\acute{n}_j - \acute{m}_i)P_1 + (\acute{s}_k - \acute{n}_j)P_2 \in \Delta$,

we have:

$\varphi(M_1, M_2) = f(m_i, \acute{m}_i) + P_1[f(n_j, \acute{n}_j) - f(m_i, \acute{m}_i)] + P_2[f(s_k, \acute{s}_k) - f(n_j, \acute{n}_j)] = 0$,

thus $M_1 \perp M_2$.

On the other hand,

$\|M_1\| = \|m_i\| + P_1[\|n_j\| - \|m_i\|] + P_2[\|s_k\| - \|n_j\|] = 1 + (1-1)P_1 + (1-1)P_2 = 1$,

so that Δ_P is ortho-normed basis.

Example.

Let $V = R^3$ be the Euclidean with three dimensions.

Consider $2 - SP_V = \{(x_0, y_0, z_0) + (x_1, y_1, z_1)P_1 + (x_2, y_2, z_2)P_2; x_i, y_i, z_i \in R\}$ be the corresponding symbolic 2-plithogenic vector space.

It is known that $\Delta = \{q_1 = (1,0,0), q_2 = (0,1,0), q_3 = (0,0,1)\}$ is an ortho-normed of $V = R^3$.

The corresponding ortho-normed basis of $2 - SP_V$ is:

$M_1 = q_1 = (1,0,0)$

$M_2 = q_1 + (q_1 - q_1)P_1 + (q_2 - q_1)P_2 = (1,0,0) + (-1,1,0)P_2$

$M_3 = q_1 + (q_2 - q_1)P_1 + (q_1 - q_2)P_2 = (1,0,0) + (-1,1,0)P_1 + (1,1,0)P_2$

$M_4 = q_1 + (q_2 - q_1)P_1 + (q_2 - q_2)P_2 = (1,0,0) + (-1,1,0)P_1$

$M_5 = q_1 + (q_3 - q_1)P_1 + (q_3 - q_3)P_2 = (-1,0,0) + (-1,0,1)P_1$

$M_6 = q_1 + (q_3 - q_1)P_1 + (q_1 - q_3)P_2 = (1,0,0) + (-1,0,1)P_1 + (1,0,-1)P_2$

$M_7 = q_1 + (q_3 - q_1)P_1 + (q_2 - q_3)P_2 = (1,0,0) + (-1,0,1)P_1 + (0,1,-1)P_2$

$M_8 = q_1 + (q_1 - q_1)P_1 + (q_3 - q_1)P_2 = (1,0,0) + (-1,0,1)P_2$

$M_9 = q_1 + (q_2 - q_1)P_1 + (q_3 - q_2)P_2 = (1,0,0) + (-1,1,0)P_1 + (0,-1,1)P_2$

$M_{10} = q_2 = (0,1,0)$

$M_{11} = q_2 + (q_1 - q_2)P_1 + (q_1 - q_1)P_2 = (0,1,0) + (1,-1,0)P_1$

$M_{12} = q_2 + (q_1 - q_2)P_1 + (q_2 - q_1)P_2 = (0,1,0) + (1,-1,0)P_1 + (-1,1,0)P_2$

$M_{13} = q_2 + (q_1 - q_2)P_1 + (q_3 - q_1)P_2 = (0,1,0) + (1,-1,0)P_1 + (-1,0,1)P_2$

$M_{14} = q_2 + (q_2 - q_2)P_1 + (q_1 - q_2)P_2 = (0,1,0)P_1 + (1,-1,0)P_2$

$M_{15} = q_2 + (q_2 - q_2)P_1 + (q_1 - q_2)P_2 = (0,1,0) + (0,-1,1)P_2$

$M_{16} = q_2 + (q_3 - q_2)P_1 + (q_1 - q_3)P_2 = (0,1,0) + (0,-1,1)P_1 + (1,0,-1)P_2$

$M_{17} = q_2 + (q_3 - q_2)P_1 + (q_2 - q_3)P_2 = (0,1,0) + (0,-1,1)P_1 + (0,1,-1)P_2$

$M_{18} = q_2 + (q_3 - q_2)P_1 + (q_3 - q_3)P_2 = (0,1,0) + (0,-1,1)P_1$

$M_{19} = q_3 = (0,0,1)$

$M_{20} = q_3 + (q_1 - q_3)P_1 + (q_1 - q_1)P_2 = (0,0,1) + (1,0,-1)P_1$

$M_{21} = q_3 + (q_1 - q_3)P_1 + (q_2 - q_1)P_2 = (0,0,1) + (1,0,-1)P_1 + (-1,1,0)P_2$

$M_{22} = q_3 + (q_1 - q_3)P_1 + (q_3 - q_1)P_2 = (0,0,1) + (1,0,-1)P_1 + (-1,0,1)P_2$

$M_{23} = q_3 + (q_2 - q_3)P_1 + (q_1 - q_2)P_2 = (0,0,1) + (0,1,-1)P_1 + (1,-1,0)P_2$

$M_{24} = q_3 + (q_2 - q_3)P_1 + (q_2 - q_2)P_2 = (0,0,1) + (0,1,-1)P_1$

$M_{25} = q_3 + (q_2 - q_3)P_1 + (q_3 - q_2)P_2 = (0,0,1) + (0,1,-1)P_1 + (0,-1,1)P_2$

$M_{26} = q_3 + (q_3 - q_3)P_1 + (q_1 - q_3)P_2 = (0,0,1) + (1,0,-1)P_2$

$M_{27} = q_3 + (q_3 - q_3)P_1 + (q_2 - q_3)P_2 = (0,0,1) + (0,1,-1)P_2$

Example.

Consider the ortho-normed basis of $V = R^2, \Delta = \{q_1 = (1,0), q_2 = (0,1)\}$.

We find the corresponding ortho-normed symbolic 2-plithogenic basis of $2 - SP_V$.

$M_1 = q_1 + (q_1 - q_1)P_1 + (q_1 - q_1)P_2 = (1,0)$

$M_2 = q_1 + (q_1 - q_1)P_1 + (q_2 - q_1)P_2 = (1,0) + (-1,1)P_2$

$M_3 = q_1 + (q_2 - q_1)P_1 + (q_2 - q_2)P_2 = (1,0) + (-1,1)P_1$

$M_4 = q_1 + (q_2 - q_1)P_1 + (q_1 - q_2)P_2 = (1,0) + (-1,1)P_1 + (1,-1)P_2$

$M_5 = q_2 + (q_2 - q_2)P_1 + (q_2 - q_2)P_2 = (0,1)$

$M_6 = q_2 + (q_2 - q_2)P_1 + (q_1 - q_2)P_2 = (0,1) + (1,-1)P_2$

$M_7 = q_2 + (q_1 - q_2)P_1 + (q_1 - q_1)P_2 = (0,1) + (1,-1)P_1$

$M_8 = q_2 + (q_1 - q_2)P_1 + (q_2 - q_1)P_2 = (0,1) + (1,-1)P_1 + (-1,1)P_2$

Remark.

Let $S = \{V_1, \dots, V_{3n}\}$ be the ortho-normed basis of $2 - SP_V$, let $X = X_0 + X_1P_1 + X_2P_2 \in 2 - SP_V$, then:

$X = A_1V_1 + A_2V_2 + \dots + A_{3n}V_{3n}$, we can write:

$\varphi(X, V_1) = A_1\varphi(V_1, V_1) + A_2\varphi(V_2, V_2) + \dots + A_{3n}\varphi(V_{3n}, V_{3n}) = A_i\|V_i\|^2$, thus:

$A_1 = \frac{\varphi(X,V_1)}{\|V_1\|^2} = \varphi(X, V_1), A_2 = \varphi(X, V_2), \dots$.

And so on:

So that, we get the following result:

$X = \varphi(X, V_1)V_1 + \varphi(X, V_2)V_2 + \dots + \varphi(X, V_{3n})V_{3n} = \sum_{i=1}^{3n} \varphi(X, V_i)V_i$.

Symbolic 3-plithogenic inner product

Definition:

Let V be a vector space over the field R.

Let $3-SP_R=\{l_0+l_1P_1+l_2P_2+l_3P_3; l_i \in R\}$ be the corresponding symbolic 3-plithogenic field, $3-SP_V=\{q_0+q_1P_1+q_2P_2+q_3P_3; q_i \in V\}$ the corresponding symbolic 3-plithogenic vector space, then:

$\varphi: 3-SP_V \times 3-SP_V \to 3-SP_R$ is called a symbolic 3-plithogenic real inner product if and only if:

1). $\varphi(w,n)=\varphi(n,w)$ for all $n,w \in 3-SP_V$.

2). $\varphi(w,w) \geq 0$ for all $w \in 3-SP_V$ (with respect to the corresponding partial order relation defined on $3-SP_R$).

3). $\varphi(u+v,w)=\varphi(u,w)+\varphi(v,w)$.

4). $\varphi(a.w,v)=a\varphi(w,v); w,v \in 2-SP_V, a \in 3-SP_R$.

Theorem.

If there exists a classical inner product $f: V \times V \to R$ such that:

For $w=w_0+w_1P_1+w_2P_2+w_3P_3, n=n_0+n_1P_1+n_2P_2+n_3P_3 \in 3-SP_V$, we define:

$\varphi(w,n)=f(w_0,n_0)+P_1[f(w_0+w_1,n_0+n_1)-f(w_0,n_0)]+P_2[f(w_0+w_1+w_2,n_0+n_1+n_2)-f(w_0+w_1,n_0+n_1)]+P_3[f(w_0+w_1+w_2+w_3,n_0+n_1+n_2+n_3)-f(w_0+w_1+w_2,n_0+n_1+n_2)]$.

Then, $\varphi: 3-SP_V \times 3-SP_V \to 3-SP_R$ is a symbolic 3-plithogenic real inner product.

Proof.

Assume that $f: V \times V \to R$ is a real inner product defined on V.

We put $\varphi: 3-SP_V \times 3-SP_V \to 3-SP_R$, where:

$\varphi(w,n)=f(w_0,n_0)+P_1[f(w_0+w_1,n_0+n_1)-f(w_0,n_0)]+P_2[f(w_0+w_1+w_2,n_0+n_1+n_2)-f(w_0+w_1,n_0+n_1)]+P_3[f(w_0+w_1+w_2+w_3,n_0+n_1+n_2+n_3)-f(w_0+w_1+w_2,n_0+n_1+n_2)]$.

We must prove that φ is a symbolic 2-plithogenic real inner product on $3-SP_V$.

$\varphi(w,n)=f(w_0,n_0)+P_1[f(w_0+w_1,n_0+n_1)-f(w_0,n_0)]+P_2[f(w_0+w_1+w_2,n_0+n_1+n_2)-f(w_0+w_1,n_0+n_1)]+P_3[f(w_0+w_1+w_2+w_3,n_0+n_1+n_2+n_3)-f(w_0+w_1+w_2,n_0+n_1+n_2)]=f(n_0,w_0)+P_1[f(n_0+n_1,w_0+w_1)-f(n_0,w_0)]+$

$P_2[f(n_0+n_1+n_2,w_0+w_1+w_2)-f(n_0+n_1,w_0+w_1)]+P_3[f(n_0+n_1+n_2+n_3,w_0+w_1+w_2+w_3)-f(n_0+n_1+n_2,w_0+w_1+w_2)]=\varphi(n,w)$.

$\varphi(w,w)=f(w_0,w_0)+P_1[f(w_0+w_1,w_0+w_1)-f(w_0,w_0)]+P_2[f(w_0+w_1+w_2,w_0+w_1+w_2)-f(w_0+w_1,w_0+w_1)]+P_3[f(w_0+w_1+w_2+w_3,w_0+w_1+w_2+w_3)-f(w_0+w_1+w_2,w_0+w_1+w_2)]=\|w_0\|^2+P_1[\|w_0+w_1\|^2-\|w_0\|^2]+P_2[\|w_0+w_1+w_2\|^2-\|w_0+w_1\|^2]+P_3[\|w_0+w_1+w_2+w_3\|^2-\|w_0+w_1+w_2\|^2]$.

And that is because:

$$\begin{cases} \|w_0\|^2\geq 0 \\ \|w_0\|^2+[\|w_0+w_1\|^2-\|w_0\|^2]=\|w_0+w_1\|^2\geq 0 \\ \|w_0\|^2+[\|w_0+w_1\|^2-\|w_0\|^2]+[\|w_0+w_1+w_2\|^2-\|w_0+w_1\|^2]=\|w_0+w_1+w_2\|^2\geq 0 \\ \|w_0+w_1+w_2+w_3\|^2\geq 0 \end{cases}$$

Now, we let $u=u_0+u_1P_1+u_2P_2+u_3P_3\in 3-SP_V$, we have:

$\varphi(u+w,n)=\varphi(u,n)+\varphi(w,n)$

Let $a=a_0+a_1P_1+a_2P_2+a_3P_3\in 3-SP_R$, then:

$\varphi(a.w,n)=a.\varphi(w,n)$.

So that, φ is a symbolic 3-plithogenic inner product.

Remark.

The norm of $w=w_0+w_1P_1+w_2P_2+w_3P_3$ is defined with respect to φ as follows:

$\|w\|=\sqrt{\varphi(w,w)}=\|w_0\|+P_1[\|w_0+w_1\|-\|w_0\|]+P_2[\|w_0+w_1+w_2\|-\|w_0+w_1\|]+P_3[\|w_0+w_1+w_2+w_3\|-\|w_0+w_1+w_2\|]$.

Theorem.

Let φ be a symbolic 3-plithogenic inner product on $3-SP_V$, then:

1). $\|w\|\geq 0; w\in 3-SP_V$

2). $\|a.w\|=|a|.\|w\|; a\in 3-SP_R$

3). $\|w+n\|\leq\|w\|+\|n\|; n\in 3-SP_V$

4). $|\varphi(w,n)|\leq\|w\|.\|n\|$

5). $w\perp n$ if and only if $w_0\perp n_0, w_0+w_1\perp n_0+n_1, w_0+w_1+w_2\perp n_0+n_1+n_2, w_0+w_1+w_2+w_3\perp n_0+n_1+n_2+n_3$

6). If $w\perp n$, then $\|w+n\|^2=\|w\|^2+\|n\|^2$

Proof.

1). It holds directly from the definition of norm, and from the partial order relation defined on $3-SP_R$.

2). Let $a=a_0+a_1P_1+a_2P_2+a_3P_3\in 3-SP_R, w=w_0+w_1P_1+w_2P_2+w_3P_3\in 3-SP_V$, we have:

$\|a.w\|^2=\varphi(a.w,a.w)=a^2\varphi(w,w)$, thus $\|a.w\|=|a|.\|w\|$.

3). $\|w+n\|=\|w_0+n_0\|+P_1[\|w_0+w_1+n_0+n_1\|-\|w_0+n_0\|]+P_2[\|w_0+w_1+w_2+n_0+n_1+n_2\|-\|w_0+w_1+n_0+n_1\|]+P_3[\|w_0+w_1+w_2+w_3+n_0+n_1+n_2+n_3\|-\|w_0+w_1+w_2+n_0+n_1+n_2\|]$, we have:

$$\begin{cases}\|w_0+n_0\|\le\|w_0\|+\|n_0\|\\ \|w_0+w_1+n_0+n_1\|\le\|w_0+w_1\|+\|n_0+n_1\|\\ \|w_0+w_1+w_2+n_0+n_1+n_2\|\le\|w_0+w_1+w_2\|+\|n_0+n_1+n_2\|\\ \|w_0+w_1+w_2+w_3+n_0+n_1+n_2+n_3\|\le\|w_0+w_1+w_2+w_3\|+\|n_0+n_1+n_2+n_3\|\end{cases}$$

So that:

$\|w+n\|\le\|w\|+\|n\|$.

4). We have:

$|\varphi(w,n)|=|\varphi(w_0,n_0)|+P_1[|\varphi(w_0+w_1,n_0+n_1)|-|\varphi(w_0,n_0)|]+P_2[|\varphi(w_0+w_1+w_2,n_0+n_1+n_2)|-|\varphi(w_0+w_1,n_0+n_1)|]+P_3[|\varphi(w_0+w_1+w_2+w_3,n_0+n_1+n_2+n_3)|-|\varphi(w_0+w_1+w_2,n_0+n_1+n_2)|]$.

According to Cauchy-Shwartz inequality, we can write:

$$\begin{cases}|f(w_0,n_0)|\le\|w_0\|+\|n_0\|\\ |f(w_0+w_1,n_0+n_1)|\le\|w_0+w_1\|+\|n_0+n_1\|\\ |f(w_0+w_1+w_2,n_0+n_1+n_2)|\le\|w_0+w_1+w_2\|+\|n_0+n_1+n_2\|\\ |f(w_0+w_1+w_2+w_3,n_0+n_1+n_2+n_3)|\le\|w_0+w_1+w_2+w_3\|+\|n_0+n_1+n_2+n_3\|\end{cases}$$

So that,

$|\varphi(w,n)|\le\|w\|.\|n\|$

5). $w\perp n$ if and only if $\varphi(w,n)=0$, which is equivalent to:

$$\begin{cases}f(w_0,n_0)=0\Longrightarrow w_0\perp n_0\\ f(w_0+w_1,n_0+n_1)\Longrightarrow w_0+w_1\perp n_0+n_1\\ f(w_0+w_1+w_2,n_0+n_1+n_2)\Longrightarrow w_0+w_1+w_2\perp n_0+n_1+n_2\\ f(w_0+w_1+w_2+w_3,n_0+n_1+n_2+n_3)\Longrightarrow w_0+w_1+w_2+w_3\perp n_0+n_1+n_2+n_3\end{cases}$$

6). Assume that $w\perp n$, then $\|w+n\|^2=\varphi(w+n,w+n)=\varphi(w,w)+\varphi(n,n)+2\varphi(w,n)=\|w\|^2+\|n\|^2$.

Conclusion

In This paper we have studied the real inner product defined over symbolic 2-plithogenic vector spaces, where we discussed the concept of inner (scalar) products over the symbolic 2-plithogenic vector spaces by using the corresponding Euclidean scalar products to get theorems that describe the conditions of orthogonality in this class of spaces. Also, we illustrated many examples to explain the ortho-normed symbolic 2-plithogenic spaces.

References

[1] Taffach, N., " An Introduction to Symbolic 2-Plithogenic Vector Spaces Generated from The Fusion of Symbolic Plithogenic Sets and Vector Spaces", Neutrosophic Sets and Systems, Vol 54, 2023.

[2] Ali, R., and Hasan, Z., "An Introduction To The Symbolic 3-Plithogenic Vector Spaces", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.

[3] Ali, R., and Hasan, Z., "An Introduction To The Symbolic 3-Plithogenic Vector Spaces", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.

[4] M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, "Introduction to Symbolic 2-Plithogenic Probability Theory," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.

[5] Taffach, N., and Ben Othman, K., " An Introduction to Symbolic 2-Plithogenic Modules Over Symbolic 2-Plithogenic Rings", Neutrosophic Sets and Systems, Vol 54, 2023.

[6] Smarandache, F., " Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)", Neutrosophic Sets and Systems, vol. 53, 2023.

[7] Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.

[8] Alfahal, A.; Alhasan, Y.; Abdulfatah, R.; Mehmood, A.; Kadhim, M. On Symbolic 2-Plithogenic Real Matrices and Their Algebraic Properties. *Int. J. Neutrosophic Sci.* **2023**, *21*.

[9] Abobala, M., Hatip, A., and Bal, M., " A Review On Recent Advantages In Algebraic Theory Of Neutrosophic Matrices", International Journal of Neutrosophic Science, Vol.17, 2021.

[10] Abobala, M. On Refined Neutrosophic Matrices and Their Applications in Refined Neutrosophic Algebraic Equations. *J. Math.* **2021**, *2021*, 5531093.

[11] Merkepci, M., and Abobala, M., " On Some Novel Results About Split-Complex Numbers, The Diagonalization Problem And Applications To Public Key Asymmetric Cryptography", Journal of Mathematics, Hindawi, 2023.

[12] Abobala, M., On Refined Neutrosophic Matrices and Their Applications In Refined Neutrosophic Algebraic Equations, Journal Of Mathematics, Hindawi, 2021

[13] Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.

[14] Olgun, N., Hatip, A., Bal, M., and Abobala, M., " A Novel Approach To Necessary and Sufficient Conditions For The Diagonalization of Refined Neutrosophic Matrices", International Journal of neutrosophic Science, Vol. 16, pp. 72-79, 2021.

[15] Sankari, H., and Abobala, M., "Neutrosophic Linear Diophantine Equations With two Variables", Neutrosophic Sets and Systems, Vol. 38, pp. 22-30, 2020.

[16] Ibrahim, M., and Abobala, M., "An Introduction To Refined Neutrosophic Number Theory", Neutrosophic Sets and Systems, Vol. 45, 2021.

[17] Abobala, M., "On Some Algebraic Properties of n-Refined Neutrosophic Elements and n-Refined Neutrosophic Linear Equations", Mathematical Problems in Engineering, Hindawi, 2021.

[18] Abobala, M., "On The Characterization of Maximal and Minimal Ideals In Several Neutrosophic Rings", Neutrosophic sets and systems, Vol. 45, 2021.

[19] Abobala, M., "A Study Of Nil Ideals and Kothe's Conjecture In Neutrosophic Rings", International Journal of Mathematics and Mathematical Sciences, hindawi, 2021

[20] Abobala, M., "Neutrosophic Real Inner Product Spaces", Neutrosophic Sets and Systems, Vol. 43, 2021.

[21] Agboola, A.A.A., Akinola, A.D., and Oyebola, O.Y., " Neutrosophic Rings I" , International J.Mathcombin, Vol 4,pp 1-14. 2011

[22] Smarandache, F., and Ali, M., "Neutrosophic Triplet Group", Neural. Compute. Appl. 2019.

[23] Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Concept Of Symbolic 7-Plithogenic Real Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

[24] Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Symbolic 8-Plithogenic Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

[25] Nader Mahmoud Taffach , Ahmed Hatip.," A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations ", Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.

[26] Nader Mahmoud Taffach , Ahmed Hatip., "A Review on Symbolic 2-Plithogenic Algebraic Structures " Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.

[27] Merkepci, H., "On Novel Results about the Algebraic Properties of Symbolic 3-Plithogenic and 4-Plithogenic Real Square Matrices", Symmetry, MDPI, 2023.

[28] Hatip, A., " On The Algebraic Properties of Symbolic n-Plithogenic Matrices For n=5, n=6", Galoitica Journal of Mathematical Structures and Applications, 2023.

Received 10/6/2023, Accepted 29/9/2023

Neutrosophic Sets and Systems, Vol. 59, 2023
University of New Mexico

Plithogenic and Neutrosophic Markov Chains: Modeling Uncertainty and Ambiguity in Stochastic Processes

Suhar Massassati[1], Mohamed Bisher Zeina[2] and Yasin Karmouta[3]
[1] Faculty of Science, Dept. of Mathematical Statistics, University of Aleppo, Aleppo, Syria, e-mail: suharmassasati1987@gmail.com.
[2] Faculty of Science, Dept. of Mathematical Statistics, University of Aleppo, Aleppo, Syria, e-mail: bisher.zeina@gmail.com.
[3] Faculty of Science, Dept. of Mathematical Statistics, University of Aleppo, Aleppo, Syria, e-mail: yassinkarmouta@gmail.com.

Abstract: In this work we present for the first time the concept of literal neutrosophic markov chains and literal plithogenic markov chains. Also, we presented many theorems related to the properties of transition matrix. In literal neutrosophic markov chains we proved that a neutrosophic matrix $M = A + BI$ is a transition matrix if and only if A is a classical transition matrix and $A + B$ is a classical transition matrix. We also proved that multiplication of two neutrosophic transition matrices is again a neutrosophic transition matrix and that the power of a neutrosophic transition matrix is a neutrosophic transition matrix. Finally, we proved that the (n) step neutrosophic transition matrix is equivalent to raising the main neutrosophic transition matrix to the power n. In literal plithogenic markov chains which is a generalization of the previous case we proved that $M = A + BP_1 + CP_2$ is a plithogenic transition matrix if and only if all of the matrices $A, A + B, A + B + C$ are transition matrices in classical concept. We also proved that multiplication of two plithogenic transition matrices is a plithogenic transition matrix and that raising a plithogenic transition matrix to a power r will produce a new plithogenic transition matrix. Also, as in neutrosophic case, the (n) step plithogenic transition matrix is equivalent to the main plithogenic matrix raised to

zenodo.10031184/10.5281

the power n. Theorems were provided with suitable solved examples and problems.

Keywords: Neutrosophic; Plithogenic; Markov Chains; Transition Matrix; Chapman-Kolmogorov.

1. Introduction

In the realm of stochastic processes and probability theory, Markov chains stand as a foundational model for understanding the dynamics of sequential events. These chains provide a powerful framework for analyzing various systems, ranging from biological processes to financial markets. However, traditional Markov chains often struggle to capture the inherent uncertainties and ambiguities present in many real-world scenarios. [1]–[5]

This paper delves into the intriguing fusion of two distinct conceptual frameworks, namely plithogenic and neutrosophic, with the well-established Markov chain theory. Plithogenic and neutrosophic concepts extend the conventional notions of truth and falsity to encompass the realm of partial truth and indeterminacy, respectively.[6]–[21] This unique blend of theories offers a promising avenue to model complex systems where inherent vagueness and uncertainty play a significant role.

Throughout this paper, we aim to elucidate the theoretical foundations of plithogenic and neutrosophic Markov chains, shedding light on their mathematical underpinnings and conceptual implications. We will explore how these novel extensions can be seamlessly integrated into traditional Markov chain models which have many practical applications across diverse domains such as decision-making, risk assessment, and artificial intelligence.

By merging the realms of classical Markov chains, plithogenic reasoning, and neutrosophic logic, this paper strives to contribute to the advancement of probabilistic modeling in situations where uncertainty and ambiguity are central. Through comprehensive exploration and illustrative examples, we endeavor to demonstrate the utility and significance of these novel frameworks in tackling the intricacies of real-world systems. In doing so, we aim to provide researchers and practitioners with a deeper understanding of the capabilities and limitations of plithogenic and neutrosophic Markov chains, paving the way for more nuanced and accurate modeling in complex and uncertain scenarios.

This work can be considered as a complement to previous works in probability theory and stochastic processes built under symbolic neutrosophic structures and can be also considered as an introduction to related fields such as queueing theory, reliability theory, dynamic systems, etc.[11], [17], [22]–[41]

2. Preliminaries

Definition 2.1

Let $\boldsymbol{R(I)} = \{\boldsymbol{a + bI}; \boldsymbol{I^2 = I}\}$, we call $\mathbf{R(I)}$ the neutrosophic field of reals.

Definition 2.2

Let $R(I)$ be the neutrosophic field of reals, and let $a_N = a_1 + a_2 I, b_N = b_1 + b_2 I \in R(I)$. We can say that $a_N \geq_N b_N$ if: $a_1 \geq b_1$ and $a_1 + a_2 \geq b_1 + b_2$

Definition 2.3

One-dimensional isometry between R(I) and R×R and its inverse are defined as follows:

$\boldsymbol{T}: \boldsymbol{R(I)} \to \boldsymbol{R} \times \boldsymbol{R}; \boldsymbol{T(a + bI) = (a, a + b)}.$

$\boldsymbol{T^{-1}}: \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R(I)}; \boldsymbol{T^{-1}(a, b) = a + (b - a)I}.$

Definition 2.4

Let $\boldsymbol{R(P_1,P_2)} = \{\boldsymbol{a_0 + a_1P_1 + a_2P_2; P_1^2 = P_1, P_2^2 = P_2, P_1P_2 = P_2P_1 = P_2}\}$, we call $\boldsymbol{R(P_1,P_2)}$ Plithogenic field of reals.

Definition 2.5

Let $R(P_1,P_2)$ be the Plithogenic field of reals, and let $a_P = a_0 + a_1P_1 + a_2P_2, b_P = b_0 + b_1P_1 + b_2P_2 \in R(P_1,P_2)$. We say that $a_P \geq_P b_P$ if:

$\boldsymbol{a_0 \geq b_0, a_0 + a_1 \geq b_0 + b_1 \ and \ a_0 + a_1 + a_2 \geq b_0 + b_1 + b_2}$

Definition 2.6

One-dimensional isometry between $\boldsymbol{R(P_1,P_2)}$ and the space $\boldsymbol{R \times R \times R}$ is defined as follows:

$\boldsymbol{T: R(P_1,P_2) \rightarrow R \times R \times R; T(a_0 + a_1P_1 + a_2P_2) = (a_0, a_0 + a_1, a_0 + a_1 + a_2)}$

$T^{-1}: R \times R \times R \rightarrow R(P_1,P_2); T^{-1}(a_0,a_1,a_2) = a_0 + (a_1 - a_0)P_1 + (a_2 - a_1)P_2$

3. Literal Neutrosophic Markov chains

Definition 3.1

A set of random variables $X_0, X_1, X_2, \dots$ satisfying:

$Pr\{X_{n+1} = i_{n+1} | X_n = i_n, X_{n-1} = i_{n-1}, \dots, X_0 = i_0\} = Pr\{X_{n+1} = i_{n+1} | X_n = i_n\}$

is called a literal or symbolic neutrosophic markov chain if the last probability takes the form $Pr\{X_{n+1} = i_{n+1} | X_n = i_n\} = a + bI; 0 \leq a \leq 1, 0 \leq a + b \leq 1, I^2 = I$

Definition 3.2

We call $\boldsymbol{p_{ij}^{(n,n+1)}}_{\boldsymbol{N}} = \boldsymbol{Pr(X_{n+1} = j | X_n = i) \in R(I)}$ literal or symbolic neutrosophic one-step transition probability.

Definition 3.3

A squared neutrosophic matrix

$M_N = A + BI = [a_{ij} + b_{ij}I]_{n \times n}$

Is called a neutrosophic markov transition matrix if its elements satisfy:

1. $\sum_j a_{ij} + b_{ij}I = 1$; $i = 1,2,3, \dots, n$
2. $0 \leq_N a_{ij} + b_{ij}I \leq_N 1$; $i,j = 1,2,3, \dots, n$

Example 3.1

let′s take: $$M_N = \begin{bmatrix} 0.3I & 1 - 0.3I \\ 0.4 + 0.2I & 0.6 - 0.2I \end{bmatrix}$$

Then M_N is a neutrosophic transition matrix because:

$$0.3I + 1 - 0.3I = 1 \quad \text{and} \quad 0.4 + 0.2I + 0.6 - 0.2I = 1$$

Also, according to the definition of comparison between Neutrosophic numbers we have:

$0 + 0.3I \leq_N 1 + 0I$ because $0 \leq 1$ & $0.3 \leq 1$

$1 - 0.3I \leq_N 1 + 0I$ because $1 \leq 1$ & $0.7 \leq 1$

$0.4 + 0.2I \leq_N 1 + 0I$ because $0.4 \leq 1$ & $0.6 \leq 1$

$0.6 - 0.2I \leq_N 1 + 0I$ because $0.6 \leq 1$ & $0.4 \leq 1$

$0 + 0I \leq_N 0 + 0.3I$ because $0 \leq 0$ & $0 \leq 0.3$

$0 + 0I \leq_N 1 - 0.3I$ because $0 \leq 1$ & $0 \leq 0.7$

$0 + 0I \leq_N 0.4 + 0.2I$ because $0 \leq 0.4$ & $0 \leq 0.6$

$0 + 0I \leq_N 0.6 - 0.2I$ because $0 \leq 0.6$ & $0 \leq 0.4$

Theorem 3.1

The matrix $M_N = A + BI$ is a neutrosophic transition matrix if and only if A is a crisp transition matrix and $A + B$ is a crisp transition matrix.

Proof

Let's assume that M_N is a neutrosophic transition matrix and prove that A *and* $A + B$ are two transition matrices:

we have $0 + 0I \leq_N a_{ij} + b_{ij}I \leq_N 1 + 0I$ so $a_{ij} \leq 1$, $a_{ij} + b_{ij} \leq 1$, $0 \leq a_{ij}$ and $0 \leq a_{ij} + b_{ij}$ which means that:

$0 \leq a_{ij} \leq 1$ and $0 \leq a_{ij} + b_{ij} \leq 1$

Also, we have $\sum_j (a_{ij} + b_{ij}I) = 1 = 1 + 0I$ which means that $\sum_j b_{ij} = 0$ and $\sum_j a_{ij} = 1$

So, we can conclude that:

$0 \leq a_{ij} \leq 1 \; and \sum_j a_{ij} = 1 \Rightarrow \; A$ is a transition matrix.

$0 \leq a_{ij} + b_{ij} \leq 1 \; and \sum_j (a_{ij} + b_{ij}) = 1 \Rightarrow \; A + B$ is a transition matrix.

Now, let's assume that both A and A+B are transition matrices and prove that $M_N = A + BI$ is a neutrosophic transition matrix:

since $A, A + B$ are transition matrices then $0 \leq a_{ij} \leq 1$, $0 \leq a_{ij} + b_{ij} \leq 1$ which means that $0 \leq a_{ij} + b_{ij}I \leq 1$

Also, we have $\sum_j a_{ij} = 1$ and $\sum_j (a_{ij} + b_{ij}) = 1$ that yields to the fact that $\sum_j b_{ij} = 0$

Then we conclude that $\sum_j (a_{ij} + b_{ij}I) = 1$ and this proves the theorem.

Example 3.2

Let's take the matrix:

$$M_N = \begin{bmatrix} 0.3I & 1 - 0.3I \\ 0.4 + 0.2I & 0.6 - 0.2I \end{bmatrix}$$

that is:

$$M_N = \begin{bmatrix} 0 & 1 \\ 0.4 & 0.6 \end{bmatrix} + \begin{bmatrix} 0.3 & -0.3 \\ 0.2 & -0.2 \end{bmatrix} I$$

$$A = \begin{bmatrix} 0 & 1 \\ 0.4 & 0.6 \end{bmatrix} \; and \;\; A + B = \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix}$$

we note that A and A+B are two transition matrices fulfill conditions

$$\sum_j a_{ij} = 1 \, ; i = 1,2 \qquad\qquad 0 \leq a_{ij} \leq 1 \, ; i, j = 1,2$$

$$\sum_j a_{ij} + b_{ij} = 1 \, ; i = 1,2 \qquad 0 \leq a_{ij} + b_{ij} \leq 1 \, ; i, j = 1,2$$

Theorem 3.2

If M_1 and $M_2$are two neutrosophic transition matrices, then their multiplication is a neutrosophic transition matrix.

Proof

Let $M_1 = \begin{bmatrix} a_{11}+b_{11}I & a_{12}+b_{12}I \\ a_{21}+b_{21}I & a_{22}+b_{22}I \end{bmatrix}, M_2 = \begin{bmatrix} c_{11}+d_{11}I & c_{12}+d_{12}I \\ c_{21}+d_{21}I & c_{22}+d_{22}I \end{bmatrix}$

$M_1.M_2$

$$= \begin{bmatrix} (a_{11}+b_{11}I)(c_{11}+d_{11}I)+(a_{12}+b_{12}I)(c_{21}+d_{21}I) & (a_{11}+b_{11}I)(c_{12}+d_{12}I)+(a_{12}+b_{12}I)(c_{22} \\ (a_{21}+b_{21}I)(c_{11}+d_{11}I)+(a_{22}+b_{22}I)(c_{21}+d_{21}I) & (a_{21}+b_{21}I)(c_{12}+d_{12}I)+(a_{22}+b_{22}I)(c_{22} \end{bmatrix}$$

Let's check the first condition:

$$(a_{11}+b_{11}I)(c_{11}+d_{11}I)+(a_{12}+b_{12}I)(c_{21}+d_{21}I)+(a_{11}+b_{11}I)(c_{12}+d_{12}I)+(a_{12}+b_{12}I)(c_{22}+d_{22}I) =$$

$(a_{11}+b_{11}I)[(c_{11}+d_{11}I)+(c_{12}+d_{12}I)]+(a_{12}+b_{12}I)[(c_{21}+d_{21}I)+(c_{22}+d_{22}I)]=$

$(a_{11}+b_{11}I)+(a_{12}+b_{12}I)=1$

Similarly, we find that sum of elements of the second row of matrix $(M_1.M_2)$ is 1

Also, since all elements of the matrices M_1 and M_2 are positive and since that sum of each row of the matrix $M_1.M_2$ is 1 then we conclude that each element lays between 0 and 1

Example 3.3

Let $M_1 = \begin{bmatrix} 0.6I & 1-0.6I \\ 0.3+0.1I & 0.7-0.1I \end{bmatrix}, M_2 = \begin{bmatrix} 0.2+0.3I & 0.8-0.3I \\ 0.2I & 1-0.2I \end{bmatrix}$

$$M_1.M_2 = \begin{bmatrix} 0.6I & 1-0.6I \\ 0.3+0.1I & 0.7-0.1I \end{bmatrix}.\begin{bmatrix} 0.2+0.3I & 0.8-0.3I \\ 0.2I & 1-0.2I \end{bmatrix}$$

$$= \begin{bmatrix} 0.32I+0.06I^2 & 1-0.32I-0,06I^2 \\ 0.6+0.25I+0,01I^2 & 0.94-0.25I-0,01I^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.38I & 1-0.38I \\ 0.6+0.26I & 0.31-0.26I \end{bmatrix}$$

Note that the matrix $M_1.M_2$ It is a neutrosophic transition matrix because it satisfies the assumed conditions.

Definition 3.4

Let $M_N = A + BI$ be a neutrosophic matrix and let $r \in \mathbb{N}$, then:

$$M_N^r = A^r + I[(A+B)^r - A^r]$$

Theorem 3.3

If M_N neutrosophic transition matrix, then M_N^r is a neutrosophic transition matrix.

Proof

Straight forward by mathematical induction according to theorem 3.2.

Example 3.4:

Let $M_N = \begin{bmatrix} 0.6I & 1-0.6I \\ 0.3+0.1I & 0.7-0.1I \end{bmatrix}$

$M_N^2 = M_N.M_N = \begin{bmatrix} 0.6I & 1-0.6I \\ 0.3+0.1I & 0.7-0.1I \end{bmatrix}.\begin{bmatrix} 0.6I & 1-0.6I \\ 0.3+0.1I & 0.7-0.1I \end{bmatrix}$

$= \begin{bmatrix} 0.3-0.08I+0.3I^2 & 0.7+0.08I-0.3I^2 \\ 0.21+0.22I+0.05I^2 & 0.79-0.22I-0.05I^2 \end{bmatrix}$

$= \begin{bmatrix} 0.3-0.38I & 0.7+0.38I \\ 0.21+0.27I & 0.79-0.27I \end{bmatrix}$

Notice that M_N^2 is a neutrosophic transition matrix, also:

$M_N^3 = M_N^2.M_N = \begin{bmatrix} 0.3-0.38I & 0.7+0.38I \\ 0.21+0.27I & 0.79-0.27I \end{bmatrix}.\begin{bmatrix} 0.6I & 1-0.6I \\ 0.3+0.1I & 0.7-0.1I \end{bmatrix}$

$= \begin{bmatrix} 0.21+0.364I-0.190I^2 & 0.79-0.364I+0.190I^2 \\ 0.237+0.124I+0.135I^2 & 0.763-0.124I-0.135I^2 \end{bmatrix}$

$= \begin{bmatrix} 0.21+0.174I & 0.79-0.174I \\ 0.237+0.259I & 0.763-0.259I \end{bmatrix}$

We note that M_N^3 is also a neutrosophic transition matrix.

Theorem 3.4

Let $M_N = A + BI$ be a neutrosophic transition matrix and let $M_N^{(n)}$ be the (n) steps transition matrix then:

$$M_N^{(n)} = M_N^n$$

Proof

By takin the isometric image we have:

$T\left(M_N^{(n)}\right) = (A^{(n)}, (A+B)^{(n)})$

Since both $A^{(n)}, (A+B)^{(n)}$ are transition matrices in classical scene then by the well-known Chapman-Kolmogorov theorem we have:

$A^{(n)} = A^n, (A+B)^{(n)} = (A+B)^n$

Which means that:

$T\left(M_N^{(n)}\right) = (A^n, (A+B)^n)$

Now, taking inverse isometry yields to:

$T^{-1}\left(T\left(M_N^{(n)}\right)\right) = A^n + [(A+B)^n - A^n]I = M^n$

4. Literal Plithogenic Markov chains

Definition 4.1

A set of random variables $X_0, X_1, X_2, \ldots$ satisfying:

$Pr\,\{X_{n+1} = i_{n+1}|X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} = Pr\,\{X_{n+1} = i_{n+1}|X_n = i_n\}$

is called a literal or symbolic neutrosophic markov chain if the last probability takes the form $Pr\,\{X_{n+1} = i_{n+1}|X_n = i_n\} = a + \mathrm{b}P_1 + \mathrm{cP_2}; 0 \leq \mathrm{a} \leq 1, 0 \leq \mathrm{a} + \mathrm{b} \leq 1, 0 \leq \mathrm{a} + \mathrm{b} + \mathrm{c} \leq 1; P_1^2 = P_1, P_2^2 = P_2,$

$P_1P_2 = P_2P_1 = P_2$

Definition 4.2

We call $\boldsymbol{p_{ij}^{(n,n+1)}}_{\boldsymbol{P}} = \boldsymbol{Pr(X_{n+1} = j|X_n = i)} \in \boldsymbol{R(P_1, P_2)}$ literal or symbolic plithogenic one-step transition probability.

Definition 4.3

A squared plithogenic matrix

$M_N = A + BP_1 + CP_2 = \left[a_{ij} + b_{ij}P_1 + c_{ij}P_2\right]_{n\times n}$

Is called a plithogenic markov transition matrix if its elements satisfy:

1. $\sum_{\mathrm{j}} \mathrm{a_{ij}} + \mathrm{b_{ij}P_1} + \mathrm{c_{ij}P_2} = 1 \quad ;\ \mathrm{i} = 1,2,3, \ldots, \mathrm{n}$
2. $0 \leq_{\mathrm{p}} a_{ij} + b_{ij}P_1 + c_{ij}P_2 \leq_{\mathrm{p}} 1 \quad ;\ \mathrm{i, j} = 1,2,3, \ldots, \mathrm{n}$

Example 4.1

let's take: $M_P = \begin{bmatrix} 0.3P_1 + 0.1P_2 & 1 - 0.3P_1 - 0.1P_2 \\ 0.4 + 0.2P_1 - 0.6P_2 & 0.6 - 0.2P_1 + 0.6P_2 \end{bmatrix}$

Then $\mathrm{M_P}$ is a plithogenic transition matrix because:

$0.3P_1 + 0.1P_2 + 1 - 0.3P_1 - 0.1P_2$

$= 1 \quad \text{and} \quad 0.4 + 0.2P_1 - 0.6P_2 + 0.6 - 0.2P_1 + 0.6P_2 = 1$

Also, according to the definition of comparison between plithogenic numbers we have:

$0.3P_1 + 0.1P_2 \leq_P 1 + 0P_1 + 0P_2$ because $0 \leq 1$ & $0.3 \leq 1$

$1 - 0.3P_1 - 0.1P_2 \leq_P 1 + 0P_1 + 0P_2$ because $1 \leq 1$ & $0.7 \leq 1$

$0.4 + 0.2P_1 - 0.6P_2 \leq_P 1 + 0P_1 + 0P_2$ because $0.4 \leq 1$ & $0.6 \leq 1$

$0.6 - 0.2P_1 + 0.6P_2 \leq_P 1 + 0P_1 + 0P_2$ because $0.6 \leq 1$ & $0.4 \leq 1$

$0 + 0P_1 + 0P_2 \leq_P 0.3P_1 + 0.1P_2$ because $0 \leq 0$ & $0 \leq 0.3$

$0 + 0P_1 + 0P_2 \leq_P 1 - 0.3P_1 - 0.1P_2$ because $0 \leq 1$ & $0 \leq 0.7$

$0 + 0P_1 + 0P_2 \leq_P 0.4 + 0.2P_1 - 0.6P_2$ because $0 \leq 0.4$ & $0 \leq 0.6$

$0 + 0P_1 + 0P_2 \leq_P 0.6 - 0.2P_1 + 0.6P_2$ because $0 \leq 0.6$ & $0 \leq 0.4$

Theorem4.1

The matrix $M_P = A + BP_1 + CP_2$ is a plithogenic transition matrix if and only if A is a crisp transition matrix, $A + B$ is a crisp transition matrix and $A + B + C$ is a crisp transition matrix.

Proof

Let's assume that M_P is a plithogenic transition matrix and prove that $A, A + B$ and $A + B + C$ are transition matrices:

we have $0 + 0P_1 + 0P_2 \leq_P a_{ij} + b_{ij}P_1 + c_{ij}P_2 \leq_P 1 + 0P_1 + 0P_2$ so $a_{ij} \leq 1$, $a_{ij} + b_{ij} \leq 1$and $a_{ij} + b_{ij} + c_{ij} \leq 1$

$0 \leq a_{ij}$, $0 \leq a_{ij} + b_{ij}$ and $0 \leq a_{ij} + b_{ij} + c_{ij}$ which means that:

$0 \leq a_{ij} \leq 1$ and $0 \leq a_{ij} + b_{ij} \leq 1$

Also, we have $\sum_j (a_{ij} + b_{ij}P_1 + c_{ij}P_2) = 1 = 1 + 0P_1 + 0P_2$ which means that$\sum_j c_{ij} = 0$ $\sum_j b_{ij} = 0$ and$\sum_j a_{ij} = 1$

So, we can conclude that:

$0 \leq a_{ij} \leq 1$ and $\sum_j a_{ij} = 1 \Rightarrow A$ is a transition matrix.

$0 \leq a_{ij} + b_{ij} \leq 1$ and $\sum_j (a_{ij} + b_{ij}) = 1 \Rightarrow A + B$ is a transition matrix.

$0 \leq a_{ij} + b_{ij} + c_{ij} \leq 1$ and $\sum_j (a_{ij} + b_{ij} + c_{ij}) = 1 \Rightarrow$ A + B + C transition matrix.

Now, let's assume that both A, A+B and A+B+C are transition matrices and prove that $M_P = A + BP_1 + CP_2$ is a plithogenic transition matrix:

since $A, A + B, A + B + C$ are transition matrices then $0 \leq a_{ij} \leq 1$, $0 \leq a_{ij} + b_{ij} \leq 1$, $0 \leq a_{ij} + b_{ij} + c_{ij} \leq 1$ which means that $0 \leq_P a_{ij} + b_{ij}P_1 + c_{ij}P_2 \leq_P 1$

Also, we have $\sum_j a_{ij} = 1$, $\sum_j (a_{ij} + b_{ij}) = 1$and $\sum_j (a_{ij} + b_{ij} + c_{ij}) = 1$ that yields to the fact that $\sum_j b_{ij} = 0$ and $\sum_j c_{ij} = 0$

Then we conclude that $\sum_j (a_{ij} + b_{ij}P_1 + c_{ij}P_2) = 1$ and this proves the theorem.

Example 4.2

Let's take the matrix:

$$M_P = \begin{bmatrix} 0.3P_1 + 0.1P_2 & 1 - 0.3P_1 - 0.1P_2 \\ 0.4 + 0.2P_1 - 0.6P_2 & 0.6 - 0.2P_1 + 0.6P_2 \end{bmatrix}$$

that is:

$$M_P = \begin{bmatrix} 0 & 1 \\ 0.4 & 0.6 \end{bmatrix} + \begin{bmatrix} 0.3 & -0.3 \\ 0.2 & -0.2 \end{bmatrix} P_1 + \begin{bmatrix} 0.1 & -0.1 \\ -0.6 & 0.6 \end{bmatrix} P_2$$

$$A = \begin{bmatrix} 0 & 1 \\ 0.4 & 0.6 \end{bmatrix} \text{ and } A + B = \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix} \text{ and } A + B + C = \begin{bmatrix} 0.4 & 0.6 \\ 0 & 1 \end{bmatrix} \Rightarrow$$

we note that $A, A + B$ and $A + B + C$ are transition matrices fulfill conditions

$$\sum_j a_{ij} = 1 \,; i = 1,2 \qquad 0 \leq a_{ij} \leq 1 \,; i,j = 1,2$$

$$\sum_j a_{ij} + b_{ij} = 1 \,; i = 1,2 \qquad 0 \leq a_{ij} + b_{ij} \leq 1 \,; i,j = 1,2$$

$$\sum_j a_{ij} + b_{ij} + c_{ij} = 1 \quad ; i = 1,2 \qquad 0 \leq a_{ij} + b_{ij} + c_{ij} \leq 1 \quad ; i,j = 1,2$$

Theorem 4.2

If M_1 and $M_2$are two plithogenic transition matrices, then their multiplication is a plithogenic transition matrix.

Proof

Let

$$M_1 = \begin{bmatrix} a_{11} + b_{11}P_1 + c_{11}P_2 & a_{12} + b_{12}P_1 + c_{12}P_2 \\ a_{21} + b_{21}P_1 + c_{21}P_2 & a_{22} + b_{22}P_1 + c_{22}P_2 \end{bmatrix} \quad M_2$$

$$= \begin{bmatrix} d_{11} + e_{11}P_1 + f_{11}P_2 & d_{12} + e_{12}P_1 + f_{12}P_2 \\ d_{21} + e_{21}P_1 + f_{21}P_2 & d_{22} + e_{22}P_1 + f_{22}P_2 \end{bmatrix}$$

$$M_1.M_2 = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

Where

$$x = (a_{11} + b_{11}P_1 + c_{11}P_2)(d_{11} + e_{11}P_1 + f_{11}P_2) + (a_{12} + b_{12}P_1 + c_{12}P_2)(d_{21} + e_{21}P_1 + f_{21}P_2)$$

$$y = (a_{11} + b_{11}P_1 + c_{11}P_2)(d_{12} + e_{12}P_1 + f_{12}P_2) + (a_{12} + b_{12}P_1 + c_{12}P_2)(d_{22} + e_{22}P_1 + f_{22}P_2)$$

$$z = (a_{21} + b_{21}P_1 + c_{21}P_2)(d_{11} + e_{11}P_1 + f_{11}P_2) + (a_{22} + b_{22}P_1 + c_{22}P_2)(d_{21} + e_{21}P_1 + f_{21}P_2)$$

$$w = (a_{21} + b_{21}P_1 + c_{21}P_2)(d_{12} + e_{12}P_1 + f_{12}P_2) + (a_{22} + b_{22}P_1 + c_{22}P_2)(d_{22} + e_{22}P_1 + f_{22}P_2)$$

Let's check the condition:

$$(a_{11} + b_{11}P_1 + c_{11}P_2)(d_{11} + e_{11}P_1 + f_{11}P_2) + (a_{12} + b_{12}P_1 + c_{12}P_2)(d_{21} + e_{21}P_1 + f_{21}P_2) + (a_{11} + b_{11}P_1 + c_{11}P_2)(d_{12} + e_{12}P_1 + f_{12}P_2) + (a_{12} + b_{12}P_1 + c_{12}P_2)(d_{22} + e_{22}P_1 + f_{22}P_2)$$

$$= (a_{11} + b_{11}P_1 + c_{11}P_2)[(d_{11} + e_{11}P_1 + f_{11}P_2) + (d_{12} + e_{12}P_1 + f_{12}P_2)] + (a_{12} + b_{12}P_1 + c_{12}P_2)$$

$$[(d_{21} + e_{21}P_1 + f_{21}P_2) + (d_{22} + e_{22}P_1 + f_{22}P_2)]$$

$$= (a_{11} + b_{11}P_1 + c_{11}P_2) + (a_{12} + b_{12}P_1 + c_{12}P_2) = 1$$

Similarly, we find that sum of elements of the second row of matrix $(M_1.M_2)$ is 1

Also, since all elements of the matrices M_1 and M_2 are positive and since that sum of each row of the matrix $M_1.M_2$ is 1 then we conclude that each element lays between 0 and 1

Example 4.3

Let $M_1 = \begin{bmatrix} 0.6\mathrm{P}_1 + 0.2\mathrm{P}_2 & 1 - 0.6\mathrm{P}_1 - 0.2\mathrm{P}_2 \\ 0.3 + 0.1\mathrm{P}_1 - 0.5\mathrm{P}_2 & 0.7 - 0.1\mathrm{P}_1 + 0.5\mathrm{P}_2 \end{bmatrix}, M_2 = \begin{bmatrix} 0.2 + 0.3\mathrm{P}_1 - 0.1\mathrm{P}_2 & 0.8 - 0.3\mathrm{P}_1 + 0.1\mathrm{P}_2 \\ 0.2\mathrm{P}_1 + 0.3\mathrm{P}_2 & 1 - 0.2\mathrm{P}_1 - 0.3\mathrm{P}_2 \end{bmatrix}$

$M_1.M_2$

$$= \begin{bmatrix} 0.6\mathrm{P}_1 + 0.2\mathrm{P}_2 & 1 - 0.6\mathrm{P}_1 - 0.2\mathrm{P}_2 \\ 0.3 + 0.1\mathrm{P}_1 - 0.5\mathrm{P}_2 & 0.7 - 0.1\mathrm{P}_1 + 0.5\mathrm{P}_2 \end{bmatrix} . \begin{bmatrix} 0.2 + 0.3\mathrm{P}_1 - 0.1\mathrm{P}_2 & 0.8 - 0.3\mathrm{P}_1 + 0.1\mathrm{P}_2 \\ 0.2\mathrm{P}_1 + 0.3\mathrm{P}_2 & 1 - 0.2\mathrm{P}_1 - 0.3\mathrm{P}_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.06\mathrm{P}_1^2 + (0.32 - 0.22\mathrm{P}_2)\mathrm{P}_1 + 0.34\mathrm{P}_2 - 0.08\mathrm{P}_2^2 & -0.06\mathrm{P}_1^2 + (-0.32 + 0.22\mathrm{P}_2)\mathrm{P}_1 - 0.34\mathrm{P}_2 \\ 0.01\mathrm{P}_1^2 + (0.25 - 0.09\mathrm{P}_2)\mathrm{P}_1 + 0.06 + 0.08\mathrm{P}_2 + 0.20\mathrm{P}_2^2 & -0.01\mathrm{P}_1^2 + (-0.25 + 0.09\mathrm{P}_2)\mathrm{P}_1 + 0.94 - 0. \end{bmatrix}$$

$$= \begin{bmatrix} 0.06\mathrm{P}_1 + (0.32\mathrm{P}_1 - 0.22\mathrm{P}_2) + 0.34\mathrm{P}_2 - 0.08\mathrm{P}_2 & -0.06\mathrm{P}_1 + (-0.32\mathrm{P}_1 + 0.22\mathrm{P}_2) - 0.34\mathrm{P}_2 + \\ 0.01\mathrm{P}_1 + (0.25\mathrm{P}_1 - 0.09\mathrm{P}_2) + 0.06 + 0.08\mathrm{P}_2 + 0.20\mathrm{P}_2 & -0.01\mathrm{P}_1 + (-0.25\mathrm{P}_1 + 0.09\mathrm{P}_2) + 0.94 - 0.0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.38\mathrm{P}_1 + 0.04\mathrm{P}_2 & -0.38\mathrm{P}_1 - 0.04\mathrm{P}_2 + 1 \\ 0.26\mathrm{P}_1 + 0.06 + 0.19\mathrm{P}_2 & -0.26\mathrm{P}_1 + 0.94 - 0.19\mathrm{P}_2 \end{bmatrix}$$

Note that the matrix $M_1.M_2$ It is a plithogenic transition matrix because it satisfies the assumed conditions.

Definition 4.4

Let $M_P = A + BP_1 + CP_2$ be a plithogenic matrix and let $r \in \mathbb{N}$, then:

$$M_P^r = A^r + P_1[(A+B)^r - A^r] + P_2[(A+B+C)^r - (A+B)^r]$$

Theorem 4.3

If $\mathrm{M_P}$ plithogenic transition matrix, then $\mathrm{M_P^r}$ is a plithogenic transition matrix.

Proof

Straight forward by mathematical induction according to theorem 4.2.

Example 4.4

Let $\mathrm{M_P} = \begin{bmatrix} 0.6\mathrm{P}_1 + 0.2\mathrm{P}_2 & 1 - 0.6\mathrm{P}_1 - 0.2\mathrm{P}_2 \\ 0.3 + 0.1\mathrm{P}_1 - 0.5\mathrm{P}_2 & 0.7 - 0.1\mathrm{P}_1 + 0.5\mathrm{P}_2 \end{bmatrix}$

$\mathrm{M_P^2} = \mathrm{M_P}.\mathrm{M_P}$

$$= \begin{bmatrix} 0.6\mathrm{P}_1 + 0.2\mathrm{P}_2 & 1 - 0.6\mathrm{P}_1 - 0.2\mathrm{P}_2 \\ 0.3 + 0.1\mathrm{P}_1 - 0.5\mathrm{P}_2 & 0.7 - 0.1\mathrm{P}_1 + 0.5\mathrm{P}_2 \end{bmatrix} . \begin{bmatrix} 0.6\mathrm{P}_1 + 0.2\mathrm{P}_2 & 1 - 0.6\mathrm{P}_1 - 0.2\mathrm{P}_2 \\ 0.3 + 0.1\mathrm{P}_1 - 0.5\mathrm{P}_2 & 0.7 - 0.1\mathrm{P}_1 + 0.5\mathrm{P}_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.30\mathrm{P}_1^2 + (0.52\mathrm{P}_2 - 0.08)\mathrm{P}_1 + 0.14\mathrm{P}_2^2 + 0.3 - 0.56\mathrm{P}_2 & -0.30\mathrm{P}_1^2 + (0.08 - 0.52\,\mathrm{P}_2)\mathrm{P}_1 + 0.56\mathrm{P}_2 - \\ 0.05\mathrm{P}_1^2 + (0.22 - 0.18\,\mathrm{P}_2)\mathrm{P}_1 - 0.14\mathrm{P}_2 - 0.35\mathrm{P}_2^2 + 0.21 & -0.05\,\mathrm{P}_1^2 + (-0.22 + 0.18\mathrm{P}_2)\mathrm{P}_1 + 0.79 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.30\mathrm{P}_1 + (0.52\mathrm{P}_2 - 0.08\mathrm{P}_1) + 0.14\mathrm{P}_2 + 0.3 - 0.56\mathrm{P}_2 & -0.30\mathrm{P}_1 + (0.08\mathrm{P}_1 - 0.52\,\mathrm{P}_2) + 0.56\mathrm{P}_2 - \\ 0.05\mathrm{P}_1 + (0.22\mathrm{P}_1 - 0.18\,\mathrm{P}_2) - 0.14\mathrm{P}_2 - 0.35\mathrm{P}_2 + 0.21 & -0.05\,\mathrm{P}_1 + (-0.22\mathrm{P}_1 + 0.18\mathrm{P}_2) + 0.79 + 0. \end{bmatrix}$$

$$= \begin{bmatrix} 0.22\mathrm{P}_1 + 0.1\mathrm{P}_2 + 0.3 & -0.22\mathrm{P}_1 + 0.\mathrm{P}_2 + 0.7 \\ 0.27\mathrm{P}_1 - 0.67\mathrm{P}_2 + 0.21 & -0.27\,\mathrm{P}_1 + 0.79 + 0.67\mathrm{P}_2 \end{bmatrix}$$

Notice that $\mathrm{M_N^2}$ is a plithogenic transition matrix.

Theorem 4.4

Let $M_P = A + BP_1 + CP_2$ be a plithogenic transition matrix and let $M_P^{(n)}$ be the (n) steps transition matrix then:

$$M_P^{(n)} = M_P^n$$

Proof

By takin the isometric image we have:

$T\left(M_P^{(n)}\right) = (A^{(n)}, (A+B)^{(n)}, (A+B+C)^{(n)})$

Since $A^{(n)}, (A+B)^{(n)}, (A+B+C)^{(n)}$ are transition matrices in classical scene then by the well-known Chapman-Kolmogorov theorem we have:

$A^{(n)} = A^n, (A+B)^{(n)} = (A+B)^n, (A+B+C)^{(n)} = (A+B+C)^n$

Which means that:

$T\left(M_P^{(n)}\right) = (A^n, (A+B)^n, (A+B+C)^n)$

Now, taking inverse isometry yields to:

$T^{-1}\left(T\left(M_P^{(n)}\right)\right) = A^n + [(A+B)^n - A^n]P_1 + [(A+B+C)^n - (A+B)^n]P_2 = M^n$

5. Conclusion

In conclusion, this paper pioneers the integration of symbolic neutrosophic and plithogenic concepts into the well-established framework of Markov chains, yielding a profound extension that encapsulates the nuances of uncertainty and ambiguity. Through a meticulous presentation of eight theorems, we have established a bridge between these novel matrices and their classical counterparts, revealing their intrinsic alignment. The introduced operations of matrix exponentiation and multiplication further amplify the versatility of these frameworks, enabling the exploration of complex system dynamics under varying degrees of indeterminacy. Moreover, the adaptation of Chapman-Kolmogorov theorem to the symbolic neutrosophic and plithogenic domains augments our ability to analyze state transitions in environments laden with partial truth. This study not only advances the theoretical frontiers of probabilistic modeling but also lays a fertile ground for practical applications across disciplines such as decision analysis, risk assessment, and artificial intelligence. As the confluence of traditional and innovative theories continues to shape the landscape of uncertainty modeling, symbolic neutrosophic and plithogenic Markov chains stand poised to offer invaluable insights into the intricate fabric of real-world systems.

Funding: This research received no external funding.

Acknowledgments: Authors are thankful for the editorial team specially the editor-in-chief professor Florentin Smarandache and the guest editor Mohamad Abobala.

Conflicts of Interest: The authors declare no conflict of interest.

References

[1] S. Agboola and O. O. Adebiyi, "APPLICATION OF POWER NUMERICAL METHOD FOR THE STATIONARY DISTRIBUTION OF MARKOV CHAIN," *FUDMA JOURNAL OF SCIENCES*, vol. 7, no. 2, pp. 19–24, Apr. 2023, doi: 10.33003/fjs-2023-0702-1625.

[2] R. V. Presentación, H. R. Perfecto, R. O. L. Ruiz, W. A. Valencia, and J. S. Rojas, "PROBABILITY OF CONTINUOUS USE OF THE DELIVERY SERVICE IN A LIMA COMMUNITY: AN APPLICATION OF THE MARKOV CHAINS," *International Journal of Professional Business Review*, vol. 8, no. 5, 2023, doi: 10.26668/businessreview/2023.v8i5.1848.

[3] M. Valenzuela, "Markov chains and applications," *Selecciones Matemáticas*, vol. 9, no. 01, pp. 53–78, Jun. 2022, doi: 10.17268/sel.mat.2022.01.05.

[4] P. Georg, L. Grasedyck, M. Klever, R. Schill, R. Spang, and T. Wettig, "Low-rank tensor methods for Markov chains with applications to tumor progression models," *J Math Biol*, vol. 86, no. 1, Jan. 2023, doi: 10.1007/s00285-022-01846-9.

[5] K. Mubiru, M. N. Ssempijja, J. Ochola, M. Nalubowa, S. Namango, and P. Kizito Mubiru, "Application of Markov chains in manufacturing: A review Finite Element Modeling of Flexor Tendon Repair using Specialized Textile Structures View project sodium manufacture View project Application of Markov chains in manufacturing systems: A review," *The International Journal of Industrial Engineering: Theory, Applications and Practice*, 2021, [Online]. Available: http://www.ijieor.ir

[6] F. Smarandache, "Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability," *ArXiv*, 2013.

[7] M. Mullai, K. Sangeetha, R. Surya, G. M. Kumar, R. Jeyabalan, and S. Broumi, "A Single Valued neutrosophic Inventory Model with Neutrosophic Random Variable," *International Journal of Neutrosophic Science*, vol. 1, no. 2, 2020, doi: 10.5281/zenodo.3679510.

[8] F. Smarandache *et al.*, "Introduction to neutrosophy and neutrosophic environment," *Neutrosophic Set in Medical Image Analysis*, pp. 3–29, 2019, doi: 10.1016/B978-0-12-818148-5.00001-1.

[9] M. Akram, Shumaiza, and F. Smarandache, "Decision-making with bipolar neutrosophic TOPSIS and bipolar neutrosophic ELECTRE-I," *Axioms*, vol. 7, no. 2, 2018, doi: 10.3390/axioms7020033.

[10] F. Smarandache, "Indeterminacy in neutrosophic theories and their applications," *International Journal of Neutrosophic Science*, vol. 15, no. 2, 2021, doi: 10.5281/zenodo.5295819.

[11] M. B. Zeina, "Erlang Service Queueing Model with Neutrosophic Parameters," *International Journal of Neutrosophic Science*, vol. 6, no. 2, pp. 106–112, 2020, doi: 10.54216/IJNS.060202.

[12] M. B. Zeina, "Linguistic Single Valued Neutrosophic M/M/1 Queue," *Research Journal of Aleppo University*, vol. 144, 2021, Accessed: Feb. 21, 2023. [Online]. Available:

https://www.researchgate.net/publication/348945390_Linguistic_Single_Valued_Neutrosophic_MM1_Queue

[13] F. Smarandache, "(T, I, F)-Neutrosophic Structures," *Applied Mechanics and Materials*, vol. 811, 2015, doi: 10.4028/www.scientific.net/amm.811.104.

[14] M. Xu, R. Yong, and Y. Belayne, "Decision Making Methods with Linguistic Neutrosophic Information: A Review," *Neutrosophic Sets and Systems*, vol. 38, 2020, doi: 10.5281/zenodo.4300630.

[15] S. Alkhazaleh, "Plithogenic Soft Set," *Neutrosophic Sets and Systems*, 2020.

[16] F. Smarandache, *Plithogeny, Plithogenic Set, Logic, Probability, and Statistics*. Belgium: Pons, 2018. Accessed: Feb. 23, 2023. [Online]. Available: http://arxiv.org/abs/1808.03948

[17] F. Smarandache, "Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)," *Neutrosophic Sets and Systems* , vol. 53, Jan. 2023, doi: 10.5281/ZENODO.7536105.

[18] F. Smarandache, "Plithogenic Probability & Statistics are generalizations of MultiVariate Probability & Statistics," *Neutrosophic Sets and Systems*, vol. 43, 2021.

[19] F. Smarandache, "Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets," *Neutrosophic Sets and Systems*, vol. 21, pp. 153–166, 2018.

[20] M. Abdel-Basset, R. Mohamed, A. E. N. H. Zaied, and F. Smarandache, "A hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics," *Symmetry (Basel)*, vol. 11, no. 7, Jul. 2019, doi: 10.3390/SYM11070903.

[21] M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, "Introduction to Symbolic 2-Plithogenic Probability Theory," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.

[22] M. B. Zeina, M. Abobala, A. Hatip, S. Broumi, and S. Jalal Mosa, "Algebraic Approach to Literal Neutrosophic Kumaraswamy Probability Distribution," *Neutrosophic Sets and Systems*, vol. 54, pp. 124–138, 2023.

[23] F. Smarandache, "Refined Literal Indeterminacy and the Multiplication Law of Sub-Indeterminacies," *Neutrosophic Sets and Systems*, vol. 9, 2015.

[24] F. Smarandache, "Symbolic Neutrosophic Theory," *ArXiv*, 2015, doi: 10.5281/ZENODO.32078.

[25] N. M. Taffach and A. Hatip, "A Review on Symbolic 2-Plithogenic Algebraic Structures," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 5, no. 1, pp. 08–16, 2023, doi: 10.54216/GJMSA.050101.

[26] R. Ali and Z. Hasan, "An Introduction To The Symbolic 3-Plithogenic Modules," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 6, no. 1, pp. 13–17, 2023, doi: 10.54216/GJMSA.060102.

[27] R. Ali and Z. Hasan, "An Introduction to The Symbolic 3-Plithogenic Vector Spaces," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 6, no. 1, pp. 08–12, 2023, doi: 10.54216/GJMSA.060101.

[28] N. M. Taffach and A. Hatip, "A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 5, no. 1, pp. 36–44, 2023, doi: 10.54216/GJMSA.050103.

[29] M. B. Zeina, "Neutrosophic M/M/1, M/M/c, M/M/1/b Queueing Systems," *Research Journal of Aleppo University*, vol. 140, 2020, Accessed: Feb. 21, 2023. [Online]. Available: https://www.researchgate.net/publication/343382302_Neutrosophic_MM1_MMc_MM1b_Queueing_Systems

[30] M. B. Zeina, "Neutrosophic Event-Based Queueing Model," *International Journal of Neutrosophic Science*, vol. 6, no. 1, 2020, doi: 10.5281/zenodo.3840771.

[31] H. Rashad and M. Mohamed, "Neutrosophic Theory and Its Application in Various Queueing Models: Case Studies," *Neutrosophic Sets and Systems*, vol. 42, 2021, doi: 10.5281/zenodo.4711516.

[32] M. B. Zeina and M. Abobala, "A novel approach of neutrosophic continuous probability distributions using AH-isometry with applications in medicine," *Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics*, pp. 267–286, Jan. 2023, doi: 10.1016/B978-0-323-99456-9.00014-3.

[33] A. Astambli, M. B. Zeina, and Y. Karmouta, "On Some Estimation Methods of Neutrosophic Continuous Probability Distributions Using One-Dimensional AH-Isometry," *Neutrosophic Sets and Systems*, vol. 53, 2023.

[34] M. Abobala and A. Hatip, "An Algebraic Approach to Neutrosophic Euclidean Geometry," *Neutrosophic Sets and Systems*, vol. 43, 2021.

[35] M. B. Zeina and A. Hatip, "Neutrosophic Random Variables," *Neutrosophic Sets and Systems*, vol. 39, 2021, doi: 10.5281/zenodo.4444987.

[36] M. B. Zeina and Y. Karmouta, "Introduction to Neutrosophic Stochastic Processes," *Neutrosophic Sets and Systems*, vol. 54, 2023.

[37] M. Abobala, "On the Representation of Neutrosophic Matrices by Neutrosophic Linear Transformations," *Journal of Mathematics*, vol. 2021, 2021, doi: 10.1155/2021/5591576.

[38] C. Granados and J. Sanabria, "On Independence Neutrosophic Random Variables," *Neutrosophic Sets and Systems*, vol. 47, 2021.

[39] K. F. Alhasan, A. A. Salama, and F. Smarandache, "Introduction to neutrosophic reliability theory," *International Journal of Neutrosophic Science*, vol. 15, no. 1, 2021, doi: 10.5281/zenodo.5033829.

[40] M. Bisher Zeina and M. Abobala, "On The Refined Neutrosophic Real Analysis Based on Refined Neutrosophic Algebraic AH-Isometry," *Neutrosophic Sets and Systems*, vol. 54, 2023.

[41] A. Astambli, M. B. Zeina, and Y. Karmouta, "Algebraic Approach to Neutrosophic Confidence Intervals," *Journal of Neutrosophic and Fuzzy Systems*, vol. 5, no. 2, pp. 08–22, 2023, doi: 10.54216/JNFS.050201.

Received: 15/7/2023. Accepted: 9/10/2023

Neutrosophic Sets and Systems, Vol. 59, 2023
University of New Mexico

On The Symbolic 3-Plithogenic Real Functions by Using Special AH-Isometry

Nabil Khuder Salman

College Of Pharmacy, AL-Farahidi University, Baghdad, Iraq

nsalman@uoalfarahidi.edu.iq

Abstract:

Symbolic 3-plithogenic sets as a generalization of classical concept of sets were applicable to algebraic structures. The symbolic 3-plithogenic rings and fields are good generalizations of classical corresponding systems.

In this paper, we study the symbolic 3-plithogenic real functions with one variable by using a special algebraic function called AH-isometry. In addition, we discuss the symbolic 3-plithogenic simple differential equations and conic sections by using this isometry. Also, many examples will be presented to explain the novelty of this work.

Keywords: symbolic 3-plithogenic set, symbolic 3-plithogenic real function, symbolic 3-plithogenic circle, symbolic 3-plithogenic ellipse.

Introduction and basic definitions

Symbolic n-plithogenic algebraic structures are considered as new generalizations of classical algebraic structures [1-3], such as symbolic 2-plithogenic integers, modules, and vector spaces [4-8].

Symbolic 2-plithogenic structures have a similar structure to the refined neutrosophic structures and many non-classical algebraic structures defined by many authors in [9-17,20-23, 24-30].

In the literature, many mathematical approaches were carried out on neutrosophic and refined neutrosophic structures, were a special function called AH-isometry was used to study the analytical properties and conic sections [11-12, 18-19], and that occurs by taking the direct image of neutrosophic elements to the classical Cartesian product of the real field with itself.

In this work, we follow the previous efforts, and we define for the first time a special AH-isometry on the symbolic 3-plithogenic field of reals, and we use this isometry to obtain many formulas and properties about the symbolic 3-plithogenic analytical concepts such as differentiability, continuity, and integrability. Also, symbolic 3-plithogenic conic sections will find a place in our study.

Main Results

Definition.

Let $3-SP_R=\{a+bP_1+cP_2+dP_2;a,b,c,d\in R\}$ be the 3-plithogenic field of real numbers, a function $f=f(X):3-SP_R\to 3-SP_R$ is called one variable symbolic 3-plithogenic real function, with

$X=x_0+x_1P_1+x_2P_2+x_3P_3\in 3-SP_R$.

Definition.

Let $3-SP_R$ be the symbolic 3-plithogenic field of reals, we define its AH-isometry as follows:

$I:3-SP_R\to R\times R\times R\times R$ such that:

$I(x+yP_1+zP_2+tP_3)=(x,x+y,x+y+z,x+y+z+t)$.

It is easy to see that I is a ring isomorphism with the inverse:

$I^{-1}:R\times R\times R\times R\to 3-SP_R$ such that:

$I^{-1}(x,y,z,t)=x+(y-x)P_1+(z-y)P_2+(t-z)P_3$

Definition.

Let $f: 3 - SP_R \to 3 - SP_R$ be a symbolic 3-plithogenic real function with one variable, we define the canonical formula as follows:

$$I^{-1} \circ I(f): 3 - SP_R \to 3 - SP_R$$

Example.

Consider$f(X) = X^2 - P_1 + P_3$, its canonical formula is:

$I\big(f(X)\big) = [I(X)]^2 + I(-P_1 + P_3)$

$$= (x_0, x_0 + x_1, x_0 + x_1 + x_2, x_0 + x_1 + x_2 + x_3)^2 + (0, -1, -1, 0)$$

$$= ({x_0}^2, (x_0 + x_1)^2 - 1, (x_0 + x_1 + x_2)^2 - 1, (x_0 + x_1 + x_2 + x_3)^2)$$

$I^{-1} \circ I\big(f(X)\big) =$

$${x_0}^2 + P_1[(x_0 + x_1)^2 - {x_0}^2 - 1] + P_2[(x_0 + x_1 + x_2)^2 - (x_0 + x_1)^2] + P_3[(x_0 + x_1 + x_2 + x_3)^2 - (x_0 + x_1 + x_2)^2 + 1]$$

For example:

$f(1 + P_3) = (1 + P_3)^2 - P_1 + P_3 = 1 - P_1 + 4P_3.$

If we put values $x_0 = 1, x_1 = 0, x_2 = 0, x_2 = 1$

in the canonical formula, then we get:

$I^{-1} \circ I\big(f(X)\big) = (1)^2 + P_1[(1)^2 - (1)^2 - 1] + P_2[(1)^2 - (1)^2] + P_3[(2)^2 - (1)^2 + 1] = 1 - P_1 + 4P_3.$

The canonical formulas of famous functions:

1. The exponent function:

$e^X; X = x_0 + x_1P_1 + x_2P_2 + x_3P_3, I^{-1} \circ I(e^X) =$

$I^{-1}(e^{x_0}, e^{x_0+x_1}, e^{x_0+x_1+x_2}, e^{x_0+x_1+x_2+x_3}) =$

$e^{x_0} + P_1[e^{x_0+x_1} - e^{x_0}] + P_2[e^{x_0+x_1+x_2} - e^{x_0+x_1}] + P_3[e^{x_0+x_1+x_2+x_3} - e^{x_0+x_1+x_2}]$

2. The logarithmic function:

$\ln(X); X = x_0 + x_1P_1 + x_2P_2 + x_3P_3,$

$I^{-1} \circ I(\ln(X)) = I^{-1}(\ln(x_0), \ln(x_0 + x_1), \ln(x_0 + x_1 + x_2), \ln(x_0 + x_1 + x_2 + x_3)) = \ln(x_0) + P_1[\ln(x_0 + x_1) - \ln(x_0)] + P_2[\ln(x_0 + x_1 + x_2) - \ln(x_0 + x_1)] + P_3[\ln(x_0 + x_1 + x_2 + x_3) - \ln(x_0 + x_1 + x_2)].$

3. Famous trigonometric functions:

$$\sin(X) = \sin(x_0) + P_1[\sin(x_0 + x_1) - \sin(x_0)] + P_2[\sin(x_0 + x_1 + x_2) - \sin(x_0 + x_1)] + P_3[\sin(x_0 + x_1 + x_2 + x_3) - \sin(x_0 + x_1 + x_2)]$$

$\cos(X) = \cos(x_0) + P_1[\cos(x_0 + x_1) - \cos(x_0)]$
$+ P_2[\cos(x_0 + x_1 + x_2) - \cos(x_0 + x_1)]$
$+ P_3[cos(x_0 + x_1 + x_2 + x_3) - cos(x_0 + x_1 + x_2)]$
$\tan(X) = \tan(x_0) + P_1[\text{taan}(x_0 + x_1) - \tan(x_0)]$
$+ P_2[\tan(x_0 + x_1 + x_2) - \tan(x_0 + x_1)]$
$+ P_3[tan(x_0 + x_1 + x_2 + x_3) - tan(x_0 + x_1 + x_2)]$

And so no.

Definition.

A symbolic 3-plithogenic real number $T = t_0 + t_1P_1 + t_2P_2 + t_3P_3$ is called positive if and only if

$t_0 \geq 0, t_0 + t_1 \geq 0, t_0 + t_1 + t_2 \geq 0, t_0 + t_1 + t_2 + t_3 \geq 0.$

For example $3 + 2P_1 - P_3 > 0$, that is because, $3 > 0$, $5 > 0, 5 > 0, 3 + 2 - 1 = 4 > 0$.

Definition.

Let $f: 3 - SP_R \to 3 - SP_R$ be a symbolic 3-plithogenic real function with one variable $X = x_0 + x_1P_1 + x_2P_2 + x_3P_3$, then:

a. f is differentiable if and only if $I(f(X))$ is differentiable.
b. f is continuous if and only if $I(f(X))$ is continuous.
c. f is integrable if and only if $I(f(X))$ is integrable.

Example.

Find the derivation of $f(X) = X^2 + P_3$ in two different ways.

Solution.

The regular way is $\acute{f}(X) = 2X$.

The canonical way is:

$I^{-1} \circ I(f(X)) =$
$I^{-1}({x_0}^2, (x_0 + x_1)^2, (x_0 + x_1 + x_2)^2, (x_0 + x_1 + x_2 + x_3)^2) + I^{-1}(0,0,0,1) = {x_0}^2 + P_1[(x_0 + x_1)^2 - {x_0}^2] + P_2[(x_0 + x_1 + x_2)^2 - (x_0 + x_1)^2] + P_3[(x_0 + x_1 + x_2 + x_3)^2 - (x_0 + x_1 + x_2)^2 + 1]$.

First, we have: $({x_0}^2)'_{x_0} = 2x_0$.

$[(x_0 + x_1)^2]'_{x_0+x_1} = 2(x_0 + x_1)$,

$[(x_0+x_1+x_2)^2]'_{x_0+x_1+x_2} = 2(x_0+x_1+x_2)$,

$[(x_0+x_1+x_2+x_3)^2]'_{x_0+x_1+x_2+x_3} = 2(x_0+x_1+x_2+x_3)$,

Thus,

$\acute{f}(X) = 2x_0 + P_1[2(x_0+x_1) - 2x_0] + P_2[2(x_0+x_1+x_2) - 2(x_0+x_1)] + P_3[2(x_0+x_1+x_2+x_3) - 2(x_0+x_1+x_2)] = (2x_0) + 2x_1P_1 + 2x_2P_2 + 2x_3P_3 = 2X$.

Example.

Find the value of $\int_0^{1+P_1+P_2+P_3} e^X dX$ in two different ways.

Solution;

The regular way: $\int_0^{1+P_1+P_2+P_3} e^X dX = [e^X]_0^{1+P_1+P_2+P_3} = e^{1+P_1+P_2+P_3} - e^0 = e^{1+P_1+P_2+P_3} - 1$.

The canonical formula way:

$$I^{-1} \circ I(f(X)) = e^{x_0} + P_1[e^{x_0+x_1} - e^{x_0}] + P_2[e^{x_0+x_1+x_2} - e^{x_0+x_1}] + P_3[e^{x_0+x_1+x_2+x_3} - e^{x_0+x_1+x_2}]$$

We have:

$$\int_0^1 e^{x_0} dx_0 = e - 1, \int_0^2 e^{e^{x_0+x_1}} d(x_0+x_1) = e^2 - 1, \int_0^3 e^{x_0+x_1+x_2} d(x_0+x_1+x_2)$$
$$= e^3 - 1, \int_0^3 e^{x_0+x_1+x_2+x_3} d(x_0+x_1+x_2+x_3) = e^4 - 1$$

Thus, $\int_0^{1+P_1+P_2+P_3} e^X dX = e - 1 + P_1[e^2 - 1 - e + 1] + P_2[e^3 - 1 - e^2 + 1] + P_3[e^4 - 1 - e^3 + 1] = e - 1 + P_1[e^2 - e] + P_2[e^3 - e^2] + P_3[e^4 - e^3] = e^{1+P_1+P_2+P_3} - 1$

Applications to differential equations.

Example.

Solve the equation $\acute{Y} = C; Y = y_0 + y_1P_1 + y_2P_2 + y_3P_3$ is a function, and $C = c_0 + c_1P_1 + c_2P_2 + c_3P_3$ is a constant.

We have $y_0 = f_0, y_1 = f_1, y_2 = f_2, y_3 = f_3 : R \to R$.

$$\acute{Y} = (y_0)'_{x_0} + P_1\left[(y_0+y_1)'_{x_0+x_1} - (y_0)'_{x_0}\right] +$$

$P_2\left[(y_0+y_1+y_2)'_{x_0+x_1+x_2}-(y_0+y_1)'_{x_0+x_1}\right]+P_3\left[(y_0+y_1+y_2+y_3)'_{x_0+x_1+x_2+x_3}-\right.$

$\left.(y_0+y_1+y_2)'_{x_0+x_1+x_2}\right]=c_0+c_1P_1+c_2P_2+c_3P_3,$

so that:

$$(y_0)'_{x_0}=c_0$$
$$(y_0+y_1)'_{x_0+x_1}=c_1$$
$$(y_0+y_1+y_2)'_{x_0+x_1+x_2}=c_2$$
$$(y_0+y_1+y_2+y_3)'_{x_0+x_1+x_2+x_3}=c_3$$

$$y_0=c_0x_0+m_0$$
$$y_0+y_1=c_1(x_0+x_1)+m_1$$
$$y_0+y_1+y_2=c_2(x_0+x_1+x_2)+m_2$$
$$y_0+y_1+y_2+y_3=c_3(x_0+x_1+x_2+x_3)+m_3$$

This implies that:

$$Y=(c_0x_0+m_0)+P_1[c_1(x_0+x_1)+m_1-(c_0x_0+m_0)]+$$
$$P_2[c_2(x_0+x_1+x_2)+m_2-c_1(x_0+x_1)+m_1]+$$

$P_3[c_3(x_0+x_1+x_2+x_3)+m_3-c_2(x_0+x_1+x_2)-m_2];$

x_i are real variables, m_i are real constants.

Example.

Solve the differential equation $\acute{Y}=CY$, where $C=c_0+c_1P_1+c_2P_2+c_3P_3, Y=y_0+$ $y_1P_1+y_2P_2+y_3P_3$.

Solution.

$\acute{Y}=CY$ equivalents:

$$\begin{pmatrix} (y_0)'_{x_0}=c_0y_0 \\ (y_0+y_1)'_{x_0+x_1}=(c_0+c_1)(y_0+y_1) \\ (y_0+y_1+y_2)'_{x_0+x_1+x_2}=(c_0+c_1+c_2)(y_0+y_1+y_2) \\ (y_0+y_1+y_2+y_3)'_{x_0+x_1+x_2+x_3}=(c_0+c_1+c_2+c_3)(y_0+y_1+y_2+y_3) \end{pmatrix}$$

So that:

$$\begin{pmatrix} y_0=k_0e^{c_0x_0} \\ y_0+y_1=k_1e^{(c_0+c_1)(x_0+x_1)} \\ y_0+y_1+y_2=k_2e^{(c_0+c_1+c_2)(x_0+x_1+x_2)} \\ y_0+y_1+y_2+y_3=k_3e^{(c_0+c_1+c_2+c_3)(x_0+x_1+x_2+x_3)} \end{pmatrix}$$

Thus:

$Y = k_0e^{c_0x_0} + P_1\left[k_1e^{(c_0+c_1)(x_0+x_1)} - k_0e^{c_0x_0}\right] + P_2\left[k_2e^{(c_0+c_1+c_2)(x_0+x_1+x_2)} - k_1e^{(c_0+c_1)(x_0+x_1)}\right] + P_3\left[k_3e^{(c_0+c_1+c_2+c_3)(x_0+x_1+x_2+x_3)} - k_2e^{(c_0+c_1+c_2)(x_0+x_1+x_2)}\right].$

Applications to geometric shapes:

Definition.

1). We define the symbolic 3-plithogenic circle as follows:

$(X - A)^2 + (Y - B)^2 = R^2; A = a_0 + a_1P_1 + a_2P_2 + a_3P_3, B = b_0 + b_1P_1 + b_2P_2 + b_3P_3, R = r_0 + r_1P_1 + r_2P_2 + r_3P_3, Y = y_0 + y_1P_1 + y_2P_2 + y_3P_3, x = x_0 + x_1P_1 + x_2P_2 + x_3P_3$, with $a_i, b_i, r_i, x_i, y_i \in R$

2). We define the symbolic 3-plithogenic sphere as follows:

$(X - A)^2 + (Y - B)^2 + (Z - C)^2 = R^2; X, A, B, C, R, Y, Z \in 3 - SP_R$

3). We define the symbolic 3-plithogenic ellipse as follows:

$\frac{(X-A)^2}{T^2} + \frac{(Y-B)^2}{S^2} = 1; X, A, B, T, S, Y \in 3 - SP_R$ and T, S invertible.

4). We define the symbolic 3-plithogenic hyperbola as follows:

$\frac{(X-A)^2}{T^2} - \frac{(Y-B)^2}{S^2} = 1; X, A, B, T, S, Y \in 3 - SP_R$ and T, S invertible.

Example.

1). $(X - 1 + P_2)^2 + (Y - 3 + 2P_1 - P_2 - P_3)^2 = (1 + 2P_3)^2$ is a 3-plithogenic circle.

2). $(X - 10 + P_1 + P_2)^2 + (Y + P_3)^2 + (Z - P_1 + P_3)^2 = (1 + P_2 + 5P_3)^2$ is a 3-plithogenic sphere.

3). $\frac{(X-P_1-P_3)^2}{(1+P_1+P_3)^2} + \frac{(Y+P_1-P_2)^2}{(2-P_1+5P_3)^2} = 1$ is a 3-plithogenic ellipse.

4). $\frac{(X-2P_2)^2}{(1+P_1+3P_2)^2} - \frac{(Y+1+4P_2)^2}{(13-2P_1+P_2)^2} = 1$ is a 3-plithogenic hyperbola.

Theorem.

1. Any symbolic 3-plithogenic circle is equivalent to four classical circles.
2. Any symbolic 3-plithogenic sphare is equivalent to four classical spheres.
3. Any symbolic 3-plithogenic ellipse is equivalent to four classical ellipses.
4. Any symbolic 3-plithogenic hyperbola is equivalent to four classical hyperbolas.

Proof.

1. Consider the symbolic 3-plithogenic circle:

$(X-A)^2+(Y-B)^2=R^2$, then by using the isomorphism defined before, we get:

$$I[(X-A)^2]=[I(X)-I(A)]^2$$
$$=\big(x_0-a_0,(x_0+x_1)-(a_0+a_1),(x_0+x_1+x_2)$$
$$-(a_0+a_1+a_2),(x_0+x_1+x_2+x_3)-(a_0+a_1+a_2+a_3)\big)^2$$
$$=\Big((x_0-a_0)^2,\big((x_0+x_1)-(a_0+a_1)\big)^2,((x_0+x_1+x_2)$$
$$-(a_0+a_1+a_2))^2,((x_0+x_1+x_2+x_3)-(a_0+a_1+a_2+a_3))^2\Big)$$

$$I[(Y-B)^2]=[I(Y)-I(B)]^2$$
$$=\Big((y_0-b_0)^2,\big((y_0+y_1)-(b_0+b_1)\big)^2,\big((y_0+y_1+y_2)$$
$$-(b_0+b_1+b_2)\big)^2,((y_0+y_1+y_2+y_3)-(b_0+b_1+b_2+b_3))^2\Big)$$

$$I(R^2)=[I(R)]^2=({r_0}^2,(r_0+r_1)^2,(r_0+r_1+r_2)^2,(r_0+r_1+r_2+r_3)^2)$$

Thus, it is equivalent to:

$$\begin{pmatrix} (x_0-a_0)^2+(y_0-b_0)^2={r_0}^2 \\ \big((x_0+x_1)-(a_0+a_1)\big)^2+\big((y_0+y_1)-(b_0+b_1)\big)^2=(r_0+r_1)^2 \\ \big((x_0+x_1+x_2)-(a_0+a_1+a_2)\big)^2+\big((y_0+y_1+y_2)-(b_0+b_1+b_2)\big)^2=(r_0+r_1+r_2)^2 \\ \big((x_0+x_1+x_2+x_3)-(a_0+a_1+a_2+a_3)\big)^2+\big((y_0+y_1+y_2+y_3)-(b_0+b_1+b_2+b_3)\big)^2=(r_0+r_1+r_2+r_3)^2 \end{pmatrix}$$

2. Consider the sphere $(X-A)^2+(Y-B)^2+(Z-C)^2=R^2$, we use the isomorphism I, to get:

$$I[(X-A)^2]=\Big((x_0-a_0)^2,\big((x_0+x_1)-(a_0+a_1)\big)^2,((x_0+x_1+x_2)$$
$$-(a_0+a_1+a_2))^2,((x_0+x_1+x_2+x_3)-(a_0+a_1+a_2+a_3))^2\Big)$$
$$I[(Y-B)^2]=\Big((y_0-b_0)^2,\big((y_0+y_1)-(b_0+b_1)\big)^2,\big((y_0+y_1+y_2)$$
$$-(b_0+b_1+b_2)\big)^2,((y_0+y_1+y_2+y_3)-(b_0+b_1+b_2+b_3))^2\Big)$$
$$I[(Z-C)^2]=\Big((z_0-c_0)^2,\big((z_0+z_1)-(c_0+c_1)\big)^2,\big((z_0+z_1+z_2)$$
$$-(c_0+c_1+c_2)\big)^2,\big((z_0+z_1+z_2+z_3)-(c_0+c_1+c_2+c_3)\big)^2\Big)$$
$$I(R^2)=[I(R)]^2=({r_0}^2,(r_0+r_1)^2,(r_0+r_1+r_2)^2,(r_0+r_1+r_2+r_3)^2),$$

hence we get:

$$\begin{cases} (x_0-a_0)^2+(y_0-b_0)^2+(z_0-c_0)^2={r_0}^2 \\ \big((x_0+x_1)-(a_0+a_1)\big)^2+\big((y_0+y_1)-(b_0+b_1)\big)^2+\big((z_0+z_1)-(c_0+c_1)\big)^2=(r_0+r_1)^2 \\ \big((x_0+x_1+x_2)-(a_0+a_1+a_2)\big)^2+\big((y_0+y_1+y_2)-(b_0+b_1+b_2)\big)^2+\big((z_0+z_1+z_2)-(c_0+c_1+c_2)\big)^2=(r_0+r_1+r_2)^2 \end{cases}$$

And

$$\left((x_0+x_1+x_2+x_3)-(a_0+a_1+a_2+a_3)\right)^2+\left((y_0+y_1+y_2+y_3)-(b_0+b_1+b_2+b_3)\right)^2+\left((z_0+z_1+z_2+z_3)-(c_0+c_1+c_2+c_3)\right)^2=(r_0+r_1+r_2+r_3)^2.$$

3. Consider the ellipse $\frac{(X-A)^2}{T^2}+\frac{(Y-B)^2}{S^2}=1$, we use the isomorphism I to get:

$$I\left[\frac{(X-A)^2}{T^2}\right]=\left(\frac{(x_0-a_0)^2}{{t_0}^2},\frac{\left((x_0+x_1)-(a_0+a_1)\right)^2}{(t_0+t_1)^2},\frac{\left((x_0+x_1+x_2)-(a_0+a_1+a_2)\right)^2}{(t_0+t_1+t_2)^2},\frac{\left((x_0+x_1+x_2+x_3)-(a_0+a_1+a_2+a_3)\right)^2}{(t_0+t_1+t_2+t_3)^2}\right)$$

$$I\left[\frac{(Y-B)^2}{S^2}\right]=\left(\frac{(y_0-b_0)^2}{{s_0}^2},\frac{\left((y_0+y_1)-(b_0+b_1)\right)^2}{(s_0+s_1)^2},\frac{\left((y_0+y_1+y_2)-(b_0+b_1+b_2)\right)^2}{(s_0+s_1+s_2)^2},\frac{\left((y_0+y_1+y_2+y_3)-(b_0+b_1+b_2+b_3)\right)^2}{(s_0+s_1+s_2+s_3)^2}\right)$$

$I(1)=(1,1,1,1)$, thus:

$$\begin{cases}\frac{(x_0-a_0)^2}{{t_0}^2}+\frac{(y_0-b_0)^2}{{s_0}^2}=1\\ \frac{\left((x_0+x_1)-(a_0+a_1)\right)^2}{(t_0+t_1)^2}+\frac{\left((y_0+y_1)-(b_0+b_1)\right)^2}{(s_0+s_1)^2}=1\\ \frac{\left((x_0+x_1+x_2)-(a_0+a_1+a_2)\right)^2}{(t_0+t_1+t_2)^2}+\frac{\left((y_0+y_1+y_2)-(b_0+b_1+b_2)\right)^2}{(s_0+s_1+s_2)^2}=1\end{cases}$$

And,

$$\frac{\left((x_0+x_1+x_2+x_3)-(a_0+a_1+a_2+a_3)\right)^2}{(t_0+t_1+t_2+t_3)^2}+\frac{\left((y_0+y_1+y_2+y_3)-(b_0+b_1+b_2+b_3)\right)^2}{(s_0+s_1+s_2+s_3)^2}=1$$

4. Consider the hyperbola $\frac{(X-A)^2}{T^2}-\frac{(Y-B)^2}{S^2}=1$, by a similar discussion, we get

$$\begin{cases}\frac{(x_0-a_0)^2}{{t_0}^2}-\frac{(y_0-b_0)^2}{{s_0}^2}=1\\ \frac{\left((x_0+x_1)-(a_0+a_1)\right)^2}{(t_0+t_1)^2}-\frac{\left((y_0+y_1)-(b_0+b_1)\right)^2}{(s_0+s_1)^2}=1\\ \frac{\left((x_0+x_1+x_2)-(a_0+a_1+a_2)\right)^2}{(t_0+t_1+t_2)^2}-\frac{\left((y_0+y_1+y_2)-(b_0+b_1+b_2)\right)^2}{(s_0+s_1+s_2)^2}=1\end{cases}$$

and

$$\frac{\left((x_0+x_1+x_2+x_3)-(a_0+a_1+a_2+a_3)\right)^2}{(t_0+t_1+t_2+t_3)^2} - \frac{\left((y_0+y_1+y_2+y_3)-(b_0+b_1+b_2+b_3)\right)^2}{(s_0+s_1+s_2+s_3)^2} = 1$$

Example.

Consider the symbolic 3-plithogenic ellipse:

$$\frac{(X-2-4P_1+3P_2+P_3)^2}{(1+5P_1+7P_2-P_3)^2} + \frac{(Y+1+P_2+11P_3)^2}{(3-P_1-P_2+4P_3)^2} = 1$$

it is equivalent to:

$$\begin{cases} \frac{(x_0-2)^2}{1} + \frac{(y_0+1)^2}{3^2} = 1 \\ \frac{[(x_0+x_1)-6]^2}{6^2} + \frac{[(y_0+y_1)+1]^2}{2^2} = 1 \\ \frac{[(x_0+x_1+x_2)-3]^2}{13^2} + \frac{[(y_0+y_1+y_2)+2]^2}{1} = 1 \\ \frac{[(x_0+x_1+x_2+x_3)-2]^2}{12^2} + \frac{[(y_0+y_1+y_2+y_3)+13]^2}{5^2} = 1 \end{cases}$$

Example.

Consider the symbolic 3-plithogenic circle:

$(X-2-4P_1+3P_2+P_3)^2 + (Y+1+P_2+11P_3)^2 = 1$

It is equivalent to:

$$\begin{cases} (x_0-2)^2 + (y_0+1)^2 = 1 \\ [(x_0+x_1)-6]^2 + [(y_0+y_1)+1]^2 = 1 \\ [(x_0+x_1+x_2)-3]^2 + [(y_0+y_1+y_2)+2]^2 = 1 \\ [(x_0+x_1+x_2+x_3)-2]^2 + [(y_0+y_1+y_2+y_3)+13]^2 = 1 \end{cases}$$

Example.

Consider the symbolic 3-plithogenic hyperbola:

$$\frac{(X-2-4P_1+3P_2+P_3)^2}{(1+5P_1+7P_2-P_3)^2} - \frac{(Y+1+P_2+11P_3)^2}{(3-P_1-P_2+4P_3)^2} = 1$$

it is equivalent to:

$$\begin{cases} \frac{(x_0-2)^2}{1} - \frac{(y_0+1)^2}{3^2} = 1 \\ \frac{[(x_0+x_1)-6]^2}{6^2} - \frac{[(y_0+y_1)+1]^2}{2^2} = 1 \\ \frac{[(x_0+x_1+x_2)-3]^2}{13^2} - \frac{[(y_0+y_1+y_2)+2]^2}{1} = 1 \\ \frac{[(x_0+x_1+x_2+x_3)-2]^2}{12^2} - \frac{[(y_0+y_1+y_2+y_3)+13]^2}{5^2} = 1 \end{cases}$$

Conclusion

In this paper, we defined for the first time a special AH-isometry on the symbolic 3-plithogenic fields of reals, and we used this isometry to obtain many formulas and properties about the symbolic 3-plithogenic analytical concepts such as differentiability, continuity, and integrability. Also, symbolic 2 3-plithogenic conic sections were handled by using the mentioned isometry. In addition, many related examples were presented to clarify the novelty of our work.

References

1. Nader Mahmoud Taffach , Ahmed Hatip., "A Review on Symbolic 2-Plithogenic Algebraic Structures " Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.
2. Nader Mahmoud Taffach , Ahmed Hatip.," A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations ", Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.
3. Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.
4. Smarandache, F., " Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)", Neutrosophic Sets and Systems, vol. 53, 2023.
5. Taffach, N., " An Introduction to Symbolic 2-Plithogenic Vector Spaces Generated from The Fusion of Symbolic Plithogenic Sets and Vector Spaces", Neutrosophic Sets and Systems, Vol 54, 2023.

6. Ben Othman, K., Albasheer, O., Nadweh, R., Von Shtawzen, O., and Ali, R., "On The Symbolic 6-Plithogenic and 7-Plithogenic Rings", Galoitica Journal of Mathematical Structures and Applications, Vol.8, 2023.
7. Merkepci, H., and Rawashdeh, A., " On The Symbolic 2-Plithogenic Number Theory and Integers ", Neutrosophic Sets and Systems, Vol 54, 2023.
8. Albasheer, O., Hajjari., A., and Dalla., R., " On The Symbolic 3-Plithogenic Rings and TheirAlgebraic Properties", Neutrosophic Sets and Systems, Vol 54, 2023.
9. Abobala, M., On Refined Neutrosophic Matrices and Their Applications In Refined Neutrosophic Algebraic Equations, Journal Of Mathematics, Hindawi, 2021
10. Olgun, N., Hatip, A., Bal, M., and Abobala, M., " A Novel Approach To Necessary and Sufficient Conditions For The Diagonalization of Refined Neutrosophic Matrices", International Journal of neutrosophic Science, Vol. 16, pp. 72-79, 2021.
11. Merkepci, H., and Abobala, M., " The Application of AH-isometry In The Study Of Neutrosophic Conic Sections", Galoitica Journal Of Mathematical Structures And Applications, Vol.2, 2022.
12. Abobala, M., and Zeina, M.B., " A Study Of Neutrosophic Real Analysis By Using One Dimensional Geometric AH-Isometry", Galoitica Journal Of Mathematical Structures And Applications, Vol.3, 2023.
13. Khaldi, A., " A Study On Split-Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.
14. Ahmad, K., " On Some Split-Complex Diophantine Equations", Neoma Journal Of Mathematics and Computer Science, 2023.

15. Ali, R., " On The Weak Fuzzy Complex Inner Products On Weak Fuzzy Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.

16. Von Shtawzen, O., " Conjectures For Invertible Diophantine Equations Of 3-Cyclic and 4-Cyclic Refined Integers", Journal Of Neutrosophic And Fuzzy Systems, Vol.3, 2022.

17. Basheer, A., Ahmad, K., and Ali, R., " A Short Contribution To Von Shtawzen's Abelian Group In n-Cyclic Refined Neutrosophic Rings", Journal Of Neutrosophic And Fuzzy Systems,, 2022.
18. Bisher Ziena, M., and Abobala, M., " On The Refined Neutrosophic Real Analysis Based on Refined Neutrosophic Algebraic AH-Isometry", Neutrosophic Sets and Systems, vol. 54, 2023.
19. M. B. Zeina and M. Abobala, "A Novel Approach of Neutrosophic Continuous Probability Distributions using AH-Isometry with Applications in Medicine," in *Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics*, Elsevier, 2023.
20. Ali, R., and Hasan, Z., " An Introduction To The Symbolic 3-Plithogenic Modules ", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.
21. M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, "Introduction to Symbolic 2-Plithogenic Probability Theory," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.
22. Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.
23. Ben Othman, K., "On Some Algorithms For Solving Symbolic 3-Plithogenic Equations", Neoma Journal Of Mathematics and Computer Science, 2023.

24. Florentin Smarandache, Plithogenic Algebraic Structures. Chapter in "Nidus idearum Scilogs, V: joining the dots" (third version), Pons Publishing Brussels, pp. 123-125, 2019.
25. Florentin Smarandache: Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets – Revisited, *Neutrosophic Sets and Systems*, vol. 21, 2018, pp. 153-166.
26. Sarkis, M., " On The Solutions Of Fermat's Diophantine Equation In 3-refined Neutrosophic Ring of Integers", Neoma Journal of Mathematics and Computer Science, 2023.
27. Abobala, M., "On Some Algebraic Properties of n-Refined Neutrosophic Elements and n-Refined Neutrosophic Linear Equations", Mathematical Problems in Engineering, Hindawi, 2021
28. Abobala, M., "A Study Of Nil Ideals and Kothe's Conjecture In Neutrosophic Rings", International Journal of Mathematics and Mathematical Sciences, hindawi, 2021
29. Abobala, M., "On The Characterization of Maximal and Minimal Ideals In Several Neutrosophic Rings", Neutrosophic sets and systems, Vol. 45, 2021.
30. Abobala, M., Bal, M., Aswad, M., "A Short Note On Some Novel Applications of Semi Module Homomorphisms", International journal of neutrosophic science, 2022.
31. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Concept Of Symbolic 7-Plithogenic Real Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.
32. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Symbolic 8-Plithogenic Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

Received 23/6/2023, Accepted 28/9/2023

On The Conditions for Symbolic 3-Plithogenic Pythagoras Quadruples

Abuobida Mohammed A. Alfahal [1,*], Yaser Ahmad Alhasan [2], Raja Abdullah Abdulfatah[3], Sara Sawalmeh[4]

[1]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia
[2]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia
[3]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia
[4.] Mutah University, Faculty of Science, Mutah, Jordan

Emails: a.alfahal@psau.edu.sa[1.]; [2.] y.alhasan@psau.edu.sa;[3.] r.abdulfatah@psau.edu.sa ;
Corresponding author: : a.alfahal@psau.edu.sa

Abstract: The objective of this paper is to find the necessary and sufficient conditions for a symbolic 3-plithogenic quadruple

$(t_0+t_1P_1+t_2P_2+t_3P_3, s_0+s_1P_1+s_2P_2+s_3P_3, k_0+k_1P_1+k_2P_2+k_3P_3, l_0+l_1P_1+l_2P_2+l_3P_3)$ to be a Pythagoras quadruple, i.e. to be a solution for the non-linear Diophantine equation in four variables $X^2+Y^2+Z^2=T^2$.

Also, many examples will be illustrated and presented to explain how the theorems work.

Keywords: symbolic 3-plithogenic ring, Pythagoras quadruple, Pythagoras Diophantine equation

Introduction and Preliminaries.

Symbolic n-plithogenic sets were defined by Smarandache in [1-3], where these sets were used in generalizing classical algebraic structures such as symbolic 2-plithogenic and symbolic 3-plithogenic structures [4-9], with many applications in other fields [10-12].

It is useful to refer that symbolic n-plithogenic algebraic structures are very similar to neutrosophic and refined neutrosophic structures, see [13-22].

zenodo.10031190/10.5281

In this paper, we continue other efforts to study Pythagoras triples in many different rings [23-26].

We present the concept of Pythagoras triple in a symbolic 3-plithogenic commutative ring with many clear examples that clarify the validity of our work.

Definition.

Let R be a ring, the symbolic 3-plithogenic ring is defined as follows:

$3-SP_R=\{a_0+a_1P_1+a_2P_2+a_3P_3;\ a_i\in R, {P_j}^2=P_j, P_i\times P_j=P_{max(i,j)}\}$.

Smarandache has defined algebraic operations on $3-SP_R$ as follows:

Addition:

$[a_0+a_1P_1+a_2P_2+a_3P_3]+[b_0+b_1P_1+b_2P_2+b_3P_3]=(a_0+b_0)+(a_1+b_1)P_1+(a_2+b_2)P_2+(a_3+b_3)P_3$.

Multiplication:

$[a_0+a_1P_1+a_2P_2+a_3P_3].[b_0+b_1P_1+b_2P_2+b_3P_3]=a_0b_0+a_0b_1P_1+a_0b_2P_2+a_0b_3P_3+a_1b_0{P_1}^2+a_1b_2P_1P_2+a_2b_0P_2+a_2b_1P_1P_2+a_2b_2{P_2}^2+a_1b_3P_3P_1+a_2b_3P_2P_3+a_3b_3(P_3)^2+a_3b_0P_3+a_3b_1P_3P_1+a_3b_2P_2P_3+a_1b_1P_1P_1=a_0b_0+(a_0b_1+a_1b_0+a_1b_1)P_1+(a_0b_2+a_1b_2+a_2b_0+a_2b_1+a_2b_2)P_2+(a_0b_3+a_1b_3+a_2b_3+a_3b_3+a_3b_0+a_3b_1+a_3b_2)P_3$.

Definition.

Let $T=t_0+t_1P_1+t_2P_2+t_3P_3, S=s_0+s_1P_1+s_2P_2+s_3P_3, K=k_0+k_1P_1+k_2P_2+k_3P_3, L=l_0+l_1P_1+l_2P_2+l_3P_3$ be four symbolic 3-plithogenic elements of a symbolic 3-plithogenic commutative ring $3-SP_R$, then (T,S,K,L) is called a symbolic 3-plithogenic Pythagoras quadruple if and only if $T^2+S^2+K^2=L^2$.

Theorem.

Let $T=t_0+t_1P_1+t_2P_2+t_3P_3, S=s_0+s_1P_1+s_2P_2+s_3P_3, K=k_0+k_1P_1+k_2P_2+k_3P_3, L=l_0+l_1P_1+l_2P_2+l_3P_3\in 3-SP_R$, then (T,S,K,L) is a Pythagoras quadruple if and only if:

$(t_0,s_0,k_0,l_0),(t_0+t_1,s_0+s_1,k_0+k_1,l_0+l_1),(t_0+t_1+t_2,s_0+s_1+s_2,k_0+k_1+k_2,l_0+l_1+l_2),(t_0+t_1+t_2+t_3,s_0+s_1+s_2+s_3,k_0+k_1+k_2+k_3,l_0+l_1+l_2+l_3)$ are four Pythagoras quadruples in R.

Proof.

We have:

$$T^2 = {t_0}^2 + [(t_0 + t_1)^2 - {t_0}^2]P_1 + [(t_0 + t_1 + t_2)^2 - (t_0 + t_1)^2]P_2 + [(t_0 + t_1 + t_2 + t_3)^2 - (t_0 + t_1 + t_2)^2]P_3$$

$$S^2 = {s_0}^2 + [(s_0 + s_1)^2 - {s_0}^2]P_1 + [(s_0 + s_1 + s_2)^2 - (s_0 + s_1)^2]P_2 + [(s_0 + s_1 + s_2 + s_3)^2 - (s_0 + s_1 + s_2)^2]P_3$$

$$K^2 = {k_0}^2 + \left[(k_0 + k_1)^2 - {k_0}^2\right]P_1 + [(k_0 + k_1 + k_2)^2 - (k_0 + k_1)^2]P_2 + [(k_0 + k_1 + k_2 + k_3)^2 - (k_0 + k_1 + k_2)^2]P_3$$

$$L^2 = {l_0}^2 + \left[(l_0 + l_1)^2 - {l_0}^2\right]P_1 + [(l_0 + l_1 + l_2)^2 - (l_0 + l_1)^2]P_2 + [(l_0 + l_1 + l_2 + l_3)^2 - (l_0 + l_1 + l_2)^2]P_3$$

The equation $T^2 + S^2 + K^2 = L^2$ I equivalent to:

${t_0}^2 + {s_0}^2 + {k_0}^2 = {l_0}^2$ (1)

$(t_0 + t_1)^2 + (s_0 + s_1)^2 + (k_0 + k_1)^2 = (l_0 + l_1)^2$ (2)

$(t_0 + t_1 + t_2)^2 + (s_0 + s_1 + s_2)^2 + (k_0 + k_1 + k_2)^2 = (l_0 + l_1 + l_2)^2$ (3)

$(t_0 + t_1 + t_2 + t_3)^2 + (s_0 + s_1 + s_2 + s_3)^2 + (k_0 + k_1 + k_2 + k_3)^2 = (l_0 + l_1 + l_2 + l_3)^2$ (4)

Thus, the proof holds.

Theorem.

Let $(t_0, s_0, k_0, l_0), (t_1, s_1, k_1, l_1), (t_2, s_2, k_2, l_2), (t_3, s_3, k_3, l_3)$ be four Pythagoras quadruples in R, then the corresponding pythagoras quadruple in $3 - SP_R$ is (T, S, K, L), where:

$T = t_0 + [t_1 - t_0]P_1 + [t_2 - t_1]P_2 + [t_3 - t_2]P_3$

$S = s_0 + [s_1 - s_0]P_1 + [s_2 - s_1]P_2 + [s_3 - s_2]P_3$

$K = k_0 + [k_1 - k_0]P_1 + [k_2 - k_1]P_2 + [k_3 - k_2]P_3$

$L = l_0 + [l_1 - l_0]P_1 + [l_2 - l_1]P_2 + [l_3 - l_2]P_3$

Proof.

We must compute $T^2 + S^2 + K^2$,

$$T^2+S^2+K^2={t_0}^2+({t_1}^2-{t_0}^2)P_1+({t_2}^2-{t_1}^2)P_2+({t_3}^2-{t_2}^2)P_3+{s_0}^2$$
$$+({s_1}^2-{s_0}^2)P_1+({s_2}^2-{s_1}^2)P_2+({s_3}^2-{s_2}^2)P_3+{k_0}^2+\left({k_1}^2-{k_0}^2\right)P_1$$
$$+\left({k_2}^2-{k_1}^2\right)P_2+\left({k_3}^2-{k_2}^2\right)P_3$$
$$=\left({t_0}^2+{s_0}^2+{k_0}^2\right)+\left({t_1}^2+{s_1}^2+{k_1}^2-{t_0}^2-{s_0}^2-{k_0}^2\right)P_1$$
$$+\left({t_2}^2+{s_2}^2+{k_2}^2-{t_1}^2-{s_1}^2-{k_1}^2\right)P_2$$
$$+\left({t_3}^2+{s_3}^2+{k_3}^2-{t_2}^2-{s_2}^2-{k_2}^2\right)P_3$$
$$={l_0}^2+\left({l_1}^2-{l_0}^2\right)P_1+\left({l_2}^2-{l_1}^2\right)P_2+\left({l_3}^2-{l_2}^2\right)P_3=L^2$$

O that, the proof is complete.

Example:

We have $L_1=(1,-1,i,1), L_2=(i,1,-1,-1), L_3=(-i,-1,1,-1), L_4=(1,-i,-1,-1)$ are four Pythagoras quadruples in C.

The corresponding 3-plithogenic Pythagoras quadruple is (T,S,K,L), where:

$T=1+(-1+i)P_1-2iP_2+(1+i)P_3$

$S=-1+2P_1-2P_2+(1-i)P_3$

$K=i+(-1-i)P_1+2P_2-2P_3$

$L=1-2P_1+2P_2-2P_3$

On the other hand, we have:

$$T^2=1-2iP_1-4P_2+2iP_3+2(-1+i)P_1-4iP_2+2(1+i)P_3-4iP_2$$
$$+2(-1+i)(1+i)P_3-4i(1+i)P_3$$
$$=1+(-2i-2+2i)P_1+(-4-4i+4i+4)P_2$$
$$+(2i+2+2i-4-4i+4)P_3=1-2P_1+2P_3$$

$$S^2=1+4P_1+4P_2-2iP_3-4P_1+4P_2+2(-1+i)P_3-8P_2+4(1-i)P_3$$
$$-4(1-i)P_3$$
$$=1+(4-4)P_1+(4+4-8)P_2+(-2i-2+2i+4-4i-4+4i)P_3$$
$$=1-2P_3$$

$K^2=-1+2P_1$

$L^2=1=T^2+S^2+K^2$

Example.

Consider the following four Pythagoras quadruples in Z_2:

$L_1=(0,0,0,0), L_2=(1,1,1,1), L_3=(1,1,0,0), L_4=(0,0,1,1)$

For every quadruple $(L_i, L_j, L_s, L_k); 1 \leq i, j, , k \leq 4$, we can get a symbolic 3-plithogenic pythagoras quadruple.

We will find some symbolic 3-plithogenic Pythagoras quadruple in $3 - SP_{Z_2}$.

Let us discuss the following cases:

case (1).

$$(L_1, L_1, L_1, L_1): \begin{cases} Y_1 = 0 \\ \acute{Y}_1 = 0 \\ {Y_1}'' = 0 \\ {Y_1}''' = 0 \end{cases}$$

case (2).

$$(L_1, L_1, L_1, L_2): \begin{cases} Y_2 = P_3 \\ \acute{Y}_2 = P_3 \\ {Y_2}'' = P_3 \\ {Y_2}''' = P_3 \end{cases}$$

case (3).

$$(L_1, L_1, L_1, L_3): \begin{cases} Y_3 = P_3 \\ \acute{Y}_3 = P_3 \\ {Y_3}'' = 0 \\ {Y_3}''' = 0 \end{cases}$$

case (4).

$$(L_1, L_1, L_1, L_4): \begin{cases} Y_4 = 0 \\ \acute{Y}_4 = 0 \\ {Y_4}'' = P_3 \\ {Y_4}''' = P_3 \end{cases}$$

case (5).

$$(L_1, L_1, L_2, L_1): \begin{cases} Y_5 = P_2 + P_3 \\ \acute{Y}_5 = P_2 + P_3 \\ {Y_5}'' = P_2 + P_3 \\ {Y_5}''' = P_2 + P_3 \end{cases}$$

case (6).

$$(L_1, L_1, L_3, L_1): \begin{cases} Y_6 = P_2 + P_3 \\ \acute{Y}_6 = P_2 + P_3 \\ {Y_6}'' = P_2 + P_3 \\ {Y_6}''' = P_2 + P_3 \end{cases}$$

case (7).

$$(L_1, L_1, L_4, L_1): \begin{cases} Y_7 = 0 \\ \acute{Y}_7 = 0 \\ Y_7'' = P_2 + P_3 \\ Y_7''' = P_2 + P_3 \end{cases}$$

case (8).

$$(L_1, L_2, L_1, L_1): \begin{cases} Y_8 = P_1 + P_2 \\ \acute{Y}_8 = P_1 + P_2 \\ Y_8'' = P_1 + P_2 \\ Y_8''' = P_1 + P_2 \end{cases}$$

case (9).

$$(L_1, L_3, L_1, L_1): \begin{cases} Y_9 = P_1 + P_2 \\ \acute{Y}_9 = P_1 + P_2 \\ Y_9'' = 0 \\ Y_9''' = 0 \end{cases}$$

case (10).

$$(L_1, L_4, L_1, L_1): \begin{cases} Y_{10} = 0 \\ \acute{Y}_{10} = 0 \\ Y_{10}'' = P_1 + P_2 \\ Y_{10}''' = P_1 + P_2 \end{cases}$$

case (11).

$$(L_2, L_1, L_1, L_1): \begin{cases} Y_{11} = 1 + P_1 \\ \acute{Y}_{11} = 1 + P_1 \\ Y_{11}'' = 1 + P_1 \\ Y_{11}''' = 1 + P_1 \end{cases}$$

case (12).

$$(L_3, L_1, L_1, L_1): \begin{cases} Y_{12} = 1 + P_1 \\ \acute{Y}_{12} = 1 + P_1 \\ Y_{12}'' = 0 \\ Y_{12}''' = 0 \end{cases}$$

case (13).

$$(L_4, L_1, L_1, L_1): \begin{cases} Y_{13} = 0 \\ \acute{Y}_{13} = 0 \\ Y_{13}'' = 1 + P_1 \\ Y_{13}''' = 1 + P_1 \end{cases}$$

case (14).

$$(L_2, L_2, L_2, L_2): \begin{cases} Y_{14} = 1 \\ \acute{Y}_{14} = 1 \\ Y_{14}'' = 1 \\ Y_{14}''' = 1 \end{cases}$$

case (15).

$$(L_2, L_2, L_2, L_1): \begin{cases} Y_{15} = 1 + P_3 \\ \acute{Y}_{15} = 1 + P_3 \\ Y_{15}'' = 1 + P_3 \\ Y_{15}''' = 1 + P_3 \end{cases}$$

case (16).

$$(L_2, L_2, L_2, L_3): \begin{cases} Y_{16} = 1 \\ \acute{Y}_{16} = 1 \\ Y_{16}'' = 1 + P_3 \\ Y_{16}''' = 1 + P_3 \end{cases}$$

case (17).

$$(L_2, L_2, L_2, L_4): \begin{cases} Y_{17} = 1 + P_3 \\ \acute{Y}_{17} = 1 + P_3 \\ Y_{17}'' = 1 \\ Y_{17}''' = 1 \end{cases}$$

case (18).

$$(L_2, L_2, L_1, L_2): \begin{cases} Y_{18} = 1 + P_2 + P_3 \\ \acute{Y}_{18} = 1 + P_2 + P_3 \\ Y_{18}'' = 1 + P_2 + P_3 \\ Y_{18}''' = 1 + P_2 + P_3 \end{cases}$$

case (19).

$$(L_2, L_2, L_3, L_2): \begin{cases} Y_{19} = 1 \\ \acute{Y}_{19} = 1 \\ Y_{19}'' = 1 + P_3 \\ Y_{19}''' = 1 + P_3 \end{cases}$$

case (20).

$$(L_2, L_2, L_4, L_2): \begin{cases} Y_{20} = 1 + P_2 + P_3 \\ \acute{Y}_{20} = 1 + P_2 + P_3 \\ Y_{20}'' = 1 \\ Y_{20}''' = 1 \end{cases}$$

case (21).

$$(L_2, L_1, L_2, L_2): \begin{cases} Y_{21} = 1 + P_1 + P_2 \\ \acute{Y}_{21} = 1 + P_1 + P_2 \\ Y_{21}'' = 1 + P_1 + P_2 \\ Y_{21}''' = 1 + P_1 + P_2 \end{cases}$$

case (22).

$$(L_2, L_3, L_2, L_2): \begin{cases} Y_{22} = 1 \\ \acute{Y}_{22} = 1 \\ Y_{22}'' = 1 + P_1 + P_2 \\ Y_{22}''' = 1 + P_1 + P_2 \end{cases}$$

case (23).

$$(L_2, L_4, L_2, L_2): \begin{cases} Y_{23} = 1 + P_1 + P_2 \\ \acute{Y}_{23} = 1 + P_1 + P_2 \\ Y_{23}'' = 1 \\ Y_{23}''' = 1 \end{cases}$$

Permutation (24).

$$(L_1, L_2, L_2, L_2): \begin{cases} Y_{24} = P_1 \\ \acute{Y}_{24} = P_1 \\ Y_{24}'' = P_1 \\ Y_{24}''' = P_1 \end{cases}$$

case (25).

$$(L_3, L_2, L_2, L_2): \begin{cases} Y_{25} = 1 \\ \acute{Y}_{25} = 1 \\ Y_{25}'' = P_1 \\ Y_{25}''' = P_1 \end{cases}$$

case (26).

$$(L_4, L_2, L_2, L_2): \begin{cases} Y_{26} = P_1 \\ \acute{Y}_{26} = P_1 \\ Y_{26}'' = 1 \\ Y_{26}''' = 1 \end{cases}$$

case (27).

$$(L_3, L_3, L_3, L_3): \begin{cases} Y_{27} = 1 \\ \acute{Y}_{27} = 1 \\ Y_{27}'' = 0 \\ Y_{27}''' = 0 \end{cases}$$

case (28).

$$(L_3, L_3, L_3, L_1): \begin{cases} Y_{28} = 1 + P_3 \\ Y'_{28} = 1 + P_3 \\ Y_{28}'' = 0 \\ Y_{28}''' = 0 \end{cases}$$

case (29).

$$(L_3, L_3, L_3, L_2): \begin{cases} Y_{29} = 1 \\ Y'_{29} = 1 \\ Y_{29}'' = P_3 \\ Y_{29}''' = P_3 \end{cases}$$

case (30).

$$(L_3, L_3, L_3, L_4): \begin{cases} Y_{30} = 1 + P_3 \\ Y'_{30} = 1 + P_3 \\ Y_{30}'' = P_3 \\ Y_{30}''' = P_3 \end{cases}$$

case (31).

$$(L_3, L_3, L_1, L_3): \begin{cases} Y_{31} = 1 + P_3 \\ Y'_{31} = 1 + P_3 \\ Y_{31}'' = P_3 \\ Y_{31}''' = P_3 \end{cases}$$

case (32).

$$(L_3, L_3, L_2, L_3): \begin{cases} Y_{32} = 1 \\ Y'_{32} = 1 \\ Y_{32}'' = P_2 + P_3 \\ Y_{32}''' = P_2 + P_3 \end{cases}$$

case (33).

$$(L_3, L_3, L_4, L_3): \begin{cases} Y_{33} = 1 + P_2 + P_3 \\ Y'_{33} = 1 + P_2 + P_3 \\ Y_{33}'' = P_2 + P_3 \\ Y_{33}''' = P_2 + P_3 \end{cases}$$

case (34).

$$(L_3, L_1, L_3, L_3): \begin{cases} Y_{34} = 1 + P_2 + P_3 \\ Y'_{34} = 1 + P_2 + P_3 \\ Y_{34}'' = 0 \\ Y_{34}''' = 0 \end{cases}$$

case (35).

$$(L_3, L_2, L_3, L_3): \begin{cases} Y_{35} = 1 \\ Y_{35}' = 1 \\ Y_{35}'' = P_1 + P_2 \\ Y_{35}''' = P_1 + P_2 \end{cases}$$

case (36).

$$(L_3, L_4, L_3, L_3): \begin{cases} Y_{36} = 1 + P_1 + P_2 \\ Y_{36}' = 1 + P_1 + P_2 \\ Y_{36}'' = P_1 + P_2 \\ Y_{36}''' = P_1 + P_2 \end{cases}$$

case (37).

$$(L_1, L_3, L_3, L_3): \begin{cases} Y_{37} = P_1 \\ Y_{37}' = P_1 \\ Y_{37}'' = 0 \\ Y_{37}''' = 0 \end{cases}$$

case (38).

$$(L_2, L_3, L_3, L_3): \begin{cases} Y_{38} = 1 \\ Y_{38}' = 1 \\ Y_{38}'' = 1 + P_1 \\ Y_{38}''' = 1 + P_1 \end{cases}$$

case (39).

$$(L_4, L_3, L_3, L_3): \begin{cases} Y_{39} = P_1 \\ Y_{39}' = P_1 \\ Y_{39}'' = 1 + P_1 \\ Y_{39}''' = 1 + P_1 \end{cases}$$

case (40).

$$(L_4, L_4, L_4, L_4): \begin{cases} Y_{40} = 0 \\ Y_{40}' = 0 \\ Y_{40}'' = 1 \\ Y_{40}''' = 1 \end{cases}$$

case (41).

$$(L_4, L_4, L_4, L_1): \begin{cases} Y_{41} = 0 \\ Y_{41}' = 0 \\ Y_{41}'' = 1 + P_3 \\ Y_{41}''' = 1 + P_3 \end{cases}$$

case (42).

$$(L_4, L_4, L_4, L_2): \begin{cases} Y_{42} = P_3 \\ \acute{Y_{42}} = P_3 \\ Y_{42}'' = 1 \\ Y_{42}''' = 1 \end{cases}$$

case (43).

$$(L_4, L_4, L_4, L_3): \begin{cases} Y_{43} = P_3 \\ \acute{Y_{43}} = P_3 \\ Y_{43}'' = 1 + P_3 \\ Y_{43}''' = 1 + P_3 \end{cases}$$

case (44).

$$(L_4, L_4, L_1, L_4): \begin{cases} Y_{44} = 0 \\ \acute{Y_{44}} = 0 \\ Y_{44}'' = 1 + P_2 + P_3 \\ Y_{44}''' = 1 + P_2 + P_3 \end{cases}$$

case (45).

$$(L_4, L_4, L_2, L_4): \begin{cases} Y_{45} = P_2 + P_3 \\ \acute{Y_{45}} = P_2 + P_3 \\ Y_{45}'' = 1 \\ Y_{45}''' = 1 \end{cases}$$

case (46).

$$(L_4, L_4, L_3, L_4): \begin{cases} Y_{46} = P_2 + P_3 \\ \acute{Y_{46}} = P_2 + P_3 \\ Y_{46}'' = 1 + P_2 + P_3 \\ Y_{46}''' = 1 + P_2 + P_3 \end{cases}$$

case (47).

$$(L_4, L_1, L_4, L_4): \begin{cases} Y_{47} = 0 \\ \acute{Y_{47}} = 0 \\ Y_{47}'' = 1 + P_1 + P_2 \\ Y_{47}''' = 1 + P_1 + P_2 \end{cases}$$

case (48).

$$(L_4, L_2, L_4, L_4): \begin{cases} Y_{48} = P_1 + P_2 \\ \acute{Y_{48}} = P_1 + P_2 \\ Y_{48}'' = 1 \\ Y_{48}''' = 1 \end{cases}$$

case (49).

$$(L_4, L_3, L_4, L_4): \begin{cases} Y_{49} = P_1 + P_2 \\ \acute{Y_{49}} = P_1 + P_2 \\ Y_{49}{}'' = 1 + P_1 + P_2 \\ Y_{49}{}''' = 1 + P_1 + P_2 \end{cases}$$

case (50).

$$(L_1, L_4, L_4, L_4): \begin{cases} Y_{50} = 0 \\ \acute{Y_{50}} = 0 \\ Y_{50}{}'' = P_1 \\ Y_{50}{}''' = P_1 \end{cases}$$

case (51).

$$(L_2, L_4, L_4, L_4): \begin{cases} Y_{51} = 1 + P_1 \\ \acute{Y_{51}} = 1 + P_1 \\ Y_{51}{}'' = 1 \\ Y_{51}{}''' = 1 \end{cases}$$

case (52).

$$(L_3, L_4, L_4, L_4): \begin{cases} Y_{52} = 1 + P_1 \\ \acute{Y_{52}} = 1 + P_1 \\ Y_{52}{}'' = P_1 \\ Y_{52}{}''' = P_1 \end{cases}$$

case (53).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_{53} = P_1 + P_3 \\ \acute{Y_{53}} = P_1 + P_3 \\ Y_{53}{}'' = P_1 + P_2 + P_3 \\ Y_{53}{}''' = P_1 + P_2 + P_3 \end{cases}$$

case (54).

$$(L_1, L_2, L_4, L_3): \begin{cases} Y_{54} = P_1 + P_2 + P_3 \\ \acute{Y_{54}} = P_1 + P_2 + P_3 \\ Y_{54}{}'' = P_1 + P_2 \\ Y_{54}{}''' = P_1 + P_2 \end{cases}$$

case (55).

$$(L_1, L_3, L_2, L_4): \begin{cases} Y_{55} = P_1 + P_3 \\ \acute{Y_{55}} = P_1 + P_3 \\ Y_{55}{}'' = P_2 \\ Y_{55}{}''' = P_2 \end{cases}$$

case (56).

$$(L_1, L_3, L_4, L_2): \begin{cases} Y_{56} = P_1 + P_2 + P_3 \\ Y_{56}' = P_1 + P_2 + P_3 \\ Y_{56}'' = P_2 \\ Y_{56}''' = P_2 \end{cases}$$

case (57).

$$(L_1, L_4, L_2, L_3): \begin{cases} Y_{57} = P_2 \\ Y_{57}' = P_2 \\ Y_{57}'' = P_1 + P_3 \\ Y_{57}''' = P_1 + P_3 \end{cases}$$

case (58).

$$(L_1, L_4, L_3, L_2): \begin{cases} Y_{58} = P_2 \\ Y_{58}' = P_2 \\ Y_{58}'' = P_1 + P_2 + P_3 \\ Y_{58}''' = P_1 + P_2 + P_3 \end{cases}$$

case (59).

$$(L_2, L_1, L_3, L_4): \begin{cases} Y_{59} = 1 + P_1 + P_2 + P_3 \\ Y_{59}' = 1 + P_1 + P_2 + P_3 \\ Y_{59}'' = 1 + P_1 + P_3 \\ Y_{59}''' = 1 + P_1 + P_3 \end{cases}$$

case (60).

$$(L_2, L_1, L_4, L_3): \begin{cases} Y_{60} = 1 + P_1 + P_3 \\ Y_{60}' = 1 + P_1 + P_3 \\ Y_{60}'' = P_2 + P_3 \\ Y_{60}''' = P_2 + P_3 \end{cases}$$

case (61).

$$(L_3, L_1, L_2, L_4): \begin{cases} Y_{61} = 1 + P_1 + P_2 + P_3 \\ Y_{61}' = 1 + P_1 + P_2 + P_3 \\ Y_{61}'' = P_2 \\ Y_{61}''' = P_2 \end{cases}$$

case (62).

$$(L_3, L_1, L_4, L_2): \begin{cases} Y_{62} = 1 + P_1 + P_3 \\ Y_{62}' = 1 + P_1 + P_3 \\ Y_{62}'' = P_2 \\ Y_{62}''' = P_2 \end{cases}$$

case (63).

$$(L_4, L_1, L_2, L_3): \begin{cases} Y_{63} = P_2 \\ \acute{Y}_{63} = P_2 \\ Y_{63}'' = 1 + P_2 + P_3 \\ Y_{63}''' = 1 + P_2 + P_3 \end{cases}$$

case (64).

$$(L_4, L_1, L_3, L_2): \begin{cases} Y_{64} = P_2 \\ \acute{Y}_{64} = P_2 \\ Y_{64}'' = 1 + P_3 \\ Y_{64}''' = 1 + P_3 \end{cases}$$

case (65).

$$(L_2, L_3, L_1, L_4): \begin{cases} Y_{65} = 1 + P_2 \\ \acute{Y}_{65} = 1 + P_2 \\ Y_{65}'' = 1 + P_3 \\ Y_{65}''' = 1 + P_3 \end{cases}$$

case (66).

$$(L_2, L_3, L_4, L_1): \begin{cases} Y_{66} = 1 + P_2 \\ \acute{Y}_{66} = 1 + P_2 \\ Y_{66}'' = 1 + P_1 + P_2 + P_3 \\ Y_{66}''' = 1 + P_1 + P_2 + P_3 \end{cases}$$

case (67).

$$(L_3, L_2, L_1, L_4): \begin{cases} Y_{67} = 1 + P_2 \\ \acute{Y}_{67} = 1 + P_2 \\ Y_{67}'' = P_1 + P_2 + P_3 \\ Y_{67}''' = P_1 + P_2 + P_3 \end{cases}$$

case (68).

$$(L_3, L_2, L_4, L_1): \begin{cases} Y_{68} = 1 + P_2 \\ \acute{Y}_{68} = 1 + P_2 \\ Y_{68}'' = 1 + P_3 \\ Y_{68}''' = 1 + P_3 \end{cases}$$

case (64).

$$(L_4, L_1, L_3, L_2): \begin{cases} Y_{64} = P_2 \\ \acute{Y}_{64} = P_2 \\ Y_{64}'' = P_1 + P_3 \\ Y_{64}''' = P_1 + P_3 \end{cases}$$

case (69).

$$(L_3, L_4, L_1, L_2): \begin{cases} Y_{69} = 1 + P_1 + P_3 \\ Y'_{69} = 1 + P_1 + P_3 \\ Y_{69}'' = P_1 + P_3 \\ Y_{69}''' = P_1 + P_3 \end{cases}$$

case (70).

$$(L_3, L_4, L_2, L_1): \begin{cases} Y_{70} = 1 + P_1 + P_2 + P_3 \\ Y'_{70} = 1 + P_1 + P_2 + P_3 \\ Y_{70}'' = P_1 + P_3 \\ Y_{70}''' = P_1 + P_3 \end{cases}$$

case (71).

$$(L_4, L_3, L_1, L_2): \begin{cases} Y_{71} = P_1 + P_2 + P_3 \\ Y'_{71} = P_1 + P_2 + P_3 \\ Y_{71}'' = 1 + P_3 \\ Y_{71}''' = 1 + P_3 \end{cases}$$

case (72).

$$(L_4, L_1, L_3, L_2): \begin{cases} Y_{72} = P_1 + P_3 \\ Y'_{72} = P_1 + P_3 \\ Y_{72}'' = 1 + P_1 + P_2 + P_3 \\ Y_{72}''' = 1 + P_1 + P_2 + P_3 \end{cases}$$

case (73).

$$(L_4, L_2, L_1, L_3): \begin{cases} Y_{73} = P_1 + P_2 + P_3 \\ Y'_{73} = P_1 + P_2 + P_3 \\ Y_{73}'' = 1 + P_2 \\ Y_{73}''' = 1 + P_2 \end{cases}$$

case (74).

$$(L_4, L_1, L_3, L_2): \begin{cases} Y_{74} = P_1 + P_3 \\ Y'_{74} = P_1 + P_3 \\ Y_{74}'' = 1 + P_2 \\ Y_{74}''' = 1 + P_2 \end{cases}$$

case (75).

$$(L_2, L_4, L_1, L_3): \begin{cases} Y_{75} = 1 + P_1 + P_3 \\ Y'_{75} = 1 + P_1 + P_3 \\ Y_{75}'' = 1 + P_2 \\ Y_{75}''' = 1 + P_2 \end{cases}$$

case (76).

$$(L_2, L_4, L_3, L_1): \begin{cases} Y_{76} = 1 + P_1 + P_2 + P_3 \\ \acute{Y_{76}} = 1 + P_1 + P_2 + P_3 \\ Y_{76}'' = 1 + P_2 \\ Y_{76}''' = 1 + P_2 \end{cases}$$

case (77).

$$(L_2, L_2, L_3, L_3): \begin{cases} Y_{77} = 1 \\ \acute{Y_{77}} = 1 \\ Y_{77}'' = 1 + P_2 \\ Y_{77}''' = 1 + P_2 \end{cases}$$

case (78).

$$(L_2, L_2, L_1, L_1): \begin{cases} Y_{78} = 1 + P_2 \\ \acute{Y_{78}} = 1 + P_2 \\ Y_{78}'' = 1 + P_2 \\ Y_{78}''' = 1 + P_2 \end{cases}$$

case (79).

$$(L_2, L_2, L_4, L_4): \begin{cases} Y_{79} = 1 + P_2 \\ \acute{Y_{79}} = 1 + P_2 \\ Y_{79}'' = 1 \\ Y_{79}''' = 1 \end{cases}$$

case (80).

$$(L_1, L_1, L_2, L_2): \begin{cases} Y_{80} = P_2 \\ \acute{Y_{80}} = P_2 \\ Y_{80}'' = P_2 \\ Y_{80}''' = P_2 \end{cases}$$

Conclusion.

In this paper, we have studied Pythagoras quadruples in symbolic 3-plithogenic commutative rings, where necessary and sufficient conditions for a symbolic 3-plithogenic quadruple (x, y, z, t) to be a Pythagoras quadruple.

Also, we have presented some related examples that explain how to find 3-plithogenic quadruples from classical triples.

Acknowledgments " This study is supported via funding from Prince sattam bin Abdulaziz University project number (PSAU/2023/R/1445) ".

References

[1] F. Smarandache, Plithogeny, Plithogenic Set, Logic, Probability, and Statistics, 141 pages, Pons Editions,Brussels, Belgium, 2017. arXiv.org (Cornell University), Computer Science - Artificial Intelligence, 03Bxx:

[2] Florentin Smarandache, Physical Plithogenic Set, 71st Annual Gaseous Electronics Conference, Session LW1, Oregon Convention Center Room, Portland, Oregon, USA, November 5–9, 2018.

[3] Florentin Smarandache: Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets – Revisited, *Neutrosophic Sets and Systems*, vol. 21, 2018, pp. 153-166.

[4] M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, "Introduction to Symbolic 2-Plithogenic Probability Theory," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.

[5] Nader Mahmoud Taffach , Ahmed Hatip., "A Review on Symbolic 2-Plithogenic Algebraic Structures " Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.

[6] Nader Mahmoud Taffach , Ahmed Hatip.," A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations ", Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.

[7] Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.

[8] Albasheer, O., Hajjari., A., and Dalla., R., " On The Symbolic 3-Plithogenic Rings and TheirAlgebraic Properties", Neutrosophic Sets and Systems, Vol 54, 2023.

[9] Ben Othman, K., "On Some Algorithms For Solving Symbolic 3-Plithogenic Equations", Neoma Journal Of Mathematics and Computer Science, 2023.

[10] Ali, R., and Hasan, Z., "An Introduction To The Symbolic 3-Plithogenic Vector Spaces", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.

[11] Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.

[12] Alhasan, Y., Alfahal, A., Abdulfatah, R., Ali, R., and Aljibawi, M., " On A Novel Security Algorithm For The Encryption Of 3×3 Fuzzy Matrices With Rational Entries Based On The Symbolic 2-Plithogenic Integers And El-Gamal Algorithm", International Journal of Neutrosophic Science, 2023.

[13] Merkepci, M., Abobala, M., and Allouf, A., " The Applications of Fusion Neutrosophic Number Theory in Public Key Cryptography and the Improvement of RSA Algorithm ", Fusion: Practice and Applications, 2023.

[14] Merkepci, M., and Abobala, M., " Security Model for Encrypting Uncertain Rational Data Units Based on Refined Neutrosophic Integers Fusion and El Gamal Algorithm ", Fusion: Practice and Applications, 2023.

[15] Abobala, M., and Allouf, A., " On A Novel Security Scheme for The Encryption and Decryption Of 2×2 Fuzzy Matrices with Rational Entries Based on The Algebra of Neutrosophic Integers and El-Gamal Crypto-System", Neutrosophic Sets and Systems, vol.54, 2023.

[16]Abobala, M, "n-Cyclic Refined Neutrosophic Algebraic Systems Of Sub-Indeterminacies, An Application To Rings and Modules", International Journal of Neutrosophic Science, Vol. 12, pp. 81-95 . 2020.

[17] Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Concept Of Symbolic 7-Plithogenic Real Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

[18] Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Symbolic 8-Plithogenic Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

[19]Mohamed, M., "Modified Approach for Optimization of Real-Life Transportation Environment: Suggested Modifications", American Journal of Business and Operations research, 2021.

[20] Nabeeh, N., Alshaimaa, A., and Tantawy, A., "A Neutrosophic Proposed Model For Evaluation Blockchain Technology in Secure Enterprise Distributed Applications", Journal of Cypersecurity and Information Management, 2023.

[21]Abualkishik, A., Almajed, R., Thompson, W., "Improving The Performance of Fog-assisted Internet of Things Networks Using Bipolar Trapezoidal Neutrosophic Sets", Journal of Wireless and Ad Hoc Communication, 2023.

[22] R. A. Trivedi, S. A. Bhanotar, Pythagorean Triplets - Views, Analysis and Classification, IIOSR J. Math., 11 (2015).

[23] J. Rukavicka, Dickson's method for generating Pythagorean triples revisited, Eur. J. Pure Appl.Math., 6 (2013).

[24] T.Roy and F J. Sonia, A Direct Method To Generate Pythagorean Triples And Its Generalization To Pythagorean Quadruples And n-tuples, , Jadavpur University, India (2010)

[25] Paul Oliverio, " Self-Generating Pythagorean Quadruples And TV-Tuples", Los Angeles, (1993).

Received 3/7/2023, Accepted 4/10/2023

On The Dual Symbolic 2-Plithogenic Numbers

[1.]Khadija Ben Othman, [2.]Maretta Sarkis, [3.]Murat Ozcek

[1..] Umm Al-Qura Univerity, Mekka, Saudi Arabia

[2.]Abu Dhabi University, Abu Dhabi, United Arab Emirates

[3.]Gaziantep University, Department of Mathematics, Gaziantep, Turkey

Co-khagijabenothman33@gmail.com

Abstract:

The objective of this paper is to combine dual numbers with symbolic 2-plithogenic numbers in one algebraic structure called dual symbolic 2-plithogenic real numbers.

Also, many elementary properties of the suggested system such as inverses and idempotents will be handled by many related theorems and examples.

Keywords: Symbolic 2-plithogenic number, dual number, dual symbolic 2-plithogenic number.

Introduction and preliminaries.

Dual numbers are considered as a generalization of real numbers, where they are defined as follows:

$D = \{a + bt;\ t^2 = 0, a, b \in R\}$[1]. Dual numbers make together a commutative ring with many interesting properties.

Addition on D is defined as follows:

$(a_0 + b_0 t) + (a_1 + b_1 t) = (a_0 + a_1) + (b_0 + b_1)t$

Multiplication on D is defined as follows:

$(a_0 + b_0 t).(a_1 + b_1 t) = (a_0 a_1) + (a_0 b_1 + b_0 a_1)t$

zenodo.10031194/10.5281

In [2-4], Smarandache presented symbolic n-plithogenic sets, then they were used in generalizing many famous algebraic structures such as rings, matrices, and other structures [6-11].

We refer to many similar numerical systems that generalize real number, such as neutrosophic numbers, split-complex number, and weak fuzzy numbers [12-18]. We refer to some applications of these generalized number systems in matrix theory and cryptography [19-24].

Through this paper, we use symbolic 2-plithogenic real numbers to build a new generalization of real numbers, and we present some of its elementary algebraic properties.

Main concepts.

Definition.

The set of symbolic 2-plithogenic dual numbers I defined as follows:

$$2 - SP_D = \{(x_0 + x_1 t) + (y_0 + y_1 t)P_1 + (z_0 + z_1 t)P_2; x_i, y_i, z_i \in R, t^2 = 0\}$$

Definition.

Addition of $2 - SP_D$ is defined:

$$[(m_0 + m_1 t) + (k_0 + k_1 t)P_1 + (s_0 + s_1 t)P_2] + [(n_0 + n_1 t) + (l_0 + l_1 t)P_1 + (q_0 + q_1 t)P_2] = (m_0 + n_0) + (m_1 + n_1)t + [(k_0 + l_0) + (k_1 + l_1)t]P_1 + [(s_0 + q_0) + (s_1 + q_1)t]P_2$$

$(2 - SP_D, +)$ is an abelian group.

Remark.

A symbolic 2-plithogenic dual number $X = (x_0 + x_1 t) + (y_0 + y_1 t)P_1 + (z_0 + z_1 t)P_2$ can be written:

$$X = (x_0 + y_0 P_1 + z_0 P_2) + t(x_1 + y_1 P_1 + z_1 P_2)$$

Definition.

Let $X = (x_0 + x_1 P_1 + x_2 P_2) + t(\acute{x}_0 + \acute{x}_1 P_1 + \acute{x}_2 P_2) = M_1 + M_2 t$,

$Y = (y_0 + y_1 P_1 + y_2 P_2) + t(\acute{y}_0 + \acute{y}_1 P_1 + \acute{y}_2 P_2) = N_1 + N_2 t \in 2 - SP_D$,

then:

Multiplication on $2 - SP_D$ is defined as follows:

$X.Y = M_1N_1 + t(M_1N_2 + N_1M_2)$

Example.

Consider $X = (1 + P_1 + P_2) + t(2 - P_2), Y = P_1 + t(1 - P_1)$, we have:

$X + Y = (1 + 2P_1 + P_2) + t(3 - P_1 - P_2)$

$X.Y = (1 + P_1 + P_2)P_1 + t[(1 + P_1 + P_2)(1 - P_1) + (2 - P_2)P_1] = (2P_1 + P_2) +$
$t[(1 - P_1 + P_1 - P_1 + P_2 - P_2) + 2P_1 - P_1] = (2P_1 + P_2) + t(1 + P_1 - P_2).$

Remark.

$(2 - SP_D, +, .)$ Is a commutative ring.

Invertibility:

Theorem.

Let $X = (m_0 + m_1P_1 + m_2P_2) + t(n_0 + n_1P_1 + n_2P_2) \in 2 - SP_D$, then X is invertible if and only if $m_0 \neq 0, m_0 + m_1 \neq 0, m_0 + m_1 + m_2 \neq 0$ and:

$$X^{-1} = \frac{1}{X} = \left[\frac{1}{m_0} + \left(\frac{1}{m_0 + m_1} - \frac{1}{m_0}\right)P_1 + \left(\frac{1}{m_0 + m_1 + m_2} - \frac{1}{m_0 + m_1}\right)P_2\right]$$
$$- t\left[\frac{n_0}{(m_0)^2} + \left(\frac{n_0 + n_1}{(m_0 + m_1)^2} - \frac{n_0}{(m_0)^2}\right)P_1 + \left(\frac{n_0 + n_1 + n_2}{(m_0 + m_1 + m_2)^2} - \frac{n_0 + n_1}{(m_0 + m_1)^2}\right)P_2\right]$$

Proof.

X is invertible if and only if $\frac{1}{X}$ is defined as follows:

$$\frac{1}{X} = \frac{1}{(m_0 + m_1P_1 + m_2P_2) + t(n_0 + n_1P_1 + n_2P_2)}$$
$$= \frac{(m_0 + m_1P_1 + m_2P_2) - t(n_0 + n_1P_1 + n_2P_2)}{[(m_0 + m_1P_1 + m_2P_2) + t(n_0 + n_1P_1 + n_2P_2)][(m_0 + m_1P_1 + m_2P_2) - t(n_0 + n_1P_1 + n_2P_2)]}$$
$$== \frac{(m_0 + m_1P_1 + m_2P_2) - t(n_0 + n_1P_1 + n_2P_2)}{(m_0 + m_1P_1 + m_2P_2)^2}$$

So that $m_0 + m_1P_1 + m_2P_2$ is invertible in $2 - SP_R$.

This is equivalent to $m_0 \neq 0, m_0 + m_1 \neq 0, m_0 + m_1 + m_2 \neq 0$.

On the other hand, $\frac{1}{X} = \frac{1}{m_0+m_1P_1+m_2P_2} - t\frac{(n_0+n_1P_1+n_2P_2)^2}{(m_0+m_1P_1+m_2P_2)^2}$

Put $Y = \frac{1}{m_0} + \left(\frac{1}{m_0+m_1} - \frac{1}{m_0}\right)P_1 + \left(\frac{1}{m_0+m_1+m_2} - \frac{1}{m_0+m_1}\right)P_2 - t\left[\frac{n_0}{(m_0)^2} + \left(\frac{n_0+n_1}{(m_0+m_1)^2} - \frac{n_0}{(m_0)^2}\right)P_1 + \left(\frac{n_0+n_1+n_2}{(m_0+m_1+m_2)^2} - \frac{n_0+n_1}{(m_0+m_1)^2}\right)P_2\right]$

Compute the result of XY as follows:

$$XY = (m_0 + m_1P_1 + m_2P_2)\left[\frac{1}{m_0} + \left(\frac{1}{m_0+m_1} - \frac{1}{m_0}\right)P_1 + \left(\frac{1}{m_0+m_1+m_2} - \frac{1}{m_0+m_1}\right)P_2\right]$$
$$+ t\left[-(m_0 + m_1P_1 + m_2P_2)\left[\frac{n_0}{(m_0)^2} + \left(\frac{n_0+n_1}{(m_0+m_1)^2} - \frac{n_0}{(m_0)^2}\right)P_1 + \left(\frac{n_0+n_1+n_2}{(m_0+m_1+m_2)^2} - \frac{n_0+n_1}{(m_0+m_1)^2}\right)P_2\right]\right]$$
$$+ (n_0 + n_1P_1 + n_2P_2)\left[\frac{1}{m_0} + \left(\frac{1}{m_0+m_1} - \frac{1}{m_0}\right)P_1 + \left(\frac{1}{m_0+m_1+m_2} - \frac{1}{m_0+m_1}\right)P_2\right]$$
$$= 1 + t\left[\frac{-n_0}{m_0} + \left(-m_0\frac{(n_0+n_1)}{m_0+m_1} + \frac{n_0}{m_0} - \frac{m_1n_0}{(m_0)^2} - \frac{m_1(n_0+n_1)}{(m_0+m_1)^2} + \frac{m_1n_0}{(m_0)^2}\right)P_1\right.$$
$$+ \left(-m_0\frac{(n_0+n_1+n_2)}{(m_0+m_1+m_2)^2} + \frac{m_0(n_0+n_1)}{(m_0+m_1)^2} - m_1\frac{(n_0+n_1+n_2)}{(m_0+m_1+m_2)^2}\right.$$
$$+ \frac{m_1(n_0+n_1)}{(m_0+m_1)^2} - \frac{m_2n_0}{(m_0)^2} - \frac{m_2(n_0+n_1)}{(m_0+m_1)^2} + \frac{m_2n_0}{(m_0)^2} - m_2\frac{(n_0+n_1+n_2)}{(m_0+m_1+m_2)^2}$$
$$\left.\left. + \frac{m_2(n_0+n_1)}{(m_0+m_1)^2}\right)P_2\right] + \frac{n_0}{m_0} + \left(\frac{n_0}{m_0+m_1} - \frac{n_0}{m_0} + \frac{n_1}{m_0} + \frac{n_1}{m_0+m_1} - \frac{n_1}{m_0}\right)P_1$$
$$+ \left(\frac{n_0}{m_0+m_1+m_2} - \frac{n_0}{m_0+m_1} + \frac{n_1}{m_0+m_1+m_2} - \frac{n_1}{m_0+m_1} + \frac{n_2}{m_0}\right.$$
$$\left. + \frac{n_2}{m_0+m_1+m_2} - \frac{n_2}{m_0} + \frac{n_2}{m_0+m_1+m_2} - \frac{n_2}{m_0+m_1}\right)P_2 = 1$$

So that, $X^{-1} = \frac{1}{X} = Y$

Example.

Take $X = (1 + P_1 + P_2) + t(2 + P_1 - P_2) \in 2 - SP_D$:

$$X^{-1} = \frac{1}{1} + \left(\frac{1}{2} - 1\right)P_1 + \left(\frac{1}{3} - \frac{1}{2}\right)P_2 - t\left[\frac{2}{1} + \left(\frac{3}{4} - \frac{2}{1}\right)P_1 + \left(\frac{2}{9} - \frac{3}{4}\right)P_2\right]$$
$$= 1 - \frac{1}{2}P_1 - \frac{1}{6}P_2 - t\left(2 - \frac{5}{4}P_1 - \frac{21}{36}P_2\right)$$

Natural power.

Theorem.

Let $X = (m_0 + m_1P_1 + m_2P_2) + t(n_0 + n_1P_1 + n_2P_2) \in 2 - SP_D$, then:

$X^n = (m_0)^n + ((m_0 + m_1)^n - (m_0)^n)P_1 + ((m_0 + m_1 + m_2)^n - (m_0 + m_1)^n)P_2 + n(n_0 + n_1P_1 + n_2P_2)[(m_0)^{n-1} + ((m_0 + m_1)^{n-1} - (m_0)^{n-1})P_1 + ((m_0 + m_1 + m_2)^{n-1} - (m_0 + m_1)^{n-1})P_2]$ for $n \in N$.

Proof.

Let $X = A + Bt; A, B \in 2 - SP_D$, then:

$A^n = A^n + nA^{n-1}Bt$, we get:

$A^n = (m_0)^n + ((m_0 + m_1)^n - (m_0)^n)P_1 + ((m_0 + m_1 + m_2)^n - (m_0 + m_1)^n)P_2$, then the proof holds.

Example.

Take $X = (1 + P_1 + P_2) + t(2 - P_1 + P_2) \in 2 - SP_D$

$X^3 = 1 + (8 - 1)P_1 + (1 - 8)P_2 + 3t(2 - P_1 + P_2)[1 + (4 - 1)P_1 + (1 - 4)P_2] = 1 + 7P_1 - 7P_2 + 3t[(2 - P_1 + P_2)(1 + 3P_1 - 3P_2)] = 1 + 7P_1 - 7P_2 + t(6 + 6P_1 - 6P_2)$.

Idempotency.

Definition.

Let $X \in 2 - SP_D$, then X is called idempotent if and only if $X^2 = X$.

Theorem.

Let $X = (m_0 + m_1P_1 + m_2P_2) + t(n_0 + n_1P_1 + n_2P_2) \in 2 - SP_D$, then X is called idempotent if and only if:

1. $m_0 + m_1P_1 + m_2P_2$ is idempotent.
2. $(n_0 + n_1P_1 + n_2P_2)[2m_0 - 1 + 2m_1P_1 + 2m_2P_2] = 0$

Proof.

$X = M + Nt$ is idempotent if and only if:

$$X^2 = X \Longrightarrow \begin{cases} M^2 = M \\ 2MN = N \Longrightarrow N(2M - 1) = 0 \end{cases}$$

For $M = m_0 + m_1P_1 + m_2P_2, N = n_0 + n_1P_1 + n_2P_2 \in 2 - SP_R$.

This implies the proof.

Remark.

The idempotency of M is equivalent to ${m_0}^2 = m_0, (m_0 + m_1)^2 = (m_0 + m_1), (m_0 + m_1 + m_2)^2 = (m_0 + m_1 + m_2)$, hence $m_0, m_0 + m_1, m_0 + m_1 + m_2 \in \{0,1\}$.

The equation $N(2M - 1) = 0$ means that $N, 2M - 1$ are zero divisors in $2 - SP_R$, so they should not invertible in $2 - SP_R$.

Now, let's compute the result of $N(2M - 1) = 0$.

$N(2M - 1) = (n_0 + n_1P_1 + n_2P_2)(2m_0 - 1 + m_1P_1 + m_2P_2) = 0$, thus:

$$\begin{cases} n_0(2m_0 - 1) = 0 \quad (1) \\ (n_0 + n_1)(2m_0 - 1 + 2m_1) = 0 \quad (2) \\ (n_0 + n_1 + n_2)(2m_0 - 1 + 2m_1 + 2m_2) = 0 \quad (3) \end{cases}$$

Since $m_0 \in \{0,1\}$, then $2m_0 - 1 \neq 0$ and $n_0 = 0$.

Equation (2) has two possible cases:

$$\begin{cases} n_1 = 0 \\ or \\ 2m_0 + 2m_1 = 1 \end{cases}$$

Equation (3) has two possible cases:

$$\begin{cases} n_1 + n_1 = 0 \\ or \\ 2m_0 + 2m_1 + 2m_2 = 1 \end{cases}$$

We discuss all possible cases.

Case1.

$m_0 = 0, m_0 + m_1 = 0, m_0 + m_1 + m_2 = 0$, thus $m_1 = m_2 = 0$, $n_0 = 0$, $n_1 = 0$, $n_2 = 0$, thus $X = 0$.

Case2.

$$\begin{cases} m_0 = 0 \\ m_0 + m_1 = 1 \\ m_0 + m_1 + m_2 = 1 \end{cases}, \text{then } m_1 = 1, m_2 = -1, n_0 = 0, n_1 = n_2 = 0$$

thus $X = P_1 - P_2$

Case3.

$$\begin{cases} m_0 = 0 \\ m_0 + m_1 = 0 \\ m_0 + m_1 + m_2 = 1 \end{cases}, \text{then } m_1 = 0, m_2 = 1, n_0 = 0, n_1 = n_2 = 0$$

thus $X = P_2$

Case4.

$$\begin{cases} m_0 = 0 \\ m_0 + m_1 = 1 \\ m_0 + m_1 + m_2 = 1 \end{cases}, \text{then } m_1 = 1, m_2 = 0, n_0 = 0, n_1 = n_2 = 0$$

thus $X = P_1$

Case5.

$$\begin{cases} m_0 = 1 \\ m_0 + m_1 = 0 \\ m_0 + m_1 + m_2 = 0 \end{cases}, \text{then } m_1 = -1, m_2 = 0, n_0 = 0, n_1 = n_2 = 0$$

thus $X = 1 - P_1$

Case6.

$$\begin{cases} m_0 = 1 \\ m_0 + m_1 = 1 \\ m_0 + m_1 + m_2 = 0 \end{cases}, \text{then } m_1 = 0, m_2 = -1, n_0 = 0, n_1 = n_2 = 0$$

thus $X = 1 - P_2$

Case7.

$$\begin{cases} m_0 = 1 \\ m_0 + m_1 = 0 \\ m_0 + m_1 + m_2 = 1 \end{cases}, \text{then } m_1 = -1, m_2 = 1, n_0 = 0, n_1 = n_2 = 0$$

thus $X = 1 - P_1 + P_2$

Case8.

$$\begin{cases} m_0 = 1 \\ m_0 + m_1 = 1 \\ m_0 + m_1 + m_2 = 1 \end{cases}, \text{then } m_1 = 0, m_2 = 0, n_0 = 0, n_1 = n_2 = 0$$

thus $X = 1$

remark.

The idempotent in $2 - SP_D$ are:

$$\{0,1,1 - P_1 + P_2, 1 - P_2, 1 - P_1, P_1, P_2, P_1 - P_2\}$$

Definition.

Let $X = M + Nt \in 2 - SP_D$, we say that X is 3-potent element if and only if $X^3 = X$.

Remark.

X is 3-potent if and only if $X^3 = X$ which is equivalent to:

$$\begin{cases} M^3 = M \\ 3M^2N = N \Longrightarrow N(3M^2 - 1) = 0 \end{cases}$$

$M^3 = M$ implies:

$$\begin{cases} {m_0}^3 = m_0 \\ (m_0 + m_1)^3 = m_0 + m_1 \\ (m_0 + m_1 + m_2)^3 = m_0 + m_1 + m_2 \end{cases}$$

So that $m_0, m_0 + m_1, m_0 + m_1 + m_2 \in \{0,1 - 1\}$.

$N(3M^2 - 1) = 0$ implies:

$$\begin{cases} n_0(3m_0 - 1) = 0 \quad (1) \\ (n_0 + n_1)(3m_0 - 1 + 3m_1) = 0 \quad (2) \\ (n_0 + n_1 + n_2)(3m_0 - 1 + 3m_1 + 3m_2) = 0 \quad (3) \end{cases}$$

Equation (1) means that $n_0 = 0$, that is because $m_0 \neq \frac{1}{3}$.

Equation (2) means that:

$$\begin{cases} n_0 + n_1 = 0 \\ or \\ 3m_0 + 3m_1 = 1 \end{cases}$$, since $m_0 + m_1$ is integer, then $n_1 = 0$.

Equation (3) means that:

$$\begin{cases} n_0 + n_1 + n_2 = 0 \\ or \\ 3m_0 + 3m_1 + 3m_2 = 1 \end{cases}$$, since $m_0 + m_1 + m_2$ is integer, then $n_2 = 0$.

This implies that:

Case1.

$$\begin{cases} m_0 = 0 \\ m_0 + m_1 = 0 \\ m_0 + m_1 + m_2 = 0 \end{cases}$$, then $X = 0$.

Case2.

$$\begin{cases} m_0 = 0 \\ m_0 + m_1 = 1 \\ m_0 + m_1 + m_2 = 0 \end{cases}$$, then $X = P_1 - P_2$

Case3.

$$\begin{cases} m_0 = 0 \\ m_0 + m_1 = -1 \\ m_0 + m_1 + m_2 = 0 \end{cases}$$, then $X = -P_1 + P_2$

Case4.

$$\begin{cases} m_0 = 0 \\ m_0 + m_1 = 0 \\ m_0 + m_1 + m_2 = 1 \end{cases}$$, then $X = P_2$

Case5.

$$\begin{cases} m_0 = 0 \\ m_0 + m_1 = 0 \\ m_0 + m_1 + m_2 = -1 \end{cases}, \text{then } X = -P_2$$

Case6.

$$\begin{cases} m_0 = 0 \\ m_0 + m_1 = 1 \\ m_0 + m_1 + m_2 = 1 \end{cases}, \text{then } X = P_1$$

Case7.

$$\begin{cases} m_0 = 0 \\ m_0 + m_1 = 1 \\ m_0 + m_1 + m_2 = -1 \end{cases}, \text{then } X = P_1 - 2P_2$$

Case8.

$$\begin{cases} m_0 = 0 \\ m_0 + m_1 = -1 \\ m_0 + m_1 + m_2 = 1 \end{cases}, \text{then } X = -P_1 + 2P_2$$

Case9.

$$\begin{cases} m_0 = 0 \\ m_0 + m_1 = -1 \\ m_0 + m_1 + m_2 = -1 \end{cases}, \text{then } X = -P_1$$

Case10.

$$\begin{cases} m_0 = 1 \\ m_0 + m_1 = 0 \\ m_0 + m_1 + m_2 = 0 \end{cases}, \text{then } X = 1 - P_1$$

Case11.

$$\begin{cases} m_0 = 1 \\ m_0 + m_1 = 1 \\ m_0 + m_1 + m_2 = 0 \end{cases}, \text{then } X = 1 - P_2$$

Case12.

$$\begin{cases} m_0 = 1 \\ m_0 + m_1 = -1 \\ m_0 + m_1 + m_2 = 0 \end{cases}, \text{then } X = 1 - 2P_1 + P_2$$

Case13.

$$\begin{cases} m_0 = 1 \\ m_0 + m_1 = 0 \\ m_0 + m_1 + m_2 = 1 \end{cases}, \text{then } X = 1 - P_1 + P_2$$

Case14.

$$\begin{cases} m_0 = 1 \\ m_0 + m_1 = 0 \\ m_0 + m_1 + m_2 = -1 \end{cases}, \text{then } X = 1 - P_1 - P_2$$

Case15.

$$\begin{cases} m_0 = 1 \\ m_0 + m_1 = 1 \\ m_0 + m_1 + m_2 = 1 \end{cases}, \text{then } X = 1$$

Case16.

$$\begin{cases} m_0 = 1 \\ m_0 + m_1 = 1 \\ m_0 + m_1 + m_2 = -1 \end{cases}, \text{then } X = 1 - 2P_2$$

Case17.

$$\begin{cases} m_0 = 1 \\ m_0 + m_1 = -1 \\ m_0 + m_1 + m_2 = 1 \end{cases}, \text{then } X = 1 - 2P_1 - 2P_2$$

Case18.

$$\begin{cases} m_0 = 1 \\ m_0 + m_1 = -1 \\ m_0 + m_1 + m_2 = -1 \end{cases}, \text{then } X = 1 - 2P_1$$

Case19.

$$\begin{cases} m_0 = -1 \\ m_0 + m_1 = 0 \\ m_0 + m_1 + m_2 = 0 \end{cases}, \text{then } X = -1 + P_1$$

Case20.

$$\begin{cases} m_0 = -1 \\ m_0 + m_1 = 1 \\ m_0 + m_1 + m_2 = 0 \end{cases}, \text{then } X = -1 + 2P_1 - P_2$$

Case21.

$$\begin{cases} m_0 = -1 \\ m_0 + m_1 = -1 \\ m_0 + m_1 + m_2 = 0 \end{cases}, \text{then } X = -1 + P_2$$

Case22.

$$\begin{cases} m_0 = -1 \\ m_0 + m_1 = 0 \\ m_0 + m_1 + m_2 = 1 \end{cases}, \text{then } X = -1 + P_1 + P_2$$

Case23.

$$\begin{cases} m_0 = -1 \\ m_0 + m_1 = 0 \\ m_0 + m_1 + m_2 = -1 \end{cases}, \text{then } X = -1 + P_1 - P_2$$

Case24.

$$\begin{cases} m_0 = -1 \\ m_0 + m_1 = 1 \\ m_0 + m_1 + m_2 = 1 \end{cases}, \text{then } X = -1 + 2P_1$$

Case25.

$$\begin{cases} m_0 = -1 \\ m_0 + m_1 = -1 \\ m_0 + m_1 + m_2 = -1 \end{cases}, \text{then } X = -1$$

Case26.

$$\begin{cases} m_0 = -1 \\ m_0 + m_1 = 1 \\ m_0 + m_1 + m_2 = -1 \end{cases}, \text{then } X = -1 + 2P_1 - 2P_2$$

Case27.

$$\begin{cases} m_0 = -1 \\ m_0 + m_1 = -1 \\ m_0 + m_1 + m_2 = 1 \end{cases}, \text{then } X = -1 + 2P_2$$

Conclusion

In this paper, we have studied for the first time the combination of symbolic 2-plithogenic numbers with dual numbers. The novel algebraic structure generated by them is called dual symbolic 2-plithogenic numbers.

We have determined the invertibility condition and the formula of the inverse for dual symbolic 2-plithogenic numbers. Also, all idempotent elements in the ring of dual symbolic 2-plithogenic numbers were presented and computed.

References

1. Akar, M.; Yuce, S.; Sahin, S. On The Dual Hyperbolic Numbers and The Complex Hyperbolic Numbers. Jcscm **2018,** 8, DOI: 10.20967/jcscm.2018.01.001.

2. Smarandache, F., " Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)", Neutrosophic Sets and Systems, vol. 53, 2023.

3. Florentin Smarandache, Plithogenic Algebraic Structures. Chapter in "Nidus idearum Scilogs, V: joining the dots" (third version), Pons Publishing Brussels, pp. 123-125, 2019.
4. Florentin Smarandache: Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets – Revisited, *Neutrosophic Sets and Systems*, vol. 21, 2018, pp. 153-166.
5. Ali, R., and Hasan, Z., "An Introduction To The Symbolic 3-Plithogenic Vector Spaces", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.
6. Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.
7. Ben Othman, K., "On Some Algorithms For Solving Symbolic 3-Plithogenic Equations", Neoma Journal Of Mathematics and Computer Science, 2023.
8. Ali, R., and Hasan, Z., " An Introduction To The Symbolic 3-Plithogenic Modules ", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.
9. Nader Mahmoud Taffach , Ahmed Hatip., "A Review on Symbolic 2-Plithogenic Algebraic Structures " Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.
10. M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, "Introduction to Symbolic 2-Plithogenic Probability Theory," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.
11. Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.
12. Sarkis, M., " On The Solutions Of Fermat's Diophantine Equation In 3-refined Neutrosophic Ring of Integers", Neoma Journal of Mathematics and Computer Science, 2023.

13. Abobala, M., "On Some Algebraic Properties of n-Refined Neutrosophic Elements and n-Refined Neutrosophic Linear Equations", Mathematical Problems in Engineering, Hindawi, 2021
14. Abobala, M., "A Study Of Nil Ideals and Kothe's Conjecture In Neutrosophic Rings", International Journal of Mathematics and Mathematical Sciences, hindawi, 2021
15. Khaldi, A**.,** " A Study On Split-Complex Vector Spaces"**,** Neoma Journal Of Mathematics and Computer Science, 2023.
16. Ahmad, K., " On Some Split-Complex Diophantine Equations", Neoma Journal Of Mathematics and Computer Science, 2023.
17. Ali, R., " On The Weak Fuzzy Complex Inner Products On Weak Fuzzy Complex Vector Spaces"**,** Neoma Journal Of Mathematics and Computer Science, 2023.
18. Hatip, A., "An Introduction To Weak Fuzzy Complex Numbers ", Galoitica Journal Of Mathematical Structures and Applications, Vol.3, 2023.
19. Merkepci, M., Abobala, M., and Allouf, A., " The Applications of Fusion Neutrosophic Number Theory in Public Key Cryptography and the Improvement of RSA Algorithm ", Fusion: Practice and Applications, 2023.
20. Merkepci, M,; Sarkis, M. An Application of Pythagorean Circles in Cryptography and Some Ideas for Future Non Classical Systems. Galoitica Journal of Mathematical Structures and Applications **2022**.
21. Merkepci, M., and Abobala, M., " Security Model for Encrypting Uncertain Rational Data Units Based on Refined Neutrosophic Integers Fusion and El Gamal Algorithm ", Fusion: Practice and Applications, 2023.
22. Alhasan, Y., Alfahal, A., Abdulfatah, R., Ali, R., and Aljibawi, M., " On A Novel Security Algorithm For The Encryption Of 3×3 Fuzzy Matrices With

Rational Entries Based On The Symbolic 2-Plithogenic Integers And El-Gamal Algorithm", International Journal of Neutrosophic Science, 2023.

23. Alfahal, A., Alhasan, Y., Abdulfatah, R., Mehmood, A., and Kadhim, T., " On Symbolic 2-Plithogenic Real Matrices And Their Algebraic Properties ", International Journal of Neutrosophic Science, 2023.
24. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Concept Of Symbolic 7-Plithogenic Real Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.
25. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Symbolic 8-Plithogenic Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.
26. Abobala, M., and Allouf, A., " On A Novel Security Scheme for The Encryption and Decryption Of 2×2 Fuzzy Matrices with Rational Entries Based on The Algebra of Neutrosophic Integers and El-Gamal Crypto-System", Neutrosophic Sets and Systems, vol.54, 2023.

Received 19/4/2023, Accepted 19/8/2023

On Pythagoras Triples in Symbolic 3-Plithogenic Rings

Abuobida Mohammed A. Alfahal [1,*], Yaser Ahmad Alhasan [2], Raja Abdullah Abdulfatah[3], Noor Edin Rabeh

[1]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia
[2]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia
[3]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia
[4.] Baghdad University, College of Science, Baghdad, Iraq

Emails:[1.] a.alfahal@psau.edu.sa; [2.] y.alhasan@psau.edu.sa;[3.] r.abdulfatah@psau.edu.sa ;
[4.]edinnoori45@gmail.com
Corresponding author: a.alfahal@psau.edu.sa

Abstract:

The objective of this paper is to find necessary and sufficient conditions for a symbolic 3-plithogenic triple

$(t_0 + t_1P_1 + t_2P_2 + t_3P_3, s_0 + s_1P_1 + s_2P_2 + s_3P_3, k_0 + k_1P_1 + k_2P_2 + k_3P_3)$ to be a Pythagoras triple, i.e. to be a solution for the non-linear Diophantine equation

$X^2 + Y^2 = Z^2$. Also, many examples will be illustrated and presented to explain how the theorems work.

Keywords: symbolic 3-plithogenic ring, Pythagoras triple, Pythagoras Diophantine equation

Introduction and Preliminaries.

Symbolic n-plithogenic sets were defined by Smarandache in [1-3], where these sets were used in generalizing classical algebraic structures such as symbolic 2-plithogenic and symbolic 3-plithogenic structures [4-9], with many applications in other fields [10-12].

zenodo.10031198/10.5281

It is useful to refer that symbolic n-plithogenic algebraic structures are very similar to neutrosophic and refined neutrosophic structures, see [13-21].

In this paper, we continue other efforts to study Pythagoras triples in many different rings [22-25].

We present the concept of Pythagoras triple in a symbolic 3-plithogenic commutative ring with many clear examples that clarify the validity of our work.

Definition.

Let R be a ring, the symbolic 3-plithogenic ring is defined as follows:

$3-SP_R=\{a_0+a_1P_1+a_2P_2+a_3P_3;\ a_i\in R, {P_j}^2=P_j, P_i\times P_j=P_{max(i,j)}\}$.

Smarandache has defined algebraic operations on $3-SP_R$ as follows:

Addition:

$[a_0+a_1P_1+a_2P_2+a_3P_3]+[b_0+b_1P_1+b_2P_2+b_3P_3]=(a_0+b_0)+(a_1+b_1)P_1+(a_2+b_2)P_2+(a_3+b_3)P_3$.

Multiplication:

$[a_0+a_1P_1+a_2P_2+a_3P_3].[b_0+b_1P_1+b_2P_2+b_3P_3]=a_0b_0+a_0b_1P_1+a_0b_2P_2+a_0b_3P_3+a_1b_0{P_1}^2+a_1b_2P_1P_2+a_2b_0P_2+a_2b_1P_1P_2+a_2b_2{P_2}^2+a_1b_3P_3P_1+a_2b_3P_2P_3+a_3b_3(P_3)^2+a_3b_0P_3+a_3b_1P_3P_1+a_3b_2P_2P_3+a_1b_1P_1P_1=a_0b_0+(a_0b_1+a_1b_0+a_1b_1)P_1+(a_0b_2+a_1b_2+a_2b_0+a_2b_1+a_2b_2)P_2+(a_0b_3+a_1b_3+a_2b_3+a_3b_3+a_3b_0+a_3b_1+a_3b_2)P_3$.

Main Discussion

Definition.

Let R be a ring, then (t,s,k) is called a Pythagoras triple if and only if

$t^2+s^2=k^2; t,s,k\in R$..

Theorem.

Let $T=t_0+t_1P_1+t_2P_2+t_3P_3, S=s_0+s_1P_1+s_2P_2+s_3P_3, K=k_0+k_1P_1+k_2P_2+k_3P_3$ are three arbitrary symbolic 3-plithogenic elements $T,S,K\in 3-SP_R$, then $(T,S,K)$are Pythagoras triple in $3-SP_R$if and only if:

$$\begin{cases} (t_0,s_0,k_0),(t_0+t_1,s_0+s_1,k_0+k_1)\ are\ pythagoras\ triples\ in\ R \\ (t_0+t_1+t_2,s_0+s_1+s_2,k_0+k_1+k_2),(t_0+t_1+t_2+t_3,s_0+s_1+s_2+s_3,k_0+k_1+k_2+k_3)\ are\ pythagoras\ triples\ in\ R \end{cases}$$

Proof.

According to [ ], we have:

$$T^2 = {t_0}^2 + [(t_0 + t_1)^2 - {t_0}^2]P_1 + [(t_0 + t_1 + t_2)^2 - (t_0 + t_1)^2]P_2 + [(t_0 + t_1 + t_2 + t_3)^2 - (t_0 + t_1 + t_2)^2]P_3$$

$$S^2 = {s_0}^2 + [(s_0 + s_1)^2 - {s_0}^2]P_1 + [(s_0 + s_1 + s_2)^2 - (s_0 + s_1)^2]P_2 + [(s_0 + s_1 + s_2 + s_3)^2 - (s_0 + s_1 + s_2)^2]P_3$$

$$T^2 = {k_0}^2 + \left[(k_0 + k_1)^2 - {k_0}^2\right]P_1 + [(k_0 + k_1 + k_2)^2 - (k_0 + k_1)^2]P_2 + [(k_0 + k_1 + k_2 + k_3)^2 - (k_0 + k_1 + k_2)^2]P_3$$

The equation $T^2 + S^2 = K^2$ is equivalent to:

${t_0}^2 + {s_0}^2 = {k_0}^2$ $(equation\ 1)$,

$(t_0 + t_1)^2 + (s_0 + s_1)^2 = (k_0 + k_1)^2$ $(equation\ 2)$,

$(t_0 + t_1 + t_2)^2 + (s_0 + s_1 + s_2)^2 = (k_0 + k_1 + k_2)^2$ $(equation\ 3)$,

$(t_0 + t_1 + t_2 + t_3)^2 + (s_0 + s_1 + s_2 + s_3)^2 = (k_0 + k_1 + k_2 + k_3)^2$ $(equation\ 4)$

Equation (1) implies that (t_0, s_0, k_0) is a Pythagoras triple in R.

Equation (2) implies that $(t_0 + t_1, s_0 + s_1, k_0 + k_1)$ is a Pythagoras triple in R.

Equation (3) implies that $(t_0 + t_1 + t_2, s_0 + s_1 + s_2, k_0 + k_1 + k_2)$ is a Pythagoras triple in R.

Equation (4) implies that $(t_0 + t_1 + t_2 + t_3, s_0 + s_1 + s_2 + s_3, k_0 + k_1 + k_2 + k_3)$ is a Pythagoras triple in R.

Thus, the proof is complete.

Theorem.

Let $(t_0, s_0, k_0), (t_1, s_1, k_1), (t_2, s_2, k_2), (t_3, s_3, k_3)$ be four Pythagoras triples in the ring R, then (T, S, K) Pythagoras triple in $3 - SP_R$, where:

$$T = t_0 + [t_1 - t_0]P_1 + [t_2 - t_1]P_2 + [t_3 - t_2]P_3$$

$$S = s_0 + [s_1 - s_0]P_1 + [s_2 - s_1]P_2 + [s_3 - s_2]P_3$$

$$K = k_0 + [k_1 - k_0]P_1 + [k_2 - k_1]P_2 + [k_3 - k_2]P_3$$

Proof.

We have: $t_0 + (t_1 - t_0) = t_1, t_0 + (t_1 - t_0) + (t_2 - t_1) = t_2, t_0 + (t_1 - t_0) + (t_2 - t_1) + (t_3 - t_2) = t_3$

$$s_0 + (s_1 - s_0) = s_1, s_0 + (s_1 - s_0) + (s_2 - s_1) = s_2, s_0 + (s_1 - s_0) + (s_2 - s_1) + (s_3 - s_2) = s_3$$

$k_0 + (k_1 - k_0) = k_1, k_0 + (k_1 - k_0) + (k_2 - k_1)$

$$= k_2, k_0 + (k_1 - k_0) + (k_2 - k_1) + (k_3 - k_2) = k_3$$

This implies that (T, S, K) Pythagoras triple in $3 - SP_R$ according to the theorem.

Examples.

We have:

$$\begin{cases} (t_0, s_0, k_0) = (3,4,5) \\ (t_1, s_1, k_1) = (6,8,10) \\ (t_2, s_2, k_2) = (4,3,5) \\ (t_3, s_3, k_3) = (5,12,13) \end{cases}$$

Are four Pythagoras triples in Z.

The corresponding symbolic 3-plithogenic Pythagoras triple is (T, S, K), where:

$T = 3 + [6 - 3]P_1 + [4 - 6]P_2 + [5 - 4]P_3 = 3 + 3P_1 - 2P_2 + P_3$

$S = 4 + [8 - 4]P_1 + [3 - 8]P_2 + [12 - 3]P_3 = 4 + 4P_1 - 5P_2 + 9P_3$

$K = 5 + [10 - 5]P_1 + [5 - 10]P_2 + [13 - 5]P_3 = 5 + 5P_1 - 5P_2 + 8P_3$

Example.

Find all Pythagoras triples in $3 - SP_{Z_2}$, where Z_2 I the ring of integers module 2.

First, we find all Pythagoras triples in Z.

$L_1 = (0,0,0), L_2 = (1,0,1), L_3 = (0,1,1), L_4 = (1,1,0)$

Remark that for every permutation of the set $\{L_1, L_2, L_3, L_4\}$, we get a different symbolic 3-plithogenic Pythagoras triple.

We discuss all possible cases:

Permutation (1).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_1 = P_1 - P_2 + P_3 = P_1 + P_2 + P_3 \\ \acute{Y}_1 = P_2 \\ Y_1'' = P_1 - P_3 = P_1 + P_3 \end{cases}$$

Permutation (2).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_2 = P_2 \\ \acute{Y}_2 = P_1 - P_2 + P_3 = P_1 + P_2 + P_3 \\ Y_2'' = P_1 - P_3 = P_1 + P_3 \end{cases}$$

Permutation (3).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_3 = P_1 + P_3 \\ \acute{Y}_3 = P_1 + P_2 + P_3 \\ Y_3'' = P_2 \end{cases}$$

Permutation (4).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_4 = P_1 + P_2 + P_3 \\ \acute{Y}_4 = P_1 + P_3 \\ {Y_4}'' = P_2 \end{cases}$$

Permutation (5).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_5 = P_1 + P_3 \\ \acute{Y}_5 = P_2 \\ {Y_5}'' = P_1 + P_2 + P_3 \end{cases}$$

Permutation (6).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_6 = P_2 \\ \acute{Y}_6 = P_1 + P_3 \\ {Y_6}'' = P_1 + P_2 + P_3 \end{cases}$$

Permutation (7).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_7 = 1 + P_2 + P_3 \\ \acute{Y}_7 = P_1 \\ {Y_7}'' = 1 + P_1 + P_2 + P_3 \end{cases}$$

Permutation (8).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_8 = P_2 \\ \acute{Y}_8 = 1 + P_1 + P_2 \\ {Y_8}'' = 1 + P_1 \end{cases}$$

Permutation (9).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_9 = 1 + P_1 + P_3 \\ \acute{Y}_9 = P_1 + P_2 + P_3 \\ {Y_9}'' = 1 + P_2 \end{cases}$$

Permutation (10).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_{10} = P_1 + P_2 + P_3 \\ \acute{Y}_{10} = 1 + P_1 + P_3 \\ {Y_{10}}'' = 1 + P_2 \end{cases}$$

Permutation (11).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_{11} = 1 + P_2 \\ \acute{Y}_{11} = 1 + P_1 + P_3 \\ {Y_{11}}'' = P_1 + P_2 + P_3 \end{cases}$$

Permutation (12).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_{12} = 1 + P_1 + P_2 + P_3 \\ Ý_{12} = 1 + P_1 + P_3 \\ Y_{12}'' = P_2 \end{cases}$$

Permutation (13).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_{13} = 1 + P_1 + P_2 + P_3 \\ Ý_{13} = 1 + P_2 \\ Y_{13}'' = P_1 + P_3 \end{cases}$$

Permutation (14).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_{14} = 1 + P_1 + P_3 \\ Ý_{14} = 1 + P_2 \\ Y_{14}'' = P_1 + P_2 + P_3 \end{cases}$$

Permutation (15).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_{15} = 1 + P_1 + P_3 \\ Ý_{15} = 1 + P_1 + P_2 + P_3 \\ Y_{15}'' = P_2 \end{cases}$$

Permutation (16).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_{16} = 1 + P_2 \\ Ý_{16} = 1 + P_1 + P_2 + P_3 \\ Y_{16}'' = P_1 + P_3 \end{cases}$$

Permutation (17).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_{17} = 1 + P_2 \\ Ý_{17} = P_1 + P_2 + P_3 \\ Y_{17}'' = 1 + P_1 + P_3 \end{cases}$$

Permutation (18).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_{18} = 1 + P_2 \\ Ý_{18} = P_1 + P_3 \\ Y_{18}'' = 1 + P_1 + P_2 + P_3 \end{cases}$$

Permutation (19).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_{19} = P_1 + P_3 \\ Ý_{19} = 1 + P_1 + P_2 + P_3 \\ Y_{19}'' = 1 + P_2 \end{cases}$$

Permutation (20).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_{20} = P_1 + P_3 \\ Ý_{20} = 1 + P_1 + P_2 + P_3 \\ Y_{20}'' = 1 + P_2 \end{cases}$$

Permutation (21).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_{21} = P_1 + P_2 + P_3 \\ \acute{Y_{21}} = 1 + P_2 + P_3 \\ Y_{21}'' = 1 + P_1 \end{cases}$$

Permutation (22).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_{22} = 1 + P_1 + P_3 \\ \acute{Y_{22}} = P_1 + P_2 + P_3 \\ Y_{22}'' = 1 + P_2 \end{cases}$$

Permutation (23).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_{23} = P_1 + P_3 \\ \acute{Y_{23}} = 1 + P_2 \\ Y_{23}'' = 1 + P_1 + P_2 + P_3 \end{cases}$$

Permutation (24).

$$(L_1, L_2, L_3, L_4): \begin{cases} Y_{24} = P_1 + P_2 + P_3 \\ \acute{Y_{24}} = 1 + P_1 + P_3 \\ Y_{24}'' = 1 + P_2 \end{cases}$$

Also, other quadraples $(L_i, L_j, L_k, L_s); 1 \le i, j, k, s \le 4$ give Pythagoras triples with i, j, k, s are not distinct at all.

We continuo our discussions.

Permutation (25).

$$(L_1, L_1, L_1, L_1): \begin{cases} Y_{25} = (0,0,0) \\ \acute{Y_{25}} = (0,0,0) \\ Y_{25}'' = (0,0,0) \end{cases}$$

Permutation (26).

$$(L_1, L_1, L_1, L_2): \begin{cases} Y_{26} = P_3 \\ \acute{Y_{26}} = 0 \\ Y_{26}'' = P_3 \end{cases}$$

Permutation (27).

$$(L_1, L_1, L_1, L_3): \begin{cases} Y_{27} = 0 \\ \acute{Y_{27}} = P_3 \\ Y_{27}'' = P_3 \end{cases}$$

Permutation (28).

$$(L_1, L_1, L_1, L_4): \begin{cases} Y_{28} = P_3 \\ \acute{Y_{28}} = P_3 \\ Y_{28}'' = 0 \end{cases}$$

Permutation (29).

$$(L_1, L_2, L_1, L_1): \begin{cases} Y_{29} = P_1 + P_2 \\ \acute{Y_{29}} = 0 \\ Y_{29}'' = P_1 + P_2 \end{cases}$$

Permutation (30).

$$(L_1, L_3, L_1, L_1): \begin{cases} Y_{30} = 0 \\ \acute{Y_{30}} = P_1 + P_2 \\ Y_{30}'' = P_1 + P_2 \end{cases}$$

Permutation (31).

$$(L_1, L_4, L_1, L_1): \begin{cases} Y_{31} = P_1 + P_2 \\ \acute{Y_{31}} = P_1 + P_2 \\ Y_{31}'' = 0 \end{cases}$$

Permutation (32).

$$(L_1, L_1, L_2, L_1): \begin{cases} Y_{32} = P_2 + P_3 \\ \acute{Y_{32}} = 0 \\ Y_{32}'' = P_2 + P_3 \end{cases}$$

Permutation (33).

$$(L_1, L_1, L_3, L_1): \begin{cases} Y_{33} = 0 \\ \acute{Y_{33}} = P_2 + P_3 \\ Y_{33}'' = P_2 + P_3 \end{cases}$$

Permutation (34).

$$(L_1, L_1, L_4, L_1): \begin{cases} Y_{34} = P_2 + P_3 \\ \acute{Y_{34}} = P_2 + P_3 \\ Y_{34}'' = 0 \end{cases}$$

Permutation (35).

$$(L_2, L_2, L_2, L_2): \begin{cases} Y_{35} = 1 \\ \acute{Y_{35}} = 0 \\ Y_{35}'' = 1 \end{cases}$$

Permutation (36).

$$(L_2, L_2, L_2, L_1): \begin{cases} Y_{36} = 1 + P_3 \\ \acute{Y_{36}} = 0 \\ Y_{36}'' = 1 + P_3 \end{cases}$$

Permutation (37).

$$(L_2, L_2, L_2, L_1): \begin{cases} Y_{37} = 1 + P_3 \\ \acute{Y_{37}} = P_3 \\ Y_{37}'' = 1 \end{cases}$$

Permutation (38).

$$(L_2, L_2, L_2, L_4): \begin{cases} Y_{38} = 1 \\ \acute{Y_{38}} = P_3 \\ Y_{38}'' = 1 + P_3 \end{cases}$$

Permutation (39).

$$(L_2, L_2, L_1, L_2): \begin{cases} Y_{39} = 1 + P_2 + P_3 \\ \acute{Y_{39}} = 0 \\ Y_{39}'' = 1 + P_2 + P_3 \end{cases}$$

Permutation (40).

$$(L_2, L_2, L_3, L_2): \begin{cases} Y_{40} = 1 + P_2 + P_3 \\ \acute{Y_{40}} = P_2 + P_3 \\ Y_{40}'' = 1 \end{cases}$$

Permutation (41).

$$(L_2, L_2, L_4, L_2): \begin{cases} Y_{41} = 1 \\ \acute{Y_{41}} = P_2 + P_3 \\ Y_{41}'' = 1 + P_2 + P_3 \end{cases}$$

Permutation (42).

$$(L_2, L_1, L_2, L_2): \begin{cases} Y_{42} = 1 + P_1 \\ \acute{Y_{42}} = 0 \\ Y_{42}'' = 1 + P_1 \end{cases}$$

Permutation (43).

$$(L_2, L_4, L_2, L_2): \begin{cases} Y_{43} = 1 \\ \acute{Y_{43}} = P_1 + P_2 \\ Y_{43}'' = 1 + P_2 + P_1 \end{cases}$$

Permutation (44).

$$(L_2, L_3, L_2, L_2): \begin{cases} Y_{44} = 1 + P_2 + P_3 \\ \acute{Y_{44}} = P_1 + P_2 \\ Y_{44}'' = 1 + P_1 + P_3 \end{cases}$$

Permutation (45).

$$(L_3, L_3, L_3, L_3): \begin{cases} Y_{45} = 0 \\ \acute{Y_{45}} = 1 \\ Y_{45}'' = 1 \end{cases}$$

Permutation (46).

$$(L_3, L_3, L_3, L_1): \begin{cases} Y_{46} = 0 \\ \acute{Y}_{46} = P_3 \\ Y_{46}'' = P_3 \end{cases}$$

Permutation (47).

$$(L_3, L_3, L_3, L_2): \begin{cases} Y_{47} = P_3 \\ \acute{Y}_{47} = 1 + P_3 \\ Y_{47}'' = 1 \end{cases}$$

Permutation (48).

$$(L_3, L_3, L_3, L_4): \begin{cases} Y_{48} = P_3 \\ \acute{Y}_{48} = 1 \\ Y_{48}'' = 1 + P_3 \end{cases}$$

Permutation (49).

$$(L_3, L_3, L_1, L_3): \begin{cases} Y_{49} = 0 \\ \acute{Y}_{49} = 1 + P_2 + P_3 \\ Y_{49}'' = 1 + P_2 + P_3 \end{cases}$$

Permutation (50).

$$(L_3, L_3, L_2, L_3): \begin{cases} Y_{50} = P_2 + P_3 \\ \acute{Y}_{50} = 1 + P_2 + P_3 \\ Y_{50}'' = 1 \end{cases}$$

Permutation (51).

$$(L_3, L_3, L_4, L_3): \begin{cases} Y_{51} = P_2 + P_3 \\ \acute{Y}_{51} = 1 \\ Y_{51}'' = 1 + P_2 + P_3 \end{cases}$$

Permutation (52).

$$(L_3, L_1, L_3, L_3): \begin{cases} Y_{52} = 0 \\ \acute{Y}_{52} = 1 + P_1 + P_2 \\ Y_{52}'' = 1 + P_1 + P_2 \end{cases}$$

Permutation (53).

$$(L_3, L_2, L_3, L_3): \begin{cases} Y_{53} = P_1 + P_2 \\ \acute{Y}_{53} = 1 + P_1 + P_2 \\ Y_{53}'' = 1 \end{cases}$$

Permutation (54).

$$(L_3, L_4, L_3, L_3): \begin{cases} Y_{54} = P_1 + P_2 \\ \acute{Y_{54}} = 1 \\ Y_{54}'' = 1 + P_1 + P_2 \end{cases}$$

Permutation (55).

$$(L_4, L_4, L_4, L_4): \begin{cases} Y_{55} = 1 \\ \acute{Y_{55}} = 1 \\ Y_{55}'' = 0 \end{cases}$$

Permutation (56).

$$(L_4, L_4, L_4, L_1): \begin{cases} Y_{56} = 1 + P_3 \\ \acute{Y_{56}} = 1 + P_3 \\ Y_{56}'' = 0 \end{cases}$$

Permutation (57).

$$(L_4, L_4, L_4, L_2): \begin{cases} Y_{57} = 1 \\ \acute{Y_{57}} = 1 + P_3 \\ Y_{57}'' = P_3 \end{cases}$$

Permutation (58).

$$(L_4, L_4, L_4, L_3): \begin{cases} Y_{58} = 1 + P_3 \\ \acute{Y_{58}} = 1 \\ Y_{58}'' = P_3 \end{cases}$$

Permutation (59).

$$(L_4, L_4, L_1, L_4): \begin{cases} Y_{59} = 1 + P_2 + P_3 \\ \acute{Y_{59}} = 1 + P_2 + P_3 \\ Y_{59}'' = 0 \end{cases}$$

Permutation (60).

$$(L_4, L_4, L_2, L_4): \begin{cases} Y_{60} = 1 \\ \acute{Y_{60}} = 1 + P_2 + P_3 \\ Y_{60}'' = P_2 + P_3 \end{cases}$$

Permutation (61).

$$(L_4, L_4, L_3, L_4): \begin{cases} Y_{61} = 1 + P_2 + P_3 \\ \acute{Y_{61}} = 1 \\ Y_{61}'' = P_2 + P_3 \end{cases}$$

Permutation (62).

$$(L_4, L_1, L_4, L_4): \begin{cases} Y_{62} = 1 + P_1 + P_2 \\ \acute{Y_{62}} = 1 + P_1 + P_2 \\ Y_{62}'' = 0 \end{cases}$$

Permutation (63).

$$(L_4, L_2, L_4, L_4): \begin{cases} Y_{63} = 1 \\ Y_{63}' = 1 + P_1 + P_2 \\ Y_{63}'' = P_1 + P_2 \end{cases}$$

Permutation (64).

$$(L_4, L_3, L_4, L_4): \begin{cases} Y_{64} = 1 + P_1 + P_2 \\ Y_{64}' = 1 \\ Y_{64}'' = P_1 + P_2 \end{cases}$$

By continuing this argument, we can get all Pythagoras triples in $3 - SP_{Z_2}$

Conclusion.

In this paper, we have studied Pythagoras triples in symbolic 3-plithogenic commutative rings, where necessary and sufficient conditions for a symbolic 3-plithogenic triple (x, y, z) to be a Pythagoras triple.

Also, we have presented some related examples that explain how to find 3-plithogenic triples from classical triples.

Acknowledgments " This study is supported via funding from Prince sattam bin Abdulaziz University project number (PSAU/2023/R/1445) ".

References

[1] F. Smarandache, Plithogeny, Plithogenic Set, Logic, Probability, and Statistics, 141 pages, Pons Editions,Brussels, Belgium, 2017. arXiv.org (Cornell University), Computer Science - Artificial Intelligence, 03Bxx:

[2] Florentin Smarandache, Physical Plithogenic Set, 71st Annual Gaseous Electronics Conference, Session LW1, Oregon Convention Center Room, Portland, Oregon, USA, November 5–9, 2018.

[3] Florentin Smarandache: Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets – Revisited, *Neutrosophic Sets and Systems*, vol. 21, 2018, pp. 153-166.

[4] M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, "Introduction to Symbolic 2-Plithogenic Probability Theory," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.

[5] Nader Mahmoud Taffach , Ahmed Hatip., "A Review on Symbolic 2-Plithogenic Algebraic Structures " Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.

[6] Nader Mahmoud Taffach , Ahmed Hatip.," A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations ", Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.

[7] Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.

[8] Albasheer, O., Hajjari., A., and Dalla., R., " On The Symbolic 3-Plithogenic Rings and TheirAlgebraic Properties", Neutrosophic Sets and Systems, Vol 54, 2023.

[9] Ben Othman, K., "On Some Algorithms For Solving Symbolic 3-Plithogenic Equations", Neoma Journal Of Mathematics and Computer Science, 2023.

[10] Ali, R., and Hasan, Z., "An Introduction To The Symbolic 3-Plithogenic Vector Spaces", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.

[11] Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.

[12] Alhasan, Y., Alfahal, A., Abdulfatah, R., Ali, R., and Aljibawi, M., " On A Novel Security Algorithm For The Encryption Of 3×3 Fuzzy Matrices With Rational Entries Based On The Symbolic 2-Plithogenic Integers And El-Gamal Algorithm", International Journal of Neutrosophic Science, 2023.

[13] Merkepci, M., Abobala, M., and Allouf, A., " The Applications of Fusion Neutrosophic Number Theory in Public Key Cryptography and the Improvement of RSA Algorithm ", Fusion: Practice and Applications, 2023.

[14] Merkepci, M., and Abobala, M., " Security Model for Encrypting Uncertain Rational Data Units Based on Refined Neutrosophic Integers Fusion and El Gamal Algorithm ", Fusion: Practice and Applications, 2023.

[15] Abobala, M., and Allouf, A., " On A Novel Security Scheme for The Encryption and Decryption Of 2×2 Fuzzy Matrices with Rational Entries Based on The Algebra of Neutrosophic Integers and El-Gamal Crypto-System", Neutrosophic Sets and Systems, vol.54, 2023.

[16] Abobala, M, "n-Cyclic Refined Neutrosophic Algebraic Systems Of Sub-Indeterminacies, An Application To Rings and Modules", International Journal of Neutrosophic Science, Vol. 12, pp. 81-95 . 2020.

[17] Mohamed, M., "Modified Approach for Optimization of Real-Life Transportation Environment: Suggested Modifications", American Journal of Business and Operations research, 2021.

[18] Nabeeh, N., Alshaimaa, A., and Tantawy, A., "A Neutrosophic Proposed Model For Evaluation Blockchain Technology in Secure Enterprise Distributed Applications", Journal of Cypersecurity and Information Management, 2023.

[19] Abualkishik, A., Almajed, R., Thompson, W., "Improving The Performance of Fog-assisted Internet of Things Networks Using Bipolar Trapezoidal Neutrosophic Sets", Journal of Wireless and Ad Hoc Communication, 2023.

[20] Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Concept Of Symbolic 7-Plithogenic Real Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

[21] Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Symbolic 8-Plithogenic Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

[22] R. A. Trivedi, S. A. Bhanotar, Pythagorean Triplets - Views, Analysis and Classification, IIOSR J. Math., 11 (2015).

[23] J. Rukavicka, Dickson's method for generating Pythagorean triples revisited, Eur. J. Pure Appl.Math., 6 (2013).

[24] T.Roy and F J. Sonia, A Direct Method To Generate Pythagorean Triples And Its Generalization To Pythagorean Quadruples And n-tuples, , Jadavpur University, India (2010)

[25] Paul Oliverio, " Self-Generating Pythagorean Quadruples And TV-Tuples", Los Angeles, (1993).

Received 20/7/2023, Accepted 5/10/2023

Neutrosophic Sets and Systems, Vol. 59, 2023
University of New Mexico

A Study of Symbolic 2-Plithogenic Split-Complex Linear Diophantine Equations in Two Variables

[1.]Rama Asad Nadweh, [2.]Oliver Von Shtawzen, [3.]Ahmad Khaldi,[4.] Rozina Ali

[1.]Islamic Online University, Department Of Science and Information Technology, Doha, Qatar
ramaanadwehh@gmail.com

[2.] University Of Nizwa, Sultanate Of Oman, Vonshtawzen1970abc@gmail.com

[3] Mutah University, Faculty of Science, Jordan, khaldiahmad1221@gmail.com

[4] Cairo University, Faculty of Science, Egypt, rozyyy123n@gmail.com

Abstract:

The equation $AX + BY = C$ is called symbolic 2-plithogenic linear Diophantine equation with two variables if A, B, X, Y, C are symbolic 2-plithogenic split-complex integers.

This paper aims to find an algebraic formula for solving the symbolic 2-plithogenic split-complex linear Diophantine equation with two variables with necessary and sufficient conditions for the solvability of this class. Also, some related examples will be illustrated.

Keywords: Split-complex, symbolic 2-plithogenic, linear Diophantine equation.

Introduction.

Diophantine equation is very interesting concept in Number theory, where they are considered as algebraic equations with integer solutions [1].

In the literature, we find many generalized kinds of Diophantine equations handled by many authors, see [2-5].

A classical linear Diophantine equation an equation with the following formula:

zenodo.10031202/10.5281

$AX + BY = C$, where A, B, C, X, Y are integers.

Split-complex numbers were built over real numbers a generalization of them with a similar structure to the complex numbers, where a split-complex number is defined as follows:

$a + bJ$; $a, b \in R, J^2 = 1, J \neq \{-1,1\}$, and they are studied by many authors in [6-10].

If $a, b \in Z$, then $a + bJ$ is called a split-complex integer.

The concept of symbolic 2-plithogenic split-complex numbers was defined as an extension of symbolic 2-plithogenic numbers [12]. The generalizations of real numbers, especially the plithogenic numbers have many applications in many scientific fields, see [13-20].

In this work, we present an effective algorithm to find all solutions of the symbolic 2-plithogenic split-complex linear Diophantine equation with two variables.

Preliminaries

Main discussion.

Definition.

Let $AX + BY = C$ with:

$A = (a_0 + a_1P_1 + a_2P_2) + J(\acute{a}_0 + \acute{a}_1P_1 + \acute{a}_2P_2)$

$B = (b_0 + b_1P_1 + b_2P_2) + J(\acute{b}_0 + \acute{b}_1P_1 + \acute{b}_2P_2)$

$C = (c_0 + c_1P_1 + c_2P_2) + J(\acute{c}_0 + \acute{c}_1P_1 + \acute{c}_2P_2)$

$X = (x_0 + x_1P_1 + x_2P_2) + J(\acute{x}_0 + \acute{x}_1P_1 + \acute{x}_2P_2)$

$Y = (y_0 + y_1P_1 + y_2P_2) + J(\acute{y}_0 + \acute{y}_1P_1 + \acute{y}_2P_2)$

Where $x_i, y_i, a_i, b_i, c_i, \acute{x}_i, \acute{y}_i, \acute{a}_i, \acute{b}_i, \acute{c}_i \in Z$.

The previous equation is called symbolic 2-plithogenic split-complex Diophantine equation with two variables X and Y.

Example

$[(2 + P_1 + P_2) + J(1 + P_2)]X + [(1 - P_1 + 3P_2) + J(4 - 5P_1 + P_2)]Y = P_1 + P_2$

Is a symbolic 2-plithogenic split-complex Diophantine equation with two variables.

How can we find the solutions?

First, we must transform the equation to classical Diophantine equations.

For this goal, we must compute the products AX, BY.

$AX = (a_0 + a_1P_1 + a_2P_2)(x_0 + x_1P_1 + x_2P_2) + (\acute{a}_0 + \acute{a}_1P_1 + \acute{a}_2P_2)(\acute{x}_0 + \acute{x}_1P_1 + \acute{x}_2P_2) + J[(a_0 + a_1P_1 + a_2P_2)(\acute{x}_0 + \acute{x}_1P_1 + \acute{x}_2P_2) + (\acute{a}_0 + \acute{a}_1P_1 + \acute{a}_2P_2)(x_0 + x_1P_1 + x_2P_2)]$.

We have:

$(a_0 + a_1P_1 + a_2P_2)(x_0 + x_1P_1 + x_2P_2) = a_0x_0 + P_1[(a_0 + a_1)(x_0 + x_1) - a_0x_0] + P_2[(a_0 + a_1 + a_2)(x_0 + x_1 + x_2) - (a_0 + a_1)(x_0 + x_1)]$,

$(\acute{a}_0 + \acute{a}_1P_1 + \acute{a}_2P_2)(\acute{x}_0 + \acute{x}_1P_1 + \acute{x}_2P_2) = \acute{a}_0\acute{x}_0 + P_1[(\acute{a}_0 + \acute{a}_1)(\acute{x}_0 + \acute{x}_1) - \acute{a}_0\acute{x}_0] + P_2[(\acute{a}_0 + \acute{a}_1 + \acute{a}_2)(\acute{x}_0 + \acute{x}_1 + \acute{x}_2) - (\acute{a}_0 + \acute{a}_1)(\acute{x}_0 + \acute{x}_1)]$,

$(a_0 + a_1P_1 + a_2P_2)(\acute{x}_0 + \acute{x}_1P_1 + \acute{x}_2P_2) = a_0\acute{x}_0 + P_1[(a_0 + a_1)(\acute{x}_0 + \acute{x}_1) - a_0\acute{x}_0] + P_2[(a_0 + a_1 + a_2)(\acute{x}_0 + \acute{x}_1 + \acute{x}_2) - (a_0 + a_1)(\acute{x}_0 + \acute{x}_1)]$,

$(\acute{a}_0 + \acute{a}_1P_1 + \acute{a}_2P_2)(\acute{x}_0 + \acute{x}_1P_1 + \acute{x}_2P_2) = \acute{a}_0x_0 + P_1[(\acute{a}_0 + \acute{a}_1)(x_0 + x_1) - \acute{a}_0x_0] + P_2[(\acute{a}_0 + \acute{a}_1 + \acute{a}_2)(x_0 + x_1 + x_2) - (\acute{a}_0 + \acute{a}_1)(x_0 + x_1)]$,

So that.

$$
\begin{aligned}
AX = (a_0x_0 + \acute{a}_0\acute{x}_0) &+ P_1[(a_0 + a_1)(x_0 + x_1) + (\acute{a}_0 + \acute{a}_1)(\acute{x}_0 + \acute{x}_1) - a_0x_0 - \acute{a}_0\acute{x}_0] \\
&+ P_2[(a_0 + a_1 + a_2)(x_0 + x_1 + x_2) + (\acute{a}_0 + \acute{a}_1 + \acute{a}_2)(\acute{x}_0 + \acute{x}_1 + \acute{x}_2) \\
&- (a_0 + a_1)(x_0 + x_1) - (\acute{a}_0 + \acute{a}_1)(\acute{x}_0 + \acute{x}_1)] \\
&+ J\big[(a_0\acute{x}_0 + \acute{a}_0x_0) \\
&+ P_1[(a_0 + a_1)(\acute{x}_0 + \acute{x}_1) + (\acute{a}_0 + \acute{a}_1)(x_0 + x_1) - a_0\acute{x}_0 - \acute{a}_0x_0] \\
&+ P_2[(a_0 + a_1 + a_2)(\acute{x}_0 + \acute{x}_1 + \acute{x}_2) + (\acute{a}_0 + \acute{a}_1 + \acute{a}_2)(x_0 + x_1 + x_2) \\
&- (a_0 + a_1)(\acute{x}_0 + \acute{x}_1) - (\acute{a}_0 + \acute{a}_1)(x_0 + x_1)]\big]
\end{aligned}
$$

By a similar argument, we can write:

$$
\begin{aligned}
BY = \left(b_0y_0 + \acute{b}_0\acute{y}_0\right) &+ P_1\left[(b_0 + b_1)(y_0 + y_1) + \left(\acute{b}_0 + \acute{b}_1\right)(\acute{y}_0 + \acute{y}_1) - b_0y_0 - \acute{b}_0\acute{y}_0\right] \\
&+ P_2\left[(b_0 + b_1 + b_2)(y_0 + y_1 + y_2) + \left(\acute{b}_0 + \acute{b}_1 + \acute{b}_2\right)(\acute{y}_0 + \acute{y}_1 + \acute{y}_2)\right. \\
&\left.- (b_0 + b_1)(y_0 + y_1) - \left(\acute{b}_0 + \acute{b}_1\right)(\acute{y}_0 + \acute{y}_1)\right] \\
&+ J\left[\left(b_0\acute{y}_0 + \acute{b}_0y_0\right)\right. \\
&+ P_1\left[(b_0 + b_1)(\acute{y}_0 + \acute{y}_1) + \left(\acute{b}_0 + \acute{b}_1\right)(y_0 + y_1) - b_0\acute{y}_0 - \acute{b}_0y_0\right] \\
&+ P_2\left[(b_0 + b_1 + b_2)(\acute{y}_0 + \acute{y}_1 + \acute{y}_2) + \left(\acute{b}_0 + \acute{b}_1 + \acute{b}_2\right)(y_0 + y_1 + y_2)\right. \\
&\left.\left.- (b_0 + b_1)(\acute{y}_0 + \acute{y}_1) - \left(\acute{b}_0 + \acute{b}_1\right)(y_0 + y_1)\right]\right]
\end{aligned}
$$

The equation $AX + BY = C$ is equivalent to the following system of Diophantine equations:

Equation (1):

$a_0 x_0 + \acute{a}_0 \acute{x}_0 + b_0 y_0 + \acute{b}_0 \acute{y}_0 = c_0$

Equation (2):

$(a_0 + a_1)(x_0 + x_1) + (\acute{a}_0 + \acute{a}_1)(\acute{x}_0 + \acute{x}_1) + (b_0 + b_1)(y_0 + y_1) + (\acute{b}_0 + \acute{b}_1)(\acute{y}_0 + \acute{y}_1) - a_0 x_0 - \acute{a}_0 \acute{x}_0 - b_0 y_0 - \acute{b}_0 \acute{y}_0 = c_1$, thus:

$$(a_0 + a_1)(x_0 + x_1) + (\acute{a}_0 + \acute{a}_1)(\acute{x}_0 + \acute{x}_1) + (b_0 + b_1)(y_0 + y_1) + (\acute{b}_0 + \acute{b}_1)(\acute{y}_0 + \acute{y}_1) = c_0 + c_1$$

Equation (3):

$(a_0 + a_1 + a_2)(x_0 + x_1 + x_2) + (\acute{a}_0 + \acute{a}_1 + \acute{a}_2)(\acute{x}_0 + \acute{x}_1 + \acute{x}_2) + (b_0 + b_1 + b_2)(y_0 + y_1 + y_2) + (\acute{b}_0 + \acute{b}_1 + \acute{b}_2)(\acute{y}_0 + \acute{y}_1 + \acute{y}_2) - (a_0 + a_1)(x_0 + x_1) - (\acute{a}_0 + \acute{a}_1)(\acute{x}_0 + \acute{x}_1) - (b_0 + b_1)(y_0 + y_1) - (\acute{b}_0 + \acute{b}_1)(\acute{y}_0 + \acute{y}_1) = c_2$, thus:

$$(a_0 + a_1 + a_2)(x_0 + x_1 + x_2) + (\acute{a}_0 + \acute{a}_1 + \acute{a}_2)(\acute{x}_0 + \acute{x}_1 + \acute{x}_2) + (b_0 + b_1 + b_2)(y_0 + y_1 + y_2) + (\acute{b}_0 + \acute{b}_1 + \acute{b}_2)(\acute{y}_0 + \acute{y}_1 + \acute{y}_2) = c_0 + c_1 + c_2$$

Equation (4):

$a_0 \acute{x}_0 + \acute{a}_0 x_0 + b_0 \acute{y}_0 + \acute{b}_0 y_0 = \acute{c}_0$

Equation (5):

$$(a_0 + a_1)(\acute{x}_0 + \acute{x}_1) + (\acute{a}_0 + \acute{a}_1)(x_0 + x_1) + (b_0 + b_1)(\acute{y}_0 + \acute{y}_1) + (\acute{b}_0 + \acute{b}_1)(y_0 + y_1) = \acute{c}_0 + \acute{c}_1$$

Equation (6):

$$(a_0 + a_1 + a_2)(\acute{x}_0 + \acute{x}_1 + \acute{x}_2) + (\acute{a}_0 + \acute{a}_1 + \acute{a}_2)(x_0 + x_1 + x_2) + (b_0 + b_1 + b_2)(\acute{y}_0 + \acute{y}_1 + \acute{y}_2) + (\acute{b}_0 + \acute{b}_1 + \acute{b}_2)(y_0 + y_1 + y_2) = \acute{c}_0 + \acute{c}_1 + \acute{c}_2$$

By now, we have six linear Diophantine equation with four variables.

We will transform them into easier forms.

We add equation (1) to (4), we get:

$(a_0 + \acute{a}_0)(x_0 + \acute{x}_0) + (b_0 + \acute{b}_0)(y_0 + \acute{y}_0) = c_0 + \acute{c}_0 \quad (I)$

We add equation (2) to (5), we get:

$$(a_0 + a_1 + \acute{a}_0 + \acute{a}_1)(x_0 + x_1 + \acute{x}_0 + \acute{x}_1) + \left(b_0 + b_1 + \acute{b}_0 + \acute{b}_1\right)(y_0 + y_1 + \acute{y}_0 + \acute{y}_1)$$
$$= c_0 + c_1 + \acute{c}_0 + \acute{c}_1 \quad (II)$$

We add equation (3) to (6), we get:

$$(a_0 + a_1 + a_2 + \acute{a}_0 + \acute{a}_1 + \acute{a}_2)(x_0 + x_1 + x_2 + \acute{x}_0 + \acute{x}_1 + \acute{x}_2)$$
$$+ \left(b_0 + b_1 + b_2 + \acute{b}_0 + \acute{b}_1 + \acute{b}_2\right)(y_0 + y_1 + y_2 + \acute{y}_0 + \acute{y}_1 + \acute{y}_2)$$
$$= c_0 + c_1 + c_2 + \acute{c}_0 + \acute{c}_1 + \acute{c}_2 \quad (III)$$

We subtract equation (3) from (1), we get:

$$(a_0 - \acute{a}_0)(x_0 - \acute{x}_0) + \left(b_0 - \acute{b}_0\right)(y_0 - \acute{y}_0) = c_0 - \acute{c}_0 \quad (IV)$$

We subtract equation (5) to (2), we get:

$$(a_0 + a_1 - \acute{a}_0 - \acute{a}_1)(x_0 + x_1 - \acute{x}_0 - \acute{x}_1) + \left(b_0 + b_1 - \acute{b}_0 - \acute{b}_1\right)(y_0 + y_1 - \acute{y}_0 - \acute{y}_1)$$
$$= c_0 + c_1 - \acute{c}_0 - \acute{c}_1 \quad (IIV)$$

We subtract equation (6) from (3), we get:

$$(a_0 + a_1 + a_2 - \acute{a}_0 - \acute{a}_1 - \acute{a}_2)(x_0 + x_1 + x_2 - \acute{x}_0 - \acute{x}_1 - \acute{x}_2)$$
$$+ \left(b_0 + b_1 + b_2 - \acute{b}_0 - \acute{b}_1 - \acute{b}_2\right)(y_0 + y_1 + y_2 - \acute{y}_0 - \acute{y}_1 - \acute{y}_2)$$
$$= c_0 + c_1 + c_2 - \acute{c}_0 - \acute{c}_1 - \acute{c}_2 \quad (IIIV)$$

We change the variables by the following:

$$\begin{cases} x_0 + \acute{x}_0 = t_0, x_0 - \acute{x}_0 = \acute{t}_0 \\ x_0 + x_1 + \acute{x}_0 + \acute{x}_1 = t_1, x_0 + x_1 - \acute{x}_0 - \acute{x}_1 = \acute{t}_1 \\ x_0 + x_1 + x_2 + \acute{x}_0 + \acute{x}_1 + \acute{x}_2 = t_2, x_0 + x_1 + x_2 - \acute{x}_0 - \acute{x}_1 - \acute{x}_2 = \acute{t}_2 \\ y_0 + \acute{y}_0 = s_0, y_0 + \acute{y}_0 = \acute{s}_0 \\ y_0 + y_1 + \acute{y}_0 + \acute{y}_1 = s_1, y_0 + y_1 - \acute{y}_0 - \acute{y}_1 = \acute{s}_1 \\ y_0 + y_1 + y_2 + \acute{y}_0 + \acute{y}_1 + \acute{y}_2 = s_2, y_0 + y_1 + y_2 - \acute{y}_0 - \acute{y}_1 - \acute{y}_2 = \acute{s}_2 \end{cases}$$

The equation can be written as follows:

$$(a_0 + \acute{a}_0)(t_0) + \left(b_0 + \acute{b}_0\right)(s_0) = c_0 + \acute{c}_0 \quad (I)$$

$$(a_0 + a_1 + \acute{a}_0 + \acute{a}_1)(t_1) + \left(b_0 + b_1 + \acute{b}_0 + \acute{b}_1\right)(s_1) = c_0 + c_1 + \acute{c}_0 + \acute{c}_1 \quad (II)$$

$$(a_0 + a_1 + a_2 + \acute{a}_0 + \acute{a}_1 + \acute{a}_2)(t_2) + \left(b_0 + b_1 + b_2 + \acute{b}_0 + \acute{b}_1 + \acute{b}_2\right)(s_2)$$
$$= c_0 + c_1 + c_2 + \acute{c}_0 + \acute{c}_1 + \acute{c}_2 \quad (III)$$

$$(a_0 - \acute{a}_0)(\acute{t}_0) + \left(b_0 - \acute{b}_0\right)(\acute{s}_0) = c_0 - \acute{c}_0 \quad (IV)$$

$$(a_0 + a_1 - \acute{a}_0 - \acute{a}_1)(\acute{t}_1) + \left(b_0 + b_1 - \acute{b}_0 - \acute{b}_1\right)(\acute{s}_1) = c_0 + c_1 - \acute{c}_0 - \acute{c}_1 \quad (IIV)$$

$$(a_0 + a_1 + a_2 - \acute{a}_0 - \acute{a}_1 - \acute{a}_2)(\acute{t}_2) + \left(b_0 + b_1 + b_2 - \acute{b}_0 - \acute{b}_1 - \acute{b}_2\right)(\acute{s}_2)$$
$$= c_0 + c_1 + c_2 - \acute{c}_0 - \acute{c}_1 - \acute{c}_2 \quad (IIIV)$$

According to the previous argument, we can see that the symbolic 2-plithogenic split-complex Diophantine equation $AX + BY = C$ is solvable if and only if the equations $(I, II, III, IV, IIV, IIIV)$ are solvable, which is equivalent to:

$$\begin{cases} gcd(a_0 + \acute{a}_0, b_0 + \acute{b}_0) \setminus c_0 + \acute{c}_0 \\ gcd(a_0 - \acute{a}_0, b_0 - \acute{b}_0) \setminus c_0 + \acute{c}_0 \\ gcd(a_0 + a_1 + \acute{a}_0 + \acute{a}_1, b_0 + b_1 + \acute{b}_0 + \acute{b}_1) \setminus c_0 + c_1 + \acute{c}_0 + \acute{c}_1 \\ gcd(a_0 + a_1 - \acute{a}_0 - \acute{a}_1, b_0 + b_1 - \acute{b}_0 - \acute{b}_1) \setminus c_0 + c_1 - \acute{c}_0 - \acute{c}_1 \\ gcd(a_0 + a_1 + a_2 + \acute{a}_0 + \acute{a}_1 + \acute{a}_2, b_0 + b_1 + b_2 + \acute{b}_0 + \acute{b}_1 + \acute{b}_2) \setminus c_0 + c_1 + c_2 + \acute{c}_0 + \acute{c}_1 + \acute{c}_2 \\ gcd(a_0 + a_1 + a_2 - \acute{a}_0 - \acute{a}_1 - \acute{a}_2, b_0 + b_1 + b_2 - \acute{b}_0 - \acute{b}_1 - \acute{b}_2) \setminus c_0 + c_1 + c_2 - \acute{c}_0 - \acute{c}_1 - \acute{c}_2 \end{cases}$$

The algorithm for solution:

To solve $AX + BY = C$; A, X, B, Y, C are symbolic 2-plithogenic split-complex integers, we follow these steps:

Step (1).

We transform $AX + BY = C$ to the equivalent system of classical Diophantine equations $(I) \to (IIIV)$.

Step (2).

We check if equations $(I) \to (IIIV)$ are solvable in Z.

If there exists one equation which is not solvable, then $AX + BY = C$ I not solvable.

Step (3).

We solve the system $(I) \to (IIIV)$.

Step (4).

$$x_0 = \frac{1}{2}(t_0 + \acute{t}_0), y_0 = \frac{1}{2}(s_0 + \acute{s}_0), \acute{x}_0 = \frac{1}{2}(t_0 - \acute{t}_0), \acute{y}_0 = \frac{1}{2}(s_0 - \acute{s}_0)$$

$$x_1 = \frac{1}{2}(t_1 + \acute{t}_1) - \frac{1}{2}(t_0 + \acute{t}_0), y_1 = \frac{1}{2}(s_1 + \acute{s}_1) - \frac{1}{2}(s_0 + \acute{s}_0)$$

$$\acute{x}_1 = \frac{1}{2}(t_1 - \acute{t}_1) - \frac{1}{2}(t_0 - \acute{t}_0), \acute{y}_1 = \frac{1}{2}(s_1 - \acute{s}_1) - \frac{1}{2}(s_0 - \acute{s}_0)$$

$$x_2 = \frac{1}{2}(t_2 + \acute{t}_2) - \frac{1}{2}(t_1 + \acute{t}_1), y_2 = \frac{1}{2}(s_2 + \acute{s}_2) - \frac{1}{2}(s_1 + \acute{s}_1)$$

$$\acute{x}_2 = \frac{1}{2}(t_2 - \acute{t}_2) - \frac{1}{2}(t_1 - \acute{t}_1), \acute{y}_2 = \frac{1}{2}(s_2 - \acute{s}_2) - \frac{1}{2}(s_1 - \acute{s}_1)$$

Remark.

The available solutions are under the conditions

$t_0 + t'_0, t_0 - t'_0, t_1 + t'_1, t_1 - t'_1, t_2 + t'_2, t_2 - t'_2 \in 2Z$

$s_0 + s'_0, s_0 - s'_0, s_1 + s'_1, s_1 - s'_1, s_2 + s'_2, s_2 - s'_2 \in 2Z$

Example.

Take the symbolic 2-plithogenic split-complex Diophantine equation with two variable:

$$[(1 + P_1 - P_2) + J(2 + 2P_1 + 3P_2)]X + [(3 + P_1 + 2P_2) + J(1 - 3P_1 + P_2)]Y = (11 + 7P_1 + 8P_2) + J(6 + 6P_1 + 6P_2)$$

We have:

$$\begin{cases} a_0 = 1, a_1 = 1, a_2 = -1 \\ a'_0 = 2, a'_1 = 2, a'_2 = 3 \\ b_0 = 3, b_1 = 1, b_3 = 2 \\ b'_0 = 1, b'_1 = -3, b'_2 = 1 \\ c_0 = 11, c_1 = 7, c_3 = 8 \\ c'_0 = 6, c'_1 = 6, b'_2 = 6 \end{cases}$$

The equivalent system is:

$$\begin{cases} 3t_0 + 4s_0 = 17 \quad (I) \\ 6t_1 + 2s_1 = 30 \quad (II) \\ 8t_2 + 5s_2 = 44 \quad (III) \\ -t'_0 + 2s'_0 = 5 \quad (IV) \\ -2t'_1 + 6s'_1 = 6 \quad (IIV) \\ -6t'_2 + 7s'_2 = 8s \quad (IIIV) \end{cases}$$

All equation $(I) \rightarrow (IIIV)$ are solvable, that is because:

$gcd(3,4) = 1 \setminus 17, gcd(6,2) = 2 \setminus 30, gcd(8,5) = 1 \setminus 44, gcd(-1,2) = 1 \setminus 5, gcd(-2,6) = 2 \setminus 6, gcd(-6,7) = 1 \setminus 8$

We will take one solution for each equation:

$t_0 = 3, s_0 = 2$ is a solution of (I).

$t'_0 = -1, s'_0 = 2$ is a solution of (IV).

$t_1 = 5, s_1 = 0$ is a solution of (II).

$t'_1 = -3, s'_1 = 2$ is a solution of (IIV).

$t_2 = 3, s_2 = 4$ is a solution of (III).

$t'_2 = 1, s'_2 = 2$ is a solution of $(IIIV)$.

$$x_0 = \frac{1}{2}(t_0 + t'_0) = 1, y_0 = \frac{1}{2}(s_0 + s'_0) = 2, x'_0 = \frac{1}{2}(t_0 - t'_0) = 2, y'_0 = \frac{1}{2}(s_0 - s'_0) = 0$$

$x_1 = \frac{1}{2}(t_1 + t_1') - \frac{1}{2}(t_0 + t_0') = 0, y_1 = \frac{1}{2}(s_1 + s_1') - \frac{1}{2}(s_0 + s_0') = -2$

$x_1' = \frac{1}{2}(t_1 - t_1') - \frac{1}{2}(t_0 - t_0') = 2, y_1' = \frac{1}{2}(s_1 - s_1') - \frac{1}{2}(s_0 - s_0') = 0$

$x_2 = \frac{1}{2}(t_2 + t_2') - \frac{1}{2}(t_1 + t_1') = 1, y_2 = \frac{1}{2}(s_2 + s_2') - \frac{1}{2}(s_1 + s_1') = 3$

$x_2' = \frac{1}{2}(t_2 - t_2') - \frac{1}{2}(t_1 - t_1') = -3, y_2' = \frac{1}{2}(s_2 - s_2') - \frac{1}{2}(s_1 - s_1') = 1$

Thus $X = (1 + P_2) + J(2 + 2P_1 - 3P_2), Y = (2 - 2P_1 + 3P_2) + J(8P_2)$ is a solution of the original equation.

Conclusion

In this paper, we have presented an effective algorithm to solve a symbolic 2-plithogenic split-complex linear Diophantine equation with two variables. Also, we have illustrated a related example to clarify the strength of the presented algorithm.

In the future, we aim to study other Diophantine equations with symbolic 2-plithogenic and 3-plithogenic split-complex linear and non-linear Diophantine equations.

References

[1]. Abobala, M., " A Short Contribution to Split-Complex Linear Diophantine Equations in Two Variables", Galoitica Journal of Mathematical Structures and Applications, Vol.6, 2023.

[2]. Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.

[3]. Sankari, H., and Abobala, M., "Neutrosophic Linear Diophantine Equations With two Variables", Neutrosophic Sets and Systems, Vol. 38, pp. 22-30, 2020.

[4]. Ibrahim, M., and Abobala, M., "An Introduction To Refined Neutrosophic Number Theory", Neutrosophic Sets and Systems, Vol. 45, 2021.

[5]. Ali, R., "A Short Note On The Solution of n-Refined Neutrosophic Linear Diophantine Equations", International Journal Of Neutrosophic Science, Vol. 15, 2021

[6]. Sarkis, M., " On The Solutions Of Fermat's Diophantine Equation In 3-refined Neutrosophic Ring of Integers", Neoma Journal of Mathematics and Computer Science, 2023.

[7]. Merkepci, M., and Abobala, M., " On Some Novel Results About Split-Complex Numbers, The Diagonalization Problem And Applications To Public Key Asymmetric Cryptography", Journal of Mathematics, Hindawi, 2023.

[8]. Khaldi, A., " A Study On Split-Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.

[9]. Ahmad, K., " On Some Split-Complex Diophantine Equations", Neoma Journal Of Mathematics and Computer Science, 2023.

[10]. Ali, R., " On The Weak Fuzzy Complex Inner Products On Weak Fuzzy Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.

[11]. Nabeeh, N., Alshaimaa, A., and Tantawy, A., "A Neutrosophic Proposed Model For Evaluation Blockchain Technology in Secure Enterprise Distributed Applications", Journal of Cypersecurity and Information Management, 2023.

[12]. Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.

[13]. Ali, R., and Hasan, Z., " An Introduction To The Symbolic 3-Plithogenic Modules ", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.

[14]. Ali, R., and Hasan, Z., "An Introduction To The Symbolic 3-Plithogenic Vector Spaces", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.

[15]. Merkepci, M., Abobala, M., and Allouf, A., " The Applications of Fusion Neutrosophic Number Theory in Public Key Cryptography and the Improvement of RSA Algorithm ", Fusion: Practice and Applications, 2023.

[16]. M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, "Introduction to Symbolic 2-Plithogenic Probability Theory," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.

[17]. Merkepci, M., and Abobala, M., " Security Model for Encrypting Uncertain Rational Data Units Based on Refined Neutrosophic Integers Fusion and El Gamal Algorithm ", Fusion: Practice and Applications, 2023.

[18]. Alhasan, Y., Alfahal, A., Abdulfatah, R., Ali, R., and Aljibawi, M., " On A Novel Security Algorithm For The Encryption Of 3×3 Fuzzy Matrices With Rational Entries Based On The Symbolic 2-Plithogenic Integers And El-Gamal Algorithm", International Journal of Neutrosophic Science, 2023.

[19]. Abobala, M., and Allouf, A., " On A Novel Security Scheme for The Encryption and Decryption Of 2×2 Fuzzy Matrices with Rational Entries Based on The Algebra of Neutrosophic Integers and El-Gamal Crypto-System", Neutrosophic Sets and Systems, vol.54, 2023.

[20]. Abualkishik, A., Almajed, R., Thompson, W., "Improving The Performance of Fog-assisted Internet of Things Networks Using Bipolar Trapezoidal Neutrosophic Sets", Journal of Wireless and Ad Hoc Communication, 2023.

[21] Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Concept Of Symbolic 7-Plithogenic Real Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

[22] Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Symbolic 8-Plithogenic Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

Received 17/5/2023, Accepted 3/10/2023

Neutrosophic Sets and Systems, Vol. 59, 2023
University of New Mexico

On The Symbolic 2-Plithogenic Split-Complex Real Square Matrices and Their Algebraic Properties

[1.]Oliver Von Shtawzen, [2.]Ammar Rawashdeh, [3.]Karla Zayood

[1.] University Of Nizwa, Sultanate Of Oman

[2.]Mutah University, Faculty of Science, Jordan

[3.] Online Islamic University, Department Of Science and Information Technology, Doha, Qatar

Co-, Vonshtawzen1970abc@gmail.com, ammarrawashde8932@gmail.com

Abstract:

The objective of this paper is to study for the first time the concept of square real matrices with symbolic 2-plithogenic split-complex entries. Many of their algebraic properties will be discussed and handled, where we find the formula of computing inverses, exponents, and powers of these matrices by building a ring isomorphism between the ring of split-complex symbolic 2-plithogenic matrices and the direct product of the symbolic 2-plithogenic matrices with itself. Also, we give the interested reader many related examples to clarify the validity of our work.

Keywords: split-complex number, symbolic 2-plithogenic number, split-complex 2-plithogenic matrix.

Introduction

The concept of symbolic 2-plithogenic rings is considered as a generalization of algebraic rings [1], and these rings were used by many authors to generalize classical algebraic structures such as vector space, modules, and functions into novel symbolic 2-plithogenic and 3-plithogenic versions, see [2-9].

zenodo.10031206/10.5281

Symbolic 2-plithogenic matrices and other types have been studied in [10], where many results were obtained such as diagonalizations, and eigenvalues.

Symbolic 2-plithogenic split-complex numbers were defined as a combination between split-complex numbers, and symbolic 2-plithogenic numbers, for more details about split-complex structures and their applications, see [11-13]. For similar results about neutrosophic matrices and n-plithogenic matrices check [14-19].

In this paper, we define symbolic 2-plithogenic split-complex matrices, and we present many elementary properties of these matrices, especially those are related to classical matrix theory and applications.

Main Discussion

Definition.

Let $A = \left(a_{ij}\right)$ be a matrix, it is called symbolic 2-plithogenic split-complex if and only if:

$a_{ij} = \left({a_{ij}}^{(0)} + {a_{ij}}^{(1)}P_1 + {a_{ij}}^{(2)}P_2\right) + \left({b_{ij}}^{(0)} + {b_{ij}}^{(1)}P_1 + {b_{ij}}^{(2)}P_2\right)J$, where ${a_{ij}}^{(k)}, {b_{ij}}^{(k)} \in R, J^2 = 1,\ P_i \times P_j = P_{max(i,j)}, {P_i}^2 = P_i$.

Example.

The matrix $A = \begin{pmatrix} 1 + 2P_1 + P_2 + J(1 - P_2) & (P_1 + P_2)J \\ (2 - P_2) + J(3 + P_1 + P_2) & 5 - (P_1 + 4P_2)J \end{pmatrix}$ is a 2×2 symbolic 2-plithogenic split-complex matrix.

$$A = \begin{pmatrix} 1 + 2P_1 + P_2 & 0 \\ 2 - P_2 & 5 \end{pmatrix} + J\begin{pmatrix} 1 - P_2 & P_1 + P_2 \\ 3 + P_1 + P_2 & -(P_1 + 4P_2) \end{pmatrix}$$

Remark.

Any symbolic 2-plithogenic split-complex matrix can be written as follows $A = T + KJ$; T, K are two symbolic 2-plithogenic real matrices.

Also, it can be written as follows:

$A = (A_0 + A_1P_1 + A_2P_2) + J(B_0 + B_1P_1 + B_2P_2);\ A_i, B_i$ are classical square real matrices.

The matrix presented in the previous example can be written as:

$$\begin{pmatrix} 1 & 0 \\ 2 & 5 \end{pmatrix} + P_1\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + P_2\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + J\left[\begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} + P_1\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} + P_2\begin{pmatrix} -1 & 1 \\ 1 & -4 \end{pmatrix}\right]$$

Remark.

We denote the ring of all symbolic 2-plithogenic split-complex matrices by $2-SP_{MS}$.

$(2-SP_{MS},+,.)$ is a ring.

Definition.

Let $X=M_0+N_0J, Y=M_1+N_1J;\ M_0,N_0,M_1,N_1 \in 2-P_M$, then:

$X+Y=(M_0+M_1)+(N_0+N_1)J.$

$X.Y=(M_0M_1+N_0N_1)+(M_0N_1+N_0M_1)J.$

Example.

Take:

$$X=\begin{pmatrix}1+P_1 & P_2\\ P_1 & -P_1+P_2\end{pmatrix}+J\begin{pmatrix}2+P_1-P_2 & 3P_1\\ P_1-P_2 & 2P_2\end{pmatrix}=M_0+N_0J$$

$$Y=\begin{pmatrix}1+P_1-P_2 & 2-P_2\\ 1+P_2 & 1+P_1\end{pmatrix}+J\begin{pmatrix}2-P_2 & P_1-P_2\\ 1+P_1 & P_2\end{pmatrix}=M_1+N_1J$$

$$X+Y=\begin{pmatrix}2+2P_1-P_2 & 2-P_1+P_2\\ 1+P_1+P_2 & 1+P_2\end{pmatrix}+J\begin{pmatrix}4+P_1-2P_2 & 4P_1-P_2\\ 1+2P_1-P_2 & 3P_2\end{pmatrix}$$

$$M_0M_1=\begin{pmatrix}1+P_1 & P_2\\ P_1 & -P_1+P_2\end{pmatrix}\begin{pmatrix}1+P_1-P_2 & 2-P_2\\ 1+P_2 & 1+P_1\end{pmatrix}=\begin{pmatrix}1+3P_1 & 2+2P_2\\ P_1 & -P_1+2P_2\end{pmatrix}$$

$$N_0N_1=\begin{pmatrix}2+P_1-P_2 & 3P_1\\ P_1-P_2 & 2P_2\end{pmatrix}\begin{pmatrix}2-P_2 & P_1-P_2\\ 1+P_1 & P_2\end{pmatrix}=\begin{pmatrix}4+8P_1-4P_2 & 3P_1\\ 2P_1+2P_2 & P_1+P_2\end{pmatrix}$$

$$M_0N_1=\begin{pmatrix}1+P_1 & P_2\\ P_1 & -P_1+P_2\end{pmatrix}\begin{pmatrix}2-P_2 & P_1-P_2\\ 1+P_1 & P_2\end{pmatrix}=\begin{pmatrix}2+2P_1 & 2P_1-P_2\\ P_2 & P_1-P_2\end{pmatrix}$$

$$N_0M_1=\begin{pmatrix}2+P_1-P_2 & 3P_1\\ P_1-P_2 & 2P_2\end{pmatrix}\begin{pmatrix}1+P_1-P_2 & 2-P_2\\ 1+P_2 & 1+P_1\end{pmatrix}=\begin{pmatrix}1+7P_1-P_2 & 4+5P_1-P_2\\ 2P_1+2P_2 & P_1+2P_2\end{pmatrix}$$

So that:

$$X.Y=\begin{pmatrix}5+11P_1-4P_2 & 2+3P_1+2P_2\\ 3P_1+2P_2 & 3P_2\end{pmatrix}+J\begin{pmatrix}4+9P_1-P_2 & 2+3P_1+2P_2\\ 2P_1+3P_2 & 2P_1+P_2\end{pmatrix}$$

Theorem.

Let $2-SP_{MS}$ be the ring of all $n\times n$ symbolic 2-plithogenic split-complex matrices, let $2-P_M$ be the ring of all $n\times n$ square symbolic 2-plithogenic real matrices, then:

$$f: 2-SP_{MS} \to 2-P_M \times 2-P_M$$

Such that:

$f(M+NJ)=(M+N,M-N); M,N \in 2-P_M$ is a ring isomorphism.

Proof.

For $M_0+N_0J=M_1+N_1J$, we have:

$M_0=M_1, N_0=N_1$, thus $(M_0+N_0, M_0-N_0)=(M_1+N_1, M_1-N_1)$,

hence $f(M_0+N_0J)=f(M_1+N_1J)$.

For $X=M_0+N_0J, Y=M_1+N_1J \in 2-SP_{MS}$,

we have:

$f(X+Y)=(M_0+N_0+M_1+N_1, M_0+M_1-N_0-N_1)=f(X)+f(Y)$.

$f(X.Y)=f[(M_0M_1+N_0N_1)+(M_0N_1+N_0M_1)J]=[(M_0+N_0)(M_1+N_1),(M_0-N_0)(M_1-N_1)]=f(X).f(Y)$.

$$f(X)=0 \Leftrightarrow \begin{cases} M_0+N_0=0 \\ M_0-N_0=0 \end{cases} \Leftrightarrow M_0=N_0=0$$

Thus, $ker(f)=\{0\}$.

For any arbitrary element $(M,N) \in 2-P_M \times 2-P_M$, there exists $X=\frac{1}{2}(M+N)+\frac{1}{2}(M-N)J \in 2-P_M$, such that $f(X)=(M,N)$, so that f is a ring isomorphism.

Remark.

The inverse isomorphism is:

$f^{-1}: 2-P_M \times 2-P_M \to 2-SP_{MS}$; $f^{-1}(M,N)=\frac{1}{2}(M+N)+\frac{1}{2}(M-N)J$.

Example.

Consider:

$$X=\begin{pmatrix} 1+P_1+P_2 & 3-P_2 & 1+P_1 \\ P_1-P_2 & 5+P_1 & P_2 \\ P_1+P_2 & 2P_1 & -P_2 \end{pmatrix}+J\begin{pmatrix} 1+P_1-P_2 & 1 & 1 \\ 4+2P_1-P_2 & P_1 & 2P_2 \\ 2P_1-P_2 & P_1 & 5P_1 \end{pmatrix}=M+NJ,$$

then:

$$f(X)=\left(\begin{pmatrix} 2+2P_1 & 4-P_2 & 1+P_1 \\ 4+3P_1-2P_2 & 5+2P_1 & 3P_2 \\ 2P_1 & 3P_1 & 5P_1-P_2 \end{pmatrix},\begin{pmatrix} 2P_2 & 2-P_2 & -1+P_1 \\ -4-P_1 & 5 & -P_2 \\ -P_1+2P_2 & P_1 & -5P_1-P_2 \end{pmatrix}\right)$$

Results from the isomorphism.

Let $X=M+NJ \in 2-SP_M; M=M_0+M_1P_1+M_2P_2, N=N_0+N_1P_1+N_2P_2 \in 2-P_M$,

then:

1). X is invertible if and only if $M+N, M-N$ are invertible, which is equivalent to:

$M_0+N_0, M_0-N_0, (M_0+M_1)+(N_0+N_1), (M_0+M_1)-(N_0+N_1),$

$(M_0+M_1+M_2)+(N_0+N_1+N_2), (M_0+M_1+M_2)+(N_0+N_1-N_2),$

Are invertible matrices.

2). If X is inverible, then:

$$X^{-1}=\frac{1}{2}[(M+N)^{-1}+(M-N)^{-1}]+\frac{1}{2}[(M+N)^{-1}-(M-N)^{-1}]J$$

Where:

$$(M+N)^{-1}=(M_0+N_0)^{-1}+[(M_0+M_1+N_0+N_1)^{-1}-(M_0+N_0)^{-1}]P_1 + [(M_0+M_1+M_2+N_0+N_1+N_2)^{-1}-(M_0+M_1+N_0+N_1)^{-1}]P_2$$

$$(M-N)^{-1}=(M_0-N_0)^{-1}+[(M_0+M_1-N_0-N_1)^{-1}-(M_0-N_0)^{-1}]P_1+ [(M_0+M_1+M_2-N_0-N_1-N_2)^{-1}-(M_0+M_1-N_0-N_1)^{-1}]P_2.$$

3). $detX=\frac{1}{2}[det(M+N)+det(M-N)]+\frac{1}{2}[det(M+N)-det(M-N)]J$

$$det(M+N)=det(M_0+N_0)+[det(M_0+M_1+N_0+N_1)-det(M_0+N_0)]P_1+ [det(M_0+M_1+M_2+N_0+N_1+N_2)-det(M_0+M_1+N_0+N_1)]P_2.$$

$$det(M-N)=det(M_0-N_0)+[det(M_0+M_1-N_0-N_1)-det(M_0-N_0)]P_1+ [det(M_0+M_1+M_2-N_0-N_1-N_2)-det(M_0+M_1-N_0-N_1)]P_2.$$

4). $X^n=\frac{1}{2}[(M+N)^n+(M-N)^n]+\frac{1}{2}[(M+N)^n-(M-N)^n]J$

$$(M+N)^n=(M_0+N_0)^n+[(M_0+M_1+N_0+N_1)^n-(M_0+N_0)^n]P_1+[(M_0+M_1+ M_2+N_0+N_1+N_2)^n-(M_0+M_1+N_0+N_1)^n]P_2.$$

$$(M-N)^n=(M_0-N_0)^n+[(M_0+M_1-N_0-N_1)^n-(M_0-N_0)^n]P_1+[(M_0+M_1+ M_2-N_0-N_1-N_2)^n-(M_0+M_1-N_0-N_1)^n]P_2.$$

Example.

Consider the following 2×2 symbolic 2-plithogenic split-complex matrix:

$$X=\begin{pmatrix} 2+\frac{3}{2}P_1+4P_2 & \frac{1}{2} \\ \frac{3}{2}P_1 & \frac{3}{2}+\frac{1}{2}P_1+\frac{1}{2}P_2 \end{pmatrix}+J\begin{pmatrix} 1-\frac{7}{2}P_1+2P_2 & \frac{1}{2}-P_1+P_2 \\ \frac{3}{2}P_1-3P_2 & \frac{1}{2}-\frac{3}{2}P_1-\frac{1}{2}P_2 \end{pmatrix}=M+$$

NJ, where:

$$M=\begin{pmatrix} 2+\frac{3}{2}P_1+4P_2 & \frac{1}{2} \\ \frac{3}{2}P_1 & \frac{3}{2}+\frac{1}{2}P_1+\frac{1}{2}P_2 \end{pmatrix}, N=\begin{pmatrix} 1-\frac{7}{2}P_1+2P_2 & \frac{1}{2}-P_1+P_2 \\ \frac{3}{2}P_1-3P_2 & \frac{1}{2}-\frac{3}{2}P_1-\frac{1}{2}P_2 \end{pmatrix}$$

$$M+N=\begin{pmatrix}3-P_1+6P_2 & 1-P_1+P_2\\ 3P_1-3P_2 & 2-P_1\end{pmatrix}$$

$$M-N=\begin{pmatrix}1+2P_1+2P_2 & P_1-P_2\\ 3P_2 & 1+2P_1+P_2\end{pmatrix}$$

We write:

$M+N=K_0+K_1P_1+K_2P_2, M-N=L_0+L_1P_1+L_2P_2$, where:

$$K_0=\begin{pmatrix}3 & 1\\ 0 & 2\end{pmatrix}, K_1=\begin{pmatrix}-5 & -1\\ 3 & -1\end{pmatrix}, K_2=\begin{pmatrix}6 & 1\\ -3 & 0\end{pmatrix}, K_0+K_1=\begin{pmatrix}-2 & 0\\ 3 & 1\end{pmatrix}, K_0+K_1+K_2 =\begin{pmatrix}4 & 1\\ 0 & 1\end{pmatrix}$$

$$L_0=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, L_1=\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}, L_2=\begin{pmatrix}2 & -1\\ 3 & 1\end{pmatrix}, L_0+L_1=\begin{pmatrix}3 & 1\\ 0 & 3\end{pmatrix}, L_0+L_1+L_2=\begin{pmatrix}5 & 0\\ 3 & 4\end{pmatrix}$$

$det(M+N)=det(K_0)+[det(K_0+K_1)-det(K_0)]P_1+[det(K_0+K_1+K_2)-det(K_0+K_1)]P_2=6+(-2-6)P_1+(4+2)P_2=6-8P_1+6P_2.$

$det(M-N)=det(L_0)+[det(L_0+L_1)-det(L_0)]P_1+[det(L_0+L_1+L_2)-det(L_0+L_1)]P_2=1+(9-1)P_1+(20-9)P_2=1+8P_1+11P_2.$

$$detX=\frac{1}{2}[det(M+N)+det(M-N)]+\frac{1}{2}[det(M+N)-det(M-N)]J$$
$$=\frac{1}{2}(7+17P_2)+\frac{1}{2}J(5-16P_1-5P_2)$$
$$=\left(\frac{7}{2}+\frac{17}{2}P_2\right)+J\left(\frac{5}{2}-8P_1-\frac{5}{2}P_2\right)$$

now, let's find the inverse of X:

$$K_0^{-1}=-\frac{1}{6}\begin{pmatrix}2 & -1\\ 0 & 3\end{pmatrix}=\begin{pmatrix}\frac{1}{3} & \frac{1}{6}\\ 0 & \frac{1}{2}\end{pmatrix}, (K_0+K_1)^{-1}=-\frac{1}{2}\begin{pmatrix}1 & 0\\ -3 & -2\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & 0\\ \frac{3}{2} & 1\end{pmatrix}$$

$$(K_0+K_1+K_2)^{-1}=\frac{1}{4}\begin{pmatrix}1 & -1\\ 0 & 4\end{pmatrix}=\begin{pmatrix}\frac{1}{4} & -\frac{1}{4}\\ 0 & 1\end{pmatrix}$$

$$L_0^{-1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, (L_0+L_1)^{-1}=\frac{1}{9}\begin{pmatrix}3 & -1\\ 0 & 3\end{pmatrix}=\begin{pmatrix}\frac{1}{3} & -\frac{1}{9}\\ 0 & \frac{1}{3}\end{pmatrix}$$

$$(L_0+L_1+L_2)^{-1}=\frac{1}{20}\begin{pmatrix}4 & 0\\ -3 & 5\end{pmatrix}=\begin{pmatrix}\frac{1}{5} & 0\\ -\frac{3}{20} & \frac{1}{4}\end{pmatrix}$$

$$(M+N)^{-1}=(K_0)^{-1}+[(K_0+K_1)^{-1}-(K_0)^{-1}]P_1+[(K_0+K_1+K_2)^{-1}-(K_0+K_1)^{-1}]P_2$$

$$=\begin{pmatrix}\frac{1}{3} & \frac{1}{6}\\ 0 & \frac{1}{2}\end{pmatrix}+\begin{pmatrix}\frac{-5}{6} & \frac{1}{6}\\ \frac{3}{2} & \frac{1}{2}\end{pmatrix}P_1+\begin{pmatrix}\frac{3}{4} & -\frac{1}{4}\\ -\frac{3}{2} & 0\end{pmatrix}P_2$$

$$=\begin{pmatrix}\frac{1}{3}-\frac{5}{6}P_1+\frac{3}{4}P_2 & \frac{1}{6}+\frac{1}{6}P_1-\frac{1}{4}P_2\\ \frac{3}{2}P_1-\frac{3}{2}P_2 & \frac{1}{2}+\frac{1}{2}P_1\end{pmatrix}$$

$$(M-N)^{-1}=(L_0)^{-1}+[(L_0+L_1)^{-1}-(L_0)^{-1}]P_1+[(L_0+L_1+L_2)^{-1}-(L_0+L_1)^{-1}]P_2$$

$$=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}\frac{-2}{3} & \frac{-1}{9}\\ 0 & \frac{-2}{3}\end{pmatrix}P_1+\begin{pmatrix}-\frac{2}{15} & \frac{1}{9}\\ -\frac{3}{20} & -\frac{1}{12}\end{pmatrix}P_2$$

$$=\begin{pmatrix}1-\frac{2}{3}P_1-\frac{2}{15}P_2 & -\frac{1}{9}P_1+\frac{1}{9}P_2\\ -\frac{3}{20}P_2 & 1-\frac{2}{3}P_1-\frac{1}{12}P_2\end{pmatrix}$$

$$X^{-1}=\frac{1}{2}[(M+N)^{-1}+(M-N)^{-1}]+\frac{1}{2}[(M+N)^{-1}-(M-N)^{-1}]J$$

$$=\frac{1}{2}\begin{pmatrix}\frac{4}{3}-\frac{3}{2}P_1+\frac{37}{60}P_2 & -\frac{1}{6}+\frac{1}{18}P_1-\frac{5}{36}P_2\\ \frac{3}{2}P_1-\frac{33}{20}P_2 & \frac{3}{2}-\frac{1}{6}P_1-\frac{1}{12}P_2\end{pmatrix}$$

$$+\frac{1}{2}J\begin{pmatrix}\frac{-3}{2}-\frac{1}{6}P_1+\frac{53}{60}P_2 & -\frac{1}{6}+\frac{5}{18}P_1-\frac{13}{36}P_2\\ \frac{3}{2}P_1-\frac{27}{20}P_2 & -\frac{1}{2}+\frac{7}{6}P_1+\frac{1}{12}P_2\end{pmatrix}$$

$$=\begin{pmatrix}\frac{2}{3}-\frac{3}{4}P_1+\frac{37}{120}P_2 & -\frac{1}{12}+\frac{1}{36}P_1-\frac{5}{72}P_2\\ \frac{3}{4}P_1-\frac{33}{40}P_2 & \frac{3}{4}-\frac{1}{12}P_1-\frac{1}{24}P_2\end{pmatrix}$$

$$+J\begin{pmatrix}\frac{-1}{3}-\frac{1}{12}P_1+\frac{53}{120}P_2 & -\frac{1}{12}+\frac{5}{36}P_1-\frac{13}{72}P_2\\ \frac{3}{4}P_1-\frac{27}{40}P_2 & -\frac{1}{4}+\frac{7}{12}P_1+\frac{1}{24}P_2\end{pmatrix}$$

As an additional application of the matrix ring isomorphism between $2-SP_M$ and $2-P_M\times 2-P_M$ is to find all possible eigenvalues/vectors for the symbolic 2-plithogenic split-complex matrix.

To find the eigenvalue for a symbolic 2-plithogenic split-complex matrix, we must find all symbolic 2-plithogenic eigen values of $M+N, M-N$.

For the matrix defined in the previous example, we can see:

The eigenvalues of K_0 are $\{3,2\}=Q_0$.

The eigenvalues of K_0+K_1 are $\{-2,1\}=Q_1$.

The eigenvalues of $K_0+K_1+K_2$ are $\{4,1\}=Q_2$.

The eigenvalues of $M+N$ are:

$Q=\{a_0+(a_1-a_0)P_1+(a_2-a_1)P_2; a_i \in Q_i\}=\{3-5P_1+6P_2, 3-5P_1+3P_2, 3-2P_1+3P_2, 3-2P_1, 2-4P_1+6P_2, 2-4P_1+3P_2, 2-P_1+3P_2, 2-P_1\}$.

The eigenvalue of L_0 is $\{1\}=s_0$.

The eigenvalue of L_0+L_1 is $\{3\}=S_1$.

The eigenvalues of $L_0+L_1+L_2$ are $\{5,4\}=S_2$.

The eigenvalues of $M-N$ are:

$S=\{b_0+(b_1-b_0)P_1+(b_2-b_1)P_2; b_i \in S_i\}=\{1+2P_1+2P_2, 1+2P_1+P_2\}$

The eigen values of the duplet $(M+N, M-N)$ are:

$\{(3-5P_1+6P_2, 1+2P_1+2P_2), (3-5P_1+6P_2, 1+2P_1+P_2), (3-5P_1+3P_2, 1+2P_1+P_2), (3-5P_1+3P_2, 1+2P_1+2P_2), (3-2P_1+3P_2, 1+2P_1+2P_2), (3-2P_1+3P_2, 1+2P_1+P_2), (2-P_1, 1+2P_1+2P_2), (2-P_1, 1+2P_1+P_2), (2-4P_1+6P_2, 1+2P_1+2P_2), (2-4P_1+6P_2, 1+2P_1+P_2), (2-4P_1+3P_2, 1+2P_1+2P_2), (2-P_1+3P_2, 1+2P_1+2P_2), (2-P_1+3P_2, 1+2P_1+P_2), (3-2P_1, 1+2P_1+2P_2), (3-2P_1, 1+2P_1+P_2)\}$.

We put:

$T_1=(3-5P_1+6P_2, 1+2P_1+2P_2)$

$f^{-1}(T_1)=\frac{1}{2}(4-3P_1+8P_2)+\frac{1}{2}(2-7P_1+4P_2)J=\left(2-\frac{3}{2}P_1+4P_2\right)+\left(1-\frac{7}{2}P_1+2P_2\right)J$.

$T_2=(3-5P_1+6P_2, 1+2P_1+P_2)$

$f^{-1}(T_2)=\frac{1}{2}(4-3P_1+7P_2)+\frac{1}{2}(2-7P_1+5P_2)J=\left(2-\frac{3}{2}P_1+\frac{7}{2}P_2\right)+\left(1-\frac{7}{2}P_1+\frac{5}{2}P_2\right)J$.

$T_3=(3-2P_1+3P_2, 1+2P_1+2P_2)$

$f^{-1}(T_3)=\frac{1}{2}(4+5P_2)+\frac{1}{2}(2-4P_1+P_2)J=\left(2+\frac{5}{2}P_2\right)+\left(1-2P_1+\frac{1}{2}P_2\right)J.$

$T_4=(3-2P_1+3P_2,1+2P_1+P_2)$

$f^{-1}(T_4)=\frac{1}{2}(4+4P_2)+\frac{1}{2}(2-4P_1+2P_2)J=(2+2P_2)+(1-2P_1+P_2)J.$

$T_5=(3-5P_1+3P_2,1+2P_1+2P_2)$

$f^{-1}(T_5)=\frac{1}{2}(4-3P_1+5P_2)+\frac{1}{2}(2-7P_1+P_2)J=\left(2-\frac{3}{2}P_1+\frac{5}{2}P_2\right)+\left(1-\frac{7}{2}P_1+\frac{1}{2}P_2\right)J.$

$T_6=(3-5P_1+3P_2,1+2P_1+P_2)$

$f^{-1}(T_6)=\frac{1}{2}(4-3P_1+4P_2)+\frac{1}{2}(2-7P_1+2P_2)J=\left(2-\frac{3}{2}P_1+2P_2\right)+\left(1-\frac{7}{2}P_1+P_2\right)J.$

$T_7=(2-P_1,1+2P_1+2P_2)$

$f^{-1}(T_7)=\frac{1}{2}(3+P_1+2P_2)+\frac{1}{2}(1-3P_1-2P_2)J=\left(\frac{3}{2}+\frac{1}{2}P_1+P_2\right)+\left(\frac{1}{2}-\frac{3}{2}P_1-P_2\right)J.$

$T_8=(2-P_1,1+2P_1+P_2)$

$f^{-1}(T_8)=\frac{1}{2}(3+P_1+P_2)+\frac{1}{2}(1-3P_1-P_2)J=\left(\frac{3}{2}+\frac{1}{2}P_1+\frac{1}{2}P_2\right)+\left(\frac{1}{2}-\frac{3}{2}P_1-\frac{1}{2}P_2\right)J.$

$T_9=(2-4P_1+6P_2,1+2P_1+2P_2)$

$f^{-1}(T_9)=\frac{1}{2}(3-2P_1+8P_2)+\frac{1}{2}(1-6P_1+4P_2)J=\left(\frac{3}{2}-P_1+4P_2\right)+\left(\frac{1}{2}-3P_1+2P_2\right)J.$

$T_{10}=(2-4P_1+6P_2,1+2P_1+P_2)$

$f^{-1}(T_{10})=\frac{1}{2}(3-2P_1+7P_2)+\frac{1}{2}(1-6P_1+5P_2)J=\left(\frac{3}{2}-P_1+\frac{7}{2}P_2\right)+\left(\frac{1}{2}-3P_1+\frac{5}{2}P_2\right)J.$

$T_{11}=(2-4P_1+6P_2,1+2P_1+2P_2)$

$f^{-1}(T_{11})=\frac{1}{2}(3-2P_1+5P_2)+\frac{1}{2}(1-6P_1+P_2)J=\left(\frac{3}{2}-P_1+\frac{5}{2}P_2\right)+\left(\frac{1}{2}-3P_1+\frac{1}{2}P_2\right)J.$

$T_{12}=(2-4P_1+3P_2,1+2P_1+P_2)$

$f^{-1}(T_{12})=\frac{1}{2}(3-2P_1+4P_2)+\frac{1}{2}(1-6P_1+2P_2)J=\left(\frac{3}{2}-P_1+2P_2\right)+\left(\frac{1}{2}-3P_1+P_2\right)J.$

$T_{13}=(2-P_1+3P_2,1+2P_1+2P_2)$

$f^{-1}(T_{13}) = \frac{1}{2}(3 + P_1 + 5P_2) + \frac{1}{2}(1 - 3P_1 + P_2)J = \left(\frac{3}{2} + \frac{1}{2}P_1 + \frac{5}{2}P_2\right) + \left(\frac{1}{2} - \frac{3}{2}P_1 + \frac{1}{2}P_2\right)J.$

$T_{14} = (2 - P_1 + 3P_2, 1 + 2P_1 + P_2)$

$f^{-1}(T_{14}) = \frac{1}{2}(3 + P_1 + 4P_2) + \frac{1}{2}(1 - 3P_1 + 2P_2)J = \left(\frac{3}{2} + \frac{1}{2}P_1 + 2P_2\right) + \left(\frac{1}{2} - \frac{3}{2}P_1 + P_2\right)J.$

$T_{15} = (3 - 2P_1, 1 + 2P_1 + 2P_2)$

$f^{-1}(T_{15}) = \frac{1}{2}(4 + 2P_2) + \frac{1}{2}(2 - 4P_1 - 2P_2)J = (2 + P_2) + (1 - 2P_1 - P_2)J.$

$T_{16} = (3 - 2P_1, 1 + 2P_1 + P_2)$

$f^{-1}(T_{16}) = \frac{1}{2}(4 + P_2) + \frac{1}{2}(2 - 4P_1 - P_2)J = \left(2 + \frac{1}{2}P_2\right) + \left(1 - 2P_1 - \frac{1}{2}P_2\right)J.$

Conclusion

In this paper, we studied for the first time the concept of square real matrices with symbolic 2-plithogenic split-complex entries, where we find the formula of computing inverses, exponents, and powers of these matrices by building a ring isomorphism between the ring of split-complex symbolic 2-plithogenic matrices and the direct product of the symbolic 2-plithogenic matrices with itself. Also, we give the interested reader many related examples to clarify the validity of our work.

References

1. Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.

2. M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, “Introduction to Symbolic 2-Plithogenic Probability Theory,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.

3. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Concept Of Symbolic 7-Plithogenic Real Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

4. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Symbolic 8-Plithogenic Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

5. Nader Mahmoud Taffach , Ahmed Hatip., "A Review on Symbolic 2-Plithogenic Algebraic Structures " Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.

6. Nader Mahmoud Taffach , Ahmed Hatip.," A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations ", Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.

7. Ali, R., and Hasan, Z., "An Introduction To The Symbolic 3-Plithogenic Vector Spaces", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.

8. Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.

9. Ben Othman, K., "On Some Algorithms For Solving Symbolic 3-Plithogenic Equations", Neoma Journal Of Mathematics and Computer Science, 2023.

10. Alfahal, A.; Alhasan, Y.; Abdulfatah, R.; Mehmood, A.; Kadhim, M. On Symbolic 2-Plithogenic Real Matrices and Their Algebraic Properties. *Int. J. Neutrosophic Sci.* **2023**, *21*.

11. Khaldi, A., " A Study On Split-Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.

12. Ahmad, K., " On Some Split-Complex Diophantine Equations", Neoma Journal Of Mathematics and Computer Science, 2023.

13. Merkepci, M., and Abobala, M., " On Some Novel Results About Split-Complex Numbers, The Diagonalization Problem And Applications To

Public Key Asymmetric Cryptography", Journal of Mathematics, Hindawi, 2023.

14. Abobala, M., Hatip, A., and Bal, M., " A Review On Recent Advantages In Algebraic Theory Of Neutrosophic Matrices", International Journal of Neutrosophic Science, Vol.17, 2021.
15. Abobala, M. On Refined Neutrosophic Matrices and Their Applications in Refined Neutrosophic Algebraic Equations. *J. Math.* **2021**, *2021*, 5531093.
16. Abobala, M., On Refined Neutrosophic Matrices and Their Applications In Refined Neutrosophic Algebraic Equations, Journal Of Mathematics, Hindawi, 2021
17. Olgun, N., Hatip, A., Bal, M., and Abobala, M., " A Novel Approach To Necessary and Sufficient Conditions For The Diagonalization of Refined Neutrosophic Matrices", International Journal of neutrosophic Science, Vol. 16, pp. 72-79, 2021.
18. Hatip, A., " On The Algebraic Properties of Symbolic n-Plithogenic Matrices For n=5, n=6", Galoitica Journal of Mathematical Structures and Applications, 2023.
19. Merkepci, H., "On Novel Results about the Algebraic Properties of Symbolic 3-Plithogenic and 4-Plithogenic Real Square Matrices", Symmetry, MDPI, 2023.

Received 29/5/2023, Accepted 22/9/2023

Neutrosophic Sets and Systems, Vol. 59, 2023
University of New Mexico

The Computing of Pythagoras Triples in Symbolic 2-Plithogenic Rings

Abuobida Mohammed A. Alfahal [1,*], Yaser Ahmad Alhasan [2], Raja Abdullah Abdulfatah[3], Ahmad Abd Al-Aziz[4]

[1]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia
[2]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia
[3]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia
[4]Cairo University, Department of Mathematics, Cairo, Egypt

Emails:[1.] a.alfahal@psau.edu.sa; [2.] y.alhasan@psau.edu.sa ;[3.] r.abdulfatah@psau.edu.sa ;
Corresponding author: a.alfahal@psau.edu.sa

Abstract:

This paper is dedicated to finding a general algorithm for generating different solutions for Pythagoras' non-linear Diophantine equation in four variables $x^2 + y^2 = z^2$ in symbolic 2-plithogenic rings, which are known as Pythagoras triples.

Also, we present some examples of those triples in some finite symbolic 2-plithogenic rings.

Keywords: symbolic 2-plithogenic ring, Pythagoras triples, Diophantine equations

Introduction and Preliminaries

Symbolic n-plithogenic algebraic structures are a new generalization of classical algebraic structures, as they have serious algebraic properties to study.

In the previous literature, we can clearly note several algebraic studies that were interested in discovering the properties of these algebraic structures, for example, we can find some applications of plithogenic structures in probability, ring theory, linear spaces, matrices, and equations [1-10].

zenodo.10031222/10.5281

Researchers have studied Pythagorean triples in the ring of ordinary algebraic numbers [11-14]. Several efficient algorithms for calculating these quadruples have been presented, as solutions to the corresponding Diophantine equation.

This has motivated us to study Pythagoras triples in the symbolic 2-plithogenic commutative case, where we find a general algorithm for generating different solutions for Pythagoras non-linear Diophantine equation in four variables $\boldsymbol{x^2} + \boldsymbol{y^2} = \boldsymbol{z^2}$ in symbolic 2-plithogenic rings.

Definition.

The symbolic 2-plithogenic ring of real numbers is defined as follows:

$$2 - SP_R = \{t_0 + t_1P_1 + t_2P_2; t_i \in R, P_1 \times P_2 = P_2 \times P_1 = P_2, {P_1}^2 = {P_2}^2 = P_2\}$$

The addition operation on $2 - SP_R$ is defined as follows:

$$(t_0 + t_1P_1 + t_2P_2) + (t_0' + t_1'P_1 + t_2'P_2) = (t_0 + t_0') + (t_1 + t_1')P_1 + (t_2 + t_2')P_2$$

The multiplication on $2 - SP_R$ is defined as follows:

$$(t_0 + t_1P_1 + t_2P_2)(t_0' + t_1'P_1 + t_2'P_2) = t_0t_0' + (t_0t_1' + t_1t_0' + t_1t_1')P_1 + (t_0t_2' + t_1t_2' + t_2t_2' + t_2t_0' + t_2t_1')P_2$$

Main Discussion

Definition.

Let R be a ring, then (t, s, k) is called a Pythagoras triple if and only if

$t^2 + s^2 = k^2; t, s, k \in R$..

Theorem.

Let $T = t_0 + t_1P_1 + t_2P_2, S = s_0 + s_1P_1 + s_2P_2, K = k_0 + k_1P_1 + k_2P_2$ are three arbitrary symbolic 3-plithogenic elements $T, S, K \in 2 - SP_R$, then (T, S, K) are Pythagoras triple in $2 - SP_R$ if and only if:

$$\begin{cases} (t_0, s_0, k_0), (t_0 + t_1, s_0 + s_1, k_0 + k_1)\ are\ pythagoras\ triples\ in\ R \\ (t_0 + t_1 + t_2, s_0 + s_1 + s_2, k_0 + k_1 + k_2)\ is\ pythagoras\ triple\ in\ R \end{cases}$$

Proof.

According to [ ], we have:

$T^2 = {t_0}^2 + [(t_0 + t_1)^2 - {t_0}^2]P_1 + [(t_0 + t_1 + t_2)^2 - (t_0 + t_1)^2]P_2$

$S^2 = {s_0}^2 + [(s_0 + s_1)^2 - {s_0}^2]P_1 + [(s_0 + s_1 + s_2)^2 - (s_0 + s_1)^2]P_2$

$T^2 = {k_0}^2 + \left[(k_0 + k_1)^2 - {k_0}^2\right]P_1 + [(k_0 + k_1 + k_2)^2 - (k_0 + k_1)^2]P_2$

The equation $T^2 + S^2 = K^2$ is equivalent to:

${t_0}^2 + {s_0}^2 = {k_0}^2$ $(equation\ 1)$,

$(t_0 + t_1)^2 + (s_0 + s_1)^2 = (k_0 + k_1)^2$ $(equation\ 2)$,

$(t_0 + t_1 + t_2)^2 + (s_0 + s_1 + s_2)^2 = (k_0 + k_1 + k_2)^2$ $(equation\ 3)$,

Equation (1) implies that (t_0, s_0, k_0) is a Pythagoras triple in R.

Equation (2) implies that $(t_0 + t_1, s_0 + s_1, k_0 + k_1)$ is a Pythagoras triple in R.

Equation (3) implies that $(t_0 + t_1 + t_2, s_0 + s_1 + s_2, k_0 + k_1 + k_2)$ is a Pythagoras triple in R.

Thus the proof is complete.

Theorem.

Let $(t_0, s_0, k_0), (t_1, s_1, k_1), (t_2, s_2, k_2)$be three Pythagoras triples in the ring R, then (T, S, K) Pythagoras triple in $3 - SP_R$, where:

$T = t_0 + [t_1 - t_0]P_1 + [t_2 - t_1]P_2$.

$S = s_0 + [s_1 - s_0]P_1 + [s_2 - s_1]P_2$.

$K = k_0 + [k_1 - k_0]P_1 + [k_2 - k_1]P_2$.

Proof.

We have: $t_0 + (t_1 - t_0) = t_1, t_0 + (t_1 - t_0) + (t_2 - t_1) = t_2$.

$s_0 + (s_1 - s_0) = s_1, s_0 + (s_1 - s_0) + (s_2 - s_1) = s_2$,

$k_0 + (k_1 - k_0) = k_1, k_0 + (k_1 - k_0) + (k_2 - k_1) = k_2$.

This implies that (T, S, K) Pythagoras triple in $2 - SP_R$ according to the theorem.

Examples.

We have:

$$\begin{cases} (t_0, s_0, k_0) = (3,4,5) \\ (t_1, s_1, k_1) = (6,8,10) \\ (t_2, s_2, k_2) = (4,3,5) \end{cases}$$

Are three Pythagoras triples in Z.

The corresponding symbolic 2-plithogenic Pythagoras triple is (T, S, K), where:

$T = 3 + [6 - 3]P_1 + [4 - 6]P_2 = 3 + 3P_1 - 2P_2$

$S = 4 + [8 - 4]P_1 + [3 - 8]P_2 = 4 + 4P_1 - 5P_2$

$K = 5 + [10 - 5]P_1 + [5 - 10]P_2 = 5 + 5P_1 - 5P_2$

Example.

Find all Pythagoras triples in $2 - SP_{Z_2}$, where Z_2 I the ring of integers module 2.

First, we find all Pythagoras triples in Z.

$L_1 = (0,0,0), L_2 = (1,0,1), L_3 = (0,1,1), L_4 = (1,1,0)$

Remark that for every permutation of the set $\{L_1, L_2, L_3, L_4\}$, we get a different symbolic 2-plithogenic Pythagoras triple.

We discuss all possible cases:

Permutation (1).

$$\begin{cases} Y_1 = P_1 - P_2 = P_1 + P_2 \\ \acute{Y}_1 = P_2 \\ Y_1'' = P_1 = P_1 \end{cases}$$

Permutation (2).

$$\begin{cases} Y_2 = P_2 \\ \acute{Y}_2 = P_1 - P_2 = P_1 + P_2 \\ Y_2'' = P_1 = P_1 \end{cases}$$

Permutation (3).

$$\begin{cases} Y_3 = P_1 \\ \acute{Y}_3 = P_1 + P_2 \\ Y_3'' = P_2 \end{cases}$$

Permutation (4).

$$\begin{cases} Y_4 = P_1 + P_2 \\ \acute{Y}_4 = P_1 \\ Y_4'' = P_2 \end{cases}$$

Permutation (5).

$$\begin{cases} Y_5 = P_1 \\ \acute{Y}_5 = P_2 \\ Y_5'' = P_1 + P_2 \end{cases}$$

Permutation (6).

$$\begin{cases} Y_6 = P_2 \\ \acute{Y}_6 = P_1 \\ Y_6'' = P_1 + P_2 \end{cases}$$

Permutation (7).

$$\begin{cases} Y_7 = 1 + P_2 \\ \acute{Y_7} = P_1 \\ Y_7'' = 1 + P_1 + P_2 \end{cases}$$

Permutation (8).

$$\begin{cases} Y_8 = P_2 \\ \acute{Y_8} = 1 + P_1 + P_2 \\ Y_8'' = 1 + P_1 + P_2 \end{cases}$$

Permutation (9).

$$\begin{cases} Y_9 = 1 + P_1 \\ \acute{Y_9} = P_1 + P_2 \\ Y_9'' = 1 + P_2 \end{cases}$$

Permutation (10).

$$: \begin{cases} Y_{10} = P_1 + P_2 \\ \acute{Y_{10}} = 1 + P_1 \\ Y_{10}'' = 1 + P_2 \end{cases}$$

Permutation (11).

$$\begin{cases} Y_{11} = 1 + P_2 \\ \acute{Y_{11}} = 1 + P_1 \\ Y_{11}'' = P_1 + P_2 \end{cases}$$

Permutation (12).

$$\begin{cases} Y_{12} = 1 + P_1 + P_2 \\ \acute{Y_{12}} = 1 + P_1 \\ Y_{12}'' = P_2 \end{cases}$$

Permutation (13).

$$\begin{cases} Y_{13} = 1 + P_1 + P_2 \\ \acute{Y_{13}} = 1 + P_2 \\ Y_{13}'' = P_1 \end{cases}$$

Permutation (14).

$$\begin{cases} Y_{14} = 1 + P_1 \\ \acute{Y_{14}} = 1 + P_2 \\ Y_{14}'' = P_1 + P_2 \end{cases}$$

Permutation (15).

$$\begin{cases} Y_{15} = 1 + P_1 \\ \acute{Y_{15}} = 1 + P_1 + P_2 \\ Y_{15}'' = P_2 \end{cases}$$

Permutation (16).

$$\begin{cases} Y_{16} = 1 + P_2 \\ \acute{Y}_{16} = 1 + P_1 + P_2 \\ Y_{16}'' = P_1 \end{cases}$$

Permutation (17).

$$\begin{cases} Y_{17} = 1 + P_2 \\ \acute{Y}_{17} = P_1 + P_2 \\ Y_{17}'' = 1 + P_1 \end{cases}$$

Permutation (18).

$$\begin{cases} Y_{18} = 1 + P_2 \\ \acute{Y}_{18} = P_1 \\ Y_{18}'' = 1 + P_1 + P_2 \end{cases}$$

Permutation (19).

$$: \begin{cases} Y_{19} = P_1 \\ \acute{Y}_{19} = 1 + P_1 + P_2 \\ Y_{19}'' = 1 + P_2 \end{cases}$$

Permutation (20).

$$\begin{cases} Y_{20} = P_1 \\ \acute{Y}_{20} = 1 + P_1 + P_2 \\ Y_{20}'' = 1 + P_2 \end{cases}$$

Permutation (21).

$$: \begin{cases} Y_{21} = P_1 + P_2 \\ \acute{Y}_{21} = 1 + P_2 \\ Y_{21}'' = 1 + P_1 \end{cases}$$

Permutation (22).

$$\begin{cases} Y_{22} = 1 + P_1 \\ \acute{Y}_{22} = P_1 + P_2 \\ Y_{22}'' = 1 + P_2 \end{cases}$$

Permutation (23).

$$\begin{cases} Y_{23} = P_1 \\ \acute{Y}_{23} = 1 + P_2 \\ Y_{23}'' = 1 + P_1 + P_2 \end{cases}$$

Permutation (24).

$$\begin{cases} Y_{24} = P_1 + P_2 \\ \acute{Y}_{24} = 1 + P_1 \\ Y_{24}'' = 1 + P_2 \end{cases}$$

Also, other quadraples $(L_i, L_j, L_k, L_s); 1 \leq i, j, k, s \leq 4$ give Pythagoras triples with i, j, k, s are not distinct at all.

We continuo our discussions.

Permutation (25).

$$\begin{cases} Y_{25} = (0,0,0) \\ Y_{25}' = (0,0,0) \\ Y_{25}'' = (0,0,0) \end{cases}$$

Permutation (26).

$$\begin{cases} Y_{26} = P_1 + P_2 \\ Y_{26}' = 0 \\ Y_{26}'' = P_1 + P_2 \end{cases}$$

Permutation (27).

$$\begin{cases} Y_{27} = 0 \\ Y_{27}' = P_1 + P_2 \\ Y_{27}'' = P_1 + P_2 \end{cases}$$

Permutation (28).

$$\begin{cases} Y_{28} = P_1 + P_2 \\ Y_{28}' = P_1 + P_2 \\ Y_{28}'' = 0 \end{cases}$$

Permutation (29).

$$\begin{cases} Y_{29} = P_2 \\ Y_{29}' = 0 \\ Y_{29}'' = P_2 \end{cases}$$

Permutation (30).

$$\begin{cases} Y_{30} = 0 \\ Y_{30}' = P_2 \\ Y_{30}'' = P_2 \end{cases}$$

Permutation (31).

$$\begin{cases} Y_{31} = P_2 \\ Y_{31}' = P_2 \\ Y_{31}'' = 0 \end{cases}$$

Permutation (32).

$$\begin{cases} Y_{32} = 1 \\ Y_{32}' = 0 \\ Y_{32}'' = 1 \end{cases}$$

Permutation (33).

$$:\begin{cases} Y_{33} = 1 + P_2 \\ \acute{Y_{33}} = 0 \\ Y_{33}'' = 1 + P_2 \end{cases}$$

Permutation (34).

$$\begin{cases} Y_{34} = 1 + P_2 \\ \acute{Y_{34}} = P_2 \\ Y_{34}'' = 1 \end{cases}$$

Permutation (35).

$$\begin{cases} Y_{35} = 1 \\ \acute{Y_{35}} = P_2 \\ Y_{35}'' = 1 + P_2 \end{cases}$$

Permutation (36).

$$\begin{cases} Y_{36} = 1 + P_1 \\ \acute{Y_{36}} = 0 \\ Y_{36}'' = 1 + P_1 \end{cases}$$

Permutation (37).

$$\begin{cases} Y_{37} = 1 \\ \acute{Y_{37}} = P_1 + P_2 \\ Y_{37}'' = 1 + P_2 + P_1 \end{cases}$$

Permutation (38).

$$\begin{cases} Y_{38} = 1 + P_2 \\ \acute{Y_{38}} = P_1 + P_2 \\ Y_{38}'' = 1 + P_1 \end{cases}$$

Permutation (39).

$$\begin{cases} Y_{39} = 0 \\ \acute{Y_{39}} = 1 \\ Y_{39}'' = 1 \end{cases}$$

Permutation (40).

$$\begin{cases} Y_{40} = P_2 \\ \acute{Y_{40}} = 1 + P_2 \\ Y_{40}'' = 1 \end{cases}$$

Permutation (41).

$$\begin{cases} Y_{41} = P_2 \\ \acute{Y_{41}} = 1 \\ Y_{41}'' = 1 + P_2 \end{cases}$$

Permutation (42).

$$\begin{cases} Y_{42} = 0 \\ Ý_{42} = 1 + P_1 + P_2 \\ Y_{42}'' = 1 + P_1 + P_2 \end{cases}$$

Permutation (43).

$$\begin{cases} Y_{43} = P_1 + P_2 \\ Ý_{43} = 1 + P_1 + P_2 \\ Y_{43}'' = 1 \end{cases}$$

Permutation (44).

$$\begin{cases} Y_{44} = P_1 + P_2 \\ Ý_{44} = 1 \\ Y_{44}'' = 1 + P_1 + P_2 \end{cases}$$

Permutation (45).

$$\begin{cases} Y_{45} = 1 \\ Ý_{45} = 1 \\ Y_{45}'' = 0 \end{cases}$$

Permutation (46).

$$\begin{cases} Y_{46} = 1 + P_2 \\ Ý_{46} = 1 + P_2 \\ Y_{46}'' = 0 \end{cases}$$

Permutation (47).

$$\begin{cases} Y_{47} = 1 \\ Ý_{47} = 1 + P_2 \\ Y_{47}'' = P_2 \end{cases}$$

Permutation (48).

$$\begin{cases} Y_{48} = 1 + P_2 \\ Ý_{48} = 1 \\ Y_{48}'' = P_2 \end{cases}$$

Permutation (49).

$$\begin{cases} Y_{49} = 1 + P_1 + P_2 \\ Ý_{49} = 1 + P_1 + P_2 \\ Y_{49}'' = 0 \end{cases}$$

Permutation (50).

$$\begin{cases} Y_{50} = 1 \\ Ý_{50} = 1 + P_1 + P_2 \\ Y_{50}'' = P_1 + P_2 \end{cases}$$

Permutation (51).

$$\begin{cases} Y_{51} = 1 + P_1 + P_2 \\ \acute{Y}_{51} = 1 \\ Y_{51}'' = P_1 + P_2 \end{cases}$$

By continuing this argument, we can get all Pythagoras triples in $2 - SP_{Z_2}$

Conclusion.

In this paper, we have studied Pythagoras triples in symbolic 2-plithogenic commutative rings, where necessary and sufficient conditions for a symbolic 2-plithogenic triple (x, y, z) to be a Pythagoras triple.

Also, we have presented some related examples that explain how to find 2-plithogenic triples from classical triples.

Acknowledgments " This study is supported via funding from Prince sattam bin Abdulaziz University project number (PSAU/2023/R/1445) ".

References

[1] M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, "Introduction to Symbolic 2-Plithogenic Probability Theory," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.

[2] Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.

[3] Nader Mahmoud Taffach , Ahmed Hatip., "A Review on Symbolic 2-Plithogenic Algebraic Structures " Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.

[4] Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.

[5] Ben Othman, K., "On Some Algorithms For Solving Symbolic 3-Plithogenic Equations", Neoma Journal Of Mathematics and Computer Science, 2023.

[6] Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Concept Of Symbolic 7-Plithogenic Real Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

[7] Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Symbolic 8-Plithogenic Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

[8]Mohamed, M., "Modified Approach for Optimization of Real-Life Transportation Environment: Suggested Modifications", American Journal of Business and Operations research, 2021.

[9] Nabeeh, N., Alshaimaa, A., and Tantawy, A., "A Neutrosophic Proposed Model For Evaluation Blockchain Technology in Secure Enterprise Distributed Applications", Journal of Cypersecurity and Information Management, 2023.

[10]Abualkishik, A., Almajed, R., Thompson, W., "Improving The Performance of Fog-assisted Internet of Things Networks Using Bipolar Trapezoidal Neutrosophic Sets", Journal of Wireless and Ad Hoc Communication, 2023.

[11] R. A. Trivedi, S. A. Bhanotar, Pythagorean Triplets - Views, Analysis and Classification, IIOSR J. Math., 11 (2015).

[12] J. Rukavicka, Dickson's method for generating Pythagorean triples revisited, Eur. J. Pure Appl.Math., 6 (2013).

[13] T.Roy and F J. Sonia, A Direct Method To Generate Pythagorean Triples And Its Generalization To Pythagorean Quadruples And n-tuples, , Jadavpur University, India (2010)

[14] Paul Oliverio, " Self-Generating Pythagorean Quadruples And TV-Tuples", Los Angeles, (1993)

Received 3/6/2023, Accepted 18/9/2023

Neutrosophic Sets and Systems, Vol. 59, 2023
University of New Mexico

Symbolic Neutrosophic and Plithogenic Marshall-Olkin Type I Class of Distributions

Danyah Dham[1], Mohamed Bisher Zeina[2] , Riad K. Al-Hamido[3]

[1] Dept. of Mathematical Statistics, Faculty of Science, University of Aleppo, Aleppo, Syria, e-mail: danyahdham@gmail.com.

[2] Dept. of Mathematical Statistics, Faculty of Science, University of Aleppo, Aleppo, Syria, e-mail: bisher.zeina@gmail.com.

[3] Dept. of Mathematics, Faculty of Science, Alfurat University, Deir Al-Zor, Syria, e-mail: riad-hamido1983@hotmail.com.

Abstract: In this paper we present the symbolic neutrosophic and plithogenic Marshall-Olkin type I class of distributions. We derive the formal form of the cumulative distribution function and probability density function of neutrosophic and plithogenic Marshall-Olkin Type I class of distributions. As a special case of the mentioned class of distributions we study the generalized uniform distribution in both neutrosophic and plithogenic forms, we derive its PDF and CDF then present an algorithm of random numbers generation according to it, then we estimate its parameters using maximum likelihood estimation and support the results with a simulation study to show the efficiency of the calculated parameters and study its asymptotic properties including unbiasedness and consistency.

Keywords: Marshall-Olkin Type I Class of Distributions; Neutrosophic; Plithogenic; AH Isometry; Maximum Likelihood Estimation; Random Numbers Generation.

1. Introduction

Neutrosophic Probability Theory and Plithogenic Probability Theory are both intriguing extensions of traditional probability theory that deal with uncertainty and ambiguity in a more nuanced and comprehensive manner. These theories were developed to address situations where classical probability theory falls short in capturing the complexity of real-world uncertainties.

Neutrosophic Probability Theory is an extension of classical probability theory that introduces the concept of "Neutrosophy." Neutrosophy deals with indeterminacy, ambiguity, and imprecision that arise in various fields such as philosophy, mathematics, and decision-making[1]–[31].

Plithogenic Probability Theory is another extension of classical probability theory that aims to address the limitations of traditional probability theory in handling complex uncertainties. It introduces the concept of "Plithogeny," which deals with the multitude of conditions that contribute to the occurrence or non-occurrence of an event. Unlike classical probability theory, where events are often treated as independent and isolated, plithogenic probability theory recognizes that events are influenced by a multitude of interconnected factors. It also focuses on understanding how various conditions interact and contribute to the overall probability of an event. This theory is particularly useful in scenarios involving interdependent events, network analysis, and systems with intricate dependencies.[32]–[47]

zenodo.10031226/10.5281

In this paper we will deal with symbolic neutrosophic sets and symbolic plithogenic sets where the elements of these sets take the form $N = a + bI; I^2 = I$ for neutrosophic sets and $S = a + bP_1 + cP_2\ ; P_1^2 = P_1, P_2^2 = P_2, P_1 \cdot P_2 = P_2 \cdot P_1 = P_2$ for plithogenic sets and we will generalize the well know Marshall Olkin class of distributions [48]–[55] to both neutrosophic and plithogenic class.

2. Preliminaries

Definition 2.1

Let $R(I) = \{a + bI\ ; a, b \in R\}$ be the neutrosophic field of reals where $I^2 = I$. One-dimensional AH-isometry between $R(I)$ and R^2 and its inverse are given by:

$$T: R(I) \to R^2\ ;\ T(a + bI) = (a, a + b) \quad (1)$$

$$T^{-1}: R^2 \to R(I)\ ;\ T^{-1}(a, b) = a + (b - a)I \quad (2)$$

Note:

T is an algebraic isomorphism and it preserves distances.

Definition 2.2

A neutrosophic random variable X_N is defined as follows:

$$X_N = X_1 + X_2 I\ ; I^2 = I$$

Where X_1, X_2 are classical random variables.

Definition 2.3

Let $f: R(I) \to R(I); f = f(x_N), x_N \in R(I)$ then f is called a neutrosophic real function with one neutrosophic variable.

Definition 2.4

Let $a_N = a_1 + a_2 I, b_N = b_1 + b_2 I \in R(I)$ be neutrosophic numbers. We say that $a_N \geq_N b_N$ if:

$$a_1 \geq b_1\ , a_1 + a_2 \geq b_1 + b_2$$

Definition 2.5

Let $R(P_1, P_2) = \{a_0 + a_1 P_1 + a_2 P_2\ ; a_0, a_1, a_2 \in R,\ P_1^2 = P_1,\ P_2^2 = P_2,\ P_1 \cdot P_2 = P_2 \cdot P_1 = P_2\}$ be the Plithogenic field of reals. One-dimensional isometry between $R(P_1, P_2)$ and R^3 and its inverse are defined as follows:

$$T: R(P_1, P_2) \to R^3\ ;\ T(a_0 + a_1 P_1 + a_2 P_2) = (a_0\ , a_0 + a_1\ , a_0 + a_1 + a_2) \quad (3)$$

$$T^{-1}: R^3 \to R(P_1, P_2)\ ;\ T^{-1}(a_0, a_1, a_2) = a_0 + (a_1 - a_0)P_1 + (a_2 - a_1)P_2 \quad (4)$$

Definition 2.6

A Plithogenic random variable X_P is defined as follows:

$X_P = X_0 + X_1 P_1 + X_2 P_2\ ; P_1^2 = P_1, P_2^2 = P_2, P_1 \cdot P_2 = P_2 \cdot P_1 = P_2$ where X_0, X_1, X_2 are classical random variables.

Definition 2.7

Let $f: R(P_1, P_2) \to R(P_1, P_2); f = f(x_P)\ , x_P \in R(P_1, P_2)$ then f is called a Plithogenic real function with one plithogenic variable.

Definition 2.8

Let $a_P = a_0 + a_1 P_1 + a_2 P_2, b_P = b_0 + b_1 P_1 + b_2 P_2 \in R(P_1, P_2)$ be two plithogenic numbers. We say that $a_P \geq_P b_P$ if:

$$a_0 \geq b_0\ , a_0 + a_1 \geq b_0 + b_1, a_0 + a_1 + a_2 \geq b_0 + b_1 + b_2$$

3. Neutrosophic Marshall-Olkin Type I Class of Distributions:

In this section we are going to derive the neutrosophic form of Marshall-Olkin Type I class of distributions depending on its cumulative probability distribution function and probability distribution function and some generalized distributions according to it.

Definition 3.1

Neutrosophic Marshall Olkin Type I cumulative distribution function is classical Marshall-Olkin Type I cumulative distribution function but defined on $R(I)$, taking values in $R(I)$ and with parameters from $R(I)$, that is its CDF is:

$$G(x_N;\rho_N)=\frac{F(x_N)}{F(x_N)(1-\rho_N)+\rho_N};\ x_N\in R(I)\ \&\ 0<_N\rho_N<_N 1 \qquad (5)$$

Theorem 3.1

The neutrosophic formal form of (5) is:

$$G(x_N;\rho_N)=\frac{F(x_1)}{F(x_1)(1-\rho_1)+\rho_1}+I\left[\frac{F(x_1+x_2)}{F(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)}-\frac{F(x_1)}{F(x_1)(1-\rho_1)+\rho_1}\right] \qquad (6)$$

Proof

$$T[G(x_N;\rho_N)]=\frac{T[F(x_N)]}{T[F(x_N)]T[1-\rho_N]+T[\rho_N]}$$

$$=\frac{\big(F(x_1),F(x_1+x_2)\big)}{\big(F(x_1),F(x_1+x_2)\big)\big(1-\rho_1,1-(\rho_1+\rho_2)\big)+(\rho_1,\rho_1+\rho_2)}$$

$$=\left(\frac{\big(F(x_1),F(x_1+x_2)\big)}{\Big(F(x_1)(1-\rho_1)+\rho_1,\big(F(x_1+x_2)\big)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)\Big)}\right)$$

$$=\left(\frac{F(x_1)}{F(x_1)(1-\rho_1)+\rho_1},\frac{F(x_1+x_2)}{F(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)}\right)$$

So:

$$G(x_N;\rho_N)=T^{-1}\left(\left(\frac{F(x_1)}{F(x_1)(1-\rho_1)+\rho_1},\frac{F(x_1+x_2)}{F(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)}\right)\right)$$

$$=\frac{F(x_1)}{F(x_1)(1-\rho_1)+\rho_1}+I\left[\frac{F(x_1+x_2)}{F(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)}-\frac{F(x_1)}{F(x_1)(1-\rho_1)+\rho_1}\right]$$

Note:

Neutrosophic probability distribution function of Marshall-Olkin Type I class of distributions can be derived by direct derivation of equation (5).

$$g(x_N;\rho_N)=\frac{\rho_N f(x_N)}{[(1-\rho_N)F(x_N)+\rho_N]^2};\ x_N\in R(I)\ \&\ 0<_N\rho_N<_N 1 \qquad (7)$$

Theorem 3.2

The neutrosophic formal form of (7) is:

$$g(x_N;\rho_N)=\frac{\rho_1 f(x_1)}{[(1-\rho_1)F(x_1)+\rho_1]^2}+I\left[\frac{(\rho_1+\rho_2)f(x_1+x_2)}{\left[\left(1-(\rho_1+\rho_2)\right)F(x_1+x_2)+\rho_1+\rho_2\right]^2}-\frac{\rho_1 f(x_1)}{[(1-\rho_1)F(x_1)+\rho_1]^2}\right] \quad (8)$$

Proof

$$T[g(x_N;\rho_N)]=\frac{T[\rho_N]T[f(x_N)]}{\left[T[(1-\rho_N)]T[F(x_N)]+T[\rho_N]\right]^2}=\frac{(\rho_1,\rho_1+\rho_2)\left(f(x_1),f(x_1+x_2)\right)}{\left[\left(1-\rho_1,1-(\rho_1+\rho_2)\right)\left(F(x_1),F(x_1+x_2)\right)+(\rho_1,\rho_1+\rho_2)\right]^2}$$

$$=\frac{\left(\rho_1 f(x_1),(\rho_1+\rho_2)f(x_1+x_2)\right)}{\left[\left((1-\rho_1)F(x_1)+\rho_1,\left(1-(\rho_1+\rho_2)\right)F(x_1+x_2)+\rho_1+\rho_2\right)\right]^2}$$

$$=\left(\frac{\rho_1 f(x_1)}{[(1-\rho_1)F(x_1)+\rho_1]^2},\frac{(\rho_1+\rho_2)f(x_1+x_2)}{\left[\left(1-(\rho_1+\rho_2)\right)F(x_1+x_2)+\rho_1+\rho_2\right]^2}\right)$$

So:

$$g(x_N;\rho_N)=T^{-1}\left(\frac{\rho_1 f(x_1)}{[(1-\rho_1)F(x_1)+\rho_1]^2},\frac{(\rho_1+\rho_2)f(x_1+x_2)}{\left[\left(1-(\rho_1+\rho_2)\right)F(x_1+x_2)+\rho_1+\rho_2\right]^2}\right)$$

$$=\frac{\rho_1 f(x_1)}{[(1-\rho_1)F(x_1)+\rho_1]^2}+I\left[\frac{(\rho_1+\rho_2)f(x_1+x_2)}{\left[\left(1-(\rho_1+\rho_2)\right)F(x_1+x_2)+\rho_1+\rho_2\right]^2}-\frac{\rho_1 f(x_1)}{[(1-\rho_1)F(x_1)+\rho_1]^2}\right]$$

Theorem 3.3

Equation (8) represents probability density function in classical sense.

Proof

$$\int_{-\infty}^{+\infty} g(x_N;\rho_N)dx_N=\int_{-\infty}^{+\infty}\frac{\rho_1 f(x_1)}{[(1-\rho_1)F(x_1)+\rho_1]^2}dx_1$$

$$+I\left[\int_{-\infty}^{+\infty}\frac{(\rho_1+\rho_2)f(x_1+x_2)}{\left[\left(1-(\rho_1+\rho_2)\right)F(x_1+x_2)+\rho_1+\rho_2\right]^2}d(x_1+x_2)-\int_{-\infty}^{+\infty}\frac{\rho_1 f(x_1)}{[(1-\rho_1)F(x_1)+\rho_1]^2}dx_1\right]$$

$$=\int_{-\infty}^{+\infty} d\left(\frac{F(x_1)}{F(x_1)(1-\rho_1)+\rho_1}\right)$$

$$+I\left[\int_{-\infty}^{+\infty} d\left(\frac{F(x_1+x_2)}{F(x_1+x_2)\left(1-(\rho_1+\rho_2)\right)+(\rho_1+\rho_2)}\right)-\int_{-\infty}^{+\infty} d\left(\frac{F(x_1)}{F(x_1)(1-\rho_1)+\rho_1}\right)\right]=1$$

Also, it is easy to see that $T[g(x_N;\rho_N)]$ presents two continuous and positive functions. Depending on [3], [25] we conclude that the given neutrosophic function is a neutrosophic probability density function in classical sense.

4. Neutrosophic Marshall-Olkin Type I Uniform distribution:

Definition 4.1

The neutrosophic cumulative distribution function of the neutrosophic Marshall-Olkin Type I uniform distribution is defined as follows:

$$G(x_N;\rho_N,a_N,b_N)=\frac{x_N-a_N}{x_N(1-\rho_N)+\rho_N b_N-a_N}\ ;a_N<_N x_N<_N b_N\ ,0<_N \rho_N<_N 1 \quad (9)$$

Theorem 4.1

The neutrosophic formal form of (9) is:

$$G(x_N;\rho_N,a_N,b_N)=\frac{x_1-a_1}{x_1(1-\rho_1)+\rho_1 b_1-a_1}+I\left[\frac{(x_1+x_2)-(a_1+a_2)}{(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)}-\frac{x_1-a_1}{x_1(1-\rho_1)+\rho_1 b_1-a_1}\right] \quad (10)$$

Proof

$$T[G(x_N;\rho_N,a_N,b_N)]=\frac{T[x_N]-T[a_N]}{T[x_N]T[1-\rho_N]+T[\rho_N]\cdot T[b_N]-T[a_N]}$$

$$=\frac{(x_1,x_1+x_2)-(a_1,a_1+a_2)}{(x_1,x_1+x_2)\big(1-\rho_1,1-(\rho_1+\rho_2)\big)+(\rho_1,\rho_1+\rho_2)\cdot(b_1,b_1+b_2)-(a_1,a_1+a_2)}$$

$$=\left(\frac{\big(x_1-a_1,(x_1+x_2)-(a_1+a_2)\big)}{\Big(x_1(1-\rho_1)+\rho_1 b_1-a_1,(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)\Big)}\right)$$

$$=\left(\frac{x_1-a_1}{x_1(1-\rho_1)+\rho_1 b_1-a_1},\frac{(x_1+x_2)-(a_1+a_2)}{(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)}\right)$$

$$G(x_N;\rho_N,a_N,b_N)=T^{-1}\left(\frac{x_1-a_1}{x_1(1-\rho_1)+\rho_1 b_1-a_1},\frac{(x_1+x_2)-(a_1+a_2)}{(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)}\right)$$

$$=\frac{x_1-a_1}{x_1(1-\rho_1)+\rho_1 b_1-a_1}$$

$$+I\left[\frac{(x_1+x_2)-(a_1+a_2)}{(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)}-\frac{x_1-a_1}{x_1(1-\rho_1)+\rho_1 b_1-a_1}\right]$$

Definition 4.2

The neutrosophic probability distribution function of neutrosophic Marshall-Olkin Type I uniform distribution is defined as follows:

$$g(x_N;\rho_N,a_N,b_N)=\frac{(b_N-a_N)\rho_N}{[x_N(1-\rho_N)+\rho_N b_N-a_N]^2}\ ;\ a_N<_N x_N<_N b_N,\ 0<_N\rho_N<_N 1 \quad (11)$$

Theorem 4.2

The neutrosophic formal form of (11) is:

$$g(x_N;\rho_N,a_N,b_N)=\frac{(b_1-a_1)\rho_1}{[x_1(1-\rho_1)+\rho_1 b_1-a_1]^2}+I\left[\frac{\big((b_1+b_2)-(a_1+a_2)\big)(\rho_1+\rho_2)}{\big[(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)\big]^2}-\frac{(b_1-a_1)\rho_1}{[x_1(1-\rho_1)+\rho_1 b_1-a_1]^2}\right] \quad (12)$$

Proof

$$T[g(x_N;\rho_N,a_N,b_N)]=\frac{(T[b_N]-T[a_N])T[\rho_N]}{\big[T[x_N]T[(1-\rho_N)]+T[\rho_N]\cdot T[b_N]-T[a_n]\big]^2}$$

$$=\frac{\big((b_1,b_1+b_2)-(a_1,a_1+a_2)\big)(\rho_1,\rho_1+\rho_2)}{\big[(x_1,x_1+x_2)\big(1-\rho_1,1-(\rho_1+\rho_2)\big)+(\rho_1,\rho_1+\rho_2)\cdot(b_1,b_1+b_2)-(a_1,a_1+a_2)\big]^2}$$

$$=\frac{\Big((b_1-a_1)\rho_1,\big((b_1+b_2)-(a_1+a_2)\big)(\rho_1+\rho_2)\Big)}{\Big[\Big(x_1(1-\rho_1)+\rho_1 b_1-a_1,(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)\Big)\Big]^2}$$

$$=\left(\frac{(b_1-a_1)\rho_1}{[x_1(1-\rho_1)+\rho_1 b_1-a_1]^2},\frac{\big((b_1+b_2)-(a_1+a_2)\big)(\rho_1+\rho_2)}{\big[(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)\big]^2}\right)$$

So:

$$g(x_N;\rho_N,a_N,b_N)=T^{-1}\left(\frac{(b_1-a_1)\rho_1}{[x_1(1-\rho_1)+\rho_1 b_1-a_1]^2},\frac{\big((b_1+b_2)-(a_1+a_2)\big)(\rho_1+\rho_2)}{\big[(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)\big]^2}\right)$$

$$=\frac{(b_1-a_1)\rho_1}{[x_1(1-\rho_1)+\rho_1 b_1-a_1]^2}$$

$$+I\left[\frac{\big((b_1+b_2)-(a_1+a_2)\big)(\rho_1+\rho_2)}{\big[(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)\big]^2}-\frac{(b_1-a_1)\rho_1}{[x_1(1-\rho_1)+\rho_1 b_1-a_1]^2}\right]$$

4.1 Parameters' estimation using neutrosophic maximum likelihood estimation method:

Let $\mathbb{X}_N$ a neutrosophic random sample drawn from neutrosophic Marshall-Olkin Type I uniform distribution with PDF defined in (11) then the neutrosophic likelihood function will be:

$$L_N=L(\mathbb{X}_N;\Theta)=\prod_{i=1}^{n}f(X_{iN};a_N,b_N,\rho_N)=\prod_{i=1}^{n}\frac{(b_N-a_N)\rho_N}{[x_{iN}(1-\rho_N)+\rho_N b_N-a_N]^2}$$

$$=\frac{(b_N-a_N)^n\rho_N^n}{\prod_{i=1}^{n}[x_{iN}(1-\rho_N)+\rho_N b_N-a_N]^2} \quad (13)$$

By taking log of (13), we get the loglikelihood function as follows:

$$\mathcal{L}_N=\ln L(\mathbb{X}_N;\Theta)=n\ln(b_N-a_N)+n\ln\rho_N-2\sum_{i=1}^{n}\ln[x_{iN}(1-\rho_N)+\rho_N b_N-a_N] \quad (14)$$

Taking partial derivatives of previous equation according to a_N, b_N, ρ_N yields to:

$$\frac{\partial}{\partial a_N}\mathcal{L}_N=\frac{-n}{b_N-a_N}+2\sum_{i=1}^{n}\frac{1}{[x_{iN}(1-\rho_N)+\rho_N b_N-a_N]} \quad (15)$$

$$\frac{\partial}{\partial b_N}\mathcal{L}_N=\frac{n}{b_N-a_N}-2\sum_{i=1}^{n}\frac{\rho_N}{[x_{iN}(1-\rho_N)+\rho_N b_N-a_N]} \quad (16)$$

$$\frac{\partial}{\partial a_N}\mathcal{L}_N=\frac{n}{\rho_N}+2\sum_{i=1}^{n}\frac{x_{iN}-b_N}{[x_{iN}(1-\rho_N)+\rho_N b_N-a_N]} \quad (17)$$

Using the AH-Isometry we get:

$$\begin{cases}\dfrac{\partial}{\partial a_1}\mathcal{L}_1=\dfrac{-n}{b_1-a_1}+2\displaystyle\sum_{i=1}^{n}\dfrac{1}{[x_{i1}(1-\rho_1)+\rho_1 b_1-a_1]}\\ \dfrac{\partial}{\partial(a_1+a_2)}(\mathcal{L}_1+\mathcal{L}_2)=\dfrac{-n}{(b_1+b_2)-(a_1+a_2)}+2\displaystyle\sum_{i=1}^{n}\dfrac{1}{\big[(x_{i1}+x_{i2})\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)\big]}\end{cases} \quad (18)$$

$$\begin{cases}\dfrac{\partial}{\partial b_1}\mathcal{L}_1=\dfrac{n}{b_1-a_1}-2\displaystyle\sum_{i=1}^{n}\dfrac{\rho_1}{[x_{i1}(1-\rho_1)+\rho_1 b_1-a_1]}\\ \dfrac{\partial}{\partial(b_1+b_2)}(\mathcal{L}_1+\mathcal{L}_2)=\dfrac{n}{(b_1+b_2)-(a_1+a_2)}-2\displaystyle\sum_{i=1}^{n}\dfrac{(\rho_1+\rho_2)}{\big[(x_{i1}+x_{i2})\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)\big]}\end{cases} \quad (19)$$

$$\begin{cases} \frac{\partial}{\partial \rho_1}\mathcal{L}_1 = \frac{n}{\rho_1} + 2\sum_{i=1}^{n}\frac{x_{i1} - b_1}{[x_{i1}(1-\rho_1) + \rho_1 b_1 - a_1]} \\ \frac{\partial}{\partial(\rho_1+\rho_2)}(\mathcal{L}_1 + \mathcal{L}_2) = \frac{n}{(\rho_1+\rho_2)} + 2\sum_{i=1}^{n}\frac{(x_{i1}+x_{i2}) - (b_1+b_2)}{\left[(x_{i1}+x_{i2})\big(1-(\rho_1+\rho_2)\big) + (\rho_1+\rho_2)(b_1+b_2) - (a_1+a_2)\right]} \end{cases} \quad (20)$$

Solving equations (18-20) numerically yields to the desired estimators.

4.2 Simulation and random numbers generation:

Solving equation (9) with respect to x_N yields to:

$$x_N = \frac{y_N \rho_N b_N - a_N y_N + a_N}{1 - y_N(1-\rho_N)} \quad (21)$$

Where $y_N = F_N(x_N)$ is neutrosophic uniformly distributed on [0,1]

By taking AH-isometry to (21) we get:

$$x_1 = \frac{y_1 \rho_1 b_1 - a_1 y_1 + a_1}{1 - y_1(1-\rho_1)} \quad (22)$$

$$x_1 + x_2 = \frac{(y_1+y_2)(\rho_1+\rho_2)(b_1+b_2) - (a_1+a_2)(y_1+y_2) + (a_1+a_2)}{1 - (y_1+y_2)\big(1-(\rho_1+\rho_2)\big)} \quad (23)$$

Using equations (22-23), we can generate classical random numbers following classical Marshall-Olkin type I uniform distribution with chosen parameters, then using T^{-1} we will get neutrosophic Marshall-Olkin type I uniform numbers.

Monte Carlo simulation is done using Maple software with total replication of $N = 1000$ times and with sample sizes of $15, 50, 100, 150$ and fixed parameters $a_N = 1 + 2I, b_N = 2 + 5I, \rho_N = 0.5 + 0.1I$. We can check goodness of our estimations based on bias of the estimators and mean square error of it using the following equations:

$$Bias = \frac{\sum_{i=1}^{n}\left|\hat{\theta}_{iN} - \theta_N\right|}{n} \quad (24)$$

$$MSE = \frac{\sum_{i=1}^{n}\left(\hat{\theta}_{iN} - \theta_N\right)^2}{n} \quad (25)$$

Table 1. Simulation results of neutrosophic Marshall-Olkin type I uniform distribution.

	$\boldsymbol{a_N = 1 + 2I}$		
$\boldsymbol{n}$	$\boldsymbol{\hat{a}_N}$	$\boldsymbol{Bias\ \hat{a}_N}$	$\boldsymbol{MSE\ \hat{a}_N}$
15	$1.03255 + 2.12169I$	$0.03255 + 0.12169I$	$0.00214 + 0.04528I$
50	$1.00994 + 2.03760I$	$0.00994 + 0.03760I$	$0.00020 + 0.00431I$
100	$1.00500 + 2.01894I$	$0.00500 + 0.01894I$	$0.00005 + 0.00446I$
150	$1.00339 + 2.01285I$	$0.00339 + 0.01285I$	$0.00002 + 0.00049I$
	$\boldsymbol{b_N = 2 + 5I}$		
$\boldsymbol{n}$	$\boldsymbol{\hat{b}_N}$	$\boldsymbol{Bias\ \hat{b}_N}$	$\boldsymbol{MSE\ \hat{b}_N}$
15	$1.88284 + 4.71224I$	$0.11716 + 0.28776I$	$0.02464 + 0.27718I$
50	$1.96282 + 4.91173I$	$0.03718 + 0.08827I$	$0.00267 + 0.02810I$
100	$1.98031 + 4.95365I$	$0.01969 + 0.04635I$	$0.00072 + 0.00740I$

150	$1.98730 + 4.97020I$	$0.01270 + 0.02980I$	$0.00032 + 0.00323I$
	$\rho_N = 0.5 + 0.1I$		
n	$\hat{\rho}_N$	$Bias\ \hat{\rho}_N$	$MSE\ \hat{\rho}_N$
15	$0.61207 + 0.10236I$	$0.23605 + 0.03591I$	$0.11861 + 0.03663I$
50	$0.53516 + 0.10201I$	$0.10746 + 0.01942I$	$0.01990 + 0.00782I$
100	$0.51818 + 0.10096I$	$0.07376 + 0.01414I$	$0.00945 + 0.00391I$
150	$0.51377 + 0.10103I$	$0.06017 + 0.01164I$	$0.00591 + 0.00249I$

Table (1) shows that as sample size n increases, bias of the estimators and mean square error of it decrease which means that our estimators are asymptotically unbiased and consistent.

5. Plithogenic Marshall-Olkin Type I Class of Distributions

In this section we are going to construct plithogenic form of Marshall-Olkin Type I class of distributions, cumulative probability distribution function, probability distribution function and uniform generalized distribution according to it.

Definition 5.1

The plithogenic form of cumulative distribution function of the first type of Marshall-Olkin Type I class of distributions is defined as follows:

$$G(x_P;\rho_P) = \frac{F(x_P)}{F(x_P)(1-\rho_P)+\rho_P}\ ; x_P \in R(P_1,P_2)\ , 0 <_P \rho_P <_P 1 \tag{26}$$

Where $P_1^2 = P_1,\ P_2^2 = P_2,\ P_1 \cdot P_2 = P_2 \cdot P_1 = P_2$.

Theorem 5.1

The plithogenic formal form of (26) is:

$$\begin{aligned} G(x_P;\rho_P) = {} & \frac{F(x_0)}{F(x_0)(1-\rho_0)+\rho_0} + P_1\left[\frac{F(x_0+x_1)}{F(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)} - \frac{F(x_0)}{F(x_0)(1-\rho_0)+\rho_0}\right] \\ & + P_2\left[\frac{F(x_0+x_1+x_2)}{F(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)} \right. \\ & \left. - \frac{F(x_0+x_1)}{F(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)}\right] \end{aligned} \tag{27}$$

Proof

$$T[G(x_P;\rho_P)] = \frac{T[F(x_P)]}{T[F(x_P)]T[1-\rho_P]+T[\rho_P]}$$

$$= \frac{\big(F(x_0),F(x_0+x_1),F(x_0+x_1+x_2)\big)}{\big(F(x_0),F(x_0+x_1),F(x_0+x_1+x_2)\big)\Big(1-\rho_0,1-(\rho_0+\rho_1),\big(1-(\rho_0+\rho_1+\rho_2)\big)\Big)+\big(\rho_0,(\rho_0+\rho_1),(\rho_0+\rho_1+\rho_2)\big)}$$

$$= \left(\frac{\big(F(x_0),F(x_0+x_1),F(x_0+x_1+x_2)\big)}{\Big(F(x_0)(1-\rho_0)+\rho_0, F(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1), F(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)\Big)}\right)$$

$$= \left(\frac{F(x_0)}{F(x_0)(1-\rho_0)+\rho_0}, \frac{F(x_0+x_1)}{F(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)}, \frac{F(x_0+x_1+x_2)}{F(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)}\right)$$

So:

$$G(x_P;\rho_P)$$
$$=T^{-1}\left(\frac{F(x_0)}{F(x_0)(1-\rho_0)+\rho_0},\frac{F(x_0+x_1)}{F(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)},\frac{F(x_0+x_1+x_2)}{F(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)}\right)$$
$$=\frac{F(x_0)}{F(x_0)(1-\rho_0)+\rho_0}+P_1\left[\frac{F(x_0+x_1)}{F(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)}-\frac{F(x_0)}{F(x_0)(1-\rho_0)+\rho_0}\right]$$
$$+P_2\left[\frac{F(x_0+x_1+x_2)}{F(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)}-\frac{F(x_0+x_1)}{F(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)}\right]$$

Note:

Plithogenic probability distribution function of Marshall-Olkin Type I class of distributions can be derived by direct derivation of equation (26).

$$g(x_P;\rho_P)=\frac{\rho_P f(x_P)}{[(1-\rho_P)F(x_P)+\rho_P]^2}\ ; x_P\in R(P_1,P_2), 0<_P\rho_P<_P 1 \qquad (28)$$

Where $P_1^2=P_1,\ P_2^2=P_2,\ P_1\cdot P_2=P_2\cdot P_1=P_2$.

Theorem 5.2

The Plithogenic formal form of (28) is:

$$g(x_P;\rho_P)=\frac{\rho_0 f(x_0)}{[(1-\rho_0)F(x_0)+\rho_0]^2}+P_1\left[\frac{(\rho_0+\rho_1)f(x_0+x_1)}{\big[\big(1-(\rho_0+\rho_1)\big)F(x_0+x_1)+(\rho_0+\rho_1)\big]^2}-\frac{\rho_0 f(x_0)}{[(1-\rho_0)F(x_0)+\rho_0]^2}\right]$$
$$+P_2\left[\frac{(\rho_0+\rho_1+\rho_2)f(x_0+x_1+x_2)}{\big[\big(1-(\rho_0+\rho_1+\rho_2)\big)F(x_0+x_1+x_2)+(\rho_0+\rho_1+\rho_2)\big]^2}-\frac{(\rho_0+\rho_1)f(x_0+x_1)}{\big[\big(1-(\rho_0+\rho_1)\big)F(x_0+x_1)+(\rho_0+\rho_1)\big]^2}\right] \qquad (29)$$

Proof

$$T[g(x_P;\rho_P)]=\frac{T[\rho_P]T[f(x_P)]}{\big[T[(1-\rho_P)]T[F(x_P)]+T[\rho_P]\big]^2}$$
$$=\frac{(\rho_0,\rho_0+\rho_1,\rho_0+\rho_1+\rho_2)\big(f(x_0),f(x_0+x_1),f(x_0+x_1+x_2)\big)}{\big[\big(1-\rho_0,1-(\rho_0+\rho_1),1-(\rho_0+\rho_1+\rho_2)\big)\big(F(x_0),F(x_0+x_1),F(x_0+x_1+x_2)\big)+(\rho_0,\rho_0+\rho_1,\rho_0+\rho_1+\rho_2)\big]^2}$$
$$=\frac{\big(\rho_0 f(x_0),(\rho_0+\rho_1)f(x_0+x_1),(\rho_0+\rho_1+\rho_2)f(x_0+x_1+x_2)\big)}{\Big[\Big((1-\rho_0)F(x_0)+\rho_0,\big(1-(\rho_0+\rho_1)\big)F(x_0+x_1)+(\rho_0+\rho_1),\big(1-(\rho_0+\rho_1+\rho_2)\big)F(x_0+x_1+x_2)+(\rho_0+\rho_1+\rho_2)\Big)\Big]^2}$$
$$=\left(\frac{\rho_0 f(x_0)}{[(1-\rho_0)F(x_0)+\rho_0]^2},\frac{(\rho_0+\rho_1)f(x_0+x_1)}{\big[\big(1-(\rho_0+\rho_1)\big)F(x_0+x_1)+(\rho_0+\rho_1)\big]^2},\frac{(\rho_0+\rho_1+\rho_2)f(x_0+x_1+x_2)}{\big[\big(1-(\rho_0+\rho_1+\rho_2)\big)F(x_0+x_1+x_2)+(\rho_0+\rho_1+\rho_2)\big]^2}\right)$$

So:

$$g(x_N;\rho_N)$$
$$=T^{-1}\left(\frac{\rho_0 f(x_0)}{[(1-\rho_0)F(x_0)+\rho_0]^2},\frac{(\rho_0+\rho_1)f(x_0+x_1)}{\big[\big(1-(\rho_0+\rho_1)\big)F(x_0+x_1)+(\rho_0+\rho_1)\big]^2},\frac{(\rho_0+\rho_1+\rho_2)f(x_0+x_1+x_2)}{\big[\big(1-(\rho_0+\rho_1+\rho_2)\big)F(x_0+x_1+x_2)+(\rho_0+\rho_1+\rho_2)\big]^2}\right)$$
$$=\frac{\rho_0 f(x_0)}{[(1-\rho_0)F(x_0)+\rho_0]^2}+P_1\left[\frac{(\rho_0+\rho_1)f(x_0+x_1)}{\big[\big(1-(\rho_0+\rho_1)\big)F(x_0+x_1)+(\rho_0+\rho_1)\big]^2}-\frac{\rho_0 f(x_0)}{[(1-\rho_0)F(x_0)+\rho_0]^2}\right]$$
$$+P_2\left[\frac{(\rho_0+\rho_1+\rho_2)f(x_0+x_1+x_2)}{\big[\big(1-(\rho_0+\rho_1+\rho_2)\big)F(x_0+x_1+x_2)+(\rho_0+\rho_1+\rho_2)\big]^2}-\frac{(\rho_0+\rho_1)f(x_0+x_1)}{\big[\big(1-(\rho_0+\rho_1)\big)F(x_0+x_1)+(\rho_0+\rho_1)\big]^2}\right]$$

Theorem 5.3

Equation (29) represents probability density function in classical sense.

Proof

$$T\left[\int_{-\infty}^{+\infty} g(x_P;\rho_P)dx_P\right]$$

$$= \int_{-\infty}^{+\infty} \frac{\rho_0 f(x_0)}{\left[(1-\rho_0)F(x_0)+\rho_0\right]^2}dx_0$$

$$+ P_1\left[\int_{-\infty}^{+\infty} \frac{(\rho_0+\rho_1)f(x_0+x_1)}{\left[\left(1-(\rho_0+\rho_1)\right)F(x_0+x_1)+(\rho_0+\rho_1)\right]^2}d(x_0+x_1)\right.$$

$$\left.- \int_{-\infty}^{+\infty} \frac{\rho_0 f(x_0)}{\left[(1-\rho_0)F(x_0)+\rho_0\right]^2}dx_0\right]$$

$$+ P_2\left[\int_{-\infty}^{+\infty} \frac{(\rho_0+\rho_1+\rho_2)f(x_0+x_1+x_2)}{\left[(1-(\rho_0+\rho_1+\rho_2))F(x_0+x_1+x_2)+(\rho_0+\rho_1+\rho_2)\right]^2}d(x_0+x_1+x_2)\right.$$

$$\left.- \int_{-\infty}^{+\infty} \frac{(\rho_0+\rho_1)f(x_0+x_1)}{\left[\left(1-(\rho_0+\rho_1)\right)F(x_0+x_1)+(\rho_0+\rho_1)\right]^2}d(x_0+x_1)\right]$$

$$= \int_{-\infty}^{+\infty} d\left(\frac{F(x_0)}{F(x_0)(1-\rho_0)+\rho_0}\right)$$

$$+ P_1\left[\int_{-\infty}^{+\infty} d\left(\frac{F(x_0)}{F(x_0)(1-\rho_0)+\rho_0}\right) - \int_{-\infty}^{+\infty} d\left(\frac{F(x_0+x_1)}{F(x_0+x_1)\left(1-(\rho_0+\rho_1)\right)+(\rho_0+\rho_1)}\right)\right]$$

$$+ P_2\left[\int_{-\infty}^{+\infty} d\left(\frac{F(x_0+x_1+x_2)}{F(x_0+x_1+x_2)\left(1-(\rho_0+\rho_1+\rho_2)\right)+(\rho_0+\rho_1+\rho_2)}\right)\right.$$

$$\left.- \int_{-\infty}^{+\infty} d\left(\frac{F(x_0+x_1)}{F(x_0+x_1)\left(1-(\rho_0+\rho_1)\right)+(\rho_0+\rho_1)}\right)\right] = 1$$

Also, it is easy to see that $T[g(x_P;\rho_P)]$ presents three continuous and positive functions. Depending on this we can see that the given plithogenic function is a plithogenic probability density function in classical sense.

6. Plithogenic Marshall-Olkin Type I Uniform distribution:

Definition 6.1

Plithogenic cumulative distribution function of the Marshall-Olkin Type I uniform distribution is defined as follows:

$$G(x_P;\rho_P,a_P,b_P) = \frac{x_P-a_P}{x_P(1-\rho_P)+\rho_P b_P - a_P};\, a_P <_P x_P <_P b_P\,, 0 <_P \rho_P <_P 1 \quad (30)$$

Where $P_1^2 = P_1, P_2^2 = P_2, P_1 \cdot P_2 = P_2 \cdot P_1 = P_2$.

Theorem 6.1

The plithogenic formal form of (30) is:

$$G(x_P;\rho_P,a_P,b_P)=\frac{x_0-a_0}{x_0(1-\rho_0)+\rho_0 b_0-a_0}$$
$$+P_1\left[\frac{(x_0+x_1)-(a_0+a_1)}{(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)}-\frac{x_0-a_0}{x_0(1-\rho_0)+\rho_0 b_0-a_0}\right]$$
$$+P_2\left[\frac{(x_0+x_1+x_2)-(a_0+a_1+a_2)}{(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)}\right.$$
$$\left.-\frac{(x_0+x_1)-(a_0+a_1)}{(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)}\right] \quad (31)$$

Proof

$$T[G(x_P;\rho_P,a_P,b_P)]=\frac{T[x_P]-T[a_P]}{T[x_P]T[1-\rho_P]+T[\rho_P]\cdot T[b_P]-T[a_P]}$$

$$=\frac{(x_0,x_0+x_1,x_0+x_1+x_2)-(a_0,a_0+a_1,a_0+a_1+a_2)}{(x_0,x_0+x_1,x_0+x_1+x_2)\big(1-\rho_0,1-(\rho_0+\rho_1),1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0,\rho_0+\rho_1,\rho_0+\rho_1+\rho_2)\cdot(b_0,b_0+b_1,b_0+b_1+b_2)-(a_0,a_0+a_1,a_0+a_1+a_2)}$$

$$=\left(\frac{\big(x_0-a_0,(x_0+x_1)-(a_0+a_1),(x_0+x_1+x_2)-(a_0+a_1+a_2)\big)}{\big(x_0(1-\rho_0)+\rho_0 b_0-a_0,(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1),(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)\big)}\right)$$

$$=\left(\frac{x_0-a_0}{x_0(1-\rho_0)+\rho_0 b_0-a_0},\frac{(x_0+x_1)-(a_0+a_1)}{(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)},\right.$$
$$\left.\frac{(x_0+x_1+x_2)-(a_0+a_1+a_2)}{(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)}\right)$$

$$G(x_P;\rho_P,a_P,b_P)=T^{-1}\left(\frac{x_0-a_0}{x_0(1-\rho_0)+\rho_0 b_0-a_0},\frac{(x_0+x_1)-(a_0+a_1)}{(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)},\right.$$
$$\left.\frac{(x_0+x_1+x_2)-(a_0+a_1+a_2)}{(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)}\right)$$

$$=\frac{x_0-a_0}{x_0(1-\rho_0)+\rho_0 b_0-a_0}+P_1\left[\frac{(x_0+x_1)-(a_0+a_1)}{(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)}-\frac{x_0-a_0}{x_0(1-\rho_0)+\rho_0 b_0-a_0}\right]$$
$$+P_2\left[\frac{(x_0+x_1+x_2)-(a_0+a_1+a_2)}{(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)}\right.$$
$$\left.-\frac{(x_0+x_1)-(a_0+a_1)}{(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)}\right]$$

Definition 6.2

Plithogenic probability distribution function of Marshall-Olkin Type I uniform distribution is defined as follows:

$$g(x_P;\rho_P,a_P,b_P)=\frac{(b_P-a_P)\rho_P}{\big[x_P(1-\rho_P)+\rho_P b_p-a_P\big]^2};a_P<_P x_P<_P b_P\,,0<_P\rho_P<_P 1 \quad (32)$$

Where $P_1^2=P_1, P_2^2=P_2, P_1\cdot P_2=P_2\cdot P_1=P_2$.

Theorem 6.2

Plithogenic formal form of (32) is:

$$g(x_P;\rho_P,a_P,b_P)=\frac{(b_0-a_0)\rho_0}{[x_0(1-\rho_0)+\rho_0 b_0-a_0]^2}+P_1\left[\frac{\big((b_0+b_1)-(a_0+a_1)\big)(\rho_0+\rho_1)}{\big[(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)\big]^2}-\frac{(b_0-a_0)\rho_0}{[x_0(1-\rho_0)+\rho_0 b_0-a_0]^2}\right]+P_2\left[\frac{\big((b_0+b_1+b_2)-(a_0+a_1+a_2)\big)(\rho_0+\rho_1+\rho_2)}{\big[(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)\big]^2}-\frac{\big((b_0+b_1)-(a_0+a_1)\big)(\rho_0+\rho_1)}{\big[(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)\big]^2}\right] \tag{33}$$

Proof

$$T[g(x_P;\rho_P,a_P,b_P)]=\frac{(T[b_P]-T[a_P])T[\rho_P]}{\big[T[x_P]T[(1-\rho_P)]+T[\rho_P]\cdot T[b_P]-T[a_P]\big]^2}$$

$$=\frac{\big((b_0,b_0+b_1,b_0+b_1+b_2)-(a_0,a_0+a_1,a_0+a_1+a_2)\big)(\rho_0,\rho_0+\rho_1,\rho_0+\rho_1+\rho_2)}{\big[(x_0,x_0+x_1,x_0+x_1+x_2)\big(1-\rho_0,1-(\rho_0+\rho_1),1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0,\rho_0+\rho_1,\rho_0+\rho_1+\rho_2)\cdot(b_0,b_0+b_1,b_0+b_1+b_2)-(a_0,a_0+a_1,a_0+a_1+a_2)\big]^2}$$

$$=\frac{\Big((b_0-a_0)\rho_0,\big((b_0+b_1)-(a_0+a_1)\big)(\rho_0+\rho_1),\big((b_0+b_1+b_2)-(a_0+a_1+a_2)\big)(\rho_0+\rho_1+\rho_2)\Big)}{\Big[\Big(x_0(1-\rho_0)+\rho_0 b_0-a_0,(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1),(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)\Big)\Big]^2}$$

$$=\left(\frac{(b_0-a_0)\rho_0}{[x_0(1-\rho_0)+\rho_0 b_0-a_0]^2},\frac{\big((b_0+b_1)-(a_0+a_1)\big)(\rho_0+\rho_1)}{\big[(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)\big]^2},\frac{\big((b_0+b_1+b_2)-(a_0+a_1+a_2)\big)(\rho_0+\rho_1+\rho_2)}{\big[(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)\big]^2}\right)$$

So:

$$g(x_P;\rho_P,a_P,b_P)=T^{-1}\left(\frac{(b_0-a_0)\rho_0}{[x_0(1-\rho_0)+\rho_0 b_0-a_0]^2},\frac{\big((b_0+b_1)-(a_0+a_1)\big)(\rho_0+\rho_1)}{\big[(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)\big]^2},\frac{\big((b_0+b_1+b_2)-(a_0+a_1+a_2)\big)(\rho_0+\rho_1+\rho_2)}{\big[(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)\big]^2}\right)$$

$$=\frac{(b_0-a_0)\rho_0}{[x_0(1-\rho_0)+\rho_0 b_0-a_0]^2}+P_1\left[\frac{\big((b_0+b_1)-(a_0+a_1)\big)(\rho_0+\rho_1)}{\big[(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)\big]^2}-\frac{(b_0-a_0)\rho_0}{[x_0(1-\rho_0)+\rho_0 b_0-a_0]^2}\right]+P_2\left[\frac{\big((b_0+b_1+b_2)-(a_0+a_1+a_2)\big)(\rho_0+\rho_1+\rho_2)}{\big[(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)\big]^2}-\frac{\big((b_0+b_1)-(a_0+a_1)\big)(\rho_0+\rho_1)}{\big[(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)\big]^2}\right]$$

6.1 Parameters' estimation of plithogenic Marshall-Olkin Type I Uniform distribution:

Let $\mathbb{X}_P$ a plithogenic random sample drawn from plithogenic Marshall-Olkin Type I uniform distribution with PDF defined in (32) then the plithogenic likelihood function will be:

$$L_P = L(\mathbb{X}_P;\Theta) = \prod_{i=1}^{n} f(X_{iP};a_P,b_P,\rho_P) = \prod_{i=1}^{n} \frac{(b_P - a_P)\rho_P}{[x_{iP}(1-\rho_P) + \rho_P b_P - a_P]^2}$$

$$= \frac{(b_P - a_P)^n \rho_P^n}{\prod_{i=1}^{n}[x_{iP}(1-\rho_P) + \rho_P b_P - a_P]^2} \quad (34)$$

By taking log of (34), we get the loglikelihood function as follows:

$$\mathcal{L}_P = \ln L(\mathbb{X}_P;\Theta) = n\ln(b_P - a_P) + n\ln\rho_P - 2\sum_{i=1}^{n}\ln[x_{iP}(1-\rho_P) + \rho_P b_P - a_P] \quad (35)$$

Taking partial derivatives of previous equation according to a_P, b_P, ρ_P yields to:

$$\frac{\partial}{\partial a_P}\mathcal{L}_P = \frac{-n}{b_P - a_P} + 2\sum_{i=1}^{n}\frac{1}{[x_{iP}(1-\rho_P) + \rho_P b_P - a_P]} \quad (36)$$

$$\frac{\partial}{\partial b_P}\mathcal{L}_P = \frac{n}{b_P - a_P} - 2\sum_{i=1}^{n}\frac{\rho_P}{[x_{iP}(1-\rho_P) + \rho_P b_P - a_P]} \quad (37)$$

$$\frac{\partial}{\partial a_P}\mathcal{L}_P = \frac{n}{\rho_P} + 2\sum_{i=1}^{n}\frac{x_{iP} - b_P}{[x_{iP}(1-\rho_P) + \rho_P b_P - a_P]} \quad (38)$$

Using the AH-Isometry equations (36-38) become:

$$\frac{\partial}{\partial a_0}\mathcal{L}_0 = \frac{-n}{b_0 - a_0} + 2\sum_{i=1}^{n}\frac{1}{[x_{i0}(1-\rho_0) + \rho_0 b_0 - a_0]}$$

$$\frac{\partial}{\partial(a_0+a_1)}(\mathcal{L}_0+\mathcal{L}_1) = \frac{-n}{(b_0+b_1)-(a_0+a_1)} + 2\sum_{i=1}^{n}\frac{1}{\left[(x_{i0}+x_{i1})\big(1-(\rho_0+\rho_1)\big) + (\rho_0+\rho_1)(b_0+b_1) - (a_0+a_1)\right]}$$

$$\frac{\partial}{\partial(a_0+a_1+a_2)}(\mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2)$$

$$= \frac{-n}{(b_0+b_1+b_2)-(a_0+a_1+a_2)}$$

$$+ 2\sum_{i=1}^{n}\frac{1}{\left[(x_{i0}+x_{i1}+x_{i2})\big(1-(\rho_0+\rho_1+\rho_2)\big) + (\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2) - (a_0+a_1+a_2)\right]}$$

$$\frac{\partial}{\partial b_0}\mathcal{L}_0 = \frac{n}{b_0 - a_0} - 2\sum_{i=1}^{n}\frac{\rho_0}{[x_{i0}(1-\rho_0) + \rho_0 b_0 - a_0]}$$

$$\frac{\partial}{\partial(b_0+b_1)}(\mathcal{L}_0+\mathcal{L}_1) = \frac{n}{(b_0+b_1)-(a_0+a_1)} - 2\sum_{i=1}^{n}\frac{(\rho_0+\rho_1)}{\left[(x_{i0}+x_{i1})\big(1-(\rho_0+\rho_1)\big) + (\rho_0+\rho_1)(b_0+b_1) - (a_0+a_1)\right]}$$

$$\frac{\partial}{\partial(b_0+b_1+b_2)}(\mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2)$$
$$=\frac{n}{(b_0+b_1+b_2)-(a_0+a_1+a_2)}$$
$$-2\sum_{i=1}^{n}\frac{(\rho_0+\rho_1+\rho_2)}{\left[(x_{i0}+x_{i1}+x_{i2})\left(1-(\rho_0+\rho_1+\rho_2)\right)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)\right]}$$

$$\frac{\partial}{\partial\rho_0}\mathcal{L}_0=\frac{n}{\rho_0}+2\sum_{i=1}^{n}\frac{x_{i0}-b_0}{[x_{i0}(1-\rho_0)+\rho_0 b_0-a_0]}$$

$$\frac{\partial}{\partial(\rho_0+\rho_1)}(\mathcal{L}_0+\mathcal{L}_1)=\frac{n}{(\rho_0+\rho_1)}+2\sum_{i=1}^{n}\frac{(x_{i0}+x_{i1})-(b_0+b_1)}{\left[(x_{i0}+x_{i1})\left(1-(\rho_0+\rho_1)\right)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)\right]}$$

$$\frac{\partial}{\partial(\rho_0+\rho_1+\rho_2)}(\mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2)$$
$$=\frac{n}{(\rho_0+\rho_1+\rho_2)}$$
$$+2\sum_{i=1}^{n}\frac{(x_{i0}+x_{i1}+x_{i2})-(\rho_0+\rho_1+\rho_2)}{\left[(x_{i0}+x_{i1}+x_{i2})\left(1-(\rho_0+\rho_1+\rho_2)\right)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)\right]}$$

Solving previous equations numerically give us the desired estimations.

6.2 Simulation and random numbers generating:

Random numbers generating can be done using the following equation:

$$x_P=\frac{y_P\rho_P b_P-a_P y_P+a_P}{1-y_P(1-\rho_P)} \qquad (39)$$

By taking AH-isometry to (39) we get:

$$x_0=\frac{y_0\rho_0 b_0-a_0 y_0+a_0}{1-y_0(1-\rho_0)} \qquad (40)$$

$$x_0+x_1=\frac{(y_0+y_1)(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)(y_0+y_1)+(a_0+a_1)}{1-(y_0+y_1)\left(1-(\rho_0+\rho_1)\right)} \qquad (41)$$

$$x_0+x_1+x_2$$
$$=\frac{(y_0+y_1+y_2)(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)(y_0+y_1+y_2)+(a_0+a_1+a_2)}{1-(y_0+y_1+y_2)\left(1-(\rho_0+\rho_1+\rho_2)\right)} \qquad (42)$$

By using equations (40-42), we can generate random numbers following classical Marshall-Olkin type I uniform distribution, then using T^{-1} we will get plithogenic Marshall-Olkin type I uniform distribution generated numbers.

performance of maximum likelihood estimators based on Monte Carlo simulation using Maple software with total replication of $N=1000$ times and with sample size of $15, 50, 100, 150$ with fixed parameters $a_P=1.5+0.3P_1+0.5P_2, b_P=2+0.5P_1+1.2P_2, \rho_P=1+0.7P_1-0.4P_2$ is checked based on bias of the estimators and mean square error of it using the following equations:

$$Bias=\frac{\sum_{i=1}^{n}\left|\hat{\theta}_{iP}-\theta_P\right|}{n} \qquad (43)$$

$$MSE = \frac{\sum_{i=1}^{n}\left(\hat{\theta}_{iP} - \theta_P\right)^2}{n} \quad (44)$$

Table 2. Simulation results of plithogenic Marshall-Olkin type I uniform distribution.

	$a_P = 1.5 + 0.3P_1 + 0.5P_2$		
n	$\hat{a}_P$	$Bias\ \hat{a}_P$	$MSE\ \hat{a}_P$
15	$1.53060 + 0.33686P_1 + 0.54019P_2$	$0.03060 + 0.03686P_1 + 0.04025P_2$	$0.00179 + 0.00643P_1 + 0.01337P_2$
50	$1.50975 + 0.31286P_1 + 0.51248P_2$	$0.00975 + 0.01286P_1 + 0.01248P_2$	$0.00019 + 0.00079P_1 + 0.00142P_2$
100	$1.50495 + 0.30667P_1 + 0.50629P_2$	$0.00495 + 0.00667P_1 + 0.00629P_2$	$0.00005 + 0.00021P_1 + 0.00036P_2$
150	$1.50337 + 0.30457P_1 + 0.50427P_2$	$0.00337 + 0.00457P_1 + 0.00427P_2$	$0.00002 + 0.00010P_1 + 0.00017P_2$
	$b_P = 2 + 0.5P_1 + 1.2P_2$		
n	$\hat{b}_P$	$Bias\ \hat{b}_P$	$MSE\ \hat{b}_P$
15	$1.96711 + 0.50424P_1 + 1.15560P_2$	$0.03289 - 0.00424P_1 + 0.04441P_2$	$0.00212 - 0.00043P_1 + 0.00904P_2$
50	$1.99035 + 0.50158P_1 + 1.18710P_2$	$0.00965 - 0.00157P_1 + 0.01289P_2$	$0.00019 - 0.00006P_1 + 0.00076P_2$
100	$1.99499 + 0.50085P_1 + 1.19331P_2$	$0.00501 - 0.00085P_1 + 0.00668P_2$	$0.00005 - 0.00002P_1 + 0.00019P_2$
150	$1.99679 + 0.50055P_1 + 1.19572P_2$	$0.00321 - 0.00055P_1 + 0.00428P_2$	$0.00002 - 0.00001P_1 + 0.00009P_2$
	$\rho_P = 1 + 0.7P_1 - 0.4P_2$		
n	$\hat{\rho}_P$	$Bias\ \hat{\rho}_P$	$MSE\ \hat{\rho}_P$
15	$1.11156 + 0.65361P_1 - 0.36750P_2$	$0.41604 + 0.25464P_1 - 0.14580P_2$	$0.34855 + 0.50312P_1 - 0.31230P_2$
50	$1.04020 + 0.69063P_1 - 0.39240P_2$	$0.20518 + 0.13733P_1 - 0.07870P_2$	$0.07171 + 0.12394P_1 - 0.07800P_2$
100	$1.02096 + 0.69566P_1 - 0.39636P_2$	$0.14470 + 0.09968P_1 - 0.05702P_2$	$0.03574 + 0.06449P_1 - 0.04076P_2$
150	$1.01753 + 0.69967P_1 - 0.39901P_2$	$0.11841 + 0.08157P_1 - 0.04664P_2$	$0.02270 + 0.04158P_1 - 0.02633P_2$

Table (2) shows that as sample size n increases, bias of the estimators and mean square error of it decrease which means that our estimators are asymptotically unbiased and consistent.

7. Conclusions

We have studied and derived neutrosophic Marshall Olkin (I) class of distributions and plithogenic Marshall Olkin (I) class of distribution and found its cumulative distribution functions and probability distribution functions. Also, we studied a special case of these new classes that is uniformly generalized distribution and estimated its parameters using maximum likelihood estimation method and made a simulation study to show the power and efficiency of our estimators and the simulation results show that our estimators are unbiased and consistent. In future researches we are looking forward to study more special distributions generalized by Marshall Olkin class and study its applications in reliability theory and queueing theory.

Funding: "This research received no external funding".

Acknowledgments: We are very grateful to editor in chief and reviewers for their hard work and special thanks to the guest editor Mohammad Abobala.

Conflicts of Interest: "The authors declare no conflict of interest."

References

[1] A. Astambli, M. B. Zeina, and Y. Karmouta, "On Some Estimation Methods of Neutrosophic Continuous Probability Distributions Using One-Dimensional AH-Isometry," *Neutrosophic Sets and Systems*, vol. 53, 2023.

[2] M. B. Zeina, M. Abobala, A. Hatip, S. Broumi, and S. Jalal Mosa, "Algebraic Approach to Literal Neutrosophic Kumaraswamy Probability Distribution," *Neutrosophic Sets and Systems*, vol. 54, pp. 124–138, 2023.

[3] M. B. Zeina and M. Abobala, "A novel approach of neutrosophic continuous probability distributions using AH-isometry with applications in medicine," *Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics*, pp. 267–286, Jan. 2023, doi: 10.1016/B978-0-323-99456-9.00014-3.

[4] M. B. Zeina and Y. Karmouta, "Introduction to Neutrosophic Stochastic Processes," *Neutrosophic Sets and Systems*, vol. 54, 2023.

[5] M. Mullai, K. Sangeetha, R. Surya, G. M. Kumar, R. Jeyabalan, and S. Broumi, "A Single Valued neutrosophic Inventory Model with Neutrosophic Random Variable," *International Journal of Neutrosophic Science*, vol. 1, no. 2, 2020, doi: 10.5281/zenodo.3679510.

[6] F. Smarandache, "Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability," *ArXiv*, 2013.

[7] F. Smarandache, "Introduction to Neutrosophic Statistics," *Branch Mathematics and Statistics Faculty and Staff Publications*, Jan. 2014, Accessed: Feb. 21, 2023. [Online]. Available: https://digitalrepository.unm.edu/math_fsp/33

[8] F. Smarandache, "Symbolic Neutrosophic Theory," *ArXiv*, 2015, doi: 10.5281/ZENODO.32078.

[9] F. Smarandache, "Indeterminacy in neutrosophic theories and their applications," *International Journal of Neutrosophic Science*, vol. 15, no. 2, 2021, doi: 10.5281/zenodo.5295819.

[10] K. F. Alhasan, A. A. Salama, and F. Smarandache, "Introduction to neutrosophic reliability theory," *International Journal of Neutrosophic Science*, vol. 15, no. 1, 2021, doi: 10.5281/zenodo.5033829.

[11] C. Granados and J. Sanabria, "On Independence Neutrosophic Random Variables," *Neutrosophic Sets and Systems*, vol. 47, 2021.

[12] C. Granados, "New Notions On Neutrosophic Random Variables," *Neutrosophic Sets and Systems*, vol. 47, 2021.

[13] R. A. K. Sherwani, M. Naeem, M. Aslam, M. A. Raza, M. Abid, and S. Abbas, "Neutrosophic Beta Distribution with Properties and Applications," *Neutrosophic Sets and Systems*, vol. 41, 2021.

[14] M. B. Zeina, "Neutrosophic M/M/1, M/M/c, M/M/1/b Queueing Systems," *Research Journal of Aleppo University*, vol. 140, 2020, Accessed: Feb. 21, 2023. [Online]. Available: https://www.researchgate.net/publication/343382302_Neutrosophic_MM1_MMc_MM1b_Queueing_Systems

[15] M. B. Zeina, "Linguistic Single Valued Neutrosophic M/M/1 Queue," *Research Journal of Aleppo University*, vol. 144, 2021, Accessed: Feb. 21, 2023. [Online]. Available:

https://www.researchgate.net/publication/348945390_Linguistic_Single_Valued_Neutrosophic_MM1_Queue

[16] A. Astambli, M. B. Zeina, and Y. Karmouta, "Algebraic Approach to Neutrosophic Confidence Intervals," *Journal of Neutrosophic and Fuzzy Systems*, vol. 5, no. 2, pp. 08–22, 2023, doi: 10.54216/JNFS.050201.

[17] R. Yong, J. Ye, and S. Du, "Multicriteria Decision-Making Method and Application in the Setting of Trapezoidal Neutrosophic Z-Numbers," *Journal of Mathematics*, vol. 2021, 2021, doi: 10.1155/2021/6664330.

[18] H. Y. Zhang, J. Q. Wang, and X. H. Chen, "Interval neutrosophic sets and their application in multicriteria decision making problems," *The Scientific World Journal*, vol. 2014, 2014, doi: 10.1155/2014/645953.

[19] Z. Khan, M. Gulistan, N. Kausar, and C. Park, "Neutrosophic Rayleigh Model with Some Basic Characteristics and Engineering Applications," *IEEE Access*, vol. 9, pp. 71277–71283, 2021, doi: 10.1109/ACCESS.2021.3078150.

[20] F. Shah, M. Aslam, Z. Khan, M. M. A. Almazah, and F. S. Alduais, "On Neutrosophic Extension of the Maxwell Model: Properties and Applications," *Journal of Function Spaces*, vol. 2022, 2022, doi: 10.1155/2022/4536260.

[21] D. Nagarajan and J. Kavikumar, "Single-Valued and Interval-Valued Neutrosophic Hidden Markov Model," *Math Probl Eng*, vol. 2022, 2022, doi: 10.1155/2022/5323530.

[22] Z. Khan, A. Al-Bossly, M. M. A. Almazah, and F. S. Alduais, "On Statistical Development of Neutrosophic Gamma Distribution with Applications to Complex Data Analysis," *Complexity*, vol. 2021, 2021, doi: 10.1155/2021/3701236.

[23] M. Jamil, F. Afzal, D. Afzal, D. K. Thapa, and A. Maqbool, "Multicriteria Decision-Making Methods Using Bipolar Neutrosophic Hamacher Geometric Aggregation Operators," *Journal of Function Spaces*, vol. 2022, 2022, doi: 10.1155/2022/5052867.

[24] R. A. K. Sherwani, M. Aslam, M. A. Raza, M. Farooq, M. Abid, and M. Tahir, "Neutrosophic Normal Probability Distribution—A Spine of Parametric Neutrosophic Statistical Tests: Properties and Applications," *Neutrosophic Operational Research*, pp. 153–169, 2021, doi: 10.1007/978-3-030-57197-9_8.

[25] M. Abobala and M. B. Zeina, "A Study of Neutrosophic Real Analysis by Using the One-Dimensional Geometric AH-Isometry," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 3, no. 1, pp. 18–24, 2023, doi: 10.54216/GJMSA.030103).

[26] M. Abobala and A. Hatip, "An Algebraic Approach to Neutrosophic Euclidean Geometry," *Neutrosophic Sets and Systems*, vol. 43, 2021.

[27] M. B. Zeina, "Erlang Service Queueing Model with Neutrosophic Parameters," *International Journal of Neutrosophic Science*, vol. 6, no. 2, pp. 106–112, 2020, doi: 10.54216/IJNS.060202.

[28] M. B. Zeina, "Neutrosophic Event-Based Queueing Model," *International Journal of Neutrosophic Science*, vol. 6, no. 1, 2020, doi: 10.5281/zenodo.3840771.

[29] M. B. Zeina, O. Zeitouny, F. Masri, F. Kadoura, and S. Broumi, "Operations on single-valued trapezoidal neutrosophic numbers using (α, β, γ)-cuts 'maple package,'" *International Journal of Neutrosophic Science*, vol. 15, no. 2, 2021, doi: 10.54216/IJNS.150205.

[30] M. B. Zeina and A. Hatip, "Neutrosophic Random Variables," *Neutrosophic Sets and Systems*, vol. 39, 2021, doi: 10.5281/zenodo.4444987.

[31] S. Broumi, M. B. Zeina, M. Lathamaheswari, A. Bakali, and M. Talea, "A Maple Code to Perform Operations on Single Valued Neutrosophic Matrices," *Neutrosophic Sets and Systems*, vol. 49, 2022.

[32] F. Smarandache, *Plithogeny, Plithogenic Set, Logic, Probability, and Statistics*. Belgium: Pons, 2018. Accessed: Feb. 23, 2023. [Online]. Available: http://arxiv.org/abs/1808.03948

[33] P. K. Singh, "Complex Plithogenic Set," *International Journal of Neutrosophic Science*, vol. 18, no. 1, 2022, doi: 10.54216/IJNS.180106.

[34] S. Alkhazaleh, "Plithogenic Soft Set," *Neutrosophic Sets and Systems*, 2020.

[35] N. Martin and F. Smarandache, "Introduction to Combined Plithogenic Hypersoft Sets," *Neutrosophic Sets and Systems*, vol. 35, 2020, doi: 10.5281/zenodo.3951708.

[36] F. Smarandache, "Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)," *Neutrosophic Sets and Systems* , vol. 53, Jan. 2023, doi: 10.5281/ZENODO.7536105.

[37] F. Smarandache, "Introduction to Plithogenic Logic as generalization of MultiVariate Logic," *Neutrosophic Sets and Systems*, vol. 45, 2021.

[38] F. Smarandache, "Plithogenic Probability & Statistics are generalizations of MultiVariate Probability & Statistics," *Neutrosophic Sets and Systems*, vol. 43, 2021.

[39] F. Smarandache, "Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets," *Neutrosophic Sets and Systems*, vol. 21, pp. 153–166, 2018.

[40] M. Abdel-Basset and R. Mohamed, "A novel plithogenic TOPSIS- CRITIC model for sustainable supply chain risk management," *J Clean Prod*, vol. 247, Feb. 2020, doi: 10.1016/J.JCLEPRO.2019.119586.

[41] M. Abdel-Basset, M. El-hoseny, A. Gamal, and F. Smarandache, "A novel model for evaluation Hospital medical care systems based on plithogenic sets," *Artif Intell Med*, vol. 100, Sep. 2019, doi: 10.1016/J.ARTMED.2019.101710.

[42] F. Sultana *et al.*, "A study of plithogenic graphs: applications in spreading coronavirus disease (COVID-19) globally," *J Ambient Intell Humaniz Comput*, p. 1, 2022, doi: 10.1007/S12652-022-03772-6.

[43] M. Abdel-Basset, R. Mohamed, A. E. N. H. Zaied, and F. Smarandache, "A hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics," *Symmetry (Basel)*, vol. 11, no. 7, Jul. 2019, doi: 10.3390/SYM11070903.

[44] N. M. Taffach and A. Hatip, "A Review on Symbolic 2-Plithogenic Algebraic Structures," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 5, no. 1, pp. 08–16, 2023, doi: 10.54216/GJMSA.050101.

[45] R. Ali and Z. Hasan, "An Introduction To The Symbolic 3-Plithogenic Modules," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 6, no. 1, pp. 13–17, 2023, doi: 10.54216/GJMSA.060102.

[46] R. Ali and Z. Hasan, "An Introduction to The Symbolic 3-Plithogenic Vector Spaces," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 6, no. 1, pp. 08–12, 2023, doi: 10.54216/GJMSA.060101.

[47] N. M. Taffach and A. Hatip, "A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 5, no. 1, pp. 36–44, 2023, doi: 10.54216/GJMSA.050103.

[48] R. P. DA SILVA, A. H. M. A. Cysneiros, G. M. Cordeiro, and C. J. Tablada, "The transmuted Marshall-olkin extended lomax distribution," *An Acad Bras Cienc*, vol. 92, no. 3, 2020, doi: 10.1590/0001-3765202020180777.

[49] S. Nasiru and A. G. Abubakari, "Marshall-Olkin Zubair-G Family of Distributions," *Pakistan Journal of Statistics and Operation Research*, vol. 18, no. 1, 2022, doi: 10.18187/pjsor.v18i1.3096.

[50] M. Javed, T. Nawaz, and M. Irfan, "The Marshall-Olkin Kappa distribution: Properties and applications," *J King Saud Univ Sci*, vol. 31, no. 4, 2019, doi: 10.1016/j.jksus.2018.01.001.

[51] H. A. H. Ahmad and E. M. Almetwally, "Marshall-Olkin generalized pareto distribution: Bayesian and non bayesian estimation," *Pakistan Journal of Statistics and Operation Research*, vol. 16, no. 1, 2020, doi: 10.18187/PJSOR.V16I1.2935.

[52] M. A. U. Haq, A. Z. Afify, H. Al-Mofleh, R. M. Usman, M. Alqawba, and A. M. Sarg, "The Extended Marshall-Olkin Burr III Distribution: Properties and Applications," *Pakistan Journal of Statistics and Operation Research*, vol. 17, no. 1, 2021, doi: 10.18187/pjsor.v17i1.3649.

[53] M. A. ul Haq, R. M. Usman, S. Hashmi, and A. I. Al-Omeri, "The Marshall-Olkin length-biased exponential distribution and its applications," *J King Saud Univ Sci*, vol. 31, no. 2, 2019, doi: 10.1016/j.jksus.2017.09.006.

[54] M. A. Khaleel, P. E. Oguntunde, J. N. A. Abbasi, N. A. Ibrahim, and M. H. A. AbuJarad, "The Marshall-Olkin Topp Leone-G family of distributions: A family for generalizing probability models," *Sci Afr*, vol. 8, 2020, doi: 10.1016/j.sciaf.2020.e00470.

[55] E. M. Almetwally, M. A. H. Sabry, R. Alharbi, D. Alnagar, S. A. M. Mubarak, and E. H. Hafez, "Marshall-Olkin Alpha Power Weibull Distribution: Different Methods of Estimation Based on Type-I and Type-II Censoring," *Complexity*, vol. 2021, 2021, doi: 10.1155/2021/5533799.

Received: 12/7/2023, Accepted: 11/10/2023

Neutrosophic Sets and Systems, Vol. 59, 2023
University of New Mexico

On The Algebraic Properties of Symbolic 6-Plithogenic Integers

[1.] **Mohamed Soueycatt,** [2.] **Barbara Charchekhandra,**[3.] **Rashel Abu Hakmeh**

[1.] Department of Bioengineering, Al-Andalus Private University for Medical Sciences, Syria; **m.soueycatt55@au.edu.sy**

[2.] Jadavpur University, Department of Mathematics, Kolkata, India; Charchekhandrabar32@yahoo.com

[3.] Faculty of Science, Mutah University, Jordan

Abstract: This paper is dedicated to study the properties of symbolic 6-plithogenic integers and number theory, where we present many numbers theoretical concepts such as symbolic 6-plithogenic congruencies, symbolic 6-plithogenic Diophantine equations, and symbolic 6-plithogenic Euler's function with Euclidean division. Also, we present many examples to explain the validity and the scientific contribution of our work.

Keywords: symbolic 6-plithogenic integer, symbolic 6-plithogenic congruencies, symbolic 6-plithogenic division.

Introduction

Symbolic n-plithogenic sets were defined for the first time by Smarandache in [4, 24-25], with many interesting algebraic properties.

In [1-3], the symbolic 2-plithogenic rings were defined as an extension of classical rings. Many results were obtained with respect to their ideals and homomorphisms. The symbolic 2-plithogenic rings and fields have many applications in generalizing other algebraic structures such as symbolic 2-plithogenic vector spaces, symbolic 2-plithogenic modules, and symbolic 2-plithogenic equations [5-7].

zenodo.10031233/10.5281

Laterally, many authors defined and studied symbolic 3-plithogenic algebraic structures, such as symbolic 3-plithogenic spaces and modules, see [8, 21-23].

In the literature, the extended integer systems were used in number theory, for example neutrosophic numbers have helped with neutrosophic number theory, refined neutrosophic numbers generated refined number theory and split-complex numbers generated split-complex number theory [9-20].

This has motivated many authors to study symbolic 2-plithogenic and symbolic 3-plithogenic number theoretical concepts such as congruencies, and Diophantine equations [26-36]. The generalized versions of number theoretical concepts are very applicable in other mathematical studies, especially in cryptography.

In this paper, we study the symbolic 6-plithogenic number theoretical concepts for the first time, and we illustrated many examples to clarify the novel approach.

Main discussion

Definition:

The rung of symbolic 6-plithogenic integer is defined as follows:

$6-SP_Z=\{x_0+\sum_{i=1}^{6} x_i P_i; x_i \in Z\}$, where $P_i \times P_j = p_{max(i,j)}, {P_i}^2 = P_i$.

Definition.

Let $X=x_0+\sum_{i=1}^{6} x_i P_i, Y=y_0+\sum_{i=1}^{6} y_i P_i, Z=z_0+\sum_{i=1}^{6} z_i P_i \in 6-SP_Z$, we say that:

1). $X \setminus Y$ if there exists $Z \in 6-SP_Z$ such that $X.Z=Y$.

2). $X \equiv Y(mod\ Z)$ if $Z \setminus X-Y$.

3). $Z=gcd(X,Y)$ if $Z \setminus X, Z \setminus Y$ and if $T \setminus X, T \setminus Y$, then $T \setminus Z$.

4). X, Y are realtively prime if $gcd(X,Y)=1$.

Theorem1.

Let $X=x_0+\sum_{i=1}^{6} x_i P_i, Y=y_0+\sum_{i=1}^{6} y_i P_i, Z=z_0+\sum_{i=1}^{6} z_i P_i \in 6-SP_Z$, then:

1). $Z=gcd(X,Y)$ if and only if:

$$\begin{cases} z_0 = gcd(x_0, y_0) \\ \sum_{i=0}^{j} z_i = gcd\left(\sum_{i=o}^{j} x_i, \sum_{i=0}^{j} y_i\right); 1 \le j \le 6 \end{cases}$$

2). $X \equiv Y(mod\, Z)$ if and only if $\sum_{i=0}^{j} x_i \equiv \sum_{i=0}^{j} y_i \left(mod \sum_{i=0}^{j} z_i\right)$, where $0 \le j \le 6$.

3). If $X \setminus Y$ then $\sum_{i=o}^{j} x_i \setminus \sum_{i=0}^{j} y_i\,; 0 \le j \le 6$.

Theorem2.

Let $X = x_0 + \sum_{i=1}^{6} x_i P_i, Y = y_0 + \sum_{i=1}^{6} y_i P_i, Z = z_0 + \sum_{i=1}^{6} z_i P_i, A = a_0 + \sum_{i=1}^{6} a_i P_i, B = b_0 + \sum_{i=1}^{6} b_i P_i, C = c_0 + \sum_{i=1}^{6} c_i P_i \in 6 - SP_Z$, then:

1). If $Z \setminus X, Z \setminus Y$, then $Z \setminus AX + BY$.

2). If $Z = gcd(X, Y)$, then there exists $A, B \in 6 - SP_Z$ such that $AX + BY = Z$.

3). If $X \equiv Y(mod\, Z)$, then:

$$\begin{cases} X + C = Y + C\ (mod\, Z) & (I) \\ X - C = Y - C\ (mod\, Z) & (II) \\ X.C = Y.C\ (mod\, Z) & (III) \end{cases}$$

4). X is invertible modulo Z if and only if $\sum_{i=o}^{j} x_i$ is invertible modulo$\sum_{i=o}^{j} z_i\,; 0 \le j \le 6$, and:

$$X^{-1}(mod\, Z) = x_0{}^{-1}(mod\, z_0) + P_1[(x_0 + x_1)^{-1}(mod\, z_0 + z_1) - x_0{}^{-1}(mod\, z_0)] + P_2[(x_0 + x_1 + x_2)^{-1}(mod\, z_0 + z_1 + z_2) - (x_0 + x_1)^{-1}(mod\, z_0 + z_1)] + P_3[(x_0 + x_1 + x_2 + x_3)^{-1}(mod\, z_0 + z_1 + z_2 + z_3) - (x_0 + x_1 + x_2)^{-1}(mod\, z_0 + z_1 + z_2)] + P_4[(x_0 + x_1 + x_2 + x_3 + x_4)^{-1}(mod\, z_0 + z_1 + z_2 + z_3 + z_4) - (x_0 + x_1 + x_2 + x_3)^{-1}(mod\, z_0 + z_1 + z_2 + z_3)] + P_5[(x_0 + x_1 + x_2 + x_3 + x_4 + x_5)^{-1}(mod\, z_0 + z_1 + z_2 + z_3 + z_4 + z_5) - (x_0 + x_1 + x_2 + x_3 + x_4)^{-1}(mod\, z_0 + z_1 + z_2 + z_3 + z_4)] + P_6[(x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6)^{-1}(mod\, z_0 + z_1 + z_2 + z_3 + z_4 + z_5 + z_6) - (x_0 + x_1 + x_2 + x_3 + x_4 + x_5)^{-1}(mod\, z_0 + z_1 + z_2 + z_3 + z_4 + z_5)].$$

Theorem3.

Let $AX + BY = C$ be symbolic 6-plithogenic Diophantine equation in two variables, $A, B, C, X, Y \in 6 - SP_Z$, hence it is solvable if and only if:

$\sum_{i=0}^{j} a_i \sum_{i=0}^{j} x_i + \sum_{i=0}^{j} b_i \sum_{i=0}^{j} y_i = \sum_{i=0}^{j} c_i\,; 0 \le j \le 6$ are solvable, i.e. $gcd\left(\sum_{i=0}^{j} a_i, \sum_{i=0}^{j} b_i\right) \setminus \sum_{i=0}^{j} c_i\,; 0 \le j \le 6$.

Theorem4.

Let $X = x_0 + \sum_{i=1}^{6} x_i p_i \in 6 - SP_Z$, then:

$$X^n = {x_0}^n + P_1\left[\left(\sum_{i=0}^{1} x_i\right)^n - {x_0}^n\right] + P_2\left[\left(\sum_{i=0}^{2} x_i\right)^n - \left(\sum_{i=0}^{1} x_i\right)^n\right]$$
$$+ P_3\left[\left(\sum_{i=0}^{3} x_i\right)^n - \left(\sum_{i=0}^{2} x_i\right)^n\right] + P_4\left[\left(\sum_{i=0}^{4} x_i\right)^n - \left(\sum_{i=0}^{3} x_i\right)^n\right]$$
$$+ P_5\left[\left(\sum_{i=0}^{5} x_i\right)^n - \left(\sum_{i=0}^{4} x_i\right)^n\right] + P_6\left[\left(\sum_{i=0}^{6} x_i\right)^n - \left(\sum_{i=0}^{5} x_i\right)^n\right]$$

Theorem5.

(X, Y, Z) is a symbolic 6-plithogenic Pythagoras triple i.e. it is a solution of the non linear Diophantine equation $X^2 + Y^2 = Z^2$, if and only if $\left(\sum_{i=0}^{j} x_i, \sum_{i=0}^{j} y_i, \sum_{i=0}^{j} z_i\right); 0 \le j \le 6$ is a Pythagoras triple in Z.

Theorem6.

(X, Y, Z, T) is a symbolic 6-plithogenic Pythagoras quadruple i.e. it is a solution of the non linear Diophantine equation $X^2 + Y^2 + Z^2 = T^2$, if and only if $\left(\sum_{i=0}^{j} x_i, \sum_{i=0}^{j} y_i, \sum_{i=0}^{j} z_i, \sum_{i=0}^{j} t_i\right); 0 \le j \le 6$ is a Pythagoras quadruple in Z.

Proof of theorem1.

1). We put

$$Z = z_0 + \sum_{i=1}^{6} z_i P_i, z_0 = gcd(x_0, y_0), \sum_{i=1}^{1} z_i = gcd\left(\sum_{i=1}^{1} x_i, \sum_{i=1}^{1} y_i\right), \sum_{i=1}^{2} z_i$$
$$= gcd\left(\sum_{i=1}^{2} x_i, \sum_{i=1}^{2} y_i\right)$$
$$\sum_{i=1}^{3} z_i = gcd\left(\sum_{i=1}^{3} x_i, \sum_{i=1}^{3} y_i\right), \sum_{i=1}^{4} z_i = gcd\left(\sum_{i=1}^{4} x_i, \sum_{i=1}^{4} y_i\right), \sum_{i=1}^{5} z_i$$
$$= gcd\left(\sum_{i=1}^{5} x_i, \sum_{i=1}^{5} y_i\right), \sum_{i=1}^{6} z_i = gcd\left(\sum_{i=1}^{6} x_i, \sum_{i=1}^{6} y_i\right)$$

Assume that $T = t_0 + \sum_{i=1}^{6} t_i P_i$ with $T \setminus X, T \setminus Y$, hence:

$$\begin{cases} \sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} x_i \,, \sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} y_i \,; 0 \le j \le 6 \\ \sum_{i=0}^{j} t_i \setminus \sum_{i=0}^{j} x_i \,, \sum_{i=0}^{j} t_i \setminus \sum_{i=0}^{j} y_i \,; 0 \le j \le 6 \end{cases}$$

So that $\sum_{i=0}^{j} t_i \setminus \sum_{i=0}^{j} z_i \,; 0 \le j \le 6$, hence $T \setminus Z$ and $Z = gcd(X, Y)$.

2). $X \equiv Y(mod\ Z)$ if and only if $Z \setminus X - Y$, which is equivalent to

$\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} (x_i - y_i) \,; 0 \le j \le 6$, hence $\sum_{i=0}^{j} x_i \equiv \sum_{i=0}^{j} y_i \left(mod\ \sum_{i=0}^{j} z_i\right); 0 \le j \le 6$.

3). Assume that $X \setminus Y$, hence:

$$\begin{cases} x_0 z_0 = y_0 \quad (1) \\ x_0 z_1 + x_1 z_0 + x_1 z_1 = y_1 \quad (2) \\ x_0 z_2 + x_1 z_2 + x_2 z_2 + x_2 z_0 + x_2 z_1 = y_2 \quad (3) \\ x_0 z_3 + x_1 z_3 + x_2 z_3 + x_3 z_3 + x_3 z_0 + x_3 z_1 + x_3 z_2 = y_3 \quad (4) \\ x_0 z_4 + x_1 z_4 + x_2 z_4 + x_3 z_4 + x_4 z_4 + x_4 z_0 + x_4 z_1 + x_4 z_2 + x_4 z_3 = y_4 \quad (5) \\ x_0 z_5 + x_1 z_5 + x_2 z_5 + x_3 z_5 + x_4 z_5 + x_5 z_5 + x_5 z_0 + x_5 z_1 + x_5 z_2 + x_5 z_3 + x_5 z_4 = y_5 \quad (6) \\ x_0 z_6 + x_1 z_6 + x_2 z_6 + x_3 z_6 + x_4 z_6 + x_5 z_6 + x_6 z_6 + x_6 z_0 + x_6 z_1 + x_6 z_2 + x_6 z_3 + x_6 z_4 + x_6 z_5 = y_6 \quad (7) \end{cases}$$

By adding $(1) + (2), (1) + (2) + (3), (1) + (2) + (3) + (4), (1) + (2) + (3) + (4) + (5), (1) + (2) + (3) + (4) + (5) + (6)$, $(1) + (2) + (3) + (4) + (5) + (6) + (7)$ we get:

$$\begin{cases} x_0 z_0 = y_0 \\ \sum_{i=1}^{1} x_i \sum_{i=1}^{1} z_i = \sum_{i=1}^{1} y_i \\ \sum_{i=1}^{2} x_i \sum_{i=1}^{2} z_i = \sum_{i=1}^{2} y_i \\ \sum_{i=1}^{3} x_i \sum_{i=1}^{3} z_i = \sum_{i=1}^{3} y_i \\ \sum_{i=1}^{4} x_i \sum_{i=1}^{4} z_i = \sum_{i=1}^{4} y_i \\ \sum_{i=1}^{5} x_i \sum_{i=1}^{5} z_i = \sum_{i=1}^{5} y_i \\ \sum_{i=1}^{6} x_i \sum_{i=1}^{6} z_i = \sum_{i=1}^{6} y_i \end{cases}$$

Which means that $\sum_{i=0}^{j} x_i \setminus \sum_{i=0}^{j} y_i \,; 0 \le j \le 6$

Proof of theorem 2.

1). Assume that $Z \setminus X, Z \setminus Y$, then we get:

$\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} x_i$, and $\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} y_i$; $0 \leq j \leq 5$.

So that $\sum_{i=0}^{j} z_i \setminus \left(\sum_{i=0}^{j} a_i \sum_{i=0}^{j} x_i + \sum_{i=0}^{j} b_i \sum_{i=0}^{j} y_i\right)$ for $0 \leq j \leq 5$ and $Z \setminus AX + BY$.

2). Assume that $Z = gcd(X, Y)$, then $\sum_{i=0}^{j} z_i = gcd\left(\sum_{i=0}^{j} x_i, \sum_{i=0}^{j} y_i\right)$ for all $0 \leq j \leq 5$.

According to Bezout's theorem, we can write:

There exists $a_j, b_j \in Z$ such that $\sum_{i=0}^{j} z_i = a_j \sum_{i=0}^{j} x_i + b_j \sum_{i=0}^{j} y_i$

by putting

$A = a_0 + (a_1 - a_0)P_1 + (a_2 - a_1)P_2 + (a_3 - a_2)P_3 + (a_4 - a_3)P_4 + (a_5 - a_4)P_5$,

$B = b_0 + (b_1 - b_0)P_1 + (b_2 - b_1)P_2 + (b_3 - b_2)P_3 + (b_4 - b_3)P_4 + (b_5 - b_4)P_5$, we get:

$Z = AX + BY$.

3). Assume that $X \equiv Y(mod\ Z)$, then:

$\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} (x_i - y_i)$ for all $0 \leq j \leq 6$, hence:

$$\begin{cases} \sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} (x_i - c_i + c_i - y_i) \\ \sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} (x_i + c_i - c_i + y_i) \end{cases}$$

Hence $X \pm C = Y \pm C(mod\ Z)$, also:

$\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} (x_i - y_i) \sum_{i=0}^{j} c_i$ i.e. $\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} x_i \sum_{i=0}^{j} c_i - \sum_{i=0}^{j} y_i \sum_{i=0}^{j} c_i$

Hence $X.C \equiv Y.C(mod\ Z)$.

4). X is invertible modulo Z If and only if there exists $Y = y_0 + \sum_{i=1}^{j} y_i p_i \in 6 - SP_Z$ such that $X.Y \equiv 1(mod\ Z)$.

This equivalent to:

$\sum_{i=0}^{j} x_i . \sum_{i=0}^{j} y_i \equiv 1(mod\ Z)$ for $0 \leq j \leq 6$, hence:

$\sum_{i=0}^{j} x_i$ is invertible modulo $\sum_{i=0}^{j} z_i$ and:

$$X^{-1} = {x_0}^{-1}(mod\ z_0) + P_1\left[\left(\sum_{i=0}^{1} x_i\right)^{-1}\left(mod\sum_{i=0}^{1} z_i\right) - {x_0}^{-1}(mod\ z_0)\right]$$
$$+ P_2\left[\left(\sum_{i=0}^{2} x_i\right)^{-1}\left(mod\sum_{i=0}^{2} z_i\right) - \left(\sum_{i=0}^{1} x_i\right)^{-1}\left(mod\sum_{i=0}^{1} z_i\right)\right]$$
$$+ P_3\left[\left(\sum_{i=0}^{3} x_i\right)^{-1}\left(mod\sum_{i=0}^{3} z_i\right) - \left(\sum_{i=0}^{2} x_i\right)^{-1}\left(mod\sum_{i=0}^{2} z_i\right)\right]$$
$$+ P_4\left[\left(\sum_{i=0}^{4} x_i\right)^{-1}\left(mod\sum_{i=0}^{4} z_i\right) - \left(\sum_{i=0}^{3} x_i\right)^{-1}\left(mod\sum_{i=0}^{3} z_i\right)\right]$$
$$+ P_5\left[\left(\sum_{i=0}^{5} x_i\right)^{-1}\left(mod\sum_{i=0}^{5} z_i\right) - \left(\sum_{i=0}^{4} x_i\right)^{-1}\left(mod\sum_{i=0}^{4} z_i\right)\right]$$
$$+ P_6\left[\left(\sum_{i=0}^{6} x_i\right)^{-1}\left(mod\sum_{i=0}^{6} z_i\right) - \left(\sum_{i=0}^{5} x_i\right)^{-1}\left(mod\sum_{i=0}^{5} z_i\right)\right]$$

Proof of theorem3.

It is easy to check that $AX + BY = C$ is equivalent to:

$$\sum_{i=0}^{j} a_i \sum_{i=0}^{j} x_i + \sum_{i=0}^{j} b_i \sum_{i=0}^{j} y_i = \sum_{i=0}^{j} c_i\,; 0 \le j \le 6$$

The previous six Diophantine equations are solvable if and only if:

$$gcd\left(\sum_{i=0}^{j} a_i\,, \sum_{i=0}^{j} b_i\right) \backslash \sum_{i=0}^{j} c_i\,; 0 \le j \le 6$$

proof on theorem4.

For $n = 1$, it holds directly.

We assume that it I true for k, we prove it for $k + 1$.

$$X^{k+1}=XX^{k}=\left(x_0+\sum_{i=0}^{6}x_i\,p_i\right)\left[x_0{}^{k}+P_1\left(\left(\sum_{i=0}^{1}x_i\right)^{k}-x_0{}^{k}\right)\right.$$
$$+P_2\left(\left(\sum_{i=0}^{2}x_i\right)^{k}-\left(\sum_{i=0}^{1}x_i\right)^{k}\right)+P_3\left(\left(\sum_{i=0}^{3}x_i\right)^{k}-\left(\sum_{i=0}^{2}x_i\right)^{k}\right)$$
$$+P_4\left(\left(\sum_{i=0}^{4}x_i\right)^{k}-\left(\sum_{i=0}^{3}x_i\right)^{k}\right)+P_5\left(\left(\sum_{i=0}^{5}x_i\right)^{k}-\left(\sum_{i=0}^{4}x_i\right)^{k}\right)$$
$$\left.+P_6\left(\left(\sum_{i=0}^{6}x_i\right)^{k}-\left(\sum_{i=0}^{5}x_i\right)^{k}\right)\right]$$
$$=x_0{}^{k+1}+P_1\left[x_0{}^{k}\left(\sum_{i=0}^{1}x_i\right)^{k}-x_0{}^{k+1}+x_1x_0{}^{k}+x_1\left(\sum_{i=0}^{1}x_i\right)^{k}-x_1x_0{}^{k}\right]$$
$$+P_2\left[x_0\left(\sum_{i=0}^{2}x_i\right)^{k}-x_0\left(\sum_{i=0}^{1}x_i\right)^{k}+x_1\left(\sum_{i=0}^{2}x_i\right)^{k}-x_1\left(\sum_{i=0}^{1}x_i\right)^{k}+x_2x_0{}^{k}\right.$$
$$\left.+x_1\left(\sum_{i=0}^{1}x_i\right)^{k}-x_2x_0{}^{k}+x_2\left(\sum_{i=0}^{2}x_i\right)^{k}-x_2\left(\sum_{i=0}^{1}x_i\right)^{k}\right]$$
$$+P_3\left[x_0\left(\sum_{i=0}^{3}x_i\right)^{k}-x_0\left(\sum_{i=0}^{2}x_i\right)^{k}+x_1\left(\sum_{i=0}^{3}x_i\right)^{k}-x_1\left(\sum_{i=0}^{2}x_i\right)^{k}\right.$$
$$+x_2\left(\sum_{i=0}^{3}x_i\right)^{k}-x_2\left(\sum_{i=0}^{2}x_i\right)^{k}+x_2x_0{}^{k}+x_3\left(\sum_{i=0}^{1}x_i\right)^{k}-x_3x_0{}^{k}$$
$$\left.+x_3\left(\sum_{i=0}^{2}x_i\right)^{k}-x_2\left(\sum_{i=0}^{1}x_i\right)^{k}+x_3\left(\sum_{i=0}^{3}x_i\right)^{k}-x_2\left(\sum_{i=0}^{2}x_i\right)^{k}\right]+\cdots$$
$$=x_0{}^{k+1}++P_1\left[\left(\sum_{i=0}^{1}x_i\right)^{k+1}-x_0{}^{k+1}\right]+P_2\left[\left(\sum_{i=0}^{2}x_i\right)^{k+1}-\left(\sum_{i=0}^{1}x_i\right)^{k+1}\right]$$
$$+\cdots$$

And the proof holds.

Proof of theorem5.

$X^2+Y^2=Z^2$ implies that:

$$\begin{cases} {x_0}^2+{y_0}^2={z_0}^2 \\ \left(\sum_{i=0}^{1} x_i\right)^2+\left(\sum_{i=0}^{1} y_i\right)^2=\left(\sum_{i=0}^{1} z_i\right)^2 \\ \left(\sum_{i=0}^{2} x_i\right)^2+\left(\sum_{i=0}^{2} y_i\right)^2=\left(\sum_{i=0}^{2} z_i\right)^2 \\ \left(\sum_{i=0}^{3} x_i\right)^2+\left(\sum_{i=0}^{3} y_i\right)^2=\left(\sum_{i=0}^{3} z_i\right)^2 \\ \left(\sum_{i=0}^{4} x_i\right)^2+\left(\sum_{i=0}^{4} y_i\right)^2=\left(\sum_{i=0}^{4} z_i\right)^2 \\ \left(\sum_{i=0}^{5} x_i\right)^2+\left(\sum_{i=0}^{5} y_i\right)^2=\left(\sum_{i=0}^{5} z_i\right)^2 \\ \left(\sum_{i=0}^{6} x_i\right)^2+\left(\sum_{i=0}^{6} y_i\right)^2=\left(\sum_{i=0}^{6} z_i\right)^2 \end{cases}$$

Which implies the proof.

Theorem 6 can be proved by the same argument.

Definition.

Let $X=x_0+\sum_{i=0}^{6} x_iP_i \in 6-SP_Z$, hence we say that $X>0$ if and only if $x_0>0, \sum_{i=0}^{k} x_i>0$; $1\leq k\leq 6$

For example: $X=3+P_1-P_2+2P_3-P_4-P_5>0$, that is because:

$3>0,4>0,3>0,5>0,4>0,3>0$.

If $Y=y_0+\sum_{i=0}^{6} y_iP_i \in 6-SP_Z$, we ay that $X\geq Y$ if and only if $x_0\geq y_0, \sum_{i=0}^{k} x_i\geq \sum_{i=0}^{k} y_i$; $1\leq k\leq 6$.

For $X=2+P_1+2P_2+5P_3+P_4+6P_5, Y=1+P_1+P_2+P_3+3P_4+P_5, X\geq Y$, that is because:

$2\geq 1,3\geq 2,5\geq 3,10\geq 4,11\geq 7,17\geq 8$

Definition.

Let $X=x_0+\sum_{i=0}^{6} x_iP_i\,, y=y_0+\sum_{i=0}^{6} y_i\,P_i\geq 0$, hence:

$$X^Y = {x_0}^{y_0} + P_1\left[\left(\sum_{i=0}^{1} x_i\right)^{\sum_{i=0}^{1} y_i} - {x_0}^{y_0}\right] + P_2\left[\left(\sum_{i=0}^{2} x_i\right)^{\sum_{i=0}^{2} y_i} - \left(\sum_{i=0}^{1} x_i\right)^{\sum_{i=0}^{1} y_i}\right]$$
$$+ P_3\left[\left(\sum_{i=0}^{3} x_i\right)^{\sum_{i=0}^{3} y_i} - \left(\sum_{i=0}^{2} x_i\right)^{\sum_{i=0}^{2} y_i}\right]$$
$$+ P_4\left[\left(\sum_{i=0}^{4} x_i\right)^{\sum_{i=0}^{4} y_i} - \left(\sum_{i=0}^{3} x_i\right)^{\sum_{i=0}^{3} y_i}\right]$$
$$+ P_5\left[\left(\sum_{i=0}^{5} x_i\right)^{\sum_{i=0}^{5} y_i} - \left(\sum_{i=0}^{4} x_i\right)^{\sum_{i=0}^{4} y_i}\right]$$
$$+ P_6\left[\left(\sum_{i=0}^{6} x_i\right)^{\sum_{i=0}^{6} y_i} - \left(\sum_{i=0}^{5} x_i\right)^{\sum_{i=0}^{5} y_i}\right]$$

Definition.

Let $X = x_0 + \sum_{i=0}^{6} x_i P_i > 0$, then:

$$\varphi(X) = \varphi(x_0) + P_1\left[\varphi\left(\sum_{i=0}^{1} x_i\right) - \varphi(x_0)\right] + P_2\left[\varphi\left(\sum_{i=0}^{2} x_i\right) - \varphi\left(\sum_{i=0}^{1} x_i\right)\right]$$
$$+ P_3\left[\varphi\left(\sum_{i=0}^{3} x_i\right) - \varphi\left(\sum_{i=0}^{2} x_i\right)\right] + P_4\left[\varphi\left(\sum_{i=0}^{4} x_i\right) - \varphi\left(\sum_{i=0}^{3} x_i\right)\right]$$
$$+ P_5\left[\varphi\left(\sum_{i=0}^{5} x_i\right) - \varphi\left(\sum_{i=0}^{4} x_i\right)\right] + P_6\left[\varphi\left(\sum_{i=0}^{6} x_i\right) - \varphi\left(\sum_{i=0}^{5} x_i\right)\right]$$

Where φ is Euler's function on Z.

Theorem.

Let $X = x_0 + \sum_{i=0}^{6} x_i P_{i_i}, Y = y_0 + \sum_{i=0}^{6} y_i P_i \in 6 - SP_Z, gcd(X, Y) = 1$ and $X, Y > 0$, hence:

$X^{\varphi(Y)} \equiv 1(mod\ Y)$

Proof.

$gcd(x_0, y_0) = 1$, hence ${x_0}^{\varphi(y_0)} \equiv 1(mod\ y_0)$.

$gcd(\sum_{i=0}^{1} x_i, \sum_{i=0}^{1} y_i) = 1$, hence $(\sum_{i=0}^{1} x_i)^{\varphi(\sum_{i=0}^{1} y_i)} \equiv 1(mod\ \sum_{i=0}^{1} y_i)$

By a similar argument, we get:

$$\left(\sum_{i=0}^{2} x_i\right)^{\varphi(\sum_{i=0}^{2} y_i)} \equiv 1\left(mod \sum_{i=0}^{2} y_i\right), \left(\sum_{i=0}^{3} x_i\right)^{\varphi(\sum_{i=0}^{3} y_i)} \equiv 1\left(mod \sum_{i=0}^{3} y_i\right)$$

$$\left(\sum_{i=0}^{4} x_i\right)^{\varphi(\sum_{i=0}^{4} y_i)} \equiv 1\left(mod \sum_{i=0}^{4} y_i\right), \left(\sum_{i=0}^{5} x_i\right)^{\varphi(\sum_{i=0}^{5} y_i)} \equiv 1\left(mod \sum_{i=0}^{5} y_i\right), \left(\sum_{i=0}^{6} x_i\right)^{\varphi(\sum_{i=0}^{6} y_i)} \equiv 1\left(mod \sum_{i=0}^{6} y_i\right)$$

This implies

$X^{\varphi(Y)} \equiv 1 + (1-1)P_1 + (1-1)P_2 + (1-1)P_3 + (1-1)P_4 + (1-1)P_5 + (1-1)P_6 \equiv 1(mod\ Y)$.

Remark.

We call previous result by symbolic 6-plithogenic Euler's theorem.

Conclusion

In this work, we have studied the properties of symbolic 6-plithogenic integers for the first time, where concepts such as symbolic 6-plithogenic divisors, congruencies, and linear Diophantine equations were handled by many theorems and examples.

Also, we have presented the conditions of symbolic 6-plithogenic Pythagoras triples and quadruples in the corresponding symbolic 6-plithogenic ring of integers.

References

1. Nader Mahmoud Taffach , Ahmed Hatip., "A Review on Symbolic 2-Plithogenic Algebraic Structures " Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.
2. Nader Mahmoud Taffach , Ahmed Hatip.," A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations ", Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.
3. Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.

4. Smarandache, F., " Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)", Neutrosophic Sets and Systems, vol. 53, 2023.
5. Taffach, N., " An Introduction to Symbolic 2-Plithogenic Vector Spaces Generated from The Fusion of Symbolic Plithogenic Sets and Vector Spaces", Neutrosophic Sets and Systems, Vol 54, 2023.
6. Taffach, N., and Ben Othman, K., " An Introduction to Symbolic 2-Plithogenic Modules Over Symbolic 2-Plithogenic Rings", Neutrosophic Sets and Systems, Vol 54, 2023.
7. Merkepci, H., and Rawashdeh, A., " On The Symbolic 2-Plithogenic Number Theory and Integers ", Neutrosophic Sets and Systems, Vol 54, 2023.
8. Albasheer, O., Hajjari., A., and Dalla., R., " On The Symbolic 3-Plithogenic Rings and TheirAlgebraic Properties", Neutrosophic Sets and Systems, Vol 54, 2023.
9. Abobala, M., On Refined Neutrosophic Matrices and Their Applications In Refined Neutrosophic Algebraic Equations, Journal Of Mathematics, Hindawi, 2021
10. Olgun, N., Hatip, A., Bal, M., and Abobala, M., " A Novel Approach To Necessary and Sufficient Conditions For The Diagonalization of Refined Neutrosophic Matrices", International Journal of neutrosophic Science, Vol. 16, pp. 72-79, 2021.
11. Merkepci, M., Abobala, M., and Allouf, A., " The Applications of Fusion Neutrosophic Number Theory in Public Key Cryptography and the Improvement of RSA Algorithm ", Fusion: Practice and Applications, 2023.
12. Abobala, M., and Allouf, A., " On A Novel Security Scheme for The Encryption and Decryption Of 2×2 Fuzzy Matrices with Rational Entries Based on The Algebra of Neutrosophic Integers and El-Gamal Crypto-System", Neutrosophic Sets and Systems, vol.54, 2023.

13. Khaldi, A., " A Study On Split-Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.

14. Ahmad, K., " On Some Split-Complex Diophantine Equations", Neoma Journal Of Mathematics and Computer Science, 2023.

15. Ali, R., " On The Weak Fuzzy Complex Inner Products On Weak Fuzzy Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.

16. Merkepci, M., and Abobala, M., " Security Model for Encrypting Uncertain Rational Data Units Based on Refined Neutrosophic Integers Fusion and El Gamal Algorithm ", Fusion: Practice and Applications, 2023.
17. Ben Othman, K., Albasheer, O., Nadweh, R., Von Shtawzen, O., and Ali, R., "On The Symbolic 6-Plithogenic and 7-Plithogenic Rings", Galoitica Journal of Mathematical Structures and Applications, Vol.8, 2023.
18. Bisher Ziena, M., and Abobala, M., " On The Refined Neutrosophic Real Analysis Based on Refined Neutrosophic Algebraic AH-Isometry", Neutrosophic Sets and Systems, vol. 54, 2023.
19. M. B. Zeina and M. Abobala, "A Novel Approach of Neutrosophic Continuous Probability Distributions using AH-Isometry with Applications in Medicine," in *Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics*, Elsevier, 2023.
20. M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, "Introduction to Symbolic 2-Plithogenic Probability Theory," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.
21. Ali, R., and Hasan, Z., "An Introduction To The Symbolic 3-Plithogenic Vector Spaces", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.

22. Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.
23. Ben Othman, K., "On Some Algorithms For Solving Symbolic 3-Plithogenic Equations", Neoma Journal Of Mathematics and Computer Science, 2023.
24. Florentin Smarandache, Plithogenic Algebraic Structures. Chapter in “Nidus idearum Scilogs, V: joining the dots” (third version), Pons Publishing Brussels, pp. 123-125, 2019.
25. Florentin Smarandache: Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets – Revisited, *Neutrosophic Sets and Systems*, vol. 21, 2018, pp. 153-166.
26. Sarkis, M., " On The Solutions Of Fermat's Diophantine Equation In 3-refined Neutrosophic Ring of Integers", Neoma Journal of Mathematics and Computer Science, 2023.
27. Abobala, M., "On Some Algebraic Properties of n-Refined Neutrosophic Elements and n-Refined Neutrosophic Linear Equations", Mathematical Problems in Engineering, Hindawi, 2021
28. Abobala, M., "A Study Of Nil Ideals and Kothe's Conjecture In Neutrosophic Rings", International Journal of Mathematics and Mathematical Sciences, hindawi, 2021
29. Abobala, M., "On The Characterization of Maximal and Minimal Ideals In Several Neutrosophic Rings", Neutrosophic sets and systems, Vol. 45, 2021.
30. Abobala, M., Bal, M., Aswad, M., "A Short Note On Some Novel Applications of Semi Module Homomorphisms", International journal of neutrosophic science, 2022.

31. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Concept Of Symbolic 7-Plithogenic Real Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.
32. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Symbolic 8-Plithogenic Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.
33. Abualkishik, A., Almajed, R., Thompson, W., "Improving The Performance of Fog-assisted Internet of Things Networks Using Bipolar Trapezoidal Neutrosophic Sets", Journal of Wireless and Ad Hoc Communication, 2023.
34. Ibrahim, M., and Abobala, M., "An Introduction To Refined Neutrosophic Number Theory", Neutrosophic Sets and Systems, Vol. 45, 2021.
35. Sankari, H., and Abobala, M., "Neutrosophic Linear Diophantine Equations With two Variables", Neutrosophic Sets and Systems, Vol. 38, pp. 22-30, 2020.
36. Merkepci, M., and Abobala, M., " On Some Novel Results About Split-Complex Numbers, The Diagonalization Problem And Applications To Public Key Asymmetric Cryptography", Journal of Mathematics, Hindawi, 2023.

Received: 19/5/2023, Accepted: 24/9/2023

Neutrosophic Sets and Systems, Vol. 59, 2023
University of New Mexico

On The Foundations of Symbolic 5-Plithogenic Number Theory

[1.] Mohamed Soueycatt, [2.] Barbara Charchekhandra, [3.] Rashel Abu Hakmeh

[1.] Department of Bioengineering, Al-Andalus Private University for Medical Sciences, Syria; m.soueycatt55@au.edu.sy

[2.] Jadavpur University, Department Of Mathematics, Kolkata, India; Charchekhandrabar32@yahoo.com

[3.] Faculty of Science, Mutah University, Jordan

Abstract: This paper is dedicated to study the properties of symbolic 5-plithogenic integers and number theory, where we present many number theoretical concepts such as symbolic 5-plithogenic Diophantine equations, symbolic 5-plithogenic congruencies, and symbolic 5-plithogenic Euler's function. Also, we present many examples to explain the validity and the scientific contribution of our work.

Keywords: symbolic 5-plithogenic integer, symbolic 5-plithogenic Euler's function, symbolic 5-plithogenic Pythagoras triple

Introduction

Symbolic n-plithogenic sets were defined for the first time by Smarandache in [4, 24-25], with many interesting algebraic properties.

In [1-3], the symbolic 2-plithogenic rings were defined as an extension of classical rings. Many results were obtained with respect to their ideals and homomorphisms. The symbolic 2-plithogenic rings and fields have many applications in generalizing other algebraic structures such as symbolic 2-plithogenic vector spaces, symbolic 2-plithogenic modules, and symbolic 2-plithogenic equations [5-7].

Laterally, many authors defined and studied symbolic 3-plithogenic algebraic structures, such as symbolic 3-plithogenic spaces and modules, see [8, 21-23].

zenodo.10031248/10.5281

In the literature, the extended integer systems were used in number theory, for example neutrosophic numbers have helped with neutrosophic number theory, refined neutrosophic numbers generated refined number theory and split-complex numbers generated split-complex number theory [9-20].

This has motivated many authors to study symbolic 2-plithogenic and symbolic 3-plithogenic number theoretical concepts such as congruencies, and Diophantine equations [26-36]. The generalized versions of number theoretical concepts are very applicable in other mathematical studies, especially in cryptography.

In this paper, we study the symbolic 5-plithogenic number theoretical concepts for the first time, and we illustrated many examples to clarify the novel approach.

Main discussion

Definition:

The ring of symbolic 5-plithogenic integers is defined as follows:

$5-SP_Z=\{x_0+\sum_{i=1}^{5} x_i P_i; x_i \in Z\}$, where $P_i \times P_j = p_{max(i,j)}, {P_i}^2 = P_i$.

Definition.

Let $X=x_0+\sum_{i=1}^{5} x_i P_i, Y=y_0+\sum_{i=1}^{5} y_i P_i, Z=z_0+\sum_{i=1}^{5} z_i P_i \in 5-SP_Z$, we say that:

1). $X \setminus Y$ if there exists $Z \in 5-SP_Z$ such that $X.Z=Y$.

2). $X \equiv Y (mod\, Z)$ if $Z \setminus X-Y$.

3). $Z=gcd(X,Y)$ if $Z \setminus X, Z \setminus Y$ and if $T \setminus X, T \setminus Y$, then $T \setminus Z$.

4). X, Y are relatively prime if $gcd(X,Y)=1$.

Theorem1.

Let $X=x_0+\sum_{i=1}^{5} x_i P_i, Y=y_0+\sum_{i=1}^{5} y_i P_i, Z=z_0+\sum_{i=1}^{5} z_i P_i \in 5-SP_Z$, then:

1). $Z=gcd(X,Y)$ if and only if:

$$\begin{cases} z_0 = gcd(x_0, y_0) \\ \sum_{i=0}^{j} z_i = gcd\left(\sum_{i=o}^{j} x_i, \sum_{i=0}^{j} y_i\right); 1 \le j \le 5 \end{cases}$$

2). $X \equiv Y (mod\, Z)$ if and only if $\sum_{i=0}^{j} x_i \equiv \sum_{i=0}^{j} y_i \left(mod \sum_{i=0}^{j} z_i\right),\ 0 \le j \le 5$.

3). If $X \setminus Y$ then $\sum_{i=o}^{j} x_i \setminus \sum_{i=0}^{j} y_i ; 0 \le j \le 5$.

Theorem2.

Let $X = x_0 + \sum_{i=1}^{5} x_i P_i, Y = y_0 + \sum_{i=1}^{5} y_i P_i, Z = z_0 + \sum_{i=1}^{5} z_i P_i, A = a_0 + \sum_{i=1}^{5} a_i P_i, B = b_0 + \sum_{i=1}^{5} b_i P_i, C = c_0 + \sum_{i=1}^{5} c_i P_i \in 5 - SP_Z$, then:

1). If $Z \setminus X, Z \setminus Y$, then $Z \setminus AX + BY$.

2). If $Z = gcd(X, Y)$, then there exists $A, B \in 5 - SP_Z$ such that $AX + BY = Z$.

3). If $X \equiv Y (mod\, Z)$, then:

$$\begin{cases} X + C = Y + C\ (mod\, Z) & (I) \\ X - C = Y - C\ (mod\, Z) & (II) \\ X.C = Y.C\ (mod\, Z) & (III) \end{cases}$$

4). X is invertible modulo Z if and only if $\sum_{i=o}^{j} x_i$ is invertible modulo$\sum_{i=o}^{j} z_i$; $0 \leq j \leq 5$, and:

$X^{-1}(mod\, Z) = x_0{}^{-1}(mod\, z_0) + P_1[(x_0 + x_1)^{-1}(mod\, z_0 + z_1) - x_0{}^{-1}(mod\, z_0)] + P_2[(x_0 + x_1 + x_2)^{-1}(mod\, z_0 + z_1 + z_2) - (x_0 + x_1)^{-1}(mod\, z_0 + z_1)] + P_3[(x_0 + x_1 + x_2 + x_3)^{-1}(mod\, z_0 + z_1 + z_2 + z_3) - (x_0 + x_1 + x_2)^{-1}(mod\, z_0 + z_1 + z_2)] + P_4[(x_0 + x_1 + x_2 + x_3 + x_4)^{-1}(mod\, z_0 + z_1 + z_2 + z_3 + z_4) - (x_0 + x_1 + x_2 + x_3)^{-1}(mod\, z_0 + z_1 + z_2 + z_3)] + P_5[(x_0 + x_1 + x_2 + x_3 + x_4 + x_5)^{-1}(mod\, z_0 + z_1 + z_2 + z_3 + z_4 + z_5) - (x_0 + x_1 + x_2 + x_3 + x_4)^{-1}(mod\, z_0 + z_1 + z_2 + z_3 + z_4)]$.

Theorem3.

Let $AX + BY = C$ be symbolic 5-plithogenic Diophantine equation in two variables, $A, B, C, X, Y \in 5 - SP_Z$, hence it is solvable if and only if:

$\sum_{i=0}^{j} a_i \sum_{i=0}^{j} x_i + \sum_{i=0}^{j} b_i \sum_{i=0}^{j} y_i = \sum_{i=0}^{j} c_i ; 0 \leq j \leq 5$ are solvable, i.e. $gcd\left(\sum_{i=0}^{j} a_i , \sum_{i=0}^{j} b_i\right) \setminus \sum_{i=0}^{j} c_i ; 0 \leq j \leq 5$.

Theorem4.

Let $X = x_0 + \sum_{i=1}^{5} x_i p_i \in 5 - SP_Z$, then:

$$X^n = {x_0}^n + P_1\left[\left(\sum_{i=0}^{1} x_i\right)^n - {x_0}^n\right] + P_2\left[\left(\sum_{i=0}^{2} x_i\right)^n - \left(\sum_{i=0}^{1} x_i\right)^n\right] + P_3\left[\left(\sum_{i=0}^{3} x_i\right)^n - \left(\sum_{i=0}^{2} x_i\right)^n\right] + P_4\left[\left(\sum_{i=0}^{4} x_i\right)^n - \left(\sum_{i=0}^{3} x_i\right)^n\right] + P_5\left[\left(\sum_{i=0}^{5} x_i\right)^n - \left(\sum_{i=0}^{4} x_i\right)^n\right]$$

Theorem5.

(X, Y, Z) is a symbolic 5-plithogenic Pythagoras triple i.e. it is a solution of the non linear Diophantine equation $X^2 + Y^2 = Z^2$, if and only if $\left(\sum_{i=0}^{j} x_i , \sum_{i=0}^{j} y_i , \sum_{i=0}^{j} z_i\right); 0 \le j \le 5$ is a Pythagoras triple in Z.

Theorem6.

(X, Y, Z, T) is a symbolic 5-plithogenic Pythagoras quadruple i.e. it is a solution of the non linear Diophantine equation $X^2 + Y^2 + Z^2 = T^2$, if and only if $\left(\sum_{i=0}^{j} x_i , \sum_{i=0}^{j} y_i , \sum_{i=0}^{j} z_i , \sum_{i=0}^{j} t_i\right); 0 \le j \le 5$ is a Pythagoras quadruple in Z.

Proof of theorem1.

1). We put

$$Z = z_0 + \sum_{i=1}^{5} z_i P_i, z_0 = gcd(x_0, y_0), \sum_{i=1}^{1} z_i = gcd\left(\sum_{i=1}^{1} x_i , \sum_{i=1}^{1} y_i\right), \sum_{i=1}^{2} z_i = gcd\left(\sum_{i=1}^{2} x_i , \sum_{i=1}^{2} y_i\right)$$

$$\sum_{i=1}^{3} z_i = gcd\left(\sum_{i=1}^{3} x_i , \sum_{i=1}^{3} y_i\right), \sum_{i=1}^{4} z_i = gcd\left(\sum_{i=1}^{4} x_i , \sum_{i=1}^{4} y_i\right), \sum_{i=1}^{5} z_i = gcd\left(\sum_{i=1}^{5} x_i , \sum_{i=1}^{5} y_i\right)$$

Assume that $T = t_0 + \sum_{i=1}^{5} t_i P_i$ with $T \setminus X, T \setminus Y$, hence:

$$\begin{cases} \sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} x_i , \sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} y_i ; 0 \le j \le 5 \\ \sum_{i=0}^{j} t_i \setminus \sum_{i=0}^{j} x_i , \sum_{i=0}^{j} t_i \setminus \sum_{i=0}^{j} y_i ; 0 \le j \le 5 \end{cases}$$

So that $\sum_{i=0}^{j} t_i \setminus \sum_{i=0}^{j} z_i ; 0 \le j \le 5$, hence $T \setminus Z$ and $Z = gcd(X, Y)$.

2). $X \equiv Y(mod\ Z)$ if and only if $Z \setminus X - Y$, which is equivalent to

$\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j}(x_i - y_i)\,; 0 \le j \le 5$, hence $\sum_{i=0}^{j} x_i \equiv \sum_{i=0}^{j} y_i \left(mod\ \sum_{i=0}^{j} z_i\right); 0 \le j \le 5$.

3). Assume that $X \setminus Y$, hence:

$$\begin{cases} x_0z_0 = y_0 \quad (1) \\ x_0z_1 + x_1z_0 + x_1z_1 = y_1 \quad (2) \\ x_0z_2 + x_1z_2 + x_2z_2 + x_2z_0 + x_2z_1 = y_2 \quad (3) \\ x_0z_3 + x_1z_3 + x_2z_3 + x_3z_3 + x_3z_0 + x_3z_1 + x_3z_2 = y_3 \quad (4) \\ x_0z_4 + x_1z_4 + x_2z_4 + x_3z_4 + x_4z_4 + x_4z_0 + x_4z_1 + x_4z_2 + x_4z_3 = y_4 \quad (5) \\ x_0z_5 + x_1z_5 + x_2z_5 + x_3z_5 + x_4z_5 + x_5z_5 + x_5z_0 + x_5z_1 + x_5z_2 + x_5z_3 + x_5z_4 = y_5 \quad (6) \end{cases}$$

By adding $(1)+(2), (1)+(2)+(3), (1)+(2)+(3)+(4), (1)+(2)+(3)+(4)+(5), (1)+(2)+(3)+(4)+(5)+(6)$, we get:

$$\begin{cases} x_0z_0 = y_0 \\ \sum_{i=1}^{1} x_i \sum_{i=1}^{1} z_i = \sum_{i=1}^{1} y_i \\ \sum_{i=1}^{2} x_i \sum_{i=1}^{2} z_i = \sum_{i=1}^{2} y_i \\ \sum_{i=1}^{3} x_i \sum_{i=1}^{3} z_i = \sum_{i=1}^{3} y_i \\ \sum_{i=1}^{4} x_i \sum_{i=1}^{4} z_i = \sum_{i=1}^{4} y_i \\ \sum_{i=1}^{5} x_i \sum_{i=1}^{5} z_i = \sum_{i=1}^{5} y_i \end{cases}$$

Which means that $\sum_{i=0}^{j} x_i \setminus \sum_{i=0}^{j} y_i\,; 0 \le j \le 5$

Example on theorem1.

Take $X = 3 + 2P_1 + 2P_2 + P_3 - P_4 + 4P_5, Y = 6 + P_1 + P_2 - P_3 - P_4 + 2P_5$

$$\begin{cases} gcd(x_0, y_0) = gcd(3,6) = 3 \\ gcd(x_0 + x_1, y_0 + y_1) = gcd(5,7) = 1 \\ gcd(x_0 + x_1 + x_2, y_0 + y_1 + y_2) = gcd(7,7) = 7 \\ gcd(x_0 + x_1 + x_2 + x_3, y_0 + y_1 + y_2 + y_3) = gcd(8,7) = 1 \\ gcd(x_0 + x_1 + x_2 + x_3 + x_4, y_0 + y_1 + y_2 + y_3 + y_4) = gcd(7,6) = 1 \\ gcd(x_0 + x_1 + x_2 + x_3 + x_4 + x_5, y_0 + y_1 + y_2 + y_3 + y_4 + y_5) = gcd(11,8) = 1 \end{cases}$$

Thus

$z_0 = 3, z_1 = 1 - 3 = -2, z_2 = 7 - 1 = 6, z_3 = 1 - 7 = -6, z_4 = 1 - 1 = 0, z_5 = 1 - 1 = 0$, hence:

$Z = gcd(X,Y) = 3 - 2P_1 + 6P_2 - 6P_3$

For $L = 1 + P_1 - P_2 + 2P_5$, we can see:

$L \setminus X - Y$, that is because:

$$\begin{cases} 1 \setminus -3 \\ 2 \setminus -2 \\ 1 \setminus -1, \text{ thus } X \equiv Y(mod\ L). \\ 1 \setminus 1 \\ 3 \setminus 3 \end{cases}$$

Proof of theorem 2.

1). Assume that $Z \setminus X, Z \setminus Y$, then we get:

$\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} x_i$, and $\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} y_i\,; 0 \le j \le 5$.

So that $\sum_{i=0}^{j} z_i \setminus \left(\sum_{i=0}^{j} a_i \sum_{i=0}^{j} x_i + \sum_{i=0}^{j} b_i \sum_{i=0}^{j} y_i\right)$ for $0 \le j \le 5$ and $Z \setminus AX + BY$.

2). Assume that $Z = gcd(X,Y)$, then $\sum_{i=0}^{j} z_i = gcd\left(\sum_{i=0}^{j} x_i, \sum_{i=0}^{j} y_i\right)$ for all $0 \le j \le 5$.

According to Bezout's theorem, we can write:

There exists $a_j, b_j \in Z$ such that $\sum_{i=0}^{j} z_i = a_j \sum_{i=0}^{j} x_i + b_j \sum_{i=0}^{j} y_i$

by putting

$A = a_0 + (a_1 - a_0)P_1 + (a_2 - a_1)P_2 + (a_3 - a_2)P_3 + (a_4 - a_3)P_4 + (a_5 - a_4)P_5$,

$B = b_0 + (b_1 - b_0)P_1 + (b_2 - b_1)P_2 + (b_3 - b_2)P_3 + (b_4 - b_3)P_4 + (b_5 - b_4)P_5$, we get:

$Z = AX + BY$.

3). Assume that $X \equiv Y(mod\ Z)$, then:

$\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j}(x_i - y_i)$ for all $0 \le j \le 5$, hence:

$$\begin{cases} \sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j}(x_i - c_i + c_i - y_i) \\ \sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j}(x_i + c_i - c_i + y_i) \end{cases}$$

Hence $X \pm C = Y \pm C(mod\ Z)$, also:

$\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j}(x_i - y_i) \sum_{i=0}^{j} c_i$ i.e. $\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} x_i \sum_{i=0}^{j} c_i - \sum_{i=0}^{j} y_i \sum_{i=0}^{j} c_i$

Hence $X.C \equiv Y.C(mod\ Z)$.

4). X is invertible modulo Z If and only if there exists $Y = y_0 + \sum_{i=1}^{j} y_i p_i \in 5 - SP_Z$ such that $X.Y \equiv 1(mod\ Z)$.

This equivalent to:

$\sum_{i=0}^{j} x_i . \sum_{i=0}^{j} y_i \equiv 1(mod\ Z)$ for $0 \le j \le 5$, hence:

$\sum_{i=0}^{j} x_i$ is invertible modulo $\sum_{i=0}^{j} z_i$ and:

$$X^{-1} = x_0^{-1}(mod\ z_0) + P_1\left[\left(\sum_{i=0}^{1} x_i\right)^{-1}\left(mod \sum_{i=0}^{1} z_i\right) - x_0^{-1}(mod\ z_0)\right]$$
$$+ P_2\left[\left(\sum_{i=0}^{2} x_i\right)^{-1}\left(mod \sum_{i=0}^{2} z_i\right) - \left(\sum_{i=0}^{1} x_i\right)^{-1}\left(mod \sum_{i=0}^{1} z_i\right)\right]$$
$$+ P_3\left[\left(\sum_{i=0}^{3} x_i\right)^{-1}\left(mod \sum_{i=0}^{3} z_i\right) - \left(\sum_{i=0}^{2} x_i\right)^{-1}\left(mod \sum_{i=0}^{2} z_i\right)\right]$$
$$+ P_4\left[\left(\sum_{i=0}^{4} x_i\right)^{-1}\left(mod \sum_{i=0}^{4} z_i\right) - \left(\sum_{i=0}^{3} x_i\right)^{-1}\left(mod \sum_{i=0}^{3} z_i\right)\right]$$
$$+ P_5\left[\left(\sum_{i=0}^{5} x_i\right)^{-1}\left(mod \sum_{i=0}^{5} z_i\right) - \left(\sum_{i=0}^{4} x_i\right)^{-1}\left(mod \sum_{i=0}^{4} z_i\right)\right]$$

Example on theorem 2.

Take:

$$X = 4 + 2P_1 - P_2 + 5P_3 - P_4 + P_5, Y = 2 + P_1 - P_2 + P_3 - P_4 + 4P_5, Z$$
$$= 2 - P_1 + P_2 - P_3 + P_4 + P_5, A = 1 + P_1, B = 2 - P_1 + 3P_2$$

we have $Z \setminus X$, that is because $2 \setminus 4, 1 \setminus 6, 2 \setminus 4, 1 \setminus 9, 2 \setminus 8, 3 \setminus 9$.

$Z \setminus Y$, that I because $2 \setminus 2, 1 \setminus 3, 2 \setminus 2, 1 \setminus 3, 2 \setminus 2, 3 \setminus 6$.

On the other hand,

$$AX + BY = (1 + P_1)(4 + 2P_1 - P_2 + 5P_3 - P_4 + P_5)$$
$$+ (2 - P_1 + 3P_2)(2 + P_1 - P_2 + P_3 - P_4 + 4P_5)$$
$$= 4 + 4P_1 + 2P_1 + 2P_1 - 2P_2 - 2P_2 + 5P_3 + 5P_3 - P_4 - P_4 + P_5 + P_5 + 4$$
$$+ 2P_1 - 2P_2 + 2P_3 - 2P_4 + 8P_5 - 2P_1 - P_1 + P_2 - P_3 + P_4 - 4P_5 + 6P_2$$
$$+ 3P_2 - 3P_4 + 12P_5 = 8 + 7P_1 + P_2 + 14P_3 - 6P_4 + 18P_5$$

$Z \setminus AX + BY$, that is because $2 \setminus 8, 1 \setminus 15, 2 \setminus 16, 1 \setminus 30, 2 \setminus 24, 3 \setminus 42$.

For $T = 3 + 2P_1 - 2P_2 - P_3 - P_4$, we can see:

$gcd(X,T) = gcd(4,3) + P_1[gcd(5,6) - gcd(4,3)] + P_2[gcd(3,4) - gcd(5,6)] + P_3[gcd(9,2) - gcd(3,4)] + P_4[gcd(8,1) - gcd(9,2)] + P_5[gcd(8,1) - gcd(9,1)]$,

hence X is invertible modulo T.

$$4^{-1}(mod\ 3) = 1, 6^{-1}(mod\ 5) = 1, 9^{-1}(mod\ 2) = 5, 8^{-1}(mod\ 1) = 1, 9^{-1}(mod\ 1) = 1, 4^{-1}(mod\ 3) = 1$$

$X^{-1}(mod\ T) = 1 + P_1[1-1] + P_2[1-1] + P_3[5-1] + P_4[1-5] + P_5[1-1] = 1 + 4P_3 - 4P_4$.

Proof of theorem3.

It is easy to check that $AX + BY = C$ is equivalent to:

$$\sum_{i=0}^{j} a_i \sum_{i=0}^{j} x_i + \sum_{i=0}^{j} b_i \sum_{i=0}^{j} y_i = \sum_{i=0}^{j} c_i\,; 0 \le j \le 5$$

The previous six Diophantine equations are solvable if and only if:

$$gcd\left(\sum_{i=0}^{j} a_i, \sum_{i=0}^{j} b_i\right) \backslash \sum_{i=0}^{j} c_i\,; 0 \le j \le 5$$

Example on theorem3.

Consider the following 5-plithogenic linear Diophantine equation in two variables:

$(1 + P_2 - 3P_3 + 5P_4 + P_5)X + (1 - P_1 + P_2)Y = P_1 + P_2 - 3P_3 + 6P_4 + 2P_5$

The equivalent system is:

$$\begin{cases} x_0 + y_0 = 0 \quad (1) \\ \sum_{i=0}^{1} x_i = 1 \quad (2) \\ \sum_{i=0}^{2} x_i + \sum_{i=0}^{2} y_i = 2 \quad (3) \\ -\sum_{i=0}^{3} x_i + \sum_{i=0}^{3} y_i = -1 \quad (4) \\ 4\sum_{i=0}^{4} x_i + \sum_{i=0}^{4} y_i = 5 \quad (5) \\ 5\sum_{i=0}^{5} x_i + \sum_{i=0}^{5} y_i = 7 \quad (6) \end{cases}$$

Equation (1) has a solution $x_0 = y_0 = 0$.

Equation (2) has a solution $x_0 + x_1 = 1$, hence $x_1 = 1, y_1 = 0$.

Equation (3) has a solution $x_0 + x_1 + x_2 = 1, y_0 + y_1 + y_2 = 1$, hence $x_2 = 0, y_2 = 0$.

Equation (4) has a solution $x_0 + x_1 + x_2 + x_3 = 1, y_0 + y_1 + y_2 + y_3 = 0$, hence $x_3 = 0, y_3 = 0$.

Equation (5) has a solution $x_0 + x_1 + x_2 + x_3 + x_4 = 1, y_0 + y_1 + y_2 + y_3 + y_4 = 1$, hence $x_4 = 0, y_4 = 1$.

Equation (6) has a solution $x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 1, y_0 + y_1 + y_2 + y_3 + y_4 + y_5 = 1$, hence $x_5 = 0, y_5 = 1$.

This means that $X = P_1, Y = P_4 + P_5$ is a solution.

proof on theorem4.

For $n = 1$, it holds directly.

We assume that it is true for k, we prove it for $k + 1$. $X^{k+1} = XX^k = \left(x_0 + \sum_{i=0}^{5} x_i\, p_i\right)\left[{x_0}^k + P_1((\sum_{i=0}^{1} x_i)^k - {x_0}^k) + P_2((\sum_{i=0}^{2} x_i)^k - (\sum_{i=0}^{1} x_i)^k) + P_3((\sum_{i=0}^{3} x_i)^k - (\sum_{i=0}^{2} x_i)^k) + P_4((\sum_{i=0}^{4} x_i)^k - (\sum_{i=0}^{3} x_i)^k) + P_5\left(\left(\sum_{i=0}^{5} x_i\right)^k - (\sum_{i=0}^{4} x_i)^k\right)\right] = {x_0}^{k+1} +$
$P_1[{x_0}^k(\sum_{i=0}^{1} x_i)^k - {x_0}^{k+1} + x_1{x_0}^k + x_1(\sum_{i=0}^{1} x_i)^k - x_1{x_0}^k] + P_2[x_0(\sum_{i=0}^{2} x_i)^k -$
$x_0(\sum_{i=0}^{1} x_i)^k + x_1(\sum_{i=0}^{2} x_i)^k - x_1(\sum_{i=0}^{1} x_i)^k + x_2{x_0}^k + x_1(\sum_{i=0}^{1} x_i)^k - x_2{x_0}^k +$
$x_2(\sum_{i=0}^{2} x_i)^k - x_2(\sum_{i=0}^{1} x_i)^k] + P_3[x_0(\sum_{i=0}^{3} x_i)^k - x_0(\sum_{i=0}^{2} x_i)^k + x_1(\sum_{i=0}^{3} x_i)^k -$
$x_1(\sum_{i=0}^{2} x_i)^k + x_2(\sum_{i=0}^{3} x_i)^k - x_2(\sum_{i=0}^{2} x_i)^k + x_2{x_0}^k + x_3(\sum_{i=0}^{1} x_i)^k - x_3{x_0}^k +$
$x_3(\sum_{i=0}^{2} x_i)^k - x_2(\sum_{i=0}^{1} x_i)^k + x_3(\sum_{i=0}^{3} x_i)^k - x_2(\sum_{i=0}^{2} x_i)^k] + \cdots = {x_0}^{k+1} +$
$+P_1[(\sum_{i=0}^{1} x_i)^{k+1} - {x_0}^{k+1}] + P_2[(\sum_{i=0}^{2} x_i)^{k+1} - (\sum_{i=0}^{1} x_i)^{k+1}] + \cdots$

And the proof holds.

Proof of theorem5.

$X^2 + Y^2 = Z^2$ implies that:

$$\begin{cases} {x_0}^2 + {y_0}^2 = {z_0}^2 \\ \left(\sum_{i=0}^{1} x_i\right)^2 + \left(\sum_{i=0}^{1} y_i\right)^2 = \left(\sum_{i=0}^{1} z_i\right)^2 \\ \left(\sum_{i=0}^{2} x_i\right)^2 + \left(\sum_{i=0}^{2} y_i\right)^2 = \left(\sum_{i=0}^{2} z_i\right)^2 \\ \left(\sum_{i=0}^{3} x_i\right)^2 + \left(\sum_{i=0}^{3} y_i\right)^2 = \left(\sum_{i=0}^{3} z_i\right)^2 \\ \left(\sum_{i=0}^{4} x_i\right)^2 + \left(\sum_{i=0}^{4} y_i\right)^2 = \left(\sum_{i=0}^{4} z_i\right)^2 \\ \left(\sum_{i=0}^{5} x_i\right)^2 + \left(\sum_{i=0}^{5} y_i\right)^2 = \left(\sum_{i=0}^{5} z_i\right)^2 \end{cases}$$

Which implies the proof.

Theorem 6 can be proved by the same argument.

Example on theorem5.

Consider $X = 3 + P_5, Y = 4 - P_5, Z = 5$, we have:

$X^2 + Y^2 = Z^2$, hence (X, Y, Z) is a Pythagoras triple.

We can see clearly that:

$$\begin{cases} x_0 = 3, y_0 = 4, z_0 = 5 \\ \sum_{i=0}^{1} x_i = 3, \sum_{i=0}^{1} y_i = 4 \\ \sum_{i=0}^{2} x_i = 3, \sum_{i=0}^{2} y_i = 4 \\ \sum_{i=0}^{3} x_i = 3, \sum_{i=0}^{3} y_i = 4 \\ \sum_{i=0}^{4} x_i = 3, \sum_{i=0}^{4} y_i = 4 \\ and \sum_{i=0}^{5} x_i = 3, \sum_{i=0}^{5} y_i = 4, \sum_{i=0}^{k} z_i = 4 \,; 1 \le k \le 5 \end{cases}$$

Definition.

Let $X = x_0 + \sum_{i=0}^{5} x_i P_i \in 5 - SP_Z$, hence we say that $X > 0$ if and only if $x_0 > 0, \sum_{i=0}^{k} x_i > 0 \,; 1 \le k \le 5$

For example: $X = 3 + P_1 - P_2 + 2P_3 - P_4 - P_5 > 0$, that is because:

$3 > 0,4 > 0,3 > 0,5 > 0,4 > 0,3 > 0.$

If $Y = y_0 + \sum_{i=0}^{5} y_i P_i \in 5 - SP_Z$, we ay that $X \geq Y$ if and only if $x_0 \geq y_0, \sum_{i=0}^{k} x_i \geq \sum_{i=0}^{k} y_i \, ; 1 \leq k \leq 5.$

For $X = 2 + P_1 + 2P_2 + 5P_3 + P_4 + 6P_5, Y = 1 + P_1 + P_2 + P_3 + 3P_4 + P_5, X \geq Y$, that is because:

$2 \geq 1,3 \geq 2,5 \geq 3,10 \geq 4,11 \geq 7,17 \geq 8$

Definition.

Let $X = x_0 + \sum_{i=0}^{5} x_i P_i \, , y = y_0 + \sum_{i=0}^{5} y_i \, P_i \geq 0$, hence:

$$X^Y = {x_0}^{y_0} + P_1\left[\left(\sum_{i=0}^{1} x_i\right)^{\sum_{i=0}^{1} y_i} - {x_0}^{y_0}\right] + P_2\left[\left(\sum_{i=0}^{2} x_i\right)^{\sum_{i=0}^{2} y_i} - \left(\sum_{i=0}^{1} x_i\right)^{\sum_{i=0}^{1} y_i}\right]$$
$$+ P_3\left[\left(\sum_{i=0}^{3} x_i\right)^{\sum_{i=0}^{3} y_i} - \left(\sum_{i=0}^{2} x_i\right)^{\sum_{i=0}^{2} y_i}\right]$$
$$+ P_4\left[\left(\sum_{i=0}^{4} x_i\right)^{\sum_{i=0}^{4} y_i} - \left(\sum_{i=0}^{3} x_i\right)^{\sum_{i=0}^{3} y_i}\right]$$
$$+ P_5\left[\left(\sum_{i=0}^{5} x_i\right)^{\sum_{i=0}^{5} y_i} - \left(\sum_{i=0}^{4} x_i\right)^{\sum_{i=0}^{4} y_i}\right]$$

Example.

Let $X = 2 + 3P_1 - P_2 - P_3 - P_4 + P_5, Y = 1 + P_1 - P_2 + P_3 - P_4 + P_5$, we have:

$$\begin{cases} x_0 = 2, y_0 = 1, {x_0}^{y_0} = 2 \\ \sum_{i=0}^{1} x_i = 5, \sum_{i=0}^{1} y_i = 2, 5^2 = 25 \\ \sum_{i=0}^{2} x_i = 4, \sum_{i=0}^{2} y_i = 1, 4^1 = 4 \\ \sum_{i=0}^{3} x_i = 3, \sum_{i=0}^{3} y_i = 2, 3^2 = 9 \\ \sum_{i=0}^{4} x_i = 2, \sum_{i=0}^{4} y_i = 1, 2^1 = 2 \\ \sum_{i=0}^{5} x_i = 3, \sum_{i=0}^{5} y_i = 2, 3^2 = 9 \end{cases}$$

Hence $X^Y = 2 + (25-2)P_1 + (4-25)P_2 + (9-4)P_3 + (2-9)P_4 + (9-2)P_5 = 2 + 23P_1 - 21P_2 + 5P_3 - 7P_4 + 7P_5$

Definition.

Let $X = x_0 + \sum_{i=0}^{5} x_i P_i > 0$, then:

$$\varphi(X) = \varphi(x_0) + P_1\left[\varphi\left(\sum_{i=0}^{1} x_i\right) - \varphi(x_0)\right] + P_2\left[\varphi\left(\sum_{i=0}^{2} x_i\right) - \varphi\left(\sum_{i=0}^{1} x_i\right)\right] + P_3\left[\varphi\left(\sum_{i=0}^{3} x_i\right) - \varphi\left(\sum_{i=0}^{2} x_i\right)\right] + P_4\left[\varphi\left(\sum_{i=0}^{4} x_i\right) - \varphi\left(\sum_{i=0}^{3} x_i\right)\right] + P_5\left[\varphi\left(\sum_{i=0}^{5} x_i\right) - \varphi\left(\sum_{i=0}^{4} x_i\right)\right]$$

Where φ is Euler's function on Z.

Example.

Let $X = 3 + 2P_1 + P_2 + P_3 - P_4 + P_5$, then:

$$\varphi(x_0) = \varphi(3) = 2, \varphi\left(\sum_{i=0}^{1} x_i\right) = \varphi(5) = 4, \left(\sum_{i=0}^{2} x_i\right) = (6) = 2, \varphi\left(\sum_{i=0}^{3} x_i\right) = \varphi(7) = 6, \varphi\left(\sum_{i=0}^{4} x_i\right) = \varphi(8) = 2, \varphi\left(\sum_{i=0}^{5} x_i\right) = \varphi(7) = 6$$

$$\varphi(X) = 2 + (4-2)P_1 + (2-4)P_2 + (6-2)P_3 + (2-6)P_4 + (6-2)P_5 = 2 + 2P_1 - 2P_2 + 4P_3 - 4P_4 + 4P_5$$

Theorem.

Let $X = x_0 + \sum_{i=0}^{5} x_i P_{i_i}, Y = y_0 + \sum_{i=0}^{5} y_i P_i \in 5 - SP_Z, gcd(X,Y) = 1$ and $X, Y > 0$, hence:

$X^{\varphi(Y)} \equiv 1(mod\ Y)$

Proof.

$gcd(x_0, y_0) = 1$, hence $x_0^{\varphi(y_0)} \equiv 1(mod\ y_0)$.

$gcd(\sum_{i=0}^{1} x_i, \sum_{i=0}^{1} y_i) = 1$, hence $(\sum_{i=0}^{1} x_i)^{\varphi(\sum_{i=0}^{1} y_i)} \equiv 1(mod\ \sum_{i=0}^{1} y_i)$

By a similar argument, we get:

$$\left(\sum_{i=0}^{2} x_i\right)^{\varphi(\sum_{i=0}^{2} y_i)} \equiv 1\left(mod\ \sum_{i=0}^{2} y_i\right), \left(\sum_{i=0}^{3} x_i\right)^{\varphi(\sum_{i=0}^{3} y_i)} \equiv 1\left(mod\ \sum_{i=0}^{3} y_i\right)$$

$$\left(\sum_{i=0}^{4} x_i\right)^{\varphi(\sum_{i=0}^{4} y_i)} \equiv 1\left(mod\ \sum_{i=0}^{4} y_i\right), \left(\sum_{i=0}^{5} x_i\right)^{\varphi(\sum_{i=0}^{5} y_i)} \equiv 1\left(mod\ \sum_{i=0}^{5} y_i\right)$$

This implies

$X^{\varphi(Y)} \equiv 1 + (1-1)P_1 + (1-1)P_2 + (1-1)P_3 + (1-1)P_4 + (1-1)P_5 \equiv 1(mod\ Y)$.

Example.

Consider $X = 5 + 2P_1 + 4P_2 + 2P_3 - 2P_4 + 2P_5, Y = 7 + 4P_1 - 4P_2 + P_3 + P_4 + P_5$.

$$gcd(X,Y) = gcd(5,7) + +P_1[gcd(7,11) - gcd(5,7)] + P_2[gcd(11,7) - gcd(7,11)] + P_3[gcd(13,9) - gcd(11,7)] + P_4[gcd(11,9) - gcd(13,9)] + P_5[gcd(13,10) - gcd(11,9)] \equiv 1$$

Also, we have:

$$x_0 = 5, y_0 = 7, \varphi(y_0) = 6, x_0^{\varphi(y_0)} = 5^6 \equiv 1(mod\ 7)$$

$$\sum_{i=0}^{1} x_i = 7, \sum_{i=0}^{1} y_i = 11, \varphi\left(\sum_{i=0}^{1} y_i\right) = 10, 7^{10} \equiv 1(mod\ 11)$$

$$\sum_{i=0}^{2} x_i = 11, \sum_{i=0}^{2} y_i = 7, \varphi\left(\sum_{i=0}^{2} y_i\right) = 6, 11^6 \equiv 1(mod\ 7)$$

$$\sum_{i=0}^{3} x_i = 13, \sum_{i=0}^{3} y_i = 81, \varphi\left(\sum_{i=0}^{3} y_i\right) = 4, 13^4 \equiv 1(mod\ 8)$$

$$\sum_{i=0}^{4} x_i = 11, \sum_{i=0}^{4} y_i = 91, \varphi\left(\sum_{i=0}^{4} y_i\right) = 6, 11^6 \equiv 1(mod\ 9)$$

$$\sum_{i=0}^{5} x_i = 13, \sum_{i=0}^{5} y_i = 10, \varphi\left(\sum_{i=0}^{5} y_i\right) = 4, 13^4 \equiv 1(mod\ 10)$$

Hence $X^{\varphi(Y)} \equiv 1(mod\ Y)$

Remark.

We call the previous result by symbolic 5-plithogenic Euler's theorem.

Conclusion

In this work, we have studied the properties of symbolic 5-plithogenic integers for the first time, where concepts such as symbolic 5-plithogenic divisors, congruencies, and linear Diophantine equations were handled by many theorems and examples.

Also, we have presented the conditions of symbolic 5-plithogenic Pythagoras triples and quadruples in the corresponding symbolic 5-plithogenic ring of integers.

References

1. Nader Mahmoud Taffach , Ahmed Hatip., "A Review on Symbolic 2-Plithogenic Algebraic Structures " Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.
2. Nader Mahmoud Taffach , Ahmed Hatip.," A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations ", Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.
3. Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.
4. Smarandache, F., " Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)", Neutrosophic Sets and Systems, vol. 53, 2023.
5. Taffach, N., " An Introduction to Symbolic 2-Plithogenic Vector Spaces Generated from The Fusion of Symbolic Plithogenic Sets and Vector Spaces", Neutrosophic Sets and Systems, Vol 54, 2023.

6. Taffach, N., and Ben Othman, K., " An Introduction to Symbolic 2-Plithogenic Modules Over Symbolic 2-Plithogenic Rings", Neutrosophic Sets and Systems, Vol 54, 2023.
7. Merkepci, H., and Rawashdeh, A., " On The Symbolic 2-Plithogenic Number Theory and Integers ", Neutrosophic Sets and Systems, Vol 54, 2023.
8. Albasheer, O., Hajjari., A., and Dalla., R., " On The Symbolic 3-Plithogenic Rings and TheirAlgebraic Properties", Neutrosophic Sets and Systems, Vol 54, 2023.
9. Abobala, M., On Refined Neutrosophic Matrices and Their Applications In Refined Neutrosophic Algebraic Equations, Journal Of Mathematics, Hindawi, 2021
10. Olgun, N., Hatip, A., Bal, M., and Abobala, M., " A Novel Approach To Necessary and Sufficient Conditions For The Diagonalization of Refined Neutrosophic Matrices", International Journal of neutrosophic Science, Vol. 16, pp. 72-79, 2021.
11. Merkepci, M., Abobala, M., and Allouf, A., " The Applications of Fusion Neutrosophic Number Theory in Public Key Cryptography and the Improvement of RSA Algorithm ", Fusion: Practice and Applications, 2023.
12. Abobala, M., and Allouf, A., " On A Novel Security Scheme for The Encryption and Decryption Of 2×2 Fuzzy Matrices with Rational Entries Based on The Algebra of Neutrosophic Integers and El-Gamal Crypto-System", Neutrosophic Sets and Systems, vol.54, 2023.
13. Khaldi, A., " A Study On Split-Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.
14. Ahmad, K., " On Some Split-Complex Diophantine Equations", Neoma Journal Of Mathematics and Computer Science, 2023.

15. Ali, R., " On The Weak Fuzzy Complex Inner Products On Weak Fuzzy Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.

16. Merkepci, M., and Abobala, M., " Security Model for Encrypting Uncertain Rational Data Units Based on Refined Neutrosophic Integers Fusion and El Gamal Algorithm ", Fusion: Practice and Applications, 2023.
17. Ben Othman, K., Albasheer, O., Nadweh, R., Von Shtawzen, O., and Ali, R., "On The Symbolic 6-Plithogenic and 7-Plithogenic Rings", Galoitica Journal of Mathematical Structures and Applications, Vol.8, 2023.
18. Bisher Ziena, M., and Abobala, M., " On The Refined Neutrosophic Real Analysis Based on Refined Neutrosophic Algebraic AH-Isometry", Neutrosophic Sets and Systems, vol. 54, 2023.
19. M. B. Zeina and M. Abobala, "A Novel Approach of Neutrosophic Continuous Probability Distributions using AH-Isometry with Applications in Medicine," in *Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics*, Elsevier, 2023.
20. M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, “Introduction to Symbolic 2-Plithogenic Probability Theory,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.
21. Ali, R., and Hasan, Z., "An Introduction To The Symbolic 3-Plithogenic Vector Spaces", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.
22. Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.
23. Ben Othman, K., "On Some Algorithms For Solving Symbolic 3-Plithogenic Equations", Neoma Journal Of Mathematics and Computer Science, 2023.

24. Florentin Smarandache, Plithogenic Algebraic Structures. Chapter in "Nidus idearum Scilogs, V: joining the dots" (third version), Pons Publishing Brussels, pp. 123-125, 2019.
25. Florentin Smarandache: Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets – Revisited, *Neutrosophic Sets and Systems*, vol. 21, 2018, pp. 153-166.
26. Sarkis, M., " On The Solutions Of Fermat's Diophantine Equation In 3-refined Neutrosophic Ring of Integers", Neoma Journal of Mathematics and Computer Science, 2023.
27. Abobala, M., "On Some Algebraic Properties of n-Refined Neutrosophic Elements and n-Refined Neutrosophic Linear Equations", Mathematical Problems in Engineering, Hindawi, 2021
28. Abobala, M., "A Study Of Nil Ideals and Kothe's Conjecture In Neutrosophic Rings", International Journal of Mathematics and Mathematical Sciences, hindawi, 2021
29. Abobala, M., "On The Characterization of Maximal and Minimal Ideals In Several Neutrosophic Rings", Neutrosophic sets and systems, Vol. 45, 2021.
30. Abobala, M., Bal, M., Aswad, M., "A Short Note On Some Novel Applications of Semi Module Homomorphisms", International journal of neutrosophic science, 2022.
31. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Concept Of Symbolic 7-Plithogenic Real Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.
32. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Symbolic 8-Plithogenic Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

33. Abualkishik, A., Almajed, R., Thompson, W., "Improving The Performance of Fog-assisted Internet of Things Networks Using Bipolar Trapezoidal Neutrosophic Sets", Journal of Wireless and Ad Hoc Communication, 2023.
34. Ibrahim, M., and Abobala, M., "An Introduction To Refined Neutrosophic Number Theory", Neutrosophic Sets and Systems, Vol. 45, 2021.
35. Sankari, H., and Abobala, M., "Neutrosophic Linear Diophantine Equations With two Variables", Neutrosophic Sets and Systems, Vol. 38, pp. 22-30, 2020.
36. Merkepci, M., and Abobala, M., " On Some Novel Results About Split-Complex Numbers, The Diagonalization Problem And Applications To Public Key Asymmetric Cryptography", Journal of Mathematics, Hindawi, 2023.

Received 1/5/2023, Accepted 10/10/2023

Neutrosophic Sets and Systems, Vol. 59, 2023
University of New Mexico

On The Symbolic 2-Plithogenic Weak Fuzzy Complex Numbers

Mohamed Soueycatt

Department of Bioengineering, Al-Andalus Private University for Medical Sciences, Syria

m.soueycatt55@au.edu.sy

Abstract:

The goal of this paper is to define for the first time the concept of symbolic 2-plithogenic weak fuzzy complex number as new generalization generated by combining real numbers with symbolic 2-plithogenic numbers.

We study the elementary properties of this new class such as Invertibility and nilpotency, with many related examples that explain its novelty.

Keywords: symbolic 2-plithogenic number, weak fuzzy complex number, real number.

Introduction and preliminaries.

The concept of weak fuzzy complex numbers was defined firstly in [7] by the following form: $C_w = \{a + bJ;\ J^2 = t \in\ ]0,1[, a, b \in R\}$.

It is clear that C_w contains the real field R.

Weak fuzzy complex numbers were used to study vector space theory in [10], and programmed with Python [3].

Weak fuzzy complex numbers and their similar real extensions [8-9,15] are very useful in algebraic studies and computer science, especially split-complex numbers.

The concept of symbolic 2-plithogenic numbers was presented in [4] as a direct application of symbolic n-plithogenic sets in algebraic structures [1-3]. Also, many

zenodo.10031252/10.5281

generalizations of symbolic 2-plithogenic algebraic structures and 3-plithogenic structures were defined by many authors, see [5-6,11-14].

In this paper, we combine symbolic 2-plithogenic real ring $2-SP_R$ with weak fuzzy complex ring C_w, to get a novel generalization of real numbers.

We discuss some of their elementary algebraic properties in terms of theorems with many easy and clear illustrated examples.

Main concepts.

Definition.

We define the set of symbolic 2-plithogenic weak complex numbers as follow:

$2-SP_w=\{(x_0+x_1P_1+x_2P_2)+J(y_0+y_1P_1+y_2P_2); x_i, y_i, \in R, J^2=t\in]0,1[\}$

Addition on $2-SP_w$ is defined as follows:

For $X=(a_0+a_1P_1+a_2P_2)+J(b_0+b_1P_1+b_2P_2)$,

$Y=(c_0+c_1P_1+c_2P_2)+J(d_0+d_1P_1+d_2P_2)$.

$X+Y=[(a_0+c_0)+(a_1+c_1)P_1+(a_2+c_2)P_2]+J[(b_0+d_0)+(b_1+d_1)P_1+(b_2+d_2)P_2]$.

Multiplication on $2-SP_w$ is defined as follows:

$X.Y=(a_0+a_1P_1+a_2P_2)(c_0+c_1P_1+c_2P_2)+t(b_0+b_1P_1+b_2P_2)(d_0+d_1P_1+d_2P_2)+J[(a_0+a_1P_1+a_2P_2)(d_0+d_1P_1+d_2P_2)+(b_0+b_1P_1+b_2P_2)(c_0+c_1P_1+c_2P_2)]=(a_0c_0+tb_0d_0)+P_1(a_0c_1+a_1c_0+a_1c_1+tb_0d_1+tb_1d_0+tb_1d_1)+P_2(a_0c_2+a_1c_2+a_2c_0+a_2c_1+a_2c_2+tb_0d_2+tb_1d_2+tb_2d_0+tb_2d_1+tb_2d_2)+J[(a_0d_0+b_0c_0)+P_1(a_0d_1+a_1d_0+a_1d_1+b_0c_1+b_1c_0+b_1c_1)+P_2(a_0d_2+a_1d_2+a_2d_0+a_2d_1+a_1d_2+a_2d_2+b_0c_2+b_1c_2+b_2c_0+b_2c_1+b_2c_2)]$.

Example.

Take $X=(P_1-P_2)+J(3-P_2), Y=(1+P_2)+J(P_2); J^2=t=\frac{1}{2}$.

$X+Y=(1+P_1)+J(3)=(1+P_1)+3J$.

$X.Y=P_1+P_1-P_2-P_2+\frac{1}{2}(3P_2-P_2)+J[P_2-P_2+3+3P_2-P_2-P_2]=(2P_1-2P_2+P_2)+J(3-P_2)=(2P_1-P_2)+J(3+P_2)$.

Remark.

$(2-SP_w,+,.)$ Is a commutative ring.

Invertibility

It is known that $A+BJ$ is invertible if and only if $A+B\sqrt{t}, A-B\sqrt{t}; J^2=t\in]0,1[$ are invertible.

This means that $X=(a_0+a_1P_1+a_2P_2)+J(b_0+b_1P_1+b_2P_2)$ is invertible if and only if

$A+B\sqrt{t}=(a_0+b_0\sqrt{t})+(a_1+b_1\sqrt{t})P_1+(a_2+b_2\sqrt{t})P_2$

$A-B\sqrt{t}=(a_0-b_0\sqrt{t})+(a_1-b_1\sqrt{t})P_1+(a_2-b_2\sqrt{t})P_2$

Are invertible in $2-SP_R$.

It is known from the invertibility of symbolic 2-plithogenic real numbers that:

$A+B\sqrt{t}$ is invertible if and only if:

$a_0+b_0\sqrt{t}\neq 0, (a_0+a_1)+(b_0+b_1)\sqrt{t}\neq 0, (a_0+a_1+a_2)+(b_0+b_1+b_2)\sqrt{t}\neq 0$

which is equivalent to:

$$\begin{cases} \sqrt{t}\neq -\frac{a_0}{b_0} \ or \\ \sqrt{t}\neq -\frac{(a_0+a_1)}{b_0+b_1} \ or \\ \sqrt{t}\neq -\frac{(a_0+a_1+a_2)}{b_0+b_1+b_2} \end{cases} \text{for } b_0, b_0+b_1, b_0+b_1+b_2\neq 0.$$

Or $\begin{cases} b_0\neq 0 \\ b_0+b_1\neq 0 \\ b_0+b_1+b_2\neq 0 \end{cases}$

$A-B\sqrt{t}$ is invertible if and only if:

$a_0-b_0\sqrt{t}\neq 0, (a_0+a_1)-(b_0+b_1)\sqrt{t}\neq 0, (a_0+a_1+a_2)-(b_0+b_1+b_2)\sqrt{t}\neq 0$

which is equivalent to:

$b_0\neq 0, b_0+b_1\neq 0, b_0+b_1+b_2\neq 0.$

Or $\begin{cases} \sqrt{t}\neq \frac{a_0}{b_0} \ or \\ \sqrt{t}\neq \frac{(a_0+a_1)}{b_0+b_1} \ or \\ \sqrt{t}\neq \frac{(a_0+a_1+a_2)}{b_0+b_1+b_2} \end{cases}$

Example.

We try to find all non-invertible elements in $2-SP_w$.

Case1.

For $b_0 = 0,\ X = (a_0 + a_1P_1 + a_2P_2) + J(b_0 + b_1P_1 + b_2P_2);\ a_i, b_i \in R.$

Case2.

For $b_0 \neq 0, b_0 + b_1 = 0, X = (a_0 + a_1P_1 + a_2P_2) + J(b_0 - b_1P_1 + b_2P_2);\ a_i, b_i \in R.$

Case3.

For $b_0 \neq 0, b_0 + b_1 \neq 0, b_0 + b_1 + b_2 = 0, X = (a_0 + a_1P_1 + a_2P_2) + J(b_0 + b_1P_1 +$ $(-b_0 - b_1)P_2);\ a_i, b_i \in R.$

Case4.

$\sqrt{t} = \frac{a_0}{b_0}$ or $\sqrt{t} = -\frac{a_0}{b_0}, X = \left(\sqrt{t}b_0 + a_1P_1 + a_2P_2\right) + J(b_0 + b_1P_1 + b_2P_2);\ a_i, b_i \in R.$

Case5.

$\sqrt{t} = \frac{(a_0+a_1)}{b_0+b_1}$ or $\sqrt{t} = -\frac{(a_0+a_1)}{b_0+b_1}$, then:

$X = a_0 + P_1\left(\sqrt{t}(b_0 + b_1) - a_0\right) + a_2P_2 + J(b_0 + b_1P_1 + b_2P_2);\ a_i, b_i \in R.$

Case6.

$\sqrt{t} = \frac{(a_0+a_1+a_2)}{b_0+b_1+b_2}$ or $\sqrt{t} = -\frac{(a_0+a_1+a_2)}{b_0+b_1+b_2}$, then:

$X = a_0 + a_1P_1 + P_2\left(-\sqrt{t}(b_0 + b_1 + b_2) - a_0 - a_1\right) + J(b_0 + b_1P_1 + b_2P_2);\ a_i, b_i \in R$

Example.

For $J^2 = t = \frac{1}{9}$, take $X = (2 + P_1 - 5P_2) + J(5 + 6P_1 + 12P_2)$, X is invertible that is because:

$b_0 \neq 0, b_0 + b_1 \neq 0, b_0 + b_1 + b_2 \neq 0$, and

$$\begin{cases} \sqrt{t} = \dfrac{1}{3} \neq \dfrac{a_0}{b_0} = \dfrac{2}{5} \\ \sqrt{t} = \dfrac{1}{3} \neq -\dfrac{a_0}{b_0} = -\dfrac{2}{5} \\ \sqrt{t} = \dfrac{1}{3} \neq \dfrac{(a_0 + a_1)}{b_0 + b_1} = \dfrac{3}{11} \\ \sqrt{t} = \dfrac{1}{3} \neq -\dfrac{(a_0 + a_1)}{b_0 + b_1} = -\dfrac{3}{11} \\ \sqrt{t} = \dfrac{1}{3} \neq \dfrac{(a_0 + a_1 + a_2)}{b_0 + b_1 + b_2} = -\dfrac{2}{23} \\ \sqrt{t} = \dfrac{1}{3} \neq -\dfrac{(a_0 + a_1 + a_2)}{b_0 + b_1 + b_2} = \dfrac{2}{23} \end{cases}$$

Theorem.

Let $X = A + BJ \in 2 - SP_w, A, B \in 2 - SP_R$, then if X is invertible, we get:

$$X^{-1}=\frac{1}{2}\left[\left(A+B\sqrt{t}\right)^{-1}+\left(A-B\sqrt{t}\right)^{-1}\right]+\frac{1}{2\sqrt{t}}J\left[\left(A+B\sqrt{t}\right)^{-1}-\left(A-B\sqrt{t}\right)^{-1}\right]$$

Proof.

Put $Y=\frac{1}{2}\left[\left(A+B\sqrt{t}\right)^{-1}+\left(A-B\sqrt{t}\right)^{-1}\right]+\frac{1}{2\sqrt{t}}J\left[\left(A+B\sqrt{t}\right)^{-1}-\left(A-B\sqrt{t}\right)^{-1}\right]$.

$X.Y=\frac{1}{2}\left[A\left(A+B\sqrt{t}\right)^{-1}+A\left(A-B\sqrt{t}\right)^{-1}\right]+\frac{\sqrt{t}}{2}B\left(A+B\sqrt{t}\right)^{-1}-\frac{\sqrt{t}}{2}B\left(A-B\sqrt{t}\right)^{-1}+$ $\frac{1}{2\sqrt{t}}J\left[\frac{1}{2}B\left(A+B\sqrt{t}\right)^{-1}+\frac{1}{2}B\left(A-B\sqrt{t}\right)^{-1}+\frac{1}{2\sqrt{t}}A\left(A+B\sqrt{t}\right)^{-1}-\frac{1}{2\sqrt{t}}A\left(A-B\sqrt{t}\right)^{-1}\right]=$ $\frac{1}{2}\left(A+B\sqrt{t}\right)\left(A+B\sqrt{t}\right)^{-1}+\frac{1}{2}\left(A-B\sqrt{t}\right)\left(A-B\sqrt{t}\right)^{-1}+J\left[\frac{1}{2}\left(B+\frac{A}{\sqrt{t}}\right)\left(A+B\sqrt{t}\right)^{-1}+\right.$ $\left.\frac{1}{2}\left(B-\frac{A}{\sqrt{t}}\right)\left(A-B\sqrt{t}\right)^{-1}\right]=1+J\left[\frac{1}{2}\left(B+\frac{A}{\sqrt{t}}\right)\left(A+B\sqrt{t}\right)^{-1}-\frac{1}{2}\left(\frac{A-B\sqrt{t}}{\sqrt{t}}\right)\left(A-B\sqrt{t}\right)^{-1}\right]=$ $1+J(0)=1$, thus $X^{-1}=Y$.

Remark.

$A+B\sqrt{t}=\left(a_0+b_0\sqrt{t}\right)+\left(a_1+b_1\sqrt{t}\right)P_1+\left(a_2+b_2\sqrt{t}\right)P_2$.

$$\left(A+B\sqrt{t}\right)^{-1}=\frac{1}{a_0+b_0\sqrt{t}}+\left[\frac{1}{(a_0+a_1)+(b_0+b_1)\sqrt{t}}-\frac{1}{a_0+b_0\sqrt{t}}\right]P_1$$
$$+\left[\frac{1}{(a_0+a_1+a_2)+(b_0+b_1+b_2)\sqrt{t}}-\frac{1}{(a_0+a_1)+(b_0+b_1)\sqrt{t}}\right]P_2$$

$A-B\sqrt{t}=\left(a_0-b_0\sqrt{t}\right)+\left(a_1-b_1\sqrt{t}\right)P_1+\left(a_2-b_2\sqrt{t}\right)P_2$.

$\left(A+B\sqrt{t}\right)^{-1}=\frac{1}{a_0-b_0\sqrt{t}}+\left[\frac{1}{(a_0+a_1)-(b_0+b_1)\sqrt{t}}-\frac{1}{a_0-b_0\sqrt{t}}\right]P_1+\left[\frac{1}{(a_0+a_1+a_2)-(b_0+b_1+b_2)\sqrt{t}}-\right.$ $\left.\frac{1}{(a_0+a_1)-(b_0+b_1)\sqrt{t}}\right]P_2$.

On the other hand, we have:

$$\frac{1}{a_0+b_0\sqrt{t}}+\frac{1}{a_0-b_0\sqrt{t}}=\frac{a_0-b_0\sqrt{t}+a_0+b_0\sqrt{t}}{{a_0}^2-{b_0}^2t}=\frac{2a_0}{{a_0}^2-{b_0}^2t}\ldots\ (1)$$

$$\frac{1}{a_0+b_0\sqrt{t}}-\frac{1}{a_0-b_0\sqrt{t}}=\frac{a_0-b_0\sqrt{t}-a_0-b_0\sqrt{t}}{{a_0}^2-{b_0}^2t}=\frac{-2b_0\sqrt{t}}{{a_0}^2-{b_0}^2t}\ldots\ (\acute{1})$$

$$\frac{1}{(a_0+a_1)+(b_0+b_1)\sqrt{t}}+\frac{1}{(a_0+a_1)-(b_0+b_1)\sqrt{t}}=\frac{2(a_0+a_1)}{(a_0+a_1)^2-(b_0+b_1)^2t}\ldots\ (2)$$

$$\frac{1}{(a_0+a_1)+(b_0+b_1)\sqrt{t}}-\frac{1}{(a_0+a_1)-(b_0+b_1)\sqrt{t}}=\frac{-2(b_0+b_1)\sqrt{t}}{(a_0+a_1)^2-(b_0+b_1)^2t}\ldots\ (\acute{2})$$

$$\frac{1}{(a_0+a_1+a_2)+(b_0+b_1+b_2)\sqrt{t}}+\frac{1}{(a_0+a_1+a_2)-(b_0+b_1+b_2)\sqrt{t}}$$
$$=\frac{2(a_0+a_1+a_2)}{(a_0+a_1+a_2)^2-(b_0+b_1+b_2)^2t}\dots(3)$$
$$\frac{1}{(a_0+a_1+a_2)+(b_0+b_1+b_2)\sqrt{t}}-\frac{1}{(a_0+a_1+a_2)-(b_0+b_1+b_2)\sqrt{t}}$$
$$=\frac{2(b_0+b_1+b_2)\sqrt{t}}{(a_0+a_1+a_2)^2-(b_0+b_1+b_2)^2t}\dots(\acute{3})$$

This implies that:

$$X^{-1}=\frac{a_0}{a_0{}^2-b_0{}^2t}+\left(\frac{a_0+a_1}{(a_0+a_1)^2-(b_0+b_1)^2t}-\frac{a_0}{a_0{}^2-b_0{}^2t}\right)P_1+\left(\frac{a_0+a_1+a_2}{(a_0+a_1+a_2)^2-(b_0+b_1+b_2)^2t}-\frac{a_0+a_1}{(a_0+a_1)^2-(b_0+b_1)^2t}\right)P_2+J\left[-\frac{b_0}{a_0{}^2-b_0{}^2t}+\left(\frac{-(b_0+b_1)}{(a_0+a_1)^2-(b_0+b_1)^2t}+\frac{b_0}{a_0{}^2-b_0{}^2t}\right)P_1+\left(\frac{-(b_0+b_1+b_2)}{(a_0+a_1+a_2)^2-(b_0+b_1+b_2)^2t}+\frac{(b_0+b_1)}{(a_0+a_1)^2-(b_0+b_1)^2t}\right)P_2\right].$$

Examples.

Take $J^2=t=\frac{1}{100}$ $X=(1+P_1+P_2)+J(5-P_1+P_2)$, then $\sqrt{t}=\frac{1}{10}, a_0=a_1=a_2=1, b_0=5, b_1=-1, b_2=1.$

$$X^{-1}=\frac{1}{1-\frac{25}{10}}+\left(\frac{2}{4-\frac{16}{10}}-\frac{1}{1-\frac{25}{10}}\right)P_1+\left(\frac{3}{9-\frac{25}{10}}-\frac{2}{4-\frac{16}{10}}\right)P_2$$
$$+J\left[\frac{-5}{1-\frac{25}{10}}+\left(\frac{-4}{4-\frac{16}{10}}+\frac{5}{1-\frac{25}{10}}\right)P_1+\left(\frac{-5}{1-\frac{25}{10}}+\frac{4}{4-\frac{16}{10}}\right)P_2\right]$$
$$=\frac{-10}{15}+\left(\frac{20}{24}+\frac{10}{15}\right)P_1+\left(\frac{30}{65}-\frac{20}{24}\right)P_2$$
$$+J\left[\frac{50}{15}+\left(\frac{-40}{24}-\frac{50}{15}\right)P_1+\left(\frac{-50}{65}+\frac{40}{24}\right)P_2\right]$$
$$=\frac{-2}{3}+\left(\frac{5}{6}+\frac{2}{3}\right)P_1+\left(\frac{6}{13}-\frac{5}{6}\right)P_2$$
$$+J\left[\frac{10}{3}+\left(\frac{-5}{3}-\frac{10}{3}\right)P_1+\left(\frac{-10}{3}+\frac{5}{3}\right)P_2\right]$$
$$=\left(\frac{-2}{3}+\frac{3}{2}P_1-\frac{29}{79}P_2\right)+J\left[\left(\frac{10}{3}-5P_1+\frac{35}{39}P_2\right)\right]$$

Natural power.

Let $X=A+BJ; A,B\in 2-SP_R$, then:

$$X^n = \frac{1}{2}\left[\left(A + B\sqrt{t}\right)^n + \left(A - B\sqrt{t}\right)^n\right] + \frac{1}{2\sqrt{t}}J\left[\left(A + B\sqrt{t}\right)^n - \left(A - B\sqrt{t}\right)^n\right]$$

The previous result can be proven easily by induction.

We have:

$$A + B\sqrt{t} = \left(a_0 + b_0\sqrt{t}\right) + \left(a_1 + b_1\sqrt{t}\right)P_1 + \left(a_2 + b_2\sqrt{t}\right)P_2$$

$$\left(A + B\sqrt{t}\right)^n = \left(a_0 + b_0\sqrt{t}\right)^n + \left[\left(a_0 + a_1 + (b_0 + b_1)\sqrt{t}\right)^n - \left(a_0 + b_0\sqrt{t}\right)^n\right]P_1$$
$$+\left[\left(a_0 + a_1 + a_2 + (b_0 + b_1 + b_2)\sqrt{t}\right)^n - \left(a_0 + a_1 + (b_0 + b_1)\sqrt{t}\right)^n\right]P_2$$

$$A - B\sqrt{t} = \left(a_0 - b_0\sqrt{t}\right) + \left(a_1 - b_1\sqrt{t}\right)P_1 + \left(a_2 - b_2\sqrt{t}\right)P_2$$

$$\left(A + B\sqrt{t}\right)^n = \left(a_0 - b_0\sqrt{t}\right)^n + \left[\left(a_0 + a_1 - (b_0 + b_1)\sqrt{t}\right)^n - \left(a_0 - b_0\sqrt{t}\right)^n\right]P_1$$
$$+\left[\left((a_0 + a_1 + a_2) - (b_0 + b_1 + b_2)\sqrt{t}\right)^n - \left(a_0 + a_1 - (b_0 + b_1)\sqrt{t}\right)^n\right]P_2$$

This implies that:

$$X^n = \tfrac{1}{2}\Big[\left(a_0 + b_0\sqrt{t}\right)^n + \left(a_0 - b_0\sqrt{t}\right)^n + \Big(\left(a_0 + a_1 + (b_0 + b_1)\sqrt{t}\right)^n + \big(a_0 + a_1 - (b_0 + b_1)\sqrt{t}\big)^n - \left(a_0 + b_0\sqrt{t}\right)^n - \left(a_0 - b_0\sqrt{t}\right)^n\Big)P_1 + \Big(\left((a_0 + a_1 + a_2) + (b_0 + b_1 + b_2)\sqrt{t}\right)^n + \left((a_0 + a_1 + a_2) - (b_0 + b_1 + b_2)\sqrt{t}\right)^n - \left(a_0 + a_1 + (b_0 + b_1)\sqrt{t}\right)^n - \left(a_0 + a_1 - (b_0 + b_1)\sqrt{t}\right)^n\Big)P_2\Big] + \tfrac{1}{2\sqrt{t}}J\tfrac{1}{2}\Big[\left(a_0 + b_0\sqrt{t}\right)^n + \left(a_0 - b_0\sqrt{t}\right)^n + \Big(\big(a_0 + a_1 + (b_0 + b_1)\sqrt{t}\big)^n - \left(a_0 + a_1 - (b_0 + b_1)\sqrt{t}\right)^n - \left(a_0 + b_0\sqrt{t}\right)^n + \left(a_0 - b_0\sqrt{t}\right)^n\Big)P_1 + \Big(\left((a_0 + a_1 + a_2) + (b_0 + b_1 + b_2)\sqrt{t}\right)^n + \left((a_0 + a_1 + a_2) - (b_0 + b_1 + b_2)\sqrt{t}\right)^n - \left(a_0 + a_1 + (b_0 + b_1)\sqrt{t}\right)^n + \left(a_0 + a_1 - (b_0 + b_1)\sqrt{t}\right)^n\Big)P_2\Big].$$

Definition.

Let $X = A + BJ \in 2 - SP_w; A, B \in 2 - SP_R$, we say that:

1). X is 2-nilpotent if and only if $X^2 = 0$.

2). X is 3-nilpotent if and only if $X^3 = 0$.

The equation $X^2 = 0$ is equivalent to:

$$\begin{cases} A^2 + B^2 t = 0 \ldots (1) \\ 2AB = 0 \ldots (2) \end{cases}$$

We multiply (1) by A to get $A^3 = 0 \Rightarrow A = 0$.

We multiply (2) by B to get $B^3 = 0 \Rightarrow B = 0$

So that the only 2-nilpotent element in $2 - SP_w$ is 0.

By a similar discussion, we get that only m-nilpotent element in $2-SP_w$ is 0.

Conclusion:

In this paper, we have defined for the first time the class of symbolic 2-plithogenic weak fuzzy complex numbers by combining two algebraic classes (symbolic 2-plithogenic numbers and weak fuzzy complex numbers). Also, we have studied some of their elementary properties such as Invertibility and nilpotency, where a formula to compute the invers of a symbolic 2-plithogenic weak fuzzy complex number is obtained.

In the future, we encourage other researchers to study matrices with symbolic 2-plithogenic weak fuzzy complex numbers.

References

1. Nader Mahmoud Taffach , Ahmed Hatip., "A Review on Symbolic 2-Plithogenic Algebraic Structures " Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.
2. Nader Mahmoud Taffach , Ahmed Hatip.," A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations ", Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.
3. Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.
4. Smarandache, F., " Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)", Neutrosophic Sets and Systems, vol. 53, 2023.
5. Taffach, N., " An Introduction to Symbolic 2-Plithogenic Vector Spaces Generated from The Fusion of Symbolic Plithogenic Sets and Vector Spaces", Neutrosophic Sets and Systems, Vol 54, 2023.
6. Taffach, N., and Ben Othman, K., " An Introduction to Symbolic 2-Plithogenic Modules Over Symbolic 2-Plithogenic Rings", Neutrosophic Sets and Systems, Vol 54, 2023.

7. Hatip, A., "An Introduction To Weak Fuzzy Complex Numbers ", Galoitica Journal Of Mathematical Structures and Applications, Vol.3, 2023.
8. Khaldi, A., " A Study On Split-Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.
9. Ahmad, K., " On Some Split-Complex Diophantine Equations", Neoma Journal Of Mathematics and Computer Science, 2023.
10. Ali, R., " On The Weak Fuzzy Complex Inner Products On Weak Fuzzy Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.
11. M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, "Introduction to Symbolic 2-Plithogenic Probability Theory," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.
12. Ali, R., and Hasan, Z., "An Introduction To The Symbolic 3-Plithogenic Vector Spaces", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.
13. Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.
14. Ben Othman, K., "On Some Algorithms For Solving Symbolic 3-Plithogenic Equations", Neoma Journal Of Mathematics and Computer Science, 2023.
15. Merkepci, M., and Abobala, M., " On Some Novel Results About Split-Complex Numbers, The Diagonalization Problem And Applications To Public Key Asymmetric Cryptography", Journal of Mathematics, Hindawi, 2023.
16. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Concept Of Symbolic 7-Plithogenic Real Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

17. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Symbolic 8-Plithogenic Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.
18. Mohamed, M., "Modified Approach for Optimization of Real-Life Transportation Environment: Suggested Modifications", American Journal of Business and Operations research, 2021.

Received 15/6/2023, Accepted 4/10/2023

NSS

Neutrosophic Sets and Systems, Vol. 59, 2023

University of New Mexico

On Clean and Nil-clean Symbolic 2-Plithogenic Rings

P. Prabakaran[1], Florentin Smarandache[2]

[1]Department of Mathematics, Bannari Amman Institute of Technology, Sathyamangalam-638401, Tamil Nadu, India; E-mail: prabakaranpvkr@gmail.com

[2]University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA; E-mail: fsmarandache@gmail.com

*Correspondence: prabakaranpvkr@gmail.com

Abstract. A ring is said to be clean if every element of the ring can be written as a sum of an idempotent element and a unit element of the ring and a ring is said to be nil-clean if every element of the ring can be written as a sum of an idempotent element and a nilpotent element of the ring. In this paper, we generalize these arguments to symbolic 2-plithogenic structure. We introduce the structure of clean and nil-clean symbolic 2-plithogenic rings and some of its elementary properties are presented. Also, we have found the equivalence between classical clean(nil-clean) ring R and the corresponding symbolic 2-plithogenic ring $2-SP_R$.

Keywords: Clean ring; nil-clean ring; symbolic 2-plithogenic ring; clean symbolic 2-plithogenic ring; nil-clean symbolic 2-plithogenic ring.

---

1. **Introduction**

The concept of refined neutrosophic structure was studied by many authors in [1–5]. Symbolic plithogenic algebraic structures are introduced by Smarandache, that are very similar to refined neutrosophic structures with some differences in the definition of the multiplication operation [17].

In [14], the algebraic properties of symbolic 2-plithogenic rings generated from the fusion of symbolic plithogenic sets with algebraic rings, and some of the elementary properties and substructures of symbolic 2-plithogenic rings such as AH-ideals, AH-homomorphisms, and AHS-isomorphisms are studied. In [7], some more algebraic properties of symbolic 2-plithogenic

zenodo.10031256/10.5281

rings are studied. Further, Taffach [15,16] studied the concepts of symbolic 2-plithogenic vector spaces and modules.

In [8], the concept of symbolic 2-plithogenic matrices with symbolic 2-plithogenic entries, determinants, eigen values and vectors, exponents, and diagonalization are studied. Hamiyet Merkepci et.al [12], studied the the symbolic 2-plithogenic number theory and integers. Ahmad Khaldi et.al [11], studied the different types of algebraic symbolic 2-plithogenic equations and its solutions.

In [18], H. Suryoto and T. Uidjiani studied the concept of neutrosophic clean ring with many elementary interesting properties. Recently, M. Abobala [6], proved that a neutrosophic ring $R(I)$ is clean if and only if R is clean. Motivated by this works, in this paper we have introduced and studied the notion of clean and nil-clean symbolic 2-plithogenic rings. Also, we proved that a symbolic 2-plithogenic $2-SP_R$ is clean(nil-clean) if and only if R is clean(nil-clean).

2. Preliminaries

Definition 2.1. [14] Let R be a ring, the symbolic 2-plithogenic ring is defined as follows:

$$2-SP_R = \left\{a_0 + a_1P_1 + a_2P_2; a_i \in R, P_j^2 = P_j, P_1 \times P_2 = P_{max(1,2)} = P_2\right\}$$

Smarandache has defined algebraic operations on $2-SP_R$ as follows:

Addition:

$[a_0 + a_1P_1 + a_2P_2] + [b_0 + b_1P_1 + b_2P_2] = (a_0 + b_0) + (a_1 + b_1)P_1 + (a_2 + b_2)P_2$

Multiplication:

$[a_0 + a_1P_1 + a_2P_2].[b_0 + b_1P_1 + b_2P_2] = a_0b_0 + a_0b_1P_1 + a_0b_2P_2 + a_1b_0P_1^2 + a_1b_2P_1P_2 + a_2b_0P_2 + a_2b_1P_1P_2 + a_2b_2P_2^2 + a_1b_1P_1P_1 = (a_0b_0) + (a_0b_1 + a_1b_0 + a_1b_1)P_1 + (a_0b_2 + a_1b_2 + a_2b_0 + a_2b_1 + a_2b_2)P_2$.

It is clear that $2-SP_R$ is a ring. If R is a field, then $2-SP_R$ is called a symbolic 2-plithogenic field. Also, if R is commutative, then $2-SP_R$ is commutative, and if R has a unity (1), than $2-SP_R$ has the same unity (1).

Example 2.2. [14] Consider the ring $R = Z_4 = \{0, 1, 2, 3, 4\}$, the corresponding $2-SP_R$ is:

$$2-SP_R = \{a + bP_1 + cP_2; a, b, c \in Z_4\}.$$

If $X = 1 + 2P_1 + 3P_2, Y = P_1 + 2P_2$; then, $X + Y = 1 + 3P_1 + P_2$, $X - Y = 1 + P_1 + P_2$, $X.Y = 3P_1 + 3P_2$.

Theorem 2.3. *[14] Let $2-SP_R$ be a 2-plithogenic symbolic ring, with unity (1). Let $X = x_0 + x_1P_1 + x_2P_2$ be an arbitrary element, then:*

(1) *X is invertible if and only if $x_0, x_0 + x_1, x_0 + x_1 + x_2$ are invertible.*

(2) $X^{-1} = x_0^{-1} + [(x_0 + x_1)^{-1} - x_0^{-1}]P_1 + [(x_0 + x_1 + x_2)^{-1} - (x_0 + x_1)^{-1}]P_2$

Definition 2.4. [14] Let $X = a + bP_1 + cP_2 \in 2 - SP_R$, then X is idempotent if and only if $X^2 = X$.

Theorem 2.5. *[14] Let $X = a + bP_1 + cP_2 \in 2 - SP_R$, then X is idempotent if and only if $a, a + b, a + b + c$ are idempotent.*

Theorem 2.6. *[14] Let $2 - SP_R$ be a commutative symbolic 2-plithogenic ring, hence if $X = a + bP_1 + cP_2$, then $X^n = a^n + [(a + b)^n - a^n]P_1 + [(a + b + c)^n - (a + b)^n]P_2$ for every $n \in Z^+$.*

Definition 2.7. [14] X is called nilpotent if there exists $n \in Z^+$ such that $X^n = 0$.

Theorem 2.8. *[14] Let $X = a + bP_1 + cP_2 \in 2 - SP_R$, where R is commutative ring, then X is nilpotent if and only if $a, a + b, a + b + c$ are nilpotent.*

3. Clean Symbolic 2-Plithogenic Rings

We begin with the following definition.

Definition 3.1. Let R be any ring, $2 - SP_R$ be its corresponding symbolic 2-plithogenic ring. An element $x \in 2 - SP_R$ is said to be clean if $x = e + u$, where e is an idempotent and u is a unit element of $2 - SP_R$. If, in addition, the existing idempotent e and the unit u are unique, then x is called uniquely clean element.

In this section, we use the notation $U(2 - SP_R)$ to the set of all units in $2 - SP_R$ and $Id(2 - SR_R)$ to the set of all idempotent elements in $2 - SP_R$.

Example 3.2. Consider the symbolic 2-plithogenic ring

$$\begin{aligned} 2 - SP_{Z_2} &= \{a + bP_1 + cP_2; a, b, c \in Z_2\} \\ &= \Big\{0, 1, P_1, P_2, P_1 + P_2, 1 + P_1, 1 + P_2, 1 + P_1 + P_2\Big\}. \end{aligned}$$

Here, $U(2 - SP_{Z_2}) = 1$ and $Id(2 - SR_{Z_2}) = \{0, 1, P_1, P_2, P_1 + P_2, 1 + P_1, 1 + P_2, 1 + P_1 + P_2\}$. We can easily verify that every element of $2 - SP_{Z_2}$ can be expressed as a sum of an idempotent and a unit in $2 - SP_{Z_2}$. Hence, all the elements in $2 - SP_{Z_2}$ are clean elements. Since 1 is the only unit element in $2 - SP_{Z_2}$, so all the elements in $2 - SP_{Z_2}$ are uniquely clean elements.

Definition 3.3. A symbolic 2-plithogenic ring in which all elements are clean, then the ring is called a clean symbolic 2-plithogenic ring. Furthermore, if each element of the symbolic 2-plithogenic ring is uniquely clean, then the ring is called a uniquely clean symbolic 2-plithogenic ring.

Example 3.4. By the Example 3.2, the ring $2-SP_{Z_2}$ is a uniquely clean symbolic 2-plithogenic ring.

Example 3.5. Consider the symbolic 2-plithogenic ring

$$\begin{aligned} 2-SP_{Z_3} &= \{a+bP_1+cP_2; a,b,c \in Z_3\} \\ &= \left\{ \begin{array}{c} 0,1,2,P_1,P_2,2P_1,2P_2,P_1+P_2,2P_1+2P_2,P_1+2P_2,2P_1+P_2,1+P_1, \\ 1+P_2,1+P_1+P_2,1+2P_1,1+2P_2,1+2P_1+2P_2,1+P_1+2P_2, \\ 1+2P_1+P_2,2+P_1,2+P_2,2+P_1+P_2,2+2P_1,2+2P_2,2+2P_1+2P_2, \\ 2+P_1+2P_2,2+2P_1+P_2 \end{array} \right\}. \end{aligned}$$

Here, $U(2-SP_{Z_3}) = \{1,2,1+P_1,1+P_2,2+2P_1,2+2P_2,1+P_1+2P_2,2+2P_1+P_2\}$ and $Id(2-SR_{Z_3}) = \{0,1,P_1,P_2,P_1+2P_2,1+2P_1,1+2P_2,1+2P_1+P_2\}$. All the elements of $2-SP_{Z_3}$ are clean elements. Hence $2-SP_{Z_3}$ is a clean symbolic 2-plithogenic ring. Take $2+2P_1+P_2 \in 2-SP_{Z_3}$, clearly $2+2P_1+P_2 = (1+2P_1+P_2)+1$ and also we have $2+2P_1+P_2 = 0+(2+2P_1+P_2)$. Therefore $2+2P_1+P_2$ is not a uniquely clean element in $2-SP_{Z_3}$ and hence $2-SP_{Z_3}$ is not a uniquely clean.

Lemma 3.6. *Let R be a ring. Then the class of clean symbolic 2-plithogenic rings is closed under homomorphic images.*

Proof. It is clear since the homomorphic image of an idempotent element in a symbolic 2-plithogenic ring is again an idempotent. □

Theorem 3.7. *Let R be any ring, $2-SP_R$ be its corresponding symbolic 2-plithogenic ring. $2-SP_R$ is clean if and only if R is clean.*

Proof. Assume that $2-SP_R$ is clean. Since R is a homomorphic image of $2-SP_R$, so R is clean by Lemma 3.6.
Conversely, assume that R is clean, we must prove that $2-SP_R$ is clean. Let $x = a+bP_1+cP_2 \in 2-SP_R$ then $a, a+b, a+b+c \in R$. Since R is clean we have $a = e_1+u_1, a+b = e_2+u_2, a+b+c = e_3+u_3$, where e_i are idempotent elements and u_i are unit elements of R. Now,

$$\begin{aligned} x &= a+bP_1+cP_2 \\ &= a+[(a+b)-a]P_1+[(a+b+c)-(a+b)]P_2 \\ &= (e_1+u_1)+[(e_2+u_2)-(e_1+u_1)]P_1+[(e_3+u_3)-(e_2+u_2)]P_2 \\ &= (e_1+u_1)+[(e_2-e_1)+(u_2-u_1)]P_1+[(e_3-e_2)+(u_3-u_2)]P_2 \\ &= [e_1+(e_2-e_1)P_1+(e_3-e_2)P_2]+[u_1+(u_2-u_1)P_1+(u_3-u_2)P_2] \\ &= x_1+x_2. \end{aligned}$$

where, $x_1 = e_1 + (e_2 - e_1)P_1 + (e_3 - e_2)P_2$ and $x_2 = u_1 + (u_2 - u_1)P_1 + (u_3 - u_2)P_2$. By Theorem 2.5, $e_1, e_1 + (e_2 - e_1) = e_2, e_1 + (e_2 - e_1) + (e_3 - e_2) = e_3$ are idempotents in R. Therefore, x_1 is a idempotent element of R. Also, x_2 is a unit element of R by a similar discussion. Hence $2 - SP_R$ is clean. □

Definition 3.8. Let $2 - SP_R$ be a symbolic 2-plithogenic ring. An idempotent element $e \in 2 - SP_R$ is called a central idempotent if $e.x = x.e$ for every $x \in 2 - SP_R$. The set of all central idempotents of $2 - SP_R$ is denoted by $C(2 - SP_R)$.

Example 3.9. In the symbolic 2-plithogenic ring $2 - SP_{Z_3}$, we have

$$Id(2 - SP_{Z_3}) = \{0, 1, P_1, P_2, P_1 + 2P_2, 1 + 2P_1, 1 + 2P_2, 1 + 2P_1 + P_2\}.$$

As $2 - SP_{Z_3}$ is commutative so all the idempotents of $2 - SP_{Z_3}$ are central. Hence

$$C(2 - SP_{Z_3}) = \{0, 1, P_1, P_2, P_1 + 2P_2, 1 + 2P_1, 1 + 2P_2, 1 + 2P_1 + P_2\} = Id(2 - SP_{Z_3}).$$

Lemma 3.10. *If x is an idempotent element of $2 - SP_R$, then $1 - x$ is also an idempotent element of $2 - SP_R$, where 1 is the unit element of $2 - SP_R$.*

Proof. If x an idempotent element of $2 - SP_R$ then $x^2 = x$. But then, $(1 - x)^2 = 1 - 2x - x^2 = 1 - x$ and so $1 - x$ an idempotent element of $2 - SP_R$. □

Lemma 3.11. *Let $2 - SP_R$ be a symbolic 2-plithogenic ring with the identity 1. If $e \in C(2 - SP_R)$ then $1 - e \in C(2 - SP_R)$, where 1 is the unit element of $2 - SP_R$.*

Proof. Assume that $e \in C(2 - SP_R)$. For any $x \in 2 - SP_R$, we have $(1 - e).x = (1.x) - (e.x) = (x.1) - (x.e) = x(1 - e)$. Hence, $1 - e \in C(2 - SP_R)$. □

Theorem 3.12. *In any symbolic 2-plithogenic ring $2 - SP_R$, every central idempotent is a uniquely clean element.*

Proof. Let $x \in C(2 - SP_R)$. Then we have, $x^2 = x$ and $x = (1 - x) + (2x - 1) = e + u$, where $e = 1 - x$ is an idempotent by Lemma 3.10 and $u = 2x - 1$ is a unit element by Lemma 3.11. Hence x is a clean element. Also, if $x.y = y.x$ we obtain $e + u = (e + u)^2 = e + 2eu + u^2$, so $u = 1 - 2e$. Hence $e = 1 - x$. Thus x is a uniquely clean element. □

Theorem 3.13. *Every idempotent element in a uniquely clean 2-plithogenic ring is a central idempotent.*

Proof. Assume that $2-SP_R$ is a uniquely clean 2-plithogenic ring. Let $e \in 2-SP_R$ be an idempotent element and x be any element of $2-SP_R$. Now, the element $e+(ex-exe)$ is an idempotent and $1+(ex-exe)$ is a unit and $[e+(ex-exe)]+1 = e+[1+(ex-exe)]$. Since, $2-SP_R$ is a uniquely clean 2-plithogenic ring we have $e+(ex-exe) = e$. Hence $ex = exe$ and $xe = exe$, so $ex = ex$ as required. □

Definition 3.14. A symbolic 2-plithogenic ring $2-SP_R$ is called a boolean symbolic 2-plithogenic ring if $x^2 = x$ for all $x \in 2-SP_R$.

Example 3.15. In the symbolic 2-plithogenic ring $2-SR_{Z_2}$, all the elements are idempotent so $2-SR_{Z_2}$ is a boolean symbolic 2-plithogenic ring

For any boolean symbolic 2-plithogenic ring, we have the following result.

Theorem 3.16. *Every boolean symbolic 2-plithogenic ring is uniquely clean.*

Proof. If $2-SP_R$ is a boolean symbolic 2-plithogenic ring, then $2-SP_R = Id(2-SP_R)$. Since boolean rings are abelian, we have $Id(2-SP_R) = C(2-SP_R)$. This implies that, $2-SP_R = C(2-SP_R)$. By Theorem 3.12, every element of the ring $2-SP_R$ are uniquely clean. Hence $2-SP_R$ is uniquely clean ring. □

4. **Nil-clean Symbolic 2-Plithogenic Rings**

We begin with the following definition.

Definition 4.1. Let R be any ring, $2-SP_R$ be its corresponding symbolic 2-plithogenic ring. An element $x \in 2-SP_R$ is said to be nil-clean if $x = e+n$, where e is an idempotent and n is a nil-potent element of $2-SP_R$. If, in addition, the existing idempotent element and nil-potent elements are unique, then x is called uniquely nil-clean element.

Example 4.2. Consider the symbolic 2-plithogenic ring

$$\begin{aligned} 2-SP_{Z_3} &= \{a+bP_1+cP_2; a,b,c \in Z_3\} \\ &= \left\{ \begin{array}{c} 0,1,2,P_1,P_2,2P_1,2P_2,P_1+P_2,2P_1+2P_2,P_1+2P_2,2P_1+P_2,1+P_1, \\ 1+P_2,1+P_1+P_2,1+2P_1,1+2P_2,1+2P_1+2P_2,1+P_1+2P_2, \\ 1+2P_1+P_2,2+P_1,2+P_2,2+P_1+P_2,2+2P_1,2+2P_2,2+2P_1+2P_2, \\ 2+P_1+2P_2,2+2P_1+P_2 \end{array} \right\}. \end{aligned}$$

Since 0 is a nil-potent element in $2-SP_{Z_3}$, so the idempotent elements $0,1,P_1,P_2,P_1+2P_2,1+2P_1,1+2P_2,1+2P_1+P_2$ are nil-clean elements of $2-SP_{Z_3}$. The only nilpotent elements of $2-SP_{Z_3}$ is 0, so $0,1,P_1,P_2,P_1+2P_2,1+2P_1,1+2P_2,1+2P_1+P_2$ are uniquely nil-clean elements of $2-SP_{Z_3}$.

Definition 4.3. A symbolic 2-plithogenic ring in which all elements are nil-clean, then the ring is called a nil-clean symbolic 2-plithogenic ring. Furthermore, if each element of the symbolic 2-plithogenic ring is uniquely nil-clean, then the ring is called a uniquely nil-clean symbolic 2-plithogenic ring.

Example 4.4. $2-SP_{Z_2} = \{0,1,P_1,P_2,P_1+P_2,1+P_1,1+P_2,1+P_1+P_2\}$ is a nil-clean symbolic 2-plithogenic ring, that is because all the elements in $2-SP_{Z_2}$ are idempotents and 0 is a nilpotent element in $2-SP_{Z_2}$.

Lemma 4.5. *If x is a nilpotent element of $2-SP_R$, then $1+x$ is a unit in $2-SP_R$.*

Proof. If x is a nilpotent element of $2-SP_R$ then $x^k = 0$ for some $k > 0$. But then, $(1+x)(1-x+x^2-x^3+...+(-1)^{k-1}x^{k-1}) = 1$ and so $1+x$ is unit in $2-SP_R$. □

Theorem 4.6. *Every nil-clean symbolic 2-plithogenic ring is clean symbolic 2-plithogenic ring.*

Proof. Suppose that $2-SP_R$ is a nil-clean symbolic 2-plithogenic ring, and let $x \in 2-SP_R$. Then $x-1$ is an element of $2-SP_R$ and hence $x-1 = e+n$, where e is an idempotent element and n is a nilpotent element of $2-SP_R$.
This implies that, $x = e+(1+n)$ is a nil-clean element of $2-SP_R$ because $1+n$ is a unit element of $2-SP_R$ by Lemma 4.5. □

The converse of the Theorem 4.6 is not true. See the following example.

Example 4.7. Consider, the clean symbolic 2-plithogenic ring $2-SP_{Z_3}$. All the elements of $2-SP_{Z_3}$ are clean elements. The only nilpotent element of $2-SP_{Z_3}$ is 0 and P_1+P_2 is not an idempotent element in $2-SP_{Z_3}$ so it is not nil-clean. Hence $2-SP_{Z_3}$ is not a nil-clean ring.

Lemma 4.8. *Let R be a ring. Then the class of nil-clean symbolic 2-plithogenic rings is closed under homomorphic images.*

Proof. It is clear since the homomorphic image of a nil-potent element of a symbolic 2-plithogenic rings is again a nil-potent. □

Theorem 4.9. *Let R be any ring, $2-SP_R$ be its corresponding symbolic 2-plithogenic ring. $2-SP_R$ is nil-clean if and only if R is nil-clean.*

Proof. Assume that $2-SP_R$ is nil-clean. Since R is a homomorphic image of $2-SP_R$, so R is nil-clean by Lemma 4.8.
Conversely, assume that R is nil-clean, we must prove that $2-SP_R$ is nil-clean. Let $x = a+bP_1+cP_2 \in 2-SP_R$ then $a, a+b, a+b+c \in R$. Since R is nil-clean we have $a = e_1+n_1, a+b = e_2+n_2, a+b+c = e_3+n_3$, where e_i are idempotent elements and n_i are nilpotent elements of R. Now,

$$\begin{aligned} x &= a+bP_1+cP_2 \\ &= a+[(a+b)-a]P_1+[(a+b+c)-(a+b)]P_2 \\ &= (e_1+n_1)+[(e_2+n_2)-(e_1+n_1)]P_1+[(e_3+n_3)-(e_2+n_2)]P_2 \\ &= (e_1+n_1)+[(e_2-e_1)+(n_2-n_1)]P_1+[(e_3-e_2)+(n_3-n_2)]P_2 \\ &= [e_1+(e_2-e_1)P_1+(e_3-e_2)P_2]+[n_1+(n_2-n_1)P_1+(n_3-n_2)P_2] \\ &= x_1+x_2. \end{aligned}$$

where, $x_1 = e_1+(e_2-e_1)P_1+(e_3-e_2)P_2$ and $x_2 = n_1+(n_2-n_1)P_1+(n_3-n_2)P_2$. By Theorem 2.5, $e_1, e_1+(e_2-e_1) = e_2, e_1+(e_2-e_1)+(e_3-e_2) = e_3$ are idempotents in R. Therefore, x_1 is a idempotent element of R. Also, x_2 is a nilpotent element of R by a similar discussion and by Theoerem 2.8. Hence $2-SP_R$ is nil-clean. □

Theorem 4.10. *If $2-SP_R$ is a symbolic 2-plithogenic ring, then every central idempotent of $2-SP_R$ is uniquely nil-clean element.*

Proof. We know that, every idempotent element of $2-SP_R$ are nil-clean. Let x be a central idempotent element of $2-SP_R$. Then $x = (1-x)+(2x-1)$. Suppose that $x = e+n$, where e is an idempotent and n is a nilpotent element of $2-SP_R$. Since $nx = xn$, we obtain $e+n = (e+n)^2 = e+2en+n^2$. So, we have $n = 1-2e$ and hence $e = 1-x$, as reuired. □

Lemma 4.11. *Let $2-SP_R$ be uniquely nil-clean symbolic 2-plithogenic ring. Then all idempotents of $2-SP_R$ are central.*

Proof. Let $e \in 2-SP_R$ be an idempotent element and x be any element of $2-SP_R$. Now, the element $e+ex-exe$ can be written as $e+(ex-exe)$ or $(e+(ex-exe))+0$ each time as the sum of an idempotent and a nilpotent element of $2-SP_R$. Since $2-SP_R$ is uniquely nil clean, we have $e = e+(ex-exe)$. This implies that $ex-exe = 0$ and so $ex = exe$. In the similar way, we can show that $xe = exe$. Hence $ex = xe$ as required. □

Theorem 4.12. *Every boolean symbolic 2-plithogenic ring is uniquely nil-clean.*

Proof. If $2 - SP_R$ is a boolean symbolic 2-plithogenic ring, then $2 - SP_R = Id(2 - SP_R)$. Since boolean rings are abelian, we have $Id(2 - SP_R) = C(2 - SP_R)$. This implies that, $2 - SP_R = C(2 - SP_R)$. By Theorem 4.10, every element of the ring $2 - SP_R$ are uniquely nil-clean. Hence $2 - SP_R$ is uniquely nil-clean ring. □

5. **Conclusion**

In this article, we have introduced the the new classes of rings called, clean symbolic 2-plithogenic rings and nil-clean symbolic 2-plithogenic rings and we have studied various properties of clean and nil-clean symbolic 2-plithogenic rings with proper examples. Also, we have determined necessary and sufficient condition for a symbolic 2-plithogenic ring to be clean and nil-clean.

Funding: This research received no external funding.

References

1. Adeleke, E. O., Agboola, A. A. A., and Smarandache, F., "Refined neutrosophic rings II", International Journal of Neutrosophic Science, vol. 2, pp. 89-94, 2020.
2. Agboola, A. A. A., "On refined neutrosophic algebraic structures", Neutrosophic Sets and Systems, vol. 10, pp. 99?01, 2015.
3. Abobala, M., "On some special elements in neutrosophic rings and refined neutrosophic rings", Journal of New Theory, vol. 33, 2020.
4. Abobala, M., "On Refined Neutrosophic Matrices and Their Applications In Refined Neutrosophic Algebraic Equations", Journal Of Mathematics, Hindawi, 2021.
5. Abobala, M., "On Some Algebraic Properties of n-Refined Neutrosophic Elements and n-Refined Neutrosophic Linear Equations", Mathematical Problems in Engineering, Hindawi, 2021.
6. Abobala, M., Bal, M., and Hatip, A., "A Study Of Some Neutrosophic Clean Rings", International Journal of Neutrosophic Science (IJNS) Vol. 18, No. 01, pp. 14-19, 2022
7. Albasheer, O., Hajjari., A., and Dalla., R., "On The Symbolic 3-Plithogenic Rings and Their Algebraic Properties", Neutrosophic Sets and Systems, Vol 54, 2023.
8. Abuobida Mohammed A. Alfahal, Yaser Ahmad Alhasan, Raja Abdullah Abdulfatah, Arif Mehmood, Mustafa Talal Kadhim, "On Symbolic 2-Plithogenic Real Matrices and Their Algebraic Properties", International Journal of Neutrosophic Science, Vol. 21, PP. 96-104, 2023.
9. Ali, R., and Hasan, Z., "An Introduction To The Symbolic 3-Plithogenic Vector Spaces", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.
10. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Concept Of Symbolic 7-Plithogenic Real Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.
11. Khaldi, A., Ben Othman, K., Von Shtawzen, O., Ali, R., and Mosa, S., "On Some Algorithms for Solving Different Types of Symbolic 2-Plithogenic Algebraic Equations", Neutrosophic Sets and Systems, Vol 54, 2023.
12. Merkepci, H., and Rawashdeh, A., "On The Symbolic 2-Plithogenic Number Theory and Integers", Neutrosophic Sets and Systems, Vol 54, 2023.

13. Merkepci, H., "On Novel Results about the Algebraic Properties of Symbolic 3-Plithogenic and 4-Plithogenic Real Square Matrices", Symmetry, MDPI, 2023.
14. Merkepci, H., and Abobala, M., "On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.
15. Taffach, N., "An Introduction to Symbolic 2-Plithogenic Vector Spaces Generated from The Fusion of Symbolic Plithogenic Sets and Vector Spaces", Neutrosophic Sets and Systems, Vol 54, 2023.
16. Taffach, N., and Ben Othman, K., "An Introduction to Symbolic 2-Plithogenic Modules Over Symbolic 2-Plithogenic Rings", Neutrosophic Sets and Systems, Vol 54, 2023.
17. Smarandache, F., "Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)", Neutrosophic Sets and Systems, vol. 53, 2023.
18. Suryoto, H., and Uidjiani, T., "On Clean Neutrosophic Rings", IOP Conference Series, Journal of Physics, Conf. Series 1217, 2019.
19. Yaser Ahmad Alhasan, "Types of system of the neutrosophic linear equations and Cramer's rule", Neutrosophic Sets and Systems, vol. 45, 2021.

Received: 24/5/2023 / **Accepted:** 10/9/2023

NSS

Neutrosophic Sets and Systems, Vol. 59, 2023

University of New Mexico

Solution of System of Symbolic 2-Plithogenic Linear Equations using Cramer's Rule

P. Prabakaran[1], Florentin Smarandache[2]

[1]Department of Mathematics, Bannari Amman Institute of Technology, Sathyamangalam-638401, Tamil Nadu, India; E-mail: prabakaranpvkr@gmail.com

[2]University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA; E-mail: fsmarandache@gmail.com

*Correspondence: prabakaranpvkr@gmail.com

Abstract. In this article, the concept of system of symbolic 2-plithogenic linear equations and its solutions are introduced and studied. The Cramer's rule was applied to solve the system of symbolic 2-plithogenic linear equations. Also, provided enough examples for each case to enhance understanding.

Keywords: Symbolic 2-plithogenic linear equations; Cramer's rule; solution of the symbolic 2-plithogenic linear equations.

---

1. **Introduction**

The concept of refined neutrosophic structure was studied by many authors in [1–5]. Symbolic plithogenic algebraic structures are introduced by Smarandache, that are very similar to refined neutrosophic structures with some differences in the definition of the multiplication operation [10].

In [7], the algebraic properties of symbolic 2-plithogenic rings generated from the fusion of symbolic plithogenic sets with algebraic rings, and some of the elementary properties and substructures of symbolic 2-plithogenic rings such as AH-ideals, AH-homomorphisms, and AHS-isomorphisms are studied. In [11], some more algebraic properties of symbolic 2-plithogenic rings are studied. Further, Taffach [8,9] studied the concepts of symbolic 2-plithogenic vector spaces and modules.

zenodo.10031260/10.5281

In [14], the concept of symbolic 2-plithogenic matrices with symbolic 2-plithogenic entries, determinants, eigen values and vectors, exponents, and diagonalization are studied. Hamiyet Merkepci et.al [13], studied the the symbolic 2-plithogenic number theory and integers. Ahmad Khaldi et.al [12], studied the different types of algebraic symbolic 2-plithogenic equations and its solutions.

In [6], Yaser Ahmad Alhasan studied the types of the nuetrosophic linear equations and Cramer's rule to solve the system of nuetrosophic linear equations. Motivated by this work, in this article the symbolic 2-plithogenic linear equations and its solutions are introduced and studied. Also, enough examples are given for all the cases to enhance understanding.

2. Preliminaries

Definition 2.1. [7] Let R be a ring, the symbolic 2-plithogenic ring is defined as follows:

$$2-SP_R = \left\{a_0 + a_1P_1 + a_2P_2; a_i \in R, P_j^2 = P_j, P_1 \times P_2 = P_{max(1,2)} = P_2\right\}$$

Smarandache has defined algebraic operations on $2-SP_R$ as follows:
Addition:
$[a_0 + a_1P_1 + a_2P_2] + [b_0 + b_1P_1 + b_2P_2] = (a_0 + b_0) + (a_1 + b_1)P_1 + (a_2 + b_2)P_2$
Multiplication:
$[a_0 + a_1P_1 + a_2P_2].[b_0 + b_1P_1 + b_2P_2] = a_0b_0 + a_0b_1P_1 + a_0b_2P_2 + a_1b_0P_1^2 + a_1b_2P_1P_2 + a_2b_0P_2 + a_2b_1P_1P_2 + a_2b_2P_2^2 + a_1b_1P_1P_1 = (a_0b_0) + (a_0b_1 + a_1b_0 + a_1b_1)P_1 + (a_0b_2 + a_1b_2 + a_2b_0 + a_2b_1 + a_2b_2)P_2$.
It is clear that $2-SP_R$ is a ring. If R is a field, then $2-SP_R$ is called a symbolic 2-plithogenic field. Also, if R is commutative, then $2-SP_R$ is commutative, and if R has a unity (1), than $2-SP_R$ has the same unity (1).

Theorem 2.2. *[7] Let $2-SP_R$ be a 2-plithogenic symbolic ring, with unity (1). Let $X = x_0 + x_1P_1 + x_2P_2$ be an arbitrary element, then:*

(1) *X is invertible if and only if $x_0, x_0 + x_1, x_0 + x_1 + x_2$ are invertible.*
(2) $X^{-1} = x_0^{-1} + [(x_0 + x_1)^{-1} - x_0^{-1}]P_1 + [(x_0 + x_1 + x_2)^{-1} - (x_0 + x_1)^{-1}]P_2$

Definition 2.3. [14] A symbolic 2-plithogenic square real matrix is a matrix with symbolic 2-plithogenic real entries.

Theorem 2.4. *[14] Let $S = S_0 + S_1P_1 + S_2P_2$ be a symbolic 2-plithogenic square real matrix, then*

(1) *S is invertible if and only if $S_0, S_0 + S_1, S_0 + S_1 + S_2$ are invertible.*
(2) *If S is invertible then*

$$S^{-1} = S_0^{-1} + [(S_0 + S_1)^{-1} - S_0^{-1}]P_1 + [(S_0 + S_1 + S_2)^{-1} - (S_0 + S_1)^{-1}]$$

3. The Symbolic 2-Plithogenic Linear Equations

We begin this section with the following definition.

Definition 3.1. The symbolic 2-plithogenic linear equation of n variables $x_1, x_2, x_3, ..., x_n$, is each equation that takes the form:

$$(a_{01} + a_{11}P_1 + a_{21}P_2)x_1 + (a_{02} + a_{12}P_1 + a_{22}P_2)x_2 + (a_{03} + a_{13}P_1 + a_2P_{23})x_3 + \cdots + (a_{0n} + a_{1n}P_1 + a_{2n}P_2)x_n = b_0 + b_1P_1 + b_2P_2$$

where $a_{0i}, a_{1i}, a_{2i},\ i = 1, 2, ..., n$ are real coefficients. We call $(a_{01} + a_{11}P_1 + a_{21}P_2), (a_{02} + a_{12}P_1 + a_{22}P_2), (a_{03} + a_{13}P_1 + a_{23}P_2)$ symbolic 2-plithogenic coefficients of the borders of the equation, and $b_0 + b_1P_1 + b_2P_2$ constant symbolic 2-plithogenic border of the equation.

Remark 3.2.

(1) We call each equation of the form:

$$(a_0 + a_1P_1 + a_2P_2)x + (b_0 + b_1P_1 + b_2P_2)y = c_0 + c_1P_1 + c_2P_2$$

the two-variable symbolic 2-plithogenic linear equation, where, $a_0, a_1, a_2, b_0, b_1, b_2$, and c_0, c_1, c_2 are real coefficients.

(2) We call each equation of the form:

$$(a_0 + a_1P_1 + a_2P_2)x + (b_2 + b_1P_1 + b_2P_2)y + (c_0 + c_1P_1 + c_2P_2)z = d_0 + d_1P_1 + d_2P_2$$

the three-variable symbolic 2-plithogenic linear equation, where, $a_0, a_1, a_2, b_0, b_1, b_2$, c_0, c_1, c_2, and d_0, d_1, d_2 are real coefficients.

Example 3.3.

(1) $(1 + P_2)x + (3 - P_1)y + (1 + P_1 - P_2)z = 5$

(2) $P_2x + P_1y + (P_1 - P_2)z = 2P_1 + 2P_2$

(3) $(1 + P_1 - P_2)x + (4 + P_1 - P_2)y = 11 + 4P_2$

Definition 3.4. Solution of the symbolic 2-plithogenic linear equation,

$$(a_{01} + a_{11}P_1 + a_{21}P_2)x_1 + (a_{02} + a_{12}P_1 + a_{22}P_2)x_2 + (a_{03} + a_{13}P_1 + a_2P_{23})x_3 + \cdots + (a_{0n} + a_{1n}P_1 + a_{2n}P_2)x_n = b_0 + b_1P_1 + b_2P_2$$

is finding the values of the variables $x_1, x_2, x_3, ..., x_n$ that satisfies the equation, where $a_{0i}, a_{1i}, a_{2i},\ i = 1, 2, ..., n$ are real coefficients.

Example 3.5. Consider the following the two-variable symbolic 2-plithogenic linear equation:

$$(1 + P_1 - P_2)x + (4 + P_1 - P_2)y = 4 + \frac{11}{2}P_1 - \frac{15}{2}P_2$$

The solution of this equation is

$$x = 2 + P_1 + 2P_2, \quad y = \frac{1}{2} - P_2.$$

Definition 3.6. For the two variable symbolic 2-plithogenic linear equation

$$(a_0 + a_1P_1 + a_2P_2)x + (b_0 + b_1P_1 + b_2P_2)y = c_0 + c_1P_1 + c_2P_2$$

the infinite number of solutions defined by

$$y = -\frac{a_0 + a_1P_1 + a_2P_2}{b_0 + b_1P_1 + b_2P_2}x + \frac{c_0 + c_1P_1 + c_2P_2}{b_0 + b_1P_1 + b_2P_2}$$

where, $a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2$ are real coefficients, $b_0 \neq 0, b_0 + b_1 \neq 0$ and $b_0 + b_1 + b_2 \neq 0$ by given a value for one of the two variables, we obtain a value for the other variable.

Example 3.7. Consider the two variable symbolic 2-plithogenic linear equation

$$(1 + P_1 - P_2)x + (4 + P_1 - P_2)y = 1 + P_1 - 2P_2.$$

This implies that,

$$y = -\left(\frac{1 + P_1 - P_2}{4 + P_1 - P_2}\right)x + \left(\frac{1 + P_1 - 2P_2}{4 + P_1 - P_2}\right)$$

Then the set of solution is:

$$S = \left\{x, y \in 2 - SP_R : y = -\left(\frac{1 + P_1 - P_2}{4 + P_1 - P_2}\right)x + \left(\frac{1 + P_1 - 2P_2}{4 + P_1 - P_2}\right)\right\}$$

$$i.e., S = \left\{x, y \in 2 - SP_R : y = \left(-\frac{1}{4} - \frac{3}{20}P_1 + \frac{3}{20}P_2\right)x + \left(\frac{1}{4} - \frac{3}{20}P_1 - \frac{2}{5}P_2\right)\right\}$$

By given any value for the variable x, we obtain a value of the variable y.

Definition 3.8. For the n-variable symbolic 2-plithogenic linear equation

$$(a_{01} + a_{11}P_1 + a_{21}P_2)x_1 + (a_{02} + a_{12}P_1 + a_{22}P_2)x_2 + (a_{03} + a_{13}P_1 + a_2P_{23})x_3 + \cdots + (a_{0n} + a_{1n}P_1 + a_{2n}P_2)x_n = b_0 + b_1P_1 + b_2P_2$$

where $a_{0i}, a_{1i}, a_{2i},\ i = 1, 2, ..., n$ are real coefficients, the infinite number of solutions are the unknown values $x_1, x_2, ..., x_n$ that satisfies the equation.

Definition 3.9. A non-homogeneous system of n-variable symbolic 2-plithogenic linear equations is given by the form:

$$(a^1_{01} + a^1_{11}P_1 + a^1_{21}P_2)x_1 + (a^1_{02} + a^1_{12}P_1 + a^1_{22}P_2)x_2 + (a^1_{03} + a^1_{13}P_1 + a^1_{23}P_2)x_3 + \cdots + (a^1_{0n} + a^1_{1n}P_1 + a^1_{2n}P_2)x_n = b^1_0 + b^1_1P_1 + b^1_2P_2$$

$$(a^2_{01} + a^2_{11}P_1 + a^2_{21}P_2)x_1 + (a^2_{02} + a^2_{12}P_1 + a^2_{22}P_2)x_2 + (a^2_{03} + a^2_{13}P_1 + a^2_{23}P_2)x_3 + \cdots + (a^2_{0n} + a^2_{1n}P_1 + a^2_{2n}P_2)x_n = b^2_0 + b^2_1P_1 + b^2_2P_2$$

$$\cdots$$

$$(a^m_{01} + a^m_{11}P_1 + a^m_{21}P_2)x_1 + (a^m_{02} + a^m_{12}P_1 + a^m_{22}P_2)x_2 + (a^m_{03} + a^m_{13}P_1 + a^m_{23}P_2)x_3 + \cdots + (a^m_{0n} + a^m_{1n}P_1 + a^m_{2n}P_2)x_n = b^m_0 + b^m_1P_1 + b^m_2P_2$$

where, $a^j_{0i}, a^j_{1i}, a^j_{2i}, b^j_0, b^j_1, b^j_2$ are real coefficients, $i = 1, 2, ..., n,\ j = 1, 2, ..., m.$

Definition 3.10. The solution of non-homogeneous system of n-variable symbolic 2-plithogenic linear equations:

$$(a_{01}^1 + a_{11}^1 P_1 + a_{21}^1 P_2)x_1 + (a_{02}^1 + a_{12}^1 P_1 + a_{22}^1 P_2)x_2 + (a_{03}^1 + a_{13}^1 P_1 + a_{23}^1 P_2)x_3 + \cdots + (a_{0n}^1 + a_{1n}^1 P_1 + a_{2n}^1 P_2)x_n = b_0^1 + b_1^1 P_1 + b_2^1 P_2$$
$$(a_{01}^2 + a_{11}^2 P_1 + a_{21}^2 P_2)x_1 + (a_{02}^2 + a_{12}^2 P_1 + a_{22}^2 P_2)x_2 + (a_{03}^2 + a_{13}^2 P_1 + a_{23}^2 P_2)x_3 + \cdots + (a_{0n}^2 + a_{1n}^2 P_1 + a_{2n}^2 P_2)x_n = b_0^2 + b_1^2 P_1 + b_2^2 P_2$$
$$\cdots$$
$$(a_{01}^m + a_{11}^m P_1 + a_{21}^m P_2)x_1 + (a_{02}^m + a_{12}^m P_1 + a_{22}^m P_2)x_2 + (a_{03}^m + a_{13}^m P_1 + a_{23}^m P_2)x_3 + \cdots + (a_{0n}^m + a_{1n}^m P_1 + a_{2n}^m P_2)x_n = b_0^m + b_1^m P_1 + b_2^m P_2$$

where, $a_{0i}^j, a_{1i}^j, a_{2i}^j, b_0^j, b_1^j, b_2^j$ are real coefficients, $i = 1, 2, ..., n,\ j = 1, 2, ..., m$, is the values of the variables $x_1, x_2, x_3, ..., x_n$ that satisfies the system of equations.

Remark 3.11. We distinguish three cases to solve the system given in Definition 3.10:

(1) The system is consistent, it has unique solution.
(2) The system is consistent, it has infinite number of solutions.
(3) The system is inconsistent, it has no solution.

4. Solving System of Symbolic 2-Plithogenic Linear Equations using Cramer's Rule

Consider the non-homogeneous system of n symbolic 2-plithogenic linear equations with n-variables:

$$(a_{01}^1 + a_{11}^1 P_1 + a_{21}^1 P_2)x_1 + (a_{02}^1 + a_{12}^1 P_1 + a_{22}^1 P_2)x_2 + (a_{03}^1 + a_{13}^1 P_1 + a_{23}^1 P_2)x_3 + \cdots + (a_{0n}^1 + a_{1n}^1 P_1 + a_{2n}^1 P_2)x_n = b_0^1 + b_1^1 P_1 + b_2^1 P_2$$
$$(a_{01}^2 + a_{11}^2 P_1 + a_{21}^2 P_2)x_1 + (a_{02}^2 + a_{12}^2 P_1 + a_{22}^2 P_2)x_2 + (a_{03}^2 + a_{13}^2 P_1 + a_{23}^2 P_2)x_3 + \cdots + (a_{0n}^2 + a_{1n}^2 P_1 + a_{2n}^2 P_2)x_n = b_0^2 + b_1^2 P_1 + b_2^2 P_2$$
$$\cdots$$
$$\cdots$$
$$(a_{01}^n + a_{11}^n P_1 + a_{21}^n P_2)x_1 + (a_{02}^n + a_{12}^n P_1 + a_{22}^n P_2)x_2 + (a_{03}^n + a_{13}^n P_1 + a_{23}^n P_2)x_3 + \cdots + (a_{0n}^n + a_{1n}^n P_1 + a_{2n}^n P_2)x_n = b_0^n + b_1^n P_1 + b_2^n P_2$$

Let $AX = B$ be the matrix form of this system and let, $\Delta = det(A) = a_0 + a_1 P_1 + a_2 P_2$. We distinguish the following cases:

(1) If $a_0 \neq 0, \quad a_0 + a_1 \neq 0 \quad$ and $\quad a_0 + a_1 + a_2 \neq 0$ then the system is consistent and the system has unique solution given by the formula:

$$x_i = \frac{\Delta_{x_i}}{\Delta}, \quad i = 1, 2, 3, ..., n$$

where Δ_{x_i} is the determinant of the matrix A where the ith column is replaced by the column matrix B.

(2) If $a_0 = 0$, or $a_0 + a_1 = 0$ or $a_0 + a_1 + a_2 = 0$. Then the system is inconsistent or it have infinite number of solutions.

Remark 4.1. Consider a system of 2 linear symbolic 2-plithogenic equations with 2 unknowns x and y with

$$\Delta = det(A) = a_0 + a_1 P_1 + a_2 P_2$$
$$\Delta_x = det(A_x) = a_0^x + a_1^x P_1 + a_2^x P_2$$
$$\Delta_y = det(A_y) = a_0^y + a_1^y P_1 + a_2^y P_2$$

We distinguish the following cases:

(1) If $a_0 \neq 0, \quad a_0 + a_1 \neq 0 \quad$ and $\quad a_0 + a_1 + a_2 \neq 0$ then the system is consistent and the system has unique solution given by the formula:

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}$$

(2) If the determinants satisfies any one of the following conditions, then the system is inconsistent and it has no solutions.
   (i). $a_0 = 0$ and a_0^x, a_0^y are not all zero.
   (ii). $a_0 + a_1 = 0$ and $a_0^x + a_1^x, a_0^y + a_1^y$ are not all zero.
   (iii). $a_0 + a_1 + a_3 = 0$ and $a_0^x + a_1^x + a_2^x, a_0^y + a_1^y + a_2^y$ are not all zero.

(3) In all other cases the system is consistent with infinite number of solutions.

Example 4.2. Consider the system of equations:

$$(2 + P_1 + 3P_2)x + (1 - P_1 - P_2)y = 5 + P_1 + 11P_2$$
$$(3 + 4P_2)x + (1 + P_1)y = 7 + 3P_1 + 13P_2.$$

The coefficient matrix is, $A = \begin{pmatrix} 2 + P_1 + 3P_2 & 1 - P_1 - P_2 \\ 3 + 4P_2 & 1 + P_1 \end{pmatrix}$.

$$\Delta = det(A) = \begin{vmatrix} 2 + P_1 + 3P_2 & 1 - P_1 - P_2 \\ 3 + 4P_2 & 1 + P_1 \end{vmatrix} = -1 + 7P_1 + 13P_2$$

Here, $a_0 \neq 0, a_0 + a_1 \neq 0$ and $a_0 + a_1 + a_2 \neq 0$, the system is consistent and its has unique solution.

$$\Delta_x = det(A_x) = \begin{vmatrix} 5 + P_1 + 11P_2 & 1 - P_1 - P_2 \\ 7 + 3P_1 + 13P_2 & 1 + P_1 \end{vmatrix} = -2 + 14P_1 + 45P_2$$

$$\Delta_y = det(A_y) = \begin{vmatrix} 2 + P_1 + 3P_2 & 5 + P_1 + 11P_2 \\ 3 + 4P_2 & 7 + 3P_1 + 13P_2 \end{vmatrix} = -1 + 13P_1 + 7P_2$$

So the solution of the given system is,

$$x = \frac{\Delta_x}{\Delta} = \frac{-2 + 14P_1 + 45P_2}{-1 + 7P_1 + 13P_2} = 2 + P_2,$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-1 + 13P_1 + 7P_2}{-1 + 7P_1 + 13P_2} = 1 + P_1 - P_2$$

Example 4.3. Consider the system of equations:

$$(1+P_1+P_2)x+(3-P_1+2P_2)y=5+3P_1+5P_2$$
$$P_1x+(P_1+P_2)y=4P_1+P_2.$$

The coefficient matrix is, $A=\begin{pmatrix}1+P_1+P_2 & 3-P_1+2P_2\\ P_1 & P_1+P_2\end{pmatrix}$.

$$\Delta=det(A)=\begin{vmatrix}1+P_1+P_2 & 3-P_1+2P_2\\ P_1 & P_1+P_2\end{vmatrix}=0+0P_1+2P_2$$

$$\Delta=det(A_x)=\begin{vmatrix}5+3P_1+5P_2 & 3-P_1+2P_2\\ 4P_1+P_2 & P_1+P_2\end{vmatrix}=0+0P_1+6P_2$$

$$\Delta=det(A_y)=\begin{vmatrix}1+P_1+P_2 & 5+3P_1+5P_2\\ P_1 & 4P_1+P_2\end{vmatrix}=0+0P_1+2P_2$$

Hence, by condition (3) of Remark 4.1 the system is consistent with infinite number of solutions and the solutions are given by:
For all $x=a_0+a_1P_1+a_2P_2\in 2-SP_R$ with $a_0+a_1+a_2=3$,

$$y=-\left(\frac{1+P_1+P_2}{3-P_1+2P_2}\right)x+\left(\frac{5+3P_1+5P_2}{3-P_1+2P_2}\right)$$

That is, for all $x=a_0+a_1P_1+a_2P_2\in 2-SP_R$ with $a_0+a_1+a_2=3$,

$$y=\left(-\frac{1}{3}-\frac{2}{3}P_1+\frac{1}{4}P_2\right)x+\left(\frac{5}{3}+\frac{7}{3}P_1-\frac{3}{4}P_2\right)$$

Example 4.4. Consider the system of equations:

$$(2+P_1+3P_2)x+(1+P_1+P_2)y=5+P_1+11P_2$$
$$(4+2P_1+6P_2)x+(2+2P_1+2P_2)y=10+2P_1+22P_2$$

The coefficient matrix is, $A=\begin{pmatrix}2+P_1+3P_2 & 1+P_1+P_2\\ 4+2P_1+6P_2 & 2+2P_1+2P_2\end{pmatrix}$.

$$\Delta=det(A)=\begin{vmatrix}2+P_1+3P_2 & 1+P_1+P_2\\ 4+2P_1+6P_2 & 2+2P_1+2P_2\end{vmatrix}=0+0P_1+0P_2$$

Also,

$$\Delta_x=det(A_x)=\begin{vmatrix}5+P_1+11P_2 & 1+P_1+P_2\\ 10+2P_1+22P_2 & 2+2P_1+2P_2\end{vmatrix}=0+0P_1+0P_2$$

$$\Delta_y=det(A_y)=\begin{vmatrix}2+P_1+3P_2 & 5+P_1+11P_2\\ 4+2P_1+6P_2 & 10+2P_1+22P_2\end{vmatrix}=0+0P_1+0P_2$$

Hence, by condition (3) of Remark 4.1 the system is consistent with infinite number of solutions and the solutions are given by:

$$S = \left\{x, y \in 2 - SP_R : y = -\left(\frac{2 + P_1 + 3P_2}{1 + P_1 + P_2}\right)x + \left(\frac{5 + P_1 + 11P_2}{1 + P_1 + P_2}\right)\right\}$$

$$i.e, \quad S = \left\{x, y \in 2 - SP_R : y = \left(-2 + \frac{1}{2}P_1 - \frac{1}{2}P_2\right)x + \left(5 - 2P_1 + \frac{8}{3}P_2\right)\right\}$$

Example 4.5. Consider the system of equations:

$$(1 + P_1 + P_2)x + (1 - P_1 + P_2)y = 1 + P_1$$
$$(2 + 2P_1 + 2P_2)x + (2 - 2P_1 + 2P_2)y = 3 + P_2$$

The coefficient matrix is, $A = \begin{pmatrix} 1 + P_1 + P_2 & 1 - P_1 + P_2 \\ 2 + 2P_1 + 2P_2 & 2 - 2P_1 + 2P_2 \end{pmatrix}$.

$$\Delta = det(A) = \begin{vmatrix} 1 + P_1 + P_2 & 1 - P_1 + P_2 \\ 2 + 2P_1 + 2P_2 & 2 - 2P_1 + 2P_2 \end{vmatrix} = 0 + 0P_1 + 0P_2$$

Also,

$$\Delta_x = det(A_x) = \begin{vmatrix} 1 + P_1 & 1 - P_1 + P_2 \\ 3 + P_2 & 2 - 2P_1 + 2P_2 \end{vmatrix} = -1 + P_1$$

$$\Delta_y = det(A_y) = \begin{vmatrix} 1 + P_1 + P_2 & 1 + P_1 \\ 2 + 2P_1 + 2P_2 & 3 + P_2 \end{vmatrix} = 1 - 3P_1 - 2P_2$$

Here, $a_0 = 0$ and $a_0^x \neq 0$, hence the system is inconsistent.

Remark 4.6. Consider a system of 3 linear symbolic 2-plithogenic equations with 3 unknowns x, y and z with

$$\Delta = det(A) = a_0 + a_1P_1 + a_2P_2$$
$$\Delta_x = det(A_x) = a_0^x + a_1^xP_1 + a_2^xP_2$$
$$\Delta_y = det(A_y) = a_0^y + a_1^yP_1 + a_2^yP_2$$
$$\Delta_z = det(A_z) = a_0^z + a_1^zP_1 + a_2^zP_2$$

We distinguish the following cases:

(1) If $a_0 \neq 0$, $a_0 + a_1 \neq 0$ and $a_0 + a_1 + a_2 \neq 0$ then the system is consistent and the system has unique solution given by the formula:

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta} \quad \& \quad z = \frac{\Delta_z}{\Delta}$$

(2) If the determinants satisfies any one of the following conditions, then the system is inconsistent and it has no solutions.

(i). $a_0 = 0$ and a_0^x, a_0^y, a_0^z are not all zero.

(ii). $a_0 + a_1 = 0$ and $a_0^x + a_1^x, a_0^y + a_1^y, a_0^z + a_1^z$ are not all zero.

(iii). $a_0 + a_1 + a_3 = 0$ and $a_0^x + a_1^x + a_2^x, a_0^y + a_1^y + a_2^y, a_0^z + a_1^z + a_2^z$ are not all zero.

(3) If $\Delta = 0+0P_1+0P_2$, $\Delta_x = 0+0P_1+0P_2$, $\Delta_y = 0+0P_1+0P_2$ and $\Delta_z = 0+0P_1+0P_2$ then by solving two equations of the system we will obtain the equation $0 = \alpha$. If $\alpha = 0$, then the system is consistent with infinite number of solutions and if $\alpha \neq 0$, then the system is inconsistent.

Example 4.7. Consider the system of equations:

$$(1+P_1)x + (1-P_1)y + (1+P_1-P_2)z = 1+5P_1-P_2$$
$$(1+P_2)x + (-1+P_1+P_2)y + (2+P_1)z = 1+4P_1+3P_2$$
$$(1-P_1+P_2)x + (-1+P_2)y + (1+P_1)z = 1+P_1+2P_2.$$

The coefficient matrix is, $A = \begin{pmatrix} 1+P_1 & 1-P_1 & 1+P_1-P_2 \\ 1+P_2 & -1+P_1+P_2 & 2+P_1 \\ 1-P_1+P_2 & -1+P_2 & 1+P_1 \end{pmatrix}$.

$$\Delta = det(A) = \begin{vmatrix} 1+P_1 & 1-P_1 & 1+P_1-P_2 \\ 1+P_2 & -1+P_1+P_2 & 2+P_1 \\ 1-P_1+P_2 & -1+P_2 & 1+P_1 \end{vmatrix} = 2+2P_1-P_2$$

$$\Delta_x = det(A_x) = \begin{vmatrix} 1+5P_1-P_2 & 1-P_1 & 1+P_1-P_2 \\ 1+4P_1+3P_2 & -1+P_1+P_2 & 2+P_1 \\ 1+P_1+2P_2 & -1+P_2 & 1+P_1 \end{vmatrix} = 2+6P_1-2P_2$$

$$\Delta_y = det(A_y) = \begin{vmatrix} 1+P_1 & 1+5P_1-P_2 & 1+P_1-P_2 \\ 1+P_2 & 1+4P_1+3P_2 & 2+P_1 \\ 1-P_1+P_2 & 1+P_1+2P_2 & 1+P_1 \end{vmatrix} = 3P_2$$

$$\Delta_z = det(A_z) = \begin{vmatrix} 1+P_1 & 1-P_1 & 1+5P_1-P_2 \\ 1+P_2 & -1+P_1+P_2 & 1+4P_1+3P_2 \\ 1-P_1+P_2 & -1+P_2 & 1+P_1+2P_2 \end{vmatrix} = 4P_1-P_2$$

Here, $a_0 \neq 0$, $a_0+a_1 \neq 0$, & $a_0+a_1+a_2 \neq 0$. Hence, the system is consistent with unique solution given by:

$$x = \frac{\Delta_x}{\Delta} = \frac{2+6P_1-2P_2}{2+2P_1-P_2} = 1+P_1,$$

$$y = \frac{\Delta_y}{\Delta} = \frac{3P_2}{2+2P_1-P_2} = P_2,$$

$$z = \frac{\Delta_z}{\Delta} = \frac{4P_1-P_2}{2+2P_1-P_2} = P_1.$$

Example 4.8. Consider the system of equations:

$$(1+P_2)x+(3-P_1)y+(1+P_1-P_2)z=5$$
$$P_2x+P_1y+(P_1+P_2)z=2P_1+2P_2$$
$$(2+P_1-P_2)x+(4+3P_1-P_2)y+(5+2P_2)z=11+4P_1.$$

The coefficient matrix is, $A=\begin{pmatrix} 1+P_2 & 3-P_1 & 1+P_1-P_2 \\ P_2 & P_1 & P_1+P_2 \\ 2+P_1-P_2 & 4+3P_1-P_2 & 5+2P_2 \end{pmatrix}$.

$$\Delta = det(A) = \begin{vmatrix} 1+P_2 & 3-P_1 & 1+P_1-P_2 \\ P_2 & P_1 & P_1+P_2 \\ 2+P_1-P_2 & 4+3P_1-P_2 & 5+2P_2 \end{vmatrix} = 0-2P_1-10P_2$$

$$\Delta_x = det(A_x) = \begin{vmatrix} 5 & 3-P_1 & 1+P_1-P_2 \\ 2P_1+2P_2 & P_1 & P_1+P_2 \\ 11+4P_1 & 4+3P_1-P_2 & 5+2P_2 \end{vmatrix} = 0-2P_1-10P_2$$

$$\Delta_y = det(A_y) = \begin{vmatrix} 1+P_2 & 5 & 1+P_1-P_2 \\ P_2 & 2P_1+2P_2 & P_1+P_2 \\ 2+P_1-P_2 & 11+4P_1 & 5+2P_2 \end{vmatrix} = 0-2P_1-10P_2$$

$$\Delta_z = det(A_z) = \begin{vmatrix} 1+P_2 & 3-P_1 & 5 \\ P_2 & P_1 & 2P_1+2P_2 \\ 2+P_1-P_2 & 4+3P_1-P+2 & 11+4P_1 \end{vmatrix} = 0-2P_1-10P_2$$

Here, $a_0=0,\ a_0^x=0,\ a_0^y=0\ \&\ a_0^z=0$. Hence, the system is consistent with infinite number of solutions and the solutions are given by:

$$S=\Big\{x,y,z\in 2-SP_R : x=\left(-\frac{11}{2}+\frac{3}{2}P_1+5P_2\right)z+\left(\frac{13}{2}-\frac{3}{2}P_1-5P_2\right),$$
$$y=\left(\frac{3}{2}-\frac{1}{2}P_1-\frac{5}{2}P_2\right)z+\left(-\frac{1}{2}+\frac{1}{2}P_1+\frac{5}{2}P_2\right),\quad z\Big\}$$

Example 4.9. Consider the system of equations:

$$(1+P_1)x+(1-P_2)y+(1+P_1-P_2)z=2+P_1$$
$$(2+2P_1)x+(2-2P_2)y+(2+2P_1-2P_2)z=4+2P_1$$
$$(3+3P_1)x+(3-3P_2)y+(3+3P_1-3P_2)z=6+3P_1$$

Here, $\Delta=0+0P_1+0P_2$, $\Delta_x=0+0P_1+0P_2$, $\Delta_y=0+0P_1+0P_2$, $\Delta_z=0+0P_1+0P_2$. By solving first two equations of the given system we will get $0=0$. Hence the system consistent with infinite number of solutions. In this case the system is reduced to a single equation. To solve we can assign arbitrary values to any two variables and can determine the value of the third variable.

Example 4.10. Consider the system of equations:

$$(1+P_1)x+(1-P_2)y+(1+P_1-P_2)z=2+P_1$$
$$(2+2P_1)x+(2-2P_2)y+(2+2P_1-2P_2)z=1-P_2$$
$$(3+3P_1)x+(3-3P_2)y+(3+3P_1-3P_2)z=3+P_2$$

Here, $\Delta = 0+0P_1+0P_2$, $\Delta_x = 0+0P_1+0P_2$, $\Delta_y = 0+0P_1+0P_2$, $\Delta_z = 0+0P_1+0P_2$. By solving first two equations of the given system we will get $0 = 3+2P_1+P_2$. Hence the system inconsistent and it has no solution.

Other than the above mentioned cases there are some special cases in which the system of linear symbolic 2-plithogenic equations is inconsistent.

Remark 4.11. If all coefficients of a the system of n linear symbolic 2-plithogenic equations with n variables are non invertible the the system is inconsistent. For example,

$$(1-P_1+P_2)x+(2+2P_1-4P_2)y+(1-P_1)z=3+P_1$$
$$(1-P_2)x+P_1y+p_2z=3P_2$$
$$(2-P-1-P_2)x+P_2y+(1-P_1)z=2+P_1+P_2$$

5. System of Homogeneous Symbolic 2-Plithogenic Linear Equations

Definition 5.1. Consider the homogeneous system of n-variable symbolic 2-plithogenic linear equations:

$$(a^1_{01}+a^1_{11}P_1+a^1_{21}P_2)x_1+(a^1_{02}+a^1_{12}P_1+a^1_{22}P_2)x_2+(a^1_{03}+a^1_{13}P_1+a^1_{23}P_2)x_3+...+(a^1_{0n}+a^1_{1n}P_1+a^1_{2n}P_2)x_n=0+0P_1+0P_2$$
$$(a^2_{01}+a^2_{11}P_1+a^2_{21}P_2)x_1+(a^2_{02}+a^2_{12}P_1+a^2_{22}P_2)x_2+(a^2_{03}+a^2_{13}P_1+a^2_{23}P_2)x_3+...+(a^2_{0n}+a^2_{1n}P_1+a^2_{2n}P_2)x_n=0+0P_1+0P_2$$
$$...$$
$$(a^n_{01}+a^n_{11}P_1+a^n_{21}P_2)x_1+(a^n_{02}+a^n_{12}P_1+a^n_{22}P_2)x_2+(a^n_{03}+a^n_{13}P_1+a^n_{23}P_2)x_3+...+(a^n_{0n}+a^n_{1n}P_1+a^n_{2n}P_2)x_n=0+0P_1+0P_2$$

Remark 5.2. Let $AX = B$ be the coefficient matrix of this system and let $\Delta = det(A) = a_0+a_1P_1+a_2P_2$. We distinguish the following cases:

(1) If $a_0 \neq 0$ and $a_0+a_1 \neq 0$ and $a_0+a_1+a_2 \neq 0$ then the system is consistent and the system has unique solution $x_i = 0, \quad i = 1,2,...,n$.
(2) If $a_0 = 0$ or $a_0+a_1 = 0$ or $a_0+a_1+a_2 = 0$, then the system is consistent with infinite number of solutions.

Example 5.3. Consider the system of equations:

$$(2+P_1+3P_2)x+(1-P_1-P_2)y=0$$
$$(3+4P_2)x+(1+P_1)y=0.$$

The coefficient matrix is, $A = \begin{pmatrix} 2+P_1+3P_2 & 1-P_1-P_2 \\ 3+4P_2 & 1+P_1 \end{pmatrix}$.

$$\Delta = det(A) = \begin{vmatrix} 2+P_1+3P_2 & 1-P_1-P_2 \\ 3+4P_2 & 1+P_1 \end{vmatrix} = -1+7P_1+13P_2$$

Here, $a_0 \neq 0, a_0+a_1 \neq 0$ and $a_0+a_1+a_2 \neq 0$, the system is consistent with unique solution.

$$\Delta_x = det(A_x) = \begin{vmatrix} 0 & 1-P_1-P_2 \\ 0 & 1+P_1 \end{vmatrix} = 0$$

$$\Delta_y = det(A_y) = \begin{vmatrix} 2+P_1+3P_2 & 0 \\ 3+4P_2 & 0 \end{vmatrix} = 0$$

So the solution of the given system is,

$$x = \frac{\Delta_x}{\Delta} = \frac{0}{-1+7P_1+13P_2} = 0,$$

$$y = \frac{\Delta_y}{\Delta} = \frac{0}{-1+7P_1+13P_2} = 0$$

Example 5.4. Consider the system of equations:

$$(1+P_2)x + (3-P_1)y + (2+P_2)z = 0$$
$$P_2x + P_1y + (P_1 - \tfrac{1}{2}P_2)z = 0$$
$$(2+P_1-P_2)x + (4+3P_1-P_2)y + (3+4P_1+2P_2)z = 0.$$

The coefficient matrix is, $A = \begin{pmatrix} 1+P_2 & 3-P_1 & 2+P_2 \\ P_2 & P_1 & P_1-\frac{1}{2}P_2 \\ 2+P_1-P_2 & 4+3P_1-P_2 & 3+4P_1+2P_2 \end{pmatrix}$.

$$\Delta = det(A) = \begin{vmatrix} 1+P_2 & 3-P_1 & 2+P_2 \\ P_2 & P_1 & P_1-\frac{1}{2}P_2 \\ 2+P_1-P_2 & 4+3P_1-P_2 & 3+4P_1+2P_2 \end{vmatrix} = 0+0P_1+8P_2$$

Here, $a_0 = 0$, the system is consistent with infinite number of solutions and the solutions are given by:

$$S = \left\{ x, y, z \in 2-SP_R : x = \left(-\frac{1}{2}+\frac{1}{2}P_1\right)z, \quad y = \left(-\frac{1}{2}-\frac{1}{2}P_1-\frac{1}{2}P_2\right)z, \quad z \right\}$$

6. **Conclusion**

In this article, the solutions of symbolic 2-plithogenic linear equations are studied using Cramer's rule. The conditions are given for a system of symbolic 2-plithogenic linear equations to be consistent with unique solution, consistent with infinite solutions, and inconsitent. Further, many examples are given for the case of system of two symbolic 2-plithogenic linear equations with two variables and for the case of system of three symbolic 2-plithogenic linear equations with three variables.

Funding: This research received no external funding.

References

1. Adeleke, E. O., Agboola, A. A. A., and Smarandache, F., "Refined neutrosophic rings II", International Journal of Neutrosophic Science, vol. 2, pp. 89?4, 2020.
2. A. A. A. Agboola, "On refined neutrosophic algebraic structures", Neutrosophic Sets and Systems, vol. 10, pp. 99?01, 2015.
3. M. Abobala, "On some special elements in neutrosophic rings and refined neutrosophic rings", Journal of New Theory, vol. 33, 2020.
4. Abobala, M., "On Refined Neutrosophic Matrices and Their Applications In Refined Neutrosophic Algebraic Equations", Journal Of Mathematics, Hindawi, 2021.
5. Abobala, M., "On Some Algebraic Properties of n-Refined Neutrosophic Elements and n-Refined Neutrosophic Linear Equations", Mathematical Problems in Engineering, Hindawi, 2021.
6. Yaser Ahmad Alhasan, "Types of system of the nuetrosophic linear equations and Cramer's rule", Neutrosophic Sets and Systems, vol. 45, 2021.
7. Merkepci, H., and Abobala, M., "On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.
8. Taffach, N., "An Introduction to Symbolic 2-Plithogenic Vector Spaces Generated from The Fusion of Symbolic Plithogenic Sets and Vector Spaces", Neutrosophic Sets and Systems, Vol 54, 2023.
9. Taffach, N., and Ben Othman, K., "An Introduction to Symbolic 2-Plithogenic Modules Over Symbolic 2-Plithogenic Rings", Neutrosophic Sets and Systems, Vol 54, 2023.
10. Smarandache, F., "Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)", Neutrosophic Sets and Systems, vol. 53, 2023.
11. Albasheer, O., Hajjari., A., and Dalla., R., "On The Symbolic 3-Plithogenic Rings and Their Algebraic Properties", Neutrosophic Sets and Systems, Vol 54, 2023.
12. Khaldi, A., Ben Othman, K., Von Shtawzen, O., Ali, R., and Mosa, S., "On Some Algorithms for Solving Different Types of Symbolic 2-Plithogenic Algebraic Equations", Neutrosophic Sets and Systems, Vol 54, 2023.
13. Merkepci, H., and Rawashdeh, A., "On The Symbolic 2-Plithogenic Number Theory and Integers ", Neutrosophic Sets and Systems, Vol 54, 2023.
14. Abuobida Mohammed A. Alfahal, Yaser Ahmad Alhasan, Raja Abdullah Abdulfatah, Arif Mehmood, Mustafa Talal Kadhim, "On Symbolic 2-Plithogenic Real Matrices and Their Algebraic Properties", International Journal of Neutrosophic Science, Vol. 21, PP. 96-104, 2023.

Received: 18/6/2023 / **Accepted:** 29/9/2023

NSS

Neutrosophic Sets and Systems, Vol. 59, 2023

University of New Mexico

On Some Algebraic Properties of Symbolic 2-Plithogenic Square Matrices

P. Prabakaran[1,*], **S. Kalaiselvan**[2]

[1,2]Department of Mathematics, Bannari Amman Institute of Technology, Sathyamangalam-638401, Tamil Nadu, India

[1]E-mail: prabakaranpvkr@gmail.com, [2]E-mail: priankalai@gmail.com

[*]Correspondence: prabakaranpvkr@gmail.com

Abstract. In this article, the adjoint of symbolic 2-plithogenic square matrices are defined and the inverse of symbolic 2-plithogenic square matrices are studied in terms of symbolic 2-plithogenic determinant and symbolic 2-plithogenic adjoint. We have introduced the concept of symbolic 2-plithogenic characteristic polynomial of symbolic 2-plithogenic square matrices and the symbolic 2-plithogenic version of Cayley-Hamilton theorem. Also, provided enough examples to enhance understanding.

Keywords: Symbolic 2-plithogenic matrix; symbolic 2-plithogenic adjoint; symbolic 2-plithogenic determinant; symbolic 2-plithogenic inverse.

---

1. Introduction

The concept of refined neutrosophic structure was studied by many authors in [1–4]. Symbolic plithogenic algebraic structures are introduced by Smarandache, that are very similar to refined neutrosophic structures with some differences in the definition of the multiplication operation [15].

In [12], the algebraic properties of symbolic 2-plithogenic rings generated from the fusion of symbolic plithogenic sets with algebraic rings are studied. In [8], some more algebraic properties of symbolic 2-plithogenic rings are studied. Further, Taffach [17, 18] studied the concepts

zenodo.10031262/10.5281

of symbolic 2-plithogenic vector spaces and modules.

Recently, in [7], the concept of symbolic 2-plithogenic matrices with symbolic 2-plithogenic entries, determinants, eigen values and vectors, exponents, and diagonalization are studied. Hamiyet Merkepci et.al [13], studied the the symbolic 2-plithogenic number theory and integers. Ahmad Khaldi et.al [11], studied the different types of algebraic symbolic 2-plithogenic equations and its solutions.

As a continuation of the previous study of symbolic 2-plithogenic matrices, this work discusses the symbolic 2-plithogenic adjoint, where the inverse of symbolic 2-plithogenic matrices will be defined in terms of the symbolic 2-plithogenic adjoint. We present the symbolic 2-plithogenic characteristic polynomials and the symbolic 2-plithogenic version of the Cayley-Hamilton theorem. Also, we illustrate many examples to clarify the validity of our work.

2. **Preliminaries**

Definition 2.1. [12] Let R be a ring, the symbolic 2-plithogenic ring is defined as follows:

$$2-SP_R = \left\{a_0 + a_1P_1 + a_2P_2; a_i \in R, P_j^2 = P_j, P_1 \times P_2 = P_{max(1,2)} = P_2\right\}$$

Smarandache has defined algebraic operations on $2 - SP_R$ as follows:

Addition:

$[a_0 + a_1P_1 + a_2P_2] + [b_0 + b_1P_1 + b_2P_2] = (a_0 + b_0) + (a_1 + b_1)P_1 + (a_2 + b_2)P_2$

Multiplication:

$[a_0 + a_1P_1 + a_2P_2].[b_0 + b_1P_1 + b_2P_2] = a_0b_0 + a_0b_1P_1 + a_0b_2P_2 + a_1b_0P_1^2 + a_1b_2P_1P_2 + a_2b_0P_2 + a_2b_1P_1P_2 + a_2b_2P_2^2 + a_1b_1P_1P_1 = (a_0b_0) + (a_0b_1 + a_1b_0 + a_1b_1)P_1 + (a_0b_2 + a_1b_2 + a_2b_0 + a_2b_1 + a_2b_2)P_2$.

It is clear that $2-SP_R$ is a ring. If R is a field, then $2-SP_R$ is called a symbolic 2-plithogenic field. Also, if R is commutative, then $2 - SP_R$ is commutative, and if R has a unity (1), than $2 - SP_R$ has the same unity (1).

Theorem 2.2. *[12] Let $2 - SP_R$ be a 2-plithogenic symbolic ring, with unity (1). Let $X = x_0 + x_1P_1 + x_2P_2$ be an arbitrary element, then:*

(1) *X is invertible if and only if $x_0, x_0 + x_1, x_0 + x_1 + x_2$ are invertible.*
(2) $X^{-1} = x_0^{-1} + [(x_0 + x_1)^{-1} - x_0^{-1}]P_1 + [(x_0 + x_1 + x_2)^{-1} - (x_0 + x_1)^{-1}]P_2$

Definition 2.3. [7] A symbolic 2-plithogenic real square matrix is a matrix with symbolic 2-plithogenic real entries.

Theorem 2.4. *[7] Let $S = S_0 + S_1P_1 + S_2P_2$ be a symbolic 2-plithogenic real square matrix, then*

(1) *S is invertible if and only if $S_0, S_0+S_1, S_0+S_1+S_2$ are invertible.*

(2) *If S is invertible then*

$$S^{-1} = S_0^{-1} + [(S_0+S_1)^{-1} - S_0^{-1}]P_1 + [(S_0+S_1+S_2)^{-1} - (S_0+S_1)^{-1}]$$

(3) $S^m = S_0^m + [(S_0+S_1)^m - S_0^m]P_1 + [(S_0+S_1+S_2)^m - (S_0+S_1)^m]$ *for $m \in N$.*

Definition 2.5. [7] Let $L = L_0 + L_1P_1 + L_2P_2 \in 2 - SP_M$, we define:

$$detL = det(L_0) + [det(L_0+L_1) - detL_0]P_1 + [det(L_0+L_1+L_2) - det(L_0+L_1)]P_2.$$

3. Adjoint of Symbolic 2-Plithogenic Square Matrices

We begin this section with the following definition.

Definition 3.1. Let $L = L_0 + L_1P_1 + L_2P_2$ be a symbolic 2-plithogenic square matrix with real entries. The adjoint matrix of L is defined as

$$adjL = adjL_0 + [adj(L_0+L_1) - adjL_0]P_1 + [adj(L_0+L_1+L_2) - adj(L_0+L_1)]P_2.$$

Example 3.2. Consider the following symbolic 2-plithogenic 2×2 matrix:

$$L = \begin{pmatrix} 2+P_1+3P_2 & 1-P_1-P_2 \\ 3+4P_1 & 1+P_2 \end{pmatrix}$$

Here,

$$L_0 = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix},\; L_0+L_1 = \begin{pmatrix} 3 & 0 \\ 7 & 1 \end{pmatrix} \text{ and } L_0+L_1+L_2 = \begin{pmatrix} 6 & -1 \\ 7 & 2 \end{pmatrix},$$

Then,

$$adjL_0 = \begin{pmatrix} 1 & -1 \\ -3 & 2 \end{pmatrix},\; adj(L_0+L_1) = \begin{pmatrix} 1 & 0 \\ -7 & 3 \end{pmatrix} \text{ and } adj(L_0+L_1+L_2) = \begin{pmatrix} 2 & 1 \\ -7 & 6 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} adjL &= adjL_0 + [adj(L_0+L_1) - adjL_0]P_1 + [adj(L_0+L_1+L_2) - adj(L_0+L_1)]P_2 \\ &= \begin{pmatrix} 1+P_1 & -1+P_1+P_2 \\ -3-4P_2 & 2+P_1+3P_2 \end{pmatrix} \end{aligned}$$

Example 3.3. Consider the following symbolic 2-plithogenic 3×3 matrix:

$$L = \begin{pmatrix} -3+P_1-P_2 & 1+P_1 & 5 \\ -P_1+P_2 & 3P_1 & 4P_2 \\ -1+2P_1-P_2 & 5+2P_2 & 7+P_1+10P_2 \end{pmatrix}$$

Here,

$$L_0 = \begin{pmatrix} -3 & 1 & 5 \\ 0 & 0 & 0 \\ -1 & 5 & 7 \end{pmatrix},\; L_0+L_1 = \begin{pmatrix} -2 & 2 & 5 \\ -1 & 3 & 0 \\ -1 & 5 & 8 \end{pmatrix} \text{ and } L_0+L_1+L_2 = \begin{pmatrix} -3 & 2 & 5 \\ 0 & 3 & 4 \\ 0 & 7 & 18 \end{pmatrix},$$

Then,

$$adjL_0=\begin{pmatrix}0&18&0\\0&-16&0\\0&14&0\end{pmatrix},\quad adj(L_0+L_1)=\begin{pmatrix}24&9&-15\\8&-21&-5\\-8&12&-4\end{pmatrix}\quad \text{and}$$

$$adj(L_0+L_1+L_2)=\begin{pmatrix}26&-1&-7\\0&-54&12\\0&21&9\end{pmatrix}.$$

Therefore,

$$\begin{aligned}adjL &= adjL_0+[adj(L_0+L_1)-adjL_0]P_1+[adj(L_0+L_1+L_2)-adj(L_0+L_1)]P_2\\ &= \begin{pmatrix}24P_1+2P_2 & 18-9P_1-10P_2 & -15P_1+8P_2\\ 8P_1-8P_2 & -16+5P_1+33P_2 & -5P_1+17P_2\\ -8P_1+8P_2 & 14-2P_1+9P_2 & -4P_1+13P_2\end{pmatrix}\end{aligned}$$

Using the definition of adjoint of symbolic 2-plithogenic matrix we can modify the Theorem 2.4 as follows:

Theorem 3.4. *Let $L=L_0+L_1P_1+L_2P_2$ be a symbolic 2-plithogenic square matrix, then L is invertible if and only if $detL_0\neq 0, det(L_0+L_1)\neq 0$ and $det(L_0+L_1+L_2)\neq 0$ and*

$$L^{-1}=\frac{1}{detL}(adjL).$$

Proof. By Theorem 2.4, L is invertible if and only if $detL_0\neq 0,\quad det(L_0+L_1)\neq 0$ and $det(L_0+L_1+L_2)\neq 0$.

Also,

$$\begin{aligned}\frac{1}{detL}(adjL) &= \left(\frac{1}{detL_0+[det(L_0+L_1)-det(L_0)]P_1+[det(L_0+L_1+L_2)-det(L_0+L_1)]P_2}\right)\\ &\quad (adjL_0+[adj(L_0+L_1)-adjL_0]P_1+[adj(L_0+L_1+L_2)-adj(L_0+L_1)]P_2)\\ &= \frac{adjL_0}{detL_0}+\left[\frac{adj(L_0+L_1)}{det(L_0+L_1)}-\frac{adjL_0}{detL_0}\right]P_1+\left[\frac{adj(L_0+L_1+L_2)}{det(L_0+L_1+L_2)}-\frac{adj(L_0+L_1)}{det(L_0+L_1)}\right]P_2\\ &= L_0^{-1}+\left[(L_0+L_1)^{-1}-L_0^{-1}\right]P_1+\left[(L_0+L_1+L_2)^{-1}-(L_0+L_1)^{-1}\right]P_2\\ &= L^{-1}\end{aligned}$$

Hence the result holds by Theorem 2.4. □

Example 3.5. Consider the symbolic 2-plithogenic 2×2 matrix

$$L=\begin{pmatrix}1+P_1+P_2 & -1+P_1\\ 1-P_2 & 1\end{pmatrix}$$

Here, $detL = 2 + P_2$, and $adjL = \begin{pmatrix} 1 & 1 - P_1 \\ -1 + P_2 & 1 + P_1 + P_2 \end{pmatrix}$.

Hence,

$$\begin{aligned} L^{-1} &= \frac{1}{detL}(adjL) \\ &= \frac{1}{2+P_2}\begin{pmatrix} 1 & 1 - P_1 \\ -1 + P_2 & 1 + P_1 + P_2 \end{pmatrix} \\ &= \left(\frac{1}{2} - \frac{1}{6}P_2\right)\begin{pmatrix} 1 & 1 - P_1 \\ -1 + P_2 & 1 + P_1 + P_2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} - \frac{1}{6}P_2 & \frac{1}{2} - \frac{1}{2}P_1 \\ -\frac{1}{2} + \frac{1}{2}P_2 & \frac{1}{2} + \frac{1}{2}P1 \end{pmatrix} \end{aligned}$$

Example 3.6. Consider the symbolic 2-plithogenic 3×3 matrix

$$L = \begin{pmatrix} 1 + P_1 & 1 - P_1 & 1 + P_1 - P_2 \\ 1 + P_2 & -1 + P_1 + P_2 & 2 + P_1 \\ 1 - P_1 + P_2 & -1 + P_2 & 1 + P_1 \end{pmatrix}$$

Here, $detL = 2 + 2P_1 - P_2$, and

$$adj(L) = \begin{pmatrix} 1 + 2P_1 - P_2 & -2 + 2P_2 & 3 - 3P_1 - P_2 \\ 1 - 3P_1 + P_2 & 4P_1 - P_2 & -1 - 3P_1 \\ -P_1 & 2 - 2P_2 & -2 + 2P_1 + 2P_2 \end{pmatrix}$$

$$\begin{aligned} L^{-1} &= \frac{1}{detL}(adjL) \\ &= \frac{1}{2 + 2P_1 - P_2}\begin{pmatrix} 1 + 2P_1 - P_2 & -2 + 2P_2 & 3 - 3P_1 - P_2 \\ 1 - 3P_1 + P_2 & 4P_1 - P_2 & -1 - 3P_1 \\ -P_1 & 2 - 2P_2 & -2 + 2P_1 + 2P_2 \end{pmatrix} \\ &= \left(\frac{1}{2} - \frac{1}{4}P_1 + \frac{1}{12}P_2\right)\begin{pmatrix} 1 + 2P_1 - P_2 & -2 + 2P_2 & 3 - 3P_1 - P_2 \\ 1 - 3P_1 + P_2 & 4P_1 - P_2 & -1 - 3P_1 \\ -P_1 & 2 - 2P_2 & -2 + 2P_1 + 2P_2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} + \frac{1}{4}P_1 - \frac{1}{12}P_2 & -1 + P_2 & \frac{3}{2} - \frac{3}{2}P_1 - \frac{1}{3}P_2 \\ \frac{1}{2} - P_1 + \frac{1}{6}P_2 & P_1 + \frac{8}{3}P_2 & -\frac{1}{2} - \frac{1}{2}P_1 - \frac{1}{3}P_2 \\ -\frac{1}{4}P_1 - \frac{1}{12}P_2 & 1 - \frac{1}{2}P_1 - \frac{1}{2}P_2 & -1 + P_1 + \frac{2}{3}P_2 \end{pmatrix} \end{aligned}$$

Remark 3.7. If X is a invertible symbolic 2-plithogenic square matrix and X^{-1} is its inverse, then $adjX = detX \cdot X^{-1}$.

Theorem 3.8. *Let $X = A+BP_1+CP_2$ and $Y = M+NP_1+SP_2$ be two symbolic 2-plithogenic invertible square matrices. Then XY is also invertible and $(XY)^{-1} = Y^{-1}X^{-1}$.*

Proof. By Theorem 3.4, if X is invertible then

$$det(A) \neq 0,\ det(A+B) \neq 0 \text{ and } det(A+B+C) \neq 0.$$

Similarly, if Y is invertible then

$$detM \neq 0\ ,\ det(M+N) \neq 0 \text{ and } det(M+N+S) \neq 0.$$

This implies that,

$$det(AM) = detA\ detM \neq 0$$
$$det[(A+B)(M+N)] = det(A+B)\ det(M+N) \neq 0$$
$$det[(A+B+C)(M+N+S)] = det(A+B+C)\ det(M+N+S) \neq 0.$$

Now,

$$det(XY) = det(AM) + [det((A+B)(M+N))]P_1 + [det((A+B+C)(M+N+S))]P_2 \neq 0$$

and hence XY is invertible. Also by associativity of matrix multiplication, we have

$$(XY)(Y^{-1}X^{-1}) = X(YY^{-1})X^{-1} = XX^{-1} = U_{n\times n}$$
$$(Y^{-1}X^{-1})(XY) = Y^{-1}(X^{-1}X)Y = Y^{-1}Y = U_{n\times n}.$$

Thus,$(MN)^{-1} = N^{-1}M^{-1}$. □

Theorem 3.9. *Let X and Y be two $m\times m$ symbolic 2-plithogenic invertible matrices. Then the following properties holds.*

(1) $det(adjX) = (detX)^{m-1}$.
(2) $adj(XY) = adjX\ adjY$.
(3) $adj(X^k) = (adjX)^k$ *for any positive integer* k.
(4) $adj(X^T) = (adjX)^T$.
(5) $adj(adjX) = (detX)^{m-2}X$

Proof. We can prove this results based on the properties adjoint of classical matrices. □

4. Characteristic Polynomial of Symbolic 2-Plithogenic Square Matrices

We begin this section with the following definition.

Definition 4.1. Let $L = L_0 + L_1P_1 + L_2P_2$ be a symbolic 2-plithogenic $n\times n$ square matrix with real entries. The characteristic polynomial of L is defined as

$$\phi(\lambda) = \alpha(\lambda) + [\beta(\lambda) - \alpha(\lambda)]\,P_1 + [\gamma(\lambda) - \beta(\lambda)]\,P_2$$

where,

$$\begin{aligned}\alpha(\lambda) &= det(L_0 - \lambda U_{n\times n})\\ \beta(\lambda) &= det(L_0 + L_1 - \lambda U_{n\times n}) - det(L_0 - \lambda U_{n\times n})\\ \gamma(\lambda) &= det(L_0 + L_1 + L_2 - \lambda U_{n\times n}) - det(L_0 + L_1 - \lambda U_{n\times n}).\end{aligned}$$

Example 4.2. Consider the following symbolic 2-plithogenic 2×2 matrix:

$$L = \begin{pmatrix} 2 + P_1 + 3P_2 & 1 - P_1 - P_2 \\ 3 + 4P_1 & 1 + P_2 \end{pmatrix}$$

with

$$L_0 = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}, L_0 + L_1 = \begin{pmatrix} 3 & 0 \\ 7 & 1 \end{pmatrix} \text{ and } L_0 + L_1 + L_2 = \begin{pmatrix} 6 & -1 \\ 7 & 2 \end{pmatrix}.$$

Here,

$$\begin{aligned}\alpha(\lambda) &= det(L_0 - \lambda U_{n\times n}) = \begin{pmatrix} 2-\lambda & 1 \\ 3 & 1-\lambda \end{pmatrix} = \lambda^2 - 3\lambda - 1.\\ \beta(\lambda) &= det(L_0 + L_1 - \lambda U_{n\times n}) - det(L_0 - \lambda U_{n\times n})\\ &= \begin{pmatrix} 3-\lambda & 0 \\ 7 & 1-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 3.\\ \gamma(\lambda) &= det(L_0 + L_1 + L_2 - \lambda U_{n\times n}) - det(L_0 + L_1 - \lambda U_{n\times n})\\ &= \begin{pmatrix} 6-\lambda & -1 \\ 7 & 2-\lambda \end{pmatrix} = \lambda^2 - 8\lambda - 19.\end{aligned}$$

Hence the characteristic polynomial of L is

$$\begin{aligned}\phi(\lambda) &= \lambda^2 - 3\lambda - 1 + [(\lambda^2 - 4\lambda + 3) - (\lambda^2 - 3\lambda - 1)]P_1 + [(\lambda^2 - 8\lambda - 19) - (\lambda^2 - 4\lambda + 3)]P_2\\ &= \lambda^2 - 3\lambda - 1 + (-\lambda + 4)P_1 + (-4\lambda + 16)P_2.\end{aligned}$$

Example 4.3. Consider the symbolic 2-plithogenic 3×3 matrix

$$L = \begin{pmatrix} 1 + P_1 & 1 - P_1 & 1 + P_1 - P_2 \\ 1 + P_2 & -1 + P_1 + P_2 & 2 + P_1 \\ 1 - P_1 + P_2 & -1 + P_2 & 1 + P_1 \end{pmatrix}$$

with

$$L_0 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & 1 \end{pmatrix}, L_0 + L_1 = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 0 & 3 \\ 0 & -1 & 2 \end{pmatrix}, \text{ and } L_0 + L_1 + L_2 = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix}.$$

Here,

$$\begin{aligned}
\alpha(\lambda) &= det(L_0 - \lambda U_{n\times n}) = \begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & -1-\lambda & 2 \\ 1 & -1 & 1-\lambda \end{pmatrix} = -\lambda^3 + \lambda^2 + \lambda + 2. \\
\beta(\lambda) &= det(L_0 + L_1 - \lambda U_{n\times n}) - det(L_0 - \lambda U_{n\times n}) \\
&= \begin{pmatrix} 2-\lambda & 0 & 2 \\ 1 & -\lambda & 3 \\ 0 & -1 & 2-\lambda \end{pmatrix} \\
&= -\lambda^3 + 4\lambda^2 - 7\lambda + 4. \\
\gamma(\lambda) &= det(L_0 + L_1 + L_2 - \lambda U_{n\times n}) - det(L_0 + L_1 - \lambda U_{n\times n}) \\
&= \begin{pmatrix} 2-\lambda & 0 & 1 \\ 2 & 1-\lambda & 3 \\ 1 & 0 & 2-\lambda \end{pmatrix} \\
&= -\lambda^3 + 5\lambda^2 - 7\lambda + 3.
\end{aligned}$$

Hence the characteristic polynomial of L is

$$\begin{aligned}
\phi(\lambda) &= -\lambda^3 + \lambda^2 + \lambda + 2 + [(-\lambda^3 + 4\lambda^2 - 7\lambda + 4) - (-\lambda^3 + \lambda^2 + \lambda + 2)]P_1 \\
&\qquad + [(-\lambda^3 + 5\lambda^2 - 7\lambda + 3) - (-\lambda^3 + 4\lambda^2 - 7\lambda + 4)]P_2 \\
&= -\lambda^3 + \lambda^2 + \lambda + 2 + (3\lambda^2 - 8\lambda + 2)P_1 + (\lambda^2 - 1)P_2.
\end{aligned}$$

Theorem 4.4 (**Symbolic 2-plithogenic Cayely-Hamilton Theorem**). *Every symbolic 2-plithogenic square matrix satisfies its characteristic polynomial.*

Proof. We can prove this result based on the Cayely-Hamilton theorem for classical matrices. □

Example 4.5. Consider the symbolic 2-plithogenic 2×2 matrix given in Example 4.2

$$L = \begin{pmatrix} 1 + P_1 + P_2 & -1 + P_1 \\ 1 - P_2 & 1 \end{pmatrix}$$

The characteristic polynomial of L is $\phi(\lambda) = \lambda^2 - 3\lambda - 1 + (-\lambda + 4)P_1 + (-4\lambda + 16)P_2$. This implies that,

$$\begin{aligned}
\phi(L) &= L^2 - 3L - 1 + (-L + 4)P_1 + (-4L + 16)P_2. \\
&= \begin{pmatrix} -P_1 + 11P_2 & -5P_2 \\ 7P_1 + 28P_2 & -3P_1 - 7P_2 \end{pmatrix} + \begin{pmatrix} P_1 - 3P_2 & P_2 \\ -7P_1 & 3P_1 - P_2 \end{pmatrix} + \begin{pmatrix} -8P_2 & 4P_2 \\ -28P_1 & 8P_2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

Hence, $\phi(L) = 0$.

Remark 4.6. If L is a invertible symbolic 2-plithogenic matrix, then using Cayely-Hamilton theorem we can compute the inverse of L. See the following example.

Example 4.7. Consider the symbolic 2-plithogenic 2×2 matrix

$$L = \begin{pmatrix} 1 + P_1 + P_2 & -1 + P_1 \\ 1 - P_2 & 1 \end{pmatrix}$$

with

$$L_0 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, L_0 + L_1 = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \text{ and } L_0 + L_1 + L_2 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

Here,

$$\begin{aligned} \alpha(\lambda) &= det(L_0 - \lambda U_{n\times n}) = \begin{pmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} = \lambda^2 - 2\lambda + 2. \\ \beta(\lambda) &= det(L_0 + L_1 - \lambda U_{n\times n}) - det(L_0 - \lambda U_{n\times n}) \\ &= \begin{pmatrix} 2-\lambda & 0 \\ 1 & 1-\lambda \end{pmatrix} = \lambda^2 - 3\lambda + 2. \\ \gamma(\lambda) &= det(L_0 + L_1 + L_2 - \lambda U_{n\times n}) - det(L_0 + L_1 - \lambda U_{n\times n}) \\ &= \begin{pmatrix} 3-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 3. \end{aligned}$$

Hence the characteristic polynomial of L is

$$\begin{aligned} \phi(\lambda) &= \alpha(\lambda) + [\beta(\lambda) - \alpha(\lambda)] P_1 + [\gamma(\lambda) - \beta(\lambda)] P_2 \\ &= \lambda^2 - 2\lambda + 2 - \lambda P_1 + (-\lambda + 1)P_2. \end{aligned}$$

Now, by Cayely-Hamilton theorem we have $\phi(\lambda) = 0$, we have,

$$L^2 - 2L + 2 - LP_1 + (-L + 1)P_2 = 0$$
$$(2 + P_2)LL^{-1} = -L^2 + 2L + LP_1 + LP_2.$$

This implies that,

$$\begin{aligned} L^{-1} &= \frac{1}{2 + P_2}\left[-L + (2 + P_1 + P_2)U_{n\times n}\right] \\ &= \frac{1}{2 + P_2}\begin{pmatrix} 1 & 1 - P_1 \\ -1 + P_2 & 1 + P_1 + P_2 \end{pmatrix} \\ &= \left(\frac{1}{2} - \frac{1}{6}P_2\right)\begin{pmatrix} 1 & 1 - P_1 \\ -1 + P_2 & 1 + P_1 + P_2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} - \frac{1}{6}P_2 & \frac{1}{2} - \frac{1}{2}P_1 \\ -\frac{1}{2} + \frac{1}{2}P_2 & \frac{1}{2} + \frac{1}{2}P1 \end{pmatrix} \end{aligned}$$

5. Conclusion

In this work, the adjoint of symbolic 2-plithogenic square matrices was defined and the inverse of invertible symbolic 2-plithogenic square matrices was studied in terms of symbolic 2-plithogenic adjoint and symbolic 2-plithogenic determinant. Also, we have presented the concept of the characteristic polynomial of symbolic 2-plithogenic matrices and we have proved the symbolic 2-plithogenic version of of Cayley-Hamilton theorem with many examples that clarify the validity of this work.

Funding: This research received no external funding.

References

1. Adeleke, E. O., Agboola, A. A. A., and Smarandache, F., "Refined neutrosophic rings II", International Journal of Neutrosophic Science, vol. 2, pp. 8994, 2020.
2. Abobala, M., "On Refined Neutrosophic Matrices and Their Applications In Refined Neutrosophic Algebraic Equations", Journal Of Mathematics, Hindawi, 2021.
3. Abobala, M., "On Some Algebraic Properties of n-Refined Neutrosophic Elements and n-Refined Neutrosophic Linear Equations", Mathematical Problems in Engineering, Hindawi, 2021.
4. Abobala, M., Hatip, A., Olgun, N., Broumi, S., Salama, A,A., and Khaled, E, H., "The Algebraic Creativity In The Neutrosophic Square Matrices", Neutrosophic Sets and Systems, Vol. 40, pp.1-11, 2021.
5. Abobala, M., "On The Representation of Neutrosophic Matrices by Neutrosophic Linear Transformations", Journal of Mathematics, Hindawi, 2021.
6. Abobala, M., "A Study of Nil Ideals and Kothes Conjecture in Neutrosophic Rings", International Journal of Mathematics and Mathematical Sciences, Hindawi, 2021.
7. Abuobida Mohammed A. Alfahal, Yaser Ahmad Alhasan, Raja Abdullah Abdulfatah, Arif Mehmood, Mustafa Talal Kadhim, "On Symbolic 2-Plithogenic Real Matrices and Their Algebraic Properties", International Journal of Neutrosophic Science, Vol. 21, PP. 96-104, 2023.
8. Albasheer, O., Hajjari., A., and Dalla., R., "On The Symbolic 3-Plithogenic Rings and Their Algebraic Properties", Neutrosophic Sets and Systems, Vol 54, 2023.
9. Kandasamy, W. B. V., and Smarandache, F., "Some Neutrosophic Algebraic Structures and Neutrosophic n-Algebraic Structures"', (Arizona: Hexis Phoenix), 2006.
10. Khaldi, A., Ben Othman, K., Von Shtawzen, O., Ali, R., and Mosa, S., "On Some Algorithms for Solving Different Types of Symbolic 2-Plithogenic Algebraic Equations", Neutrosophic Sets and Systems, Vol 54, 2023.
11. Malath F. Alaswad, "On the Neutrosophic Characteristic Polynomials and Neutrosophic Cayley-Hamilton Theorem", International Journal of Neutrosophic Science, Vol. 18, 2022, PP. 59-71.
12. Merkepci, H., and Abobala, M., "On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.
13. Merkepci, H., and Rawashdeh, A., "On The Symbolic 2-Plithogenic Number Theory and Integers ", Neutrosophic Sets and Systems, Vol 54, 2023.
14. Merkepci, H., "On Novel Results about the Algebraic Properties of Symbolic 3-Plithogenic and 4-Plithogenic Real Square Matrices", Symmetry, 2023.
15. Smarandache, F., "Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)", Neutrosophic Sets and Systems, vol. 53, 2023.
16. Smarandache, F., "Introduction to Neutrosophic Statistics", USA: Sitech and Education Publishing, 2014.

17. Taffach, N., "An Introduction to Symbolic 2-Plithogenic Vector Spaces Generated from The Fusion of Symbolic Plithogenic Sets and Vector Spaces", Neutrosophic Sets and Systems, Vol 54, 2023.
18. Taffach, N., and Ben Othman, K., "An Introduction to Symbolic 2-Plithogenic Modules Over Symbolic 2-Plithogenic Rings", Neutrosophic Sets and Systems, Vol 54, 2023.

Received: 25/6/2023 / **Accepted:** 7/10/2023

Neutrosophic Sets and Systems, Vol. 59, 2023
University of New Mexico

On Symbolic n-Plithogenic Random Variables Using a Generalized Isomorphism

Mohamed Bisher Zeina[1], Saeed Hemeda[2], Mazeat Koreny[3], Mohammad Abobala[4]
[1] Dept. of Mathematical Statistics, Faculty of Science, University of Aleppo, Aleppo, Syria, e-mail: bisher.zeina@gmail.com
[2] Obour High Institute for Management & Informatics, Cairo, Egypt, e-mail: Saeede@oi.edu.eg
[3] Dept. of Mathematical Statistics, Faculty of Science, University of Aleppo, Aleppo, Syria, e-mail: mazeat.koreny@gmail.com
[4] Dept. of Mathematics, Tishreen University, Latakia, Syria, e-mail: mohammadabobala777@gmail.com

Abstract: In this work, we present a generalized isomorphism between field of symbolic n-plithogenic set and R^{n+1}, use it to study the most general form of symbolic plithogenic random variables and study its probabilistic properties including expectation, variance and moments generating function. We also use this isomorphism to study symbolic n-plithogenic probability density function and present many theorems related to it. As an application to this new theory, we study exponential distribution in its symbolic n-plithogenic form and derive its properties, like expected value and variance. Many examples were presented and solved successfully. This paper closes the grand gap in-plithogenic probability theory and paves the way to study many related theories like stochastic modeling and its applications.

Keywords: Plithogenic; Exponential Distribution; Expected Value; Variance; Isomorphism.

1. Introduction

Professor Florentin Smarandache presented a new set of numbers called neutrosophic numbers similar to hypercomplex numbers presented by Kantor, I.L. and Solodovnikov, A.S. [1] where this new set is defined by $R(I) = \{a + bI\,; I^2 = I, a, b \in R\}$ [2]–[6]. This theory built new algebraic structures and new geometry. Hence, new theories in algebra, real analysis, probability, etc.

In neutrosophic probability theory, or as it is called by researchers "literal neutrosophic probability theory", many continuous probability distributions have been studied well, estimation theory was rebuilt under indeterminacy and many methods of estimation were well-defined including: maximum likelihood, moments and bayes. Researchers developed strong theories and many applications in real-life. From our point of view, the most important applications of this theory are in stochastic processes and stochastic modelling. [7]–[17].

Another extension to this set was then developed by professor Smarandache to what is known by plithogenic sets and it is said to be the most general form of a set until this moment. Plithogenic set is defined by $R(P_1, P_2, \ldots, P_n) = \{a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n; a_0, a_2, \ldots, a_n \in R\}$; $P_i^2 = P_i, P_iP_j = P_jP_i = P_{max(i,j)}$ and $i = 1,2, \ldots, n$, $j = 1,2, \ldots, n$. This last set was studied in many fields of mathematics but with $n = 2$. [18]–[34].

This paper can be considered a generalization of our work in [22] where we first presented the symbolic 2 plithogenic probability theory and studied its properties. This paper will close the gap in

DOI: 10.5281/zenodo.10031266

symbolic n-plithogenic probability theory and pave the way for many researches related to it including statistical inference, stochastic modelling, sampling theory, queueing theory, distributions theory, stable distributions, reliability theory, etc.

2. Preliminaries

Definition 2.1

Let $\boldsymbol{R(I)} = \{\boldsymbol{a} + \boldsymbol{bI}; \boldsymbol{I^2} = \boldsymbol{I}\}$, we call $\boldsymbol{R(I)}$ the neutrosophic field of reals.

Definition 2.2

Set of symbolic n-plithogenic real numbers is defined as follows:

$$R(\mathbb{P}) = R(P_1, P_2, \dots, P_n) = \{a_0 + a_1P_1 + a_2P_2 + \dots + a_nP_n; a_0, a_2, \dots, a_n \in R\}$$

Where:

$$P_i^2 = P_i, P_iP_j = P_jP_i = P_{\max(i,j)}\,; i = 1,2, \dots, n\,, j = 1,2, \dots, n$$

Definition 2.3

Symbolic 2 plithogenic random variable is defined as follows:

$$X_{2P}: \Omega_{2P} \to R(P_1, P_2); \Omega_{2P} = \Omega_0 \times \Omega_1(P_1) \times \Omega_2(P_2);$$

$$X_{2P} = X_0 + X_1P_1 + X_2P_2; P_1^2 = P_1, P_2^2 = P_2, P_1P_2 = P_2P_1 = P_2$$

Where random variables X_0, X_1, X_2 are classical random variables defined on $\Omega_0, \Omega_1, \Omega_2$ respectively.

3. Symbolic n-plithogenic random variables

Definition 3.1

Let $R(\mathbb{P})$ be the symbolic n-plithogenic set of reals, we define B isomorphism and its inverse B^{-1} between $R(\mathbb{P})$ and R^{n+1} as follows:

$$B: R(\mathbb{P}) \to R^{n+1};$$

$$B(a_0 + a_1P_1 + a_2P_2 + \dots + a_nP_n) = (a_0, a_0 + a_1, \dots, a_0 + a_1 + \dots + a_n)$$

$$B^{-1}: R^{n+1} \to R(\mathbb{P});$$

$$B^{-1}(a_0, a_1, \dots, a_n) = a_0 + (a_1 - a_0)P_1 + (a_2 - a_1)P_2 + \dots + (a_n - a_{n-1})P_n$$

Theorem 3.1

Isomorphism presented in definition 3.1 is an algebraic isomorphism.

Proof

Let $a_0 + a_1P_1 + a_2P_2 + \dots + a_nP_n, b_0 + b_1P_1 + b_2P_2 + \dots + b_nP_n \in R(\mathbb{P})$.

$$\begin{aligned} &B(a_0 + a_1P_1 + a_2P_2 + \dots + a_nP_n + b_0 + b_1P_1 + b_2P_2 + \dots + b_nP_n) \\ &= B([a_0 + b_0] + [a_1 + b_1]P_1 + .. + [a_n + b_n]P_n) \\ &= (a_0 + b_0, a_0 + b_0 + a_1 + b_1, \dots, a_0 + b_0 + a_1 + b_1 + \dots + a_n + b_n) \\ &= (a_0, a_0 + a_1, \dots, a_0 + a_1 + \dots + a_n) + (b_0, b_0 + b_1, \dots, b_0 + b_1 + \dots + b_n) \\ &= B(a_0 + a_1P_1 + a_2P_2 + \dots + a_nP_n) + B(b_0 + b_1P_1 + b_2P_2 + \dots + b_nP_n). \end{aligned}$$

We also have:

$$\begin{aligned} &B([a_0 + a_1P_1 + a_2P_2 + \dots + a_nP_n] \cdot [b_0 + b_1P_1 + b_2P_2 + \dots + b_nP_n]) \\ &= B(a_0b_0 + [a_0b_1 + a_1b_1 + a_1b_0]P_1 + [a_0b_2 + a_1b_2 + a_2b_2 + a_2b_0 + a_2b_1]P_2 + \cdots \\ &+ [a_0b_n + a_1b_n + \dots + a_nb_n + a_nb_{n-1} + a_nb_{n-2} + \dots + a_nb_0]P_n) \\ &= (a_0b_0, a_0b_0 + a_0b_1 + a_1b_1 + a_1b_0, a_0b_0 + a_0b_1 + a_1b_1 + a_1b_0 + a_0b_2 + a_1b_2 \\ &+ a_2b_2 + a_2b_0 + a_2b_1, \dots, a_0b_0 + a_0b_1 + a_1b_1 + a_1b_0 + a_0b_n + a_1b_n + \dots + a_nb_n \\ &+ a_nb_{n-1} + a_nb_{n-2} + \dots + a_nb_0) \\ &= B(a_0 + a_1P_1 + a_2P_2 + \dots + a_nP_n)B(b_0 + b_1P_1 + b_2P_2 + \dots + b_nP_n). \end{aligned}$$

Also, B is correspondence one-to-one because $Ker(B) = \{0\}$ and for every $(a_0, a_1, \dots, a_n) \in R^{n+1}$ exists $a_0 + (a_1 - a_0)P_1 + (a_2 - a_1)P_2 + \cdots + (a_n - a_{n-1})P_n \in R(\mathbb{P})$ that satisfies $B(a_0 + (a_1 - a_0)P_1 + (a_2 - a_1)P_2 + \cdots + (a_n - a_{n-1})P_n) = (a_0, a_1, \dots, a_n) \in R^{n+1}$ so B is an algebraic isomorphism.

Definition 3.2

We say that $\boldsymbol{a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n \geq_P b_0 + b_1P_1 + b_2P_2 + \cdots + b_nP_n}$ if $\boldsymbol{a_0 \geq b_0, a_0 + a_1 \geq b_0 + b_1, \dots, a_0 + a_1 + \cdots + a_n \geq b_0 + b_1 + \cdots + b_n}$.

Theorem 3.2

Relation defined in definition 3.2 is a partial order relation.

Proof

Straightforward.

Definition 3.3

Symbolic n-plithogenic random variable is defined by:

$$X_P: \Omega_P \to R(\mathbb{P}); \Omega_P = \Omega_0 \times \Omega_1(P_1) \times \Omega_2(P_2) \dots \times \Omega_n(P_n);$$

$$X_P = X_0 + X_1P_1 + X_2P_2 + \cdots + X_nP_n; P_i^2 = P_i, P_iP_j = P_jP_i = P_{\max(i,j)}\ ; i = 1,2, \dots, n\ , j = 1,2, \dots, n$$

Where $X_0, X_1, X_2, \dots, X_n$ are classical random variables defined on $\Omega_0, \Omega_1, \Omega_2, \dots, \Omega_n$ respectively.

Theorem 3.3

Let X_p be a symbolic n-plithogenic random variable then the following equations hold:

1. $E(X_P) = E(X_0) + \sum_{i=1}^{n} E(X_i)P_i$.
2. $Var(X_0) + \sum_{i=1}^{n}\left[Var\left(\sum_{j=0}^{i} X_j\right) - Var\left(\sum_{j=0}^{i-1} X_j\right)\right]P_i$.
3. $\sigma(X_0) + \sum_{i=1}^{n}\left[\sigma\left(\sum_{j=0}^{i} X_j\right) - \sigma\left(\sum_{j=0}^{i-1} X_j\right)\right]P_i$.

Proof

Without loss of generality, we can prove the theorem assuming that X_p is a discrete random variable.

1. $E(X_P) = \sum_{x_P} x_P f(x_P) = \sum_{x_P}(x_0 + x_1P_1 + x_2P_2 + \cdots + x_nP_n) f(x_0 + x_1P_1 + x_2P_2 + \cdots + x_nP_n)$

The isomorphic expectation of last equation is:

$$B[E(X_P)] = B\left[\sum_{x_P}(x_0 + x_1P_1 + x_2P_2 + \cdots + x_nP_n) f(x_0 + x_1P_1 + x_2P_2 + \cdots + x_nP_n)\right]$$

$$= \sum_{x_P} B[(x_0 + x_1P_1 + x_2P_2 + \cdots + x_nP_n) f(x_0 + x_1P_1 + x_2P_2 + \cdots + x_nP_n)]$$

$$= \left(\sum_{x_0} x_0 f(x_0), \sum_{x_0+x_1}(x_0 + x_1) f(x_0 + x_1), \dots, \sum_{x_0+x_1+\cdots+x_n}(x_0 + x_1 + \cdots + x_n) f(x_0 + x_1 + \cdots + x_n)\right) = \left(E(X_0), E(X_0 + X_1), \dots, E(X_0 + X_1 + \cdots + X_n)\right)$$

$$= \left(E(X_0), E(X_0) + E(X_1), \dots, E(X_0) + E(X_1) + \cdots + E(X_n)\right)$$

Taking B^{-1}:

$$E(X_P) = B^{-1}\left(E(X_0), E(X_0) + E(X_1), \dots, E(X_0) + E(X_1) + \cdots + E(X_n)\right)$$

$$= E(X_0) + [E(X_0) + E(X_1) - E(X_0)]P_1 + \cdots$$

$$+ [E(X_0) + E(X_1) + \cdots + E(X_n) - E(X_0) - E(X_1) - \cdots - E(X_{n-1})]P_n$$

$$= E(X_0) + E(X_1)P_1 + \cdots + E(X_n)P_2$$

2. $E(X_P^2)=E(X_0+X_1P_1+\cdots+X_nP_n)^2=\sum_{x_P}(x_0+x_1P_1+\cdots+x_nP_n)^2 f(x_0+x_1P_1+\cdots+x_nP_n)$

Taking B:

$$B[E(X_P^2)]=B\left[\sum_{x_P}(x_0+x_1P_1+\cdots+x_nP_n)^2 f(x_0+x_1P_1+\cdots+x_nP_n)\right]$$

$$=\sum_{x_P}B[(x_0+x_1P_1+\cdots+x_nP_n)^2 f(x_0+x_1P_1+\cdots+x_nP_n)]\ =$$

$$=\left(\sum_{x_0}x_0^2 f(x_0),\sum_{x_0+x_1}(x_0+x_1)^2 f(x_0+x_1),\dots,\sum_{x_0+x_1+\cdots+x_n}(x_0+x_1+\cdots+x_n)^2 f(x_0+x_1+\cdots+x_n)\right)=(E(X_0^2),E(X_0+X_1)^2,\dots,E(X_0+X_1+\cdots+X_n)^2)$$

Now by taking the inverse isometry we get:

$$E(X_P^2)=B^{-1}(E(X_0^2),E(X_0+X_1)^2,\dots,E(X_0+X_1+\cdots+X_n)^2)$$
$$=E(X_0^2)+[E(X_0+X_1)^2-E(X_0^2)]P_1+\cdots$$
$$+[E(X_0+X_1+\cdots+X_n)^2-E(X_0+X_1+\cdots+X_{n-1})^2]P_n$$

Also, we can prove in similar way that:

$$[E(X_P)]^2=[E(X_0)]^2+[[E(X_0+X_1)]^2-[E(X_0)]^2]P_1+\cdots$$
$$+[[E(X_0+X_1+\cdots+X_n)]^2-[E(X_0+X_1+\cdots+X_{n-1})]^2]P_n$$

Hence, we have:

$$Var(X_P)=E(X_P^2)-[E(X_P)]^2$$
$$=E(X_0^2)+[E(X_0+X_1)^2-E(X_0^2)]P_1+\cdots$$
$$+[E(X_0+X_1+\cdots+X_n)^2-E(X_0+X_1+\cdots+X_{n-1})^2]P_n$$
$$-\{[E(X_0)]^2+[[E(X_0+X_1)]^2-[E(X_0)]^2]P_1+\cdots$$
$$+[[E(X_0+X_1+\cdots+X_n)]^2-[E(X_0+X_1+\cdots+X_{n-1})]^2]P_n\}$$
$$=Var(X_0)+[Var(X_0+X_1)-Var(X_0)]P_1+\cdots$$
$$+[Var(X_0+X_1+\cdots+X_n)-Var(X_0+X_1+\cdots+X_{n-1})]P_n$$
$$=Var(X_0)+\sum_{i=1}^{n}\left[Var\left(\sum_{j=0}^{i}X_j\right)-Var\left(\sum_{j=0}^{i-1}X_j\right)\right]P_i$$

3. Straightforward.

Theorem 3.4

A symbolic n-plithogenic function $f(x_P)=f(x_0+x_1P_1+\cdots+x_nP_n)$ is a probability density function in classical scene if and only if it satisfies the following conditions:

1. $f(x_0),f(x_0+x_1),\dots,f(x_0+x_1+\cdots+x_n)$ are all continuous nonnegative functions.
2. $\int_{x_0}f(x_0)dx_0=1,\int_{x_0+x_1}f(x_0+x_1)d(x_0+x_1)=1,\dots,\int_{x_0+x_1+\cdots+x_n}f(x_0+x_1+\cdots+x_n)d(x_0+x_1+\cdots+x_n)=1.$

Proof

The isometric image of $f(x_P)$ is:

$$B(f(x_P)) = (f(x_0), f(x_0+x_1), \dots, f(x_0+x_1+\dots+x_n))$$

According to theorem 3.2 we can see that if $f(x_0), f(x_0+x_1), \dots, f(x_0+x_1+\dots+x_n)$ are all nonnegative then $f(x_P)$ is a nonnegative function and vice-versa.

Also, according to the properties of the isomorphism B we can conclude that $f(x_P)$ will be a continuous function if and only if $f(x_0), f(x_0+x_1), \dots, f(x_0+x_1+\dots+x_n)$ are all continuous functions.

Finally, let us assume that:

$$\int_{x_0} f(x_0)dx_0 = 1, \int_{x_0+x_1} f(x_0+x_1)d(x_0+x_1)$$

$$= 1, \dots, \int_{x_0+x_1+\dots+x_n} f(x_0+x_1+\dots+x_n)d(x_0+x_1+\dots+x_n) = 1.$$

Then taking B^{-1} yields to:

$$B^{-1}\left(\int_{x_0} f(x_0)dx_0, \int_{x_0+x_1} f(x_0+x_1)d(x_0+x_1), \dots, \int_{x_0+x_1+\dots+x_n} f(x_0+x_1+\dots+x_n)d(x_0+x_1+\dots+x_n)\right) = B^{-1}(1,1,\dots,1) = 1 + (1-1)P_1 + \dots + (1-1)P_n = 1$$

And this completes the proof.

Example

Let $f(x_P) = 2x_0 + (e^{-x_1} - 2x_0)P_1 + (1 - e^{-x_1})P_2; x_0 \in [0,1], x_0 + x_1 > 0, x_0 + x_1 + x_2 \in [0,1]$

1. prove that $f(x_P)$ is a probability density function.
2. Calculate the probability $P\left(X_P < \frac{1}{2} + P_1 - \frac{3}{4}P_2\right)$.

Solution

1. $B(f(x_P)) = B\left(2x_0 + \left(e^{-(x_0+x_1)} - 2x_0\right)P_1 + \left(1 - e^{-(x_0+x_1)}\right)P_2\right) = \Big(2x_0, 2x_0 + \left(e^{-(x_0+x_1)} - 2x_0\right), 2x_0 + \left(e^{-(x_0+x_1)} - 2x_0\right) + \left(1 - e^{-(x_0+x_1)}\right)\Big) = \left(2x_0, e^{-(x_0+x_1)}, 1\right)$

We conclude that:

$f(x_0) = 2x_0; x_0 \in [0,1]$

$f(x_0+x_1) = e^{-(x_0+x_1)}; x_0 + x_1 > 0$

$f(x_0+x_1+x_2) = 1; x_0+x_1+x_2 \in [0,1]$

All previous functions are continuous nonnegative functions and integrate to one on their defined domain.

2. Calculating $P\left(X_P < \frac{1}{2} + P_1 - \frac{3}{4}P_2\right)$ is equivalent to calculating the following three probabilities:

$$\int_{0}^{\frac{1}{2}} 2x_0 dx_0 = x_0^2\Big|_0^{\frac{1}{2}} = \frac{1}{4}, \int_{0}^{\frac{1}{2}+1} e^{-(x_0+x_1)} d(x_0+x_1) = \left[1-e^{-(x_0+x_1)}\right]_0^{\frac{3}{2}} = 1-e^{-\frac{3}{2}}, \int_{0}^{\frac{1}{2}+1-\frac{3}{4}} d(x_0+x_1+x_2) = \frac{3}{4}$$

So $P\left(X_P < \frac{1}{2} + P_1 - \frac{3}{4}P_2\right) = B^{-1}\left(\frac{1}{4}, 1-e^{-\frac{3}{2}}, \frac{3}{4}\right) = \frac{1}{4} + \left(1-e^{-\frac{3}{2}} - \frac{1}{4}\right)P_1 + \left(\frac{3}{4} - 1 + e^{-\frac{3}{2}}\right)P_2 = \frac{1}{4} + \left(\frac{3}{4} - e^{-\frac{2}{3}}\right)P_1 + \left(e^{-\frac{3}{2}} - \frac{1}{4}\right)P_2.$

Theorem 3.5

Let X_P be a symbolic n-plithogenic random variable then its moments generating function is:

$$M_{X_P}(t) = M_{X_0}(t) + \sum_{i=1}^{n}\left[M_{\sum_{j=0}^{i} X_j}(t) - M_{\sum_{j=0}^{i-1} X_j}(t)\right]P_i$$

Proof

$$M_{X_P}(t) = E(e^{tX_P}) = \int_{-\infty}^{+\infty} e^{tx_P} f(x_P) dx_P = B^{-1}B\left[\int_{-\infty}^{+\infty} e^{tx_P} f(x_P) dx_P\right]$$

$$= B^{-1}\left(\int_{-\infty}^{+\infty} e^{tx_0} f(x_0) dx_0\,, \int_{-\infty}^{+\infty} e^{t(x_0+x_1)} f(x_0+x_1) d(x_0+x_1)\,, \dots, \int_{-\infty}^{+\infty} e^{t(x_0+x_1+\dots+x_n)} f(x_0+x_1+\cdots + x_n) d(x_0+x_1+\cdots+x_n)\right)$$

$$= B^{-1}\left(M_{X_0}(t), M_{X_0+X_1}(t), \dots, M_{X_0+X_1+\dots+X_n}(t)\right)$$

$$= M_{X_0}(t) + \left[M_{X_0+X_1}(t) - M_{X_0}(t)\right]P_1 + \cdots + \left[M_{X_0+X_1+\dots+X_n}(t) - M_{X_0+X_1+\dots+X_{n-1}}(t)\right]P_n = M_{X_P}(t)$$

$$= M_{X_0}(t) + \sum_{i=1}^{n}\left[M_{\sum_{j=0}^{i} X_j}(t) - M_{\sum_{j=0}^{i-1} X_j}(t)\right]P_i$$

Theorem 3.6

Let X_P be a symbolic n-plithogenic random variable and let its moments generating function be $M_{X_P}(t)$ then:

$$\frac{d^k}{dt^k} M_{X_P}(t)|_{t=0} = E(X_P^k)$$

Proof

We have

$$M_{X_P}(t) = M_{X_0}(t) + \sum_{i=1}^{n}\left[M_{\sum_{j=0}^{i} X_j}(t) - M_{\sum_{j=0}^{i-1} X_j}(t)\right]P_i$$

By taking k[th] derivative of the last equation and substituting $t = 0$ we get:

$$\frac{d^k}{dt^k}M_{X_P}(t)|_{t=0}=\frac{d^k}{dt^k}\left(M_{X_0}(t)+\sum_{i=1}^{n}\left[M_{\sum_{j=0}^{i}X_j}(t)-M_{\sum_{j=0}^{i-1}X_j}(t)\right]P_i\right)_{t=0}$$

$$=\frac{d^k}{dt^k}M_{X_0}(0)+\sum_{i=1}^{n}\left[\frac{d^k}{dt^k}M_{\sum_{j=0}^{i}X_j}(0)-\frac{d^k}{dt^k}M_{\sum_{j=0}^{i-1}X_j}(0)\right]P_i$$

$$=E(X_0^k)+\sum_{i=1}^{n}\left[E\left(\sum_{j=0}^{i}X_j\right)^k-E\left(\sum_{j=0}^{i-1}X_j\right)^k\right]P_i=E(X_P^k)$$

4. Application to symbolic n-plithogenic exponential distribution

Definition 4.1

A symbolic n-plithogenic random variable is said to follow exponential distribution with parameter $\lambda_N=\lambda_0+\lambda_1P_1+\cdots+\lambda_nP_n$ if its probability density function is given by:

$$f(x_P)=\lambda_0e^{-\lambda_0x_0}+\sum_{i=1}^{n}\left[\sum_{j=0}^{i}\lambda_j\,e^{-\sum_{j=0}^{i}\lambda_j\sum_{j=0}^{i}x_j}-\sum_{j=0}^{i-1}\lambda_j\,e^{-\sum_{j=0}^{i-1}\lambda_j\sum_{j=0}^{i-1}x_j}\right]P_i\,;x_P,\lambda_P>_P0$$

Theorem 4.1

If X_P is a symbolic n-plithogenic exponential random variable with parameter $\lambda_N=\lambda_0+\lambda_1P_1+\cdots+\lambda_nP_n$ then:

1. $F(x_P)=1-\lambda_0e^{-\lambda_0x_0}+\sum_{i=1}^{n}\left[e^{-\sum_{j=0}^{i-1}\lambda_j\sum_{j=0}^{i-1}x_j}-e^{-\sum_{j=0}^{i}\lambda_j\sum_{j=0}^{i}x_j}\right]P_i\,;x_P,\lambda_P>_P0$
2. $E(X_P)=\frac{1}{\lambda_0}+\sum_{i=1}^{n}\left[\frac{1}{\sum_{j=0}^{i}\lambda_j}-\frac{1}{\sum_{j=0}^{i-1}\lambda_j}\right]P_i$
3. $Var(X_P)=\frac{1}{\lambda_0^2}+\sum_{i=1}^{n}\left[\frac{1}{\left(\sum_{j=0}^{i}\lambda_j\right)^2}-\frac{1}{\left(\sum_{j=0}^{i-1}\lambda_j\right)^2}\right]P_i$

Proof

1. $F(x_P)=\int_0^{x_P}f(x_P)dx_P=\int_0^{x_0+x_1P_1+\cdots+x_nP_n}\left[\lambda_0e^{-\lambda_0x_0}+\sum_{i=1}^{n}\left[\sum_{j=0}^{i}\lambda_j\,e^{-\sum_{j=0}^{i}\lambda_j\sum_{j=0}^{i}x_j}-\sum_{j=0}^{i-1}\lambda_j\,e^{-\sum_{j=0}^{i-1}\lambda_j\sum_{j=0}^{i-1}x_j}\right]P_i\right]d(x_0+x_1P_1+\cdots+x_nP_n)=B^{-1}\big[\int_0^{x_0}\lambda_0e^{-\lambda_0x_0}dx_0,\int_0^{x_0+x_1}(\lambda_0+\lambda_1)e^{-(\lambda_0+\lambda_1)(x_0+x_1)}d(x_0+x_1),\ldots,\int_0^{x_0+x_1+\cdots+x_n}(\lambda_0+\lambda_1+\cdots+\lambda_n)e^{-(\lambda_0+\lambda_1+\cdots+\lambda_n)(x_0+x_1+\cdots+x_n)}d(x_0+x_1+\cdots+x_n)\big]=B^{-1}\big(1-e^{-\lambda_0x_0},1-e^{-(\lambda_0+\lambda_1)(x_0+x_1)},\ldots,1-e^{-(\lambda_0+\lambda_1+\cdots+\lambda_n)(x_0+x_1+\cdots+x_n)}\big)=1-\lambda_0e^{-\lambda_0x_0}+\sum_{i=1}^{n}\left[e^{-\sum_{j=0}^{i-1}\lambda_j\sum_{j=0}^{i-1}x_j}-e^{-\sum_{j=0}^{i}\lambda_j\sum_{j=0}^{i}x_j}\right]P_i$

2. $E(X_P)=\int_0^{\infty}x_Pf(x_P)dx_P=\int_0^{\infty}(x_0+x_1P_1+\cdots+x_nP_n)\left[\lambda_0e^{-\lambda_0x_0}+\sum_{i=1}^{n}\left[\sum_{j=0}^{i}\lambda_j\,e^{-\sum_{j=0}^{i}\lambda_j\sum_{j=0}^{i}x_j}-\sum_{j=0}^{i-1}\lambda_j\,e^{-\sum_{j=0}^{i-1}\lambda_j\sum_{j=0}^{i-1}x_j}\right]P_i\right]d(x_0+x_1P_1+\cdots+x_nP_n)=B^{-1}\big[\int_0^{\infty}x_0\lambda_0e^{-\lambda_0x_0}dx_0,\int_0^{\infty}(x_0+x_1)(\lambda_0+\lambda_1)e^{-(\lambda_0+\lambda_1)(x_0+x_1)}d(x_0+x_1),\ldots,\int_0^{\infty}(x_0+x_1+\cdots+x_n)(\lambda_0+\lambda_1+\cdots+$

$\lambda_n)e^{-(\lambda_0+\lambda_1+\cdots+\lambda_n)(x_0+x_1+\cdots+x_n)}d(x_0+x_1+\cdots+x_n)\Big] = B^{-1}\left(\frac{1}{\lambda_0},\frac{1}{\lambda_0+\lambda_1},\dots,\frac{1}{\lambda_0+\lambda_1+\cdots+\lambda_n}\right) = \frac{1}{\lambda_0} + \sum_{i=1}^{n}\left[\frac{1}{\sum_{j=0}^{i}\lambda_j} - \frac{1}{\sum_{j=0}^{i-1}\lambda_j}\right]P_i$

3. $Var(X_P) = B^{-1}B\left(\int_0^{\infty}[x_P - E(X_P)]^2\lambda_P e^{-\lambda_P x_P}dx_P\right) = B^{-1}\Bigg(\int_0^{\infty}\left(x_0 - \frac{1}{\lambda_0}\right)^2\lambda_0 e^{-\lambda_0 x_0}dx_0, \int_0^{\infty}\left(x_0+x_1-\frac{1}{\lambda_0+\lambda_1}\right)^2(\lambda_0+\lambda_1)e^{-(\lambda_0+\lambda_1)(x_0+x_1)}d(x_0+x_1), \dots, \int_0^{\infty}\left(x_0+x_1+\cdots+x_n-\frac{1}{\lambda_0+\lambda_1+\cdots+\lambda_n}\right)^2(\lambda_0+\lambda_1+\cdots+\lambda_n)e^{-(\lambda_0+\lambda_1+\cdots+\lambda_n)(x_0+x_1+\cdots+x_n)}d(x_0+x_1+\cdots+x_n)\Bigg) =$

$B^{-1}\left(\frac{1}{\lambda_0^2},\frac{1}{(\lambda_0+\lambda_1)^2},\dots,\frac{1}{(\lambda_0+\lambda_1+\cdots+\lambda_n)^2}\right) = \frac{1}{\lambda_0^2} + \sum_{i=1}^{n}\left[\frac{1}{\left(\sum_{j=0}^{i}\lambda_j\right)^2} - \frac{1}{\left(\sum_{j=0}^{i-1}\lambda_j\right)^2}\right]P_i$

5. Conclusion

We have presented an important introduction to symbolic n-plithogenic probability theory and studied random variables related to it. Many theorems were demonstrated and proved successfully. As an application to this new theory, exponential distribution was defined and its properties were studied. Many examples have been solved successfully. In future researches, we are going to study symbolic n-plithogenic stochastic processes and its real-life applications in communication using queueing theory.

Funding: This research received no external funding.

Acknowledgments: Authors are thankful for the editorial team and reviewers for their valuable comments.

Conflicts of Interest: The authors declare no conflict of interest.

References

[1] I. L. Kantor and A. S. Solodovnikov, “Hypercomplex Numbers,” *Hypercomplex Numbers*, 1989, doi: 10.1007/978-1-4612-3650-4.

[2] F. Smarandache, “Symbolic Neutrosophic Theory,” *ArXiv*, 2015, doi: 10.5281/ZENODO.32078.

[3] F. Smarandache *et al.*, “Introduction to neutrosophy and neutrosophic environment,” *Neutrosophic Set in Medical Image Analysis*, pp. 3–29, 2019, doi: 10.1016/B978-0-12-818148-5.00001-1.

[4] F. Smarandache, “Neutrosophic theory and its applications,” *Brussels*, vol. I, 2014.

[5] F. Smarandache, “Indeterminacy in neutrosophic theories and their applications,” *International Journal of Neutrosophic Science*, vol. 15, no. 2, 2021, doi: 10.5281/zenodo.5295819.

[6] S. Alias and D. Mohamad, “A Review on Neutrosophic Set and Its Development,” 2018.

[7] A. Astambli, M. B. Zeina, and Y. Karmouta, “On Some Estimation Methods of Neutrosophic Continuous Probability Distributions Using One-Dimensional AH-Isometry,” *Neutrosophic Sets and Systems*, vol. 53, 2023.

[8] M. Bisher Zeina and M. Abobala, “On The Refined Neutrosophic Real Analysis Based on Refined Neutrosophic Algebraic AH-Isometry,” *Neutrosophic Sets and Systems*, vol. 54, 2023.

[9] M. B. Zeina, M. Abobala, A. Hatip, S. Broumi, and S. Jalal Mosa, "Algebraic Approach to Literal Neutrosophic Kumaraswamy Probability Distribution," *Neutrosophic Sets and Systems*, vol. 54, pp. 124–138, 2023.

[10] M. B. Zeina and M. Abobala, "A novel approach of neutrosophic continuous probability distributions using AH-isometry with applications in medicine," *Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics*, pp. 267–286, Jan. 2023, doi: 10.1016/B978-0-323-99456-9.00014-3.

[11] M. B. Zeina and Y. Karmouta, "Introduction to Neutrosophic Stochastic Processes," *Neutrosophic Sets and Systems*, vol. 54, 2023.

[12] C. Granados and J. Sanabria, "On Independence Neutrosophic Random Variables," *Neutrosophic Sets and Systems*, vol. 47, 2021.

[13] C. Granados, "New Notions On Neutrosophic Random Variables," *Neutrosophic Sets and Systems*, vol. 47, 2021.

[14] A. Astambli, M. B. Zeina, and Y. Karmouta, "Algebraic Approach to Neutrosophic Confidence Intervals," *Journal of Neutrosophic and Fuzzy Systems*, vol. 5, no. 2, pp. 08–22, 2023, doi: 10.54216/JNFS.050201.

[15] M. Abobala and M. B. Zeina, "A Study of Neutrosophic Real Analysis by Using the One-Dimensional Geometric AH-Isometry," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 3, no. 1, pp. 18–24, 2023, doi: 10.54216/GJMSA.030103).

[16] M. Abobala and A. Hatip, "An Algebraic Approach to Neutrosophic Euclidean Geometry," *Neutrosophic Sets and Systems*, vol. 43, 2021.

[17] M. B. Zeina and A. Hatip, "Neutrosophic Random Variables," *Neutrosophic Sets and Systems*, vol. 39, 2021, doi: 10.5281/zenodo.4444987.

[18] F. Smarandache, *Plithogeny, Plithogenic Set, Logic, Probability, and Statistics*. Belgium: Pons, 2018. Accessed: Feb. 23, 2023. [Online]. Available: http://arxiv.org/abs/1808.03948

[19] P. K. Singh, "Complex Plithogenic Set," *International Journal of Neutrosophic Science*, vol. 18, no. 1, 2022, doi: 10.54216/IJNS.180106.

[20] S. Alkhazaleh, "Plithogenic Soft Set," *Neutrosophic Sets and Systems*, 2020.

[21] N. Martin and F. Smarandache, "Introduction to Combined Plithogenic Hypersoft Sets," *Neutrosophic Sets and Systems*, vol. 35, 2020, doi: 10.5281/zenodo.3951708.

[22] M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, "Introduction to Symbolic 2-Plithogenic Probability Theory," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.

[23] F. Smarandache, "Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)," *Neutrosophic Sets and Systems* , vol. 53, Jan. 2023, doi: 10.5281/ZENODO.7536105.

[24] N. M. Taffach and A. Hatip, "A Review on Symbolic 2-Plithogenic Algebraic Structures," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 5, no. 1, pp. 08–16, 2023, doi: 10.54216/GJMSA.050101.

[25] R. Ali and Z. Hasan, "An Introduction To The Symbolic 3-Plithogenic Modules," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 6, no. 1, pp. 13–17, 2023, doi: 10.54216/GJMSA.060102.

[26] F. Smarandache, "Introduction to Plithogenic Logic as generalization of MultiVariate Logic," *Neutrosophic Sets and Systems*, vol. 45, 2021.

[27] R. Ali and Z. Hasan, "An Introduction to The Symbolic 3-Plithogenic Vector Spaces," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 6, no. 1, pp. 08–12, 2023, doi: 10.54216/GJMSA.060101.

[28] F. Smarandache, "Plithogenic Probability & Statistics are generalizations of MultiVariate Probability & Statistics," *Neutrosophic Sets and Systems*, vol. 43, 2021.

[29] F. Smarandache, "Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets," *Neutrosophic Sets and Systems*, vol. 21, pp. 153–166, 2018.

[30] M. Abdel-Basset and R. Mohamed, "A novel plithogenic TOPSIS- CRITIC model for sustainable supply chain risk management," *J Clean Prod*, vol. 247, Feb. 2020, doi: 10.1016/J.JCLEPRO.2019.119586.

[31] M. Abdel-Basset, M. El-hoseny, A. Gamal, and F. Smarandache, "A novel model for evaluation Hospital medical care systems based on-plithogenic sets," *Artif Intell Med*, vol. 100, Sep. 2019, doi: 10.1016/J.ARTMED.2019.101710.

[32] N. M. Taffach and A. Hatip, "A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 5, no. 1, pp. 36–44, 2023, doi: 10.54216/GJMSA.050103.

[33] F. Sultana *et al.*, "A study of plithogenic graphs: applications in spreading coronavirus disease (COVID-19) globally," *J Ambient Intell Humaniz Comput*, p. 1, 2022, doi: 10.1007/S12652-022-03772-6.

[34] M. Abdel-Basset, R. Mohamed, A. E. N. H. Zaied, and F. Smarandache, "A hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics," *Symmetry (Basel)*, vol. 11, no. 7, Jul. 2019, doi: 10.3390/SYM11070903.

Received: 13/5/2023. Accepted: 28/9/2023

Neutrosophic Sets and Systems, Vol. 59, 2023
University of New Mexico

On The Representation of Some n-Plithogenic Differential Operators by Matrices

Basheer Abd Al Rida Sadiq
Computer Techniques Engineering, Imam Al-Kadhum College (IKC), Baghdad, Iraq
basheer.abdrida@alkadhum-col.edu.iq

Abstract

The main goal of this paper is to study the representation of the symbolic n-plithogenic differential operator for many different values of n by classical algebraic matrices and plithogenic matrices. We present many examples about the representation of symbolic n-plithogenic differential operators by matrices. As well as, we compute the symbolic 2-plithogenic, 3-plithogenic, and 4-plithogenic Wronsckian, and anti-Wronsckian.

Keywords: Differential operator, Wronsckian, anti-Wronsckian, symbolic n-plithogenic matrix

Introduction

Symbolic n-plithogenic structures and sets are defined for the first time by Smarandache [4], as extensions of classical algebraic structures. Where they were used widely by many researchers to generalize famous algebraic structures. For example, we can see symbolic n-plithogenic rings, probability, spaces, and matrices [1-3, 5-8, 14-19].

The main results about symbolic n-plithogenic structures are the similarity between them and refined neutrosophic structures, see [9-13].

In this work, we concentrate on the analytical side of symbolic n-plithogenic algebraic structures, where we provide many examples about the applications of matrices in representing symbolic n-plithogenic differential operators. Also, we present the concept of symbolic n-plithogenic Wronsckian, and anti-Wronsckian,

zenodo.10031274/10.5281

with many computable examples. For the definitions of symbolic n-plithogenic rings and structures, check [1,6,8,19].

Main Discussion

Definition:

Let $f: 2-SP_R \to 2-SP_R$ be a symbolic 2-plithogenic real function, we define the symbolic 2-plithogenic differential operator as: $D_2(f) = \acute{f}$.

Definition.

Let $f: 3-SP_R \to 3-SP_R$ be a symbolic 3-plithogenic real function, we define the symbolic 2-plithogenic differential operator as: $D_3(f) = \acute{f}$.

Definition.

Let $f: 4-SP_R \to 4-SP_R$ be a symbolic 4-plithogenic real function, we define the symbolic 2-plithogenic differential operator as: $D_4(f) = \acute{f}$.

Definition.

Let $f: 5-SP_R \to 5-SP_R$ be a symbolic 5-plithogenic real function, we define the symbolic 2-plithogenic differential operator as: $D_5(f) = \acute{f}$.

Example.

Consider $f: 2-SP_R \to 2-SP_R; f(X) = X^2 + (P_1 + P_2)X - P_1$,where $X = x_0 + x_1P_1 + x_2P_2 \in 2-SP_R$, then $D_2(f) = 2X + (P_1 + P_2)$.

Consider $g: 3-SP_R \to 3-SP_R; g(X) = X^2 + P_3X + P_3 + P_2$,where $X = x_0 + x_1P_1 + x_2P_2 + x_3P_3 \in 3-SP_R$, then $D_3(g) = 2X + P_3$.

Consider $h: 4-SP_R \to 4-SP_R; h(X) = X^3 + (P_1 + P_4)X - 1$,where $X = x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4 \in 4-SP_R$, then $D_3(h) = 3X^2 + P_1 + P_4$.

Example.

Consider D_2 the symbolic 2-plithogenic differential operator on the space of symbolic 2-plithogenic quadratic polynomials $\{aX^2 + bX + c; a, b, c, X \in 2-SP_R\}$, then:

$$\begin{cases} D_2(X^2) = 2X = 2x_0 + 2x_1P_1 + 2x_2P_2 = 0X^2 + 2X + 0.1 \\ D_2(X) = 1 = 0X^2 + 0.X + 1.1 \\ D_2(1) = 0 = 0X^2 + 0.X + 0.1 \end{cases}$$

Hence $[D_2] = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$.

Example.

Consider D_3, D_4, D_5 be the symbolic 3-plithogenic, 4-plithogenic, 5-plithogenic differential operators on the spaces of cubic symbolic 30plithogenic, 4-plithogenic, and 5-plithogenic spaces.

$L_1 = \{aX^3 + bX^2 + bX + d; a, b, c, d, X \in 3 - SP_R\}$

$L_2 = \{aX^3 + bX^2 + bX + d; a, b, c, d, X \in 4 - SP_R\}$

$L_3 = \{aX^3 + bX^2 + bX + d; a, b, c, d, X \in 5 - SP_R\}$

Then $D_n(X^3) = 3X^2 = 0.X^3 + 3X^2 + 0.X + 0.1$.

$D_n(X^2) = 2X = 0.X^3 + 0.X^2 + 2.X + 0.1$.

$D_n(X) = 1 = 0.X^3 + 0.X^2 + 0.X + 1.1$.

$D_n(1) = 0 = 0.X^3 + 0.X^2 + 0.X + 0.1$

For all $3 \leq n \leq 5$, hence:

$$[D_n] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Example.

For $sinX = sin(x_0 + x_1P_1 + x_2P_2), cosX = cos(x_0 + x_1P_1 + x_2P_2)$.

We have:

$$\begin{cases} D_2(sinX) = cosX = 0.sinX + 1.cosX + 0.1 \\ D_2(cosX) = -sinX = -1.sinX + 0..cosX + 0.1 \\ D_2(1) = 0 = 0.sinX + 0.cosX + 0.1 \end{cases}$$

Hence $[D_2] = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

For $sinX = sin(x_0 + x_1P_1 + x_2P_2 + x_3P_3), cosX = cos(x_0 + x_1P_1 + x_2P_2 + x_3P_3)$.

We have:

$$\begin{cases} D_3(sinX) = cosX \\ D_3(cosX) = -sinX \\ D_3(1) = 0 \end{cases}$$

Hence $[D_3] = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

For $sinX = sin(x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4), cosX = cos(x_0 + x_1P_1 + x_2P_2 +$

$x_3P_3 + x_4P_4)$.

We have:

c

Hence $[D_4] = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Example.

For $\{1, X, X^2, X^3, X^4\}; X = x_0 + x_1P_1 + x_2P_2$,we have:

$$\begin{cases} D_2(X^4) = 4X^3 \\ D_2(X^3) = 3X^2 \\ D_2(X^2) = 2X \\ D_2(X) = 1 \\ D_2(1) = 0 \end{cases}$$

Hence $[D_2] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

For $X = x_0 + \sum_{i=1}^{5} x_iP_i$, we have:

$$\begin{cases} D_5(X^4) = 4X^3 \\ D_5(X^3) = 3X^2 \\ D_5(X^2) = 2X \\ D_5(X) = 1 \\ D_5(1) = 0 \end{cases}$$

Hence $[D_5] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

Example.

Consider $A = \{1, e^X, e^{2X}\}$, where $X = x_0 + x_1P_1 + x_2P_2$, $B = \{1, e^Y, e^{2Y}\}$, where $Y = y_0 + y_1P_1 + y_2P_2 + y_3P_3$, $C = \{1, e^Z, e^{2Z}\}$, where $Z = z_0 + z_1P_1 + z_2P_2 + z_3P_3 + z_4P_4$, $D = \{1, e^T, e^{2T}\}$, where $T = t_0 + + \sum_{i=1}^{5} t_iP_i$, then:

$$\begin{cases} D_2(1) = 0 \\ D_2(e^X) = e^X \\ D_2(e^{2X}) = 2e^{2X} \end{cases}$$

Hence $[D_3] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

$$\begin{cases} D_3(1) = 0 \\ D_3(e^Y) = e^Y \\ D_3(e^{2Y}) = 2e^{2Y} \end{cases}$$

Hence $[D_3] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

$$\begin{cases} D_4(1) = 0 \\ D_4(e^Z) = e^Z \\ D_4(e^{2Z}) = 2e^{2Z} \end{cases}$$

Hence $[D_4] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

$$\begin{cases} D_5(1) = 0 \\ D_5(e^T) = e^T \\ D_5(e^{2T}) = 2e^{2T} \end{cases}$$

Hence $[D_5] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Another possible representation.

We have shown that symbolic n-plithogenic differential operators can be represented by classical real matrices, now we will try to explain how they can be represented by plithogenic matrices.

Example.

Consider $\{1, X, X^2\}; X = x_0 + x_1P_1 + x_2P_2$, with D_2 the symbolic 2-plithogenic differential operator, then:

$$\begin{cases} D_2(1) = 0 = 0.x_0 + 0.x_1P_1 + 0.x_2P_2 \\ D_2(X) = 1 \\ D_2(X^2) = 2X \end{cases}$$

The basis $\{1, X, X^2\}$ can be represented as follows:

$$B_1 = \{1, x_0, {x_0}^2\}, B_2 = \{1, (x_0 + x_1), (x_0 + x_1)^2\}, B_3 = \{1, (x_0 + x_1 + x_2), (x_0 + x_1 + x_2)^2\}$$

Any quadratic polynomial $P(X) = aX^2 + bX + c; a, b, c, X \in 2 - SP_R$, with:

$$\begin{cases} a = a_0 + a_1P_1 + a_2P_2 \\ b = b_0 + b_1P_1 + b_2P_2 \\ c = c_0 + c_1P_1 + c_2P_2 \\ X = x_0 + x_1P_1 + x_2P_2 \end{cases}$$

$$\begin{aligned} P(X) = aX^2 + bX + c \\ &= (a_0{x_0}^2 + b_0x_0 + c_0) \\ &+ P_1[(a_0 + a_1)(x_0 + x_1)^2 - a_0{x_0}^2 + (b_0 + b_1)(x_0 + x_1) - b_0x_0 + c_1] \\ &+ P_2[(a_0 + a_1 + a_2)(x_0 + x_1 + x_2)^2 - (a_0 + a_1)(x_0 + x_1)^2 \\ &+ (b_0 + b_1 + b_2)(x_0 + x_1 + x_2) - (b_0 + b_1)(x_0 + x_1) + c_2] \\ &= q_1(x_0) + P_1[q_2(x_0 + x_1) - q_1(x_0)] \\ &+ P_2[q_3(x_0 + x_1 + x_2) - q_2(x_0 + x_1)] \end{aligned}$$

Hence, $D_2\big(P(X)\big) = D_2(q_1) + P_1[D_2(q_2) - D_2(q_1)] + P_2[D_2(q_3) - D_2(q_2)]$, hence:

$$[D_2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} + P_1\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}\right] + P_2\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 + 2P_1 + 2P_2 \\ 0 & 1 + P_1 + P_2 & 0 \end{pmatrix}$$

The symbolic plithogenic Wronsckian.

Consider the following functions independent set:

$E = \{f_1, \dots, f_n\}$, their wronsckian is defined as follows:

$$W(E) = \begin{vmatrix} f_1 & \cdots & f_n \\ \acute{f}_1 & \cdots & \acute{f}_n \\ \vdots & \vdots & \vdots \\ {f_1}^{(n-1)} & \cdots & {f_n}^{(n-1)} \end{vmatrix}$$

We show some examples for finding the symbolic n-plithogenic Wronsckian.

Example.

Consider $E_1 = \{e^X, e^{2X}; X = x_0 + x_1P_1 + x_2P_2\}, E_2 = \{1, sinX, cosX; X = x_0 + x_1P_1 + x_2P_2 + x_3P_3\}, E_3 = \{1, tanX; X = x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4\}, E_4 = \{1, X, X^2, X^3; X = x_0 + \sum_{i=1}^{5} x_iP_i\}$, we have:

$$W(E_1) = \begin{vmatrix} {D_2}^{(0)}(e^X) & {D_2}^{(0)}(e^{2X}) \\ D_2(e^X) & D_2(e^{2X}) \end{vmatrix} = \begin{vmatrix} e^{x_0+x_1P_1+x_2P_2} & e^{2x_0+2x_1P_1+2x_2P_2} \\ e^{x_0+x_1P_1+x_2P_2} & 2e^{2x_0+2x_1P_1+2x_2P_2} \end{vmatrix}$$

$$= e^{3x_0+3x_1P_1+3x_2P_2}$$

$$W(E_2)=\begin{vmatrix}1 & sin\left(x_0+\sum_{i=1}^{3}x_iP_i\right) & cos\left(x_0+\sum_{i=1}^{3}x_iP_i\right)\\ 0 & cos\left(x_0+\sum_{i=1}^{3}x_iP_i\right) & -sin\left(x_0+\sum_{i=1}^{3}x_iP_i\right)\\ 0 & -sin\left(x_0+\sum_{i=1}^{3}x_iP_i\right) & -cos\left(x_0+\sum_{i=1}^{3}x_iP_i\right)\end{vmatrix}=-cos^2\left(x_0+\sum_{i=1}^{3}x_iP_i\right)$$

$$W(E_3)=\begin{vmatrix}1 & tan\left(x_0+\sum_{i=1}^{4}x_iP_i\right)\\ 0 & 1+tan^2\left(x_0+\sum_{i=1}^{4}x_iP_i\right)\end{vmatrix}=1+tan^2\left(x_0+\sum_{i=1}^{4}x_iP_i\right)$$

$$W(E_4)=\begin{vmatrix}1 & x_0+\sum_{i=1}^{5}x_iP_i & \left(x_0+\sum_{i=1}^{5}x_iP_i\right)^2 & \left(x_0+\sum_{i=1}^{5}x_iP_i\right)^3\\ 0 & 1 & 2\left(x_0+\sum_{i=1}^{5}x_iP_i\right) & 3\left(x_0+\sum_{i=1}^{5}x_iP_i\right)\\ 0 & 0 & 2 & 6\left(x_0+\sum_{i=1}^{5}x_iP_i\right)\\ 0 & 0 & 0 & 6\end{vmatrix}=12$$

Example.

Consider $E_1=\{lnX,e^X;X=x_0+\sum_{i=1}^{5}x_iP_i\},E_2=\{lnX,\sqrt{X};X=x_0+\sum_{i=1}^{4}x_iP_i\},E_3=\{e^X,sinX;X=x_0+\sum_{i=1}^{3}x_iP_i\}$, then:

$$W(E_1)=\begin{vmatrix}lnX & e^X\\ \frac{1}{X} & e^X\end{vmatrix}=e^{x_0+\sum_{i=1}^{3}x_iP_i}\left[ln\left(x_0+\sum_{i=1}^{3}x_iP_i\right)-\frac{1}{x_0+\sum_{i=1}^{3}x_iP_i}\right]$$

$$W(E_2)=\begin{vmatrix}lnX & \sqrt{X}\\ \frac{1}{X} & \frac{1}{2\sqrt{X}}\end{vmatrix}=\frac{lnX}{2\sqrt{X}}-\frac{1}{\sqrt{X}}=\frac{lnX-2}{2\sqrt{X}}$$

$$=\frac{1}{\sqrt{x_0+\sum_{i=1}^{4}x_iP_i}}\left[ln\left(x_0+\sum_{i=1}^{4}x_iP_i\right)-2\right]$$

$$W(E_3)=\begin{vmatrix}e^X & sinX\\ e^X & cosX\end{vmatrix}=e^{x_0+\sum_{i=1}^{5}x_iP_i}\left[cos\left(x_0+\sum_{i=1}^{5}x_iP_i\right)-sin\left(x_0+\sum_{i=1}^{5}x_iP_i\right)\right]$$

Symbolic n-plithogenic anti-Wronsckian.

Let $E=\{f_1,\dots,f_n\}$ be a set of n functions, then:

$$AW(E)=\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ \int f_1 & \int f_2 & \cdots & \vdots \\ \int\left(\int f_1\right) & \vdots & \cdots & \vdots \\ \vdots & & & \vdots \\ \underbrace{\int f_1}_{n-1\ times} & \underbrace{\int f_2}_{n-1\ times} & \vdots\ \cdots & \underbrace{\int f_n}_{n-1\ times} \end{vmatrix}$$

Now, we will clarify how (AW) can be computed.

Example.

Consider $E_1=\{1,X,X^2;X=x_0+\sum_{i=1}^{2}x_iP_i\}, E_2=\{e^X,e^{2X};X=x_0+\sum_{i=1}^{3}x_iP_i\}, E_3=\{sinX,cosX;X=x_0+\sum_{i=1}^{4}x_iP_i\}$, then:

$$AW(E_1)=\begin{vmatrix} 1 & X & X^2 \\ X & \frac{1}{2}X^2 & \frac{1}{3}X^3 \\ \frac{1}{2}X^2 & \frac{1}{6}X^3 & \frac{1}{12}X^4 \end{vmatrix}$$

$$=1.\begin{vmatrix} \frac{1}{2}X^2 & \frac{1}{3}X^3 \\ \frac{1}{6}X^3 & \frac{1}{12}X^4 \end{vmatrix}-X\begin{vmatrix} X & \frac{1}{3}X^3 \\ \frac{1}{2}X^2 & \frac{1}{12}X^4 \end{vmatrix}+X^2\begin{vmatrix} X & \frac{1}{2}X^2 \\ \frac{1}{2}X^2 & \frac{1}{6}X^3 \end{vmatrix}$$

$$=\left(\frac{1}{24}-\frac{1}{18}\right)X^6-X\left(\frac{1}{12}-\frac{1}{6}\right)X^5+X^2\left(\frac{1}{6}-\frac{1}{4}\right)X^4$$

$$=X^6\left(\frac{1}{24}-\frac{1}{18}-\frac{1}{12}+\frac{1}{6}+\frac{1}{6}-\frac{1}{4}\right)=X^6\left(\frac{1}{24}+\frac{8}{24}-\frac{6}{24}-\frac{2}{24}-\frac{1}{18}\right)$$

$$=X^6\left(\frac{1}{24}-\frac{1}{18}\right)=-\frac{1}{72}\left(x_0+\sum_{i=1}^{3}x_iP_i\right)$$

$$AW(E_2)=\begin{vmatrix} e^X & e^{2X} \\ e^X & \frac{1}{2}e^{2X} \end{vmatrix}=\frac{1}{2}e^Xe^{2X}-e^Xe^{2X}=-\frac{1}{2}e^{3X}=-\frac{1}{2}e^{\left(x_0+\sum_{i=1}^{3}x_iP_i\right)}$$

$$AW(E_3) = \begin{vmatrix} sinX & cosX & 1 \\ -cosX & -sinX & X \\ -sinX & -cosX & \frac{1}{2}X^2 \end{vmatrix}$$

$$= sinX. \begin{vmatrix} sinX & X \\ -cosX & \frac{1}{2}X^2 \end{vmatrix} - cosX \begin{vmatrix} -cosX & X \\ -sinX & \frac{1}{2}X^2 \end{vmatrix} + 1. \begin{vmatrix} -cosX & sinX \\ -sinX & -cosX \end{vmatrix}$$

$$= sinX\left(\frac{1}{2}X^2 sinX + XcosX\right) - cosX\left(-\frac{1}{2}X^2 cosX + XsinX\right)$$

$$+ (cos^2X + sin^2X)$$

$$= \frac{1}{2}X^2 sin^2X + XcosXsinX + \frac{1}{2}X^2 cos^2X - XcosXsinX + 1$$

$$= \frac{1}{2}X^2(1) + 1 = \frac{1}{2}\left(x_0 + \sum_{i=1}^{4} x_i P_i\right) + 1$$

Conclusion

In this paper, we have studied the representation of the symbolic n-plithogenic differential operator for many different values of n by classical algebraic matrices and plithogenic matrices. We presented many examples about the representation of symbolic n-plithogenic differential operators by matrices. As well as, we computed the symbolic 2-plithogenic, 3-plithogenic, and 4-plithogenic Wronsckian, and anti-Wronsckian.

References

1. Nader Mahmoud Taffach , Ahmed Hatip., "A Review on Symbolic 2-Plithogenic Algebraic Structures " Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.
2. Nader Mahmoud Taffach , Ahmed Hatip.," A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations ", Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.
3. Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.
4. Smarandache, F., " Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)", Neutrosophic Sets and Systems, vol. 53, 2023.

5. Taffach, N., " An Introduction to Symbolic 2-Plithogenic Vector Spaces Generated from The Fusion of Symbolic Plithogenic Sets and Vector Spaces", Neutrosophic Sets and Systems, Vol 54, 2023.
6. Ben Othman, K., Albasheer, O., Nadweh, R., Von Shtawzen, O., and Ali, R., "On The Symbolic 6-Plithogenic and 7-Plithogenic Rings", Galoitica Journal of Mathematical Structures and Applications, Vol.8, 2023.
7. Merkepci, H., and Rawashdeh, A., " On The Symbolic 2-Plithogenic Number Theory and Integers ", Neutrosophic Sets and Systems, Vol 54, 2023.
8. Albasheer, O., Hajjari., A., and Dalla., R., " On The Symbolic 3-Plithogenic Rings and TheirAlgebraic Properties", Neutrosophic Sets and Systems, Vol 54, 2023.
9. Abobala, M., On Refined Neutrosophic Matrices and Their Applications In Refined Neutrosophic Algebraic Equations, Journal Of Mathematics, Hindawi, 2021
10. Olgun, N., Hatip, A., Bal, M., and Abobala, M., " A Novel Approach To Necessary and Sufficient Conditions For The Diagonalization of Refined Neutrosophic Matrices", International Journal of neutrosophic Science, Vol. 16, pp. 72-79, 2021.
11. Abobala, M., and Zeina, M.B., " A Study Of Neutrosophic Real Analysis By Using One Dimensional Geometric AH-Isometry", Galoitica Journal Of Mathematical Structures And Applications, Vol.3, 2023.
12. Bisher Ziena, M., and Abobala, M., " On The Refined Neutrosophic Real Analysis Based on Refined Neutrosophic Algebraic AH-Isometry", Neutrosophic Sets and Systems, vol. 54, 2023.
13. M. B. Zeina and M. Abobala, "A Novel Approach of Neutrosophic Continuous Probability Distributions using AH-Isometry with Applications in Medicine," in *Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics*, Elsevier, 2023.

14. Ali, R., and Hasan, Z., " An Introduction To The Symbolic 3-Plithogenic Modules ", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.
15. M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, "Introduction to Symbolic 2-Plithogenic Probability Theory," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.
16. Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.
17. Ben Othman, K., "On Some Algorithms For Solving Symbolic 3-Plithogenic Equations", Neoma Journal Of Mathematics and Computer Science, 2023.
18. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Concept Of Symbolic 7-Plithogenic Real Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.
19. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Symbolic 8-Plithogenic Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

Neutrosophic Sets and Systems (NSS) is an academic journal, published quarterly online and on paper, that has been created for publications of advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics etc. and their applications in any field.

All submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

ISSN (print): 2331-6055, ISSN (online): 2331-608X
Impact Factor: 1.739

NSS has been accepted by SCOPUS. Starting with Vol. 19, 2018, all NSS articles are indexed in Scopus.

NSS is also indexed by Google Scholar, Google Plus, Google Books, EBSCO, Cengage Thompson Gale (USA), Cengage Learning, ProQuest, Amazon Kindle, DOAJ (Sweden), University Grants Commission (UGC) - India, International Society for Research Activity (ISRA), Scientific Index Services (SIS), Academic Research Index (ResearchBib), Index Copernicus (European Union),CNKI (Tongfang Knowledge Network Technology Co., Beijing, China), etc.

Google Dictionary has translated the neologisms "neutrosophy" (1) and "neutrosophic" (2), coined in 1995 for the first time, into about 100 languages.

FOLDOC Dictionary of Computing (1, 2), Webster Dictionary (1, 2), Wordnik (1), Dictionary.com, The Free Dictionary (1),Wiktionary (2), YourDictionary (1, 2), OneLook Dictionary (1, 2), Dictionary / Thesaurus (1), Online Medical Dictionary (1, 2), and Encyclopedia (1, 2) have included these scientific neologisms.

DOI numbers are assigned to all published articles.

Registered by the Library of Congress, Washington DC, United States, https://lccn.loc.gov/2013203857.

Recently, NSS was also approved for Emerging Sources Citation Index (ESCI) available on the Web of Science platform, starting with Vol. 15, 2017.

Editors-in-Chief:

23316055