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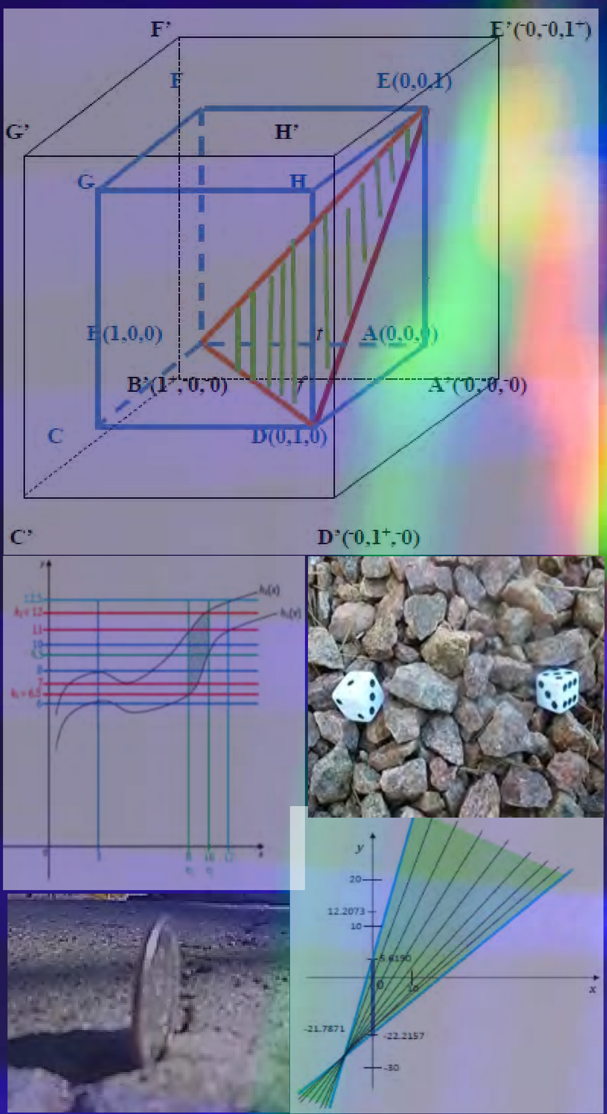

<A> <neutA> <antiA>

Florentin Smarandache . Mohamed Abdel-Basset . Said Broumi
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"Neutrosophic Sets and Systems" has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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Contents

University of New Mexico

Ordered subalgebras of ordered BCI-algebras based on the MBJ-neutrosophic structure

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Abstract: The neutrosophic set consists of three fuzzy sets called true membership function, false membership function and indeterminate membership function. MBJ-neutrosophic structure is a structure constructed using interval-valued fuzzy set instead of indeterminate membership function in the neutrosophic set. In general, the indeterminate part appears in a wide range. So instead of treating the indeterminate part as a single value, it is treated as an interval value, allowing a much more comprehensive processing. In an attempt to apply the MBJ-neutrosophic structure to ordered BCI-algebras, the notion of MBJ-neutrosophic (ordered) subalgebras is introduced and their properties are studied. The relationship between MBJ-neutrosophic subalgebra and MBJ-neutrosophic ordered subalgebra is established, and MBJ-neutrosophic ordered subalgebra is formed using (intuitionistic) fuzzy ordered subalgebra. Given an MBJ-neutrosophic set, its $(q, \tilde{\mathfrak{c}}, p)$-translative MBJ-neutrosophic set is introduced and its characterization is considered. An MBJ-neutrosophic ordered subalgebra is created using $(q, \tilde{\mathfrak{c}}, p)$-translative MBJ-neutrosophic set.

Keywords: Ordered BCI-algebra, ordered subalgebra, MBJ-neutrosophic ordered subalgebra, MBJ-ordered subalgebras, $(q, \tilde{\mathfrak{c}}, p)$-translative MBJ-neutrosophic set.

2020 Mathematics Subject Classification. 06F35, 03G25, 08A72.

1 Introduction

The classical set theory contains only two components: true and false, which means that an element can either belong to the set (true) or not belong to the set (false). However, in many cases, there is an indeterminate state which means it is not clear that elements are in or out of a set. In other words, there is a lot of incomplete or uncertain information, which is an indeterminate state that cannot be expressed as true or false. Neutrosophic logic is an extension of classical and fuzzy logic, which is a good tool for processing uncertain and indeterminate information in a more versatile way. The neutrosophic set is a mathematical concept introduced by Florentin Smarandash in the late 1990s, which is particularly useful for dealing with the indeterminate states that classical set theory cannot address. It is applied to various fields such as artificial intelligence, decision-making, expert systems, and information management that require coping with ambiguity and inaccuracy. In addition, neutrosophic set theory is applied to various algebraic structures including logical algebras (see [1], [3], [4], [5], [6], [7],[9], [13], [16], [17]). The neutrosophic set gives three values, i.e., membership degree (T), non-membership degree (F), and indeterminacy degree (I), for each element. In [11], Mohseni Takallo et al. extended the indeterminacy degree (F) to the interval value to introduce the MBJ-n-set, and it is applied to several logical algebras (see [2], [8], [10], [12], [15]). The "MBJ" stands for the initials of the three researchers, R. A. Borzooei, M. Mohseni Takallo and Y. B. Jun.

This paper applies the MBJ-neutrosophic structure to OBCI-algebras. We first introduce the notion of MBJ-neutrosophic (ordered) subalgebras in OBCI-algebras and then investigate their properties. We look at the relations of the MBJ-neutrosophic subalgebra to the MBJ-neutrosophic ordered subalgebra (O-subalgebra for simplicity). Using (intuitionistic) fuzzy ordered subalgebras, we establish an MBJ-neutrosophic ordered subalgebra. We discuss the characterization of MBJ-neutrosophic O-subalgebras. Given an MBJ-n-set, we introduce its $(q, \tilde{a_{13}}, p)$-translative MBJ-n-set and consider its characterization. We use $(q, \tilde{a_{13}}, p)$-translative MBJ-n-set to generate an MBJ-neutrosophic O-subalgebra.

2 Preliminaries

Definition 2.1 ([18])**.** An *OBCI-algebra* is defined to be a set W together with a binary relation " $\leq_W$ ", a constant "$\check{e}$" and a binary operation " $\rightsquigarrow$ " that satisfies:

$$\check{e} \leq_W (a_{11} \rightsquigarrow a_{12}) \rightsquigarrow ((a_{12} \rightsquigarrow a_{13}) \rightsquigarrow (a_{11} \rightsquigarrow a_{13})), \tag{2.1}$$

$$\check{e} \leq_W a_{11} \rightsquigarrow ((a_{11} \rightsquigarrow a_{12}) \rightsquigarrow a_{12}), \tag{2.2}$$

$$\check{e} \leq_W a_{11} \rightsquigarrow a_{11}, \tag{2.3}$$

$$\check{e} \leq_W a_{11} \rightsquigarrow a_{12},\ \check{e} \leq_W a_{12} \rightsquigarrow a_{11} \ \Rightarrow \ a_{11} = a_{12}, \tag{2.4}$$

$$a_{11} \leq_W a_{12} \ \Leftrightarrow \ \check{e} \leq_W a_{11} \rightsquigarrow a_{12}, \tag{2.5}$$

$$\check{e} \leq_W a_{11},\ a_{11} \leq_W a_{12} \ \Rightarrow \ \check{e} \leq_W a_{12} \tag{2.6}$$

for all $a_{11}, a_{12}, a_{13} \in W$.

Obviously $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ with $W = \{\check{e}\}$ is an OBCI-algebra, which is said to be the *trivial OBCI-algebra*.

Proposition 2.2 ([18]). *If* $\boldsymbol{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ *is an OBCI-algebra, it satisfies:*

$$\check{e} \rightsquigarrow a_{11} = a_{11}, \tag{2.7}$$
$$a_{13} \rightsquigarrow (a_{12} \rightsquigarrow a_{11}) = a_{12} \rightsquigarrow (a_{13} \rightsquigarrow a_{11}), \tag{2.8}$$
$$\check{e} \leq_W a_{11} \rightsquigarrow a_{12} \;\Rightarrow\; \check{e} \leq_W (a_{12} \rightsquigarrow a_{13}) \rightsquigarrow (a_{11} \rightsquigarrow a_{13}), \tag{2.9}$$
$$\check{e} \leq_W a_{11} \rightsquigarrow a_{12},\, \check{e} \leq_W a_{12} \rightsquigarrow a_{13} \;\Rightarrow\; \check{e} \leq_W a_{11} \rightsquigarrow a_{13}, \tag{2.10}$$
$$\check{e} \leq_W (a_{13} \rightsquigarrow (a_{12} \rightsquigarrow a_{11})) \rightsquigarrow (a_{12} \rightsquigarrow (a_{13} \rightsquigarrow a_{11})), \tag{2.11}$$
$$\check{e} \leq_W a_{13} \rightsquigarrow (a_{12} \rightsquigarrow a_{11}) \;\Rightarrow\; \check{e} \leq_W a_{12} \rightsquigarrow (a_{13} \rightsquigarrow a_{11}), \tag{2.12}$$
$$((a_{11} \rightsquigarrow a_{12}) \rightsquigarrow a_{12}) \rightsquigarrow a_{12} = a_{11} \rightsquigarrow a_{12}, \tag{2.13}$$
$$(a_{11} \rightsquigarrow a_{11}) \rightsquigarrow a_{11} = a_{11}, \tag{2.14}$$
$$\check{e} \leq_W (a_{12} \rightsquigarrow a_{13}) \rightsquigarrow ((a_{11} \rightsquigarrow a_{12}) \rightsquigarrow (a_{11} \rightsquigarrow a_{13})), \tag{2.15}$$
$$\check{e} \leq_W a_{11} \rightsquigarrow a_{12} \;\Rightarrow\; \check{e} \leq_W (a_{13} \rightsquigarrow a_{11}) \rightsquigarrow (a_{13} \rightsquigarrow a_{12}) \tag{2.16}$$

for all $a_{11}, a_{12}, a_{13} \in W$.

Definition 2.3 ([18]). A subset A of W is said to be

- a *subalgebra* of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ if it satisfies:

$$(\forall a_{11}, a_{12} \in W)(a_{11}, a_{12} \in A \;\Rightarrow\; a_{11} \rightsquigarrow a_{12} \in A). \tag{2.17}$$

- an *ordered subalgebra* (briefly, O-subalgebra) of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ if it satisfies:

$$(\forall a_{11}, a_{12} \in W)(a_{11}, a_{12} \in A,\, \check{e} \leq_W a_{11},\, \check{e} \leq_W a_{12} \;\Rightarrow\; a_{11} \rightsquigarrow a_{12} \in A). \tag{2.18}$$

A function $\mu : W \to [0, 1]$ is said to be a *fuzzy set* (f-set for brevity) in a set W.

Definition 2.4 ([19]). An f-set μ in W is said to be a *fuzzy ordered subalgebra* (briefly, FO-subalgebra) of an OBCI-algebra $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ if it satisfies:

$$(\forall x_{11}, x_{12} \in W)(\tilde{e} \leq_W x_{11},\, \tilde{e} \leq_W x_{12} \;\Rightarrow\; \mu(x_{11} \rightsquigarrow x_{12}) \geq \min\{\mu(x_{11}), \mu(x_{12})\}). \tag{2.19}$$

Definition 2.5 ([14]). An intuitionistic f-set $\mathcal{I} := \{\langle x_{11}, \mu_I, \nu_I\rangle \mid x_{11} \in W\}$ is said to be an *intuitionistic fuzzy ordered subalgebra* (briefly, IFO-subalgebra) of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ if it satisfies:

$$(\forall x_{11}, x_{12} \in W)\left(\begin{array}{l} \tilde{e} \leq_W x_{11},\; \tilde{e} \leq_W x_{12} \\ \quad \Rightarrow \left\{\begin{array}{l} \mu_I(x_{11} \rightsquigarrow x_{12}) \geq \min\{\mu_I(x_{11}), \mu_I(x_{12})\} \\ \nu_I(x_{11} \rightsquigarrow x_{12}) \leq \max\{\nu_I(x_{11}), \nu_I(x_{12})\} \end{array}\right. \end{array}\right). \tag{2.20}$$

The neutrosophic set (n-set for brevity) is an extension of traditional set theory and has the advantage of handling uncertain, indeterminate and inconsistent information in a more flexible way than classical sets.

Given a non-empty set W, an *n-set* in W is a structure of the form:

$$\mathcal{M}_{\circledcirc} := \{\langle x_{11}; \mathcal{M}_T^{\circledcirc}(x_{11}), \mathcal{M}_F^{\circledcirc}(x_{11}), \mathcal{M}_I^{\circledcirc}(x_{11})\rangle \mid x_{11} \in W\}$$

where $\mathcal{M}_T^{\circledcirc} : W \to [0,1]$ is a true membership function, $\mathcal{M}_F^{\circledcirc} : W \to [0,1]$ is a false membership function, and $\mathcal{M}_I^{\circledcirc} : W \to [0,1]$ is an indeterminate membership function. For brevity, we use the symbol $\mathcal{M}_{\circledcirc} := (\mathcal{M}_T^{\circledcirc}, \mathcal{M}_I^{\circledcirc}, \mathcal{M}_F^{\circledcirc})$ for the n-set

$$\mathcal{M}_{\circledcirc} := \{\langle x_{11}; \mathcal{M}_T^{\circledcirc}(x_{11}), \mathcal{M}_I^{\circledcirc}(x_{11}), \mathcal{M}_F^{\circledcirc}(x_{11})\rangle \mid x_{11} \in W\}.$$

Given an n-set $\mathcal{M}_{\circledcirc} := (\mathcal{M}_T^{\circledcirc}, \mathcal{M}_I^{\circledcirc}, \mathcal{M}_F^{\circledcirc})$ in W, we consider the following sets.

$$\begin{aligned}
\mathcal{W}(\mathcal{M}_T^{\circledcirc}; \alpha) &:= \{x_{11} \in W \mid \mathcal{M}_T^{\circledcirc}(x_{11}) \geq \alpha\},\\
\mathcal{W}(\mathcal{M}_I^{\circledcirc}; \beta) &:= \{x_{11} \in W \mid \mathcal{M}_I^{\circledcirc}(x_{11}) \geq \beta\},\\
\mathcal{W}(\mathcal{M}_F^{\circledcirc}; \gamma) &:= \{x_{11} \in W \mid \mathcal{M}_F^{\circledcirc}(x_{11}) \leq \gamma\},
\end{aligned}$$

which are said to be *neutrosophic level subsets* of W where $\alpha, \beta, \gamma \in [0,1]$.

The interval-valued f-set is an extension of f-set theory and is a mathematical tool that serves to more subtly address uncertainty.

By an *interval number*, a closed subinterval $\tilde{c} = [c^-, c^+]$ of I, where $0 \leq c^- \leq c^+ \leq 1$, is meant and by $[I]$, the set of all interval numbers is denoted. We define what is known as a *refined minimum* (briefly, rmin) and a *refined maximum* (briefly, rmax) of two elements in $[I]$, and the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $[I]$. Take two interval numbers $\tilde{c}_1 := \left[c_1^-, c_1^+\right]$ and $\tilde{c}_2 := \left[c_2^-, c_2^+\right]$. Then

$$\begin{aligned}
&\operatorname{rmin}\{\tilde{c}_1, \tilde{c}_2\} = \left[\min\left\{c_1^-, c_2^-\right\}, \min\left\{c_1^+, c_2^+\right\}\right],\\
&\operatorname{rmax}\{\tilde{c}_1, \tilde{c}_2\} = \left[\max\left\{c_1^-, c_2^-\right\}, \max\left\{c_1^+, c_2^+\right\}\right],\\
&\tilde{c}_1 \succeq \tilde{c}_2 \ \Leftrightarrow\ c_1^- \geq c_2^-,\ c_1^+ \geq c_2^+.
\end{aligned}$$

Analogously one has $\tilde{c}_1 \preceq \tilde{c}_2$ and $\tilde{c}_1 = \tilde{c}_2$. By $\tilde{c}_1 \succ \tilde{c}_2$ (resp. $\tilde{c}_1 \prec \tilde{c}_2$), $\tilde{c}_1 \succeq \tilde{c}_2$ and $\tilde{c}_1 \neq \tilde{c}_2$ (resp. $\tilde{c}_1 \preceq \tilde{c}_2$ and $\tilde{c}_1 \neq \tilde{c}_2$) are meant. For $\tilde{c}_i \in [I]$, where $i \in \Lambda$, define:

$$\operatorname*{rinf}_{i\in\Lambda} \tilde{c}_i = \left[\inf_{i\in\Lambda} c_i^-, \inf_{i\in\Lambda} c_i^+\right] \quad \text{and} \quad \operatorname*{rsup}_{i\in\Lambda} \tilde{c}_i = \left[\sup_{i\in\Lambda} c_i^-, \sup_{i\in\Lambda} c_i^+\right].$$

Given a nonempty set X, a mapping $A : X \to [I]$ is said to be an *interval-valued f-set* (briefly, an *IVF set*) in X. By $[I]^X$, we denote the set of all IVF sets in X. For all $A \in [I]^X$ and $a_{12} \in X$, $A(a_{12}) = [A^-(a_{12}), A^+(a_{12})]$ is said to be the *degree* of membership of an element x to A, where $A^- : X \to I$ and $A^+ : X \to I$ are f-sets in X which are said to be a *lower f-set* and an *upper f-et* in X, respectively. For brevity, we denote $A(a_{12}) = [A^-(a_{12}), A^+(a_{12})]$ by $A = [A^-, A^+]$.

Definition 2.6 ([11])**.** For a non-empty set W, an *MBJ-neutrosophic set* (briefly, MBJn-set) in W is defined to be a structure of the form:

$$\mathcal{G} := \{\langle a_{11}; M_G(a_{11}), \tilde{B}_G(a_{11}), J_G(a_{11})\rangle \mid a_{11} \in W\}$$

where M_G and J_G are f-sets in W, which are said to be a true membership function and a false membership function, respectively, and $\tilde{B}_G$ is an IVF set in W which is said to be an indeterminate interval-valued membership function.

For brevity, we use the symbol $\mathcal{G} = (M_G,\, \tilde{B}_G,\, J_G)$ for the MBJn-set

$$\mathcal{G} := \{\langle a_{11}; M_G(a_{11}), \tilde{B}_G(a_{11}), J_G(a_{11})\rangle \mid a_{11} \in W\}.$$

In an MBJn-set $\mathcal{G} = (M_G,\, \tilde{B}_G,\, J_G)$ in W, if we take

$$\tilde{B}_G : W \to [I],\ a_{12} \mapsto [B_G^-(a_{12}), B_G^+(a_{12})]$$

with $B_G^-(a_{12}) = B_G^+(a_{12})$, then $\mathcal{G} = (M_G,\, \tilde{B}_G,\, J_G)$ is an n-set in W.

Given an MBJn-set $\mathcal{G} = (M_G,\, \tilde{B}_G,\, J_G)$ in a set W, we consider the following sets.

$$\begin{aligned} &U(M_G; s) := \{x_{11} \in W \mid M_G(x_{11}) \geq s\},\\ &U(\tilde{B}_G; [\varepsilon_1, \varepsilon_2]) := \{x_{11} \in W \mid \tilde{B}_G(x_{11}) \succeq [\varepsilon_1, \varepsilon_2]\},\\ &L(J_G; t) := \{x_{11} \in W \mid J_G(x_{11}) \leq t\} \end{aligned}$$

where $s, t \in [0, 1]$ and $[\varepsilon_1, \varepsilon_2] \in [I]$.

3 MBJ-neutrosophic O-subalgebras

Unless specified otherwise, in what follows, $\mathbf{W} := (W,\, \rightsquigarrow,\, \tilde{e},\, \leq_W)$ denotes an OBCI-algebra.

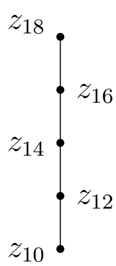
Definition 3.1. An MBJn-set $\mathcal{G} = (M_G,\, \tilde{B}_G,\, J_G)$ in $\mathbf{W} := (W,\, \rightsquigarrow,\, \tilde{e},\, \leq_W)$ is said to be an *MBJ-neutrosophic subalgebra* (briefly, MBJn-subalgeba) of $\mathbf{W} := (W,\, \rightsquigarrow,\, \tilde{e},\, \leq_W)$ if it satisfies:

$$(\forall x_{11}, x_{12} \in W) \begin{pmatrix} M_G(x_{11} \rightsquigarrow x_{12}) \geq \min\{M_G(x_{11}), M_G(x_{12})\},\\ \tilde{B}_G(x_{11} \rightsquigarrow x_{12}) \succeq \text{rmin}\{\tilde{B}_G(x_{11}), \tilde{B}_G(x_{12})\},\\ J_G(x_{11} \rightsquigarrow x_{12}) \leq \max\{J_G(x_{11}), J_G(x_{12})\}. \end{pmatrix} \tag{3.1}$$

Definition 3.2. An MBJn-set $\mathcal{G} = (M_G,\, \tilde{B}_G,\, J_G)$ in $\mathbf{W} := (W,\, \rightsquigarrow,\, \tilde{e},\, \leq_W)$ is said to be an *MBJ-neutrosophic O-subalgebra* (briefly, MBJnO-subalgebra) of $\mathbf{W} := (W,\, \rightsquigarrow,\, \tilde{e},\, \leq_W)$ if it satisfies:

$$(\forall x_{11}, x_{12} \in W) \left(\tilde{e} \leq_W x_{11},\ \tilde{e} \leq_W x_{12}\ \Rightarrow \begin{cases} M_G(x_{11} \rightsquigarrow x_{12}) \geq \min\{M_G(x_{11}), M_G(x_{12})\},\\ \tilde{B}_G(x_{11} \rightsquigarrow x_{12}) \succeq \text{rmin}\{\tilde{B}_G(x_{11}), \tilde{B}_G(x_{12})\},\\ J_G(x_{11} \rightsquigarrow x_{12}) \leq \max\{J_G(x_{11}), J_G(x_{12})\}. \end{cases} \right). \tag{3.2}$$

Example 3.3. Let $W = \{z_{10}, z_{12}, z_{14}, z_{16}, z_{18}\}$ be a set with the Hasse diagram and Table as follows:



Hassee diagram of $(W, \leq_W)$

$\rightsquigarrow$	z_{10}	z_{12}	z_{14}	z_{16}	z_{18}
z_{10}	z_{18}	z_{18}	z_{18}	z_{18}	z_{18}
z_{12}	z_{10}	z_{16}	z_{16}	z_{16}	z_{18}
z_{14}	z_{10}	z_{12}	z_{14}	z_{16}	z_{18}
z_{16}	z_{10}	z_{12}	z_{14}	z_{16}	z_{18}
z_{18}	z_{10}	z_{10}	z_{10}	z_{10}	z_{18}

Table for " $\rightsquigarrow$ "

Then $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$, where $\tilde{e} = z_{12}$, is an OBCI-algebra (see [18]).

(i) Let $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ be an MBJn-set in $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ provided by Table 1.

Table 1: Table for $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$

W	$M_G(w)$	$\tilde{B}_G(w)$	$J_G(w)$
z_{18}	0.93	$[0.56, 0.89]$	0.25
z_{16}	0.67	$[0.47, 0.56]$	0.42
z_{14}	0.54	$[0.38, 0.47]$	0.58
z_{12}	0.44	$[0.29, 0.42]$	0.69
z_{10}	0.87	$[0.51, 0.82]$	0.36

It can be eaily verified that $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ is an MBJn-subalgeba of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$.

(ii) Let $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ be an MBJn-set in $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ given by Table 2.

Table 2: Table for $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$

W	$M_G(w)$	$\tilde{B}_G(w)$	$J_G(w)$
z_{18}	0.37	$[0.53, 0.85]$	0.81
z_{16}	0.94	$[0.65, 0.94]$	0.28
z_{14}	0.82	$[0.49, 0.76]$	0.46
z_{12}	0.66	$[0.37, 0.67]$	0.53
z_{10}	0.54	$[0.53, 0.85]$	0.62

It can be eaily verified that $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ is an MBJnO-subalgebra of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$.

It is certain that every MBJn-subalgeba is an MBJnO-subalgebra, but the converse may not be true as shown in the following example. In light of this view, it can be said that the MBJnO-subalgebra is a generalization of the MBJ-neutrosophic subalgebra.

Example 3.4. (i) The MBJnO-subalgebra $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ in Example 3.3(ii) is not an MBJn-subalgeba of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ since

$$M_G(z_{10} \rightsquigarrow z_{10}) = M_G(z_{11}) = 0.37 \ngeq 0.54 = \min\{M_G(z_{10}), M_G(z_{10})\}$$

and/or $J_G(z_{10} \rightsquigarrow z_{10}) = J_G(z_{11}) = 0.81 \nleq 0.62 = \max\{J_G(z_{10}), J_G(z_{10})\}$.

(ii) Consider the OBCI-algebra $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ in Example 3.3, and let $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ be an MBJn-set in $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ given by Table 3.

Table 3: Table for $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$

W	$M_G(w)$	$\tilde{B}_G(w)$	$J_G(w)$
z_{10}	0.67	$[0.57, 0.88]$	0.28
z_{18}	0.35	$[0.36, 0.77]$	0.61
z_{16}	0.67	$[0.57, 0.88]$	0.28
z_{14}	0.35	$[0.36, 0.77]$	0.61
z_{12}	0.35	$[0.36, 0.77]$	0.61

It can be easily checked that $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ is an MBJn-subalgeba of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$. But it is not an MBJnO-subalgebra of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ since

$$\tilde{B}_G(z_{10} \rightsquigarrow z_{12}) = \tilde{B}_G(z_{11}) = [0.36, 0.77] \nsucceq [0.57, 0.88] = \text{rmin}\{\tilde{B}_G(z_{10}), \tilde{B}_G(z_{12})\}$$

and/or $J_G(z_{10} \rightsquigarrow z_{12}) = J_G(z_{11}) = 0.61 \nleq 0.28 = \max\{J_G(z_{10}), J_G(z_{12})\}$.

Using (intuitionistic) fuzzy O-subalgebras, we establish an MBJnO-subalgebra.

Theorem 3.5. *Given an MBJn-set $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ in $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$, if (M_G, J_G) is an IFO-subalgebra of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$, and $\tilde{B}_G^-$ and $\tilde{B}_G^+$ are FO-subalgebras of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$, then $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ is an MBJnO-subalgebra of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$.*

Proof. Let $x_{11}, x_{12} \in W$ be such that $\check{e} \leq_W x_{11}$ and $\check{e} \leq_W x_{12}$. Then

$$\begin{aligned}\tilde{B}_G(x_{11} \rightsquigarrow x_{12}) &= [B_G^-(x_{11} \rightsquigarrow x_{12}), B_G^+(x_{11} \rightsquigarrow x_{12})] \\ &\succeq [\min\{B_G^-(x_{11}), B_G^-(x_{12})\}, \min\{B_G^+(x_{11}), B_G^+(x_{12})\}] \\ &= \text{rmin}\{[B_G^-(x_{11}), B_G^+(x_{11})], [B_G^-(x_{12}), B_G^+(x_{12})] \\ &= \text{rmin}\{\tilde{B}_G(x_{11}), \tilde{B}_G(x_{12})\}.\end{aligned}$$

Since (M_G, J_G) is an IFO-subalgebra of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$, it is clear that

$$M_G(x_{11} \rightsquigarrow x_{12}) \geq \min\{M_G(x_{11}), M_G(x_{12})\} \text{ and } J_G(x_{11} \rightsquigarrow x_{12}) \leq \max\{J_G(x_{11}), M_G(x_{12})\}.$$

Therefore $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ is an MBJn-subalgeba of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$. □

If $\mathcal{G} = (M_G,\ \tilde{B}_G,\ J_G)$ is an MBJnO-subalgebra of $\mathbf{W} := (W,\ \rightsquigarrow,\ \tilde{e},\ \leq_W)$, then

$$\begin{aligned}[B^-_G(x_{11} \rightsquigarrow x_{12}), B^+_G(x_{11} \rightsquigarrow x_{12})] &= \tilde{B}_G(x_{11} \rightsquigarrow x_{12}) \succeq \text{rmin}\{\tilde{B}_G(x_{11}), \tilde{B}_G(x_{12})\} \\ &= \text{rmin}\{[B^-_G(x_{11}), B^+_G(x_{11}), [B^-_G(x_{12}), B^+_G(x_{12})]\} \\ &= [\min\{B^-_G(x_{11}), B^-_G(x_{12})\}, \min\{B^+_G(x_{11}), B^+_G(x_{12})\}]\end{aligned}$$

for all $x_{11}, x_{12} \in W$ with $\check{e} \leq_W x_{11}$ and $\check{e} \leq_W x_{12}$. It follows that

$$B^-_G(x_{11} \rightsquigarrow x_{12}) \geq \min\{B^-_G(x_{11}), B^-_G(x_{12})\} \text{ and } B^+_G(x_{11} \rightsquigarrow x_{12}) \geq \min\{B^+_G(x_{11}), B^+_G(x_{12})\}.$$

Thus B^-_G and B^+_G are FO-subalgebras of $\mathbf{W} := (W,\ \rightsquigarrow,\ \tilde{e},\ \leq_W)$. But (M_G, J_G) is not an IFO-subalgebra of $\mathbf{W} := (W,\ \rightsquigarrow,\ \tilde{e},\ \leq_W)$ as one can see in Example 3.3(ii). It verifies that the converse of Theorem 3.5 is not true.

We can observe that if $\mathcal{G} = (M_G,\ \tilde{B}_G,\ J_G)$ is an MBJnO-subalgebra of $\mathbf{W} := (W,\ \rightsquigarrow,\ \tilde{e},\ \leq_W)$ that satisfies $M_G(x_{11}) + J_G(x_{11}) \leq 1$ for all $x_{11} \in W$, then (M_G, J_G) is an IFO-subalgebra of $\mathbf{W} := (W,\ \rightsquigarrow,\ \tilde{e},\ \leq_W)$.

Theorem 3.6. *An MBJn-set $\mathcal{G} = (M_G,\ \tilde{B}_G,\ J_G)$ in $\mathbf{W} := (W,\ \rightsquigarrow,\ \tilde{e},\ \leq_W)$ is an MBJnO-subalgebra of $\mathbf{W} := (W,\ \rightsquigarrow,\ \tilde{e},\ \leq_W)$ if and only if the non-empty sets $U(M_G; s)$, $U(\tilde{B}_G; [\varepsilon_1, \varepsilon_2])$ and $L(J_G; t)$ are O-subalgebras of $\mathbf{W} := (W,\ \rightsquigarrow,\ \tilde{e},\ \leq_W)$ for all $s, t \in [0, 1]$ and $[\varepsilon_1, \varepsilon_2] \in [I]$.*

The O-subalgebras $U(M_G; s)$, $U(\tilde{B}_G; [\varepsilon_1, \varepsilon_2])$ and $L(J_G; t)$ are said to be *MBJ-ordered subalgebras* of $\mathcal{G} = (M_G,\ \tilde{B}_G,\ J_G)$.

Proof. Suppose that $\mathcal{G} = (M_G,\ \tilde{B}_G,\ J_G)$ is an MBJnO-subalgebra of W. Let $s, t \in [0, 1]$ and $[\varepsilon_1, \varepsilon_2] \in [I]$ be such that $U(M_G; s)$, $U(\tilde{B}_G; [\varepsilon_1, \varepsilon_2])$ and $L(J_G; t)$ are non-empty. Let $x_{11}, x_{12} \in W$ be such that $\check{e} \leq_W x_{11}$ and $\check{e} \leq_W x_{12}$. If $x_{11}, x_{12} \in U(M_G; s) \cap U(\tilde{B}_G; [\varepsilon_1, \varepsilon_2]) \cap L(J_G; t)$, then

$$\begin{aligned}&M_G(x_{11} \rightsquigarrow x_{12}) \geq \min\{M_G(x_{11}), M_G(x_{12})\} \geq \min\{s, s\} = s, \\ &\tilde{B}_G(x_{11} \rightsquigarrow x_{12}) \succeq \text{rmin}\{\tilde{B}_G(x_{11}), \tilde{B}_G(x_{12})\} \succeq \text{rmin}\{[\varepsilon_1, \varepsilon_2], [\varepsilon_1, \varepsilon_2]\} = [\varepsilon_1, \varepsilon_2], \\ &J_G(x_{11} \rightsquigarrow x_{12}) \leq \max\{J_G(x_{11}), J_G(x_{12})\} \leq \min\{t, t\} = t,\end{aligned}$$

and so $x_{11} \rightsquigarrow x_{12} \in U(M_G; s) \cap U(\tilde{B}_G; [\varepsilon_1, \varepsilon_2]) \cap L(J_G; t)$. Therefore $U(M_G; s)$, $U(\tilde{B}_G; [\varepsilon_1, \varepsilon_2])$ and $L(J_G; t)$ are O-subalgebras of $\mathbf{W} := (W,\ \rightsquigarrow,\ \tilde{e},\ \leq_W)$.

Conversely, suppose that the non-empty sets $U(M_G; s)$, $U(\tilde{B}_G; [\varepsilon_1, \varepsilon_2])$ and $L(J_G; t)$ are O-subalgebras of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ for all $s, t \in [0, 1]$ and $[\varepsilon_1, \varepsilon_2] \in [I]$. If $M_G(a_{11} \rightsquigarrow a_{12}) < \min\{M_G(a_{11}), M_G(a_{12})\}$ for some $a_{11}, a_{12} \in W$ with $\check{e} \leq_W a_{11}$ and $\check{e} \leq_W a_{12}$, then $a_{11}, a_{12} \in U(M_G; s_0)$ but $a_{11} \rightsquigarrow a_{12} \notin U(M_G; s_0)$ for $s_0 := \min\{M_G(a_{11}), M_G(a_{12})\}$. This is a contradiction, and thus $M_G(x_{11} \rightsquigarrow x_{12}) \geq \min\{M_G(x_{11}), M_G(x_{12})\}$ for all $x_{11}, x_{12} \in W$ with $\check{e} \leq_W x_{11}$ and $\check{e} \leq_W x_{12}$. Similarly, we can show that $J_G(x_{11} \rightsquigarrow x_{12}) \leq \max\{J_G(x_{11}), J_G(x_{12})\}$ for all $x_{11}, x_{12} \in W$ with $\check{e} \leq_W x_{11}$ and $\check{e} \leq_W x_{12}$. Suppose that $\tilde{B}_G(a_{11} \rightsquigarrow a_{12}) \prec \text{rmin}\{\tilde{B}_G(a_{11}), \tilde{B}_G(a_{12})\}$ for some $a_{11}, a_{12} \in W$ with $\check{e} \leq_W a_{11}$ and $\check{e} \leq_W a_{12}$. Let $\tilde{B}_G(a_{11}) = [\varrho_1, \varrho_2]$, $\tilde{B}_G(a_{12}) = [\varrho_3, \varrho_4]$ and $\tilde{B}_G(a_{11} \rightsquigarrow a_{12}) = [\varepsilon_1, \varepsilon_2]$. Then

$$[\varepsilon_1, \varepsilon_2] \prec \text{rmin}\{[\varrho_1, \varrho_2], [\varrho_3, \varrho_4]\} = [\min\{\varrho_1, \varrho_3\}, \min\{\varrho_2, \varrho_4\}],$$

and so $\varepsilon_1 < \min\{\varrho_1, \varrho_3\}$ and $\varepsilon_2 < \min\{\varrho_2, \varrho_4\}$. Taking

$$[\lambda_1, \lambda_2] := \tfrac{1}{2}\left(\tilde{B}_G(a_{11} \rightsquigarrow a_{12}) + \text{rmin}\{\tilde{B}_G(a_{11}), \tilde{B}_G(a_{12})\}\right)$$

implies that

$$\begin{aligned}[\lambda_1, \lambda_2] &= \tfrac{1}{2}\left([\varepsilon_1, \varepsilon_2] + [\min\{\varrho_1, \varrho_3\}, \min\{\varrho_2, \varrho_4\}]\right) \\ &= \left[\tfrac{1}{2}(\varepsilon_1 + \min\{\varrho_1, \varrho_3\}, \tfrac{1}{2}(\varepsilon_2 + \min\{\varrho_2, \varrho_4\}\right].\end{aligned}$$

It follows that $\min\{\varrho_1, \varrho_3\} > \lambda_1 = \frac{1}{2}(\varepsilon_1 + \min\{\varrho_1, \varrho_3\} > \varepsilon_1$ and

$$\min\{\varrho_2, \varrho_4\} > \lambda_2 = \tfrac{1}{2}(\varepsilon_2 + \min\{\varrho_2, \varrho_4\} > \varepsilon_2.$$

Hence $[\min\{\varrho_1, \varrho_3\}, \min\{\varrho_2, \varrho_4\}] \succ [\lambda_1, \lambda_2] \succ [\varepsilon_1, \varepsilon_2] = \tilde{B}_G(a_{11} \rightsquigarrow a_{12})$, and therefore $a_{11} \rightsquigarrow a_{12} \notin U(\tilde{B}_G; [\lambda_1, \lambda_2])$. On the other hand, $\tilde{B}_G(a_{11}) = [\varrho_1, \varrho_2] \succeq [\min\{\varrho_1, \varrho_3\}, \min\{\varrho_2, \varrho_4\}] \succ [\lambda_1, \lambda_2]$ and

$$\tilde{B}_G(a_{12}) = [\varrho_3, \varrho_4] \succeq [\min\{\varrho_1, \varrho_3\}, \min\{\varrho_2, \varrho_4\}] \succ [\lambda_1, \lambda_2],$$

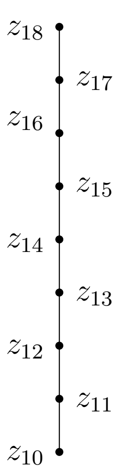
that is, $a_{11}, a_{12} \in U(\tilde{B}_G; [\lambda_1, \lambda_2])$, a contradiction. Therefore

$$\tilde{B}_G(x_{11} \rightsquigarrow x_{12}) \succeq \text{rmin}\{\tilde{B}_G(x_{11}), \tilde{B}_G(x_{12})\}$$

for all $x_{11}, x_{12} \in W$ with $\breve{e} \leq_W x_{11}$ and $\breve{e} \leq_W x_{12}$. Consequently $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ is an MBJnO-subalgebra of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$. □

The example below illustrates Theorem 3.6.

Example 3.7. Let $W = \{z_{10}, z_{11}, z_{12}, z_{13}, z_{14}, z_{15}, z_{16}, z_{17}, z_{18}\}$ be a set with the Hasse diagram and Table as follows:



Hassee diagram of $(W, \leq_W)$

$\rightsquigarrow$	z_{10}	z_{11}	z_{12}	z_{13}	z_{14}	z_{15}	z_{16}	z_{17}	z_{18}
z_{10}	z_{10}	z_{18}	z_{18}	z_{18}	z_{18}	z_{18}	z_{18}	z_{18}	z_{18}
z_{11}	z_{10}	z_{17}	z_{17}	z_{17}	z_{17}	z_{17}	z_{17}	z_{17}	z_{18}
z_{12}	z_{10}	z_{11}	z_{16}	z_{16}	z_{16}	z_{16}	z_{16}	z_{17}	z_{18}
z_{13}	z_{10}	z_{11}	z_{12}	z_{15}	z_{15}	z_{15}	z_{16}	z_{17}	z_{18}
z_{14}	z_{10}	z_{11}	z_{12}	z_{13}	z_{14}	z_{15}	z_{16}	z_{17}	z_{18}
z_{15}	z_{10}	z_{11}	z_{12}	z_{13}	z_{13}	z_{15}	z_{16}	z_{17}	z_{18}
z_{16}	z_{10}	z_{11}	z_{12}	z_{12}	z_{12}	z_{12}	z_{16}	z_{17}	z_{18}
z_{17}	z_{10}	z_{11}	z_{11}	z_{11}	z_{11}	z_{11}	z_{11}	z_{17}	z_{18}
z_{18}	z_{10}	z_{10}	z_{10}	z_{10}	z_{10}	z_{10}	z_{10}	z_{10}	z_{18}

Table for " $\rightsquigarrow$ "

Then $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$, where $\tilde{e} = z_{14}$, is an OBCI-algebra. Let $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ be an MBJn-set in $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ given by Table 4.

Table 4: Table for $\mathcal{G} = (M_G,\ \tilde{B}_G,\ J_G)$

W	$M_G(w)$	$\tilde{B}_G(w)$	$J_G(w)$
z_{10}	0.36	$[0.58, 0.79]$	0.61
z_{11}	0.49	$[0.52, 0.73]$	0.67
z_{12}	0.53	$[0.47, 0.66]$	0.46
z_{13}	0.75	$[0.39, 0.58]$	0.27
z_{14}	0.87	$[0.63, 0.85]$	0.27
z_{15}	0.75	$[0.39, 0.58]$	0.27
z_{16}	0.53	$[0.47, 0.66]$	0.46
z_{17}	0.49	$[0.52, 0.73]$	0.67
z_{18}	0.36	$[0.58, 0.79]$	0.61

Then $\mathcal{G} = (M_G,\ \tilde{B}_G,\ J_G)$ is an MBJnO-subalgebra of $\mathbf{W} := (W,\ \rightsquigarrow,\ \tilde{e},\ \leq_W)$. The sets $U(M_G; s)$, $U(\tilde{B}_G; [\varepsilon_1, \varepsilon_2])$ and $L(J_G; t)$ are given as follows:

$$U(M_G; s) = \begin{cases} \emptyset & \text{if } \ 0.87 < s \leq 1, \\ \{z_{14}\} & \text{if } \ 0.75 < s \leq 0.87, \\ \{z_{14}, z_{15}, z_{13}\} & \text{if } \ 0.53 < s \leq 0.75, \\ \{z_{14}, z_{15}, z_{13}, z_{16}, z_{12}\} & \text{if } \ 0.49 < s \leq 0.53, \\ \{z_{14}, z_{15}, z_{13}, z_{16}, z_{12}, z_{17}, z_{11}\} & \text{if } \ 0.36 < s \leq 0.49, \\ W & \text{if } \ 0.00 \leq s \leq 0.36, \end{cases}$$

$$U(\tilde{B}_G; [\varepsilon_1, \varepsilon_2]) = \begin{cases} \emptyset & \text{if } \ [0.63, 0.85] \prec [\varepsilon_1, \varepsilon_2] \preceq [1, 1], \\ \{z_{14}\} & \text{if } \ [0.58, 0.79] \prec [\varepsilon_1, \varepsilon_2] \preceq [0.63, 0.85], \\ \{z_{14}, z_{18}, z_{10}\} & \text{if } \ [0.52, 0.73] \prec [\varepsilon_1, \varepsilon_2] \preceq [0.58, 0.79], \\ \{z_{14}, z_{18}, z_{10}, z_{17}, z_{11}\} & \text{if } \ [0.47, 0.66] \prec [\varepsilon_1, \varepsilon_2] \preceq [0.52, 0.73], \\ \{z_{14}, z_{18}, z_{10}, z_{17}, z_{11}, z_{16}, z_{12}\} & \text{if } \ [0.39, 0.58] \prec [\varepsilon_1, \varepsilon_2] \preceq [0.47, 0.66], \\ W & \text{if } \ [0, 0] \prec [\varepsilon_1, \varepsilon_2] \preceq [0.39, 0.58], \end{cases}$$

and

$$L(J_G; t) = \begin{cases} W & \text{if } \ 0.67 \leq t \leq 1, \\ \{z_{14}, z_{15}, z_{13}, z_{16}, z_{12}, z_{18}, z_{10}\} & \text{if } \ 0.61 \leq t < 0.67, \\ \{z_{14}, z_{15}, z_{13}, z_{16}, z_{12}\} & \text{if } \ 0.46 \leq t < 0.61, \\ \{z_{14}, z_{15}, z_{13}\} & \text{if } \ 0.27 \leq t < 0.46, \\ \emptyset & \text{if } \ 0.00 \leq t < 0.27. \end{cases}$$

It is routine to verify that $U(M_G; s)$, $U(\tilde{B}_G; [\varepsilon_1, \varepsilon_2])$ and $L(J_G; t)$ are O-subalgebras of $\mathbf{W} := (W,\ \rightsquigarrow,\ \tilde{e},$

$\leq_W$) for all $s, t \in [0,1]$ and $[\varepsilon_1, \varepsilon_2] \in [I]$ whenever they are nonempty.

Given a non-empty subset A of W, consider an MBJn-set $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ in $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ defined by

$$M_G(x_{11}) = \begin{cases} s & \text{if } x_{11} \in A, \\ 0 & \text{otherwise,} \end{cases} \quad \tilde{B}_G(x_{11}) = \begin{cases} [\varepsilon_1, \varepsilon_2] & \text{if } x_{11} \in A, \\ [0,0] & \text{otherwise,} \end{cases} \quad J_G(x_{11}) = \begin{cases} t & \text{if } x_{11} \in A, \\ 1 & \text{otherwise,} \end{cases} \tag{3.3}$$

where $(s,t) \in (0,1] \times [0,1)$ and $\varepsilon_1, \varepsilon_2 \in (0,1]$ with $\varepsilon_1 < \varepsilon_2$.

Theorem 3.8. *Given a non-empty subset A of W, the MBJn-set $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ given in (3.3) is an MBJnO-subalgebra of $\boldsymbol{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ if and only if A is an O-subalgebra of $\boldsymbol{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$. Moreover $U(M_G; s) = A$, $U(\tilde{B}_G; [\varepsilon_1, \varepsilon_2]) = A$ and $L(J_G; t) = A$.*

Proof. Assume that $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ is an MBJnO-subalgebra of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$. Let $x_{11}, x_{12} \in W$ be such that $\check{e} \leq_W x_{11}$, $\check{e} \leq_W x_{12}$ and $x_{11}, x_{12} \in A$. Then $M_G(x_{11}) = s = M_G(x_{12})$, $\tilde{B}_G(x_{11}) = [\varepsilon_1, \varepsilon_2] = \tilde{B}_G(x_{12})$, and $J_G(x_{11}) = t = J_G(x_{12})$. Hence $M_G(x_{11} \rightsquigarrow x_{12}) \geq \min\{M_G(x_{11}), M_G(x_{12})\} = s$, $\tilde{B}_G(x_{11} \rightsquigarrow x_{12}) \succeq \operatorname{rmin}\{\tilde{B}_G(x_{11}), \tilde{B}_G(x_{12})\} = [\varepsilon_1, \varepsilon_2]$, $J_G(x_{11} \rightsquigarrow x_{12}) \leq \max\{J_G(x_{11}), J_G(x_{12})\} = t$, which imply that $M_G(x_{11} \rightsquigarrow x_{12}) = s$, $\tilde{B}_G(x_{11} \rightsquigarrow x_{12}) = [\varepsilon_1, \varepsilon_2]$ and $J_G(x_{11} \rightsquigarrow x_{12}) = t$. Hence $x_{11} \rightsquigarrow x_{12} \in A$, and therefore A is an O-subalgebra of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$.

Suppose conversely that A is an O-subalgebra of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$. Let $x_{11}, x_{12} \in W$ be such that $\check{e} \leq_W x_{11}$ and $\check{e} \leq_W x_{12}$. If $x_{11}, x_{12} \in A$, then $x_{11} \rightsquigarrow x_{12} \in A$ and thus

$$\begin{aligned} &M_G(x_{11} \rightsquigarrow x_{12}) = s = \min\{M_G(x_{11}), M_G(x_{12})\}, \\ &\tilde{B}_G(x_{11} \rightsquigarrow x_{12}) = [\varepsilon_1, \varepsilon_2] = \operatorname{rmin}\{[\varepsilon_1, \varepsilon_2], [\varepsilon_1, \varepsilon_2]\} = \operatorname{rmin}\{\tilde{B}_G(x_{11}), \tilde{B}_G(x_{12})\}, \\ &J_G(x_{11} \rightsquigarrow x_{12}) = t = \max\{J_G(x_{11}), J_G(x_{12})\}. \end{aligned}$$

If $x_{11}, x_{12} \notin A$, then $M_G(x_{11}) = 0 = M_G(x_{12})$, $\tilde{B}_G(x_{11}) = [0,0] = \tilde{B}_G(x_{12})$ and $J_G(x_{11}) = 1 = J_G(x_{12})$. Hence

$$\begin{aligned} &M_G(x_{11} \rightsquigarrow x_{12}) \geq 0 = \min\{0, 0\} = \min\{M_G(x_{11}), M_G(x_{12})\}, \\ &\tilde{B}_G(x_{11} \rightsquigarrow x_{12}) \succeq [0,0] = \operatorname{rmin}\{[0,0], [0,0]\} = \operatorname{rmin}\{\tilde{B}_G(x_{11}), \tilde{B}_G(x_{12})\}, \\ &J_G(x_{11} \rightsquigarrow x_{12}) \leq 1 = \max\{1, 1\} = \max\{J_G(x_{11}), J_G(x_{12})\}. \end{aligned}$$

If $x_{11} \in A$ and $x_{12} \notin A$, then $M_G(x_{11}) = s$, $\tilde{B}_G(x_{11}) = [\varepsilon_1, \varepsilon_2]$, $J_G(x_{11}) = t$, $M_G(x_{12}) = 0$, $\tilde{B}_G(x_{12}) = [0,0]$, and $J_G(x_{12}) = 1$. Thus

$$\begin{aligned} &M_G(x_{11} \rightsquigarrow x_{12}) \geq 0 = \min\{s, 0\} = \min\{M_G(x_{11}), M_G(x_{12})\}, \\ &\tilde{B}_G(x_{11} \rightsquigarrow x_{12}) \succeq [0,0] = \operatorname{rmin}\{[\varepsilon_1, \varepsilon_2], [0,0]\} = \operatorname{rmin}\{\tilde{B}_G(x_{11}), \tilde{B}_G(x_{12})\}, \\ &J_G(x_{11} \rightsquigarrow x_{12}) \leq 1 = \max\{t, 1\} = \max\{J_G(x_{11}), J_G(x_{12})\}. \end{aligned}$$

Similarly, if $x_{11} \notin A$ and $x_{12} \in A$, then $M_G(x_{11} \rightsquigarrow x_{12}) \geq \min\{M_G(x_{11}), M_G(x_{12})\}$, $\tilde{B}_G(x_{11} \rightsquigarrow x_{12}) \succeq \operatorname{rmin}\{\tilde{B}_G(x_{11}), \tilde{B}_G(x_{12})\}$, and $J_G(x_{11} \rightsquigarrow x_{12}) \leq \max\{J_G(x_{11}), J_G(x_{12})\}$. Therefore $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ is an MBJnO-subalgebra of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$. □

Let $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ be an MBJn-set in $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$. We denote

$$\bot := \inf\{J_G(a_{11}) \mid a_{11} \in W\},$$
$$\top := 1 - \sup\{M_G(a_{11}) \mid a_{11} \in W\},$$
$$\Omega := [1,1] - \operatorname{rsup}\{\tilde{B}_G(a_{11}) \mid a_{11} \in W\}.$$

For any $q \in [0, \top]$, $\tilde{a_{13}} \in [[0,0], \Omega]$ and $p \in [0, \bot]$, we define $\mathcal{G}^T = (M_G^q, \tilde{B}_G^{\tilde{a_{13}}}, J_G^p)$ by $M_G^q(a_{11}) = M_G(a_{11}) + q$, $\tilde{B}_G^{\tilde{a_{13}}}(a_{11}) = \tilde{B}_G(a_{11}) + \tilde{a_{13}}$ and $J_G^p(a_{11}) = J_G(a_{11}) - p$. Then $\mathcal{G}^T = (M_G^q, \tilde{B}_G^{\tilde{a_{13}}}, J_G^p)$ is an MBJn-set in W, and it is said to be a $(q, \tilde{a_{13}}, p)$-*translative MBJn-set* of $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$.

Theorem 3.9. *An MBJn-set $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ in $\boldsymbol{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ is an MBJnO-subalgebra of $\boldsymbol{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ if and only if its $(q, \tilde{a_{13}}, p)$-translative MBJn-set $\mathcal{G}^T = (M_G^q, \tilde{B}_G^{\tilde{a_{13}}}, J_G^p)$ is an MBJnO-subalgebra of $\boldsymbol{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ for $q \in [0, \top]$, $\tilde{a_{13}} \in [[0,0], \Omega]$ and $p \in [0, \bot]$.*

Proof. Assume that $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ is an MBJnO-subalgebra of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$, and let $q \in [0, \top]$, $\tilde{a_{13}} \in [[0,0], \Omega]$ and $p \in [0, \bot]$. For every $x_{11}, x_{12} \in W$ with $\check{e} \leq_W x_{11}$ and $\check{e} \leq_W x_{12}$, we obtain

$$\begin{aligned} M_G^q(x_{11} \rightsquigarrow x_{12}) &= M_G(x_{11} \rightsquigarrow x_{12}) + q \geq \min\{M_G(x_{11}), M_G(x_{12})\} + q \\ &= \min\{M_G(x_{11}) + q, M_G(x_{12}) + q\} = \min\{M_G^q(x_{11}), M_G^q(x_{12})\}, \end{aligned}$$

$$\begin{aligned} \tilde{B}_G^{\tilde{a_{13}}}(x_{11} \rightsquigarrow x_{12}) &= \tilde{B}_G(x_{11} \rightsquigarrow x_{12}) + \tilde{a_{13}} \succeq \operatorname{rmin}\{\tilde{B}_G(x_{11}), \tilde{B}_G(x_{12})\} + \tilde{a_{13}} \\ &= \operatorname{rmin}\{\tilde{B}_G(x_{11}) + \tilde{a_{13}}, \tilde{B}_G(x_{12}) + \tilde{a_{13}}\} = \operatorname{rmin}\{\tilde{B}_G^{\tilde{a_{13}}}(x_{11}), \tilde{B}_G^{\tilde{a_{13}}}(x_{12})\}, \end{aligned}$$

and

$$\begin{aligned} J_G^p(x_{11} \rightsquigarrow x_{12}) &= J_G(x_{11} \rightsquigarrow x_{12}) - p \leq \max\{J_G(x_{11}), J_G(x_{12})\} - p \\ &= \max\{J_G(x_{11}) - p, J_G(x_{12}) - p\} = \max\{J_G^p(x_{11}), J_G^p(x_{12})\}. \end{aligned}$$

Hence $\mathcal{G}^T = (M_G^q, \tilde{B}_G^{\tilde{a_{13}}}, J_G^p)$ is an MBJnO-subalgebra of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$.

Suppose conversely that the $(q, \tilde{a_{13}}, p)$-translative MBJn-set $\mathcal{G}^T = (M_G^q, \tilde{B}_G^{\tilde{a_{13}}}, J_G^p)$ of $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ is an MBJnO-subalgebra of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ for all $q \in [0, \top]$, $\tilde{a_{13}} \in [[0,0], \Omega]$ and $p \in [0, \bot]$. Let $x_{11}, x_{12} \in W$ be such that $\check{e} \leq_W x_{11}$ and $\check{e} \leq_W x_{12}$. Then

$$\begin{aligned} M_G(x_{11} \rightsquigarrow x_{12}) + q = M_G^q(x_{11} \rightsquigarrow x_{12}) &\geq \min\{M_G^q(x_{11}), M_G^q(x_{12})\} \\ &= \min\{M_G(x_{11}) + q, M_G(x_{12}) + q\} \\ &= \min\{M_G(x_{11}), M_G(x_{12})\} + q, \end{aligned}$$

$$\begin{aligned} \tilde{B}_G(x_{11} \rightsquigarrow x_{12}) + \tilde{a_{13}} = \tilde{B}_G^{\tilde{a_{13}}}(x_{11} \rightsquigarrow x_{12}) &\succeq \operatorname{rmin}\{\tilde{B}_G^{\tilde{a_{13}}}(x_{11}), \tilde{B}_G^{\tilde{a_{13}}}(x_{12})\} \\ &= \operatorname{rmin}\{\tilde{B}_G(x_{11}) + \tilde{a_{13}}, \tilde{B}_G(x_{12}) + \tilde{a_{13}}\} \\ &= \operatorname{rmin}\{\tilde{B}_G(x_{11}), \tilde{B}_G(x_{12})\} + \tilde{a_{13}}, \end{aligned}$$

and

$$\begin{aligned} J_G(x_{11} \rightsquigarrow x_{12}) - p = J_G^p(x_{11} \rightsquigarrow x_{12}) &\leq \max\{J_G^p(x_{11}), J_G^p(x_{12})\} \\ &= \max\{J_G(x_{11}) - p, J_G(x_{12}) - p\} \\ &= \max\{J_G(x_{11}), J_G(x_{12})\} - p. \end{aligned}$$

It follows that $M_G(x_{11} \rightsquigarrow x_{12}) \geq \min\{M_G(x_{11}), M_G(x_{12})\}$, $\tilde{B}_G(x_{11} \rightsquigarrow x_{12}) \succeq \text{rmin}\{\tilde{B}_G(x_{11}), \tilde{B}_G(x_{12})\}$ and $J_G(x_{11} \rightsquigarrow x_{12}) \leq \max\{J_G(x_{11}), J_G(x_{12})\}$. Therefore $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ is an MBJnO-subalgebra of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$. □

Theorem 3.10. *Given an MBJn-set $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ in $\boldsymbol{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$, consider the following sets:*

$$\begin{aligned} &U_q(M_G; s) := \{x_{11} \in W \mid M_G(x_{11}) \geq s - q\}, \\ &U_{\tilde{a_{13}}}(\tilde{B}_G; [\varepsilon_1, \varepsilon_2]) := \{x_{11} \in W \mid \tilde{B}_G(x_{11}) \succeq [\varepsilon_1, \varepsilon_2] - \tilde{a_{13}}\}, \\ &L_p(J_G; t) := \{x_{11} \in W \mid J_G(x_{11}) \leq t + p\} \end{aligned}$$

where $s, t \in [0, 1]$, $[\varepsilon_1, \varepsilon_2] \in [I]$, $q \in [0, \top]$, $\tilde{a_{13}} \in [[0, 0], \Omega]$ and $p \in [0, \bot]$ such that $p \leq t$, $[\varepsilon_1, \varepsilon_2] \succeq \tilde{a_{13}}$ and $q \geq s$. Then the $(q, \tilde{a_{13}}, p)$-translative MBJn-set of $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ is an MBJnO-subalgebra of $\boldsymbol{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ if and only if $U_q(M_G; s)$, $U_{\tilde{a_{13}}}(\tilde{B}_G; [\varepsilon_1, \varepsilon_2])$ and $L_p(J_G; t)$ are O-subalgebras of $\boldsymbol{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ for all $s \in Im(M_G)$, $[\varepsilon_1, \varepsilon_2] \in Im(\tilde{B}_G)$ and $t \in Im(J_G)$ satisfying $s \geq q$, $[\varepsilon_1, \varepsilon_2] \succeq \tilde{a_{13}}$ and $t \leq p$.

Proof. Let the $(q, \tilde{a_{13}}, p)$-translative MBJn-set of $\mathcal{G} = (M_G, \tilde{B}_G, J_G)$ be an MBJ-neutro-sophic O-subalgebra of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$. Let's have $s \in Im(M_G)$, $[\varepsilon_1, \varepsilon_2] \in Im(\tilde{B}_G)$ and $t \in Im(J_G)$ that satisfy $s \geq q$, $[\varepsilon_1, \varepsilon_2] \succeq \tilde{a_{13}}$ and $t \leq p$, respectively. Let $x_{11}, x_{12} \in W$ be such that $\check{e} \leq_W x_{11}$ and $\check{e} \leq_W x_{12}$. If $x_{11}, x_{12} \in U_q(M_G; s)$, then $M_G(x_{11}) \geq s - q$ and $M_G(x_{12}) \geq s - q$, which imply that $M_G^q(x_{11}) \geq s$ and $M_G^q(x_{12}) \geq s$. It follows that

$$M_G^q(x_{11} \rightsquigarrow x_{12}) \geq \min\{M_G^q(x_{11}), M_G^q(x_{12})\} \geq s.$$

Hence $M_G(x_{11} \rightsquigarrow x_{12}) \geq s - q$, and so $x_{11} \rightsquigarrow x_{12} \in U_q(M_G; s)$. If $x_{11}, x_{12} \in U_{\tilde{a_{13}}}(\tilde{B}_G; [\varepsilon_1, \varepsilon_2])$, then $\tilde{B}_G(x_{11}) \succeq [\varepsilon_1, \varepsilon_2] - \tilde{a_{13}}$ and $\tilde{B}_G(x_{12}) \succeq [\varepsilon_1, \varepsilon_2] - \tilde{a_{13}}$. Hence

$$\tilde{B}_G^{\tilde{a_{13}}}(x_{11} \rightsquigarrow x_{12}) \succeq \text{rmin}\{\tilde{B}_G^{\tilde{a_{13}}}(x_{11}), \tilde{B}_G^{\tilde{a_{13}}}(x_{12})\} \succeq [\varepsilon_1, \varepsilon_2],$$

and so $\tilde{B}_G(x_{11} \rightsquigarrow x_{12}) \succeq [\varepsilon_1, \varepsilon_2] - \tilde{a_{13}}$. Thus $x_{11} \rightsquigarrow x_{12} \in U_{\tilde{a_{13}}}(\tilde{B}_G; [\varepsilon_1, \varepsilon_2])$. If $x_{11}, x_{12} \in L_p(J_G; t)$, then $J_G(x_{11}) \leq t + p$ and $J_G(x_{12}) \leq t + p$. It follows that

$$J_G^p(x_{11} \rightsquigarrow x_{12}) \leq \max\{J_G^p(x_{11}), J_G^p(x_{12})\} \leq t,$$

that is, $J_G(x_{11} \rightsquigarrow x_{12}) \leq t + p$. Thus $x_{11} \rightsquigarrow x_{12} \in L_p(J_G; t)$. Therefore $U_q(M_G; s)$, $U_{\tilde{a_{13}}}(\tilde{B}_G; [\varepsilon_1, \varepsilon_2])$ and $L_p(J_G; t)$ are O-subalgebras of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$.

Conversely, suppose that $U_q(M_G; s)$, $U_{\tilde{a_{13}}}(\tilde{B}_G; [\varepsilon_1, \varepsilon_2])$ and $L_p(J_G; t)$ are ordered subalgebras of $\mathbf{W} := (W, \rightsquigarrow, \tilde{e}, \leq_W)$ for all $s \in Im(M_G)$, $[\varepsilon_1, \varepsilon_2] \in Im(\tilde{B}_G)$ and $t \in Im(J_G)$ with $s \geq q$, $[\varepsilon_1, \varepsilon_2] \succeq \tilde{a_{13}}$ and $t \leq p$. Assume that $M_G^q(a_{11} \rightsquigarrow a_{12}) < \min\{M_G^q(a_{11}), M_G^q(a_{12})\}$ for some $a_{11}, a_{12} \in W$ with $\check{e} \leq_W a_{11}$ and

$\check{e} \leq_W a_{12}$. Then $a_{11}, a_{12} \in U_q(M_G; s_0)$ and $a_{11} \rightsquigarrow a_{12} \notin U_q(M_G; s_0)$ for $s_0 = \min\{M_G^q(a_{11}), M_G^q(a_{12})\}$. This is a contradiction, and so $M_G^q(x_{11} \rightsquigarrow x_{12}) \geq \min\{M_G^q(x_{11}), M_G^q(x_{12})\}$ for all $x_{11}, x_{12} \in W$ with $\check{e} \leq_W x_{11}$ and $\check{e} \leq_W x_{12}$. If $\tilde{B}_G^{\tilde{a_{13}}}(a_{11} \rightsquigarrow a_{12}) \prec \mathrm{rmin}\{\tilde{B}_G^{\tilde{a_{13}}}(a_{11}), M_G^{\tilde{a_{13}}}(a_{12})\}$ for some $a_{11}, a_{12} \in W$ with $\check{e} \leq_W a_{11}$ and $\check{e} \leq_W a_{12}$, then there exists $\tilde{a_{12}} \in [I]$ such that $\tilde{B}_G^{\tilde{a_{13}}}(a_{11} \rightsquigarrow a_{12}) \prec \tilde{a_{12}} \preceq \mathrm{rmin}\{\tilde{B}_G^{\tilde{a_{13}}}(a_{11}), M_G^{\tilde{a_{13}}}(a_{12})\}$. Hence $a_{11}, a_{12} \in U_{\tilde{a_{13}}}(\tilde{B}_G; \tilde{a_{12}})$ but $a_{11} \rightsquigarrow a_{12} \notin U_{\tilde{a_{13}}}(\tilde{B}_G; \tilde{a_{12}})$, which is a contradiction. Thus $\tilde{B}_G^{\tilde{a_{13}}}(x_{11} \rightsquigarrow x_{12}) \succeq \mathrm{rmin}\{\tilde{B}_G^{\tilde{a_{13}}}(x_{11}), M_G^{\tilde{a_{13}}}(x_{12})\}$ for all $x_{11}, x_{12} \in W$ with $\check{e} \leq_W x_{11}$ and $\check{e} \leq_W x_{12}$. Suppose that $J_G^p(a_{11} \rightsquigarrow a_{12}) > \max\{J_G^p(a_{11}), J_G^p(a_{12})\}$ for some $a_{11}, a_{12} \in W$ with $\check{e} \leq_W a_{11}$ and $\check{e} \leq_W a_{12}$. Taking $t_0 := \max\{J_G^p(a_{11}), J_G^p(a_{12})\}$ implies that $J_G(a_{11}) \leq t_0 + p$ and $J_G(a_{12}) \leq t_0 + p$ but $J_G(a_{11} \rightsquigarrow a_{12}) > t_0 + p$. This shows that $a_{11}, a_{12} \in L_p(J_G; t_0)$ and $a_{11} \rightsquigarrow a_{12} \notin L_p(J_G; t_0)$. This is a contradiction, and therefore $J_G^p(x_{11} \rightsquigarrow x_{12}) \leq \max\{J_G^p(x_{11}), J_G^p(x_{12})\}$ for all $x_{11}, x_{12} \in W$ with $\check{e} \leq_W x_{11}$ and $\check{e} \leq_W x_{12}$. Consequently, the $(q, \tilde{a_{13}}, p)$-translative MBJn-set $\mathcal{G}^T = (M_G^q,\ \tilde{B}_G^{\tilde{a_{13}}},\ J_G^p)$ of $\mathcal{G} = (M_G,\ \tilde{B}_G,\ J_G)$ is an MBJnO-subalgebra of $\mathbf{W} := (W,\ \rightsquigarrow,\ \tilde{e},\ \leq_W)$. □

Before we conclude this paper, we raise the following question.

Question. *Is the inverse of Theorem 3.5 true or false?*

4 Conclusion

In a classical set, an element can either belong to the set (true) or not belong to the set (false). Neutrosophy is a branch of philosophy that deals with the study of indeterminacy and includes three components: true, false, and indeterminate. Neutrosophy introduces the idea that an element can have an indeterminate state, which means it is not clear that the element is in or out of a set. The neutrosophic set consists of three f-sets called true membership function, false membership function and indeterminate membership function. MBJ-neutrosophic structure is a structure constructed using interval-valued f-set instead of indeterminate membership function in the n-set. In general, the indeterminate part appears in a wide range. So instead of treating the indeterminate part as a single value, it is treated as an interval value, allowing a much more comprehensive processing.

The aim of this study is to apply such MBJ-neutrosophic structure to logical algebra, especially OBCI-algebra. We first introduced the notion of MBJ-neutrosophic (ordered) subalgebras in OBCI-algebras and then addressed severl elated properties. We looked at the relationship between the MBJn-subalgeba and the MBJ-neutrosophic ordered subalgebra. We established an MBJ-neutrosophic ordered subalgebra by using (intuitionistic) fuzzy ordered subalgebras, and discussed the characterization of MBJ-neutrosophic ordered subalgebras. We introduced the $(q, \tilde{a_{13}}, p)$-translative MBJn-set based on a MBJn-set, and considered its characterization. We generated an MBJ-neutrosophic ordered subalgebra by using $(q, \tilde{a_{13}}, p)$-translative MBJn-set. The ideas and results covered in this paper will be applied to logical algebras in the future and contribute to producing various results.

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INTRODUCTION OF NEUTROSOPHIC SOFT LIE ALGEBRAS

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Abstract. We introduce the concept of neutrosophic soft Lie subalgebras of a Lie algebra and investigate some of their properties. The Cartesian product of neutrosophic soft Lie subalgebras will be discussed. In particular, the homomorphisms of neutrosophic soft Lie algebras is introduced and investigated some of their properties.

Keywords: Lie algebra, subalgebra, neutrosophic soft set, neutrosophic soft Lie Algebras.

1. **Introduction**

The contribution of mathematics to the present-day technology in reaching to a fast trend cannot be ignored. The theories presented differently from classical methods in studies such as fuzzy set [25], intuitionistic fuzzyset [9], soft set [18], neutrosophic set [17, 20], etc. The algebraic structure of set theories dealing with uncertainties has also been studied by some authors. After Molodtsov's work, some different applications of soft sets were studied in [18]. Maji et al. [16] presented the concept of fuzzy soft set. This kind of fuzzy sets have now gained a wide recognition as useful tool,in modeling of some uncertain phenomena, computer science, mathematics, medicine, chemistry, economics, astronomy etc. Smarandache [20,21] introduced the concept of neutrosophic set which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data. Rosenfeld [19] proposed the concept of fuzzy groups in order to establish the algebraic structures of fuzzy sets. Definition of fuzzy module is given by some authors. Qiu- Mei Sun et al. defined soft modules and investigated their basic properties. Fuzzy soft modules and intuitionistic fuzzy soft modules was given and researched by C. Gunduz(Aras) and S. Bayramov [11, 12]. Neutrosophic soft modules are introduction in [23].

Lie algebras were first discovered by Sophus Lie (1842-1899) when he attempted to classify certain smoothsubgroups of general linear groups [13]. The groups he considered are now called Lie groups. By taking the tangent space at the identity element of such a group, he obtained the Lie algebra and hence the problems on groups can be reduced to problems on Lie algebras so that it becomes more tractable. There are many applications of Lie algebras in many branches of mathematics and physics. In [1–8, 10, 14, 15, 22, 24] there is an introduction the concept of fuzzy Lie subalgebras and investigation of some of their properties.

In this paper we have introduced the concept of neutrosophic soft Lie subalgebras of a Lie algebra and investigated some of their properties. The Cartesian product of neutrosophic soft Lie subalgebras will be discussed. In particular, the homomorphisms of neutrosophic soft Lie algebras is introduced and investigated some of their properties.

2. Preliminaries

In this section, we first review some elementary aspects that are necessary for this paper.

Definition 2.1. [24] An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ on L is called an intuitionistic fuzzy Lie subalgebra if the following conditions are satisfied:

$$\mu_A(x+y) \geq min(\mu_A(x), \ \ \mu_A(y)) \text{ and } \lambda_A(x+y) \leq \max\ (\lambda_A(x), \lambda_A(y)), \quad (1)$$

$$\mu_A(\alpha x) \geq \mu_A(x) \text{ and } \lambda_A(\alpha x)) \leq \ \lambda_A(x) \quad (2)$$

$$\mu_A([x,y]) \geq min\{\mu_A(x), \mu_A(y)\} \text{ and } \lambda_A([x,y]) \leq max\ \{\lambda_A(x), \lambda_A(y)\} \quad (3)$$

for all $x, y \in L$ and $\alpha \in F$.

Definition 2.2. [22] A neutrosophic set A on the universe of X is defined as
$A = \{< x, T_A(x), I_A(x), F_A(x) >, x \in X\}$, where $T, I, F : X \to]^-0, 1^+[$ and

$$^-0 < 0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$$

Definition 2.3. [17] Let X be an initial universe set and E be a set of parametres. Let $P(X)$ denote the set of all neutrosophic sets of X. Then, a neutrosophic soft set $\left(\tilde{F}\ , E\right)$ over X is a set defined by a set valued function $\tilde{F}$ representing a mapping $\tilde{F}\ : E \to P(X)$ where $\tilde{F}$ is called approximate function of the neutrosophic soft set $\left(\tilde{F}, E\right)$. In other words, the neutrosophic soft set is a parameterized family of some elements of the set $P(X)$ and therefore it can be written as a set of ordered pairs,

$$(\tilde{F}\ : E) = \left\{\left(e, \left\langle x, \ \ T_{\tilde{F}(e)}(x), I_{\tilde{F}(e)}(x), F_{\tilde{F}(e)}(x)\right\rangle\ : x\epsilon X\right)\ : e \in E\right\}$$

where $T_{\tilde{F}(e)}(x), I_{\tilde{F}(e)}(x), F_{\tilde{F}(e)}(x) \in [0,1]$ respectively called the truth-membership, indeterminacy-membership, falsity-membership function of $\tilde{F}(e)$. Since supremum of each $T,\ I,\ F$ is 1 so the inequality $0 \leq T_{\tilde{F}(e)}(x), I_{\tilde{F}(e)}(x), F_{\tilde{F}(e)}(x) \leq 3$ is obvious.

Definition 2.4. [17] Let $\left(\tilde{F}, E\right)$ be neutrosophic soft set over the common universe X. The complement of $\left(\tilde{F}, E\right)$ is denoted by $\left(\tilde{F}, E\right)^c$ and is defined by:

$$\left(\tilde{F}, E\right)^c = \left\{ \left(e, \left\langle x, F_{\tilde{F}(e)}(x), 1 - I_{\tilde{F}(e)}(x), T_{\tilde{F}(e)}(x), \right\rangle : x \in X\right) : e \in E\right\}.$$

Obvious that, $\left(\left(\tilde{F}, E\right)^c\right)^c = \left(\tilde{F}, E\right)$.

Definition 2.5. [17] Let $\left(\tilde{F}, E\right)$ and $\left(\tilde{G}, E\right)$ be two neutrosophic soft sets over the common universe X. $\left(\tilde{F}, E\right)$ is said to be neutrosophic soft subset of $\left(\tilde{G}, E\right)$ if $\mathrm{T}_{\tilde{F}(e)}(x) \leq \mathrm{T}_{\tilde{G}(e)}(x), I_{\tilde{F}(e)}(x) \leq I_{\tilde{G}(e)}(x), F_{\tilde{F}(e)}(x) \geq F_{\tilde{G}(e)}(x), \quad \forall e \in E,\ \forall x \in X$. It is denoted by $\left(\tilde{F}, E\right) \subseteq \left(\tilde{G}, E\right)$.

Definition 2.6. [17] Let $\left(\tilde{F}_1, E\right)$ and $\left(\tilde{F}_2, E\right)$ be two neutrosophic soft sets over the common universe X. Then their union is denoted by $\left(\tilde{F}_1, E\right) \bigcup \left(\tilde{F}_2, E\right) = \left(\tilde{F}_3, E\right)$ and is defined by:

$$\left(\tilde{F}_3, E\right) = \left\{ \left(e, \left\langle x, \mathrm{T}_{\tilde{F}_3(e)}(x), I_{\tilde{F}_3(e)}(x), F_{\tilde{F}_3(e)}(x), \right\rangle : x \in X\right) : e \in E\right\}$$

where

$$\begin{aligned}
\mathrm{T}_{\tilde{F}_3(e)}(x) &= \max\left\{\mathrm{T}_{\tilde{F}_1(e)}(x), \mathrm{T}_{\tilde{F}_2(e)}(x)\right\},\\
I_{\tilde{F}_3(e)}(x) &= \max\left\{I_{\tilde{F}_1(e)}(x), I_{\tilde{F}_2(e)}(x)\right\},\\
F_{\tilde{F}_3(e)}(x) &= \min\left\{F_{\tilde{F}_1(e)}(x), F_{\tilde{F}_2(e)}(x)\right\}.
\end{aligned}$$

Definition 2.7. [17] Let $\left(\tilde{F}_1, E\right)$ and $\left(\tilde{F}_2, E\right)$ be two neutrosophic soft sets over the common universe X. Then their intersection is denoted by $\left(\tilde{F}_1, E\right) \bigcap \left(\tilde{F}_2, E\right) = \left(\tilde{F}_3, E\right)$ and is defined by:

$$\left(\tilde{F}_3, E\right) = \left\{ \left(e, \left\langle x, \mathrm{T}_{\tilde{F}_3(e)}(x), I_{\tilde{F}_3(e)}(x), F_{\tilde{F}_3(e)}(x), \right\rangle : x \in X\right) : e \in E\right\}$$

where

$$\begin{aligned}
\mathrm{T}_{\tilde{F}_3(e)}(x) &= \min\left\{\mathrm{T}_{\tilde{F}_1(e)}(x), \mathrm{T}_{\tilde{F}_2(e)}(x)\right\},\\
I_{\tilde{F}_3(e)}(x) &= \min\left\{I_{\tilde{F}_1(e)}(x), I_{\tilde{F}_2(e)}(x)\right\},\\
F_{\tilde{F}_3(e)}(x) &= \max\left\{F_{\tilde{F}_1(e)}(x), F_{\tilde{F}_2(e)}(x)\right\}.
\end{aligned}$$

Definition 2.8. [17] Let $\left(\tilde{F}_1, E\right)$ and $\left(\tilde{F}_2, E\right)$ be two neutrosophic soft sets over the common universe X. Then " AND" operation on them is denoted by $\left(\tilde{F}_1, E\right) \wedge \left(\tilde{F}_2, E\right) = \left(\tilde{F}_3, E \times E\right)$ and is defined by:

$$\left(\tilde{F}_3, E \times E\right) =$$

$$\left\{\left((e_1,e_2),\left\langle x,\mathrm{T}_{\tilde{F}_{3(e_1,e_2)}}(x),I_{\tilde{F}_{3(e_1,e_2)}}(x),F_{\tilde{F}_{3(e_1,e_2)}}(x),\right\rangle:x\in X\right):(e_1,e_2)\in E\times E\right\}$$

where

$$\begin{aligned}\mathrm{T}_{\tilde{F}_{3(e_1,e_2)}}(x)&=\min\left\{\mathrm{T}_{\tilde{F}_{1(e_1)}}(x),\mathrm{T}_{\tilde{F}_{2(e_2)}}(x)\right\},\\ I_{\tilde{F}_{3(e_1,e_2)}}(x)&=\min\left\{I_{\tilde{F}_{1(e_1)}}(x),I_{\tilde{F}_{2(e_2)}}(x)\right\},\\ F_{\tilde{F}_{3(e_1,e_2)}}(x)&=\max\left\{F_{\tilde{F}_{1(e_1)}}(x),F_{\tilde{F}_{2(e_2)}}(x)\right\}.\end{aligned}$$

3. Introduction of neutrosophic Soft Lie algebras

Definition 3.1. Let E be a set of all parameters, L be Lie algebra and $P(L)$ denotes all neutrosophic sets over L. Then a pair $\left(\tilde{F},E\right)$ is called a neutrosophic soft Lie algebra over L, where, $\tilde{F}$ is a mapping given by $\tilde{F}:E\to P(L)$, if for $\forall e\in E$, $\tilde{F}(e)=\left(T_{\tilde{F}}(e),I_{\tilde{F}}(e),F_{\tilde{F}}(e)\right)$ is a neutrosophic Lie algebra over L, i.e:

$$\begin{aligned}T_{\tilde{F}}(e)(x,y)&\geq\min\left(T_{\tilde{F}}(e)(x),T_{\tilde{F}}(e)(y)\right)\\ I_{\tilde{F}}(e)(x,y)&\geq\min\left(I_{\tilde{F}}(e)(x),I_{\tilde{F}}(e)(y)\right)\\ F_{\tilde{F}}(e)(x,y)&\leq\max\left(F_{\tilde{F}}(e)(x),F_{\tilde{F}}(e)(y)\right)\end{aligned}\tag{4}$$

$$\begin{aligned}T_{\tilde{F}}(e)(\alpha x)&\geq T_{\tilde{F}}(e)(x)\\ I_{\tilde{F}}(e)(\alpha x)&\geq I_{\tilde{F}}(e)(x)\\ F_{\tilde{F}}(e)(\alpha x)&\leq F_{\tilde{F}}(e)(x)\end{aligned}\tag{5}$$

$$\begin{aligned}T_{\tilde{F}}(e)([x+y])&\geq\min\left(T_{\tilde{F}}(e)(x),T_{\tilde{F}}(e)(y)\right)\\ I_{\tilde{F}}(e)([x+y])&\geq\min\left(I_{\tilde{F}}(e)(x),I_{\tilde{F}}(e)(y)\right)\\ F_{\tilde{F}}(e)([x+y])&\leq\max\left(F_{\tilde{F}}(e)(x),F_{\tilde{F}}(e)(y)\right)\end{aligned}$$

Definition 3.2. A neutrosophicsoft set $\left(\widetilde{F},E\right)$ on L is called neustrosophic soft Lie ideal if it satisfied the conditions (3.1), (3.2) and the following additional condition:
For each $e\in E$

$$T_{\widetilde{F}}(e)([x,y])\geq T_{\widetilde{F}}(e)\ (x)$$

$$I_{\widetilde{F}}(e)([x,y])\geq I_{\widetilde{F}}(e)\ (x)$$

$$F_{\widetilde{F}}(e)([x,y])\leq F_{\widetilde{F}}(e)\ (x)$$

For all $x,y\in L$

$$T_{\widetilde{F}}(e)(0)\geq T_{\widetilde{F}}(e)(x)\qquad T_{\widetilde{F}}(e)(-x)\geq T_{\widetilde{F}}(e)(x)$$

$$I_{\widetilde{F}}(e)(0)\geq I_{\widetilde{F}}(e)(x)\qquad I_{\widetilde{F}}(e)(-x)\geq I_{\widetilde{F}}(e)(x)$$

$$F_{\widetilde{F}}(e)(0)\leq F_{\widetilde{F}}(e)(x)\qquad F_{\widetilde{F}}(e)(-x)\leq F_{\widetilde{F}}(e)(x)$$

Proposition 3.1. Every neutrosophic soft Lie ideal is a neutrosophic soft Lie subalgebra.

Theorem 3.3. *Let* $\left(\widetilde{F},E\right)$ *be a neutrosophic soft Lie subalgebraover* L*.Then* $\left(\widetilde{F},E\right)$ *is a neutrosophic soft Lie subalgebra of* L *if and only if for each* $e\in E$ *the non-empty upper s-level cut.*

$$U_{T_{\widetilde{F}(e)}}(s)=\left\{x\in L\ /T_{\widetilde{F}}(e)\ (x)\geq s\right\}$$

$$U_{I_{\widetilde{F}(e)}}(s)=\left\{x\in L\ /I_{\widetilde{F}}(e)\ (x)\geq s\right\}$$

and the non-empty Lower s- lewel cut

$V_{F_{\widetilde{F}(e)}}(s)=\left\{x\in L/F_{\widetilde{F}}(e)(x)\leq s\right\}$ *are Lie subalgebras of* L *, for all* $s\in[0,1]$

Proof. Assume that $\left(\widetilde{F}\ ,E\right)$ is a neutrosophic soft Lie subalgebra of L and let $s\in[0,1]$ be such that $U_{T_{\widetilde{F}(e)}}(s)\neq\emptyset$. Let $x,y\in L$ be such that $x\in U_{T_{\widetilde{F}(e)}}(s)$ and $y\in U_{T_{\widetilde{F}(e)}}(s)$. It follows that

$$T_{\widetilde{F}}(e)\,(x+y)\geq\min\left(T_{\widetilde{F}}(e)\,(x)\,,T_{\widetilde{F}}(e)\,(y)\right)\geq s,$$

$$T_{\widetilde{F}}(e)\,(\alpha x\ )\geq T_{\widetilde{F}}(e)\,(x)\geq s,$$

$$T_{\widetilde{F}}\,(e)\,([x,y])\geq\min(T_{\widetilde{F}}\,(e)\,(x)\,,T_{\widetilde{F}}(e)(y))\geq s$$

and hence, $x+y\in U_{T_{\widetilde{F}(e)}}(s)$, $\alpha x\in U_{T_{\widetilde{F}(e)}}(s)$, and $[x,y]\in U_{T_{\widetilde{F}(e)}}$. Thus, $U_{T_{\widetilde{F}(e)}}(s)$, forms a Lie subalgebra of L. For the case $U_{I_{\widetilde{F}(e)}}\,(s)$ and $V_{F_{\widetilde{F}(e)}}\,(s)$ the proof is analogously.

Conversely, suppose that $U_{T_{\widetilde{F}(e)}}(s)\neq\emptyset$, is a Lie subalgebra of L for every $s\in[0,1]$ and $e\in E$.

Assume that

$$T_{\widetilde{F}}\,(e)\,(x+y)<\min\left\{T_{\widetilde{F}}(e)\,(x)\,,T_{\widetilde{F}}(e)\,(y)\right\}$$

For same $x,y\in L$ now, taking

$$S_O:=\frac{1}{2}\left\{T_{\widetilde{F}}(e)(x+y),\min\left\{T_{\widetilde{F}}(e)(x),T_{\widetilde{F}}(e)(y)\right\}\right\},$$

Then we have

$$T_{\widetilde{F}}\,(e)\,(x+y)<S_0\ <\min\left\{T_{\widetilde{F}}\,(e)\,(x)\,,T_{\widetilde{F}}\,(e)\,(y)\right\}$$

And hence $x+y\in U_{T_{\widetilde{F}(e)}}\ (s)$ and $x\in U_{T_{\widetilde{F}(e)}}(s)$ and $y\in U_{T_{\widetilde{F}(e)}}(s)$.

However, this is clearly a contradiction.

Therefore,

$$T_{\widetilde{F}}\,(e)\,(x+y)\geq\min\left\{T_{\widetilde{F}}(e)\,(x)\,,T_{\widetilde{F}}(e)\,(y)\right\}$$

For all $x,y\in L$ similarly we can show that:

$$T_{\widetilde{F}}\,(e)\,(\alpha x)\geq T_{\widetilde{F}}\,(e)\,(x)\,,$$

$$T_{\widetilde{F}}\,(e)\,([x,y])\geq\min\left\{T_{\widetilde{F}}\,(e)\,(x)\,,T_{\widetilde{F}}\,(e)\,(y)\right\}$$

For the case $U_{I_{\widetilde{F}(e)}}(\ s)$ and $V_{F_{\widetilde{F}(e)}}\,(s)$ the proof is similar. □

Theorem 3.4. *If* $\left(\widetilde{F}^1, E_1\right)$ *and* $\left(\widetilde{F}^2, E_2\right)$ *be two neutrosophic soft Lie subalgebra over* L, *then intersection* $\left(\widetilde{F}^1, E_1\right) \cap \left(\widetilde{F}^2, E_2\right) = (\widetilde{F}^3, E_1 \cap E_2)$ *is a neutrosophic soft Lie subalgebra over* L.

Proof. For each $x, y \in L$, $e \in E_1 \cap E_2$,

$$T_{\widetilde{F}^3}(e)(x+y) = \min\left\{T_{\widetilde{F}^1}(e)(x+y), T_{\widetilde{F}^2}(e)(x+y)\right\} \geq$$
$$\geq \min\left\{\min\left\{T_{\widetilde{F}^1}(e)(x), T_{\widetilde{F}^1}(e)(y)\right\}, \min\left\{T_{\widetilde{F}^2}(e)(x), T_{\widetilde{F}^2}(e)(y)\right\}\right\} =$$
$$= \min\{\min\{T_{\widetilde{F}^1}(e)(x), T_{\widetilde{F}^2}(e)(x)\}, \min\{T_{\widetilde{F}^1}(e)(y), T_{\widetilde{F}^2}(e)(y)\}\} =$$
$$= \min\{T_{\widetilde{F}^3}(e)(x), T_{\widetilde{F}^3}(e)(y)\}$$
$$I_{\widetilde{F}^3}(e)(x+y) = \min\left\{I_{\widetilde{F}^1}(e)(x+y), I_{\widetilde{F}^2}(e)(x+y)\right\} \geq$$
$$\geq \min\left\{\min\left\{I_{\widetilde{F}^1}(e)(x), I_{\widetilde{F}^1}(e)(y)\right\}, \min\left\{I_{\widetilde{F}^2}(e)(x), I_{\widetilde{F}^2}(e)(y)\right\}\right\} =$$
$$= \min\left\{\min\left\{I_{\widetilde{F}^1}(e)(x), I_{\widetilde{F}^2}(e)(x)\right\}, \min\left\{I_{\widetilde{F}^1}(e)(y), I_{\widetilde{F}^2}(e)(y)\right\}\right\} =$$
$$= \min\left\{I_{\widetilde{F}^3}(e)(x), I_{\widetilde{F}^3}(e)(y)\right\}$$
$$F_{\widetilde{F}^3}(e)(x+y) = \max\left\{F_{\widetilde{F}^1}(e)(x+y), F_{\widetilde{F}^2}(e)(x+y)\right\} \leq$$
$$\leq \max\{\max\{F_{\widetilde{F}^1}(e)(x), F_{\widetilde{F}^1}(e)(y)\}, \max\{F_{\widetilde{F}^2}(e)(x), F_{\widetilde{F}^2}(e)(y)\}\}$$
$$= \max\left\{\max\{F_{\widetilde{F}^1}(e)(x), F_{\widetilde{F}^2}(e)(x)\right\}, \max\{F_{\widetilde{F}^1}(e)(y), F_{\widetilde{F}^2}(e)(y)\}\} =$$
$$= \max\left\{F_{\widetilde{F}^3}(e)(x), F_{\widetilde{F}^3}(e)(y)\right\}$$
$$T_{\widetilde{F}^3}(e)(\alpha x) = \min\left\{T_{\widetilde{F}^1}(e)(\alpha x), T_{\widetilde{F}^2}(e)(\alpha x)\right\} \geq$$
$$\geq \min\left\{T_{\widetilde{F}^1}(e)(x), T_{\widetilde{F}^2}(e)(x)\right\} = T_{\widetilde{F}^3}(e)(x)$$
$$I_{\widetilde{F}^3}(e)(\alpha x) = \min\left\{I_{\widetilde{F}^1}(e)(\alpha x), I_{\widetilde{F}^2}(e)(\alpha x)\right\} \geq$$
$$\geq \min\left\{I_{\widetilde{F}^1}(e)(x), I_{\widetilde{F}^2}(e)(x)\right\} = I_{\widetilde{F}^3}(e)(x)$$
$$F_{\widetilde{F}^3}(e)(\alpha x) = \max\left\{F_{\widetilde{F}^1}(e)(\alpha x), F_{\widetilde{F}^2}(e)(\alpha x)\right\} \leq$$
$$\leq \max\left\{F_{\widetilde{F}^1}(e)(x), F_{\widetilde{F}^2}(e)(x)\right\} = F_{\widetilde{F}^3}(e)(x)$$

$$\begin{aligned}
&T_{\tilde{F}^3}(e)[x,y] = \min\left\{T_{\tilde{F}^1}(e)[x,y], T_{\tilde{F}^2}(e)[x,y]\right\} \geq \\
&\geq \min\left\{\min\left\{T_{\tilde{F}^1}(e)(x), T_{\tilde{F}^1}(e)(y)\right\}, \min\left\{T_{\tilde{F}^2}(e)(x), T_{\tilde{F}^2}(e)(y)\right\}\right\} = \\
&= \min\left\{\min\left\{T_{\tilde{F}^1}(e)(x), T_{\tilde{F}^2}(e)(x)\right\}, \min\left\{T_{\tilde{F}^1}(e)(y), T_{\tilde{F}^2}(e)(y)\right\}\right\} = \\
&= \min\left\{T_{\tilde{F}^3}(e)(x), T_{\tilde{F}^3}(e)(y)\right\}
\end{aligned}$$

$$\begin{aligned}
&I_{\tilde{F}^3}(e)[x,y] = \min\left\{I_{\tilde{F}^1}(e)[x,y], I_{\tilde{F}^2}(e)[x,y]\right\} \geq \\
&\geq \min\left\{\min\left\{I_{\tilde{F}^1}(e)(x), I_{\tilde{F}^1}(e)(y)\right\}, \min\left\{I_{\tilde{F}^2}(e)(x), I_{\tilde{F}^2}(e)(y)\right\}\right\} = \\
&= \min\left\{\min\left\{I_{\tilde{F}^1}(e)(x), I_{\tilde{F}^2}(e)(x)\right\}, \min\left\{I_{\tilde{F}^1}(e)(y), I_{\tilde{F}^2}(e)(y)\right\}\right\} = \\
&= \min\left\{I_{\tilde{F}^3}(e)(x), I_{\tilde{F}^3}(e)(y)\right\}
\end{aligned}$$

$$F_{\tilde{F}^3}(e)[x,y] = \max\left\{F_{\tilde{F}^1}(e)[x,y], F_{\tilde{F}^2}(e)[x,y]\right\} \leq$$
$$\leq \max\left\{\max\left\{F_{\tilde{F}^1}(e)(x), F_{\tilde{F}^1}(e)(y)\right\}, \max\left\{F_{\tilde{F}^2}(e)(x), F_{\tilde{F}^2}(e)(y)\right\}\right\} =$$
$$= \max\left\{\max\left\{F_{\tilde{F}^1}(e)(x), F_{\tilde{F}^2}(e)(x)\right\}, \max\left\{F_{\tilde{F}^1}(e)(y), F_{\tilde{F}^2}(e)(y)\right\}\right\} =$$
$$= \max\left\{F_{\tilde{F}^3}(e)(x), F_{\tilde{F}^3}(e)(y)\right\}$$

□

Theorem 3.5. *Let* $\left(\widetilde{F}^1, E_1\right)$ *and* $\left(\widetilde{F}^2, E_2\right)$ *be two neutrosophic soft Lie subalgebra over* L. *If* $E_1 \cap E_2 = \emptyset$, *then union* $\left(\widetilde{F}^1, E_1\right) \cup \left(\widetilde{F}^2, E_2\right) = (\widetilde{F}^3, E_1 \cup E_2)$ *is a neutrosophic soft Lie subalgebra over* L.

Proof. Since $E_1 \cap E_2 = \emptyset$, it follows that either $e \in E_1$ or $e \in E_2$ for all $e \in E_3$. If $e \in E_1$, then $(\widetilde{F}^3, E_1 \cup E_2) = (\widetilde{F}^1, E_1)$ is a neutrosophic soft Lie subalgebra of L, and if $e \in E_2$, then $(\widetilde{F}^3, E_1 \cup E_2) = (\widetilde{F}^2, E_2)$ is a neutrosophic soft Lie subalgebra of L. Hence $\left(\widetilde{F}^1, E_1\right) \cup \left(\widetilde{F}^2, E_2\right)$ is a neutrosophic soft Lie subalgebraover L. □

Theorem 3.6. $\left(\widetilde{F}, E\right)$ *be a neutrosophic soft Lie subalgebra over* L, *and let* $\left[\left(\widetilde{F}_i, E_i\right)\right]_{i\epsilon I}$ *be nonempty family of neutrosophic soft Lie subalgebra of* L. *Then*
1) $\prod_{i\epsilon I} \left(\widetilde{F}_i, E_i\right)$ *is a neutrosophic soft Lie subalgebra of* L,
2) If $E_i \cap E_j = \emptyset$, *for all* $i, j\epsilon I$, *then* $\bigcup_{i\in I} \left(\tilde{F}_i, E_i\right)$ *is a neutrosophic soft Lie subalgebra of* L.

Theorem 3.7. *Let* $\left(\widetilde{F}^1, E_1\right)$ *and* $\left(\widetilde{F}^2, E_2\right)$ *be two neutrosophic soft Lie algebras over* L_1 *and* L_2 *respectively. Then* $\left(\widetilde{F}^1, E_1\right) \wedge \left(\widetilde{F}^2, E_2\right) = \left(\tilde{F}^3, E_1 \times E_2\right)$ *is a neutrosophic soft Lie algebra over* L.

Proof. For each $x, y \in L$, $e_1 \in E_1, e_2 \in E_2$, $\alpha \in K$,

$$T_{\tilde{F}^3}(e_1, e_2)(x+y) = T_{\tilde{F}^1}(e_1)(x+y) \wedge T_{\tilde{F}^2}(e_2)(x+y) \geq$$
$$\geq \left\{\min\left(T_{\tilde{F}^1}(e_1)(x), T_{\tilde{F}^1}(e_1)(y)\right)\right\} \wedge \left\{\min\left(T_{\tilde{F}^2}(e_2)(x), T_{\tilde{F}^2}(e_2)(y)\right)\right\} =$$
$$= \min\left\{T_{\tilde{F}^1}(e_1)(x), T_{\tilde{F}^2}(e_2)(x)\right\} \wedge \min\left\{T_{\tilde{F}^1}(e_1)(y), T_{\tilde{F}^2}(e_2)(y)\right\} =$$
$$= T_{\tilde{F}^3}(e_1, e_2)(x) \wedge T_{\tilde{F}^3}(e_1, e_2)(y)$$

$$I_{\tilde{F}^3}(e_1, e_2)(x+y) = I_{\tilde{F}^1}(e_1)(x+y) \wedge I_{\tilde{F}^2}(e_2)(x+y) \geq$$
$$\geq \left\{\min\left(I_{\tilde{F}^1}(e_1)(x), I_{\tilde{F}^1}(e_1)(y)\right)\right\} \wedge \left\{\min\left(I_{\tilde{F}^2}(e_2)(x), I_{\tilde{F}^2}(e_2)(y)\right)\right\} =$$
$$= \min\left\{I_{\tilde{F}^1}(e_1)(x), I_{\tilde{F}^2}(e_2)(x)\right\} \wedge \min\left\{I_{\tilde{F}^1}(e_1)(y), I_{\tilde{F}^2}(e_2)(y)\right\} =$$
$$= I_{\tilde{F}^3}(e_1, e_2)(x) \wedge I_{\tilde{F}^3}(e_1, e_2)(y)$$

$$F_{\tilde{F}^3}(e_1,e_2)(x+y) = F_{\tilde{F}^1}(e_1)(x+y) \vee F_{\tilde{F}^2}(e_2)(x+y) \leq$$
$$\leq \left\{\max\left(F_{\tilde{F}^1}(e_1)(x), F_{\tilde{F}^1}(e_1)(y)\right)\right\} \vee \left\{\max\left(F_{\tilde{F}^2}(e_2)(x), F_{\tilde{F}^2}(e_2)(y)\right)\right\} =$$
$$= \max\left\{F_{\tilde{F}^1}(e_1)(x), F_{\tilde{F}^2}(e_2)(x)\right\} \vee \max\left\{F_{\tilde{F}^1}(e_1)(y), F_{\tilde{F}^2}(e_2)(y)\right\} =$$
$$= F_{\tilde{F}^3}(e_1,e_2)(x) \vee F_{\tilde{F}^3}(e_1,e_2)(y)$$

$$T_{\tilde{F}^3}(e_1,e_2)(\alpha(x)) = \left(T_{\tilde{F}^1}(e_1) \wedge T_{\tilde{F}^2}(e_2)\right)(\alpha(x)) =$$
$$\left(T_{\tilde{F}^1}(e_1) \wedge T_{\tilde{F}^2}(e_2)\right)(\alpha x) = \min\left(T_{\tilde{F}^1}(e_1)(\alpha x), T_{\tilde{F}^2}(e_2)(\alpha x)\right) \geq$$
$$\min\left(T_{\tilde{F}^1}(e_1)(x), T_{\tilde{F}^2}(e_2)(x)\right) = \left(T_{\tilde{F}^1}(e_1) \wedge T_{\tilde{F}^2}(e_2)\right)(x) = T_{\tilde{F}^3}(e_1,e_2)(x),$$

$$I_{\tilde{F}^3}(e_1,e_2)(\alpha(x)) = \left(I_{\tilde{F}^1}(e_1) \wedge I_{\tilde{F}^2}(e_2)\right)(\alpha(x)) =$$
$$\left(I_{\tilde{F}^1}(e_1) \wedge I_{\tilde{F}^2}(e_2)\right)(\alpha x) = \min\left(I_{\tilde{F}^1}(e_1)(\alpha x), I_{\tilde{F}^2}(e_2)(\alpha x)\right) \geq$$
$$\min\left(I_{\tilde{F}^1}(e_1)(x), I_{\tilde{F}^2}(e_2)(x)\right) = \left(I_{\tilde{F}^1}(e_1) \wedge I_{\tilde{F}^2}(e_2)\right)(x) = I_{\tilde{F}^3}(e_1,e_2)(x),$$

$$F_{\tilde{F}^3}(e_1,e_2)(\alpha(x)) = \left(F_{\tilde{F}^1}(e_1) \vee F_{\tilde{F}^2}(e_2)\right)(\alpha(x)) =$$
$$\left(F_{\tilde{F}^1}(e_1) \vee F_{\tilde{F}^2}(e_2)\right)(\alpha x) = \max\left(F_{\tilde{F}^1}(e_1)(\alpha x), F_{\tilde{F}^2}(e_2)(\alpha x)\right) \leq$$
$$\max\left(F_{\tilde{F}^1}(e_1)(x), F_{\tilde{F}^2}(e_2)(x)\right) = \left(F_{\tilde{F}^1}(e_1) \vee F_{\tilde{F}^2}(e_2)\right)(x) = F_{\tilde{F}^3}(e_1,e_2)(x),$$

$$T_{\tilde{F}^3}(e_1,e_2)([x,y]) = T_{\tilde{F}^1}(e_1)([x,y]) \wedge T_{\tilde{F}^2}(e_2)([x,y]) \geq$$
$$\left\{\min\left(T_{\tilde{F}^1}(e_1)(x), T_{\tilde{F}^1}(e_1)(y)\right)\right\} \wedge \left\{\min\left(T_{\tilde{F}^2}(e_2)(x), T_{\tilde{F}^2}(e_2)(y)\right)\right\} =$$
$$\min\left\{T_{\tilde{F}^1}(e_1)(x), T_{\tilde{F}^2}(e_2)(x)\right\} \wedge \min\left\{T_{\tilde{F}^1}(e_1)(y), T_{\tilde{F}^2}(e_2)(y)\right\} =$$
$$T_{\tilde{F}^3}(e_1,e_2)(x) \wedge T_{\tilde{F}^3}(e_1,e_2)(y),$$

$$I_{\tilde{F}^3}(e_1,e_2)([x,y]) = I_{\tilde{F}^1}(e_1)([x,y]) \wedge I_{\tilde{F}^2}(e_2)([x,y]) \geq$$
$$\left\{\min\left(I_{\tilde{F}^1}(e_1)(x), I_{\tilde{F}^1}(e_1)(y)\right)\right\} \wedge \left\{\min\left(I_{\tilde{F}^2}(e_2)(x), I_{\tilde{F}^2}(e_2)(y)\right)\right\} =$$
$$\min\left\{I_{\tilde{F}^1}(e_1)(x), I_{\tilde{F}^2}(e_2)(x)\right\} \wedge \min\left\{I_{\tilde{F}^1}(e_1)(y), I_{\tilde{F}^2}(e_2)(y)\right\} =$$
$$I_{\tilde{F}^3}(e_1,e_2)(x) \wedge I_{\tilde{F}^3}(e_1,e_2)(y),$$

$$F_{\tilde{F}^3}(e_1,e_2)([x,y]) = F_{\tilde{F}^1}(e_1)([x,y]) \vee F_{\tilde{F}^2}(e_2)([x,y]) \leq$$
$$\leq \left\{\max\left(F_{\tilde{F}^1}(e_1)(x), F_{\tilde{F}^1}(e_1)(y)\right)\right\} \vee \left\{\max\left(F_{\tilde{F}^2}(e_2)(x), F_{\tilde{F}^2}(e_2)(y)\right)\right\} =$$
$$\max\left\{F_{\tilde{F}^1}(e_1)(x), F_{\tilde{F}^2}(e_2)(x)\right\} \vee \max\left\{F_{\tilde{F}^1}(e_1)(y), F_{\tilde{F}^2}(e_2)(y)\right\} =$$
$$F_{\tilde{F}^3}(e_1,e_2)(x) \vee F_{\tilde{F}^3}(e_1,e_2)(y).$$

□

Definition 3.8. Let $\left(\widetilde{F}^1, E_1\right)$ and $\left(\widetilde{F}^2, E_2\right)$ be two neutrosophic soft sets on a set L. Then the generalized Cartesian product $\left(\widetilde{F}^1, E\right) \times \left(\widetilde{F}^2, E\right) = (\widetilde{F}^1 \times \widetilde{F}^2, E_1 \times E_2)$ is defined as follow:

$$\widetilde{F}^1 \times \widetilde{F}^2 \; : E_1 \times E_2 \to NS\,(L)$$

$$\widetilde{F}^1 \times \widetilde{F}^2\,(e_1,e_2) = (T_{\widetilde{F}^1}(e_1) \times (T_{\widetilde{F}^2}(e_2), (I_{\widetilde{F}^1}(e_1) \times (I_{\widetilde{F}^2}(e_2), (F_{\widetilde{F}^1}(e_1) \times (F_{\widetilde{F}^2}(e_2)),$$

where,

$$(T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2)(x,y) = \min(T_{\widetilde{F}^1}(e_1)(x),\; T_{\widetilde{F}^2}(e_2)(y)),$$

$$(I_{\widetilde{F}^1}(e_1) \times (I_{\widetilde{F}^2}(e_2))(x,y) = \min(I_{\widetilde{F}^1}(e_1)(x), I_{\widetilde{F}^2}(e_2)(y)),$$

$$(F_{\widetilde{F}^1}(e_1) \times (F_{\widetilde{F^2}}(e_2))(x,y) = \max(F_{\widetilde{F}^1}(e_1)(x), F_{\widetilde{F}^2}(e_2)(y)).$$

For each $(e_1, e_2) \in E_1 \times E_2$

Theorem 3.9. *Let* $\left(\widetilde{F}^1\ ,E_1\right)$ *and* $\left(\widetilde{F}^2\ ,E_2\right)$ *be two neutrosophic soft Lie subalgebras of* L*, then is* $\left(\widetilde{F}^1\ ,E_1\right) \times \left(\widetilde{F}^2\ ,E_2\right)$ *is neutrosophic soft Lie subalgebra of* $L \times L$

Proof. Let $x = (x_1 x_2)$ and $y = (y_1 y_2) \in L \times L$. Then for each $(e_1 e_2) \in E_1 \times E_2$.

$$(T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2))(x+y) = (T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2)((x_1,x_2)+(y_1,y_2)) =$$
$$= (T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2))((x_1+y_1, x_2+y_2)) = \min(T_{\widetilde{F}^1}(e_1)(x_1+y_1), T_{\widetilde{F}^2}(e_2)(x_2+y_2))$$
$$\geq \min(\min(T_{\widetilde{F}^1}(e_1)(x_1), T_{\widetilde{F}^1}(e_1)(y_1)), \min(T_{\widetilde{F}^2}(e_2)(x_2), T_{\widetilde{F}^2}(e_2)(y_2))) =$$
$$= \min((T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2))(x_1,x_2)), \left(\left(T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2)\right)(y_1,y_2))\right) =$$
$$= \min\left((T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2))(x), \left(T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2)\right)(y))\right),$$
$$\left(I_{\widetilde{F}^1}(e_1) \times I_{\widetilde{F}^2}(e_2)\right)(x+y) = \left(I_{\widetilde{F}^1}(e_1) \times I_{\widetilde{F}^2}(e_2)\right)((x_1,x_2)+(y_1,y_2)) =$$
$$= (I_{\widetilde{F}^1}(e_1) \times I_{\widetilde{F}^2}(e_2))((x_1+y_1, x_2+y_2)) = \min(I_{\widetilde{F}^1}(e_1)(x_1+y_1), I_{\widetilde{F}^2}(e_2)(x_2+y_2))$$
$$\geq \min(\min(I_{\widetilde{F}^1}(e_1)(x_1), I_{\widetilde{F}^1}(e_1)(y_1)), \min(I_{\widetilde{F}^2}(e_2)(x_2), I_{\widetilde{F}^2}(e_2)(y_2))) =$$
$$= \min((I_{\widetilde{F}^1}(e_1) \times I_{\widetilde{F}^2}(e_2))(x_1 x_2)), \left(\left(I_{\widetilde{F}^1}(e_1) \times I_{\widetilde{F}^2}(e_2)\right)(y_1,y_2))\right) =$$
$$= \min\left((I_{\widetilde{F}^1}(e_1) \times I_{F^2}(e_2))(x), \left(I_{\widetilde{F}^1}(e_1) \times I_{\widetilde{F}^2}(e_2)\right)(y)\right),$$
$$\left(F_{\widetilde{F}^1}(e_1) \times F_{\widetilde{F}^2}(e_2)\right)(x+y) = \left(F_{\widetilde{F}^1}(e_1) \times F_{\widetilde{F}^2}(e_2)\right)((x_1,x_2)+(y_1,y_2))) =$$
$$= \left(F_{\widetilde{F}^1}(e_1) \times F_{\widetilde{F}^2}(e_2)\right)((x_1+y_1, x_2+y_2)) =$$
$$= \max(F_{\widetilde{F}^1}(e_1)(x_1+y_1), F_{\widetilde{F}^2}(e_2)(x_2+y_2)) \leq \max(\max(F_{\widetilde{F}^1}(e_1)(x_1), F_{\widetilde{F}^1}(e_1)(y_1)),$$
$$\max(F_{\widetilde{F}^2}(e_2)(x_2), F_{\widetilde{F}^2}(e_2)(y_2))) = \max((F_{\widetilde{F}^1}(e_1) \times F_{\widetilde{F}^2}(e_2))(x_1,x_2)),$$
$$\left(\left(F_{\widetilde{F}^1}(e_1) \times F_{\widetilde{F}^2}(e_2)\right)(y_1,y_2))\right) = \max\left((F_{\widetilde{F}^1}(e_1) \times F_{\widetilde{F}^2}(e_2))(x), \left(F_{\widetilde{F}^1}(e_1) F_{\widetilde{F}^2}(e_2)\right)(y))\right).$$
$$\left(T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2)\right)(\alpha x) = \left(T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2)\right)(\alpha(x_1,x_2)) =$$
$$\left(T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2)\right)((\alpha x_1, \alpha x_2)) = \min\left(T_{\widetilde{F}^1}(e_1)(\alpha x_1), T_{\widetilde{F}^2}(e_2)(\alpha x_1)\right) \geq$$
$$\geq \min(T_{\widetilde{F}^1}(e_1)(x_1), T_{\widetilde{F}^2}(e_2)(x_2)) = (T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2))(x_1,x_2))$$
$$= \left(T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2)\right)(x),$$
$$(I_{\tilde{F}^1}(e_1) \times I_{\tilde{F}^2}(e_2))(\alpha x) = (I_{\tilde{F}^1}(e_1) \times I_{\tilde{F}^2}(e_2))(\alpha(x_1,x_2)) = (I_{\tilde{F}^1}(e_1) \times I_{\tilde{F}^2}(e_2))((\alpha x_1, \alpha x_2)) =$$
$$= \min(I_{\tilde{F}^1}(e_1)(\alpha x_1), I_{\tilde{F}^2}(e_2)(\alpha x_1)) \geq \min(I_{\tilde{F}^1}(e_1)(x_1), I_{\tilde{F}^2}(e_2)(x_2)) =$$
$$= \left(I_{\tilde{F}^1}(e_1) \times I_{\tilde{F}^2}(e_2)\right)(x_1 x_2) = \left(I_{\tilde{F}^1}(e_1) \times I_{\tilde{F}^2}(e_2)\right)(x),$$
$$\left(F_{\tilde{F}^1}(e_1) \times F_{\tilde{F}^2}(e_2)\right)(\alpha x) = \left(F_{\tilde{F}^1}(e_1) \times F_{\tilde{F}^2}(e_2)\right)(\alpha(x_1,x_2)) =$$

$$= \left(F_{\tilde{F}^1}(e_1) \times F_{\tilde{F}^2}(e_2)\right)((\alpha x_1, \alpha x_2)) = \max\left(F_{\tilde{F}^1}\left(e_1\right)(\alpha x_1), F_{\tilde{F}^2}(e_2)\left(\alpha x_1\right)\right) \leq$$

$$\leq \max\left(F_{\tilde{F}^1}\left(e_1\right)(x_1), F_{\tilde{F}^2}(e_2)\left(x_2\right)\right) = \left(F_{\tilde{F}^1}(e_1) \times F_{\tilde{F}^2}(e_2)\right)(x_1, x_2) =$$

$$= \left(F_{\tilde{F}^1}(e_1) \times F_{\tilde{F}^2}\ (e_2)\right)(x),$$

$$\left(T_{\tilde{F}^1}\ (e_1) \times T_{\tilde{F}^2}\ (e_2)\right)([x,y]) = \left(T_{\tilde{F}^1}\ (e_1) \times T_{\tilde{F}^2}\ (e_2)\right)([(x_1,x_2)+(y_1,y_2)]) \geq$$

$$\geq \min(\min(T_{\tilde{F}^1}(e_1)(x_1), T_{\tilde{F}^1}(e_2)(x_2)), \min(T_{\tilde{F}^1}(e_1)(y_1), T_{\tilde{F}^2}(e_2)(y_2))) =$$

$$\min\left(\left(T_{\tilde{F}^1}(e_1) \times T_{\tilde{F}^2}(e_2)\right)(x_1,x_2), \left(T_{\tilde{F}^1}\ (e_1) \times T_{\tilde{F}^2}\ (e_2)\right)(y_1,y_2)\right) =$$

$$= \min((T_{\tilde{F}^1}(e_1) \times T_{\tilde{F}^2}(e_2))(x), \left(T_{\tilde{F}^1}(e_1) \times T_{\tilde{F}^2}(e_2)\right)(y)),$$

$$\left(I_{\tilde{F}^1}(e_1) \times I_{\tilde{F}^2}(e_2)\right)([x,y]) = \left(I_{\tilde{F}^1}(e_1) \times I_{\tilde{F}^2}(e_2)\right)([(x_1,x_2)+(y_1,y_2)]) \geq$$

$$\geq \min(\min(I_{\tilde{F}^1}(e_1)\left(x_1\right), I_{\tilde{F}^2}(e_2)\left(x_2\right)), \min(I_{\tilde{F}^1}(e_1)\left(y_1\right), I_{\tilde{F}^2}(e_2)\left(y_2\right)) =$$

$$= \min((I_{\tilde{F}^1}(e_1) \times I_{\tilde{F}^2}(e_2))(x_1,x_2), (I_{\tilde{F}^1}(e_1) \times I_{\tilde{F}^2}(e_2))(y_1,y_2)) =$$

$$= \min\left(\left(I_{\tilde{F}^1(e_1}) \times I_{\tilde{F}^2}\ (e_2)\right)(x), \left(I_{\tilde{F}^1}(e_1) \times I_{\tilde{F}^2}(e_2)\right)(y)\right),$$

$$\left(F_{\tilde{F}^1}(e_1) \times F_{\tilde{F}^2}(e_2)\right)([x,y]) = \left(F_{\tilde{F}^1}\ (e_1) \times F_{\tilde{F}^2}\ (e_2)\right)([(x_1),x_2)+(y_1,y_2)]) \leq$$

$$\leq \max(\max(F_{\tilde{F}^1}(e_1)\left(x_1\right), F_{\tilde{F}^2}(e_2)\left(x_2\right)), \max(F_{\tilde{F}^1}(e_1)\left(y_1\right), F_{\tilde{F}^2}\ (e_2)\left(y_2\right)) =$$

$$= \max((F_{\tilde{F}^1}(e_1) \times F_{\tilde{F}^2}(e_2))(x_1,x_2), (F_{\tilde{F}^1}(e_1) \times F_{\tilde{F}^2}(e_2))(y_1,y_2)) =$$

$$\max((F_{\tilde{F}^1}(e_1) \times F_{\tilde{F}^2}(e_2))(x)), (F_{\tilde{F}^1}(e_1) \times F_{\tilde{F}^2}(e_2))(y)).$$

□

This shows that $\left(\tilde{F}^1, E_1\right) \times \left(\tilde{F}^2, E_2\right)$ is a neutrosophic soft Lie subalgebra of $L \times L$.

Definition 3.10. Let $\left(\tilde{F}^1, E_1\right)$ and $\left(\tilde{F}^2, E_2\right)$be two neutrosophic soft Lie algebras over L_1 and L_2 respectively, and let $f : L_1 \to L_2$ be a homomorphism of Lie algebras, and let $g : E_1 \to E_2$ be a mapping of sets. Then we say that $(f,g) : \left(L_1, \left(\tilde{F}^1, E_1\right)\right) \to \left(L_2, \left(\tilde{F}^2, E_2\right)\right)$ is a neutrosophic soft Lie homomorphism of neutrosophic soft Lie algebras, if the following condition is satisfied:

$$f\left(T_{\tilde{F}^1}\left(e\right)\right) = \tilde{F}^2\left(g\left(e\right)\right) = T_{\tilde{F}^2}\left(g\left(e\right)\right)$$
$$f\left(I_{\tilde{F}^1}\left(e\right)\right) = \tilde{F}^2\left(g\left(e\right)\right) = I_{\tilde{F}^2}\left(g\left(e\right)\right)$$
$$f\left(F_{\tilde{F}^1}\left(e\right)\right) = \tilde{F}^2\left(g\left(e\right)\right) = F_{\tilde{F}^2}\left(g\left(e\right)\right)$$

For the Lie algebras L_1 and L_2 it can be easily observed that if $f : L_1 \to L_2$ is a Lie homomorphism $g : E \to E'$ map of sets and $\left(\tilde{F}, E'\right)$ is neutrosophic soft Lie subalgebra of L_2, then the neutrosophic soft set $f^{-1}\left(\tilde{F}, E\right)$ of L_1 is also a neutrosophic soft Lie subalgebra, where,

$$f^{-1}\left(T_{\tilde{F}}\left(e\right)\right)(x) = T_{\tilde{F}}\left(e\right)\left(f\left(x\right)\right)$$
$$f^{-1}\left(I_{\tilde{F}}\left(e\right)\right)(x) = I_{\tilde{F}}\left(e\right)\left(f\left(x\right)\right)$$
$$f^{-1}\left(F_{\tilde{F}}\left(e\right)\right)(x) = F_{\tilde{F}}\left(e\right)\left(f\left(x\right)\right)$$

Definition 3.11. Let L_1 and L_2 be two Lie algebras, $f : L_1 \to L_2$ is a Lie homomorphism,and let $\left(\tilde{F}, E\right)$ be neutrosophic soft set over L_1, $g : E \to E'$ map of sets then image of (f, g) is defined by:

$$f\left(T_{\tilde{F}}\right)(e)(y) = \sup\left\{T_{\tilde{F}}(e)(x) : x \in f^{-1}(y), e \in g^{-1}(e^{'})\right\}$$
$$f\left(I_{\tilde{F}}\right)(e)(y) = \sup\left\{I_{\tilde{F}}(e)(x) : x \in f^{-1}(y), e \in g^{-1}(e^{'})\right\}$$
$$f\left(F_{\tilde{F}}\right)(e)(y) = \inf\left\{F_{\tilde{F}}(e)(x) : x \in f^{-1}(y), e \in g^{-1}(e^{'})\right\}$$

for $\forall e \in E$, $\forall\, y \in Y$.

Theorem 3.12. *Let $f : L_1 \to L_2$ ephimorfizm of Lie algebras and $\left(\tilde{F}, E\right)$ neutrosophic soft Lie subalgebra of L_1,then the homomorphic image of $\left(\tilde{F}, E\right)$ is neutrosophic soft Lie subalgebra of L_2.*

Proof. Let $y_1, y_2 \in L_2$. Then,

$$\left\{x \mid x{\in}f^{-1}(y_1{+}y_2)\right\} \supseteq \left\{x_1{+}x_2 \mid x_1{\in}f^{-1}(y_1) \text{ and } x_2{\in}f^{-1}(y_2)\right\}.$$

Now, we have, for each $e \in E$

$$f\left(T_{\tilde{F}}(e^{'})\right)(y_1 + y_2) = \sup\left\{T_{\tilde{F}}(e)(x) \mid x \in f^{-1}(y_1 + y_2), e \in g^{-1}(e^{'})\right\}$$
$$\geq \sup\left\{T_{\tilde{F}}(e)(x_1 + x_2), \mid x_1 \in f^{-1}(y_1) \text{ and } \quad x_2 \in f^{-1}(y_2), e \in g^{-1}(e')\right\} \geq$$
$$\geq \sup\left\{\min\{T_{\tilde{F}}(e)(x_1), T_{\tilde{F}}(e)(x_2)\} \mid x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2), e \in g^{-1}(e')\right\} =$$
$$= \min\{\{\sup\left(T_{\tilde{F}}(e)\right)(x_1) \mid x_1 \in f^{-1}(y_1)\, e \in g^{-1}(e')\}, \{\sup T_{\tilde{F}}(e)(x_2) | x_2 \in f^{-1}(y_2), e \in g^{-1}(e^{'})\}\} =$$
$$= \min\left\{f(T_{\tilde{F}}(e'))(y_1), f(T_{\tilde{F}}(e'))(y_2)\right\}$$

For $y_2 \in L_2$ and $\alpha \in K$ we have

$$\left\{x \mid x{\in}f^{-1}(\alpha y)\right\} \supseteq \left\{\alpha x \mid x{\in}f^{-1}(y)\right\}.$$

$$f\left(T_{\tilde{F}}(e)\right)(\alpha y) = \sup\{T_{\tilde{F}}(e)(\alpha x) \mid x \in f^{-1}(y), e \in g^{-1}(e')\}$$
$$\geq \sup\left\{T_{\tilde{F}}(e)(x), |x \in f^{-1}(y), e \in g^{-1}(e^{'})\right\} = f\left(T_{\tilde{F}}(e^{'})\right)(y).$$

If $y_1, y_2 \in L_2$ then

$$\left\{x | x \in f^{-1}([y_1, y_2])\right\} \supseteq \left\{[x_1, x_2] | x_1 \in f^{-1}(y_1),\ x_2 \in f^{-1}(y_2)\right\}$$

Now

$$f\left(T_{\tilde{F}}(e')\right)([y_1, y_2]) = \sup\{\, T_{\tilde{F}}(e)(x) \;|x{\in}f^{-1}([y_1, y_2]), e \in g^{-1}(e')\}$$
$$\geq \sup\{T_{\tilde{F}}(e)[x_1, x_2], | x_1{\in}f^{-1}(y_1) \text{ and } x_2{\in}f^{-1}(y_2), e \in g^{-1}(e')\}$$
$$\geq \sup\left\{\min\left\{T_{\tilde{F}}(e)(x_1), T_{\tilde{F}}(e)(x_2)\right\} \mid x_1{\in}f^{-1}(y_1) \text{ and } x_2{\in}f^{-1}(y_2), e \in g^{-1}(e')\right\}$$
$$= \min\{\{\sup T_{\widetilde{F}}(e)(x_1) | x_1 \in f^{-1}(y_1), e \in g^{-1}(e')\}, \{\sup T_{\widetilde{F}}(e)(x_2) | x_2 \in f^{-1}(y_2), e \in g^{-1}(e')\}\}$$
$$= \min\{f\left(T_{\tilde{F}}(e)\right)(y_1), f\left(T_{\tilde{F}}(e)\right)(y_2)\}.$$

Now , we can easily proof for $f\left(I_{\tilde{F}}(e')\right)(y_1+y_2)\geq\min\{f\left(I_{\tilde{F}}(e')\right)(y_1),\ f\left(I_{\tilde{F}}(e')\right)(y_2)\}$

$$f(I_{\tilde{F}}(e'))(\alpha y)\geq f(I_{\tilde{F}}(e'))(y)$$

$$f\left(I_{\tilde{F}}(e')\right)([y_1,y_2])\geq\min\{f\left(I_{\tilde{F}}(e')\right)(y_1),f\left(I_{\tilde{F}}(e')\right)(y_2)\}$$

$$f\left(F_{\tilde{F}}(e')\right)(y_1+y_2)=\inf\{\ F_{\tilde{F}}(e)(x)\ \ |x\in f^{-1}(y_1+y_2),e\in g^{-1}(e')\}$$

$$\leq\inf\{F_{\tilde{F}}(e')(x_1+x_2),|x_1\in f^{-1}(y_1)\ \ ,x_2\in f^{-1}(y_2),e\in g^{-1}(e')\}$$

$$\leq\inf\left\{\max\{F_{\tilde{F}}(e)(x_1),F_{\tilde{F}}(e)(x_2)\}\ \middle|\ x_1\in f^{-1}(y_1)\ \ ,x_2\in f^{-1}(y_2),e\in g^{-1}(e')\right\}$$

$$=\max\left\{\left\{\inf F_{\tilde{F}}(e)(x_1),|x_1\in f^{-1}(y_1),e\in g^{-1}(e')\right\},\left\{\inf F_{\tilde{F}}(e)(x_2)|x_2\in f^{-1}(y_2),e\in g^{-1}(e')\right\}\right\}$$

$$=\max\{f\left(F_{\tilde{F}}(e')\right)(y_1),f\left(F_{\tilde{F}}(e')\right)(y_2)\}$$

For $y\in L_2$ and $\alpha\in K$ we have

$$f\left(F_{\tilde{F}}(e')\right)(\alpha y)=\inf\ \{\ F_{\tilde{F}}(e)(\alpha x)\ \ |x\in f^{-1}(y),e\in g^{-1}(e')\}$$

$$\leq\inf\left\{F_{\tilde{F}}(e)(x),\middle|\ x\in f^{-1}(y),e\in g^{-1}(e')\right\}=f\left(F_{\widetilde{F}}(e')\right)(y)$$

Now

$$f\left(F_{\widetilde{F}}(e')\right)([y_1,y_2])=\inf\{\ \mathrm{F}_{\widetilde{\mathrm{F}}}(\mathrm{e})(\mathrm{x})\ \ |\mathrm{x}\in\mathrm{f}^{-1}([\mathrm{y}_1,\mathrm{y}_2],e\in g^{-1}(e')\}$$

$$\leq\inf\left\{\mathrm{F}_{\widetilde{\mathrm{F}}}(\mathrm{e})[\mathrm{x}_1,\mathrm{x}_2],\middle|\ \mathrm{x}_1\in\mathrm{f}^{-1}(\mathrm{y}_1)\ \ ,\mathrm{x}_2\in\mathrm{f}^{-1}(\mathrm{y}_2),e\in g^{-1}(e')\right\}$$

$$\leq\inf\left\{\max\{\mathrm{F}_{\widetilde{\mathrm{F}}}(\mathrm{e})(\mathrm{x}_1),\mathrm{F}_{\widetilde{\mathrm{F}}}(\mathrm{e})(\mathrm{x}_2)\}\ \middle|\ \ \mathrm{x}_1\in\mathrm{f}^{-1}(\mathrm{y}_1),\ \ \mathrm{x}_2\in\mathrm{f}^{-1}(\mathrm{y}_2),e\in g^{-1}(e')\right\}$$

$$=\max\{\{\inf(F_{\widetilde{F}(e)}(x_1)|x_1\in f^{-1}(y_1),e\in g^{-1}(e')\},\{\inf F_{\widetilde{F}(e)}(x_2))|x_2\in f^{-1}(y_2),e\in g^{-1}(e')\}$$

$$=\max\{f(F_{\widetilde{F}}(e)(y_1),f(F_{\widetilde{F}}(e)(y_2)\}$$

Thus $f\left(\left(\widetilde{F},E\right)\right)=\left(f\left(T_{\widetilde{F}}(e')\right),f\left(I_{\widetilde{F}}(e')\right),f\left(F_{\widetilde{F}}(e')\right)\right)$ is a neutrosophic soft Lie algebra of L_2 . □

4. Conclusion

There we have introduced the concept of neutrosophic soft Lie subalgebras of a Lie algebra and investigated some of their properties.

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Neutrosophic Spherical Cubic Sets

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Abstract: In this paper, a new concept of Neutrosophic Spherical Cubic Set (NSCS) is introduced as an amalgamation of sets such as Neutrosophic, Interval valued, cubic and spherical sets. We studied the concepts of internal and external neutrosophic spherical cubic sets and discussed their basic properties. Further P-order, P-union, P-intersection as well as R-order, R-union, R-intersection are discussed for NSCSs.

Keywords: Neutrosophic set(NS); Neutrosophic spherical set (NSS),Neutrosophic cubic set(NCS);Neutrosophic spherical cubic sets(NSCSs)internalneutrosophic spherical cubic set (IntNSCS) and externalneutrosophic spherical cubic set(ExtNSCS). Truth Internal/External- $\mathcal{R}-$ Int/Ext , Inderterminacy Internal/External- $\mathcal{J}$ Int/Ext ,Falsity Internal/External $-\mathcal{S}$ Int/Ext

1. Introduction

Zadeh [12] established the fuzzy set notion in 1965 to cope with probabilistic uncertainty associated with inaccuracy of events, observations and desires.By the idea of fuzziness, the value of 1 is allocated to an object that is fully within the set and value of 0 is allocated to an object that is totally outside the set, then

$\circ$

any item partially inside the set will have a value ranging between 0 and 1, Fuzzy set along with it generalizations has many real life applications [9,10,11].

Jun et al. [2] proposed cubic set which is a hybrid of fuzzy sets and interval valued fuzzy sets. They also examined internal (external) cubic sets. By adding the falsehood (f), the degree of non-membership, and various properties.

In 1995,Smarandache [7,8] presented the concept of neutrosophicsets and neutrosophic logic. Neutrosophy lays the groundwork for plenty of new mathematical theories that encompass classical and fuzzy analogues. There are three defining functions in neutrosophic set they are truth T, indeterminate I and false membership function F all of which are defined on a universe of discourse X.These three functions are totally self-contained. The formation, nature, and extent of neutralities are all investigated in the Neutrosophicset. The idea ofneutrosophic set is a generalization of idea of a classical fuzzy set and so on.

KutluGundogu, Fatmaa, Kahraman, Cengiz [5,6] developed spherical fuzzy sets and spherical fuzzy TOPSIS method. They introduced generalized three dimensional spherical fuzzy sets (SFS) including some essential differences from the other fuzzy sets.

The spherical fuzzy set is a more dominant structures for coping with these situations. The idea behind spherical fuzzy set is to let decision makers to generalize other extensions of fuzzy sets by defining a membership function on a spherical surfaces and independently assign the parameters of that function with a larger domain.

The motive of the paper is to introduce a new concept called NSCSs and to study theINSCSs and ENSCSs that is truth, indeterminacy,falsity internaland truth, indeterminacy, falsity external respectively. Also, we have investigated their properties. We showed that P-union and the P-intersection of INSCSs are also theINSCSs. Examples are given to show that P-union and the P-intersection of ENSCSs may not be ENSCSs. R-union and the R- intersection of INSCSsmaynotbe INSCS.Also, we have given theconditionsfortheR-unionoftwoT-INSCSs(resp. I-INSCSsandF-INSCSs)tobeaT-INSCSs(resp.I-INSCSs and F-INSCSs) NSCS.Some of the fundamental properties of NSCSs were also investigated.

2. PRELIMINARIES

FUZZY SET [12]

A fuzzy set in a set S is defined to be a function $\mu : S \to [0,1]$. Define a relation $\leq$ on $[0,1]^S$ as follows $\left(\forall \mu, \lambda \in [0,1]^S\right), \left(\mu \leq \lambda \Leftrightarrow (\forall s \in S)(\mu(s) \leq \lambda(s)\right)$.

NEUTROSOPHIC SET [7]

Let *S*be a non-empty set. A neutrosophic set (NS)isastructureoftheform:

$$\Omega = \{\langle s, \gamma_T(s), \gamma_I(s), \gamma_F(s)\rangle / s \in S\}$$

where $\gamma_T : S \to [0,1]$ is a truth membership function, $\underset{\to}{\gamma_I} : S \to [0,1]$ is an indeterminate membership function, and $\gamma_F : S \to [0,1]$is a false membership function.

CUBIC SETS [2]

Let Sbe a non-empty set. A cubic set in Y is a structure of the form

$$\mathscr{C} = \{y, C(y), \lambda(y) / y \in S\}$$

where A is an interval valued fuzzy set in S and λ is a fuzzy set in S.

NEUTROSOPHIC CUBIC SETS [4]

Let S be a non-empty set. A neutrosophic cubic set (NCS) in S is a pair $\mathscr{A} = (C, \lambda)$ where $C = \{\langle s, C_T(s), C_I(s), C_F(s)\rangle / s \in S\}$ is an intervalneutrosophic set in S and $\lambda = \{\langle s; \lambda_T(s), \lambda_I(s), \lambda_F(s)\rangle / s \in X\}$ is a neutrosophic set inS.

SPHERICAL FUZZY SETS [5]

A Spherical Fuzzy Set $\mathscr{A}_s^{\sim}$ of the universe U is given by

$$\mathscr{A}_s^{\sim} = \left\{\left\langle u, \mu_{A_s^{\sim}}(u), \gamma_{A_s^{\sim}}(u), \pi_{A_s^{\sim}}(u)\right\rangle / u \in U\right\} \text{where } \mu_{A_s^{\sim}}, \gamma_{A_s^{\sim}}, \pi_{A_s^{\sim}}, U \to [0,1]$$

and $0 \leq \mu^2_{A_s^{\sim}}(u) + \gamma^2_{A_s^{\sim}}(u) + \pi^2_{A_s^{\sim}} \leq 1 \ \forall u \in U$.

3. NEUTROSOPHIC SPHERICAL SETS

DEFINITION 3.1

Let S be a non-empty set. A Neutrosophic Spherical set in NS is of the form

$$A_s = \{\langle s: T_{A_s}(s), I_{A_S}(s), F_{A_s}(s)\rangle / s \in S\}$$

where T_{A_s} is truth degree membership

I_{A_s} is indeterminate degree membership

F_{A_s} is false degree membership.

where

$$\mathrm{T}_{\mathrm{A_S}}(s), \mathrm{I}_{\mathrm{A_S}}(s), \mathrm{F}_{\mathrm{A_S}}(s) / s \in S \rightarrow [0,1]$$

$$0 \leq [\mathrm{T}_{\mathrm{A_S}}(s)]^2 + [\mathrm{I}_{\mathrm{A_S}}(s)]^2 + [\mathrm{F}_{\mathrm{A_S}}(s)]^2 \leq \sqrt{3}$$

INTERVAL VALUED NEUTROSOPHIC SPHERICAL SETS

DEFINITION 3.2

Let S be a non-empty set. An interval-valued Neutrosophic Spherical set is of the form

$$\mathscr{A}_s = \left\{ s : \left[T^-_{A_s(s)}, T^+_{A_s(s)} \right] \left[I^-_{A_s(s)}, I^+_{A_s(s)} \right] \left[F^-_{A_s(s)}, F^+_{A_s(s)} \right] \Big/ s \in S \right\}$$

Where $T^-_{A_s}(s), I^-_{A_s}(s), F^-_{A_s}(s) / s \in S \rightarrow [0,1]$

$$0 \leq [T^-_{A_s}(s)]^2 + [I^-_{A_s}(s)]^2 + [F^-_{A_s}(s)]^2 \leq \sqrt{3}$$

and$T^+_{A_s}(s), I^+_{A_s}(s), F^+_{A_s}(s) / s \in S \rightarrow [0,1]$

$$0 \leq [T^+_{A_s}(s)]^2 + [I^+_{A_s}(s)]^2 + [F^+_{A_s}(s)]^2 \leq \sqrt{3}$$

NEUTROSOPHIC SPHERICAL CUBIC SETS

DEFINITION 3.3

A non-empty set $\mathcal{V}_{NSC}$ of NSCS is defined by

$$\mathcal{C}_{NSC} = \{\langle v, A_s(v), \lambda_s(v)\rangle / v \in \mathcal{V}_{NSC}\}$$

where $A_s(v)$ is an IVNSS in $\mathcal{V}_{NSC}$ and $\lambda_s(v)$ is a NSS in $\mathcal{V}_{NSC}$.

EXAMPLE 3.1

For $\mathcal{V}_{NSC} = \{v_1, v_2, v_3\}$, the pair $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ with the tabular representation in Table 0.2 is an NSCS in $\mathcal{V}_{NSC}$.

Table 1: $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$

$\mathcal{V}_{NSC}$	$A_s(v)$	$\lambda_s(v)$
v_1	([0.3,0.4], [0.4,1.0], [0.3,0.5])	(0.2,0.4,0.4)
v_2	([0.4,0.7], [0.2,1.0], [0.2,0.4])	(0.5,0.2,0.3)
v_3	([0.7,0.6], [0.0,1.0], [0.3,0.8])	(0.4,0.1,0.5)

DEFINITION 3.4

A non-empty set of $\mathcal{V}_{NSC}$ of NSCS, $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ in $\mathcal{V}_{NSC}$ is said to

- Truth Int (briefly $\mathcal{R} -$ Int) is defined by

$$(\forall \ v \in \mathcal{V}_{NSC})(\mathcal{R}^-_{A_s}(v) \leq \mathcal{R}_{\lambda_s}(v) \leq \mathcal{R}^+_{A_s}(v)) \tag{1}$$

- Indeterminacy-Int (briefly $\mathcal{J} -$ Int) is defined by

$$(\forall \ v \in \mathcal{V}_{NSC})(\mathcal{J}^-_{A_s}(v) \leq \mathcal{J}_{\lambda_s}(v) \leq \mathcal{J}^+_{A_s}(v)) \tag{2}$$

- Falsity-int(briefly $\mathcal{S}$ Int) is defined by

$$(\forall \ v \in \mathcal{V}_{NSC})(\mathcal{S}^-_{A_s}(v) \leq \mathcal{S}_{\lambda_s}(v) \leq \mathcal{S}^+_{A_s}(v)) \tag{3}$$

If a NSCS, $\mathcal{C}_{NSC}$ in $\mathcal{V}_{NSC}$ satisfies above inequalities then $\mathcal{C}_{NSC}$ is an Int NSCS in $\mathcal{V}_{NSC}$.

EXAMPLE 3.2

For $\mathcal{V}_{NSC} = \{v_1, v_2, v_3\}$, the pair $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ with the tabular representation in Table 0.4 is an Int NSCS in $\mathcal{V}_{NSC}$.

Table 2: $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$

$\mathcal{V}_{NSC}$	$A_s(v)$	$\lambda_s(v)$
v_1	([0.3,0.4], [0.2,1.0], [0.5,0.6])	(0.35,0.2,0.55)
v_2	([0.5,0.6], [0.1,1.0], [0.4,0.6])	(0.5,0.1,0.4)

v_3	$([0.6,0.7],[0.1,1.0],[0.2,0.4])$	$(0.65,0.1,0.25)$

DEFINITION 3.5

A non-empty set $\mathcal{V}_{NSC}$ of NSCS, $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ is said to

- Truth Ext (briefly $\mathcal{R}$ Ext) is defined by

$$(\forall\ \ v \in \mathcal{V}_{NSC})(\mathcal{R}_{\lambda_s}(v) \not\subseteq (\mathcal{R}^-_{A_s}(v), \mathcal{R}^+_{A_s}(v)) \quad (4)$$

- Indeterminacy-Ext (briefly $\mathcal{J}$ Ext) is defined by

$$(\forall\ \ v \in \mathcal{V}_{NSC})(\mathcal{J}_{\lambda_s}(v) \not\subseteq (\mathcal{J}^-_{A_s}(v), \mathcal{J}^+_{A_s}(v)) \quad (5)$$

- Falsity-Ext (briefly $\mathcal{S}$ Ext) is defined by

$$(\forall\ \ v \in \mathcal{V}_{NSC})(\mathcal{S}_{\lambda_s}(v) \not\subseteq (\mathcal{S}^-_{A_s}(v), \mathcal{S}^+_{A_s}(v)) \quad (6)$$

If a NSCS, $\mathcal{C}_{NSC}$ in $\mathcal{V}_{NSC}$ satisfies above inequalities then $\mathcal{C}_{NSC}$ is an Ext NSCS in $\mathcal{V}_{NSC}$.

EXAMPLE 3.3

For $\mathcal{V}_{NSC} = \{v_1, v_2, v_3\}$, the pair $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ with the tabular representation in Table 0.6 is an Ext NSCS in $\mathcal{V}_{NSC}$.

Table 3: $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$

$\mathcal{V}_{NSC}$	$A_s(v)$	$\lambda_s(v)$
v_1	$([0.3,0.4],[0.2,1.0],[0.4,0.5])$	$(0.45,0.1,0.65)$
v_2	$([0.5,0.6],[0.1,1.0],[0.4,0.6])$	$(0.4,0.0,0.7)$
v_3	$([0.6,0.7],[0.1,1.0],[0.2,0.4])$	$(0.5,0.0,0.45)$

Theorem 3.4 Let $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ be a NSCS in $\mathcal{V}_{NSC}$ is not Ext then there exists $v \in \mathcal{V}_{NSC}$ such that $\mathcal{R}_{\lambda_s}(v) \in (\mathcal{R}^-_{A_s}(v), \mathcal{R}^+_{A_s}(v))$, $\mathcal{J}_{\lambda_s}(v) \in (\mathcal{J}^-_{A_s}(v), \mathcal{J}^+_{A_s}(v))$ or $\mathcal{S}_{\lambda_s}(v) \in (\mathcal{S}^-_{A_s}(v), \mathcal{S}^+_{A_s}(v))$.

Proof. From the definition of an Ext NSCS ,

$$\mathcal{R}_{\lambda_s}(v) \notin [\mathcal{R}^-_{A_s}(v), \mathcal{R}^+_{A_s}(v)],$$
$$\mathcal{J}_{\lambda_s}(v) \notin [\mathcal{J}^-_{A_s}(v), \mathcal{J}^+_{A_s}(v)],$$
$$\mathcal{S}_{\lambda_s}(v) \notin [\mathcal{S}^-_{A_s}(v), \mathcal{S}^+_{A_s}(v)]$$

for $v \in \mathcal{V}_{NSC}$. But given that $\mathcal{C}_{NSC}$ is not Ext NSCS, such that

$$\mathcal{R}^{-}_{A_s}(v) \leq \mathcal{R}_{\lambda_s}(v) \leq \mathcal{R}^{+}_{A_s}(v)$$
$$\mathcal{J}^{-}_{A_s}(v) \leq \mathcal{J}_{\lambda_s}(v) \leq \mathcal{J}^{+}_{A_s}(v)$$
$$\mathcal{S}^{-}_{A_s}(v) \leq \mathcal{S}_{\lambda_s}(v) \leq \mathcal{S}^{+}_{A_s}(v)$$

Hence the result.

Theorem 3.5 Let $\mathcal{C}_{NSC} = (A_s(v), \lambda_s)(v)$ be a NSCS in $\mathcal{V}_{NSC}$, if $\mathcal{C}_{NSC}$ is both $\mathcal{R}$ Int and $\mathcal{R}$ Ext then

$$(\forall \ \ v \in \mathcal{V}_{NSC})(\mathcal{R}_{\lambda_s}(v) \in \{\mathcal{R}^{-}_{A_s}(v)/v \in \mathcal{V}_{NSC}\} \cup \{\mathcal{R}^{+}_{A_s}(v)/v \in \mathcal{V}_{NSC}\}).$$

Proof. Two conditions (1) and (4) which implies that

$$\mathcal{R}^{-}_{A_s}(v) \leq \mathcal{R}_{\lambda_s}(v) \leq \mathcal{R}^{+}_{A_s}(v) \qquad and$$

$$\mathcal{R}_{\lambda_s}(v) \not\subseteq (\mathcal{R}^{-}_{A_s}(v), \mathcal{R}^{+}_{A_s}(v))\forall \ \ v \in \mathcal{V}_{NSC}.$$

Then $\mathcal{R}_{\lambda_s}(x) = (\mathcal{R}^{-}_{A_s}(v)$ or $\mathcal{R}_{\lambda_s}(v) = (\mathcal{R}^{+}_{A_s}(v)$ so that

$$\mathcal{R}_{\lambda_s}(v) \in \{\mathcal{R}^{-}_{A_s}(v)/v \in \mathcal{V}_{NSC}\} \cup \{\mathcal{R}^{+}_{A_s}(v)/v \in \mathcal{V}_{NSC}\}.$$

Hence proved.

Theorem 3.6 Let $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ be a NSCS in $\mathcal{V}_{NSC}$, if $\mathcal{C}_{NSC}$ is both $\mathcal{J}$ Int and $\mathcal{J}$ Ext then

$$(\forall \ \ v \in \mathcal{V}_{NSC})(\mathcal{J}_{\lambda_s}(v) \in \{\mathcal{J}^{-}_{A_s}(v)/v \in \mathcal{V}_{NSC}\} \cup \{\mathcal{J}^{+}_{A_s}(v)/v \in \mathcal{V}_{NSC}\}).$$

Proof. Two conditions (2) and (5) which implies that

$$\mathcal{J}^{-}_{A_s}(v) \leq \mathcal{J}_{\lambda_s}(v) \leq \mathcal{J}^{+}_{A_s}(v) \qquad and$$

$$\mathcal{J}_{\lambda_s}(v) \not\subseteq (\mathcal{J}^{-}_{A_s}(v), \mathcal{J}^{+}_{A_s}(v))\forall \ \ v \in \mathcal{V}_{NSC}.$$

Then $\mathcal{J}_{\lambda_s}(x) = (\mathcal{J}^{-}_{A_s}(v)$ or $\mathcal{J}_{\lambda_s}(v) = (\mathcal{J}^{+}_{A_s}(v)$ so that

$$\mathcal{J}_{\lambda_s}(v) \in \{\mathcal{J}^{-}_{A_s}(v)/v \in \mathcal{V}_{NSC}\} \cup \{\mathcal{J}^{+}_{A_s}(v)/v \in \mathcal{V}_{NSC}\}.$$

Hence proved.

Theorem 3.7 Let $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ be a NSCS $\mathcal{V}_{NSC}$, if $\mathcal{C}_{NSC}$ is both $\mathcal{S}$ Int and $\mathcal{S}$ Ext then

$$(\forall \ \ v \in \mathcal{V}_{NSC})(\mathcal{S}_{\lambda_s}(v) \in \{\mathcal{S}^{-}_{A_s}(v)/v \in \mathcal{V}_{NSC}\} \cup \{\mathcal{S}^{+}_{A_s}(v)/v \in \mathcal{V}_{NSC}\}).$$

Proof. Two conditions (3) and (6) which implies that

$$\mathcal{S}^{-}_{A_s}(v) \leq \mathcal{S}_{\lambda_s}(v) \leq \mathcal{S}^{+}_{A_s}(v) \qquad and$$

$$\mathcal{S}_{\lambda_s}(v) \not\subseteq (\mathcal{S}^-_{A_s}(v), \mathcal{S}^+_{A_s}(v)) \forall \ \ v \in \mathcal{V}_{NSC}.$$

Then $\mathcal{S}_{\lambda_s}(x) = (\mathcal{S}^-_{A_s}(v)$ or $\mathcal{S}_{\lambda_s}(v) = (\mathcal{S}^+_{A_s}(v)$ so that

$$\mathcal{S}_{\lambda_s}(v) \in \{\mathcal{S}^-_{A_s}(v)/v \in \mathcal{V}_{NSC}\} \cup \{\mathcal{S}^+_{A_s}(v)/v \in \mathcal{V}_{NSC}\}.$$

Hence proved.

Definition 3.8 Let $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ and $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$ be a NSCS in $\mathcal{V}_{NSC}$ where

$$\mathcal{C}_{NSC} = \{\mathcal{V}_{NSC}: [\mathcal{R}^-_{A_s(v)}, \mathcal{R}^+_{A_s(v)}][\mathcal{J}^-_{A_s(v)}, \mathcal{J}^+_{A_s(v)}][\mathcal{S}^-_{A_s(v)}, \mathcal{S}^+_{A_s(v)}]/v \in \mathcal{V}_{NSC}\}$$

$$\lambda_s: \{(\mathcal{V}_{NSC}, \mathcal{R}_{\lambda_s}(v), \mathcal{J}_{\lambda_s}(v), \mathcal{S}_{\lambda_s}(v))/v \in \mathcal{V}_{NSC}\}$$

$$\mathcal{B}_{NSC} = \{\mathcal{V}_{NSC}: [\mathcal{R}^-_{B_s(v)}, \mathcal{R}^+_{B_s(v)}][\mathcal{J}^-_{B_s(v)}, \mathcal{J}^+_{B_s(v)}][\mathcal{S}^-_{B_s(v)}, \mathcal{S}^+_{B_s(v)}]/v \in \mathcal{V}_{NSC}\}$$

$$\psi_s: \{(\mathcal{V}_{NSC}, \mathcal{R}_{\psi_s}(v), \mathcal{J}_{\psi_s}(v), \mathcal{S}_{\psi_s}(v))/v \in \mathcal{V}_{NSC}\}.$$

Then

$\mathcal{C}_{NSC} = \mathcal{B}_{NSC}$ iff

$A_s(v) = B_s(v)$ and $\lambda_s(v) = \psi_s(v)$ for all $v \in \mathcal{V}_{NSC}$

If $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ and $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$ be any two NSCSs, then P −order is defined by

$\mathcal{C}_{NSC} \subseteq_P \mathcal{B}_{NSC}$ iff $A_s(v) \subseteq B_s(v)$ and $\lambda_s(v) \leq \psi_s(v)$ for all $v \in \mathcal{V}_{NSC}$

If $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ and $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$ be any two NSCSs, then R −order is defined by

$\mathcal{C}_{NSC} \subseteq_R \mathcal{B}_{NSC}$ iff $A_s(v) \subseteq B_s(v)$ and $\lambda_s(v) \geq \psi_s(v)$ for all $v \in \mathcal{V}_{NSC}$.

Definition3.9 Let $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ be two NSCS in $\mathcal{V}_{NSC}$, then P-union is defined by

$$\mathcal{C}_{NSC} \cup_P \mathcal{B}_{NSC} = \{\langle v, max(A_s(v), B_s(v)), (\lambda_s(v) \vee_P \psi_s(v))\rangle : v \in \mathcal{V}_{NSC}\}$$

where $A_s(v), B_s(v)$ represent IVNSSs and $\lambda_s(v), \psi_s(v)$ represent NSSs. Hence

$$\mathcal{R}_{\mathcal{C}_{NSC}} \vee_P \mathcal{R}_{\mathcal{B}_{NSC}} = \{\langle v, max(\mathcal{R}_{A_s(v)}, B_s(v)), (\mathcal{R}_{\lambda_s}(v) \vee_P \mathcal{R}_{\psi_s}(v))\rangle : v \in \mathcal{V}_{NSC}\}$$

$$\mathcal{J}_{\mathcal{C}_{NSC}} \vee_P \mathcal{J}_{\mathcal{B}_{NSC}} = \{\langle v, max(\mathcal{J}_{A_s(v)}, B_s(v)), (\mathcal{J}_{\lambda_s}(v) \vee_P \mathcal{J}_{\psi_s}(v))\rangle : v \in \mathcal{V}_{NSC}\}$$

$$\mathcal{S}_{\mathcal{C}_{NSC}} \vee_P \mathcal{S}_{\mathcal{B}_{NSC}} = \{\langle v, max(\mathcal{S}_{A_s(v)}, B_s(v)), (\mathcal{S}_{\lambda_s}(v) \vee_P \mathcal{S}_{\psi_s}(v))\rangle : v \in \mathcal{V}_{NSC}\}$$

Definition 3.10 Let $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ be two NSCS in $\mathcal{V}_{NSC}$, then P-intersection is defined by

$$\mathcal{C}_{NSC} \cap_P \mathcal{B}_{NSC} = \{\langle v, min(A_s(v), B_s(v)), (\lambda_s(v) \wedge_P \psi_s(v))\rangle : v \in \mathcal{V}_{NSC}\}$$

where $A_s(v), B_s(v)$ represent IVNSSs and $\lambda_s(v), \psi_s(v)$ represent NSSs. Hence

$$\mathcal{R}_{\mathcal{C}_{NSC}} \wedge_P \mathcal{R}_{\mathcal{B}_{NSC}} = \{\langle v, min(\mathcal{R}_{A_s(v)}, B_s(v)), (\mathcal{R}_{\lambda_s}(v) \wedge_P \mathcal{R}_{\psi_s}(v))\rangle : v \in \mathcal{V}_{NSC}\}$$
$$\mathcal{J}_{\mathcal{C}_{NSC}} \wedge_P \mathcal{J}_{\mathcal{B}_{NSC}} = \{\langle v, min(\mathcal{J}_{A_s(v)}, B_s(v)), (\mathcal{J}_{\lambda_s}(v) \wedge_P \mathcal{J}_{\psi_s}(v))\rangle : v \in \mathcal{V}_{NSC}\}$$
$$\mathcal{S}_{\mathcal{C}_{NSC}} \wedge_P \mathcal{S}_{\mathcal{B}_{NSC}} = \{\langle v, min(\mathcal{S}_{A_s(v)}, B_s(v)), (\mathcal{S}_{\lambda_s}(v) \wedge_P \mathcal{S}_{\psi_s}(v))\rangle : v \in \mathcal{V}_{NSC}\}$$

Example 3.11 For $\mathcal{V}_{NSC} = \{v_1, v_2, v_3\}$, let $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ be two NSCSs over $\mathcal{V}_{NSC}$ is defined by

$\mathcal{C}_{NSC} = \{\langle v, A_s(v), \lambda_s(v)\rangle v \in \mathcal{V}_{NSC}\}$ is

Table 4: $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$

$\mathcal{V}_{NSC}$	$A_s(v)$	$\lambda_s(v)$
v_1	([0.3,0.4], [0.5,1.0], [0.2,0.4])	(0.35,0.55,0.25)
v_2	([0.2,0.3], [0.4,1.0], [0.4,0.6])	(0.35,0.45,0.55)
v_3	([0.4,0.5], [0.2,1.0], [0.4,0.6])	(0.55,0.35,0.45)

Table 5: $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$

$\mathcal{V}_{NSC}$	$A_s(v)$	$\psi_s(v)$
v_1	([0.1,0.2], [0.5,1.0], [0.4,0.6])	(0.45,0.55,0.45)
v_2	([0.2,0.4], [0.3,1.0], [0.5,0.7])	(0.45,0.65,0.65)
v_3	([0.3,0.5], [0.4,1.0], [0.3,0.5])	(0.65,0.55,0.75)

Table 6: $\mathcal{C}_{NSC} \cup_P \mathcal{B}_{NSC} = (A_s(v) \cup_P B_s(v), \lambda_s(v) \vee_P \psi_s(v))$

$\mathcal{V}_{NSC}$	$(A_s(v) \cup_P B_s(v)$	$(\lambda_s(v) \vee_P \psi_s(v)$
v_1	([0.3,0.4], [0.5,1.0], [0.4,0.6])	(0.45,0.55,0.45)
v_2	([0.2,0.4], [0.4,1.0], [0.5,0.7])	(0.45,0.65,0.65)
v_3	([0.4,0.5], [0.4,1.0], [0.4,0.6])	(0.65,0.55,0.75)

Table 7: $\mathcal{C}_{NSC} \cap_P \mathcal{B}_{NSC} = (A_s(v) \cap_P B_s(v), \lambda_s(v) \wedge_P \psi_s(v))$

$\mathcal{V}_{NSC}$	$(A_s(v) \cap_P B_s(v)$	$(\lambda_s(v) \wedge_P \psi_s(v)$
v_1	([0.1,0.2], [0.5,1.0], [0.2,0.4])	(0.35,0.55,0.15)
v_2	([0.2,0.3], [0.3,1.0], [0.4,0.6])	(0.35,0.45,0.55)
v_3	([0.3,0.5], [0.2,1.0], [0.3,0.5])	(0.55,0.35,0.5)

Definition 3.12 Let $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ be two NSCS in $\mathcal{V}_{NSC}$, then R-union is defined by

$$\mathcal{C}_{NSC} \cup_R \mathcal{B}_{NSC} = \{\langle v, max(A_s(v), B_s(v)), (\lambda_s(v) \vee_R \psi_s(v))\rangle : v \in \mathcal{V}_{NSC}\}$$

where $A_s(v), B_s(v)$ represent IVNSSs and $\lambda_s(v), \psi_s(v)$ represent NSSs. Hence

$$\mathcal{R}_{\mathcal{C}_{NSC}} \vee_R \mathcal{R}_{\mathcal{B}_{NSC}} = \{\langle v, max(\mathcal{R}_{A_s(v)}, B_s(v)), (\mathcal{R}_{\lambda_s}(v) \vee_R \mathcal{R}_{\psi_s}(v))\rangle : v \in \mathcal{V}_{NSC}\}$$
$$\mathcal{J}_{\mathcal{C}_{NSC}} \vee_R \mathcal{J}_{\mathcal{B}_{NSC}} = \{\langle v, max(\mathcal{J}_{A_s(v)}, B_s(v)), (\mathcal{J}_{\lambda_s}(v) \vee_R \mathcal{J}_{\psi_s}(v))\rangle : v \in \mathcal{V}_{NSC}\}$$
$$\mathcal{S}_{\mathcal{C}_{NSC}} \vee_R \mathcal{S}_{\mathcal{B}_{NSC}} = \{\langle v, max(\mathcal{S}_{A_s(v)}, B_s(v)), (\mathcal{S}_{\lambda_s}(v) \vee_R \mathcal{S}_{\psi_s}(v))\rangle : v \in \mathcal{V}_{NSC}\}$$

Definition 3.13 Let $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ be two NSCS in $\mathcal{V}_{NSC}$, then R-intersection is defined by

$$\mathcal{C}_{NSC} \cap_R \mathcal{B}_{NSC} = \{\langle v, min(A_s(v), B_s(v)), (\lambda_s(v) \wedge_R \psi_s(v))\rangle : v \in \mathcal{V}_{NSC}\}$$

where $A_s(v), B_s(v)$ represent IVNSSs and $\lambda_s(v), \psi_s(v)$ represent NSSs. Hence

$$\mathcal{R}_{\mathcal{C}_{NSC}} \wedge_R \mathcal{R}_{\mathcal{B}_{NSC}} = \{\langle v, min(\mathcal{R}_{A_s(v)}, B_s(v)), (\mathcal{R}_{\lambda_s}(v) \wedge_R \mathcal{R}_{\psi_s}(v))\rangle : v \in \mathcal{V}_{NSC}\}$$
$$\mathcal{J}_{\mathcal{C}_{NSC}} \wedge_R \mathcal{J}_{\mathcal{B}_{NSC}} = \{\langle v, min(\mathcal{J}_{A_s(v)}, B_s(v)), (\mathcal{J}_{\lambda_s}(v) \wedge_R \mathcal{J}_{\psi_s}(v))\rangle : v \in \mathcal{V}_{NSC}\}$$
$$\mathcal{S}_{\mathcal{C}_{NSC}} \wedge_R \mathcal{S}_{\mathcal{B}_{NSC}} = \{\langle v, min(\mathcal{S}_{A_s(v)}, B_s(v)), (\mathcal{S}_{\lambda_s}(v) \wedge_R \mathcal{S}_{\psi_s}(v))\rangle : v \in \mathcal{V}_{NSC}\}$$

Example 3.14 For $\mathcal{V}_{NSC} = \{v_1, v_2, v_3\}$. Let $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ be two NSCSs over $\mathcal{V}_{NSC}$ is defined by

$\mathcal{C}_{NSC} = \{\langle v, A_s(v), \lambda_s(v)\rangle v \in \mathcal{V}_{NSC}\}$ is

Table 8: $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$

$\mathcal{V}_{NSC}$	$A_s(v)$	$\lambda_s(v)$
v_1	([0.2,0.3], [0.4,1.0], [0.4,0.6])	(0.55,0.65,0.75)
v_2	([0.1,0.5], [0.6,1.0], [0.3,0.6])	(0.65,0.55,0.85)
v_3	([0.2,0.5], [0.4,1.0], [0.4,0.6])	(0.75,0.85,0.65)

Table 9: $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$

$\mathcal{V}_{NSC}$	$A_s(v)$	$\psi_s(v)$
v_1	([0.2,0.3], [0.4,1.0], [0.5,0.6])	(0.35,0.65,0.75)
v_2	([0.1,0.5], [0.6,1.0], [0.3,0.6])	(0.65,0.55,0.85)
v_3	([0.2,0.5], [0.4,1.0], [0.4,0.6])	(0.75,0.85,0.65)

Table 10: $\mathcal{C}_{NSC} \cup_R \mathcal{B}_{NSC} = (A_s(v) \cup_R B_s, \lambda_s(v) \vee_R \psi_s(v))$

$\mathcal{V}_{NSC}$	$(A_s(v) \cup_R B_s(v)$	$(\lambda_s(v) \vee_R \psi_s(v)$
v_1	([0.4,0.5], [0.4,1.0], [0.4,0.6])	(0.35,0.45,0.55)
v_2	([0.2,0.4], [0.6,1.0], [0.3,0.6])	(0.25,0.45,0.65)
v_3	([0.4,0.7], [0.4,1.0], [0.4,0.6])	(0.45,0.55,0.45)

Table 11: $\mathcal{C}_{NSC} \cap_R \mathcal{B}_{NSC} = (A_s(v) \cap_R B_s(v), \lambda_s(v) \wedge_R \psi_s(v))$

$\mathcal{V}_{NSC}$	$(A_s(v) \cap_R B_s(v)$	$(\lambda_s(v) \wedge_R \psi_s(v)$
v_1	([0.2,0.3], [0.3,1.0], [0.3,0.6])	(0.55,0.65,0.75)
v_2	([0.1,0.5], [0.5,1.0], [0.3,0.6])	(0.65,0.55,0.85)
v_3	([0.2,0.5], [0.4,1.0], [0.4,0.6])	(0.75,0.85,0.65)

Theorem 3.15 Let $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ be a NSCS over $\mathcal{V}_{NSC}$. [i)] If $\mathcal{C}_{NSC}$ is an Int NSCS, then the complement $\mathcal{C}_{NSC}^c$ is also an Int NSCS. [ii)]If $\mathcal{C}_{NSC}$ is an Ext NSCS, then the complement $\mathcal{C}_{NSC}^c$ is also an Ext NSCS.

Proof. [i)] Given $\mathcal{C}_{NSC} = \{\langle v, A_s(v), \lambda_s(v)\rangle : v \in \mathcal{V}_{NSC}\}$ is an Int NSCS this implies
$\mathcal{R}_{A_s}^-(v) \le \mathcal{R}_{\lambda_s}(v) \le \mathcal{R}_{A_s}^+(v)$,
$\mathcal{J}_{A_s}^-(v) \le \mathcal{J}_{\lambda_s}(v) \le \mathcal{J}_{A_s}^+(v)$,
$\mathcal{S}_{A_s}^-(v) \le \mathcal{S}_{\lambda_s}(v) \le \mathcal{S}_{A_s}^+(v)$ for all $v \in \mathcal{V}_{NSC}$
This implies $1 - \mathcal{R}_{A_s}^-(v) \le 1 - \mathcal{R}_{\lambda_s}(v) \le 1 - \mathcal{R}_{A_s}^+(v)$,
$1 - \mathcal{J}_{A_s}^-(v) \le 1 - \mathcal{J}_{\lambda_s}(v) \le 1 - \mathcal{J}_{A_s}^+(v)$,
$1 - \mathcal{S}_{A_s}^-(v) \le 1 - \mathcal{S}_{\lambda_s}(v) \le 1 - \mathcal{S}_{A_s}^+(v)$ for all $v \in \mathcal{V}_{NSC}$.

Hence $\mathcal{C}_{NSC}^c$ is an INSCS. [ii)] Given $\mathcal{C}_{NSC} = \{\langle v, A_s(v), \lambda_s(v)\rangle : v \in \mathcal{V}_{NSC}\}$ is an Ext NSCS. This implies

$\mathcal{R}_{\lambda_s}(v) \notin (\mathcal{R}_{A_s}^-(v), \mathcal{R}_{A_s}^+(v)), \mathcal{J}_{\lambda_s}(v) \notin (\mathcal{J}_{A_s}^-(v), \mathcal{J}_{A_s}^+(v)), \mathcal{S}_{\lambda_s}(v) \notin (\mathcal{S}_{A_s}^-(v), \mathcal{S}_{A_s}^+(v))$ for all $v \in \mathcal{V}_{NSC}$

Since,

$$\mathcal{R}_{\lambda_s}(v) \notin (\mathcal{R}_{A_s}^-(v), \mathcal{R}_{A_s}^+(v)),$$

and $0 \le \mathcal{R}_{A_s}^-(v) \le \mathcal{R}_{A_s}^+(v) \le 1$

$$\mathcal{J}_{\lambda_s}(v) \notin (\mathcal{J}_{A_s}^-(v), \mathcal{J}_{A_s}^+(v)),$$

and $0 \le \mathcal{J}_{A_s}^-(v) \le \mathcal{J}_{A_s}^+(v) \le 1$

$$\mathcal{S}_{\lambda_s}(v) \notin (\mathcal{S}_{A_s}^-(v), \mathcal{S}_{A_s}^+(v))$$

and $0 \le \mathcal{S}_{A_s}^-(v) \le \mathcal{S}_{A_s}^+(v) \le 1$

So we have
$\mathcal{R}_{\lambda_s}(v) \le \mathcal{R}_{A_s}^-(v)$ or $\mathcal{R}_{A_s}^+(v) \le \mathcal{R}_{\lambda_s}(v)$
$\mathcal{J}_{\lambda_s}(v) \le \mathcal{J}_{A_s}^-(v)$ or $\mathcal{J}_{A_s}^+(v) \le \mathcal{J}_{\lambda_s}(v)$
$\mathcal{S}_{\lambda_s}(v) \le \mathcal{S}_{A_s}^-(v)$ or $\mathcal{S}_{A_s}^+(v) \le \mathcal{S}_{\lambda_s}(v)$
This implies
$1 - \mathcal{R}_{\lambda_s}(v) \ge 1 - \mathcal{R}_{A_s}^-(v)$ or $1 - \mathcal{R}_{A_s}^+(v) \ge 1 - \mathcal{R}_{\lambda_s}(v)$
$1 - \mathcal{J}_{\lambda_s}(v) \ge 1 - \mathcal{J}_{A_s}^-(v)$ or $1 - \mathcal{J}_{A_s}^+(v) \ge 1 - \mathcal{J}_{\lambda_s}(v)$
$1 - \mathcal{S}_{\lambda_s}(v) \le 1 - \mathcal{S}_{A_s}^-(v)$ or $1 - \mathcal{S}_{A_s}^+(v) \le 1 - \mathcal{S}_{\lambda_s}(v)$ for all $v \in \mathcal{V}_{NSC}$.

Thus $1 - \mathcal{R}_{\lambda_s}(v) \notin (1 - \mathcal{R}_{A_s}^-(v), 1 - \mathcal{R}_{A_s}^+(v)), 1 - \mathcal{J}_{\lambda_s}(v) \notin (1 - \mathcal{J}_{A_s}^-(v), 1 - \mathcal{J}_{A_s}^+(v)), 1 - \mathcal{S}_{\lambda_s}(v) \notin (1 - \mathcal{S}_{A_s}^-(v), 1 - \mathcal{S}_{A_s}^+(v))$ for all $v \in \mathcal{V}_{NSC}$

Hence $\mathcal{C}_{NSC}^c = (A_s(v), \lambda_s(v))$ is an Ext NSCS.

Remark 3.16 The below example shows that $P-$ union and $P-$ intersection of $\mathcal{R}$ Ext (resp. $\mathcal{J}$ Ext and $\mathcal{S}$ Ext) NSCSs may not be $\mathcal{R}$ Ext (resp. $\mathcal{J}$ Ext and $\mathcal{S}$ Ext) NSCSs.

Example 3.17 Let $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ and $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$ be NSCSs in $\mathcal{V}_{NSC}$ where

$$A_s(v) = \{(v, (0.3,0.5), (0.5,0.7), (0.3,0.5))/v \in \mathcal{V}_{NSC}\}$$

$\lambda_s(v) = \{(v, 0.4,0.4,0.8)/v \in \mathcal{V}_{NSC}\}$
$B_s(v) = \{(v, (0.7,0.9), (0.6,0.7), (0.7,0.9))/v \in \mathcal{V}_{NSC}\}$
$\psi_s(v) = \{(v, 0.8,0.3,0.8)/v \in \mathcal{V}_{NSC}\}$

Then $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ are $\mathcal{S}$ Ext NSCSs in $\mathcal{V}_{NSC}$.

$\mathcal{C}_{NSC} \cup_P \mathcal{B}_{NSC} = (A_s \cup_P B_s, \lambda_s \vee_P \psi_s)$ of $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ is given as follows

$A_s(v) \cup_P B_s(v) = \{(v, (0.7,0.9), (0.6,0.7), (0.7,0.9))/v \in \mathcal{V}_{NSC}\}$
$\lambda_s(v) \vee_P \psi_s(v) = \{(v, 0.8,0.4,0.8)/v \in \mathcal{V}_{NSC}\}$

is not an $\mathcal{S}$ Ext NSCSs in $\mathcal{V}_{NSC}$.

Since

$(\mathcal{S}_{\lambda_s} \vee_P \mathcal{S}_{\psi_s})(v) = 0.8 \in (0.7,0.9)$
$= (\mathcal{S}_{A_s} \cup_P \mathcal{S}_{B_s})^-(v), (\mathcal{S}_{A_s} \cup_P \mathcal{S}_{B_s})^+(v)$

also $A_s \cap_P B_s = (A_s \cap B_s, \lambda_s \wedge_P \psi_s)$ with

$A_s \cap B_s = \{(v, (0.3,0.5), (0.4,0.7), (0.3,0.5))/v \in \mathcal{V}_{NSC}\}$
$\lambda_s \wedge \psi_s = \{\langle v, 0.4,0.3,0.8\rangle/v \in \mathcal{V}_{NSC}\}$

is not an $\mathcal{S}$ Ext NSCS in $\mathcal{V}_{NSC}$ since

$(\mathcal{S}_{\lambda_s} \wedge_P \mathcal{S}_{\psi_s})(v) = 0.4 \in (0.4,0.7)$
$= (\mathcal{S}_{A_s} \cap_P \mathcal{S}_{B_s})^-(v), (\mathcal{S}_{A_s} \cap_P \mathcal{S}_{B_s})^+(v)$

Example 3.18 For $\mathcal{V}_{NSC} = \{v_1, v_2, v_3\}$, let $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ and $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$ be NSCSs in $\mathcal{V}_{NSC}$ with the Table 0.21 and 0.21, respectively.

Table 12: $\mathcal{C}_{NSC} = (A_s(v), \lambda_s)(v)$

$\mathcal{V}_{NSC}$	$A_s(v)$	$\lambda_s(v)$
v_1	$([0.3,0.4], [0.5,1.0], [0.2,0.3])$	$(0.35,0.55,0.25)$
v_2	$([0.2,0.3], [0.4,1.0], [0.4,0.6])$	$(0.25,0.45,0.45)$
v_3	$([0.6,0.7], [0.1,1.0], [0.3,0.4])$	$(0.65,0.15,0.35)$

Table 13: $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$

$\mathcal{V}_{NSC}$	$A_s(v)$	$\lambda_s(v)$
v_1	$([0.2,0.4], [0.7,1.0], [0.1,0.2])$	$(0.20,0.75,0.15)$
v_2	$([0.5,0.6], [0.2,1.0], [0.3,0.4])$	$(0.55,0.25,0.25)$
v_3	$([0.4,0.6], [0.4,1.0], [0.2,0.4])$	$(0.55,0.45,0.25)$

Then $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ are both $\mathcal{R}$ Ext and $\mathcal{J}$ Ext NSCSs in $\mathcal{V}_{NSC}$. $\mathcal{C}_{NSC} \cup_P \mathcal{B}_{NSC} = (A_s \cup_P B_s, \lambda_s \vee_P \psi_s)$ and $\mathcal{C}_{NSC} \cap_P \mathcal{B}_{NSC} = (A_s \cap_P B_s, \lambda_s \wedge_P \psi_s)$ are given below Tables 0.21 and 0.21 .

Table 14: $\mathcal{C}_{NSC} \cup_P \mathcal{B}_{NSC} = (A_s \cup_P B_s, \lambda_s \vee_P \psi_s)$

$\mathcal{V}_{NSC}$	$(A_s \cup B_s(v)$	$(\lambda_s \vee \psi_s(v)$
v_1	([0.3,0.4], [0.7,1.0], [0.2,0.3])	(0.35,0.75,0.25)
v_2	([0.5,0.6], [0.4,1.0], [0.4,0.6])	(0.55,0.45,0.45)
v_3	([0.6,0.7], [0.4,1.0], [0.4,0.5])	(0.65,0.45,0.35)

Table 15: $\mathcal{C}_{NSC} \cap_P \mathcal{B}_{NSC} = (A_s \cap_P B_s, \lambda_s \wedge_P \psi_s)$

$\mathcal{V}_{NSC}$	$(A_s \cap_P B_s(v)$	$(\lambda_s \wedge_P \psi_s(v)$
v_1	([0.2,0.4], [0.5,1.0], [0.1,0.2])	(0.30,0.55,0.15)
v_2	([0.2,0.3], [0.2,1.0], [0.3,0.4])	(0.25,0.25,0.35)
v_3	([0.4,0.6], [0.1,1.0], [0.2,0.4])	(0.55,0.15,0.35)

Then $\mathcal{C}_{NSC} \cup_P \mathcal{B}_{NSC}$ is neither an $\mathcal{J}$ Ext NSCS nor a $\mathcal{R}$ Ext NSCS in $\mathcal{V}_{NSC}$ since

$$(\mathcal{J}_{\lambda_s} \vee_P \mathcal{J}_{\psi_s})(c) = 1.0 \in (0.2,1.0) = ((\mathcal{J}_{A_s} \cup_P \mathcal{J}_{B_s})^-(c), (\mathcal{J}_{A_s} \cup_P \mathcal{J}_{B_s})^+(c))$$

and

$$(\mathcal{R}_{\lambda_s} \vee_P \mathcal{R}_{\psi_s})(a) = 0.35 \in (0.3,0.4) = ((\mathcal{R}_{A_s} \cup_P \mathcal{R}_{B_s})^-(a), (\mathcal{R}_{A_s} \cup_P \mathcal{R}_{B_s})^+(a)).$$

Remark 3.19$R-$ union and $R-$ intersection of $\mathcal{R}$ Int (resp. $\mathcal{J}$ Int and $\mathcal{S}$ Int) NSCSs may not be $\mathcal{A}$ Int (resp. $\mathcal{J}$ Int and $\mathcal{S}$ Int) NSCSs in the below examples.

Example 3.20 Let $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ and $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$ be NSCSs in $\mathcal{V}_{NSC}$ where

$$A_s(v) = \{(v, (0.3,0.5), (0.4,1.0), (0.3,0.4))/v \in \mathcal{V}_{NSC}\}$$
$$\lambda_s(v) = \{(v, 0.4,0.2,0.4)/v \in \mathcal{V}_{NSC}\}$$
$$B_s(v) = \{(v, (0.5,0.6), (0.2,1.0), (0.3,0.2))/v \in \mathcal{V}_{NSC}\}$$
$$\psi_s(v) = \{(v, 0.5,0.3,0.2)/v \in \mathcal{V}_{NSC}\}$$

Then $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ are $\mathcal{R}$ Int NSCSs in $\mathcal{V}_{NSC}$ and $\mathcal{C}_{NSC} \cup_R \mathcal{B}_{NSC} = (A_s \cup_R B_s, \lambda_s \wedge_R \psi_s)$ with

$$A_s \cup_R B_s = \{\langle v, [0.5,0.6], [0.4,1.0], [0.3,0.4]\rangle/v \in \mathcal{V}_{NSC}\},$$
$$\lambda_s \wedge_R \psi_s = \{\langle v, 0.4,0.2,0.2\rangle/v \in \mathcal{V}_{NSC}\}.$$

Note that $(\mathcal{R}_{\lambda_s} \wedge_R \mathcal{R}_{\psi_s})(v) = 0.4 < 0.5 = (\mathcal{R}_{A_s} \cup_R \mathcal{R}_{B_s})^-(v)$ and $(\mathcal{R}_{\lambda_s} \wedge_R \mathcal{J}_{\psi_s})(v) = 0.2 < 0.3 = (\mathcal{J}_{A_s} \cup_R \mathcal{J}_{B_s})^-(v)$. Hence $\mathcal{C}_{NSC} \cup_R \mathcal{B}_{NSC} = (A_s \cup_R B_s, \lambda_s \wedge_R \psi_s)$ is neither a $\mathcal{R}$ Int NSCS nor a $\mathcal{J}$ Int NSCS in $\mathcal{V}_{NSC}$. But we know that $\mathcal{C}_{NSC} \cup_R \mathcal{B}_{NSC} = (A_s \cup B_s, \lambda_s \wedge \psi_s)$ is an $\mathcal{S}$ Int NSCS in $\mathcal{V}_{NSC}$.

The $R-$intersection $\mathcal{C}_{NSC} \cap_R \mathcal{B}_{NSC} = (A_s \cap B_s, \lambda_s \vee \psi_s)$ with

$$A_s \cap_R B_s = \{\langle v, [0.3,0.5], [0.2,1.0], [0.3,0.2]\rangle/v \in \mathcal{V}_{NSC}\},$$
$$\lambda_s \vee_R \psi_s = \{\langle v, 0.5,0.3,0.4\rangle/v \in \mathcal{V}_{NSC}\}.$$

Since $(\mathcal{J}_{A_s} \cap_R \mathcal{J}_{B_s})^-(v) \leq (\mathcal{J}_{\lambda_s} \vee_R \mathcal{J}_{\psi_s})(v) \leq (\mathcal{J}_{A_s} \cap_R \mathcal{J}_{B_s})^+(v)$ for all $v \in \mathcal{V}_{NSC}$. $\mathcal{C}_{NSC} \cap_R \mathcal{B}_{NSC} = (A_s \cap B_s, \lambda_s \vee_R \psi_s)$ is an $\mathcal{J}$ Int NSCS in $\mathcal{V}_{NSC}$.

But neither a $\mathcal{R}$ Int NSCS $\mathcal{S}$ Int NSCSs in $\mathcal{V}_{NSC}$.

Example 3.21 Let $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ and $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$ be NSCSs in $\mathcal{V}_{NSC}$ where

$$A_s = \{(v, (0.3,0.5), (0.5,1.0), (0.2,0.4))/v \in \mathcal{V}_{NSC}\}$$
$$\lambda_s = \{(v, 0.4,0.2,0.4)/v \in \mathcal{V}_{NSC}\}$$
$$B_s = \{(v, (0.1,0.5), (0.6,1.0), (0.3,0.5))/v \in \mathcal{V}_{NSC}\}$$
$$\psi_s = \{(v, 0.3,0.2,0.5)/v \in \mathcal{V}_{NSC}\}$$

Then $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ are $\mathcal{J}$ Int NSCSs in $\mathcal{V}_{NSC}$ and $\mathcal{C}_{NSC} \cup_R \mathcal{B}_{NSC} = (A_s \cup_R B_s, \lambda_s \wedge_R \psi_s)$ with

$$A_s \cup_R B_s = \{\langle v, [0.3,0.5], [0.6,1.0], [0.3,0.5]\rangle/v \in \mathcal{V}_{NSC}\},$$
$$\lambda_s \wedge_R \psi_s = \{\langle v, 0.3,0.2,0.4\rangle/v \in \mathcal{V}_{NSC}\}.$$

Since $(\mathcal{J}_{\lambda_s} \wedge_R \mathcal{J}_{\psi_s})(v) = 0.2 < 0.6 = (\mathcal{J}_A \cup_R \mathcal{J}_B)^-(v)$ we know that and $\mathcal{C}_{NSC} \cup_R \mathcal{B}_{NSC}$ is not an $\mathcal{J}$ Int NSCS in $\mathcal{V}_{NSC}$.

Also, the $R-$intersection $\mathcal{C}_{NSC} \cap_R \mathcal{B}_{NSC} = (A_s \cap B_s, \lambda_s \vee_R \psi_s)$ with

$$A_s \cap_R B_s = \{\langle v, [0.1,0.5], [0.5,1.0], [0.2,0.4]\rangle/v \in \mathcal{V}_{NSC}\},$$

$\lambda_s \vee_R \psi_s = \{\langle v, 0.4, 0.2, 0.5\rangle / v \in \mathcal{V}_{NSC}\}$

and it is not an $\mathcal{J}$ Int NSCS in $\mathcal{V}_{NSC}$.

Example 3.22 Let $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ and $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$ be NSCSs in $\mathcal{V}_{NSC}$ where

$$A_s = \{(v, (0.2,0.3), (0.4,1.0), (0.4,0.5))/v \in \mathcal{V}_{NSC}\}$$
$$\lambda_s = \{(v, 0.4,0.2,0.2)/v \in \mathcal{V}_{NSC}\}$$
$$B_s = \{(v, (0.4,0.6), (0.3,1.0), (0.3,0.4))/v \in \mathcal{V}_{NSC}\}$$
$$\psi_s = \{(v, 0.3,0.2,0.1)/v \in \mathcal{V}_{NSC}\}$$

Then $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ are $\mathcal{S}$ Int NSCSs in $\mathcal{V}_{NSC}$ and $\mathcal{C}_{NSC} \cup_R \mathcal{B}_{NSC} = (A_s \cup_R B_s, \lambda_s \wedge_R \psi_s)$ with

$$A_s \cup_R B_s = \{\langle v, [0.4,0.6], [0.4,1.0], [0.4,0.5]\rangle / v \in \mathcal{V}_{NSC}\},$$
$$\lambda_s \wedge_R \psi_s = \{\langle v, 0.3,0.2,0.1\rangle / v \in \mathcal{V}_{NSC}\}.$$

which is not an $\mathcal{S}$ Int NSCS in $\mathcal{V}_{NSC}$. If $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ and $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$ be NSCSs in $\mathcal{R}$ where

$$A_s(v) = \{(v, (0.2,0.3), (0.6,1.0), (0.2,0.4))/v \in \mathcal{V}_{NSC}\}$$
$$\lambda_s(v) = \{(v, 0.5,0.4,0.1)/v \in \mathcal{R}\}$$
$$B_s(v) = \{(v, (0.1,0.2), (0.5,1.0), (0.4,0.5))/v \in \mathcal{V}_{NSC}\}$$
$$\psi_s(v) = \{(v, 0.6,0.2,0.2)/v \in \mathcal{V}_{NSC}\}$$

then $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ are $\mathcal{S}$ Int NSCSs in $\mathcal{V}_{NSC}$ and the $R-$ intersection $\mathcal{C}_{NSC} \cap_R \mathcal{B}_{NSC} = (A_s \cap_R B_s, \lambda_s \vee_R \psi_s)$ of $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ which is given as follows:

$$A_s \cap_R B_s = \{\langle x, [0.1,0.2], [0.5,1.0], [0.2,0.4]\rangle / v \in \mathcal{V}_{NSC}\},$$
$$\lambda_s \vee_R \psi_s = \{\langle v, 0.6,0.4,0.2\rangle / v \in \mathcal{V}_{NSC}\}$$

and it is not an $\mathcal{S}$ Int NSCS in $\mathcal{V}_{NSC}$.

Theorem 3.23 Let $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ and $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$ be $\mathcal{R}$ Int NSCSs in $\mathcal{V}_{NSC}$ such that

$$(\forall\ v \in \mathcal{V}_{NSC})(\max\{\mathcal{R}_A^-(v), \mathcal{R}_B^-(v)\} \le (\mathcal{R}_{\lambda_s} \wedge_R \mathcal{R}_{\psi_s})(v)). \tag{7}$$

Then the $R-$ union of $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ is a $\mathcal{R}$ Int NSCSs in $\mathcal{V}_{NSC}$.

Proof. Let $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ be $\mathcal{R}$ Int NSCSs in $\mathcal{V}_{NSC}$ it satisfy the condition (7). Then $\mathcal{R}_{A_s}^-(v) \le \mathcal{R}_{\lambda_s}(v) \le \mathcal{R}_{A_s}^+(v)$ and $\mathcal{R}_{B_s}^-(v) \le \mathcal{R}_{\psi_s}(v) \le \mathcal{R}_{B_s}^+(v)$.

And so, $(\mathcal{R}_{\lambda_s} \wedge_R \mathcal{R}_{\psi_s})(v) \le (\mathcal{R}_{A_s} \cup_R \mathcal{R}_{B_s})^+(v)$.

It follows from (7) that,

$$(\mathcal{R}_{A_s} \cup_R \mathcal{R}_{B_s})^-(v) = \max\{\mathcal{R}_{A_s})^-(v), \mathcal{R}_{B_s})^-(v)\}$$
$$\le (\mathcal{R}_{\lambda_s} \wedge_R \mathcal{R}_{\psi_s})(v) \le (\mathcal{R}_{A_s} \cup_R \mathcal{R}_{B_s})^+(v).$$

Hence $\mathcal{C}_{NSC} \cup_R \mathcal{B}_{NSC} = (A_s \cup_R B_s, \lambda_s \wedge_R \psi_s)$ is a $\mathcal{R}$ Int NSCS in $\mathcal{V}_{NSC}$.

Theorem 3.24 Let $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ and $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$ be $\mathcal{J}$ Int NSCSs in $\mathcal{V}_{NSC}$ such that

$$(\forall \ v \in \mathcal{V}_{NSC})(\max\{\mathcal{J}_A^-(v), \mathcal{J}_B^-(v)\} \leq (\mathcal{J}_{\lambda_s} \wedge_R \mathcal{J}_{\psi_s})(v)). \quad (8)$$

Then the $R-$ union of $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ is a $\mathcal{J}$ Int NSCSs in a non-empty set $\mathcal{V}_{NSC}$.

Proof. Let $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ be $\mathcal{J}$ Int NSCSs in $\mathcal{V}_{NSC}$ it satisfy the condition (8). Then $\mathcal{J}_{A_s}^-(v) \leq \mathcal{J}_{\lambda_s}(v) \leq \mathcal{J}_{A_s}^+(v)$ and $\mathcal{J}_{B_s}^-(v) \leq \mathcal{J}_{\psi_s}(v) \leq \mathcal{J}_{B_s}^+(v)$.

And so, $(\mathcal{J}_{\lambda_s} \wedge_R \mathcal{J}_{\psi_s})(v) \leq (\mathcal{J}_{A_s} \cup_R \mathcal{J}_{B_s})^+(v)$.

It follows from (8) that,

$$(\mathcal{J}_{A_s} \cup_R \mathcal{J}_{B_s})^-(v) = \max\{\mathcal{J}_{A_s})^-(v), \mathcal{J}_{B_s})^-(v)\}$$
$$\leq (\mathcal{J}_{\lambda_s} \wedge_R \mathcal{J}_{\psi_s})(v) \leq (\mathcal{J}_{A_s} \cup_R \mathcal{J}_{B_s})^+(v).$$

Hence $\mathcal{C}_{NSC} \cup_R \mathcal{B}_{NSC} = (A_s \cup_R B_s, \lambda_s \wedge_R \psi_s)$ is a $\mathcal{J}$ Int NSCS in $\mathcal{V}_{NSC}$.

Theorem 3.25 Let $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ and $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$ be $\mathcal{S}$ Int NSCSs in $\mathcal{V}_{NSC}$ such that

$$(\forall \ v \in \mathcal{V}_{NSC})(\max\{\mathcal{S}_A^-(v), \mathcal{S}_B^-(v)\} \leq (\mathcal{S}_{\lambda_s} \wedge_R \mathcal{S}_{\psi_s})(v)). \quad (9)$$

Then the $R-$ union of $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ is a $\mathcal{S}$ Int NSCSs in $\mathcal{V}_{NSC}$.

Proof. Let $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ be $\mathcal{S}$ Int NSCSs in $\mathcal{V}_{NSC}$ it satisfy the condition (9). Then $\mathcal{S}_{A_s}^-(v) \leq \mathcal{S}_{\lambda_s}(v) \leq \mathcal{S}_{A_s}^+(v)$ and $\mathcal{S}_{B_s}^-(v) \leq \mathcal{A}_{\psi_s}(v) \leq \mathcal{S}_{B_s}^+(v)$.

And so, $(\mathcal{S}_{\lambda_s} \wedge_R \mathcal{S}_{\psi_s})(v) \leq (\mathcal{S}_{A_s} \cup_R \mathcal{S}_{B_s})^+(v)$.

It follows from (9) that,

$$(\mathcal{S}_{A_s} \cup_R \mathcal{S}_{B_s})^-(v) = \max\{\mathcal{S}_{A_s})^-(v), \mathcal{S}_{B_s})^-(v)\}$$
$$\leq (\mathcal{S}_{\lambda_s} \wedge_R \mathcal{S}_{\psi_s})(v) \leq (\mathcal{S}_{A_s} \cup_R \mathcal{S}_{B_s})^+(v).$$

Hence $\mathcal{C}_{NSC} \cup_R \mathcal{B}_{NSC} = (A_s \cup_R B_s, \lambda_s \wedge_R \psi_s)$ is a $\mathcal{S}$ Int NSCS in $\mathcal{V}_{NSC}$.

Corollary 3.26 If two Int NSCSs $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ and $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$ satisfy (7) ,(8)and (9) then the $R-$ union of $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ and $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$ is an Int NSCSs.

Theorem 3.27 Let $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ and $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$ be $\mathcal{J}$ Int NSCSs in $\mathcal{V}_{NSC}$ such that

$$(\forall \ v \in \mathcal{V}_{NSC})((\mathcal{J}_{\lambda_s} \vee_R \mathcal{J}_{\psi_s})(v) \leq \min\{\mathcal{J}_A^+(v), \mathcal{J}_B^+(v)\}). \quad (10)$$

Then the $R-$ intersection of $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ is a $\mathcal{J}$ Int NSCS in $\mathcal{V}_{NSC}$.

Proof. If 10 is valid. Then $\mathcal{J}_{A_s}^-(v) \leq \mathcal{J}_{\lambda_s}(v) \leq \mathcal{J}_{A_s}^+(v)$ and $\mathcal{J}_{B_s}^-(v) \leq \mathcal{J}_{\psi_s}(v) \leq \mathcal{J}_{B_s}^+(v)$ for all $\mathcal{V}_{NSC}$. It follows from 10 that

$(\mathcal{J}_{A_s} \cap_R \mathcal{J}_{B_s})^-(v) \leq (\mathcal{J}_{\lambda_s} \cap_R \mathcal{J}_{\psi_s})^-(v) \leq \min\{\mathcal{J}^+_{A_s}(v), \mathcal{J}^+_{B_s}(v)\} \leq (\mathcal{J}_{A_s} \cap_R \mathcal{J}_{B_s})^+(v)$
for all $v \in \mathcal{V}_{NSC}$.

$\therefore\ A_s \cap_R B_s = (A_s \cap_R B_s, \lambda_s \vee_R \psi_s)$ is an $\mathcal{J}$ Int NSCS in $\mathcal{V}_{NSC}$.

Theorem 3.28 Let $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ and $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$ be $\mathcal{R}$ Int NSCSs in $\mathcal{V}_{NSC}$ such that

$$(\forall\ v \in \mathcal{V}_{NSC})((\mathcal{R}_{\lambda_s} \vee_R \mathcal{R}_{\psi_s})(v) \leq \min\{\mathcal{R}^+_A(v), \mathcal{R}^+_B(v)\}). \quad (11)$$

Then the $R-$ intersection of $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ is a $\mathcal{R}$ Int NSCS in $\mathcal{V}_{NSC}$.

Proof. If 11 is valid. Then $\mathcal{R}^-_{A_s}(v) \leq \mathcal{R}_{\lambda_s}(v) \leq \mathcal{R}^+_{A_s}(v)$ and $\mathcal{R}^-_{B_s}(v) \leq \mathcal{R}_{\psi_s}(v) \leq \mathcal{R}^+_{B_s}(v)$ for all $\mathcal{V}_{NSC}$. It follows from 11 that

$(\mathcal{R}_{A_s} \cap \mathcal{R}_{B_s})^-(v) \leq (\mathcal{R}_{\lambda_s} \cap_R \mathcal{R}_{\psi_s})^-(v) \leq \min\{\mathcal{R}^+_{A_s}(v), \mathcal{R}^+_{B_s}(v)\} \leq (\mathcal{R}_{A_s} \cap_R \mathcal{R}_{B_s})^+(v)$
for all $v \in \mathcal{V}_{NSC}$.

$\therefore\ A_s \cap_R B_s = (A_s \cap_R B_s, \lambda_s \vee_R \psi_s)$ is an $\mathcal{R}$Int NSCS in $\mathcal{V}_{NSC}$.

Theorem 3.29 Let $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ and $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$ be $\mathcal{S}$ Int NSCSs in $\mathcal{V}_{NSC}$ such that

$$(\forall\ v \in \mathcal{V}_{NSC})((\mathcal{S}_{\lambda_s} \vee_R \mathcal{S}_{\psi_s})(v) \leq \min\{\mathcal{S}^+_A(v), \mathcal{S}^+_B(v)\}). \quad (12)$$

Then the $R-$ intersection of $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ is a $\mathcal{S}$ Int NSCS in $\mathcal{V}_{NSC}$.

Proof. If 12 is valid. Then $\mathcal{S}^-_{A_s}(v) \leq \mathcal{S}_{\lambda_s}(v) \leq \mathcal{S}^+_{A_s}(v)$ and $\mathcal{S}^-_{B_s}(v) \leq \mathcal{S}_{\psi_s}(v) \leq \mathcal{S}^+_{B_s}(v)$ for all $\mathcal{V}_{NSC}$. It follows from 12 that

$(\mathcal{S}_{A_s} \cap_R \mathcal{S}_{B_s})^-(v) \leq (\mathcal{S}_{\lambda_s} \cap \mathcal{S}_{\psi_s})^-(v) \leq \min\{\mathcal{S}^+_{A_s}(v), \mathcal{S}^+_{B_s}(v)\} \leq (\mathcal{S}_{A_s} \cap_R \mathcal{S}_{B_s})^+(v)$
for all $v \in \mathcal{V}_{NSC}$.

$\therefore\ A_s \cap_R B_s = (A_s \cap_R B_s, \lambda_s \vee_R \psi_s)$ is an $\mathcal{S}$ Int NSCS in $\mathcal{V}_{NSC}$.

Corollary 3.30 If two Int NSCSs $\mathcal{C}_{NSC} = (A_s(v), \lambda_s(v))$ and $\mathcal{B}_{NSC} = (B_s(v), \psi_s(v))$ satisfy conditions (10), (11), (12) then the $R-$intersection of $\mathcal{C}_{NSC}$ and $\mathcal{B}_{NSC}$ is an Int NSCS in $\mathcal{V}_{NSC}$.

Conclusions

In this paper we have introduced the notion of Neutrosophic spherical cubic sets .We have discussed properties of Neutrosophic spherical cubic sets. For the future prospects, we will extend this work by using topological structures and commit to exploring the real life applications.

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Properties of Co-local Function and Related Φ-operator in Ideal Neutrosophic Topological Spaces

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Abstract: In this paper, we introduce and study the notions of neutrosophic co-local function and neutrosophic complement co-local function in the context of a neutrosophic topological space equipped with a neutrosophic ideal. Moreover, we explore some new classes of neutrosophic sets defined in terms of the neutrosophic co-local function and the neutrosophic complement co-local function.

Keywords: Neutrosophic set; neutrosophic topological space; neutrosophic ideal; neutrosophic co-local function.

1. Introduction

The notion of neutrosophic set has gained much relevance in recent years due to its various applications. This notion was proposed by Smarandache [1] and has been studied by many researchers as can be seen in [2-8]. In particular, Karatas and Kuru [4] introduced new neutrosophic set operations and with them defined the concept of neutrosophic topological space. Following this line of research, Albowi and Salama [2] introduced the notion of neutrosophic ideal, which was later used by Salama and Smarandache [8] to introduce the concept of neutrosophic local function, investigate its properties and analyze the relations between different neutrosophic ideals and neutrosophic topologies. The purpose of this paper is to continue with this line of research, but this time we define the neutrosophic co-local function and the neutrosophic complement co-local function, investigate the main properties of these new neutrosophic operators with them we build new classes of neutrosophic sets in a neutrosophic topological space endowed with a neutrosophic ideal.

2. Preliminaries

Throughout this paper, let X be a nonempty set, called the universe of discourse.

Definition 2.1. [1] A neutrosophic set N on X is an object of the form

$$N = \{\langle x, \mu_N(x), \sigma_N(x), \gamma_N(x)\rangle : x \in X\},$$

where $\mu_N, \sigma_N, \gamma_N$ are functions from X to $[0,1]$ and $0 \leq \mu_N(x) + \sigma_N(x) + \gamma_N(x) \leq 3$.

We denote by $\mathcal{N}(X)$ the collection of all neutrosophic sets over X.

Definition 2.2. [4] For $N, M \in \mathcal{N}(X)$ we define the following:

(1) (Inclusion) N is called a neutrosophic subset of M, denoted by $N \sqsubseteq M$, if $\mu_N(x) \leq \mu_M(x)$, $\sigma_N(x) \geq \sigma_M(x)$ and $\gamma_N(x) \geq \gamma_M(x)$ for all $x \in X$. Also, we can say that M is a neutrosophic super set of N.

(2) (Equality) N is called neutrosophic equal to M, denoted by $N = M$, if $N \sqsubseteq M$ and $M \sqsubseteq N$.

(3) (Universal set) N is called the neutrosophic universal set, denoted by $\tilde{X}$, if $\mu_N(x) = 1, \sigma_N(x) = 0$ and $\gamma_N(x) = 0$ for all $x \in X$.

(4) (Empty set) N is called the neutrosophic empty set, denoted by $\tilde{\emptyset}$, if $\mu_N(x) = 0, \sigma_N(x) = 1$ and $\gamma_N(x) = 1$ for all $x \in X$.

(5) (Intersection) The neutrosophic intersection of N and M, denoted by $N \sqcap M$, is defined as

$$N \sqcap M = \{\langle x, \mu_N(x) \wedge \mu_M(x), \sigma_N(x) \vee \sigma_M(x), \gamma_N(x) \vee \gamma_M(x)\rangle : x \in X\}.$$

(6) (Union) The neutrosophic union of N and M, denoted by $N \sqcup M$, is defined as

$$N \sqcup M = \{\langle x, \mu_N(x) \vee \mu_M(x), \sigma_N(x) \wedge \sigma_M(x), \gamma_N(x) \wedge \gamma_M(x)\rangle : x \in X\}.$$

(7) (Complement) The neutrosophic complement of N, denoted by N^c, is defined as

$$N^c = \{\langle x, \gamma_N(x), 1 - \sigma_N(x), \mu_N(x)\rangle : x \in X\}.$$

Proposition 2.3. [4] If $N, M \in \mathcal{N}(X)$, then we have the following properties:

(1) $N \sqcap N = N$ and $N \sqcup N = N$.

(2) $N \sqcap M = M \sqcap N$ and $N \sqcup M = M \sqcup N$.

(3) $N \sqcap \tilde{\emptyset} = \tilde{\emptyset}$ and $N \sqcap \tilde{X} = N$.

(4) $N \sqcup \tilde{\emptyset} = N$ and $N \sqcup \tilde{X} = \tilde{X}$.

(5) $N \sqcap (M \sqcap O) = (N \sqcap M) \sqcap O$ and $N \sqcup (M \sqcup O) = (N \sqcup M) \sqcup O$.

(6) $(N^c)^c = N$.

Proposition 2.4. [6] Let $N, M \in \mathcal{N}(X)$. Then, $N \sqsubseteq M$ if and only if $M^c \sqsubseteq N^c$.

The union and intersection operations given in Definition 2.2 can be extended as follows.

Definition 2.5. [7] For $\{N_j : j \in J\} \subseteq \mathcal{N}(X)$ we define the following operations:

(1) (Arbitrary intersection) The arbitrary neutrosophic intersection of the collection $\{N_j : j \in J\}$, denoted by $\sqcap_{j \in J} N_j$, is defined as

$$\prod_{j \in J} N_j = \left\{\left\langle x, \inf_{j \in J} \mu_{N_j}(x), \sup_{j \in J} \sigma_{N_j}(x), \sup_{j \in J} \gamma_{N_j}(x)\right\rangle : x \in X\right\}.$$

(2) (Arbitrary union) The arbitrary neutrosophic union of the collecction $\{N_j : j \in J\}$, denoted by $\sqcup_{j \in J} N_j$, is defined as

$$\bigsqcup_{j \in J} N_j = \left\{\left\langle x, \sup_{j \in J} \mu_{N_j}(x), \inf_{j \in J} \sigma_{N_j}(x), \inf_{j \in J} \gamma_{N_j}(x)\right\rangle : x \in X\right\}.$$

Proposition 2.6. [4] If $\{N_j : j \in J\} \subseteq \mathcal{N}(X)$ and $M \in \mathcal{N}(X)$, then we have the following properties:

(1) $M \sqcap \left(\sqcup_{j \in J} N_j\right) = \sqcup_{j \in J}\left(M \sqcap N_j\right)$.

(2) $M \sqcup \left(\sqcap_{j \in J} N_j\right) = \sqcap_{j \in J}(M \sqcup N_j)$.

(3) $\left(\sqcap_{j \in J} N_j\right)^c = \sqcup_{j \in J} N_j^c$.

(4) $\left(\sqcup_{j \in J} N_j\right)^c = \sqcap_{j \in J} N_j^c$.

Definition 2.7. [4] A neutrosophic topology on a set X is a collection $\tau \subseteq \mathcal{NS}(X)$ which satisfies the following conditions:

(1) $\widetilde{\emptyset}$ and $\tilde{X}$ are in τ.

(2) The intersection of two neutrosophic sets belonging to τ is in τ.

(3) The union of any collection of neutrosophic sets belonging to τ is in τ.

A set X for which a neutrosophic topology τ has been defined is called a neutrosophic topological space and is denoted as a pair (X,τ). If $N \in \tau$, then N is called a neutrosophic open set and if $N^c \in \tau$, then N is called a neutrosophic closed set. We denote by τ^c the collection of all neutrosophic closed sets in the neutrosophic topological space (X,τ).

Proposition 2.8. [4] Let (X,τ) be a neutrosophic topological space. Then, the following conditions hold:

(1) $\widetilde{\emptyset}$ and $\tilde{X}$ are in τ^c.

(2) The union of two neutrosophic sets belonging to τ^c is in τ^c.

(3) The intersection of any collection of neutrosophic sets belonging to τ^c is in τ^c.

Definition 2.9. [4] Let (X,τ) be a neutrosophic topological space and $N \in \mathcal{N}(X)$. The neutrosophic closure of N, denoted by $Cl(N)$, is defined as

$$Cl(N) = \bigsqcap \{F \in \mathcal{N}(X) : N \sqsubseteq F \text{ and } F \in \tau^c\};$$

while the neutrosophic interior of N, denoted by $\text{Int}(N)$, is defined as

$$\text{Int}(N) = \bigsqcup \{U \in \mathcal{N}(X) : U \sqsubseteq N \text{ and } U \in \tau\}.$$

Proposition 2.10. [4] Let (X,τ) be a neutrosophic topological space and $N, M \in \mathcal{N}(X)$. Then, the following conditions hold:

(1) $N \sqsubseteq Cl(N)$ and $\text{Int}(N) \sqsubseteq N$.

(2) If $N \sqsubseteq M$, then $Cl(N) \sqsubseteq Cl(M)$ and $\text{Int}(N) \sqsubseteq \text{Int}(M)$.

(3) $N \in \tau^c$ if and only if $N = Cl(N)$.

(4) $N \in \tau$ if and only if $N = \text{Int}(N)$.

Definition 2.11. [5] A neutrosophic set $M = \{\langle x, \mu_M(x), \sigma_M(x), \gamma_M(x)\rangle : x \in X\}$ is called a neutrosophic point if for any element $y \in X, \mu_M(y) = a, \sigma_M(y) = b, \gamma_M(y) = c$ for $y = x$ and $\mu_M(y) = 0, \sigma_M(y) = 1, \gamma_M(y) = 1$ for $y \neq x$, where $a \in (0,1]$ and $b, c \in [0,1)$. In this case, the neutrosophic point M is denoted by $M^x_{a,b,c}$ or simply by $x_{a,b,c}$. Also, x is called the support of the neutrosophic point $x_{a,b,c}$. The neutrosophic point $x_{1,0,0}$ is called a neutrosophic crisp point.

Definition 2.12. [5] Let $N \in \mathcal{N}(X)$. A neutrosophic point $x_{a,b,c}$ is said to belong to N, denoted by $x_{a,b,c} \in N$, if $\mu_N(x) \geq a, \sigma_N(x) \leq b$ and $\gamma_N(x) \leq c$.

Lemma 2.13. [5] Let $N, M \in \mathcal{N}(X)$. Then, we have:

(1) $N = \sqcup\{x_{a,b,c} : x_{a,b,c} \in N\}$.

(2) If $x_{a,b,c} \in N$ and $N \sqsubseteq M$, then $x_{a,b,c} \in M$.

Proposition 2.14. Let $N, M \in \mathcal{N}(X)$. Then, the following properties are equivalent:

(1) $N \sqsubseteq M$.

(2) $x_{a,b,c} \in N$ implies that $x_{a,b,c} \in M$.

Proof. The proof follows directly from Lemma 2.13.

Remark 2.15. It is important to note that $\widetilde{\emptyset}$ is not the only neutrosophic set that does not have points belonging to it. For example, if $X = \{x, y\}$, then $N = \{\langle x, 0,0.5,1\rangle, \langle y, 0,0.4,1\rangle\}$ is a neutrosophic set over X for which there are no neutrosophic points belonging to it.

Let $\mathcal{N}_p(X) = \{N \in \mathcal{N}(X)$: there exists a neutrosophic point $x_{a,b,c} \in N\}$ and let $\mathcal{N}'(X) = \{\widetilde{\emptyset}\} \cup \mathcal{N}_p(X)$. In the remainder of this paper, we will use the definitions and results described previously, restricted to the collection $\mathcal{N}'(X)$.

Definition 2.16. [9] Let (X, τ) be a neutrosophic topological space and $N \in \mathcal{N}'(X)$. The neutrosophic point-kernel of N, denoted by $\mathrm{Ker}_p(N)$, is defined as

$$\mathrm{Ker}_p(N) = \bigsqcup \{x_{a,b,c} \in \mathcal{N}'(X) : F \sqcap N \neq \widetilde{\emptyset} \text{ for every } F \in \tau^c(x_{a,b,c})\},$$

where $\tau^c(x_{a,b,c}) = \{F \in \tau^c : x_{a,b,c} \in F\}$.

According to [9], the collection $\tau_k = \{N \in \mathcal{N}'(X) : \mathrm{Ker}_p(N^c) = N^c\}$ is a neutrosophic topology on X and Ker_p is the neutrosophic closure in the neutrosophic topological space (X, τ_k). We say that a neutrosophic set N is neutrosophic τ_k-open, if $N \in \tau_k$. The complement of a neutrosophic τ_k-open set we will call it a neutrosophic τ_k-closed set. We denote by Cok_p the neutrosophic interior in the neutrosophic topological space (X, τ_k). Let us note that M is τ_k-open neutrosophic if and only if $\mathrm{Cok}_p(M) = M$; while M is τ_k-closed neutrosophic if and only if $\mathrm{Ker}_p(M) = M$.

Definition 2.17. [2] A neutrosophic ideal on a set X is a nonempty collection $\mathcal{L} \subseteq \mathcal{N}'(X)$, which satisfies the following conditions:

(1) $N \in \mathcal{L}$ and $M \sqsubseteq N$ imply that $M \in \mathcal{L}$. (Hereditary property)

(2) $N, M \in \mathcal{L}$ imply that $N \sqcup M \in \mathcal{L}$. (Finite additivity property)

Definition 2.18. [9] An application $\Upsilon : \mathcal{N}'(X) \to \mathcal{N}'(X)$ is called a neutrosophic closure operator if it satisfies the following conditions:

(1) $N \sqsubseteq \Upsilon(N)$ (expansivity),

(2) $\Upsilon(\Upsilon(N)) = \Upsilon(N)$ (idempotency),

(3) $\Upsilon(N \sqcup M) = \Upsilon(N) \sqcup \Upsilon(M)$ (additivity),

(4) $\Upsilon(\widetilde{\emptyset}) = \widetilde{\emptyset}$ (non-spontaneous creation),

whenever $M, N \in \mathcal{N}'(X)$.

Lemma 2.19. [9] If $\Upsilon : \mathcal{N}'(X) \to \mathcal{N}'(X)$ is a neutrosophic closure operator, then the collection $\tau(\Upsilon) = \{N \in \mathcal{N}'(X) : \Upsilon(N^c) = N^c\}$ is a neutrosophic topology on X and Υ is the neutrosophic closure in the neutrosophic topological space $(X, \tau(\Upsilon))$.

3. Neutrosophic co-local function and related Φ-operator

In this section, we introduce and study the concept of neutrosophic co-local function as a natural generalization of the neutrosophic point-kernel of a set in a neutrosophic topological space. Moreover, we introduce the concept of neutrosophic complement co-local function (also called neutrosophic Φ-operator) and explore some new classes of neutrosophic sets defined in terms of the neutrosophic co-local function and the neutrosophic complement co-local function.

3.1. Neutrosophic co-local function

Definition 3.1.1. Let (X,τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For each $N \in \mathcal{N}'(X)$, we define the neutrosophic co-local function of N as follows:

$$N^{\bullet}(\mathcal{L},\tau) = \bigsqcup \{x_{a,b,c} \in \mathcal{N}'(X): F \sqcap N \notin \mathcal{L} \text{ for every } F \in \tau^c(x_{a,b,c})\}.$$

We will denote $N^{\bullet}(\mathcal{L},\tau)$ by $N^{\bullet}$ or $N^{\bullet}(\mathcal{L})$. Observe that the neutrosophic co-local function can be seen as an operator from $\mathcal{N}'(X)$ to $\mathcal{N}'(X)$; that is, $(\ )^{\bullet}: \mathcal{N}'(X) \to \mathcal{N}'(X)$, defined by $N \mapsto N^{\bullet}$.

The co-local function is not a neutrosophic closure operator, since in general, it does not satisfy $N \sqsubseteq N^{\bullet}$ for each $N \in \mathcal{N}'(X)$. In the case that $N \sqsubseteq N^{\bullet}$, we say that N is a neutrosophic $\bullet$-dense in itself set. The following example shows that, in general, $\tilde{X}^{\bullet}$ is a proper neutrosophic subset of $\tilde{X}$; that is, $\tilde{X}$ is not neutrosophic $\bullet$-dense in itself.

Example 3.1.2. Let $X = \mathbb{R}$ with the neutrosophic topology $\tau = \{\widetilde{\emptyset}, \widetilde{\mathbb{R}}, A^c\}$, where $A \neq \widetilde{\emptyset}$ is any neutrosophic subset having countable support of $\mathbb{R}$ and $\mathcal{L} = \mathcal{L}_c$ the neutrosophic ideal of all neutrosophic subsets having countable support of $\mathbb{R}$. Observe that $F_1 = \widetilde{\mathbb{R}}$ and $F_2 = A$ are the only neutrosophic closed sets such that $F_1 \neq \widetilde{\emptyset}$ and $F_2 \neq \widetilde{\emptyset}$. Since $\tilde{X} \cap F_1 = F_1 \notin \mathcal{L}_c$ and $\tilde{X} \sqcap F_2 = A \in \mathcal{L}_c$, then is clear that $\tilde{X}^{\bullet} = \widetilde{\mathbb{R}}^{\bullet} = A^c \sqsubseteq \widetilde{\mathbb{R}} = \tilde{X}$, but $\tilde{X}^{\bullet} \neq \tilde{X}$.

Proposition 3.1.3. Let (X,τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For every $N \in \mathcal{N}'(X)$, the following properties hold:

(1) If $\mathcal{L} = \{\widetilde{\emptyset}\}$, then $N^{\bullet} = \mathrm{Ker}_p(N)$.

(2) If $\mathcal{L} = \mathcal{N}'(X)$, then $N^{\bullet} = \widetilde{\emptyset}$.

Lemma 3.1.4. Let (X,τ) be a neutrosophic topological space with two arbitrary neutrosophic ideals $\mathcal{L}$ and $\mathcal{L}'$ on X. If $N, M \in \mathcal{N}'(X)$, then the following properties hold:

(1) If $N \sqsubseteq M$, then $N^{\bullet} \sqsubseteq M^{\bullet}$.

(2) If $\mathcal{L} \subseteq \mathcal{L}'$, then $N^{\bullet}(\mathcal{L}') \sqsubseteq N^{\bullet}(\mathcal{L})$.

(3) $N^{\bullet} = \mathrm{Ker}_p(N^{\bullet}) \sqsubseteq \mathrm{Ker}_p(N)$ ($N^{\bullet}$ is a neutrosophic τ_k-closed set).

(4) $(N^{\bullet})^{\bullet} \sqsubseteq N^{\bullet}$.

(5) $\widetilde{\emptyset}^{\bullet} = \widetilde{\emptyset}$.

(6) $(N \sqcup M)^{\bullet} = N^{\bullet} \sqcup M^{\bullet}$.

(7) If F is a neutrosophic closed set, then $F \sqcap N^{\bullet} = F \sqcap (F \sqcap N)^{\bullet} \sqsubseteq (F \sqcap N)^{\bullet}$.

(8) If $N \in \mathcal{L}$, then $N^{\bullet} = \widetilde{\emptyset}$.

(9) If $N \sqsubseteq N^{\bullet}$, then $N^{\bullet} = \mathrm{Ker}_p(N)$.

(10) If τ_1 and τ_2 be are two neutrosophic topologies on X such that $\tau_1 \subseteq \tau_2$, then $N^{\bullet}(\mathcal{L},\tau_2) \sqsubseteq N^{\bullet}(\mathcal{L},\tau_1)$.

(11) $N^{\bullet}(\mathcal{L} \cap \mathcal{L}') = N^{\bullet}(\mathcal{L}) \sqcup N^{\bullet}(\mathcal{L}')$.

Proof. (1) Assume that $x_{a,b,c} \in N^{\bullet}$ and let $F \in \tau^c(x_{a,b,c})$. Then, $F \sqcap N \notin \mathcal{L}$ and as $N \sqsubseteq M$, we have $F \sqcap N \sqsubseteq F \sqcap M$. By the hereditary property of $\mathcal{L}$, it follows that $F \sqcap M \notin \mathcal{L}$ and hence $x_{a,b,c} \in M^{\bullet}$.

(2) Suppose that $\mathcal{L} \subseteq \mathcal{L}'$, $x_{a,b,c} \in N^{\bullet}(\mathcal{L}')$ and let $F \in \tau^c(x_{a,b,c})$ be arbitrary. Then $N \sqcap F \notin \mathcal{L}'$ and as $\mathcal{L} \subseteq \mathcal{L}'$, it follows that $N \sqcap F \notin \mathcal{L}$, which implies that $x_{a,b,c} \in N^{\bullet}(\mathcal{L})$. Thus, we conclude that $N^{\bullet}(\mathcal{L}') \sqsubseteq N^{\bullet}(\mathcal{L})$.

(3) Let $x_{a,b,c} \in \mathrm{Ker}_p(N^{\bullet})$ and $F \in \tau^c(x_{a,b,c})$ be arbitrary. Then, $F \sqcap N^{\bullet} \neq \widetilde{\emptyset}$, so there exists a neutrosophic point $y_{u,v,w} \in F \sqcap N^{\bullet}$, which implies that $y_{u,v,w} \in F$ and $y_{u,v,w} \in N^{\bullet}$. Since $F \in$

$\tau^c(y_{u,v,w})$, it follows that $F \sqcap N \notin \mathcal{L}$ and so $x_{a,b,c} \in N^\bullet$. On the other hand, as $N^{\cdot} \sqsubseteq \mathrm{Ker}_p(N^\bullet)$, we conclude that $N^\bullet = \mathrm{Ker}_p(N^\bullet)$. Now, since $\{\widetilde{\emptyset}\} \subseteq \mathcal{L}$, by part (1) of Proposition 3.1.3, we have $N^\bullet \sqsubseteq N^\bullet(\{\widetilde{\emptyset}\}) = \mathrm{Ker}_p(N)$.

(4) By part (3), $N^\bullet = \mathrm{Ker}_p(N^\bullet) \sqsubseteq \mathrm{Ker}_p(N)$ for every $N \in \mathcal{N}'(X)$. In particular, for $N^\bullet$ we have $(N^\bullet)^\bullet \sqsubseteq \mathrm{Ker}_p(N^\bullet) = N^\bullet$.

(5) We have

$$\begin{aligned}\widetilde{\emptyset} &= \bigsqcup \{x_{a,b,c} \in \mathcal{N}'(X)\colon\ F \sqcap \widetilde{\emptyset} \notin \mathcal{L} \text{ for every } F \in \tau^c(x_{a,b,c})\} \\ &= \bigsqcup \{x_{a,b,c} \in \mathcal{N}'(X)\colon\ \widetilde{\emptyset} \notin \mathcal{L} \text{ for every } F \in \tau^c(x_{a,b,c})\} = \widetilde{\emptyset}.\end{aligned}$$

(6) By part (1), we have $N^\bullet \sqsubseteq (N \sqcup M)^\bullet$ and $M^\bullet \sqsubseteq (N \sqcup M)^\bullet$. Therefore, $N^\bullet \sqcup M^\bullet \sqsubseteq (N \sqcup M)^\bullet$. For the other inclusion, assume that $x_{a,b,c} \in (N \sqcup M)^\bullet$ and let $F \in \tau^c(x_{a,b,c})$ be arbitrary. Then, $(M \sqcup N) \sqcap F \notin \mathcal{L}$, i.e. $(M \sqcap F) \sqcup (N \sqcap F) \notin \mathcal{L}$. Accordingly, we have the cases $M \sqcap F \notin \mathcal{L}$ or $N \sqcap F \notin \mathcal{L}$. If $M \sqcap F \notin \mathcal{L}$, then we obtain that $x_{a,b,c} \in M^\bullet$, whereas if $N \sqcap F \notin \mathcal{L}$, then we have $x_{a,b,c} \in N^\bullet$. In both cases, it follows that $x_{a,b,c} \in M^\bullet \sqcup N^\bullet$.

(7) Let $F \in \tau^c, x_{a,b,c} \in F \sqcap N^\bullet$ and $G \in \tau^c(x_{a,b,c})$ be arbitrary. Then, $x_{a,b,c} \in F \sqcap G, F \sqcap G \in \tau^c$ and $x_{a,b,c} \in N^\bullet$, which implies that $G \sqcap (F \sqcap N) \notin \mathcal{L}$ and so $x_{a,b,c} \in (F \sqcap N)^\bullet$. Thus, we have $F \sqcap N^\bullet \sqsubseteq (F \sqcap N)^\bullet, F \sqcap N^\bullet \sqsubseteq F$ and we conclude that $F \sqcap N^\bullet \sqsubseteq F \sqcap (F \sqcap N)^\bullet$. On the other hand, the inclusion $F \sqcap N \sqsubseteq N$, means that $(F \sqcap N)^\bullet \sqsubseteq N^\bullet$ and $F \sqcap (F \sqcap N)^{\cdot} \sqsubseteq F \sqcap N^\bullet$. Therefore, $F \sqcap N^\bullet = F \sqcap (F \sqcap N)^\bullet \sqsubseteq (F \sqcap N)^\bullet$.

(8) Suppose that $N \in \mathcal{L}$ and $N^\bullet \neq \widetilde{\emptyset}$. Then, there exists a neutrosophic point $x_{a,b,c} \in N^\bullet$ and so, $N \sqcap F \notin \mathcal{L}$ for $F \in \tau^c(x_{a,b,c})$ being arbitrary. But the fact that $N \in \mathcal{L}$ implies that $N \sqcap F \in \mathcal{L}$ for each $F \in \tau^c(x_{a,b,c})$. Thus, we obtain a contradiction and hence, $N^\bullet = \widetilde{\emptyset}$.

(9) Assume that $N \sqsubseteq N^\bullet$. By part (3), $N^\bullet = \mathrm{Ker}_p(N^\bullet) \sqsubseteq \mathrm{Ker}_p(N)$ and by hypotheses, it follows that $\mathrm{Ker}_p(N) \sqsubseteq \mathrm{Ker}_p(N^\bullet) = N^\bullet \sqsubseteq \mathrm{Ker}_p(N)$ and hence, $N^\bullet = \mathrm{Ker}_p(N)$.

(10) Let $x_{a,b,c} \in N^\bullet(\mathcal{L}, \tau_2)$ and $F \in \tau_1^c(x_{a,b,c})$ be arbitrary. Since $\tau_1 \subseteq \tau_2$, we have $F \in \tau_2^c(x_{a,b,c})$ and so, $F \sqcap N \notin \mathcal{L}$. Therefore, $x_{a,b,c} \in N^\bullet(\mathcal{L}, \tau_1)$.

(11) Since $\mathcal{L} \cap \mathcal{L}' \subseteq \mathcal{L}$ and $\mathcal{L} \cap \mathcal{L}' \subseteq \mathcal{L}'$, by part (2), we have $N^\bullet(\mathcal{L}) \sqsubseteq N^\bullet(\mathcal{L} \cap L')$ and $N^\bullet(\mathcal{L}') \sqsubseteq N^\bullet(\mathcal{L} \cap \mathcal{L}')$. Thus, we deduce the inclusion $N^\bullet(\mathcal{L}) \sqcup N^\bullet(\mathcal{L}') \sqsubseteq N^\bullet(\mathcal{L} \cap \mathcal{L}')$. For the other inclusion, suppose that $x_{a,b,c} \in N^\bullet(\mathcal{L} \cap \mathcal{L}')$ and let $F \in \tau^c(x_{a,b,c})$ be arbitrary. Then, $N \sqcap F \notin \mathcal{L} \cap \mathcal{L}'$, which implies that $N \sqcap F \notin \mathcal{L}$ or $N \sqcap F \notin \mathcal{L}'$. If $N \sqcap F \notin \mathcal{L}$, then $x_{a,b,c} \in N^\bullet(\mathcal{L})$, whereas if $N \sqcap F \notin \mathcal{L}'$, then $x_{a,b,c} \in N^\bullet(\mathcal{L}')$. In both cases, it follows that $x_{a,b,c} \in N^\bullet(\mathcal{L}) \sqcup N^\bullet(\mathcal{L}')$. Therefore, $N^\bullet(\mathcal{L} \cap \mathcal{L}') \sqsubseteq N^\bullet(\mathcal{L}) \sqcup N^\bullet(\mathcal{L}')$.

Corollary 3.1.5. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. If $\{N_\alpha : \alpha \in \Delta\} \subseteq \mathcal{N}'(X)$, then the following properties hold:

(1) $(\sqcap_{\alpha \in \Delta} N_\alpha)^\bullet = \sqcap_{\alpha \in \Delta} {N_\alpha}^\bullet$.

(2) $(\sqcup_{\alpha \in \Delta} N_\alpha)^\bullet = \sqcup_{\alpha \in \Delta} {N_\alpha}^\bullet$, if Δ is finite.

Since the neutrosophic co-local function is not a neutrosophic closure operator, it is necessary to introduce a new concept that allows us to obtain a new neutrosophic topology from it.

Definition 3.1.6. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For each $N \in \mathcal{N}'(X)$, we define $Cl^\bullet(N) = N \sqcup N^\bullet$.

Remark 3.1.7. Let (X,τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For each $N \in \mathcal{N}'(X)$, the following properties hold:

(1) If $\mathcal{L} = \{\widetilde{\emptyset}\}$ then $Cl^{\bullet}(N) = N \sqcup N^{\bullet} = N \sqcup \mathrm{Ker}_p(N) = \mathrm{Ker}_p(N)$.

(2) If $\mathcal{L} = \mathcal{N}'(X)$, then $Cl^{\bullet}(N) = N \sqcup \widetilde{\emptyset} = N$.

Proposition 3.1.8. $Cl^{\bullet}$ is a neutrosophic closure operator.

Proof. The proof is an immediate consequence of Lemma 3.1.4.

According with Proposition 3.1.8 and Lemma 2.19, if (X,τ) is a neutrosophic topological space and $\mathcal{L}$ is a neutrosophic ideal on X, we denote by $\tau^{\bullet}(\mathcal{L})$ the neutrosophic topology generated by $Cl^{\bullet}$; that is $\tau^{\bullet}(\mathcal{L}) = \{N \in \mathcal{N}'(X): \quad Cl^{\bullet}(N^c) = N^c\}$. When there is no chance for confusion, we will simply write $\tau^{\bullet}$ for $\tau^{\bullet}(\mathcal{L})$. The elements of $\tau^{\bullet}$ are called neutrosophic $\tau^{\bullet}$-open sets and the complement of a neutrosophic $\tau^{\bullet}$-open set is called neutrosophic $\tau^{\bullet}$-closed set. Note that if $N \in \mathcal{N}'(X)$, then: N is neutrosophic $\tau^{\bullet}$-closed if and only if $N^c \in \tau^{\bullet}$ if and only if $Cl^{\bullet}((N^c)^c) = (N^c)^c$ if and only if $Cl^{\bullet}(N) = N$.

Remark 3.1.9. Since $N^{\bullet} = \mathrm{Ker}_p(N^{\bullet}) \sqsubseteq \mathrm{Ker}_p(N)$, then $Cl^{\bullet}(N) \sqsubseteq \mathrm{Ker}_p(N)$ for each $N \in \mathcal{N}'(X)$. Therefore, if N is a neutrosophic τ_k-closed set, then N is neutrosophic $\tau^{\bullet}$-closed. It follows that each neutrosophic τ_k-open set is neutrosophic $\tau^{\bullet}$-open; that is $\tau_k \subseteq \tau^{\bullet}$. Moreover, from Remark 3.1.7 it follows that $\tau^{\bullet}(\{\widetilde{\emptyset}\}) = \tau_k$ and $\tau^{\bullet}(\mathcal{N}'(X)) = \mathcal{N}'(X)$.

Proposition 3.1.10. Let (X,τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. If $\{N_\alpha : \alpha \in \Delta\}$ is a collection of neutrosophic $\tau^{\bullet}$-closed sets, then the following properties hold:

(1) $\sqcap \quad \{N_\alpha : \alpha \in \Delta'\}$ is a neutrosophic $\tau^{\bullet}$-closed set for any subset Δ' of Δ.

(2) $\sqcup\{N_\alpha : \alpha \in \Delta_0\}$ is a neutrosophic $\tau^{\bullet}$-closed set for any finite subset Δ_0 of Δ.

Proof. The proof is an immediate consequence of Proposition 2.6 and the duality between the notions of neutrosophic $\tau^{\bullet}$-open and neutrosophic $\tau^{\bullet}$-closed sets.

Proposition 3.1.11. Let (X,τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. Then, $N \in \mathcal{N}'(X)$ is neutrosophic $\tau^{\bullet}$-closed if and only if $N^{\bullet} \sqsubseteq N$.

Proof. Suppose that N is neutrosophic $\tau^{\bullet}$-closed. Then, $Cl^{\bullet}(N) = N$. In consequence, $N \sqcup N^{\bullet} = N$ and hence, $N^{\bullet} \sqsubseteq N$. Conversely, assume that $N^{\bullet} \sqsubseteq N$. Since $Cl^{\bullet}(N) = N \sqcup N^{\bullet}$ and $N \sqcup N^{\bullet} \sqsubseteq N$, we have $Cl^{\bullet}(N) \sqsubseteq N$. By Proposition 3.1.8, we have $N \sqsubseteq Cl^{\bullet}(N)$ and so, we conclude that $Cl^{\bullet}(N) = N$. This shows that N is neutrosophic $\tau^{\bullet}$-closed.

Proposition 3.1.12. If $\mathcal{L}$ and $\mathcal{L}'$ are neutrosophic ideals on a neutrosophic topological space (X,τ) such that $\mathcal{L} \subseteq \mathcal{L}'$, then $\tau^{\bullet}(\mathcal{L}) \subseteq \tau^{\bullet}(\mathcal{L}')$.

Proof. Consider $N \in \tau^{\bullet}(\mathcal{L})$. Then, N^c is a neutrosophic $\tau^{\bullet}(\mathcal{L})$-closed set and so, by Proposition 3.1.11, $(N^c)^{\bullet}(\mathcal{L}) \sqsubseteq N^c$. Now, by part (2) of Lemma 3.1.4, it follows that $(N^c)^{\bullet}(\mathcal{L}') \sqsubseteq (N^c)^{\bullet}(\mathcal{L}) \sqsubseteq N^c$. This shows that $(N^c)^{\bullet}(\mathcal{L}') \sqsubseteq N^c$ and N^c is a neutrosophic $\tau^{\bullet}(\mathcal{L}')$-closed set. Therefore, $N \in \tau^{\bullet}(\mathcal{L}')$.

Corollary 3.1.13. Let $\{\mathcal{J}_\alpha : \alpha \in \Delta\}$ be a collection of neutrosophic ideals on a neutrosophic topological space (X,τ). If $\mathcal{J} = \bigcap_{\alpha\in\Delta}\mathcal{J}_\alpha$ then $\tau^{\bullet}(\mathcal{J}) \subseteq \tau^{\#}$, where $\tau^{\#} = \bigcap_{\alpha\in\Delta}\tau^{\bullet}(\mathcal{J}_\alpha)$.

Proof. It is clear that $\tau^{\#}$ is a neutrosophic topology on X. Since $\mathcal{J} = \bigcap_{\alpha\in\Delta}\mathcal{J}_\alpha \subseteq \mathcal{J}_\alpha$ for every $\alpha \in \Delta$, by Proposition 3.1.12, we have $\tau^{\bullet}(\mathcal{J}) \subset \tau^{\bullet}(\mathcal{J}_\alpha)$ for every $\alpha \in \Delta$. Therefore, $\tau^{\bullet}(\mathcal{J}) \subseteq \bigcap_{\alpha\in\Delta}\tau^{\bullet}(\mathcal{J}_\alpha) = \tau^{\#}$.

Corollary 3.1.14. Suppose that (X,τ) be a neutrosophic topological space and let $\mathcal{L}$ and $\mathcal{L}'$ be two neutrosophic ideals on X. Then, $\tau^{\bullet}(\mathcal{L} \cap \mathcal{L}') = \tau^{\bullet}(\mathcal{L}) \cap \tau^{\bullet}(\mathcal{L}')$.

Proof. Let $M \in \tau^{\bullet}(\mathcal{L} \cap \mathcal{L}')$ and put $M = N^c$. Then, by part (11) of Lemma 3.4 and Proposition 3.1.11, we have:

$$M \in \tau^{\bullet}(\mathcal{L} \cap \mathcal{L}') \Leftrightarrow N \text{ is neutrosophic } \tau^{\bullet}(\mathcal{L} \cap \mathcal{L}')\text{-closed}$$
$$\Leftrightarrow N^{\bullet}(\mathcal{L}) \sqcup N^{\bullet}(\mathcal{L}') = N^{\bullet}(\mathcal{L} \cap \mathcal{L}') \sqsubseteq N$$
$$\Leftrightarrow N^{\bullet}(\mathcal{L}) \sqsubseteq N \text{ and } N^{\bullet}(\mathcal{L}') \sqsubseteq N$$
$$\Leftrightarrow M \in \tau^{\bullet}(\mathcal{L}) \text{ and } M \in \tau^{\bullet}(\mathcal{L}')$$
$$\Leftrightarrow M \in \tau^{\bullet}(\mathcal{L}) \cap \tau^{\bullet}(\mathcal{L}').$$

3.2. Neutrosophic Φ-operator and new neutrosophic sets

Definition 3.2.1. Let (X,τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For each $N \in \mathcal{N}'(X)$, we define the neutrosophic complement co-local function of N as $\Phi(N) = ((N^c)^{\cdot})^c$.

In Table 1 we summarize the main equalities related to the neutrosophic operator Φ, which are obtained by applying the neutrosophic complement operation or the co-local neutrosophic function from equation (1).

Table 1. Equalities related to the neutrosophic operator Φ.

(1) $\Phi(N) = ((N^c)^{\bullet})^c$	(2) $[\Phi(N)]^c = (N^c)^{\bullet}$
(3) $[\Phi(N)]^{\bullet} = (((N^c)^{\bullet})^c)^{\bullet}$	(4) $\Phi(N^c) = (N^{\bullet})^c$
(5) $[\Phi(N^c)]^c = N^{\bullet}$	(6) $[\Phi(N^c)]^{\bullet} = ((N^{\bullet})^c)^{\bullet}$
(7) $\Phi(N^{\bullet}) = (((N^{\bullet})^c)^{\bullet})^c$	(8) $[\Phi(N^{\bullet})]^c = ((N^{\bullet})^c)^{\bullet}$

Remark 3.2.2. From the equalities (6) and (8) of Table 1, we can deduce that $[\Phi(N^c)]^{\bullet} = [\Phi(N^{\bullet})]^c$.

In the following proposition, relevant properties related to the neutrosophic operator Φ (also called neutrosophic Φ-operator) are presented.

Proposition 3.2.3. Let (X,τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. Then, we have the following properties:

(1) If $N, M \in \mathcal{N}'(X)$ and $N \sqsubseteq M$, then $\Phi(N) \sqsubseteq \Phi(M)$. (Φ is monotone)

(2) $\Phi(N \sqcap M) = \Phi(N) \sqcap \Phi(M)$ for every $N, M \in \mathcal{N}'(X)$.

(3) $\Phi(N) \sqsubseteq \Phi(\Phi(N))$ for every $N \in \mathcal{N}'(X)$.

(4) $\Phi(\tilde{X}) = \tilde{X}$.

(5) $O \sqsubseteq \Phi(O)$ for every $O \in \tau_k$. (Φ is expansive on τ_k)

(6) $\mathrm{Cok}_p(N) \sqsubseteq \Phi(N)$ for every $N \in \mathcal{N}'(X)$.

Proof. (1) Let $N, M \in \mathcal{N}'(X)$ such that $N \sqsubseteq M$. Then, $M^c \sqsubseteq N^c$ and by part (1) of Lemma 3.1.4, $(M^c)^{\bullet} \sqsubseteq (N^c)^{\bullet}$. Therefore, $\Phi(N) = (N^c)^{\bullet})^c \sqsubseteq ((M^c)^{\bullet})^c = \Phi(M)$.

(2) If $N, M \in \mathcal{N}'(X)$, then

$$\begin{aligned}\Phi(N \sqcap M) &= (((N \sqcap M)^c)^{\bullet})^c = ((N^c \sqcup M^c)^{\bullet})^c \\ &= ((N^c)^{\bullet} \sqcup (M^c)^{\bullet})^c = ((N^c)^{\bullet})^c \sqcap ((M^c)^{\bullet})^c \\ &= \Phi(N) \sqcap \Phi(M).\end{aligned}$$

(3) Let $N \in \mathcal{N}'(X)$. By part (5) of Lemma 3.1.4, we have $((N^c)^\bullet)^\bullet \sqsubseteq (N^c)^\bullet$, which implies that $\Phi(N) = ((N^c)^\bullet)^c \sqsubseteq (((N^c)^\bullet)^\bullet)^c$. Now, by applying Definition 3.2.1 to the neutrosophic set $\Phi(N)$, we obtain that $\Phi(\Phi(N)) = (([\Phi(N)]^c)^\bullet)^c$ and by equation (2) of Table 1 , we deduce that $\Phi(\Phi(N)) = (((N^c)^\bullet)^\bullet)^c$. Hence, $\Phi(N) \sqsubseteq ((N^c)^\bullet)^\bullet)^c = \Phi(\Phi(N))$.

(4) By definition we have $\Phi(\tilde{X}) = \left(\left(\tilde{X}^c\right)^\bullet\right)^c = \left(\tilde{\emptyset}^\bullet\right)^c = \tilde{\emptyset}^c = \tilde{X}$.

(5) If $O \in \tau_k$, then O^c is a neutrosophic τ_k-closed set and so $\mathrm{Ker}_p(O^c) = O^c$. By equation (2) of Table 1 and part (3) of Lemma 3.1.4, we obtain that $[\Phi(O)]^c = (O^c)^\bullet \sqsubseteq \mathrm{Ker}_p(O^c) = O^c$ and hence, $O \sqsubseteq \Phi(O)$ for every $O \in \tau_k$.

(6) Since $\mathrm{Cok}_p(N) \in \tau_k$, by part (5), we have $\mathrm{Cok}_p(N) \sqsubseteq \Phi(\mathrm{Cok}_p(N))$ and as $\mathrm{Cok}_p(N) \sqsubseteq N$, by part (1), we deuce that $\mathrm{Cok}_p(N) \sqsubseteq \Phi(\mathrm{Cok}_p(N)) \sqsubseteq \Phi(N)$.

Definition 3.2.4. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrophic ideal on X. A subset $N \in \mathcal{N}'(X)$ is said to be:

(1) neutrosophic $\bullet$-perfect, if $N = N^\bullet$

(2) neutrosophic $\bullet$-dense, if $N^\bullet = \tilde{X}$.

(3) neutrosophic $\bullet$-condensed, if $[\Phi(N)]^\bullet = N^\bullet$.

(4) neutrosophic Φ-condensed, if $\Phi(N^\bullet) = \Phi(N)$.

(5) neutrosophic $\Phi^\bullet$-condensed, if it is neutrosophic $\bullet$-condensed and neutrosophic Φ-condensed.

(6) neutrosophic non $\Phi^\bullet$-condensed, if $\Phi(N^\bullet) = \tilde{\emptyset}$.

(7) neutrosophic $\bullet$-congruent, if $[\Phi(N)]^\bullet = N$.

(8) neutrosophic Φ-congruent, if $\Phi(N^\bullet) = N$.

(9) neutrosophic $\Phi^\bullet$-congruent, if it is neutrosophic $\bullet$-congruent and neutrosophic Φ-congruent.

Proposition 3.2.5. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. If $N \in \mathcal{N}'(X)$, then we have the following properties:

(1) If N is neutrosophic $\bullet$-perfect, then it is neutrosophic Φ-condensed.

(2) N is neutrosophic Φ-condensed if and only if N^c is neutrosophic $\bullet$-condensed.

(3) N is neutrosophic $\Phi^\bullet$-condensed if and only if N^c is neutrosophic $\Phi^\bullet$-condensed.

(4) N is neutrosophic Φ-congruent if and only if N^c is neutrosophic $\bullet$-congruent.

(5) N is neutrosophic $\Phi^\bullet$-congruent if and only if N^c is neutrosophic $\Phi^\bullet$-congruent.

(6) If N neutrosophic Φ-condensed and neutrosophic non $\Phi^\bullet$-condensed, then N^c is neutrosophic $\bullet$-dense.

(7) If N neutrosophic $\bullet$-condensed and N^c is neutrosophic non $\Phi^\bullet$-condensed, then N is neutrosophic $\bullet$-dense.

(8) If N is neutrosophic non $\Phi^\bullet$-condensed and neutrosophic $\bullet$-perfect, then N^c is neutrosopic $\bullet$-dense.

Proof. (1) From Definition 3.2.4, we have:

$$\begin{aligned} N \text{ is neutrosophic } \bullet\text{-perfect } &\Leftrightarrow N = N^\bullet \\ &\Leftrightarrow \Phi(N) = \Phi(N^\bullet) \\ &\Leftrightarrow N \text{ is neutrosophic } \Phi\text{-condensed.} \end{aligned}$$

(2) By Remark 3.2.2 and equation (2) of Table 1, we get that

$$\begin{aligned} N \text{ is neutrosophic } \Phi\text{-condensed} &\Leftrightarrow \Phi(N^\bullet) = \Phi(N) \\ &\Leftrightarrow [\Phi(N^\bullet)]^c = [\Phi(N)]^c \\ &\Leftrightarrow [\Phi(N^c)]^\bullet = (N^c)^\bullet \\ &\Leftrightarrow N^c \text{ is neutrosophic } \bullet\text{-condensed.} \end{aligned}$$

(3) The proof follows from (2).

(4) By Remark 3.2.2, we obtain that

$$\begin{aligned} N \text{ is neutrosophic } \Phi\text{-congruent} &\Leftrightarrow \Phi(N^\bullet) = N \\ &\Leftrightarrow [\Phi(N^\bullet)]^c = N^c \\ &\Leftrightarrow [\Phi(N^c)]^\bullet = N^c \\ &\Leftrightarrow N^c \text{ is neutrosophic } \bullet\text{-congruent.} \end{aligned}$$

(5) The proof follows from (4).

(6) Assume that N neutrosophic Φ-condensed and neutrosophic non $\Phi^\bullet$-condensed. Then, $\Phi(N^\bullet) = \Phi(N)$ and $\Phi(N^\bullet) = \widetilde{\emptyset}$, which implies that $\Phi(N) = \widetilde{\emptyset}$. Thus, $[\Phi(N)]^c = \tilde{X}$ and by equation (2) of Table 1, it follows that $(N^c)^\bullet = \tilde{X}$. Therefore, N^c is neutrosophic $\bullet$-densed.

(7) The proof follows from (2) and (6).

(8) Suppose that N is neutrosophic non $\Phi^\bullet$-condensed and neutrosophic $\bullet$-perfect. Then, $\Phi(N^\bullet) = \widetilde{\emptyset}$ and $N^\bullet = N$, which implies that $\Phi(N) = \Phi(N^\bullet) = \widetilde{\emptyset}$. By equation (2) of Table 1 , we deduce that $(N^c)^\bullet = [\Phi(N)]^c = \tilde{X}$ and so, N^c is neutrosophic $\bullet$-dense.

Proposition 3.2.6. Let $N \in \mathcal{N}'(X)$ and N^c be a neutrosophic $\bullet$-perfect set. Then, the following properties are equivalent:

(1) N is neutrosophic Φ-congruent

(2) N is neutrosophic Φ-condensed.

Proof. (1) $\Longrightarrow$ (2) Suppose that N is neutrosophic Φ-congruent. Then, $\Phi(N^\bullet) = N$. Since N^c is neutrosophic $\bullet$-perfect, $(N^c)^\bullet = N^c$, which implies that $\Phi(N^\bullet) = N = (N^c)^c = ((N^c)^\bullet)^c = \Phi(N)$, which shows that N is neutrosophic Φ-condensed.

(2) $\Longrightarrow$ (1) Assume that N is neutrosophic Φ-condensed. Then, $\Phi(N^\bullet) = \Phi(N)$. Since N^c is neutrosophic $\bullet$-perfect, $(N^c)^\bullet = N^c$ and by equation (2) of Table 1, it follows that $[\Phi(N)]^c = N^c$, which implies that $\Phi(N) = N$. Therefore, $\Phi(N^\bullet) = \Phi(N) = N$ and so, N is neutrosophic Φ congruent

Corollary 3.2.7. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. If $N \in \mathcal{N}'(X)$ is neutrosophic $\cdot$-perfect, then the following properties are equivalent:

(1) N is neutrosophic $\bullet$-congruent

(2) N is neutrosophic $\bullet$-condensed.

Proof. It is deduced from Proposition 3.2.6 by using parts (2) and (4) of Proposition 3.2.5.

Proposition 3.2.8. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For $N \in \mathcal{N}'(X)$, we have the following properties:

(1) If N is neutrosophic non $\Phi^\bullet$-condensed and $M \sqsubseteq N$, then M is neutrosophic non $\Phi^\bullet$-condensed.

(2) If N is neutrosophic non $\Phi^\bullet$-condensed and $M \in \mathcal{N}'(X)$, then $N \sqcap M$ is neutrosophic non $\Phi^\bullet$-condensed.

(3) If N is neutrosophic non $\Phi^{\bullet}$-condensed and $L \in \mathcal{L}$, then $N \sqcup L$ is neutrosophic non $\Phi^{\bullet}$-condensed.

(4) If N is neutrosophic non $\Phi^{\bullet}$-condensed, then $N^{\bullet}$ is neutrosophic non $\Phi^{\bullet}$-condensed.

(5) If N is neutrosophic non $\Phi^{\bullet}$-condensed, then for every $x_{a,b,c} \in \mathcal{N}'(X)$ and every $F \in \tau^c(x_{a,b,c})$, $\Phi(N^c) \sqcap F \neq \widetilde{\emptyset}$.

(6) If $\mathcal{J}$ is a neutrosophic ideal on X such that $\mathcal{J} \subseteq \mathcal{L}$ and N is neutrosophic non $\Phi^{\bullet}$-condensed, with respect to $\mathcal{J}$, then N is neutrosophic non $\Phi^{\bullet}$-condensed with respect to $\mathcal{L}$.

Proof. (1) Suppose that N is neutrosophic non $\Phi^{\bullet}$-condensed and $M \sqsubseteq N$. Then $\Phi(N^{\bullet}) = \widetilde{\emptyset}$ and $M^{\bullet} \sqsubseteq N^{\bullet}$. Thus, $\Phi(M^{\bullet}) \sqsubseteq \Phi(N^{\bullet}) = \widetilde{\emptyset}$, which means that $\Phi(M^{\bullet}) = \widetilde{\emptyset}$ and hence, M is neutrosophic non $\Phi^{\bullet}$-condensed.

(2) Since $N \sqcap M \sqsubseteq N$ for each $M \in \mathcal{N}'(X)$, the result follows from part (1).

(3) Assume that N is neutrosophic non $\Phi^{\bullet}$-condensed and $L \in \mathcal{L}$. Then $\Phi(N^{\bullet}) = \widetilde{\emptyset}$ and $L^{\bullet} = \widetilde{\emptyset}$, which implies that $(N \sqcup L)^{\bullet} = N^{\bullet} \sqcup L^{\bullet} = N^{\bullet}$ and $\Phi((N \sqcup L)^{\bullet}) = \Phi(N^{\bullet}) = \widetilde{\emptyset}$. Therefore, $N \sqcup L$ is neutrosophic non $\Phi^{\bullet}$-condensed.

(4) Suppose that N is neutrosophic non $\Phi^{\bullet}$-condensed. Then $\Phi(N^{\bullet}) = \widetilde{\emptyset}$ and $(N^{\bullet})^{\bullet} \sqsubseteq N^{\bullet}$. Hence $\Phi((N^{\bullet})^{\bullet}) \sqsubseteq \Phi(N^{\bullet}) = \widetilde{\emptyset}$ and so $N^{\bullet}$ is neutrosophic non $\Phi^{\bullet}$-condensed.

(5) Assume that N is neutrosophic non $\Phi^{\bullet}$-condensed, i.e. $\Phi(N^{\bullet}) = \widetilde{\emptyset}$. Then $[\Phi(N^{\bullet})]^c = \tilde{X}$ and so, by Remark 3.2.2, $[\Phi(N^c)]^{\bullet} = \tilde{X}$. Therefore, for every $x_{a,b,c} \in \mathcal{N}'(X)$ and every $F \in \tau^c(x_{a,b,c})$, $F \sqcap \Phi(N^c) \notin \mathcal{L}$, which implies that $F \sqcap \Phi(N^c) \neq \emptyset$, for every $x_{a,b,c} \in \mathcal{N}'(X)$ and every $F \in \tau^c(x_{a,b,c})$.

(6) Let $\mathcal{J}$ be a neutrosophic ideal on X such that $\mathcal{J} \sqsubseteq \mathcal{L}$ and N be a neutrosophic non $\Phi^{\bullet}$-condensed set with respect to $\mathcal{J}$. Then $\Phi(N^{\bullet}(\mathcal{J})) = \widetilde{\emptyset}$ and by part (2) of Lemma 3.1.4, we have $N^{\bullet}(\mathcal{L}) \sqsubseteq N^{\bullet}(\mathcal{J})$, which implies that $\Phi(N^{\bullet}(\mathcal{L})) \sqsubseteq \Phi(N^{\bullet}(\mathcal{J})) = \widetilde{\emptyset}$. Therefore, $\Phi(N^{\bullet}(\mathcal{L})) = \widetilde{\emptyset}$ and so, N is neutrosophic non $\Phi^{\bullet}$-condensed with respect to $\mathcal{L}$.

Proposition 3.2.9. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For $N \in \mathcal{N}'(X)$, we have the following properties:

(1) N is neutrosophic non $\Phi^{\bullet}$-condensed if and only if $(N^{\bullet})^c$ is neutrosophic $\bullet$-dense.

(2) N is neutrosophic non $\Phi^{\bullet}$-condensed if and only if $\Phi(N^c)$ is neutrosophic $\bullet$-dense.

(3) N^c is neutrosophic non $\Phi^{\bullet}$-condensed if and only if $\Phi(N)$ is neutrosophic $\bullet$-dense.

Proof. The proofs of (1) and (2) are obtained from Definition 3.2.4 and equation (8) of Table 1 as follows:

$$
\begin{aligned}
N \text{ is neutrosophic non } \Phi^{\bullet}\text{-condensed} \;&\Leftrightarrow \Phi(N^{\bullet}) = \widetilde{\emptyset} \\
&\Leftrightarrow [\Phi(N^{\bullet})]^c = \tilde{X} \\
&\Leftrightarrow ((N^{\bullet})^c)^{\bullet} = \tilde{X} \\
&\Leftrightarrow (N^{\bullet})^c \text{ is neutrosophic } \bullet\text{-dense} \\
&\Leftrightarrow \Phi(N^c) \text{ is neutrosophic } \bullet\text{-dense.}
\end{aligned}
$$

(3) The proof follows from (2) by changing N to N^c.

Corollary 3.2.10. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. If $\tilde{X}$ is a neutrosophic $\bullet$-dense in itself set, then every $L \in \mathcal{L}$ is a neutrosophic non $\Phi^{\bullet}$-condensed set.

Proof. Since $L \in \mathcal{L}$, we have $L^{\bullet} = \widetilde{\emptyset}$ and hence, $(L^{\bullet})^c = \tilde{X}$. According to equation (4) of Table 1, $\Phi(L^c) = \tilde{X}$ and as $\tilde{X}$ is neutrosophic $\bullet$-dense in itself, it follows that $[\Phi(L^c)]^{\bullet} = \tilde{X}^{\bullet} = \tilde{X}$ and so, $\Phi(L^c)$ is neutrosophic $\bullet$-dense. Now, by Theorem 3.2.9, we conclude that L is neutrosophic non $\Phi^{\bullet}$-condensed.

Definition 3.2.11. Let (X,τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For every $N \in \mathcal{N}'(X)$, the neutrosophic $\bullet$-frontier of N, denoted by $Fr^{\bullet}(N)$, is defined as $Fr^{\bullet}(N) = N^{\bullet} \sqcap (N^c)^{\bullet}$.

Proposition 3.2.12. Let (X,τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. If $N \in \mathcal{N}'(X)$ is neutrosophic $\bullet$-dense and $\Phi(Fr^{\bullet}(N)) = \widetilde{\emptyset}$, then N^c is neutrosophic non $\Phi^{\bullet}$-condensed.

Proof. Suppose that $N \in \mathcal{N}'(X)$ is neutrosophic $\bullet$-dense and $\Phi(Fr^{\bullet}(N)) = \widetilde{\emptyset}$. Then, $N^{\bullet} = \tilde{X}$ and $\Phi(N^{\bullet} \sqcap (N^c)^{\bullet}) = \widetilde{\emptyset}$. Hence, by parts (2) and (4) of Proposition 3.2.3, we have $\Phi(N^{\bullet}) \sqcap \Phi((N^c)^{\bullet}) = \widetilde{\emptyset}$ and $\Phi(N^{\bullet}) = \Phi(\tilde{X}) = \tilde{X}$, respectively. Thus, $\Phi((N^c)^{\bullet}) = \tilde{X} \sqcap \Phi((N^c)^{\bullet}) = \widetilde{\emptyset}$ and therefore, N^c is neutrosophic non $\Phi^{\bullet}$-condensed.

5. Conclusions

Neutrosophic topology is one of the most useful notions in neutrosophic set theory, because many of the topics studied in this branch of mathematics are done in the context of a neutrosophic topological space. In this work, we have used the notions of neutrosophic point and neutrosophic ideal to introduce and study the concepts of neutrosophic co-local function and neutrosophic complement co-local function of a subset of a neutrosophic topological space. We have established the most relevant properties of the concepts introduced and we have explored new classes of neutrosophic sets defined in terms of these concepts. Since various modifications of topology in neutrosophic set theory have recently been addressed, we consider that the notions and results given in this paper can be extended to the contexts of Refined Neutrosophic Topology, Refined Neutrosophic Crisp Topology, SuperHyperNeutrosophic Topology and Single-Valued Duplet Neutrosophic Topology, Single-Valued Neutrosophic Triplet Weak Topology and others highlighted in [10], which leave open a prominent field for future research.

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An Evaluation of Triangular Neutrosophic PERT Analysis for Real-Life Project Time and Cost Estimation

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Abstract: The textile industry sector's time and cost management issue led to a quest for contemporary tools that provide the best possible project time and cost prediction. Using a case study of Esa Textile in India, this paper assesses quantitative decision-making techniques in the textile industry. An extensive approach is provided so that specialists may utilise Triangular Neutrosophic Numbers (TNNs) to express their views about identifying features and indicators of a successful project. Determining the best approach to deal with removing interruptions that can cause delays and unnecessary expenses is also an essential responsibility. While commonly employed, traditional estimating methods like the Programme Evaluation and Review Technique (PERT) may find it difficult to adequately address the uncertainties present in real-world projects. This study examines and assesses the use of Triangular Neutrosophic PERT (TNP) analysis for project time and cost estimation in order to overcome this restriction. Neutrosophy, which allows for the depiction of inconsistent, ambiguous and partial data available in project parameters, is incorporated into the suggested TNP analysis. The efficiency of the suggested strategy has been verified by this analysis, and the network's unknown parameters are represented by triangle Neutrosophic numbers. This innovative method gives each of the three potential estimates—optimistic, most probable, and pessimistic which are all consisting of degree of membership, indeterminacy, or non-membership. This study's objective is to locate the work-network in a logical order once all of the processes at the Esa textile units have been completed. Planning is developed using the Triangular Neutrosophic Programme Evaluation and Review Techniques (TNP) even there is a time difference, which will speed up production and cut expenses. TNP provides a more thorough and adaptable depiction of uncertainty by utilizing the neutrosophic framework, which better captures the dynamic character of real-life projects.

Keywords: Neutrosophic number; Triangular Neutrosophic Number; Triangular Neutrosophic PERT, Critical Path of the Project, Scoring Function.

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1. Introduction:

The textile industry is considered a central key symbol of the comprehensiveness of the country and is an important core industry that has a significant impact on the economy of the country. Branded clothing is in high demand in many industries including chemical, electronics, civil and mechanical as the country's economy continues to grow and the standard of living of its citizens is higher. Effective project management is crucial for the successful execution of complex projects, encompassing various industries such as construction, engineering, software development, and more. Among the key challenges faced by project managers is the accurate estimation of project time and cost, as deviations from initial estimates can lead to budget overruns, schedule delays, and overall project failure. To address these challenges, researchers and practitioners have continuously sought innovative methods for project estimation that can better capture uncertainties and vagueness associated with real-life project parameters. The success of large-scale projects heavily relies on the quality of planning, scheduling, and control throughout their various phases. Without effective planning and coordination tools, even a relatively small number of phases can lead to management losing control. Project Evaluation and Review Technique (PERT) is considered the best project management tool for organizing, scheduling, and coordinating tasks in such large-scale projects. Originally designed for manufacturing projects, PERT employs a network of interconnected activities to optimize cost and time. It emphasizes the relationship between activity times, associated costs, and the overall project completion time and cost.

The production process is a major problem in implementing the production of raw materials into finished materials. A significant obstacle to turning raw resources into completed goods is the production process. Inaccuracy and completion delays will add time and money to the process. One approach is to use network analysis to foresee such a scenario. Network analysis is referred to as a network that has to be operated and is time-limited. Various real-life scenarios are being considered and expressed using Triangular Neutrosophic values. These uncertain values are then converted into crisp values using Neutrosophic Scoring functions to facilitate analysis. Next, the NPT (Neutrosophic Project Technique) approach is being employed to assess the project's time and cost estimation for the company. The primary objective is to achieve an optimal (minimum) project duration and maximize profitability while minimizing manpower requirements. By utilizing this approach, project managers can make well-informed decisions to optimize project timelines, reduce costs, and maximize profits, all while efficiently allocating resources. .

In this section, some literatures associated with the field of this study are presented. Neutrosophic sets serve as a broader concept encompassing crisp sets, fuzzy sets, and intuitionistic fuzzy sets, allowing the representation of uncertain, inconsistent, and incomplete information in real-world problems. Elements of a neutrosophic set possess truth-membership, falsity-membership, and indeterminacy membership functions. Smarandache first put forward the philosophical idea of the neutrosophic theory, which is a popularization of the fuzzy set (FS) and the IFS [1]. Traditional project estimation techniques, such as the Program Evaluation and Review Technique (PERT), have been widely used to estimate project duration and critical path analysis. PERT involves the use of three-point estimates, where the most likely, optimistic, and pessimistic time estimates are combined to derive a probabilistic estimate. Several researchers developed and implemented the concept of PERT/CPM in various real-life situations [2,3,4,5,6]. However, PERT's deterministic nature lacks the capability to handle imprecision, ambiguity, and uncertainty in project parameters.

The subtraction and division of neutrosophic numbers have been thoroughly discussed [7]. CPM and PERT theory finds practical application in project planning decision-making [8]. Building upon the Neutrosophic framework, Triangular Neutrosophic PERT (TNP) has emerged as a novel approach for project time and cost estimation, aiming to provide a more flexible and accurate model to deal with the inherent uncertainties present in real-life projects [9,10,11]. The algorithm calculates critical paths, variances, expected task times, and probabilities of completing the project within expected time frames in a more efficient manner than existing methods. The studies [12,13,14,15,16,17,18] shows the implementation of algorithm for determining the project evaluation and review technique (TNP) using neutrosophic numbers for better results of other existing methods.

Uncertainty can affect the process of assessing risks and adopting the best alternative. To overcome this problem, Abdel-Baset et.al [19] suggested the neutrosophic set as an integrated neutrosophic ANP and VIKOR method, for achieving sustainable supplier selection. The neutrosophic theory has attracted the interest of researchers in a range of fields [20,21]. Abdel-Basset [22] analyzed the uncertainty that affect the process of waste water system using Risk Assessment Model. There are several challenges that hospitals are facing according to the emergency department (ED). The study [23] suggests an integrated evaluation model assess ED under a framework of plithogenic theory. The proposed framework addressed uncertainty and ambiguity in information with an efficient manner via presenting the evaluation expression by plithogenic numbers. Abdel-Basset et al [24] studied the emission crisis in the iron and steel sector prompted the search for modern systems that contribute to reducing the resulting emissions to alleviate the growing concerns about global warming. Rahnamay Bonab et al [25] studied logistic autonomous vehicles assessment using decision support model under spherical fuzzy set integrated Choquet integral approach. Jeyaramman et al [26] studied the statistical convergences within non-Archimedean Neutrosophic normed spaces. Jdid et al [27] formulated the general model for the optimal distribution of agricultural lands using the concepts of neutrosophic science. Recently Kungumaraj. E et.al [28] investigated Indefinite integrals, Heptagonal Topology [29] and Topological Vector Spaces [30] in Neutrosophic environment.

To the best of the authors' knowledge, very little literature has been performed to evaluate project implementation in the textile sector in generic, especially by applying the Triangular Neutrosophic Pert (TNP) approaches. This study presents a TNP approach that considers uncertainty in decision-making by applying Triangular Neutrosophic numbers. The suggested methodology adopts two techniques of decision-making, which are the PERT and CPM. They are implemented under a Triangular Neutrosophic environment. The TNP method is applied to evaluate the main aspects of optimal time and cost that have an impact on the project in the Textile sector.

All in all, the primary contributions of this study are outlined below.

- To suggest a TNP approach of the determined challenges based on a Triangular Neutrosophic environment to cope with the unpredictability inherent in decision-making, this is the first study to develop a TNP approach consisting of PERT and CPM methods for evaluating optimal cost and time in the textile sector
- The proposed approach that was described may be used to get reliable answers in situations when there is a lot of uncertainty
- The organizational repercussions of this study not only have the potential to provide important guidance to the textile industry but also to decision-makers and investors in other industries
- A TNP is applied to a real life data to demonstrate the reliability, and validity of the developed approach

The main aim of this work is to elucidate the advantage of TNP method in an ESA Clothing Company, which is the primary manufacturers of garments such as t-shirts, children wear and cotton shirts. From 05.06.2023 to 04.07.2023 the time taken to manufacture the products and construction of new block in Esa clothing company has been noted. The work-network in a logical work sequence, at the time the Esa textile units' entire process is observed. The TNP method is a probabilistic technique that analyses and represents the uncertainties associated with project activities and it is an advanced technique that can be utilized in any industry. The Neutrosophic Programme Evaluation and Review

Techniques (TNP) are used to develop planning. With the help of neutrosophic framework, TNP offers a more comprehensive and flexible representation of uncertainties. This paper is organized as follows. Section 2 furnish the preliminaries and basic definitions, while section 3 present the steps involved in TNP. In section 4 real life examples were solved with the help of proposed theory. Finally, conclusion is given in the last section. Advantage of TNP method is elucidated through numerical illustrations.

2. PRELIMINARIES:

Definition 2.1. Let E be a universe. A neutrosophic set A in E is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard elements of [0,1]. It can be written as
$A = \{< x, (T_A(x), I_A(x), F_A(x)) >: x \in E; T_A(x), I_A(x), F_A(x) \in]0^-, 1^+[\}$. There is no restriction on the sum of $T_A(x), I_A(x),$ *and* $F_A(x)$. So $0 \le T_A(x), I_A(x), F_A(x) \le 3^+$.

Definition 2.2. Let E be a universe. A single valued neutrosophic set A, which can be used in real scientific and engineering applications, in E is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. $T_A(x), I_A(x), F_A(x)$ are real standard elements of [0,1]. It can be written as
$A = \{< x, (T_A(x), I_A(x), F_A(x)) >: x \in E; T_A(x), I_A(x), F_A(x) \in [0, 1] \}$.

Definition 2.3. Let $(\alpha_{\check{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}) \in [0,1]$ and $a_1, a_2, a_3 \in R$ such that $a_1 \le a_2 \le a_3$. Then a single valued triangular neutrosophic number $\check{a} = \langle (a_1, a_2, a_3); \alpha_{\check{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle$ is a special neutrosophic set on the real line set R, whose truth-membership, indeterminacy-membership and falsity-membership functions are given as follows

$$T_{\tilde{a}}(x) = \begin{cases} \alpha_{\tilde{a}}\left(\dfrac{x - a_1}{(a_2 - a_1)}\right) & if\ a_1 \le x \le a_2 \\ \alpha_{\check{a}} & if\ x = a_2 \\ \alpha_{\tilde{a}}\left(\dfrac{a_3 - x}{a_3 - a_2}\right) & if\ a < x \le a_3 \\ 0 & otherwise \end{cases}$$

$$I_{\tilde{a}}(x) = \begin{cases} \dfrac{(a_2 - x + \theta_{\tilde{a}}(x - a_1))}{(a_2 - a_1)} & if\ a_1 \le x \le a_2 \\ \theta_{\tilde{a}} & if\ x = a_2 \\ \dfrac{(x - a_2 + \theta_{\tilde{a}}(a_3 - x))}{(a_3 - a_2)} & if\ a_2 < x \le a_3 \\ 1 & otherwise \end{cases}$$

$$F_{\tilde{a}}(x) = \begin{cases} \dfrac{(a_2 - x + \beta_{\tilde{a}}(x - a_1))}{(a_2 - a_1)} & if\ a_1 \le x \le a_2 \\ \beta_{\tilde{a}} & if\ x = a_2 \\ \dfrac{(x - a_2 + \theta_{\tilde{a}}(a_3 - x))}{(a_3 - a_2)} & if\ a_2 < x \le a_3 \\ 0 & otherwise \end{cases}$$

Where $\alpha_{\check{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}$ denote the maximum truth-membership degree, minimum indeterminacy-membership degree and minimum falsity-membership degree respectively. A single valued triangular neutrosophic number $\breve{a} = \langle (a_1, a_2, a_3); \alpha_{\breve{a}}, \theta_{\breve{a}}, \beta_{\breve{a}} \rangle$ may express an ill-defined quantity about α, which is approximately equal to α.

Definition 2.4. Let $\breve{a} = \langle (a_1, a_2, a_3); \alpha_{\breve{a}}, \theta_{\breve{a}}, \beta_{\breve{a}} \rangle$ and $\breve{b} = \langle (b_1, b_2, b_3); \alpha_{\breve{a}}, \theta_{\breve{a}}, \beta_{\breve{a}} \rangle$ be two single valued triangular neutrosophic numbers and $\gamma \neq 0$ be any real number. Then,

$$\breve{a} + \breve{b} = \langle (a_1 + b_1, a_2 + b_2, a_3 + b_3); \alpha_{\breve{a}} \wedge \alpha_{\breve{b}}, \theta_{\breve{a}} \vee \theta_{\breve{b}}, \beta_{\breve{a}} \vee \beta_{\breve{b}} \rangle$$

$$\breve{a} - \breve{b} = \langle (a_1 - b_3, a_2 - b_2, a_3 - b_1); \alpha_{\breve{a}} \wedge \alpha_{\breve{b}}, \theta_{\breve{a}} \vee \theta_{\breve{b}}, \beta_{\breve{a}} \vee \beta_{\breve{b}} \rangle$$

$$\breve{a}.\breve{b} = \begin{cases} \langle (a_1b_1, a_2b_2, a_3b_3); \alpha_{\breve{a}} \wedge \alpha_{\breve{b}}, \theta_{\breve{a}} \vee \theta_{\breve{b}}, \beta_{\breve{a}} \vee \beta_{\breve{b}} \rangle \; if \; (a_3 > 0, b_3 > 0) \\ \langle (a_1b_3, a_2b_2, a_3b_1); \alpha_{\breve{a}} \wedge \alpha_{\breve{b}}, \theta_{\breve{a}} \vee \theta_{\breve{b}}, \beta_{\breve{a}} \vee \beta_{\breve{b}} \rangle \; if \; (a_3 < 0, b_3 > 0) \\ \langle (a_3b_3, a_2b_2, a_1b_1); \alpha_{\breve{a}} \wedge \alpha_{\breve{b}}, \theta_{\breve{a}} \vee \theta_{\breve{b}}, \beta_{\breve{a}} \vee \beta_{\breve{b}} \rangle \; if \; (a_3 < 0, b_3 < 0) \end{cases}$$

$$\gamma\breve{a} = \begin{cases} \langle (\gamma a_1, \gamma a_2, \gamma a_3); \alpha_{\breve{a}}, \theta_{\breve{a}}, \beta_{\breve{a}} \rangle \; if \; (\gamma > 0) \\ \langle (\gamma a_1, \gamma a_2, \gamma a_3); \alpha_{\breve{a}}, \theta_{\breve{a}}, \beta_{\breve{a}} \rangle \; if \; (\gamma > 0) \end{cases}$$

Definition 2.5. Let $\breve{a} = \langle (a_1, a_2, a_3); \alpha_{\breve{a}}, \theta_{\breve{a}}, \beta_{\breve{a}} \rangle$ be a single valued triangular neutrosophic number then

$S(\breve{a}) = \frac{1}{16}[a_1 + b_1 + c_1] \times (2 + \alpha_{\breve{a}} - \theta_{\breve{a}} - \beta_{\breve{a}})$ and

$A(\breve{a}) = \frac{1}{16}[a_1 + b_1 + c_1] \times (2 + \alpha_{\breve{a}} - \theta_{\breve{a}} + \beta_{\breve{a}})$

are called the score and accuracy degrees of $\breve{a}$ respectively.

3. METHODOLOGY

The research used an integrated evaluation design that explored conceptual and empirical references to project evaluation review techniques and critical path methods, with particular attention to work examples and analyses. Project management involves the processes necessary to ensure the timely completion of a project. The procedures are: schedule management planning, defining activities, sequencing activities, estimating activity durations, creating a schedule and managing a schedule. The next section illustrates the methodology of Project Evaluation Review Technique in Neutrosophic Environment:

3.1. PROJECT EVALUATION REVIEW TECHNIQUE IN NEUTROSOPHIC ENVIRONMENT:

Triangular Neutrosophic PERT (Project Evaluation and Review Technique) analysis is an innovative extension of the conventional PERT, which introduces triangular neutrosophic numbers to effectively handle uncertainty and indeterminacy in the management of large-scale projects. While PERT has long been a valuable tool for coordinating and optimizing tasks in various industries, real-world projects often involve imprecise and uncertain data, which can pose challenges for traditional PERT methods. Triangular neutrosophic numbers offer a more comprehensive representation of uncertainty, incorporating membership, non-membership, and indeterminacy degrees. By integrating triangular neutrosophic numbers into PERT, this advanced analysis approach empowers project managers to efficiently model, evaluate, and control projects in complex scenarios where conventional PERT techniques may be limited. This introduction lays the foundation for exploring the advantages and practical applications of Triangular Neutrosophic PERT analysis, providing insights into how it addresses the complexities of uncertain and ambiguous project environments.

PERT Calculations consisting of three timings namely Optimistic, Pessimistic and Most likely times, which are defined in neutrosophic environment as follows:

Optimistic Time ($\tilde{a}$): It refers to the minimum time required to complete an activity under the most favorable conditions or if everything proceeds smoothly without any hindrance or delay. The optimistic time serves as a baseline for calculating the expected duration and critical path in project management, providing insights into the best possible outcome for completing a specific task.

Pessimistic time($\tilde{b}$): It refers to the maximum time required to complete an activity when encountering challenges, obstacles, or delays at every stage of its execution. The pessimistic time provides a conservative estimate for project planning and risk management, allowing project managers to account for potential delays and allocate sufficient resources to handle adverse circumstances.

Most likely time($\tilde{m}$): It refers to the time required to complete an activity under normal or average conditions, without any significant favorable or unfavorable influences. The most likely time serves as a realistic estimate for project planning and scheduling, as it reflects the typical performance level and expected outcomes for the activity.

Where $\tilde{a}, \tilde{b}, \tilde{m}$, are triangular neutrosophic numbers.

In order to calculate the expected time and standard deviation of each activity based on the three-time estimates $(\tilde{a}, \tilde{b}, \tilde{m})$, it is necessary to obtain crisp values for these estimates. To achieve this, score functions and accuracy functions are utilized. By applying the score function, crisp values are obtained for each time estimate. Once the crisp values are acquired, the expected time and standard deviation of each activity can be calculated. The expected time represents the average duration for completing the activity, while the standard deviation provides a measure of the uncertainty or variability associated with the activity's completion time.

$$T_{ij} = \frac{a+4m+b}{6} \text{ and } \sigma_{ij} = \frac{b-a}{6}$$

Where a, m, b are crisp values of optimistic, most likely and pessimistic time respectively, T_{ij} is the expected time of ij activity and σ_{ij} standard deviation of ij activity.

After calculating the expected time and standard deviation of each activity, the PERT (Project Evaluation and Review Technique) network is treated similarly to the CPM (Critical Path Method) network for the purpose of calculating various network parameters. These parameters include the earliest and latest occurrence time of each activity, identifying the critical path, and determining the floats or slack times for non-critical activities.

Let a network N= $\langle E$ij$\rangle$, being a project model, is given. E is asset of events (nodes) and $A \subset E \times E$ is a set of activities. The set $E = \{1,2, \dots, n\}$ is labeled in such a way that the following condition holds: $(i, j) \in A$ and $i < j$. The activity times in the network are determined by T_{ij}.

Notations of network solution and its calculations as follows:
T_{ie} =Earliest occurrence time of predecessor event i,
T_{il} = Latest occurrence time of predecessor event i,
T_{je} =Earliest occurrence time of successor event j,
T_{jl} = Latest occurrence time of successor event j,
T_{ije} Start = Earliest start time of an activity ij,
T_{ije} Finish t=Earliest finish time of an activityij,
T_{ijl} Start=Latest start time of an Til activity ij,
T_{ijl} Finish t = Latest finish time of an activity ij,
T_{ij} = Duration time of activity ij,

Earliest and Latest occurrence time of an event:
T_{je} = maximum ($T_{je} + T_{ij}$), calculate all T_{je} for j[th] event, select maximum value.
T_{il} = minimum ($T_{jl} - T_{ij}$), calculate all T_{il} for i[th] event, select minimum value.
T_{ije} Start = T_{ie},
T_{ije} Finish t = $T_{ie} + T_{ij}$,
T_{ijl} Finish t = T_{jl},
T_{ijl} Start = $T_{jl} - T_{ij}$,

Critical path is the longest path in the network. At critical path, $T_{ie} = T_{il}$, for all i.
Slack or Float is cushion available on event/ activity by which it can be delayed without affecting the project completion time.
Slack for i[th] event = $T_{il} - T_{ie}$, for events on critical path, slack is zero.
The expected time of critical path (μ) and its variance (σ^2) calculated as follows;
$\mu = \Sigma T_{ij}$, for all ij on critical path.

3.3. TNP Algorithm

The proposed algorithm can be summarized as follows:

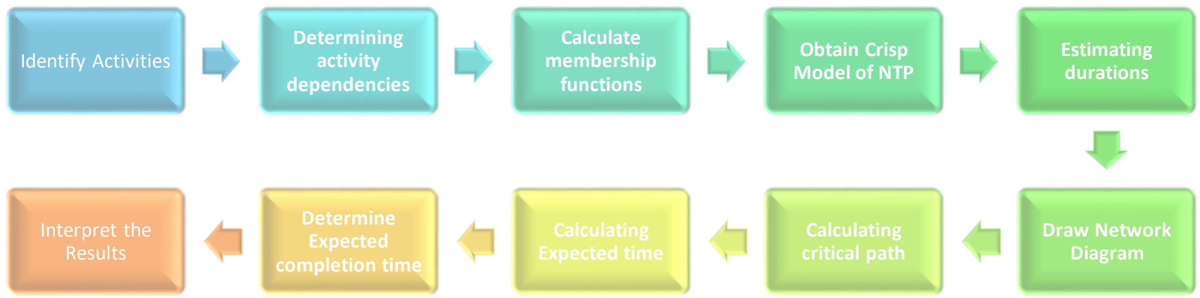


Fig.1. Flow chart

1. Addressing uncertain considered as membership $T_A(x)$ values, inconsistent mentioned as indeterminacy $I_A(x)$ and incomplete information taken as non-membership $F_A(x)$ regarding activity time involves representing the three-time estimates of the PERT technique as single-valued triangular neutrosophic numbers.
2. Calculate the membership functions for each single-valued triangular neutrosophic number using equations 1, 2, and 3.
3. Derive a crisp model of PERT three-time estimates by employing the score function equation as previously demonstrated.
4. Utilize the crisp values of the three-time estimates to compute the expected time and standard deviation for each activity.
5. Construct a PERT network diagram and calculate the project completion time for all the events using crisp values which has been taken from the single-valued triangular neutrosophic number.
6. Identify floats and determine the critical path, which represents the longest path in the network by using the formula T_{ije} Start = T_{ie}, T_{ije} Finish t = $T_{ie} + T_{ij}$, $Tijl$ Finish t = T_{jl}, T_{ijl} Start = $T_{jl} - T_{ij}$.
7. Calculate the expected time and variance of the critical path with the help of

$T_{ij} = \frac{a+4m+b}{6}$ and $\sigma_{ij} = \frac{b-a}{6}$

8. Determine the expected project completion time.
9. Assess the expected probability values for various project completion scenarios based on specific demands mentioned in the given real-life situation.

The next section illustrates the proposed algorithm with numerical example based on the real-life situation existed in a Reputed company situated in Tirupur, Tamilnadu, India.

4. APPLICATIONS

The main purpose of this section is to apply the proposed methodology in step by step process. This section is separated into three main parts. The first part offers an actual case study of the implementation of the recommended approach. The second part applies the steps of the proposed TNP approach. The third and final part discusses the results of the study.

4.1. NUMERICAL ILLUSTRATION

ESA Clothing Company established in 1997, which is the primary manufacturers of garments such as t-shirts, children wear and cotton shirts. It is infused with the aim to deal in best quality garments and the best garment solutions provider within the reach. Company made a continuous improvement in the supply of various genuine and trusted quality garments to meet the ever-increasing market requirements. They hereby introduce their company "JUBILEE TEX & ESA CLOTHING COMPANY" as one among the Leading Garment Manufacturing and Exporting Company situated at Tirupur, Tamilnadu, India, with high potential to serve and cater to the needs of the Quality conscious customers. They have a very good base in the garmenting field as their parent company was established in the year 1968 catering to the Indian domestic market. In the year 1989 their export division in the name of JUBILEE TEX was established with full focus on the export market. With a steady growth in business their new company in the name of ESA CLOTHING COMPANY was started in 2004 with wide focus on the Branded labels, Stores and importers all over Europe & U.S.

Having an initial capacity of producing 2500 Pcs per day. They have now reached a stage where they are producing 4,00,000 Pcs /month. Their focus is on the Babies, Children's, Women's, Men's wear market as this has been their prime product line since the day one of our export business. With factory spreading over 3 different premises and with 12 Lines they can dedicate each factory to different requirement of each customer. (Quantity and quality wise). They can do quantity ranging from 1,000pcs and more in three of the factories. Their factory is compliance with all Garment Factory Norms.

Between June 5, 2023, and July 4, 2023, the Esa Clothing Company diligently recorded the production timeline for their assorted products, which include boxers, track pants, and T-shirts. These products are crafted from various fabric materials such as single jersey, lycra derby, single rib, jacquard, lycra drop rib, waffle, and filament lycra jersey. Each fabric type demands a distinct duration for manufacturing. For instance, we have gathered specific data concerning the production time for T-shirts made from single rib fabric.

The manufacturing process for these T-shirts commenced on June 8, 2023, and reached completion on July 4, 2023. Notably, a substantial order of 1200 casual wear T-shirts was placed by a client in the USA. It's worth highlighting that the majority of the company's orders originate from the

USA. The entirety of the production process encompasses seven key components: knitting, dyeing, cutting, stitching, printing, ironing, and the final packing stage.

The company encounters significant challenges in securing the appropriate personnel for various roles due to a shortage of manpower. On certain days, individuals may be available for stitching tasks, while the demand lies in the packaging department, creating a similar predicament across different departments with varying availabilities. To address this uncertain, inconsistent, and indeterminacy scenario, the gathered data can be effectively presented in Table 1 using triangular neutrosophic numbers. In this context, the Triangular Neutrosophic PERT approach is adapted to optimize the projected timeline for completing the project. The project's pertinent data is presented as follows:

Table 1:

Activity	Score Function of $\underline{a}$=<(a_1,a_2,a_3,α_a,θ_a,β_a>	$S(a)$
Knitting	2=<(8,10,12).0.2,0.5,0.6>	2
Dyeing	3=<(5,8,10),0.8,0.2,0.6>	3
Cutting	4=<(9,17,25),0.3,0.6,0.4>	4
Stitching	5=<(20,25,30),0.7,0.4,0.6>	8
Printing	6=<(10,13,17).0.8,0.2,0.4>	6
Ironing	4=<(10,19,25),0.3,0.6,0.5)>	4
Packing	8=<(18,24,28).0.2,0.4,0.6>	5

Table 2:

Activity	Notation	Predecessor	Representation
Knitting	A	-	1-2
Dyeing	B	a	2-3
Cutting	C	b	3-4
Stitching	D	b	3-5
Printing	E	c	4-6
Ironing	F	d	5-6
Packing	G	e, f	6-7

In the following table t_m, t_o, t_p are optimistic, most likely and pessimistic time in neutrosophic environment, and considered as a single valued triangular neutrosophic numbers. To get the crisp values of each single valued triangular neutrosophic number, calculate score function of

$\boldsymbol{a} =< (a_1, a_2, a_3), \alpha_a, \theta_a, \beta_a >$ by using the below formula
$S(\boldsymbol{a}) = \frac{1}{16}(a_1 + a_2 + a_3) * (\alpha_a + (1 - \theta_a) + (1 - \beta_a))$.

Table 3:

ACTIVITY	t_o	t_m	t_p	t_e
a	2	4	6	4
b	2	3	4	3
c	3	4	6	4
d	4	5	6	5
e	4	6	8	6
f	2	3	4	3
g	2	4	6	4

From the calculated values in table 3 and from the given condition the network diagram with expectation time mentioned as single valued neutrosophic numbers (crisp numbers) in the following network diagram.

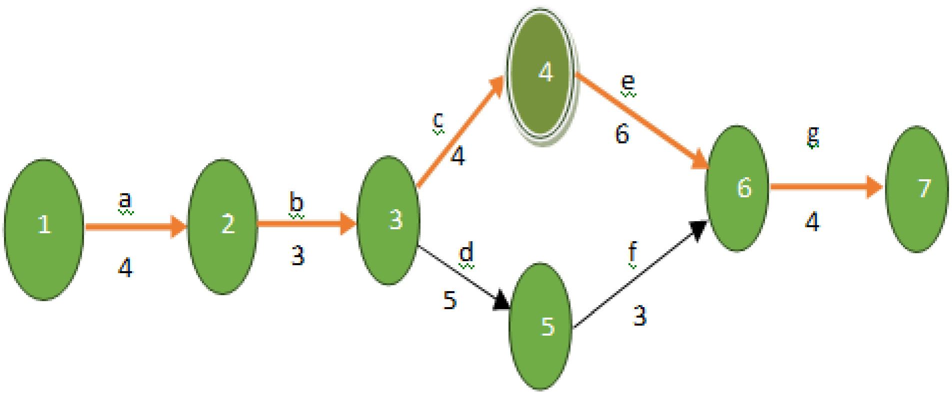


Fig2: Network Diagram

The critical path is a-b-c-e-g.
The T-shirts will be manufactured in =21 days
iii) Probability of manufacturing T shirt in 25 days

VARIANCE= $\left[\frac{6-2}{6}\right]^2 = \frac{16}{6}; \left[\frac{4-2}{6}\right]^2 = 4/6; \left[\frac{6-3}{6}\right]^2 = \frac{9}{6}; \left[\frac{8-4}{6}\right]^2 = 16/6; \left[\frac{6-2}{6}\right]^2 = \frac{16}{6}$

$\Sigma V_{critical} = \frac{61}{36}; \Sigma t_e critical = 26 \text{ days}$

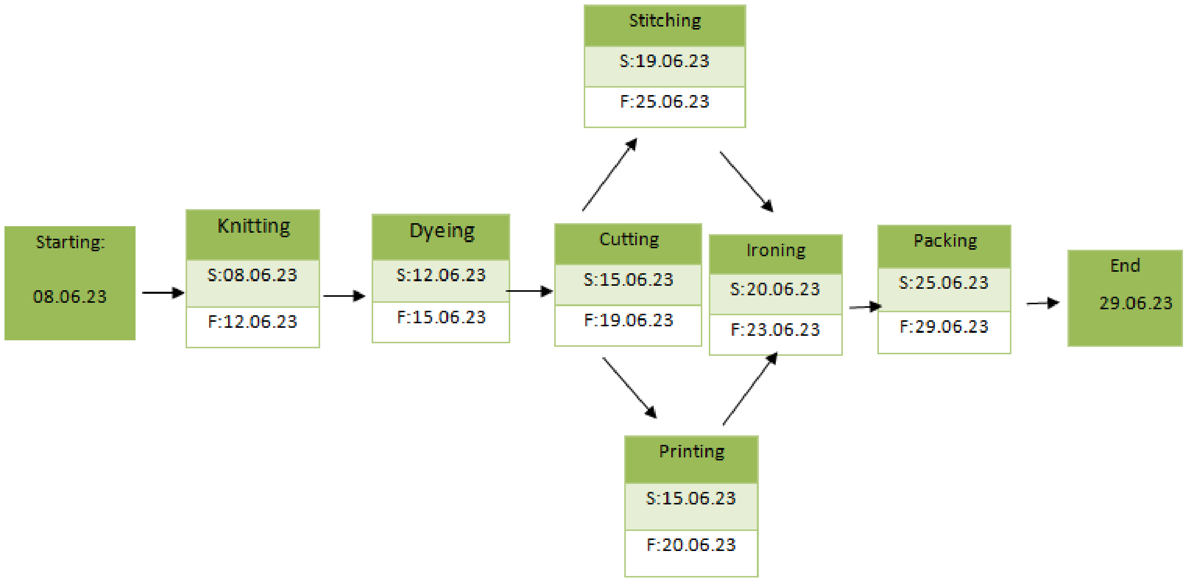
$Z = \frac{X - \Sigma T_{e critical}}{\sqrt{\Sigma v_{critical}}} = \frac{25-21}{\sqrt{1.694}} = \frac{4}{1.30} = 3.076 = 0.4989$ (from normal distribution table)

Probability=1-0.4989 =0.5011

Table 4:

ACTIVITY	ACTIVITY	t_e	ES	EF	LS	LF	SH	ST
a	1-2	4	0	4	0	4	0	0
b	2-3	3	4	7	4	7	0	0
c	3-4	4	7	11	7	11	0	0
d	3-5	5	7	12	9	14	2	0
e	4-6	6	11	17	11	17	0	0
f	5-6	3	12	15	14	17	0	2
g	6-7	4	17	21	17	21	0	0

Network of Activities:



INTERPRETATION:

Based on the current observations, crafting a single rib T-shirt takes approximately 29 days using the existing manufacturing process. However, with the implementation of the proposed algorithm and the application of the Triangular Neutrosophic PERT process, the projected completion time for manufacturing these T-shirts is reduced to 21 days. This notable enhancement shortens the timeline by 8 days compared to the existing method. This reduction in manufacturing duration inherently leads to a corresponding decrease in the production costs associated with these T-shirts. The authors have recommended the adoption of this innovative TNP approach to the Esa Clothing Company, aiming to optimize machine time, human resources, and the overall expenses tied to the T-shirt manufacturing process.

4.2 NUMERICAL ILLUSTRATION

Esa Clothing Company extends its production to encompass men's track pants, driven by a surge in demand and usage. Much like the T-shirts, track pants come in diverse fabric materials including Lycra, blended cotton, and Dry Fit Fabrics. Traditionally, companies maintain sample garment pieces as reference; for instance, a pre-existing Lycra track pant fabric was readily available. This fabric merely needed cutting and stitching to transform into a finalized product. Specifically, the provided data focuses on the stitching aspect of crafting track pants. The stitching process is further subdivided into distinct tasks, such as folding pockets, sewing bar tracks, adding waistbands, and finalizing ankle cuffs.

Table 5:

Score Function of $\underline{a}$=<($a_1,a_2,a_3,\alpha_a,\theta_a,\beta_a$>	$S(a)$
2=<(56,76,86), 0.8,0.2,0.4>	30
3=<(120,200,280), 0.7,0.5,0.6>	60
4=<(238,268,298), 0.4,0.2,0.4>	90
5=<(460,580,700), 0.2,0.5,0.6>	120
6=<(262,62,462), 0.8,0.2,0.4>	150
7=<(381,480,581), 0.6,0.2,0.4>	180
9=<(252,1005,1485), 0.4,0.6,0.4>	240
10=<(1200,1500,1800), 0.7,0.4,0.6>	340
11=<(1104,1204,1304),0.1,0.2,0.3)>	360
12=<(1388,1686,1988),0.3,0.6,0.5)>	380

Table 6:

Activity	Activity	Predecessor	Representation
Cutting	a	-	1-2
Sewing pocket	b	a	2-3
Joining pocket	c	a	2-4
Joining the sides	d	a	2-5
Add waist band and ankle case	e	b	3-5
Inset elastic	f	c	4-5
Sewing the bar tracks	g	d,e,f	5-6

A primary objective involves pinpointing the slack time for each activity and identifying potential modifications to minimize this slack period within each process. The ultimate aim is to establish the earliest feasible completion time for the project, facilitating a reduction in both process time and overall manufacturing duration. Apply the proposed algorithm to achieve this optimization for he given stitching durations (in minutes) of each process are detailed in the Table 5.

Table 7:

Activity	Activity	t_o (m's)	t_m (m's)	t_p (m's)	t_e (m's)	t_e (hrs)
a	1-2	5	7	8	180	3
b	2-3	2	3	4	60	1
c	2-4	2	3	4	60	1
d	2-5	10	11	12	360	6
e	3-5	5	9	11	240	4
f	4-5	4	5	6	120	2
g	5-6	5	9	11	240	4

Activity	Activity	t	ES	EF	LS	LF	SH	ST
a	1-2	3	0	3	0	3	0	0
b	2-3	1	3	4	4	5	0	1
c	2-4	1	3	4	6	7	0	3
d	2-5	6	3	9	3	9	0	0
e	3-5	4	4	8	5	9	1	0
f	4-5	2	4	6	7	9	3	0
g	5-6	4	9	13	9	13	0	0

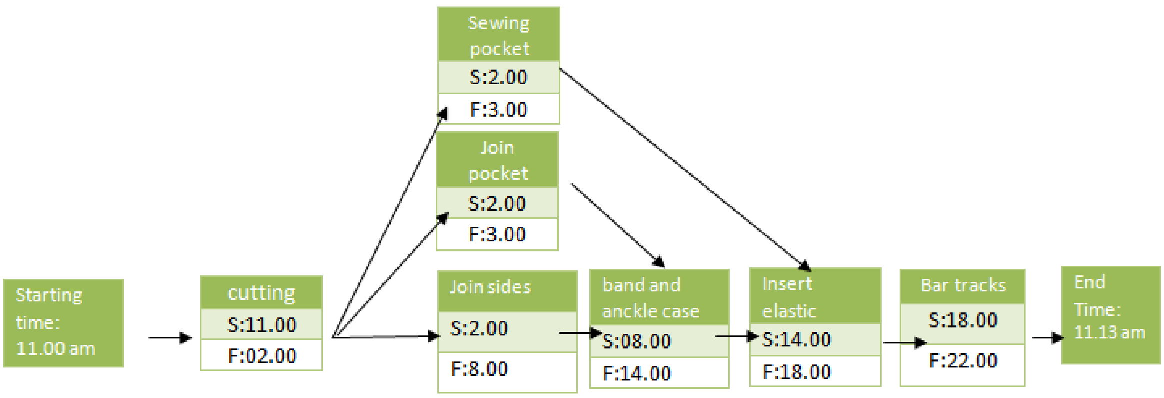


Fig 3: Network Diagram

Probability: Probability of completing the stitching process of track pants in 15 hours

Although the project is estimated to be completed within 15 hours there is no guarantee that it will actually be completed within the 15hours. If by some circumstances various activities take longer than their expected time, the project might not be completed within the desired schedule. Therefore, it will be useful to know the probability that the project deadline will be met. The first step is to find the variance and standard deviation of the total time along critical path, which is equal to the sum of the variances of activity times on the critical path.

Variance:

1) $\left[\frac{4-2}{6}\right]^2 = \frac{4}{36};\left[\frac{6-4}{6}\right]^2 = \frac{4}{36};\left[\frac{6-2}{6}\right]^2 = \frac{16}{36};$

$\sum \mathbf{t}_{\mathbf{e}_{critical}} = \mathbf{3+6+4}$ =15 hours $\sum \boldsymbol{v}_{critical}$= 0.66

$$Z = \frac{X-\Sigma T_{e_{critical}}}{\sqrt{\Sigma v_{critical}}} = \frac{13-1}{\sqrt{0.66}} = \frac{1}{0.81} = 1.2 = 0.3907 \text{(by normal distribution)}$$

$Probability = 1 - 0.3907 \;\; = \;\; 0.6093$

Thus, there is 60% chance to that the critical path will be completed in less than 15 hours.

INTERPRETATION:

As documented in the current records, the entire process currently requires 21 hours for completion. However, with the adoption of the suggested TNP methodology, the minimum time needed to finalize the process dwindles to 13 hours, thereby presenting an opportunity to economize 8 hours. Nevertheless, it's important to note that despite the potential to conclude the process within 13 hours, certain delays arise during the occurrence of events b, c, e, and f.

4.3 NUMERICAL ILLUSTRATION

Esa Clothing Company, as a manifestation of its expansion, has already established an additional production unit to meet the growing influx of orders. Presently, the company envisions the creation of yet another compact unit, dedicated to knitting activities and warehousing. To materialize this plan, an engineer has provided an estimated timeframe detailing the anticipated number of days required for the unit's completion. The construction process involves a range of activities, including basement construction, sidewall development, and roof assembly, all of which play a crucial role in the overall construction. Estimates from various companies and material quotations, based on responses received, have been compiled in a neutrosophic triangular number format, leading to the development of a new unit. In this context, the duration for executing these construction tasks extends from April 17, 2023, to July 5, 2023. To comprehensively assess the projected completion time for the project, it is imperative to consider both the factual duration and the potential timeline. This evaluation involves implementing the suggested Triangular Neutrosophic PERT (TNP) methodology, which aims to determine an optimal timeframe for achieving project culmination. The table provided outlines the specific timeframes, measured in days, allocated for each distinct construction activity.

Table: 8

Score Function of $\underline{a}$=<($a_1,a_2,a_3,\alpha_a,\theta_a,\beta_a$>	$S(a)$	Score Function of $\underline{a}$=<($a_1,a_2,a_3,\alpha_a,\theta_a,\beta_a$>	$S(a)$
1=<(8,10,12).0.2,0.5,0.6>	2	16=<(45,62,79),0.7,0.4,0.6>	17
2=<(5,8,10),0.8,0.2,0.4>	3	17=<(65,75,85),0.3,0.6,0.4>	18
3=<(10,15,20),0.3,0.6,0.2>	4	18=<(100,109,118),0.4,0.6,0.8>	20
4=<(18,24,28),0.2,0.4,0.6>	5	19=<(60,69,78),0.2,0.5,0.6>	14
5=<(10,13,17),0.8,0.2,0.4>	6	20=<(40,50,60).0.4,0.2,0.1>	20
6=<(10,19,25),0.3,0.6,0.5)>	7	21=<(49,67,84).0.8,0.4,0.6>	23
7=<(20,25,30),0.7,0.4,0.6>	8	22=<(70,74,78).0.8,0.5,0.5>	25
8=<(24,26,29),0.4,0.2,0.4>	9	23=<(55,80,105).0.7,0.2,0.5>	30
9=<(35,39,44),0.9,0.7,0.8>	10	24=<(10,13,17).0.8,0.2,0.4>	6
10=<(15,32,49),0.7,0.2,0.5>	12	25=<(70,74,78).0.8,0.5,0.5>	25
11=<(45,62,79),0.6,0.4,0.7>	17	26=<(55,80,105).0.7,0.2,0.5>	30
12=<(38,47,56),0.1,0.2,0.8>	10	27=<(75,86,97).0.8,0.2,0.4>	35
13=<(14,31,38),0.9,0.1,0.5>	12	28=<(10,19,25),0.3,0.6,0.5)>	7
14=<(65,74,83),0.2,0.5,0.6>	15	29=<(108,112,116).0.5,0.2,0.4>	40
15=<(50,52,54),0.3,0.4,0.6>	13		

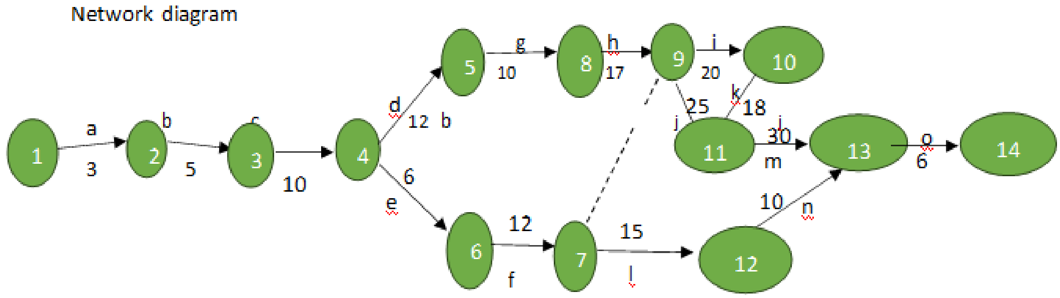
Table 9:

Activity	Activity	Predecessors	Activity
Raising base	a	-	1-2
Basement	b	a	2-3
Masonry work	c	b	3-4
Constructing roof	d	c	4-5
Fixing doors and windows	e	c	4-6
Plastering with cement	f	e	6-7
Fixing tiles	g	d	5-8
Fixing electrical lines& sanitary work	h	g	8-9
Plastering	i	f, h	9-10
Applying primer	j	f, h	9-11
Painting	k	i	10-11
Give provision to fix knitting machine	l	f	7-12
Fixing	m	j, k	11-13

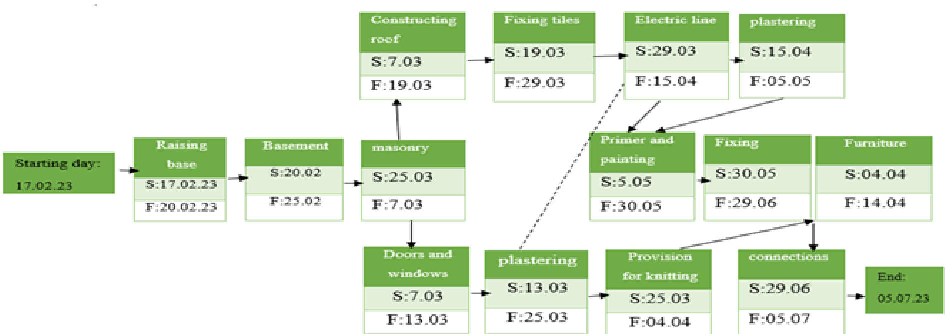
Furniture works	n	l	12-13
Giving connections for electrical works	o	m, n	13-14

Table 10

Activity	Activity	to	tm	tp	te
a	1-2	1	2	3	3
b	2-3	1	4	7	5
c	3-4	9	12	14	11
d	4-5	9	10	11	13
e	4-6	4	5	7	6
f	6-7	10	13	17	13
g	5-8	9	12	15	10
h	8-9	14	16	18	17
i	9-10	18	20	22	21
j	9-11	22	25	27	27
k	10-11	14	17	21	18
l	7-12	15	14	17	15
m	11-13	23	26	29	32
n	12-13	8	12	19	11
o	13-14	5	24	28	6



Activity	Activity	to	tm	tp	te	ES	EF	LS	LF	FS	TS
a	1-2	2	3	4	3	0	3	0	3	0	0
b	2-3	3	5	8	5	3	8	3	8	0	0
c	3-4	10	10	15	11	8	19	8	19	0	0
d	4-5	10	12	17	13	19	32	19	32	0	0
e	4-6	5	6	8	6	19	25	19	46	0	21
f	6-7	12	12	18	13	25	38	40	59	15	21
g	5-8	8	10	13	10	32	42	32	42	0	0
h	8-9	15	17	20	17	42	59	42	59	0	0
i	9-10	20	20	25	21	59	80	59	80	0	0
j	9-11	25	25	35	27	59	98	59	98	13	13
k	10-11	15	18	23	18	80	98	80	98	0	0
l	7-12	13	15	18	15	38	53	59	119	21	66
m	11-13	30	30	40	32	98	130	98	130	0	0
n	12-13	9	10	14	11	53	130	119	130	66	0
o	13-14	6	6	7	6	130	136	130	136	0	0



The depicted network diagram above illustrates the start time, completion time, and float time for each event. These outcomes were derived using the TNPERT algorithm proposed in this study.

INTERPRETATION:

The concept of TNP entails an analytical approach crafted to aid in the orderly arrangement of activities that necessitate sequential execution. Upon further scrutiny, the average time for completing the construction of both a storage facility and a knitting unit is determined to be 173 days based on the available data. By effectively organizing tasks using TNPERT techniques, the construction process for a new branch is streamlined, resulting in a reduced timeline of 131 days. This discrepancy of 42 days signifies a significant time-saving measure. Capitalizing on this time differential can effectively enhance construction efficiency and lead to diminished production expenses.

RESULT AND FUTURE WORK:

This article exemplifies the practical application of triangular neutrosophic numbers in a real-life scenario within a manufacturing company. The presence of uncertainty and ambiguity is identified, particularly stemming from a significant volume of consignments. The operational gap between the production unit and the logistics department exacerbates the uncertain and ambiguous situations within the company. The application of triangular neutrosophic numbers effectively portrays and clarifies the prevailing circumstances. Employing a scoring function, the triangular neutrosophic numbers are transformed into single-valued numbers. Subsequently, the PERT procedure is applied to ascertain both the production completion time and the associated profit. This serves as an initial exploration, and in future endeavors, considering the multifaceted nature of departments and diverse categories within such companies, the application of neutrosophic numbers holds promise for mitigating uncertainty and ambiguity. Furthermore, employing neutrosophic numbers can contribute to optimizing profits or minimizing utilization periods across various departments and categories.

CONCLUSION:

Through this data it happened to learn how a garment is manufactured and what is all the process involved in. In business it is very important to keep up the timing. To keep up the timing scheduling the works accordingly is much needed. Here using Program evaluation and review technique and critical path method we have scheduled the works and found the minimum time that will be taken to manufacture the garments. This will help the company to gain profit with less working hours and with more production. In conclusion, this research article presents the practical implementation of Triangular Neutrosophic PERT analysis. Leveraging the advantages offered by Neutrosophic numbers, the study addresses a range of issues. Esa Clothing Company's multifaceted production of garments from diverse materials and processes has been explored. Notably, the company's competitiveness has been hindered by suboptimal profit margins, partly attributed to prolonged project durations. Through the innovative application of Triangular Neutrosophic numbers, these challenges have come to light, prompting recommendations to streamline project timelines by minimizing slack and delay times across the company's endeavors. The research encompasses thorough time calculations, yielding insights into actual project completion times, projected completions, and the probabilities associated with achieving revised timeframes. As this study concludes, the adoption of Triangular Neutrosophic PERT analysis offers a strategic avenue for enhancing efficiency, ultimately contributing to improved competitiveness and profitability for Esa Clothing Company.

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University of New Mexico

An Efficient Approach for Solving Time-Dependent Shortest Path Problem under Fermatean Neutrosophic Environment

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Abstract: Efficiently determining optimal paths and calculating the least travel time within complex networks is of utmost importance in addressing transportation challenges. Several techniques have been developed to identify the most effective routes within graphs, with the Reversal Dijkstra algorithm serving as a notable variant of the classical Dijkstra's algorithm. To accommodate uncertainty within the Reversal Dijkstra algorithm, Fermatean neutrosophic numbers are harnessed. The travel time associated with the edges, which represents the connection between two nodes, can be described using fermatean neutrosophic numbers. Furthermore, the edge weights in fermatean neutrosophic graphs can be subject to temporal variations, meaning they can change over time. In this study, an extended version of the Reversal Dijkstra algorithm is employed to discover the shortest path and compute the minimum travel time within a single-source time-dependent network, where the edges are weighted using fermatean neutrosophic representations. The proposed method is exemplified, and the outcomes affirm the effectiveness of the expanded algorithm. The primary aim of this article is to serve as a reference for forthcoming shortest path algorithms designed for time-dependent fuzzy graphs

.**Keywords:** Fuzzy set theory, fermatean neutrosophic numbers, Reversal Dijikstra's Algorithm, Time- dependent Shortest Path Problem, Score Function, Shortest Travel time.

1. Introduction

The introduction should briefly place the study in a broad context and highlight why it is important. It should define the purpose of the work and its significance. The current state of the research field should be reviewed carefully and key publications cited. Please highlight controversial and diverging hypotheses when necessary. Finally, briefly mention the main aim of the work and highlight the principal conclusions. As far as possible, please keep the introduction comprehensible to scientists outside your particular field of research. References should be numbered in order of appearance and indicated by a numeral or numerals in square brackets, e.g., [1] or [2,3], or [4–6]. See the end of the document for further details on references. The shortest path problem(SPP) is a fundamental concept that finds applications in a wide range of fields, from real-life scenarios to the domain of operations research and graph theory. At its core, this problem is concerned with determining the most efficient path between two points in a network, where efficiency is typically measured in terms of minimizing a certain cost or distance metric. In real life, the shortest path problem is encountered daily in numerous ways like a delivery company optimizing its delivery routes to minimize fuel consumption and time, or a telecommunication network seeking the most efficient way to transmit data between users. Therefore, the values can be uncertain in those scenarios, to handle that Zadeh [2] introduced Fuzzy set(FS) theory which is an excellent tool to cope up imprecise data. It can expressed in terms of membership values. The concept of convexity and its applications have been extended to interval-valued fuzzy sets (IVFS) by Huidobro in their work [1]. In 1999, Atanassov introduced intuitionistic fuzzy numbers (IFN), which are defined in terms of membership and non-membership values. Additionally, Atanassov also extended the concept to interval-intuitionistic fuzzy (IVIFS) sets, which involve lower and upper bounds in relation to membership and non-membership values [3, 4]. Definitions for concentration, dilation, and characterization of Intuitionistic Fuzzy Sets (IFS) have been provided by another source [6]. The concept of interval-valued pythagorean neutrosophic sets, their operations and decision making apporach were introduced by Stephen [16] Both IFSs and IVIFS are widely applied in practical problem-solving. However, they may not adequately address situations where neutrality or a lack of knowledge is crucial. To address such cases, the concept of neutrosophic sets was introduced by Florentin Smarandache in their work [5]. Neutrosophic sets are specifically designed to handle problems that involve factors of neutrality or indeterminacy as significant components.To provide a comprehensive view of neutrosophic sets from a technical perspective, several distinct variants have been introduced in the literature. Notably, Single-valued neutrosophic fuzzy sets (SVNFS) have been proposed as a specific instance of Neutrosophic sets, which has been extensively discussed in academic works such as [11], [12], and [13]. In a parallel development, the concept of interval-valued neutrosophic fuzzy sets (IVNFS) has been put forward to represent sets within a unit interval. This innovation has led to the development of various operations and comparison techniques for interval-valued neutrosophic fuzzy sets, as extensively elaborated upon by Zhang et al. in [10]. Furthermore, Yen has contributed to the field by introducing the concept of trapezoidal neutrosophic fuzzy numbers, along with measures of similarity and operations related to them, as discussed in [14]. To expand the horizons of neutrosophic fuzzy sets, researchers have also focused on Pythagorean neutrosophic fuzzy numbers (PyNFN). The development of similarity measures for Pythagorean neutrosophic fuzzy numbers has been explored by Rajan in [31]. Fuzzy set theory has emerged as a valuable tool for managing data characterized by imprecision, inaccuracy, and vagueness. Among the challenges it addresses, one prominent problem is the Fuzzy Shortest Path Problem (FSPP), which entails finding optimal paths within a graph while optimizing an objective function in a fuzzy environment. This field has seen several significant contributions: In a pioneering effort, Dubois [17] introduced an algorithm to solve FSPP and determine optimal weights, laying the foundation for subsequent research in this domain. Klein [24] conducted an analysis of FSPP from the perspective of fuzzy mathematical programming, thereby opening the door for further exploration and extensions of the concept. Building upon this groundwork, Okada and Soper [21] introduced the Multiple Label Method tailored for large random networks, providing an effective solution for FSPP. To overcome the limitations of traditional non-interactive approaches, Okada [22] introduced the notion of the degree of possibility, a concept used to represent arc lengths using fuzzy numbers. Nayeem et al. [20] considered networks with interval-number and triangular fuzzy numbers, developing an algorithm capable of accommodating both types of uncertain numbers. Recognizing the computational complexity of FSPP, Hernandes et al. [26] presented a method that relies on a generic index ranking function to compare fuzzy numbers. This approach also accounted for graphs with negative parameters. Kumar [19] extended the scope of FSPP by addressing interval-valued fuzzy numbers and introducing an algorithm that could solve both fuzzy shortest path length and crisp shortest path length problems. Vidhya et al. [25] conducted a comparative study between the Floyd-Warshall algorithm and the rectangular algorithm in a fuzzy environment, shedding light on their performance. In a different direction, Baba [18] introduced a technique for solving the Intuitionistic Fuzzy Shortest Path Problem (IFSPP). Mukherjee [23] implemented Dijkstra's algorithm for finding the shortest path with intuitionistic fuzzy arc weights in a graph. A study on SVNF SPP was proposed Liu [28]. Broumi et al. [27] conducted a comprehensive comparative study of all existing approaches to FSPP, ultimately identifying the most suitable methods for handling uncertainty in various environments. Innovative techniques for solving the Pythagorean neutrosophic fuzzy shortest path problem have been put forth by Basha et al. in their work [30]. Additionally, Rahut's research, as presented in [32], has concentrated on fermatean neutrosophicshortest path problems, employing a similarity-based approach that has yielded optimal results for the proposed methodology. Cakir et al. suggest the time-dependent shortest path problem with bipolar neutrosophic environment [29]. Broumi et al. have introduced a novel approach for addressing the interval-valued fermatean neutrosophic shortest path problem in a related domain, as outlined in their study [33]. This approach builds upon Dijkstra's classical algorithm to navigate the complexities of this specific problem, offering valuable insights into its solution. The reversal dijikstra algorithm is a modification of standard dijikstra algorithm, which is used to find the shortest path in a weighted graph. Unlike standard Dijkstra's, which focuses on finding the shortest paths from one source to all nodes, Reversed Dijkstra's focuses on finding the shortest paths to a specific target from all nodes. To handle the fuzzy environment and time dependency, the reversal dijikstra algorithm is considered. This study extends the reversal dijkstra algorithm to find the shortest travel time along with time dependency in a fuzzy environment. In a time-dependent fuzzy graph, the concept of finding the shortest path is synonymous with identifying the shortest duration or travel time between two points in the graph. This paper combines the fermatean neutrosophic numbers with reversal dijikstra's algorithm along with time dependency. The proposed algorithm can efficiently compute both the shortest path and the corresponding shortest travel time from a starting node to every other node in a graph (or digraph) in reverse methodolgy. This graph is characterized by edges that are represented using time-dependent fermatean neutrosophic values. This paper contributes (i)the fermatean neutrosophic arc values to handle uncertainty, (ii) further, an algorithm is proposed for the reversal dijikstra algorithm with time-dependent fermatean neutrosophic numbers. (iii)the numerical examples are tracked down to show the efficieny of the proposed method.

The paper is structured as follows: Section 2 covers the essential concepts, definitions, and mathematical operations associated with fermatean neutrosophic numbers. Section 3 presents and elaborates on the algorithm proposed in this research. Section 4 provides a numerical example to illustrate the application of the proposed algorithm. Section 5 discusses analyzing the results obtained from the numerical example, offering insights and implications. Finally, Section 6 serves as the concluding segment, summarizing the main findings and the paper's overall conclusions.

2. Preliminaries.

In this section, the definitions of fermatean sets, neutrosophic sets , fermatean neutrosophic sets and their arithmetic operations are discussed.

Definition 1. [7] The Fermatean fuzzy Set (FFS) $\tilde{F}$ in the universal set X is defined by $\tilde{F} = \{\langle x, \mu_{\tilde{F}}(x), \nu_{\tilde{F}}(x)\rangle : x \in X\}$ where the membership function $\mu_{\tilde{F}}(x)$: X → [0, 1] and the non-membership function $\nu_{\tilde{F}}(x)$: X → [0, 1] satisfy the condition $[\mu_{\tilde{F}}(x)]^3 + [\nu_{\tilde{F}}(x)]^3 \leq 1$ is said to be the degree of hesitation of x to $\tilde{F}$.

Definition 2. [8] Let X be the universe of discourse. Then $N = \{\langle x, T_N(x), I_N(x), F_N(x)\rangle : x \in X\}$ is defined as Neutrosophic Fuzzy Set (NFS), where the truth-membership function is represented as $T_N(x): X \to [0,1]$ an interdeterminacy-membership function $I_N(x): X \to [0,1]$ and the falsitymembership function $F_N(x): X \to [0,1]$ which satisifes the conditions $0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3, \forall x \in X$.

Definition 3. [8] A neutrosophic fuzzy set ℓ in the universe X is the form of $\ell = \{\langle u, T_\ell(u), I_\ell(u), F_\ell(u)\rangle : u \in \ell\}$ represents the degree of truth,
indeterminacy and falisty-membership of ℓ respectively. The mapping $T_\ell(u): \ell \to [0,1]$, $I_\ell(u): \ell \to [0,1]$, $F_\ell(u): \ell \to [0,1]$ and $0 \leq T_\ell(u)^3 + I_\ell(u)^3 + F_\ell(u)^3 \leq 2$. Here $\ell = (T_\ell, I_\ell, F_\ell)$ is denoted as fermatean neutrosophic number(FNN).

Definition 4. [8] Let $\ell_1 = (T_{\ell_1}, I_{\ell_1}, F_{\ell_1})$ and $\ell_2 = (T_{\ell_2}, I_{\ell_2}, F_{\ell_2})$ be the two FNNs and $\lambda \geq 0$, then the arithmetic operations are:

1. $\ell_1 + \ell_2 = \left((\sqrt[3]{(T_{\ell_1})^3 + (T_{\ell_2})^3 - (T_{\ell_1})^3(T_{\ell_2})^3}), I_{\ell_1}I_{\ell_2}, F_{\ell_1}F_{\ell_2}\right)$
2. $\ell_1 \otimes \ell_2 = \left(T_{\ell_1}T_{\ell_2}, \sqrt[3]{(I_{\ell_1})^3 + (I_{\ell_2})^3 - (I_{\ell_1})^3(I_{\ell_2})^3}, \sqrt[3]{(F_{\ell_1})^3 + (F_{\ell_2})^3 - (F_{\ell_1})^3(F_{\ell_2})^3}\right)$
3. $\ell_1 \odot \ell_2 = \left\{\left(\sqrt[3]{\frac{(T_{\ell_1})^3 - (T_{\ell_2})^3}{1 - (T_{\ell_2})^3}}, \frac{I_{\ell_1}}{I_{\ell_2}}, \frac{F_{\ell_1}}{F_{\ell_2}}\right) \text{ if } T_{\ell_1} \geq T_{\ell_2}, I_{\ell_1} \leq I_{\ell_2}, F_{\ell_1} \leq F_{\ell_2}\right\}$
4. $\lambda \ell_1 = \left(\sqrt[3]{1 - (1 - (T_{\ell_1})^3)^\lambda}, (I_{\ell_1})^\lambda, (F_{\ell_1})^\lambda\right)$

Definition 5. [9] Let $\ell = (T_\ell, I_\ell, F_\ell)$ be the FNFS, then the score function $\Im(\ell)$ is defined by

$$\Im(\ell) = \frac{T_\ell + I_\ell + 1 - F_\ell}{3} \quad (1)$$

2.1 Advantage and Limitations of different type of fuzzy sets

The table 1 offers a detailed comparison of the advantages and limitations associated with various fuzzy set variations.

Table 1. Advantages and Restrictions with existing Approaches.

Types of Fuzzy Sets	Advantages	Restrictions
Fuzzy sets	It can employed when the weights are imprecise	Only the membership degree associated with the edge

	or uncertain in a unclear situations.	values can be utilized. It is significant for non-membership grades.
Intuitionistic Fuzzy Sets	It can be adapted with imprecise edge weights that include both membership and nonmembership values.	It becomes ineffective when the sum of membership and non-membership exceeds one.
Neutrosophic Fuzzy Sets	This set has indeterminacy as explicity quantified and truth-membership, indeterminacy membership and falsity-membership are independent.	Not applicable when the sum of truth, indeterminancy, falsity exceeds three.
Pythagorean Fuzzy Sets	It has the capability to manage imprecise arc weights, even when the combination of the acceptance grade and the rejection grade surpasses 1, subject to certain constraints.	When the sum of the squares of membership and non-membership exceeds one, it is not suitable for application. Eg:(0.8)2 + (0.7)2 ≰1.13
Pythaogrean Neutrosophic Fuzzy Sets(PNFS)	It handle when the sum of the truth, falsity and indertermincancy of the membership exceeds one	It becomes less ineffective when the sum of the sqaure of the truth, indeterminancy, falsity exceeds one.
Fermatean neutrosophic sets	It handles the situations better when the PNFS fails by cubing the turth, indeterminacny, falsity of the membership	

.

3.Reversal Dijikstra's Algorithm under fermatean neutrosophic Environment

In contrast to existing techniques, the methodology proposed in this article proves to be more effective in identifying the Shortest Path (SP). The key advantage of utilizing Fuzzy number predicted values is their ability to yield a singular value. By eliminating the need for rating FN values, this approach streamlines the decision-making process. This computational efficiency is particularly advantageous when dealing with scenarios characterized by highly uncertain parameters, making it a valuable tool for addressing Shortest Path Problems (SPPs). We argue that there are clear benefits to utilizing fermatean neutrosophic numbers (FNNs). Their ability to explicitly represent indeterminacy and differentiate between various facets of uncertainty makes them a valuable and versatile tool in these applications. FNNs provide a more impartial and nuanced insight into the functional relationships within a system. Consequently, our approach is geared towards solving the SPP within a network with fermatean neutrosophic arc lengths, bridging the source node (SN) and target node (TN). The analysis for the shortest path in fermatean neutrosophic numbers(FNN) operates as follows: We initially adapt the principles governing the prediction of values within FNNs, yielding novel and improved outcomes for predicted FNN values. We apply this modified prediction approach to solve a shortest path algorithm, such as the reversal Dijkstra algorithm. Here, the de-neutrosophication of FNNs and time-dependent FNNs associated with network arcs is executed by computing their predicted values. To calculate the shortest distance (SD) value and time-dependent shortest time, we amalgamate FNNs through a scoring function derived from the predicted FNN values. This process directly yields a crisp numerical result. In comparison to other fuzzy shortest path methods, our approach is more logically structured, robust, and straightforward to implement when dealing with fermatean neutrosophic numbers.

3.1 Proposed Algorithm.

Step 1: Assign and label [t_s,−] and permanent status to the destination node.

Step 2: calculate the labels $t_j + w_{ij}$ to the reachable node (node i) from the permanent node (node j) and assign temporary stauts.

Step 3: If node i is visited already with temporary status. choose the score function to choose the minimum node and label it as i.

Step 4: If all the nodes have become permanent status then the algorithm terminates else then go to step 2.

Step 5: Using the label information, find the shortest path by tracing it forward through the graph.

The Pseudeocode for time-dependent fermatean neutrosophic reversal-dijikstra Algorithm is present in algorithm 1.

Algorithm 1 Pseudeocode for time-dependent fermatean neutrosophic reversal dijikstra Algorithm

```
function Reversal Dijkstra(graph, target): # Initialize data structures
distance = {} # Dictionary to store the shortest distance from the target node.
priority queue = MinHeap () # MinHeap to prioritize nodes to explore # Initialize
distances
for node in graph.nodes:
distance[node] = INFINITY
distance[target] = 0 # Add the target node to the priority queue
priority queue.insert((target, 0))
while not priority queue.isEmpty():
current node, current distance = priority queue.extractMin()
# Explore neighbors of the current node
for neighbor in graph.neighbors(current node):
edge weight = graph.getEdgeWeight(current node, neighbor)
new distance = current distance + edge weight
# Relaxation step
```

```
if new distance ≤ distance[neighbor]:
distance[neighbor] = new distance
 priority queue.insert((neighbor, new distance))
   return distance.
```

4.Numerical Example

A numerical example is solved to validate the proposed algorithm's efficiency.

Example. Consider a numerical example with a network graph 1 having six nodes and eight arcs with time-dependent fermatean neutrosophic graph. The arc values are represented in the table 2. The departure time ˜ts is set as (0.2, 0.4,0.5).

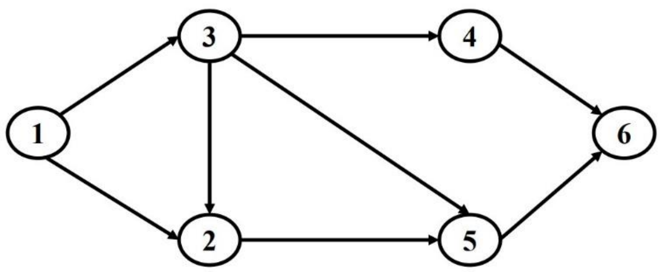

Fig .1. A Network with time-dependent fermatean neutrosophic weights

Table 2. Weight of edges for example.

Edges	Time-dependent fermatean neutrosophic Arc Values
1 → 2	(0.4, 0.6, 0.3)
1 → 3	(0.3, 0.8, 0.6)
3 → 2	(0.5,/′ 0.3, 0.2) − t
2 → 5	(0.6, 0.8, 0.4) * t
3 → 4	(0.5, 0.3, 0.7)
3 → 5	(0.8, 0.3, 0.1) + t
4 → 6	T
5 → 6	(0.7, 0.6, 0.2)

Iteration 0: Assign the destination node (6) and label is as $[t_s,-]$ and make it Permanent table 3.

Iteration 1: Calculate the distances from the targeted node (Node 6), which is the most recently marked as "Permanent", to its neighboring nodes (predecessor node of

6), specifically Nodes 5 and 4. As a result, we have established the status of these nodes in terms of being either temporary or permanent in table 4. To compare (0.70,0.24,0.1).

Table 3. Nodes that are reachable from nodes designated as "Permanent" are assigned labels and temporary status.

Edges	Label	Status
6	[(0.2, 0.4, 0.5),−]	$\mathfrak{P}$

Table 4. Nodes that are reachable from nodes designated as "Permanent" are assigned labels and temporary status.

Edges	Label	Status
6	[(0.2, 0.4, 0.5),−]	$\mathfrak{P}$
5	[(0.70, 0.24, 0.1), 6]	$\mathfrak{T}$
4	[(0.25, 0.16, 0.25), 6]	$\mathfrak{T}$

and (0.25,0.16,0.25), the definition 1 is used:

S(0.70, 0.24, 0.1) = 0.613

S(0.25, 0.16, 0.25) = 0.386

Since S(0.70, 0.24, 0.1) ≤ S(0.25, 0.16, 0.25). Therefore, [(0.25, 0.16, 0.25), 6] is marked and labeled as Permanent (P) node.

Iteration 2: Node 4 is marked as permanent node and the predecessor node for node 4 is node 3. Therefore, we maintain lists of temporary and permanent nodes in table 5. To compare (0.95,0.52,0.49) and (0.94,0.57,0.52), the definition 1 is used:

Table 5. Nodes that are reachable from nodes designated as "Permanent" are assigned labels and temporary status.

Edges	Label	Status
6	[(0.2, 0.4, 0.5),−]	$\mathfrak{P}$
5	[(0.70, 0.24, 0.1), 6]	$\mathfrak{P}$
4	[(0.25, 0.16, 0.25), 6]	$\mathfrak{P}$

S(0.95, 0.52, 0.49) = 0.65

S(0.94, 0.57, 0.52) = 0.663

Since S(0.95, 0.52, 0.49) ≤ S(0.94, 0.57, 0.52). Therefore, [(0.95, 0.52, 0.49) is marked and labeled as Permanent node.

Iteration 3: The predecessor node 5 are node 3 and node 2. Therefore, we maintain lists of temporary and permanent nodes in table 7.

Iteration 4: The predecessor of node 3 and node 2 is node 1. The list of temporary and permanent nodes are listed in table 7.

Table 6. Nodes that are reachable from nodes designated as "Permanent" are assigned labels and temporary status.

Edges	Label	Status
6	[(0.2, 0.4, 0.5),−]	𝔓
5	[(0.70, 0.24, 0.1), 6]	𝔓
4	[(0.25, 0.16, 0.25), 6]	𝔓
3	[(0.52, 0.05, 0.18), 4] (or) [(0.88, 0.03, 0.005), 5]	𝔗
2	[(0.70, 0.19, 0.06), 5]	𝔗

Table 7. Nodes that are reachable from nodes designated as "Permanent" are assigned labels and temporary status.

Edges	Label	Status
6	[(0.2, 0.4, 0.5),−]	𝔓
5	[(0.70, 0.24, 0.1), 6]	𝔓
4	[(0.25, 0.16, 0.25), 6]	𝔓
3	[(0.52, 0.05, 0.18), 4]	𝔓
2	[(0.70, 0.19, 0.06), 5]	𝔗
1	[(0.55, 0.04, 0.11), 3] (or) [(0.73, 0.11, 0.012), 2]	𝔗

Iteration 5: The predecessor node for 2 is node 3 and node 1. Therefore node 1 as Permanent node. using the label information, the network is traced and the shortest travel time from destination node to source node is 1 → 3 → 4 → 6. The shortest path from 1 to 6 is shown in Figure 2. The table 10 has been created to illustrate the efficiency of the proposed algorithm in comparison to existing approaches.

Table 8. Nodes that are reachable from nodes designated as "Permanent" are assigned labels and temporary status.

Edges	Label	Status
6	[(0.2, 0.4, 0.5),−]	𝔓
5	[(0.70, 0.24, 0.1), 6]	𝔓
4	[(0.25, 0.16, 0.25), 6]	𝔓
3	[(0.52, 0.05, 0.18), 4] (or) [(0.62, 0.04, 0.07), 2] (or) [(0.88, 0.03, 0.005), 5]	𝔗
2	[(0.70, 0.19, 0.06), 5]	𝔓
1	[(0.55, 0.04, 0.11), 3] (or) [(0.73, 0.11, 0.012), 2]	𝔗

Table 9. Nodes from destination to source

Edges	Label	Status
6	[(0.2, 0.4, 0.5),−]	𝔓
5	[(0.70, 0.24, 0.1), 6]	𝔓
4	[(0.25, 0.16, 0.25), 6]	𝔓
3	[(0.52, 0.05, 0.18), 4]	𝔓
2	[(0.70, 0.19, 0.06), 5]	𝔓
1	[(0.55, 0.04, 0.11), 3]	𝔓

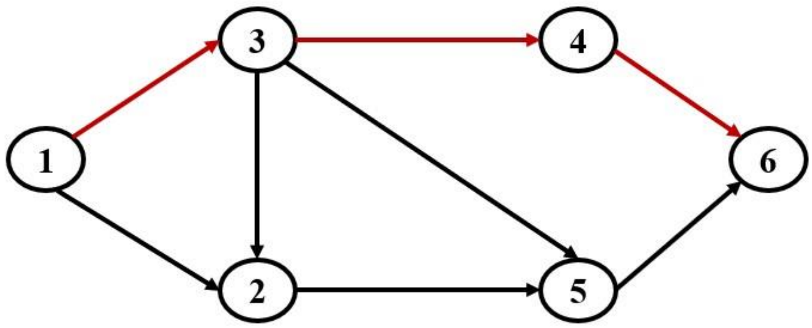

Fig .2. Shortest Path from node 1 to node 6

Table 10. Comparison with the Existing Approach

Methods with Different Neutrosophic Environment	SP	Shortest Travel Time	Score of travel time
Time-Dependent Dijkstra Algorithm with Bipolar Neutrosophic Numbers [29]	$1 \rightarrow 2 \rightarrow 5 \rightarrow 6$	(0.901,0.122,0.15,-0.078,-0.919,-0.912)	0.92
Proposed Method	$1 \rightarrow 3 \rightarrow 4 \rightarrow 6$	(0.55,0.04,0.011)	0.493

5. **Results and Discussion**

The proposed time-dependent fuzzy reversal Dijkstra's algorithm is designed to compute the shortest travel times in the context of a time-dependent fermatean neutrosophic graph. This algorithm leverages the principles of reversal Dijkstra's algorithm. In each iteration, the algorithm identifies undiscovered nodes by exploring the paths connecting them to the permanent nodes. By repeating this process, it systematically calculates and updates the shortest travel times to the starting node, accounting for the complex characteristics of the time-dependent fermatean neutrosophic graph. In this specific example, a departure time, denoted as ˜ts, has been introduced with the values (0.2, 0.4, 0.5), which represents various departure time instances. Additionally, the arrival node, which serves as the destination node, is designated as a "Permanent" node within the algorithm's execution. This means that the algorithm will consider and process these departure times and ensure that the arrival node's status remains permanent throughout the computation. Huang et al. [33] initially attempted to discover the shortest paths on time-dependent fuzzy networks by integrating the principles of fuzzy simulation and genetic optimization. In a related context, Liao et al. [34] introduced an algorithm for solving the fuzzy constrained shortest path problem, which addresses the uncertainty in both time and cost information. They also demonstrated the feasibility of the fuzzy linear programming approach for solving their problem. These methodologies have undergone thorough testing and validation on fuzzy graphs. The application presented in this article draws inspiration from these prior studies. Consequently, the application of this study holds significance when compared to previous applications documented in the existing literature. The results of the provided example underscore the applicability of an extended version of reversal Dijkstra's algorithm to time-dependent fuzzy graphs. By employing fermatean neutrosophic numbers to represent edge weights, the proposed methodology effectively addresses both the shortest path and travel time problems.

6. Conclusion

The shortest path problem plays a pivotal role and finds practical applications across a wide spectrum of fields. When dealing with uncertain situations, the vertex weights can be expressed as fuzzy numbers, enabling them to adapt to fluctuating values over time. This article focuses on the utilization of fermatean neutrosophic numbers to capture and represent uncertainty. It extends the Reversal-Dijkstra algorithm to handle time-dependent graphs with fermatean neutrosophic numbers. This extension involves the use of a scoring function to compare minimum values among the FNN and select the most favorable arc with the lowest values. In the context of a time-dependent fuzzy graph, the shortest path is defined in terms of the shortest travel time. The proposed algorithm addresses this specific scenario and includes a numerical example to demonstrate its effectiveness, ultimately yielding optimal results. For future research endeavors, we recommend the utilization of the time-dependent reversal Dijkstra's algorithm within a fuzzy environment. This approach can be further enhanced by incorporating various fuzzy extensions, such as Pythagorean fuzzy sets, spherical fuzzy sets, and more. Additionally, it would be beneficial to integrate cost, saftey values and danger factors into the analysis along side time considerations. Beyond the technical developments, these methodologies hold promise for addressing a diverse array of real-life problems. Examples include applications in cable network optimization, telecommunication routing, route planning for transportation, social network analysis, database search optimization, and traffic management for taxi services, among others.

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Neutrosophic B-spline Surface Approximation Model for 3-Dimensional Data Collection

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Abstract: Since there are three membership functions: truth, false, and indeterminacy, geometrical modeling for B-spline surface approximation including neutrosophic data is particularly difficult to construct. Using neutrosophic set theory, this study introduces a neutrosophic B-spline for 3-dimensional data collecting. The neutrosophic control net was first introduced using the neutrosophic set notion. The control net is then merged with the B-spline basis function, and the approximation approach is used to display the B-spline surface. Following this work, there is a numerical demonstration of how to create the surface. As a result, the primary goal of this study is to offer the mathematical formulation and visualization of the neutrosophic B-spline surface approximation model for 3D data collecting.

Keywords: Neutrosophic B-spline Surface; Neutrosophic Control Net; Neutrosophic Set Theory, Neutrosophic B-spline Surface Approximation

1. Introduction

In modeling and addressing real-world situations, several mathematical tools have been established. Several researchers have been drawn to the concept of fuzzy set developed by Lotfi Zadeh [1] for problems involving imprecision, ambiguity, and uncertainty because of its potential for the recreation of human thinking as well as perception using linguistic information. Numerous hypotheses were created afterward to address the issue of impreciseness but in various structural forms. When fuzzy sets and fuzzy logic proposed by Zadeh cannot express false membership data, the neutrosophic theory proposed by Florentin Smarandache [2] is newly offered as an improved alternative; meanwhile, intuitionistic fuzzy sets and intuitionistic fuzzy logic proposed by Krasimir Atanassov [3] cannot handle data indeterminacy or imperfect information [17]. In 2014, Smarandache extended his neutrosophic logic study to n-valued refined neutrosophic logic for use in physics [32]. A neutrosophic multiset is an n-valued refined neutrosophic set. The neutrosophic multiset is expanded by Chatterjee to a single-valued neutrosophic multiset [33]. Following that, a combination of neutrosophic multisets and other uncertainty methods, such as rough multisets [34] and soft multisets [35], is introduced. This is due to the information's ambiguity and impreciseness, which always combines opposing and neutral knowledge. As a result, some academics have covered a few applications in their work that use fuzzy set, intuitionistic fuzzy set, and neutrosophic set theory [26-31].

The randomness of data collection has an impact on curve and surface design. This data is used as a control point for approximate and interpolate approaches in geometric modeling [4]. The data set is required for the creation of curves and surfaces, as well as the procedure itself. Uncertainty data affecting the curve and surface is frequently disregarded or discarded. Thus, for any problem to be addressed, data sets with some variability must be filtered before being utilized to generate surfaces and curves. In geometric modeling, there are three models: Bézier, B-spline, and non-uniform rational B-splines (NURBS). However, this study focuses on the B-spline surface model. Bernstein basis, or Bézier basis, is a specific case of B-spline basis (from Basis Spline). This foundation is not worldwide [22]. B-spline surfaces are non-global because each vertex has a basis function. The B-spline basis permits changing the basis function order and surface degree without changing the control polygon vertices. Piegl and Tiller introduced the mathematical representation for the B-spline surface approximation model [22].

Atanassov enhanced the fuzzy set theory with truth, falsehood, and uncertainty degrees in 1986 [3]. As fuzzy sets only accept full membership data, they can be employed when there is inadequate data for categorization and processing [5]. To cope with uncertain data, several academics employ geometric modeling with the fuzzy set and intuitionistic fuzzy set approach [6-14]. Meanwhile, Tas and Topal [15-16] have employed a study for neutrosophic geometric modeling but only focused on the Bézier curve and surface using the approximation method generally. Rosli and Zulkifly introduced the neutrosophic B-spline curve by using the interpolation method [23], neutrosophic bicubic surface interpolation [24], and the 3-dimensional neutrosophic quartic Bézier curve approximation method [25]. However, the papers motivate the authors to produce and focus on the B-spline surface approximation method to visualize the 3-dimensional data collection. As a result, the novelty of this study is the mathematical representations of the neutrosophic B-spline surface approximation method and its visualization for truth, indeterminacy, and falsity memberships.

This project focuses on the construction of a geometric model capable of dealing with data collection; specifically, the model's primary focus will be the neutrosophic B-spline surface approximation (NB-sSA) model. The neutrosophic control point must be defined before building the NB-sSA, utilizing neutrosophic set theories and the features they provide. These control points, along with the B-spline basis function, are used to construct NB-sSA models, which are then visualized using an approximation method. This paper is organized as follows: The first section of this chapter provides background information on the topic. In the second section, the neutrosophic point relation (NPR) and the neutrosophic control net relation (NCNR) are introduced. The third section discusses the method used for the NB-sSA using NCNR. The fourth section includes a numerical example as well as a graphical representation of NB-sSA. The investigation will be completed with the fifth section as the conclusion of this study.

2. Preliminaries

In fuzzy systems, the intuitionistic set can tolerate imperfect information but not indeterminate or inconsistent information [17]. There are three membership functions in a neutrosophic set. With the addition of the parameter "indeterminacy" to the neutrosophic set (NS) specification, there are three types of membership functions: a membership function, denoted by the letter *T*; an indeterminacy membership function, denoted by the letter *I*; and a non-membership function, denoted by the letter *F*.

Definition 1 [18]
Let Y be the collection of universal space, with the element $y \in Y$. The neutrosophic set is an object in the form.

$$\hat{B} = \{\left\langle y : T_{\hat{B}(y)}, I_{\hat{B}(y)}, F_{\hat{B}(y)} \right\rangle | \ y \in Y\} \quad (1)$$

where, the functions $T, I, F : Y \to]^{-}0,1^{+}[$ define, respectively, the degree of truth membership, the degree of indeterminacy, and the degree of false membership of the element $y \in Y$ to the set $\hat{B}$ with the condition;

$$0^{-} \leq T_{\hat{B}}(y) + I_{\hat{B}}(y) + F_{\hat{B}}(y) \leq 3^{+} \quad (2)$$

There is no limit to the amount of $T_{\hat{B}}(y), I_{\hat{B}}(y)$ and $F_{\hat{B}}(y)$.

A value is chosen by NS from one of the real standard subsets or one of the non-standard subsets of $]^{-}0,1^{+}[$. The actual value of the interval $[0,1]$, on the other hand, $]^{-}0,1^{+}[$ will be utilized in technical applications since its utilization in real data, such as the resolution of scientific challenges, will be physically impossible. As a direct consequence of this, membership value utilization is increased.

$$\hat{B} = \{\langle y : T_{\hat{B}(y)}, I_{\hat{B}(y)}, F_{\hat{B}(y)} \rangle | \; y \in Y\} \text{ and } T_{\hat{B}}(y), I_{\hat{B}}(y), F_{\hat{B}}(y) \in [0,1] \quad (3)$$

There is no restriction on the sum of $T_{\hat{B}}(y), I_{\hat{B}}(y), F_{\hat{B}}(y)$ .Therefore,

$$0 \leq T_{\hat{B}}(y) + I_{\hat{B}}(y) + F_{\hat{B}}(y) \leq 3 \quad (4)$$

Definition 2 [15-16]

Let $\hat{B} = \{\langle y : T_{\hat{B}(y)}, I_{\hat{B}(y)}, F_{\hat{B}(y)} \rangle | \; y \in Y\}$ and $\hat{C} = \{\langle z : T_{\hat{C}(z)}, I_{\hat{C}(z)}, F_{\hat{C}(z)} \rangle | \; z \in Z\}$ be neutrosophic elements. Thus, $NR = \{\langle (y,z) : T_{(y,z)}, I_{(y,z)}, F_{(y,z)} \rangle | \; y \in \hat{B}, z \in \hat{C}\}$ is a neutrosophic relation between $\hat{B}$ and $\hat{C}$.

Definition 3 [15-16]

The neutrosophic set of $\hat{B}$ in space Y is neutrosophic point (NP) and $\hat{B} = \{\hat{B}_i\}$ where $i = 0,\ldots,n$ is a set of NPs where there exists $T_{\hat{B}} : Y \rightarrow [0,1]$ as truth membership, $I_{\hat{B}} : Y \rightarrow [0,1]$ as indeterminacy membership, and $F_{\hat{B}} : \hat{Y} \rightarrow [0,1]$ as false membership with,

$$T_{\hat{B}}(\hat{B}) = \begin{cases} 0 & if \; \hat{B}_i \notin \hat{B} \\ a \in (0,1) & if \; \hat{B}_i \,\hat{\in}\, \hat{B} \\ 1 & if \; \hat{B}_i \in \hat{B} \end{cases}$$

$$I_{\hat{B}}(\hat{B}) = \begin{cases} 0 & if \; \hat{B}_i \notin \hat{B} \\ b \in (0,1) & if \; \hat{B}_i \,\hat{\in}\, \hat{B} \\ 1 & if \; \hat{B}_i \in \hat{B} \end{cases} \quad (5)$$

$$F_{\hat{B}}(\hat{B}) = \begin{cases} 0 & if \; \hat{B}_i \notin \hat{B} \\ c \in (0,1) & if \; \hat{B}_i \,\hat{\in}\, \hat{B} \\ 1 & if \; \hat{B}_i \in \hat{B} \end{cases}$$

2.1 *Neutrosophic Point Relation*

Neutrosophic point relation (NPR) is based on the concept of a neutrosophic set, which was discussed in the previous section. If P,Q is a collection of Euclidean universal space points and $P,Q \in \mathbf{R}^2$, then NPR is defined as follows:

Definition 4 [23]

Let X,Y be a collection of universal space points with a non-empty set and $P,Q,I \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}$, then NPR is defined as:

$$\hat{R} = \left\{ \left\langle \left(p_i, q_j \right), T_R(p_i,q_j), I_R(p_i,q_j), F_R(p_i,q_j) \right\rangle | \; T_R(p_i,q_j), I_R(p_i,q_j), F_R(p_i,q_j) \in I \right\} \quad (6)$$

where $\left(p_i,q_j\right)$ is an ordered pair of coordinates and $\left(p_i,q_j\right)\in P\times Q$ while $T_R(p_i,q_j), I_R(p_i,q_j), F_R(p_i,q_j)$ are the truth membership, indeterminacy membership, and false membership that follow the condition of the neutrosophic set which is $0\le T_{\hat{B}}(\hat{y})+I_{\hat{B}}(\hat{y})+F_{\hat{B}}(\hat{y})\le 3$.

2.2 Neutrosophic Control Net Relation

The geometry of a spline surface can only be described by all the points required to build the surface. The control net plays an important role in the development, control, and manufacture of smooth surfaces. The neutrosophic control point relation (NCPR) is first defined in this section by using the notion of control point from the research published in [19-21] in the following way:

Definition 5 [23]
Let $\hat{R}$ be an NPR, then NCPR is defined as a set of points $n+1$ that indicates the positions and coordinates of a location and is used to describe the curve and is denoted by:

$$\begin{aligned}\hat{P}_i^T &= \left\{\hat{p}_0^T, \hat{p}_1^T, \ldots, \hat{p}_n^T\right\}\\ \hat{P}_i^I &= \left\{\hat{p}_0^I, \hat{p}_1^I, \ldots, \hat{p}_n^I\right\}\\ \hat{P}_i^F &= \left\{\hat{p}_0^F, \hat{p}_1^F, \ldots, \hat{p}_n^F\right\}\end{aligned} \tag{7}$$

where $\hat{P}_i^T$, $\hat{P}_i^I$, and $\hat{P}_i^F$ are neutrosophic control points for truth membership, indeterminacy membership, and falsity membership, and i is one less than n. Thus, the NCNR can be defined as follows.

Definition 6 [24]
Let $\hat{P}$ be an NCPR, and then define an NCNR as points $n+1$ and $m+1$ for $\hat{P}$ in their direction, and it can be denoted by $\hat{P}_{i,j}$ that represents the locations of points used to describe the surface and may be written as:

$$\hat{P}_{i,j}=\begin{bmatrix}\hat{P}_{0,0} & \hat{P}_{0,1} & \cdots & \hat{P}_{0,j}\\ \hat{P}_{1,0} & \hat{P}_{1,1} & \cdots & \hat{P}_{1,j}\\ \vdots & \vdots & \ddots & \vdots\\ \hat{P}_{i,0} & \hat{P}_{i,1} & \cdots & \hat{P}_{i,j}\end{bmatrix} \tag{8}$$

where $\hat{P}_{i,j}$ are also the points that make up a polygon's control net.

3. Neutrosophic B-spline Surface Approximation

Surface is a two-parameter vector value function that governs how the plane is projected into the Euclidean three-dimensional frame [22]. The NCNR and **Definition 1** are used to construct the neutrosophic B-spline surface approximation (NB-sSA), which is then utilized to embed the B-spline blending function in a geometric model. The model, which stands for approximation approach, is mathematically represented as follows:

Definition 7
Let $\hat{\hat{P}}_{i,j}^{T,I,F}=\left\{\hat{\hat{P}}_{i,j}^{T,I,F}\right\}_{i=0,j=0}^{n,m}$ where $i=0,1,\ldots,n$ and $j=0,1,\ldots,m$ is NCNR for truth, indeterminacy, and falsity memberships. The neutrosophic B-spline surface approximation (NB-sSA) is denoted as $BsS(u,w)$ and represented as follows:

$$BsS(u,w)=\sum_{i=0}^{n}\sum_{j=0}^{m}\hat{P}_{i,j}N_i^k(u)M_j^l(w) \tag{9}$$

where $N_i^k(u)$ and $M_j^l(w)$ are the Bernstein function in the u and w parametric directions.

$$\begin{aligned} N_i^1(u) &= \begin{cases} 1 & if \quad u_i \le u < u_{i+1} \\ 0 & otherwise \end{cases} \\ N_i^1(u) &= \frac{(u-u_i)}{u_{i+k-1}-u_i}N_i^{k-1}(u)+(7)\frac{(u_{i+k}-u)}{u_{i+k}-u_{i+1}}N_{i+1}^{k-1}(u) \end{aligned} \tag{10}$$

$$\begin{aligned} M_j^1(w) &= \begin{cases} 1 & if \quad w_j \le w < w_{j+1} \\ 0 & otherwise \end{cases} \\ M_j^1(w) &= \frac{(w-w_j)}{w_{j+l-1}-u_i}M_i^{l-1}(w)+(8)\frac{(w_{j+l}-w)}{w_{j+l}-w_{j+1}}M_{j+1}^{l-1}(w) \end{aligned} \tag{11}$$

The parametric function NB-sSA in **Equation (9)** is defined as follows and is made up of three surfaces: a membership surface, a non-membership surface, and an indeterminacy surface.

$$BsS^T(u,w)=\sum_{i=0}^{n}\sum_{j=0}^{m}\hat{P}_{i,j}{}^T N_i^k(u)M_j^l(w) \tag{12}$$

$$BsS^F(u,w)=\sum_{i=0}^{n}\sum_{j=0}^{m}\hat{P}_{i,j}{}^F N_i^k(u)M_j^l(w) \tag{13}$$

$$BsS^I(u,w)=\sum_{i=0}^{n}\sum_{j=0}^{m}\hat{P}_{i,j}{}^I N_i^k(u)M_j^l(w) \tag{14}$$

Each $BsS(u,w)$ can be expressed as a matrix product in the following way [24]:

$$BsS(u_i,w_j)=\begin{bmatrix} N_0^k(u_i) & N_1^k(u_i) & \cdots & N_i^k(u_i)\end{bmatrix}\times\begin{bmatrix}\hat{P}_{0,0} & \hat{P}_{0,1} & \cdots & \hat{P}_{0,j} \\ \hat{P}_{1,0} & \hat{P}_{1,1} & \cdots & \hat{P}_{1,j} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{P}_{i,0} & \hat{P}_{i,1} & \cdots & \hat{P}_{i,j}\end{bmatrix}\times\begin{bmatrix} M_0^l(w_j) \\ M_1^l(w_j) \\ \vdots \\ M_i^l(w_j)\end{bmatrix} \tag{15}$$

All the independent equations can be combined to form a single matrix equation:

$$BsS = N^T\hat{P}M \tag{16}$$

3.1. Properties of Neutrosophic B-Spline Surface Approximation (NB-sSA)

By using a B-spline basis to define a B-spline surface, many characteristics beyond those already mentioned become obviously clear:

- In each parametric, the degree of NB-sSA is one less than the number of NCNR vertices in that direction.
- The NCNR shape is generally followed by the NB-sSA.
- The NCNR corner point and the resulting NB-sSA coincide.
- The NCNR's shape is generally followed by the NB-sSA.
- The NB-sSA is contained within NCNR's convex hull.
- The NB-sSA has a continuity in each parametric direction that is two less than the number of NCNR vertices in that direction.
- An affine transformation does not change the NB-sSA.

- The NB-sSA lacks the variation-diminishing property. For bivariant NB-sSA, the variation-diminishing property is both undefined and unknown.

4. Numerical Example with Its Visualizations

To demonstrate a 3-dimensional neutrosophic B-spline surface using the approximation approach, suppose a four-by-four NCNR with the following degrees of membership, non-membership, and indeterminacy:

$$\begin{bmatrix} \hat{P}_{0,0} & \hat{P}_{0,1} & \hat{P}_{0,2} & \hat{P}_{0,3} \\ \hat{P}_{1,0} & \hat{P}_{1,1} & \hat{P}_{1,2} & \hat{P}_{1,3} \\ \hat{P}_{2,0} & \hat{P}_{2,1} & \hat{P}_{2,2} & \hat{P}_{2,3} \\ \hat{P}_{3,0} & \hat{P}_{3,1} & \hat{P}_{3,2} & \hat{P}_{3,3} \end{bmatrix}$$

The NB-sSA is fourth order in the u direction $(k=4)$ and third order in the w direction $(l=3)$ based on the **Equation (10)** and **Equation (11).** Therefore, by using **Equation (16)**, the NB-sSA can be derived as follows:

$$\begin{aligned} BsS &= \sum_{i=1}^{4}\sum_{j=1}^{4}\hat{P}_{i,j}N_{i,4}(u)M_{j,3}(w) \\ &= N_{1,4}(u)\left(\hat{P}_{1,1}M_{1,3}(w)+\hat{P}_{1,2}M_{2,3}(w)+\hat{P}_{1,3}M_{3,3}(w)+\hat{P}_{1,4}M_{4,3}(w)\right) \\ &+ N_{2,4}(u)\left(\hat{P}_{2,1}M_{1,3}(w)+\hat{P}_{2,2}M_{2,3}(w)+\hat{P}_{2,3}M_{3,3}(w)+\hat{P}_{2,4}M_{4,3}(w)\right) \\ &+ N_{3,4}(u)\left(\hat{P}_{3,1}M_{1,3}(w)+\hat{P}_{3,2}M_{2,3}(w)+\hat{P}_{3,3}M_{3,3}(w)+\hat{P}_{3,4}M_{4,3}(w)\right) \\ &+ N_{4,4}(u)\left(\hat{P}_{4,1}M_{1,3}(w)+\hat{P}_{4,2}M_{2,3}(w)+\hat{P}_{4,3}M_{3,3}(w)+\hat{P}_{4,4}M_{4,3}(w)\right) \end{aligned}$$

Every column is labeled $\langle T,F,I\rangle$ with its respective value and degree. Based on example below for $\hat{P}_{0,0}$ for $i=0$ and $j=0$, the value of truth membership denoted as *T* is 0.4, the value of falsity membership denoted as *F* is 0.7, and the value of indeterminacy membership denoted as *I* is 0.2.

$$\begin{bmatrix} \hat{P}_{0,0} \\ \hat{P}_{1,0} \\ \hat{P}_{2,0} \\ \hat{P}_{3,0} \end{bmatrix} = \begin{bmatrix} \langle(-16,16);0.4,0.7,0.2\rangle \\ \langle(-6,16);0.6,0.4,0.3\rangle \\ \langle(6,16);0.6,0.2,0.5\rangle \\ \langle(16,16);0.7,0.3,0.3\rangle \end{bmatrix}$$

$$\begin{bmatrix} \hat{P}_{0,1} \\ \hat{P}_{1,1} \\ \hat{P}_{2,1} \\ \hat{P}_{3,1} \end{bmatrix} = \begin{bmatrix} \langle(-16,6);0.9,0.3,0.1\rangle \\ \langle(-6,6);0.8,0.2,0.3\rangle \\ \langle(6,6);0.8,0.4,0.1\rangle \\ \langle(16,6);0.4,0.6,0.3\rangle \end{bmatrix}$$

$$\begin{bmatrix} \hat{P}_{0,2} \\ \hat{P}_{1,2} \\ \hat{P}_{2,2} \\ \hat{P}_{3,2} \end{bmatrix} = \begin{bmatrix} \langle(-16,-16);0.6,0.5,0.2\rangle \\ \langle(-6,-16);0.7,0.4,0.2\rangle \\ \langle(6,-16);0.5,0.3,0.5\rangle \\ \langle(16,-16);0.7,0.4,0.2\rangle \end{bmatrix}$$

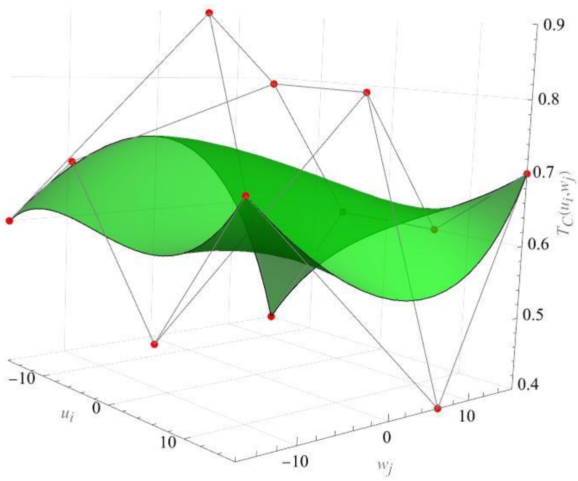

Figure 1. NB-sSA for truth membership.

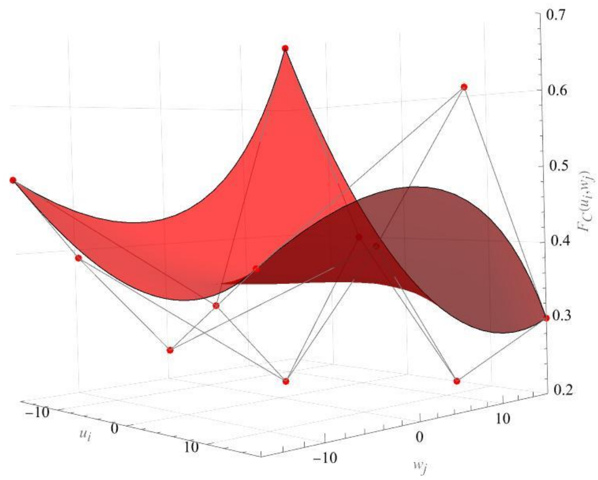

Figure 2. NB-sSA for false membership.

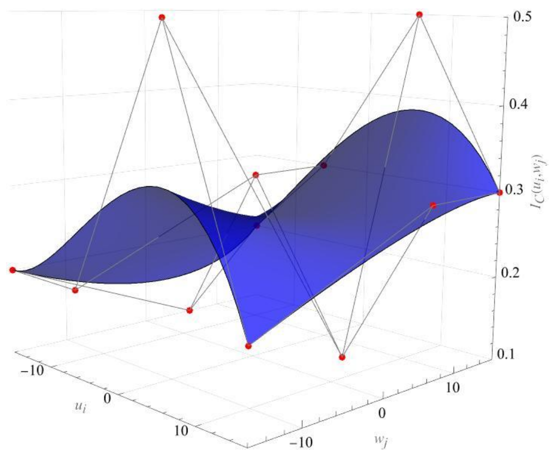

Figure 3. NB-sSA for indeterminacy membership.

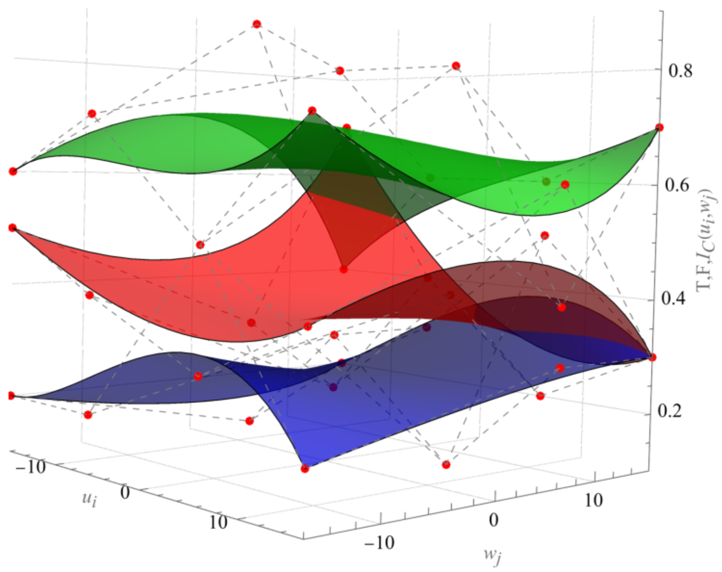

Figure 4. NB-sSA for truth, false, and indeterminacy membership.

This study employs the original formula of the B-spline basis function and then blends it with the NS theory, which uses the NCNR to approximate the surface. It differs from the interpolation approach, which requires determining the inverse of the formula B-spline basis function for the control net to find the interpolated data as introduced and visualized in Rosli and Zulkifly [23]'s study. Therefore, this study uses the B-spline surface approximation model that was introduced by Piegl and Tiller [22] and blends it with NCNR. However, one of the difficulties when constructing this model is ensuring that the random data collection adheres to the criterion of neutrosophic set theory, which is $0 \le T_{\hat{B}}(\hat{y}) + I_{\hat{B}}(\hat{y}) + F_{\hat{B}}(\hat{y}) \le 3$. The 3-dimensional neutrosophic B-spline surface approximation model is depicted in **Figures 1** to **4**. **Figure 1** shows a 3D surface for truth membership, **Figure 2** shows a 3D surface for false membership, **Figure 3** shows an indeterminacy surface, and **Figure 4** shows a 3D neutrosophic B-spline surface approximation for all membership in one axis, with green representing 3D truth membership, red representing 3D false membership, and blue representing 3D indeterminacy membership. In **Figures 1** to **4**, the red dot represents their individual NCNR, and the gray line indicates their respective control polygons that hold the NCNR. As this study uses an approximation strategy and adheres to the criterion of the neutrosophic set, each NCNR approximates its surfaces, and the memberships are not dependent on the others. An algorithm for constructing the NB-sSA will be discussed as follows:

Step 1: Introduce the NCNR by using **Definition 6.**
Step 2: Blend the NCPR with B-spline Basis function as in **Definition 7.**
Step 3: Collect the coefficients of $N_i^k(u)$ and $M_j^l(w)$. The coefficients of the parameter terms are collected and rewritten in matrix form as in the given example.
Step 4: Repeat step 1 to 3 for indeterminacy and falsity memberships cases.

5. Conclusions

By introducing NCNR, this paper introduced the NB-sSA model. This study can be expanded to produce better findings by incorporating non-uniform rational B-splines (NURBS) functions for surfaces and curves. The suggested 3-dimensional model can handle surface data visualization challenges such as modelling geographical regions with unclear borders in geoinformation systems (GIS), remote sensing, object reconstruction from an aerial laser scanner, bathymetric data visualization, and many more. Implementing this strategy has the impact or benefit of ensuring that no data is wasted throughout the data collection process in any application. The NB-sSA model can be used to address and solve difficulties characterized by uncertainty. The NCNR and NB-sSA models can provide a comprehensive analysis and description of a modelling issue in which each surface is modelled separately.

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Neutrosophic Encoding and Decoding Algorithm for ASCII Code System

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Abstract: Context and Background: This paper addresses the challenge of encoding and decoding numerical data by introducing innovative algorithms utilizing Neutrosophic ASCII codes and ASCII Neutrosophic codes. These codes serve to represent uncertain or imprecise character values through the incorporation of neutrosophic numbers, encompassing degrees of truth, falsity, and indeterminacy. **Motivation:** The study stems from the necessity to effectively represent uncertain or imprecise character values in numerical data. This is crucial in diverse applications where handling uncertain or ambiguous data is a prevalent concern. **Hypothesis:** We hypothesize that employing Neutrosophic ASCII codes and ASCII Neutrosophic codes in encoding and decoding numerical data can provide a robust solution for representing uncertain or imprecise character values. **Methods:** The encoding algorithm in this study transforms each character in the ASCII string into its corresponding ASCII code, utilizing either 7 or 8 bits based on the code type. This algorithm calculates the degree of truth, falsity, and indeterminacy for each bit, considering the uncertainty or ambiguity associated with the character. The resulting neutrosophic numbers are appended to create the Neutrosophic ASCII code or ASCII Neutrosophic code. The decoding algorithm partitions the code into groups of neutrosophic numbers, calculates the associated degrees of truth, falsity, and indeterminacy for each ASCII bit, and converts the neutrosophic numbers back to ASCII codes, reconstructing the original ASCII character string. **Results:** Our study yields a novel and effective approach for encoding and decoding numerical data, demonstrating the potential of Neutrosophic ASCII codes and ASCII Neutrosophic codes in representing uncertain or imprecise character values**. Conclusions:** The proposed algorithms offer a promising solution for handling uncertain or ambiguous data in numerical encoding and decoding. The specific and quantitative results highlight the efficacy of Neutrosophic ASCII codes and ASCII Neutrosophic codes, showcasing their potential applicability in various domains requiring robust solutions for uncertain or imprecise character representation in numerical data.

Keywords Neutrosophic ASCII Code system, ASCII Neutrosophic codes, uncertain values, neutrosophic Systems.

1. Introduction

Neutrosophic sets and their applications have been extensively studied in recent years [1–4]. Neutrosophic logic is a generalization of fuzzy logic that allows for the representation of uncertain or indeterminate information using three values: truth, falsity, and indeterminacy. This approach has been applied in various fields such as decision-making, expert systems, pattern recognition, image processing, and data analysis. In the context of information, encoding, previous studies have focused on the use of traditional coding techniques such as ASCII codes or Unicode [5]. However, these methods do not take into account the degree of uncertainty or ambiguity associated with the characters being encoded. Therefore, the proposed approach using Neutrosophic ASCII codes and ASCII Neutrosophic codes is a new and innovative approach that can address this limitation. To the best of our knowledge, there are no previous studies that have explored the use of Neutrosophic ASCII codes and ASCII Neutrosophic codes for encoding and decoding numerical data. This paper presents a novel methodology that leverages neutrosophic numbers to represent uncertain or imprecise values of characters in numerical data. The encoding and decoding algorithms proposed in this paper are designed to handle these neutrosophic numbers and convert them to or from standard ASCII codes. Therefore, this paper contributes to the field of information encoding by providing a new and innovative approach that can potentially improve the accuracy and reliability of information encoding in various applications.

The proposed methodology involves two algorithms: encoding and decoding. The encoding algorithm takes an ASCII string as input and converts each character to its corresponding ASCII code. Then, it calculates the degree of truth, falsity, and indeterminacy associated with each bit based on the degree of uncertainty or ambiguity associated with the character. These neutrosophic numbers are appended to form the Neutrosophic ASCII code or ASCII Neutrosophic code. The decoding algorithm partitions the code into groups of neutrosophic numbers and calculates the degree of truth, falsity, and indeterminacy associated with each ASCII bit. Then, it converts the neutrosophic numbers to ASCII codes and combines them to form the original ASCII character string. The specific methods used in determining the degree of truth, falsity, and indeterminacy may vary depending on the application and context. The proposed approach of using Neutrosophic ASCII codes and ASCII Neutrosophic codes for encoding and decoding numerical data is a new and innovative approach that can potentially improve the accuracy and reliability of information encoding in various applications.

1.1 Research Gap:

The existing literature predominantly focuses on traditional encoding techniques like ASCII codes or Unicode, lacking consideration for the nuanced degrees of uncertainty or ambiguity associated with characters during the encoding process. This gap underscores the need for a novel approach that can address the limitations of current methods and provide a comprehensive solution for encoding and decoding numerical data.

1.2 Reasearch Question:

How can the integration of ASCII encoding with Neutrosophic principles enhance data representation and analysis across various research domains, particularly in addressing uncertainties and ambiguities in character values? Additionally, how does this integrated approach contribute to the improvement of information processing and decoding methods?

1.3 Motivation:

The motivation behind this research stems from the necessity to overcome the limitations of conventional encoding techniques and provide a more robust solution that considers the degree of uncertainty or ambiguity associated with character values. The aim is to introduce a pioneering approach using Neutrosophic ASCII codes and ASCII Neutrosophic codes, thereby filling the research gap and advancing the field of information encoding.

1.4 Objectives:

1. Introduce a novel methodology for encoding and decoding numerical data using Neutrosophic ASCII codes and ASCII Neutrosophic codes.
2. Develop encoding and decoding algorithms specifically designed to handle neutrosophic numbers, addressing the limitations of traditional encoding techniques.
3. Explore the potential applications of Neutrosophic ASCII codes and ASCII Neutrosophic codes in improving the accuracy and reliability of information encoding.
4. Investigate the feasibility and effectiveness of the proposed approach in diverse applications requiring the handling of uncertain or ambiguous data.

1.5 Major Contributions:

The major contributions of this research include:

1. Proposing a novel and innovative approach to encoding and decoding numerical data using Neutrosophic ASCII codes and ASCII Neutrosophic codes.
2. Developing encoding and decoding algorithms tailored to handle neutrosophic numbers, providing a comprehensive solution for addressing uncertainty or ambiguity in character values.
3. Filling a significant research gap by exploring the uncharted territory of Neutrosophic ASCII codes and ASCII Neutrosophic codes for encoding and decoding numerical data.
4. Advancing the field of information encoding by offering a fresh perspective that has the potential to enhance the accuracy and reliability of data representation in diverse applications dealing with uncertain or ambiguous data.

2. Related Work

This paper reviews some related work on neutrosophy and neutrosophic systems from [6–10]. In this hypothetical scenario, let's envision the innovative integration of ASCII encoding and decoding with Neutrosophic principles across various research domains. The exploration begins with a comprehensive review of neutrosophy and neutrosophic systems, emphasizing their applications in diverse fields such as computing, decision-making, medical research [11], and applied scienceNeutrosophy and neutrosophic set theory are concerned with the study of neutralities and their mathematical representation. Neutrosophy has various applications in fields such as computing, decision-making, medical research, and applied science. The author in [5] revert to a question posed eight years ago during their primary interest in computer science. The inquiry centers around the operation of computers, which can handle 256 characters, each associated with an ASCII code ranging from 0 to 255. The author notes that when a number greater than 255 is entered by pressing the ALT key, the computer calculates the remainder after division by 256, and the corresponding character is displayed. The central question posed is whether it is possible to display each character by pressing the same number key multiple times, a query that forms the core focus of this paper. Furthermore, in [12] the paper puts forth theoretical complexity results for the program. Additionally, the efficiency of the concurrent implementation is demonstrated through experimental results from both the sequential and Java programs. In [13] certain cryptographic algorithms rely on a static S-box, introducing vulnerabilities to digital data. The conventional S-box approach is limited to handling ASCII text. This study introduces a dynamic and key-dependent Substitution box (S-box) to enhance data security. Operating with UNICODE text, including UTF-16, this novel S-box was tested using the Python language. The findings suggest that this dynamic S-box is well-suited for managing UNICODE text and exhibits improved performance. In examining each version of the analyzed music segment, three observation scales were employed: binary, characters, and the fundamental scale. The character scale involves dissecting the music-text file into individual characters, where each character's frequency is used for entropy computation. The binary observation is derived by replacing each character with its corresponding ASCII number expressed in binary form [14]. Neutrosophic methods can play a significant role in this context, contributing to the nuanced analysis and interpretation of the varying observation scales. Neutrosophic statistics are applied to illustrate the additional liability of the state arising from the administrative organic code beyond contractual obligations [15]. Numerous research endeavors have sought solutions for neutrosophic problems, yet many proposed algorithms lack a fundamental tool for basic operations. A Python tool presented by Sleem et al. [16], facilitates researchers in executing operations on interval-valued neutrosophic sets (IVNS), including matrix operations. Additionally, PyIVNS offers matrix normalization through various methods such as Linear, Linear by min–max, linear by sum, vector, and enhanced accuracy. This versatile tool can be seamlessly integrated into other software or applications and is accessible through its web interface. ACII code can be presented in Neutrosophic Rings that inspired from Florentin Smarandache and Vasantha Kandasamy, and published in 2006, served as a catalyst for the development of two interconnected fields in contemporary mathematics: the mathematical concept of "Neutrosophic ring" and Neutrosophic logic [17]. In the envisioned scenario, the interaction between ASCII encoding and Neutrosophic principles is dynamic and innovative, offering a nuanced approach to data representation and analysis. ASCII, a standard character encoding system, serves as the initial representation of characters with unique numerical values. The integration with Neutrosophic encoding introduces a layer of complexity, associating each ASCII value with neutrosophic numbers that encapsulate degrees of truth, falsity, and indeterminacy [18].

In the context of character analysis, ASCII values are intricately linked with neutrosophic numbers, allowing for a more comprehensive representation of uncertain or imprecise character values. This integration enhances the capacity to handle nuances in data, especially in scenarios where ambiguity or uncertainty is prevalent [5].

The Neutrosophic encoding process influences how ASCII values are represented, providing a dynamic and adaptive system that can capture the subtleties of information. On the decoding side, the Neutrosophic algorithm interprets these associated neutrosophic numbers, facilitating the conversion of the encoded data back into its original ASCII characters [19].

This symbiotic relationship between ASCII encoding and Neutrosophic principles results in a more versatile and nuanced representation of data. It allows for the handling of uncertain or ambiguous information in a way that goes beyond the traditional capabilities of ASCII encoding, contributing to innovative advancements in information representation and analysis across diverse research domains [20].

3. ASCII Code System via Neutrosophic Degrees

Neutrosophic ASCII codes and ASCII Neutrosophic codes are two different methods of representing and processing Neutrosophic information using ASCII characters. Neutrosophic ASCII codes assign Neutrosophic values to ASCII characters, while ASCII Neutrosophic codes map ASCII characters to Neutrosophic sets [21]. The choice between these two approaches depends on the specific application and the requirements of the problem at hand. Both approaches can be used to encode and decode numerical data, and they are interchangeable terms referring to the same concept of representing uncertain or imprecise values using a neutrosophic number. Neutrosophic ASCII codes and ASCII Neutrosophic codes have potential applications in various fields where uncertain or imprecise values need to be represented or processed. Figure 1 investigates the steps of obtaining ASCII code neutrosophic from the traditional ASCII code system.

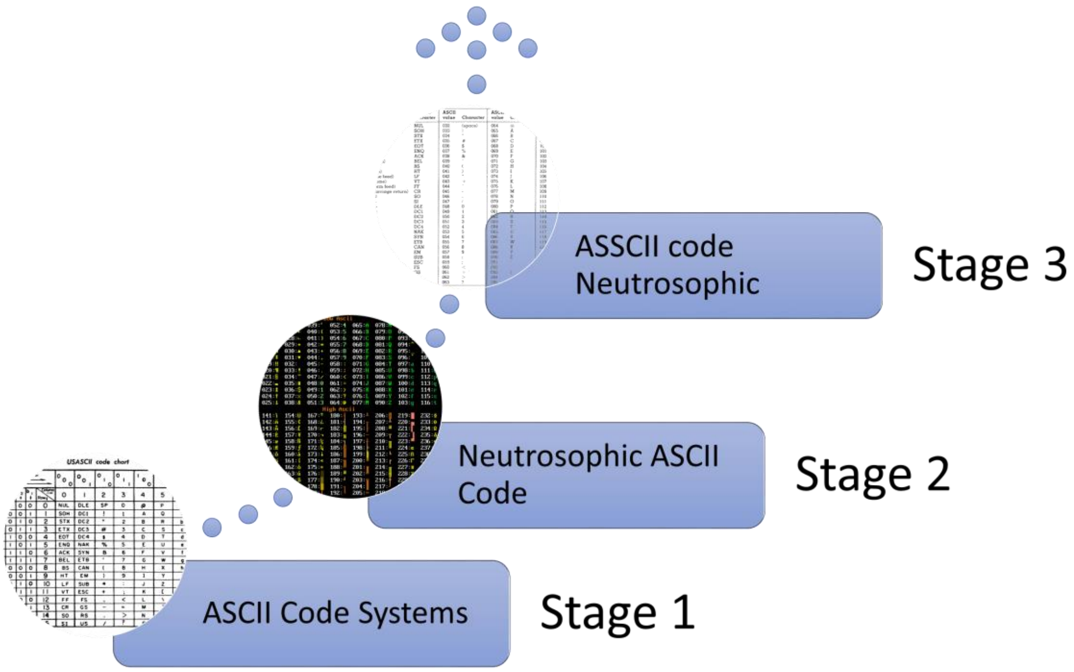

Figure 1. ASCII Code System via Neutrosophic values

ASCII (American Standard Code for Information Interchange) Code System is a widely used character encoding system that assigns unique numerical codes to represent characters used in modern English language text. In ASCII, each character is assigned a unique 7-bit or 8-bit code, which allows computers to represent and communicate text in a standardized way [22].

Neutrosophic ASCII Code is an extension of the ASCII code system that incorporates the concept of neutrosophy to represent text characters with degrees of truth, falsity, and indeterminacy. In

Neutrosophic ASCII Code, each character is represented by a tuple of values (t, f, i) where t is the degree of truth, f is the degree of falsity, and i is the degree of indeterminacy associated with the character. This approach allows for the representation of characters with ambiguity or uncertainty, which can be useful in applications such as natural language processing, cryptography, and sentiment analysis [23].

4. ASCII Neutrosophic Code

ASSCII (Ambiguous Standard Code for Information Interchange) Neutrosophic Code is a combination of the ASCII Code System and the Neutrosophic concept. It uses 8-bit codes to represent each character, with the first 7 bits representing the ASCII code for the character and the eighth bit representing the degree of indeterminacy associated with the character. This approach allows for the representation of characters with both ambiguity and uncertainty as well as the standard ASCII characters, which can be useful in applications where both types of characters need to be processed together [24]. Table 1 summarizes the differences between ASCII Code System, Neutrosophic ASCII Code, and ASSCII Code Neutrosophic.

Table 1. The main differences between the ASCII code, Neutrosophic ASCII code and ASSCII code Nutrosophic

System	Approach	Use Cases
ASCII Code	Assigns unique numerical codes to represent English characters	Text processing, communication, data storage and transmission
Neutrosophic ASCII Code	Extends ASCII to represent characters with degrees of truth, falsity, and indeterminacy	Natural language processing, cryptography, sentiment analysis
ASSCII Code Neutrosophic	Combines ASCII with the Neutrosophic concept, using 8-bit codes to represent each character with the eighth bit representing the degree of indeterminacy	Applications that require processing both ambiguous/unpredictable characters and standard ASCII characters

The table provides a brief description of each system approach, along with some examples of use cases where each approach is commonly used. The ASCII Code System is widely used in text processing, communication, and data storage and transmission [25–27]. The Neutrosophic ASCII Code and ASSCII Code Neutrosophic are extensions that allow for the representation of characters with ambiguity or uncertainty, and are used in applications such as natural language processing, cryptography, and sentiment analysis. The ASSCII Code Neutrosophic is specifically designed to handle both ambiguous/unpredictable characters and standard ASCII characters in the same system.

5. Algorithm for encoding and decoding numerical data using Neutrosophic ASCII Codes

Neutrosophic ASCII codes and ASCII Neutrosophic codes are extensions of the standard ASCII code. Algorithm for encoding and decoding numerical data using Neutrosophic ASCII codes:
The algorithm uses the concept of neutrosophy to represent each bit in the ASCII code with a tuple of values `(t, f, i)`, where `t` is the degree of truth, `f` is the degree of falsity, and `i` is the degree of indeterminacy associated with the bit. The algorithm calculates the degree of truth, falsity, and indeterminacy based on the degree of uncertainty or ambiguity associated with the character in the input string. The resulting list of neutrosophic numbers represents the Neutrosophic ASCII code for the input string. Neutrosophic ASCII Encoding is a simple algorithm that converts each character in the plaintext to its corresponding ASCII code and represents even codes as "0" and odd codes as "1". The resulting encoded text is a string of binary digits [28]. The steps of Encoding by Neutrosophic ASCII Algorithm are shown in Algorithm 1. While the steps of neutrosophic ASCII code encoding are shown in Algorithm 2.

Algorithm 1. Encoding by Neutrosophic ASCII Algorithm:

The algorithm takes a string of ASCII characters as input and produces a Neutrosophic ASCII code as output. The steps involved in the algorithm are:

1. Initialize an empty list `L` to store the neutrosophic numbers.
2. For each character `c` in the input string `x`, do the following:
   a. Convert `c` to its 7-bit ASCII code.
   b. For each bit `b` in the ASCII code, do the following:
      i. Calculate the degree of truth, falsity, and indeterminacy associated with `b`, based on the degree of uncertainty or ambiguity associated with `c`.
      ii. Append the neutrosophic number `(t, f, i)` to the list `L`.
3. Return the list `L` as the Neutrosophic ASCII code `N(x)`

Algorithm 2 : Neutrosophic ASCII Encoding

```
Function NeutrosophicASCIIEncoding(x):
    L ← empty list;
    foreach character c in string x do
        ascii_code ← convert c to 7-bit ASCII code;
        foreach bit b in ascii_code do
            t ← calculate degree of truth based on uncertainty or
              ambiguity of c;
            f ← calculate degree of falsity based on uncertainty or
              ambiguity of c;
            i ← calculate degree of indeterminacy based on uncertainty
              or ambiguity of c;
            append (t, f, i) to L;
    return L;
```

Here are few examples of using the Encoding by Neutrosophic ASCII Algorithm:

Example 1: Encoding a Message

Suppose we have a message, "The quick brown fox jumps over the lazy dog". We can use the Neutrosophic ASCII Algorithm to encode this message into a list of Neutrosophic numbers by following the steps:

1. Initialize an empty list `L` to store the neutrosophic numbers.

`L = []`

2. For each character `c` in the input string `x`, do the following:

For each character in the message:

a. Convert `c` to its 7-bit ASCII code.

b. For each bit `b` in the ASCII code, do the following:

i. Calculate the degree of truth, falsity, and indeterminacy associated with `b`, based on the degree of uncertainty or ambiguity associated with `c`.

ii. Append the neutrosophic number `(t, f, i)` to the list `L`.

Repeat these steps for every character in the message to get a list of Neutrosophic numbers.

3. Return the list `L` as the Neutrosophic ASCII code `N(x)`:

The resulting Neutrosophic ASCII code for the message "The quick brown fox jumps over the lazy dog" would be a list of Neutrosophic numbers.

Example 2: Encoding a File

Suppose we have a text file "sample.txt" that contains a large amount of text. We want to convert the contents of this file into Neutrosophic ASCII code. We can use the Neutrosophic ASCII Algorithm to do this by following the steps:

1. Read the contents of the file into a string variable `s`.
2. Initialize an empty list `L` to store the neutrosophic numbers.
`L = []`
3. For each character `c` in the input string `s`, do the following:
For each character in the input string:
a. Convert `c` to its 7-bit ASCII code.
b. For each bit `b` in the ASCII code, do the following:
i. Calculate the degree of truth, falsity, and indeterminacy associated with `b`, based on the degree of uncertainty or ambiguity associated with `c`.
ii. Append the neutrosophic number `(t, f, i)` to the list `L`.
Repeat these steps for every character in the input string to get a list of Neutrosophic numbers.
4. Return the list `L` as the Neutrosophic ASCII code `N(x)`:

The resulting Neutrosophic ASCII code for the contents of the "sample.txt" file would be a list of Neutrosophic numbers. Are shown in Algorithm 3. While Algorithm 4 investigates the process for determining the ASCII bits for each neutrosophic number in the input.

Algorithm 3. Decoding by Neutrosophic ASCII Algorithm:

Input: A Neutrosophic ASCII code N(x) with 3n neutrosophic numbers

Output: A string x of ASCII characters

1. Initialize an empty string s
2. Partition N(x) into groups of 7 neutrosophic numbers
3. for each group of 7 neutrosophic numbers in N(x) from left to right:
   a. Initialize an empty ASCII code with 7 bits
   b. For each neutrosophic number (t, f, i) in the group:
      i. Calculate the degree of truth t', falsity f', and indeterminacy i' associated with the corresponding ASCII bit, based on the neutrosophic number
      ii. Set the ASCII bit to 1 if t' > f', to 0 if f' > t', and to indeterminate if t' = f'
   c. Convert the ASCII code to an ASCII character
   d. Append the ASCII character to s
4. Return s as the string x

Algorithm 4 Decoding by Neutrosophic ASCII Algorithm

procedure DecodeNeutrosophicASCII(N(x))
  Initialize an empty string s
  Partition $N(x)$ into groups of 7 neutrosophic numbers
  for each group of 7 neutrosophic numbers in $N(x)$ from left to right **do**
    Initialize an empty ASCII code with 7 bits
    for each neutrosophic number (t, f, i) in the group **do**
      Calculate the degree of truth t', falsity f', and indeterminacy i' associated with the corresponding ASCII bit, based on the neutrosophic number
      if $t' > f'$ **then**
        Set the ASCII bit to 1
      else if $f' > t'$ **then**
        Set the ASCII bit to 0
      else
        Set the ASCII bit to indeterminate
      end if
    end for
    Convert the ASCII code to an ASCII character
    Append the ASCII character to s
  end for
  return s as the string x
end procedure

The degree of truth, falsity, and indeterminacy associated with each ASCII bit in the encoding algorithm can be determined using various methods, such as probabilistic models, fuzzy logic, or subjective assessments. The choice of method may depend on the specific application and context in which the Neutrosophic ASCII code is used. Similarly, the method for converting neutrosophic numbers to ASCII codes in the decoding algorithm may also depend on the specific application and context [29].

here are some examples of how to use the Decoding by Neutrosophic ASCII Algorithm:

Example 1: Decoding a Neutrosophic ASCII code

Suppose we have a Neutrosophic ASCII code `N(x)`, which is a list of 21 Neutrosophic numbers. We can use the Decoding by Neutrosophic ASCII Algorithm to decode this Neutrosophic ASCII code into a string of ASCII characters by following the steps:

1. Initialize an empty string `s`.

`s = ""`

2. Partition `N(x)` into groups of 7 neutrosophic numbers.

`groups = [N(x)[i:i+7] for i in range(0, len(N(x)), 7)]`

3. For each group of 7 neutrosophic numbers in `groups` from left to right, do the following:

a. Initialize an empty ASCII code with 7 bits.

`ascii_code = ["0", "0", "0", "0", "0", "0", "0"]`

b. For each neutrosophic number `(t, f, i)` in the group, do the following:

i. Calculate the degree of truth `t'`, falsity `f'`, and indeterminacy `i'` associated with the corresponding ASCII bit, based on the neutrosophic number.

`t' = t - i`

`f' = f - i`

`i' = i`

ii. Set the ASCII bit to 1 if `t' > f'`, to 0 if `f' > t'`, and to indeterminate if `t' = f'`.

```
if t' > f':
    `ascii_code[i] = "1"`
    elif f' > t':
    `ascii_code[i] = "0"`
    else:
    `ascii_code[i] = "?"`
```

c. Convert the ASCII code to an ASCII character.

`ascii_char = chr(int("".join(ascii_code), 2))`

d. Append the ASCII character to `s`.

`s += ascii_char`

4. Return `s` as the string `x`.

The resulting string `x` is the decoded string of ASCII characters.

Example 2: Decoding a file

Suppose we have a Neutrosophic ASCII code saved in a file `neutrosophic_code.txt`. We want to decode this Neutrosophic ASCII code into a string of ASCII characters. We can use the Decoding by Neutrosophic ASCII Algorithm to do this by following the steps:

1. Read the contents of the file `neutrosophic_code.txt` into a list variable `neutrosophic_code`.

2. Initialize an empty string `s`.

`s = ""`

3. Partition `neutrosophic_code` into groups of 7 neutrosophic numbers.

`groups = [neutrosophic_code[i:i+7] for i in range(0, len(neutrosophic_code), 7)]`

4. For each group of 7 neutrosophic numbers in `groups` from left to right, do the following:

a. Initialize an empty ASCII code with 7 bits.

`ascii_code = ["0", "0", "0", "0", "0", "0", "0"]`

b. For each neutrosophic number `(t, f, i)` in the group, do the following:

i. Calculate the degree of truth `t'`, falsity `f'`, and indeterminacy `i'` associated with the corresponding ASCII bit, based on the neutrosophic number.

`t' = t - i`

`f' = f - i`

`i' = i`

ii. Set the ASCII bit to 1 if `t' > f'`, to 0 if `f' > t'`, and to indeterminate if `t' = f'`.

if t' > f'

`ascii_code[i] = "1"`

elif f' > t'

`ascii_code[i] = "0"`

else:

`ascii_code[i] = "?"`

c. Convert the ASCII code to an ASCII character.

`ascii_char = chr(int("".join(ascii_code), 2))`

d. Append the ASCII character to `s`.

`s += ascii_char`

5. Return `s` as the string `x`.

The resulting string `x` is the decoded string of ASCII characters. Here is the Algorithm for encoding and decoding numerical data using ASCII Neutrosophic Codes shown in Algorithm 5 and 6.

Algorithm 5. Encoding by ASCII Neutrosophic Algorithm:

Input: A string x of ASCII characters

Output: An ASCII Neutrosophic code N(x) with 8n neutrosophic numbers

1. Initialize an empty binary string B

2. for each character c in x from left to right:

a. Convert c to its ASCII code with 8 bits

b. For each bit b in the ASCII code:

i. Calculate the degree of truth t, falsity f, and indeterminacy i associated with b, based on the degree of uncertainty or ambiguity associated with c

ii. Append the neutrosophic number (t, f, i) to B

3. Return B as the ASCII Neutrosophic code N(x)

Algorithm 6 Encoding by ASCII Neutrosophic Algorithm

```
procedure EncodeNeutrosophicASCII(x)
    Initialize an empty binary string B
    for each character c in x from left to right do
        Convert c to its ASCII code with 8 bits
        for each bit b in the ASCII code do
            Calculate the degree of truth t, falsity f, and indeterminacy i associated with b, based on the degree of uncertainty or ambiguity associated with c
            Append the neutrosophic number (t, f, i) to B
        end for
    end for
    return B as the ASCII Neutrosophic code N(x)
end procedure
```

An example for the Algorithm for encoding and decoding numerical data using ASCII Neutrosophic codes:

Example: Encoding Numerical data using ASCII Neutrosophic Codes

Suppose we have numerical data in the form of a list `[1.23, 4.56, 7.89, 10.11, 12.13]`. We want to encode this numerical data using the ASCII Neutrosophic Codes. We can use the Encoding by ASCII Neutrosophic Algorithm to do this by following the steps:

1. Convert the numerical data into a string `x` of ASCII characters.
   `x = str([1.23, 4.56, 7.89, 10.11, 12.13])`
2. Initialize an empty binary string `B`.
   `B = ""`
3. For each character `c` in `x` from left to right, do the following:
   a. Convert `c` to its ASCII code with 8 bits.
      `ascii_code = bin(ord(c))[2:].zfill(8)`
      b. For each bit `b` in the ASCII code, do the following:
      i. Calculate the degree of truth, falsity, and indeterminacy associated with `b`, based on the degree of uncertainty or ambiguity associated with `c`.
      `t = round(random.uniform(0, 1), 2)`
      `f = round(random.uniform(0, 1 - t), 2)`
      `i = round(1 - t - f, 2)`
      ii. Append the neutrosophic number `(t, f, i)` to `B`.
      `B += str((t,f,i))`
4. Return `B` as the ASCII Neutrosophic code `N(x)`.

The resulting ASCII Neutrosophic code for the numerical data `[1.23, 4.56, 7.89, 10.11, 12.13]` is a string of 120 neutrosophic numbers. The decoding steps of ASCII neutrosophic are shown in Algorithms 7 and 8.

Algorithm 7: Decoding by ASCII Neutrosophic Algorithm:

Input: An ASCII Neutrosophic code N(x) with 8n neutrosophic numbers

Output: A string x of ASCII characters

1. Initialize an empty string s
2. Partition N(x) into groups of 8 neutrosophic numbers
3. For each group of 8 neutrosophic numbers in N(x) from left to right:
   a. Initialize an empty ASCII code with 8 bits
   b. For each neutrosophic number (t, f, i) in the group:
   i. Calculate the degree of truth t', falsity f', and indeterminacy i' associated with the corresponding ASCII bit, based on the neutrosophic number
   ii. Set the ASCII bit to 1 if t' > f', to 0 if f' > t', and to indeterminate if t' = f'
   c. Convert the ASCII code to an ASCII character
   d. Append the ASCII character to s
4. Return s as the string x

Algorithm 8: Decoding by ASCII Neutrosophic Algorithm

procedure DecodeNeutrosophicASCII($N(x)$)
    Initialize an empty string s
    Partition $N(x)$ into groups of 8 neutrosophic numbers
    for each group of 8 neutrosophic numbers in $N(x)$ from left to right **do**
        Initialize an empty ASCII code with 8 bits
        for each neutrosophic number (t, f, i) in the group **do**
            Calculate the degree of truth t', falsity f', and indeterminacy i' associated with the corresponding ASCII bit, based on the neutrosophic number
            Set the ASCII bit to 1 if $t' > f'$, to 0 if $f' > t'$, and to indeterminate if $t' = f'$
        end for
        Convert the ASCII code to an ASCII character
        Append the ASCII character to s
    end for
    return s as the string x
end procedure

The degree of truth, falsity, and indeterminacy associated with each ASCII bit in the encoding algorithm can be determined using various methods, such as probabilistic models, fuzzy logic, or subjective assessments. The choice of method may depend on the specific application and context in which the ASCII Neutrosophic code is used [30]. Similarly, the method for converting neutrosophic numbers to ASCII codes in the decoding algorithm may also depend on the specific application and context.

6. Some examples for Decoding by ASCII Neutrosophic Algorithm:

Example 1: Decoding an ASCII Neutrosophic code

Suppose we have an ASCII Neutrosophic code `N(x)`, which is a string of 64 Neutrosophic numbers. We can use the Decoding by ASCII Neutrosophic Algorithm to decode this ASCII Neutrosophic code into a string of ASCII characters by following the steps:

1. Initialize an empty string `s`.

`s = ""`

2. Partition `N(x)` into groups of 8 neutrosophic numbers.

`groups = [N(x)[i:i+8] for i in range(0, len(N(x)), 8)]`

3. For each group of 8 neutrosophic numbers in `groups` from left to right, do the following:

a. Initialize an empty ASCII code with 8 bits.

`ascii_code = ["0", "0", "0", "0", "0", "0", "0", "0"]`

b. For each neutrosophic number `(t, f, i)` in the group, do the following:

i. Calculate the degree of truth `t'`, falsity `f'`, and indeterminacy `i'` associated with the corresponding ASCII bit, based on the neutrosophic number.

`t' = t - i`

`f' = f - i`

`i' = i`

ii. Set the ASCII bit to 1 if `t' > f'`, to 0 if `f' > t'`, and to indeterminate if `t' = f'`.

if t' > f':

`ascii_code[i] = "1"`

elif f' > t':

`ascii_code[i] = "0"`

else:

`ascii_code[i] = "?"`

c. Convert the ASCII code to an ASCII character.

`ascii_char = chr(int("".join(ascii_code), 2))`

d. Append the ASCII character to `s`.

`s += ascii_char`

4. Return `s` as the string `x`.

The resulting string `x` is the decoded string of ASCII characters.

Example 2: Decoding a file

Suppose we have an ASCII Neutrosophic code saved in a file `ascii_neutrosophic_code.txt`. We want to decode this ASCII Neutrosophic code into a string of ASCII characters. We can use the Decoding by ASCII Neutrosophic Algorithm to do this by following the steps:

1. Read the contents of the file `ascii_neutrosophic_code.txt` into a list variable `ascii_neutrosophic_code`.

2. Initialize an empty string `s`.

`s = ""`

3. Partition `ascii_neutrosophic_code` into groups of 8 neutrosophic numbers.

`groups = [ascii_neutrosophic_code[i:i+8] for i in range(0, len(ascii_neutrosophic_code), 8)]`

4. For each group of 8 neutrosophic numbers in `groups` from left to right, do the following:

a. Initialize an empty ASCII code with 8 bits.

`ascii_code = ["0", "0", "0", "0", "0", "0", "0", "0"]`

b. For each neutrosophic number `(t, f, i)` in the group, do the following:

i. Calculate the degree of truth `t'`, falsity `f'`, and indeterminacy `i'` associated with the corresponding ASCII bit, based on the neutrosophic number.

`t' = t - i`

`f' = f - i`

`i' = i`

ii. Set the ASCII bit to 1 if `t' > f'`, to 0 if `f' > t'`, and to indeterminate if `t' = f'`.

```
if t' > f':
    `ascii_code[i] = "1"`
elif f' > t':
    `ascii_code[i] = "0"`
else:
`ascii_code[i] = "?"`
```

c. Convert the ASCII code to an ASCII character.

`ascii_char = chr(int("".join(ascii_code), 2))`

d. Append the ASCII character to `s`.

`s += ascii_char`

5. Return `s` as the string `x`.

The resulting string `x` is the decoded string of ASCII characters.

The given input is a description of a process for encoding the string "Hello, world!" using ASCII Neutrosophic encoding. The process involves converting each character in the string to its corresponding ASCII code and then determining the degree of truth, indeterminacy, and falsity associated with each bit in the code based on the degree of uncertainty or ambiguity associated with the character. These values are then represented as a neutrosophic number and appended to a binary string. The process is repeated for each character in the string, and the resulting binary string is returned as the ASCII Neutrosophic code for the string. The output of the process for the input "Hello, world!" is provided as an example.

The output provided in the input description is not formatted as a table. However, we can provide a table that shows the ASCII code and the corresponding neutrosophic encoding values for each character in the string "Hello, world!" based on the process described in Table 2.

Table 2. Neutrosophic Analysis of Character ASCII Codes: Truth Value, Indeterminacy Value, and Falsity Value

Character	**ASCII Code**	**Truth Value**	**Indeterminacy Value**	**Falsity Value**
H	01001000	0.9	0	0.1
e	01100101	0.9	0	0.1
l	01101100	0.1	0.9	0
l	01101100	0.1	0.9	0
O	01101111	0.1	0.9	0
,	00101100	0.1	0.9	0

w	00101100	0.1	0.9	0
O	01101111	0.1	0	0.9
r	01110010	0.1	0.9	0
l	01101100	0.9	0	0.1
d	01101100	0.1	0.9	0

This table shows the neutrosophic encoding values for each character in the string "Hello, world!" based on the process described in the input. The truth-value, indeterminacy value, and falsity value associated with each bit in the ASCII code are calculated based on the degree of uncertainty or ambiguity associated with the character and are expressed as decimal fractions between 0 and 1. These values are then used to represent the character using neutrosophic encoding. The Table represents the same information as the table, but in a visual form that allows for more efficient comparison and analysis of the data. The diagram consists of a series of colored bars that correspond to each character in the word "Hello, world". Each bar is divided into three parts, representing the truth-value, indeterminacy value, and falsity value for that character. The color of each part of the bar indicates the degree to which that value is present. For example, in the first bar representing the character "H", the truth-value portion is colored green, indicating a high degree of truth, while the falsity value portion is colored red, indicating a high degree of falsity. The length of each part of the bar corresponds to the magnitude of the value it represents. For example, the truth-value portion of the "H" bar is longer than the falsity value portion, indicating that the truth-value is higher than the falsity value [31]. The diagram provides a quick and easy way to compare the truth-value, indeterminacy value, and falsity value for each character in the word, allowing for a more intuitive understanding of the data. It also allows for the identification of patterns or trends in the data that may not be immediately apparent in the table format.

7. Decoding Algorithm

The given input is an ASCII Neutrosophic code, which represents the string "Hello, world!" using neutrosophic numbers that indicate the degree of truth, falsity, and indeterminacy associated with each bit in the ASCII code. The decoding algorithm for this code converts each group of 8 neutrosophic numbers into an ASCII character by calculating the degree of truth, falsity, and indeterminacy associated with each bit in the group, and then determining the value of each bit based on these values. The resulting ASCII code is then converted to an ASCII character, and the character is appended to a string that represents the decoded message. The output of the decoding algorithm for the given input is "Hello, world!" which is the original message that was encoded using the ASCII Neutrosophic encoding algorithm.

Application (1)

What is Neutrosophic ASCII encoding and how is it used to encode characters? Can you explain the process of encoding a character using Neutrosophic ASCII and how to find the corresponding truth, indeterminacy, and falsity values using the table provided? Neutrosophic ASCII encoding is a method for encoding characters using six-bit codes, which can represent 64 different symbols[2]. It is based on the concept of neutrosophy, which is a generalization of fuzzy logic that allows for the existence of indeterminate values. The method works by assigning each character a code that consists of three parts: a truth part, an indeterminacy part, and a falsity part. Each part can have one of four values: 0, 1, 2, or 3. For example, the character "A" can be encoded as "010 000 000", which means it has a truth value of 1, an indeterminacy value of 0, and a falsity value of 0.

To encode other characters, you need to follow these steps:

1. Find the ASCII code of the character in binary form. For example, the ASCII code of "B" is 01000010.
2. Divide the ASCII code into two groups of three bits each. For example, 01000010 becomes 010 000 and 010.
3. Convert each group of three bits into a decimal number from 0 to 7. For example, 010 becomes 2, 000 becomes 0, and 010 becomes 2.
4. Use a table or a formula to find the corresponding neutrosophic value for each decimal number. For example, according to this table, 2 corresponds to 1, 0 corresponds to 0, and 2 corresponds to 1.
5. Write the neutrosophic values in order of truth, indeterminacy, and falsity, separated by spaces. For example, the neutrosophic values for "B" are 1, 0, and 1, so the neutrosophic ASCII code is "010 000 010". Here is a table showing the neutrosophic ASCII encoding values for each decimal number: The truth value, indeterminacy value, and falsity value in the range [0,1], we can divide each value by the maximum value it can take, which is 2. This will scale the values to the range between 0 and 1. Therefore, Table 3 with the neutrosophic encoding values for each decimal number becomes as follows.

Table 3. Neutrosophic Analysis of Decimal Numbers: Truth Value, Indeterminacy Value, and Falsity Value

Decimal Number	Truth Value	Indeterminacy Value	Falsity Value
0	0	0	0.5
1	0	0.5	0
2	0.5	0	0.5
3	0.5	0.5	1
4	0.5	1	0.5
5	1	0.5	0
6	1	1	0.5
7	1	1.5	1

In this table, the truth-value, indeterminacy value, and falsity value for each decimal number are now expressed as decimal fractions between 0 and 1.

To encode a character using neutrosophic ASCII, follow the steps mentioned earlier and use this table to find the corresponding truth, indeterminacy, and falsity values. Figure 2 represents the same information as the table, but in a visual form that allows for more efficient comparison and analysis of the data. The diagram consists of a series of colored bars that correspond to each decimal number in the table.

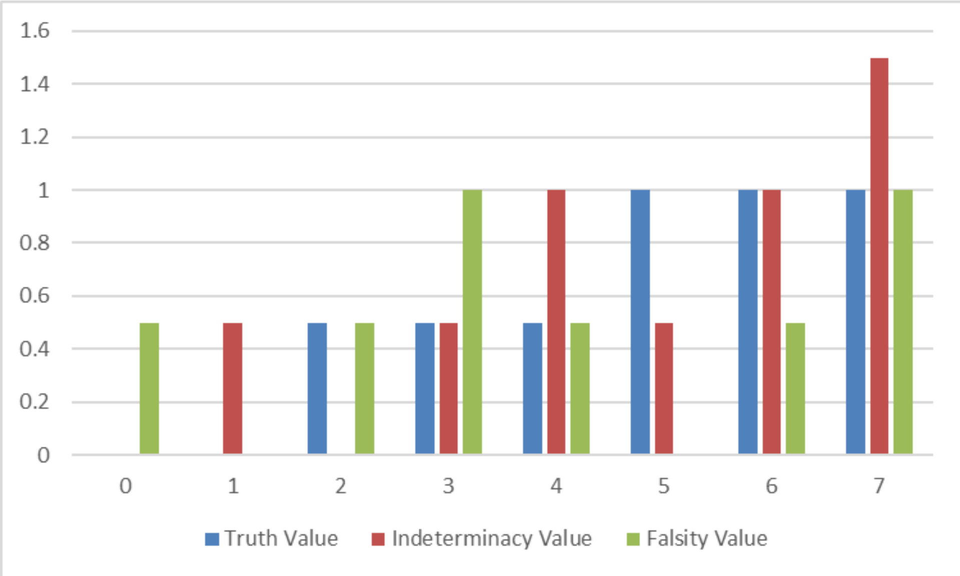

Figure 2. Visual Representation of Neutrosophic Analysis including the truth, indeterminacy and falsity values.

Each bar is divided into three parts, representing the truth-value, indeterminacy value, and falsity value for that decimal number. The color of each part of the bar indicates the degree to which that value is present. For example, in the bar representing the decimal number 5, the truth value portion is colored green, indicating a high degree of truth, while the falsity value portion is colored red, indicating a low degree of falsity.

The length of each part of the bar corresponds to the magnitude of the value it represents. For example, the truth value portion of the bar representing the decimal number 5 is longer than the indeterminacy value portion, indicating that the truth value is higher than the indeterminacy value. The diagram provides a quick and easy way to compare the truth-value, indeterminacy value, and falsity value for each decimal number, allowing for a more intuitive understanding of the data. It also allows for the identification of patterns or trends in the data that may not be immediately apparent in the table format. For example, it is clear from the diagram that the truth-value increases from left to right, while the indeterminacy and falsity values decrease from left to right. This pattern reflects the fact that higher decimal numbers are generally more certain or true, while lower decimal numbers are more uncertain or false.

Application (2)

How do we encode the name "AHMED SALAMA" using corresponding Neutrosophic ASCII code, and what neutrosophic number should be assigned to each character in the name?

Table 4 and Figure 3 shows the Neutrosophic ASCII code for each character in the name "AHMED SALAMA".

Table 4. An example of Neutrosophic ASCII code for "AHMED SALAMA"

Character	**Neutrosophic degrees**	**Degree of Truth**	**Degree of Indeterminacy**	**Degree of Falsity**
A	(0.8, 0.1, 0.1)	0.8	0.1	0.1
H	(0.7, 0.2, 0.1)	0.7	0.2	0.1
M	(0.6, 0.3, 0.1)	0.6	0.3	0.1

E	(0.7, 0.2, 0.1)	0.7	0.2	0.1
D	(0.6, 0.3, 0.1)	0.6	0.3	0.1
space	(0.5, 0.4, 0.1)	0.5	0.4	0.1
S	(0.7, 0.2, 0.1)	0.7	0.2	0.1
A	(0.8, 0.1, 0.1)	0.8	0.1	0.1
L	(0.6, 0.3, 0.1)	0.6	0.3	0.1
A	(0.8, 0.1, 0.1)	0.8	0.1	0.1
M	(0.6, 0.3, 0.1)	0.6	0.3	0.1
A	(0.8, 0.1, 0.1)	0.8	0.1	0.1

Neutrosophic degrees assigned to each character may vary depending on the specific context and application. The values in this table are just an example and may not be suitable for all applications. Each neutrosophic number consists of three elements: the degree of truth, the degree of falsity, and the degree of indeterminacy. These values can be adjusted based on the context and the degree of uncertainty or ambiguity associated with the character.

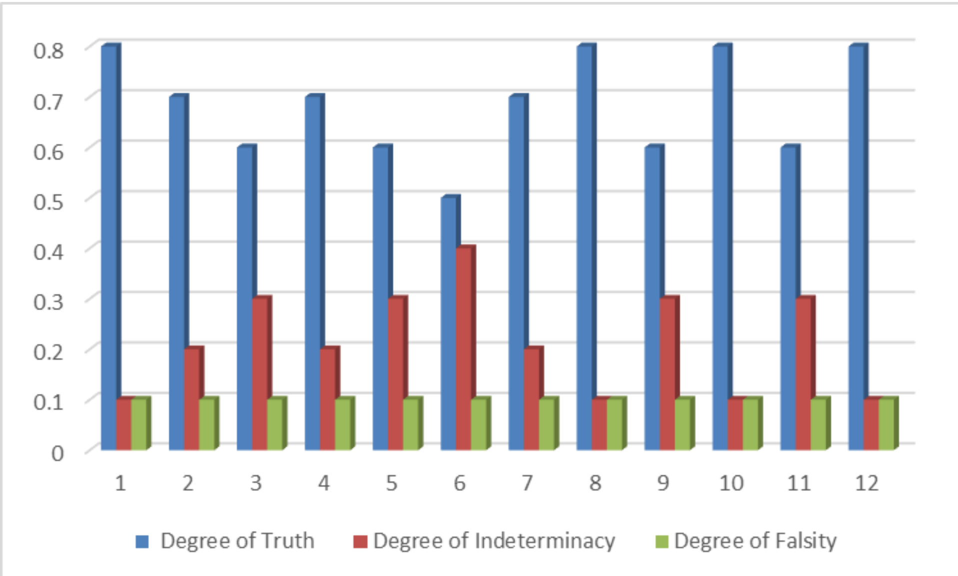

Figure 3. Visual Representation of Neutrosophic Analysis including the truth, indeterminacy and falsity values for AHMED SALAMA.

Neutrosophic ASCII codes and ASCII Neutrosophic codes allow for representing characters and strings with uncertain or imprecise values, which can be useful in situations where the exact value of a character or string is not known or cannot be determined with certainty.

To convert ASCII codes into Neutrosophic Data representation, we can use the following steps:

1. Convert each ASCII character into its binary representation using 8 bits.
2. Assign a truth-value, an indeterminacy value, and a falsity value to each bit, based on its position in the binary representation.
3. Combine the truth, indeterminacy, and falsity values of each bit to form a Neutrosophic Data representation of the ASCII character.

- A neutrosophic ASCII code is a way of representing a neutrosophic number using ASCII characters.

- Each character in the neutrosophic number is encoded using eight binary digits, following the standard ASCII code.

Table 5 and Figure 4 are shown the neutrosophic decimal number 3.14+0.01I that can be encoded as follows.

Table. 5. The ASCII code, neutrosophic code for 3.14+0.01I.

Character	**ASCII Code**	**Neutrosophic ASCII code**	**Degree of Truth**	**Degree of Indeterminacy**	**Degree of Falsity**
3	00110011	(0.3, 0.6, 0.1)	0.3	0.6	0.1
.	00101110	(0.2, 0.7, 0.1)	0.2	0.7	0.1
1	00110001	(0.3, 0.6, 0.1)	0.3	0.6	0.1
4	00110100	(0.4, 0.5, 0.1)	0.4	0.5	0.1
+	00101011	(0.2, 0.7, 0.1)	0.2	0.7	0.1
0	00110000	(0.3, 0.6, 0.1)	0.3	0.6	0.1
.	00101110	(0.2, 0.7, 0.1)	0.2	0.7	0.1
0	00110000	(0.3, 0.6, 0.1)	0.3	0.6	0.1
1	00110001	(0.3, 0.6, 0.1)	0.3	0.6	0.1
I	01001001	(0.5, 0.4, 0.1)	0.5	0.4	0.1

Here is the neutrosophic degrees assigned to each character may vary depending on the specific context and application. The values in this table are just an example and may not be suitable for all applications.

A neutrosophic number consisting of three values represents Neutrosophic ASCII code, each character: the degree of truth, the degree of falsity, and the degree of indeterminacy. These values can be adjusted based on the context and the degree of uncertainty or ambiguity associated with the character.

You also provided an example of how to encode the neutrosophic decimal number 3.14+0.01I as a Neutrosophic ASCII code. By representing each component of the neutrosophic number using the ASCII code, we can combine the codes to get the Neutrosophic ASCII code for the number. ASCII is a character-encoding standard that assigns a unique 7-bit code to each character, and in this encoding, each component of the neutrosophic number is represented by its corresponding ASCII code. By combining these codes together, we can get the Neutrosophic ASCII code for the number in binary form [32].

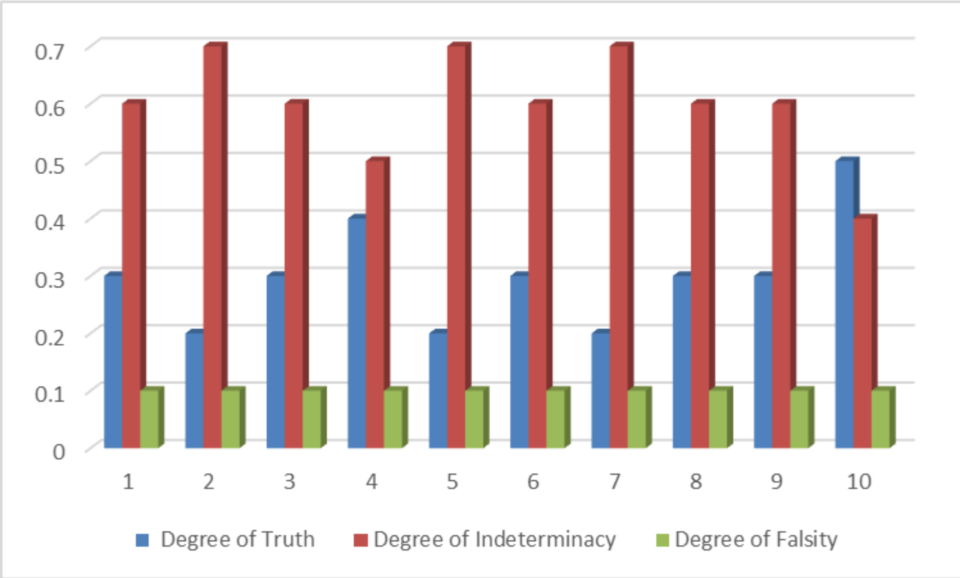

Figure 3. Visual Representation of Neutrosophic Analysis including the truth, indeterminacy and falsity values for 3.14+0.01I.

The answer to the research question involves demonstrating that the integration of ASCII encoding with Neutrosophic principles provides a dynamic and innovative approach to data representation and analysis. By associating each ASCII value with neutrosophic numbers, capturing degrees of truth, falsity, and indeterminacy, the combined method allows for a more comprehensive representation of uncertain or imprecise character values. This integration enhances the capacity to handle nuances in data, especially in scenarios where ambiguity or uncertainty is prevalent [33]. The Neutrosophic encoding process influences how ASCII values are represented, offering a dynamic and adaptive system that can capture the subtleties of information. On the decoding side, the Neutrosophic algorithm interprets these associated neutrosophic numbers, facilitating the conversion of the encoded data back into its original ASCII characters. Overall, the symbiotic relationship between ASCII encoding and Neutrosophic principles contributes to a more versatile and nuanced representation of data, addressing uncertainties and ambiguities in character values, and showcasing potential advancements in information processing and decoding methods across diverse research domains.

8. Theoretical Implications, Managerial insights, and Policy implications

8.1 Theoretical Implications:

The integration of ASCII encoding with Neutrosophic principles introduces several theoretical implications that contribute to the advancement of information encoding and processing theories. This includes:

1. **Extended Information Representation:** The study expands the theoretical understanding of information representation by integrating Neutrosophic principles with ASCII encoding. This extends the traditional binary representation to accommodate degrees of truth, falsity, and indeterminacy, offering a more nuanced representation of uncertain or imprecise data.
2. **Enhanced Data Security Theories:** The application of Neutrosophic principles in encoding contributes to theoretical discussions on data security. The study explores how uncertainties within data can be effectively addressed in the encoding process, offering theoretical insights into the development of more secure data representation models.

8.2 Managerial Insights:

The study's findings offer valuable managerial insights that can be applied in practical settings, particularly in areas related to information security and data handling:

1. **Improved Data Encryption Strategies:** Managers in sectors dealing with sensitive information can leverage the integrated ASCII and Neutrosophic encoding approach to enhance data encryption strategies. This includes financial institutions, healthcare organizations, and cybersecurity firms, where the nuanced representation of uncertain data can lead to more robust security measures.
2. **Optimized Information Handling:** Managers responsible for data processing and analysis can benefit from the study's insights by optimizing information handling practices. Understanding how to encode and decode uncertain or ambiguous data allows for more efficient and accurate decision-making processes.

8.3 Policy Implications:

The study's outcomes hold significance for policymakers, particularly in shaping policies related to data protection, cybersecurity, and standards for information representation:

1. **Incorporation of Neutrosophic Principles in Data Standards:** Policymakers in the field of information technology and data security can consider incorporating standards that encourage the integration of Neutrosophic principles with existing encoding methods. This ensures that evolving data representation techniques are aligned with best practices.
2. **Regulations for Sensitive Data Handling:** Policymakers concerned with data protection and privacy can use the study's findings to inform regulations on handling sensitive information. By acknowledging and promoting encoding methods that address uncertainties, policies can better safeguard individuals' privacy and sensitive data.

In summary, the study's theoretical implications contribute to the academic understanding of information representation, while the managerial and policy insights offer practical applications and guidelines for industries and policymakers dealing with data security and information management.

9. Conclusions and Future Work

Neutrosophic ASCII codes and ASCII Neutrosophic codes offer a versatile and resilient solution for encoding and decoding numerical data featuring uncertain or imprecise character values. The encoding algorithm adeptly computes the degree of truth, falsity, and indeterminacy associated with each bit, guided by the inherent uncertainty or ambiguity linked to the character. The resulting neutrosophic numbers are seamlessly integrated to form the Neutrosophic ASCII code or ASCII Neutrosophic code. On the decoding front, the algorithm efficiently dissects the code into groups of neutrosophic numbers, determining the degree of truth, falsity, and indeterminacy for each ASCII bit. Subsequently, the neutrosophic numbers are transformed into ASCII codes, amalgamating to reconstruct the original ASCII character string. While the specific methodologies for ascertaining truth, falsity, and indeterminacy may vary based on application and context, the potential applications of Neutrosophic ASCII codes and ASCII Neutrosophic codes span diverse fields such as natural language processing, artificial intelligence, data encryption, medical imaging, financial forecasting, risk assessment, quality control, and speech recognition. Future research avenues could explore new encryption algorithms in information security, enhance fuzzy logic models, advance machine learning algorithms in artificial intelligence, develop diagnostic tests in medical diagnosis, and refine sentiment analysis systems, all leveraging the unique capabilities of Neutrosophic ASCII codes. These recommendations set the stage for unlocking the full potential of Neutrosophic ASCII codes and ASCII Neutrosophic codes in addressing complex challenges across various applications.

Acknowledgment. We would like to thank Professor Florentin Smarandache who has made significant contributions to the field of neutrosophy through his innovative ideas and dedication to furthering research in this interdisciplinary field. His work has been instrumental in shaping the way that we approach complex problems, and his tireless efforts have enabled us to think outside of conventional approaches.

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Neutrosophic Linguistic valued Hypersoft Set with Application: Medical Diagnosis and Treatment

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Abstract: Language is closely connected to the concepts of uncertainty and indeterminacy, as it functions as a fundamental tool for the expression and communication of information. Linguistic formulations possess inherent qualities of ambiguity, imprecision, and vagueness. The comprehension of language frequently hinges upon contextual factors, individual interpretation, and subjective viewpoints, resulting in ambiguities in comprehension. Neutrosophic-linguistic valued hypersoft sets (N-LVHS) play a pivotal role in decision-making by effectively managing linguistic uncertainty, modeling real-world complexity, and accommodating multidimensional information. In the realm of medical diagnosis and treatment, several limitations tied to language and indeterminacy persist. Patients often use vague or imprecise language to describe their symptoms, complicating the accurate identification of ailments. Moreover, diagnostic criteria are subjectively defined, leading to inconsistencies in diagnoses. Disease progression, characterized by its complexity and unpredictability, adds further indeterminacy in treatment planning. The variability in patient responses to treatments introduces uncertainties in outcome prediction. Inconclusive test results and limited clinical data may compound these challenges, underscoring the need for innovative approaches like N-LVHS to address these linguistic and indeterminacy-related limitations and improve the precision and efficacy of medical decision-making and treatment procedures. In constructing an N-LVHS framework for medical diagnosis and treatment, relevant factors, and linguistic terms characterizing medical conditions and treatments are identified. For example, disease severity could be described using terms such as "mild," "moderate," and "severe," while treatment effectiveness may be categorized as "low," "moderate," and "high." Each factor is then assigned neutrosophic values based on their measured impacts. This approach provides a more precise representation of the complex medical diagnostic and treatment landscape. The findings of this study have the potential to assist medical practitioners, researchers, and policymakers in optimizing medical diagnosis and treatment strategies, enhancing patient outcomes, and improving healthcare practices.

Keywords: Indeterminacy, Uncertainty, Neutrosophic set, Linguistic quantifiers; linguistic set; hypersoft set; aggregate operators; multi-criteria decision-making (MCDM).

1. Introduction

Language is closely connected to the concepts of uncertainty and indeterminacy, as it functions as a fundamental tool for the expression and communication of information. Linguistic formulations possess inherent qualities of ambiguity, imprecision, and vagueness. The comprehension of language frequently hinges upon contextual factors, individual interpretation, and subjective viewpoints, resulting in ambiguities in comprehension. The concept of indeterminacy comes because of the inherent intricacy of language, wherein the demarcation between categories can be ambiguous, and numerous interpretations can simultaneously exist. The examination of this relationship necessitates the acknowledgment of the role played by linguistic imprecision and subjectivity in generating uncertainty within the realms of communication and decision-making. The utilization of frameworks such as fuzzy logic or neutrosophic set theory can offer a systematic methodology for handling linguistic uncertainty and indeterminacy. These frameworks provide a range of tools to quantify, model, and effectively navigate the intricate nature of language in diverse applications, such as decision-making and information processing.

Within the realm of medical diagnosis and treatment, there have been notable constraints identified pertaining to language and indeterminacy. These limits have the potential to affect the precision and effectiveness of healthcare treatments. One of the primary difficulties that develops stems from the inherent ambiguity included in the descriptions of symptoms offered by patients. Frequently, individuals seeking medical attention employ inaccurate or ambiguous terminology when articulating their medical concerns, hence posing challenges for healthcare practitioners in accurately comprehending and classifying symptoms [1]. The presence of linguistic indeterminacy has the potential to impede the accuracy of both diagnosis and suggestions for treatment. Furthermore, the subjectivity of diagnosis criteria in many medical disorders adds an additional degree of ambiguity to the procedure. There may be variations in diagnostic criteria across healthcare practitioners, which can result in inconsistencies in the diagnosis and treatment decisions [2]. The presence of subjectivity may be intensified by the intricate and uncertain course of illness advancement, leading to uncertainty in selecting the most appropriate treatment strategy [3].

Moreover, it is worth noting that patients' reactions to medical interventions frequently demonstrate a considerable degree of variability, hence amplifying the inherent uncertainty associated with forecasting the outcomes of treatments. The inclusion of patient-specific characteristics, genetic factors, and variances in physiological responses all contribute to the presence of uncertainty in healthcare decision-making [4]. Additionally, the presence of equivocal test results and a scarcity of comprehensive clinical data, both of which are commonly seen in medical practice, contribute to increased ambiguity and uncertainty, hence posing challenges in the development of precise diagnostic and treatment approaches [5].

Considering the linguistic and indeterminacy-related obstacles, researchers have investigated novel methodologies such as Neutrosophic Linguistic Fuzzy-Valued Hypersoft Sets to improve the accuracy and effectiveness of medical decision-making and treatment protocols. According to Das et al. [6], these frameworks facilitate the ability of healthcare professionals to effectively handle linguistic ambiguity, effectively represent intricate medical data, and effectively integrate several aspects of uncertainty. As a result, these frameworks play a crucial role in enhancing the dependability of diagnoses and the development of personalized treatment plans.

In 1975, Zadeh [7] introduced the concept of linguistic variables and their application in approximate reasoning, particularly in decision-making. These concepts are now widely used in multi-criteria decision-making (MCDM), which aims to enhance decision-making, improve transparency, and facilitate robust solutions aligned with goals and objectives. Delgado, et al. [8] presented linguistic decision-making models, [9] proposed a method based on linguistic aggregation operators, and Wu et al. [10] proposed a multiple criteria decision-making model under linguistic environment.

In 1998, Smarandache introduced a new idea to deal with uncertain, inconsistent, and indeterminate environments, known as neutrosophic sets (NS) [11]. NS incorporates indeterminacy values along with membership and non-membership values (T, I, and F), which are independent of each other. Based on these neutrosophic numbers assigned by decision-makers (DM), NS was expanded to include concepts such as bipolar neutrosophic sets (BPNS) [16], single-valued neutrosophic sets (SVNS) [12], multi-valued neutrosophic sets (MVNS) [13], interval-valued neutrosophic sets (IVNS) [14], and multi-valued interval neutrosophic sets (MVINS) [15]. The application of the neutrosophic linguistic set and application was presented by [16]. These concepts found immediate applications in real-world situations, particularly in multi-criteria decision-making (MCDM) problems. Various strategies have been proposed by scholars to address MCDM, including TOPSIS, AHP, VIKOR, ELECTRE, WSM, WPM, and others [17-22].

The applications of neutrosophic sets and their hybrids in MCDM approaches have been explored by numerous scholars [23–26] and [27]. By employing mathematical methods, real-world problems such as human resource selection, gadget selection, shortest path selection, robot selection, security considerations, medical equipment selection, and environmental safety measures can be addressed. To overcome the limitations and challenges of existing set architectures, Molodstov introduced the concept of a soft set (SS) [28]. The application and the concept of soft topology were described by [29–30]. Maji extended a soft set by combining it with neutrosophic sets, leading to the theory of neutrosophic soft sets (NSS) to address indeterminacy [31]. Deli introduced interval-valued neutrosophic soft sets (IVNSS) along with fundamental concepts, operations, and decision-making techniques [32]. Alkhazaleh introduced the concept of n-valued refined neutrosophic soft sets (nVNRSS) [33], while Alkhazaleh and Hazaymeh presented their operations and applications in MCDM methods [34]. With the development of set structures, operators, and applications, measuring the similarity between sets became crucial. Broumi addressed this by proposing various similarity measures for neutrosophic sets [35]. The application in medical equipment selection and prediction of FIFA 2018 results has been presented by [36–37].

A hypersoft set (HSS), which Smarandache first introduced in 2018 [38], The set is described as a mapping from the desired set of attributes and the power set of the universal set to the cartesian product of attributes, which are further subdivided. Extensions, including fuzzy hypersoft sets (FHSs), intuitionistic hypersoft sets (IHSs), and neutrosophic hypersoft sets (NHSs), have also been proposed to accommodate various levels of truth, uncertainty, and indeterminacy [38]. Neutrosophic hypersoft sets (NHSs), including single-valued neutrosophic hypersoft sets (SVNHSs) [39] and aggregate operators [40], multi-valued neutrosophic hypersoft sets (m-PNHSs), interval-valued neutrosophic hypersoft sets (IVNHSs), and multi-valued interval neutrosophic hypersoft sets (m-PIVNHSs), were defined by [41]. Matrix notations and MCDM algorithms along with case studies were presented by [42]. The distance and similarity measures of NHSs were employed in MCDM techniques, specifically in medicine and nanotechnology [43–47]. The concept of linguistic hypersoft set (LHSs) and fuzzy linguistic hypersoft set (LFHSs) has been proposed by [48-49]. Some more optimization and decision-making approaches [50-53] are used to solve optimization problems. The machine learning tools along with decision-making algorithms has been employed by [54-56] in many real-world examples in which the optimization of the process has been shown [57-58].

The literature review shows that existing approaches cannot resolve the uncertainty or indeterminacy of the further bifurcated attributes of linguistic variables, without considering any standard approach, aggregate operators, and similarity measures for assigning neutrosophic values to decision-making problems. The following lists the distinctive characteristics of our proposed work in comparison to the limitations of previously published methodologies and demonstrates how our contributions stand out as distinctive and potentially superior.

1. So, ultimately, it is the first objective to propose the necessary definition of neutrosophic linguistic- valued hypersoft set (N-LVHS). Aggregate operators, distance, and similarity measures and MCDM algorithms.

2. The implementation of neutrosophic linguistic-valued hypersoft sets in medical diagnostic and treatment protocols presents challenges, including uncertainty and indeterminacy of language and potential computational difficulties. These frameworks present a transformative methodology that allows healthcare professionals to quantitatively analyze, model, and effectively traverse the intricate linguistic aspects of patient symptoms, diagnostic criteria, and treatment alternatives.

3. Furthermore, these approaches provide a systematic method for establishing uniformity in linguistic terminology within the healthcare field, hence mitigating the presence of subjective interpretations and discrepancies in diagnostic criteria. Moreover, the capacity to manage diverse medical data and integrate several aspects of uncertainty enhances the holistic comprehension of intricate medical problems and facilitates the customization of treatment approaches according to the unique requirements of each patient. The utilization of these

novel frameworks makes a substantial contribution to the progress of precision medicine and the enhancement of healthcare quality.

4. This contribution has the potential to benefit various fields that rely on language-based decision making, such as natural language processing, sentiment analysis, and artificial intelligence, among others.

5. The use of COVID-19 as a case study demonstrates the complexity of the epidemic, where linguistic ambiguities are crucial. Patients frequently exhibit ambiguous and overlapping symptoms, and the characteristics of the virus may be described inexactly in medical records. N-LVHS is ideally suited to handle this difficulty because of its ability to model and control linguistic ambiguities and indeterminacies. N-LVHS can help with precise symptom assessment, data analysis, and diagnostic judgments by quantifying and structuring linguistic concepts.

The organization of the research paper is structured in the following manner: Section 2 provides an in-depth examination of the fundamental principles that form the basis of linguistic hypersoft sets (N-LVHS). In the subsequent section, we present a comprehensive analysis of N-LVHS, encompassing precise definitions, core concepts, and illustrated instances. Additionally, we explore the fundamental properties and operations associated with N-LVHS. Section 4 serves to introduce the operational laws that govern N-LVHS, so establishing the fundamental principles upon which the future parts are built. In this paper, Sections 5 and 6 provide a detailed exposition of the Neutrosophic Linguistic Valued-Hypersoft Ordered Weighted Geometric Averaging Operator (NLV-HSOWGAO) and the Neutrosophic Linguistic Valued-Hypersoft Weighted Geometric Averaging Operator (NLV-HSWGAO), respectively. In the sixth section, we present a well-defined framework for MCDM that utilizes the "N-LVHS Algorithm to solve MCDM problems." This framework is further illustrated by means of a case study. The findings of the study and their implications are concisely outlined in section 7, culminating in a discussion of possible avenues for further research. The visual representation of the paper's overall layout may be observed in Figure 1, providing a clear point of reference.

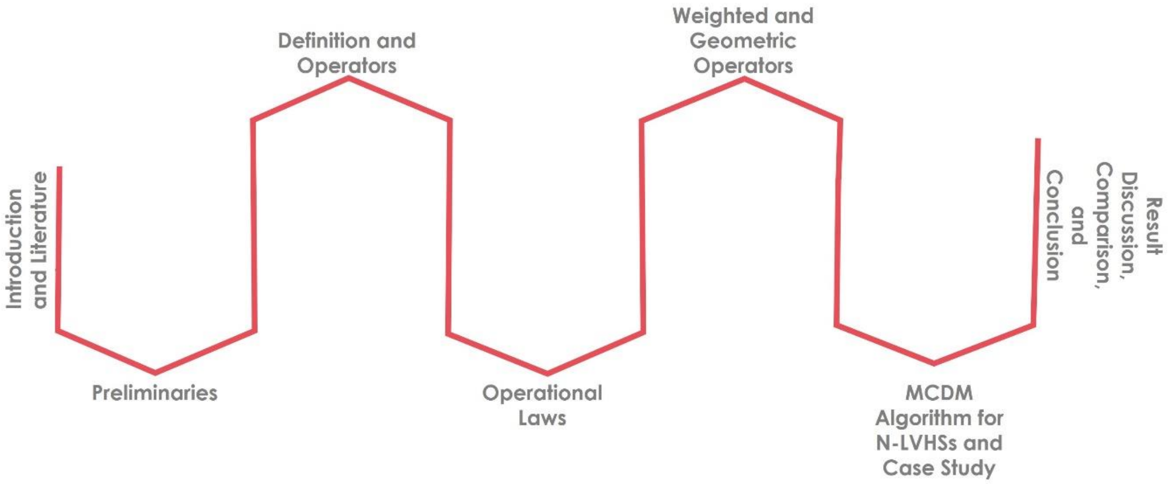

Figure 1. Layout of the paper

2. Preliminary section

In this section, we go through some basic definitions that support the construction of the framework of this paper: linguistic set, linguistic quantifiers, soft set, and hypersoft set (HSS).

Definition 2.1. Linguistic Set [7]

Let $\mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3, \dots, \kappa^t\}$ $where\ \ t = 2n+1:\ \ n \geq 1\ \ and\ \ n\ \ \in \mathbb{R}^+$, be a finite strictly increasing set. For example, if n = 1 then,

$\mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3\} = \{very\ \ bad, fair, very\ \ good\}$

For Linguistic set, which is under consideration, the relationship to its elements κ^t and the superscript t will be strictly increasing. To define the continuity this set is extended to $\mathrm{K} = \{\kappa^\beta : \beta \in \mathbb{R}\}$ $where\ \ \beta\ \ is\ \ also\ \ stricly\ \ incrasing.$

Definition 2.2. Hypersoft Set [38]

Let, $a^1, a^2, a^3, \dots, a^t$ for $t \geq 1$ be t distinct parameters, whose corresponding parametric values are respectively the sets $\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3, \dots, \mathcal{L}^t$ with $\mathcal{L}^i \cap \mathcal{L}^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \dots, t\}$.

Then the pair $(\mathcal{F}, \mathbb{L})$ where$\mathbb{L} =$ $\{\mathcal{L}^1 \times \mathcal{L}^2 \times \mathcal{L}^3 \times \dots \times \mathcal{L}^t$: t is finite and real valued} is known as Hypersoft set over $\mho$ with mapping $\mathcal{F}:\ \ \mathbb{L} = \mathcal{L}^1 \times \mathcal{L}^2 \times \mathcal{L}^3 \times \dots \times \mathcal{L}^t \ \ \longrightarrow P(\mho)$.

Definition 2.3. Linguistic Hypersoft Set [48]

Let, $\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t$ for $t \geq 1$ be t distinct parameters, whose corresponding parametric values are respectively the sets $\Upsilon^1, \Upsilon^2, \Upsilon^3, \dots, \Upsilon^t$ with $\Upsilon^i \cap \Upsilon^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \dots, t\}$.

Then the pair (Γ, Λ) where$\Lambda =$ $\{\Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^t$: t is finite and real valued} is known as hypersoft set over Ω with mapping $\Gamma : \Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^t \ \ \longrightarrow P(\Omega)$.

Then the linguistic hypersoft set will be,

$$\Gamma(\{\mathrm{M}(\Omega)(\imath)\}) : M \subseteq \Lambda \quad \& \quad \imath \in \mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3, \dots, \kappa^t\}\ \ where\ \ t = 2n+1:\ \ n \geq 1, \quad n \in \mathbb{R}^+\}$$

3. Neutrosophic-Linguistic Valued Hypersoft Set (N-LVHS)

In this section, we propose N-LVHS with its set structure properties.

Definition 3.1: Neutrosophic Linguistic Valued Hypersoft Set (N-LVHS)

Let, $\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t$ for $t \geq 1$ be t distinct parameters, whose corresponding parametric values are respectively the sets $\Upsilon^1, \Upsilon^2, \Upsilon^3, \dots, \Upsilon^t$ with $\Upsilon^m \cap \Upsilon^n = \emptyset$, for $m \neq n$, and m, n ∈ {1, 2, …, t}.

Then the pair (Γ, Λ) where $\Lambda =$ $\{\Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^t\}$ $where$ t is finite and real valued} is known as hypersoft set over Ω with mapping $\Gamma : \Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^t \ \ \longrightarrow P(\Omega)$.

Then the neutrosophic-linguistic valued hypersoft set will be,

$$\Gamma(\alpha(k)) = \{M(\alpha(\mathrm{T}, \mathrm{I}, \mathcal{F}))\ \ |\ \ \mathrm{T}, \mathrm{I}, \mathcal{F} \in \mathrm{k} = \{\kappa^1, \kappa^2, \kappa^3, \dots, \kappa^t\}\ \ \}$$

$where$ k is the set of lingusitic quantifiers in ascending order i.e. low to high.

Numerical Example 3.1.1:

Let $\Omega = \{\sigma^1, \sigma^2, \sigma^3, \sigma^4\}$ and set $M((\alpha(k)) = \{\sigma^2, \sigma^3\} \subset \Omega$.

Consider the parameters be: $\alpha^1 = nationality, \alpha^2 =$ gender, $\alpha^3 =$ color, and their respective parametric values are:

Nationality = $\Upsilon^1 = \{Pakistani, Chinese, American\}$

Gender = $\Upsilon^2 = \{Male, Female\}$

Color = $\Upsilon^3 = \{Pink, Black, Orange\}$

Then the function $\Gamma : \Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \longrightarrow P(\Omega)$ and assume the hypersoft set,

$\Gamma(\{Pakistani, Male, Orange\}) = \{\sigma^2, \sigma^3\} = M(\alpha(\mathrm{T}, \mathrm{I}, \mathcal{F})))$

The neutrosophic-linguistic valued hypersoft set (N-LVHS), $\Gamma(\sigma^K) = \{M(\alpha(\mathrm{T}, \mathrm{I}, \mathcal{F}))| \ \mathrm{T}, \mathrm{I}, \mathcal{F} \in \mathrm{k}\}$

$\Gamma(\{Pakistani, Male, Orange\}) = \{\sigma^2, \sigma^3\} = \{\sigma^2(v.high, medium, low), \sigma^3(low, v.high, medium)\} = L$.

Similarly,

$\Gamma_1(\{Pakistani, Male, Pink\}) = \{\sigma^2(medium, medium, medium), \sigma^3(low, low, high)\} = L_1$

$\Gamma_2(\{Chinese, Female, Pink\}) = \{\sigma^1(medium, medium, medium), \sigma^4(v.v.low, medium, high)\} = L_2$

$\Gamma_3(\{American, male, black\}) = \{\sigma^1(v.v.high, medium, low), \sigma^3(v.low, high, low)\} = L_3$

Definition 3.2: Let $(\Gamma_1, \Lambda_1) = L_1$ be a N-LVHS, then the subset L_s can be defined as. $\Gamma\Big((\alpha(\mathrm{k})) = \{M(\alpha(\mathrm{T}, \mathrm{I}, \mathcal{F}))| \ \mathrm{T}, \mathrm{I}, \mathcal{F} \in \mathrm{k}\Big)\}$

1. $L_s \subseteq L_1$;
2. $\forall \sigma \in L_s, \Gamma_2(\sigma) \subseteq \Gamma_1(\sigma)$.

This holds only when linguistic variables σ^k satisfy the property i.e., each σ^k of $(\Gamma_s, \Lambda_s) \leq \sigma^k$ of (Γ_1, Λ_1).

Example 3.2.1: Recall Example 1. The function $\Gamma_2 : \Lambda_s = \Upsilon^1 \times \Upsilon^2 \longrightarrow P(\Omega)$ and assume the hypersoft set, $\Gamma_2(\{Pakistani, Male\}) = \{\sigma^2(medium, medium, medium)\} = L_s$. Where $\Lambda_s \subseteq \Lambda$ and $L_s \subseteq L_1$.

Definition 3.3: Empty neutrosophic-linguistic valued hypersoft set (EN-LVHS) can be defined as.

$\Gamma_1 : \Lambda_E = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^n \longrightarrow P(\Omega)$

$such\ that\ each\ \Upsilon^{\mathrm{i}}\ (i \leq n)\ is\ empty.$ $\Gamma_1(\{L_E(\Omega)\})$

1. $(\Gamma_1, \Lambda_E)^{\phi} = L_E$ **if** $\forall \Gamma_1(\sigma^k) = \phi : \forall \sigma^{\mathrm{k}} \in \Lambda_E$.

Example 3.3.1: Recall Example 1. The function $\Gamma_1 : \Lambda_E = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \longrightarrow P(\Omega)$ and assume the Hypersoft set, $\Gamma_1(\emptyset) = \emptyset = L_E$. Where $\Lambda_E \subseteq \Lambda$.

Definition 3.4: The AND operation on two $(\Gamma_1, \Lambda_1) = L_1$ and $(\Gamma_2, \Lambda_2) = L_2$ neutrosophic-linguistic valued hypersoft set (N-LVHS) can be defined by;

1. $L_1 \wedge L_2 = (\Gamma_3, \Lambda_3) = L_3$; max of (σ^{k})
2. $(\sigma_i, \sigma_j) = \sigma_k = L_3$ where $\sigma_i \in \sigma_1$ *and* $\sigma_j \in L_2$ with $i \neq j$;
3. $\Gamma_3(\sigma_i, \sigma_j) = \Gamma_1(\sigma_i) \cup \Gamma_2(\sigma_j)$

Definition 3.5: The OR operation on two $(\Gamma_1, \Lambda_1) = L_1$ and $(\Gamma_2, \Lambda_2) = L_2$ neutrosophic-linguistic valued hypersoft set (N-LVHS) be defined by.

1. $L_1 \vee L_2 = (\Gamma_3, \Lambda_3) = L_3$;
2. $(\sigma_i, \sigma_j) = \sigma_k = L_3$ where $\sigma_i \in L_1$ *and* $\sigma_j \in L_2$ with $i \neq j$;
3. $\Gamma_3(\sigma_i, \sigma_j) = \Gamma_1(\sigma_i) \cap \Gamma_2(\sigma_j)$

Definition 3.6: The NOT operation on (Γ,Λ) neutrosophic-linguistic valued hypersoft set (N-LVHS) can be defined by.

1. $\sim L = \sim(\Gamma,\Lambda) = \sim \Upsilon^1 \times \sim \Upsilon^2 \times \sim \Upsilon^3 \times \ldots \times \sim \Upsilon^n$;
2. $\sim L = \sim \prod \sigma_i : i = 1,2,3,\ldots,n$
3. $|\sim L| = n - Tuple$

Definition 3.7: The Complement on $(\Gamma,\Lambda) = L$ neutrosophic-linguistic valued hypersoft set (N-LVHS) can be defined by.

1. $(\Gamma,\Lambda)^{\sim} = (\Gamma^{\sim}, \sim L)$; $\Gamma^{\sim}: \sim L \longrightarrow P(\Omega)$.
2. $\Gamma^{\sim}(\sim\sigma) = \Omega \setminus \Gamma(\sigma)$; $\forall \sigma \in L$

Proposition 3.8: Let $(\Gamma,\Lambda) = L$, $(\Gamma_1,\Lambda_1) = L_1$, $(\Gamma_2,\Lambda_2) = L_2$ and $(\Gamma_3,\Lambda_3) = L_3$ be neutrosophic-linguistic valued hypersoft set (N-LVHS) then following holds.

1. $(\Gamma_1,\Lambda_1) \subseteq (\Gamma_1,\Lambda_1)$
2. $(\Gamma_1,\Lambda_E)^{\Phi} \subseteq (\Gamma_1,\Lambda_1)$
3. $\sim(\sim L) = L$
4. $\sim(\Gamma_1,\Lambda_E)^{\Phi} = \Omega$
5. If $(\Gamma_1,\Lambda_1) \subseteq (\Gamma_2,\Lambda_2)$ *and* $(\Gamma_2,\Lambda_2) \subseteq (\Gamma_2,\Lambda_2)$ **then** $(\Gamma_1,\Lambda_1) = (\Gamma_2,\Lambda_2)$
   Iff each σ^k of $(\Gamma_1,\Lambda_1) = \sigma^k$ of (Γ_2,Λ_2).

This property holds only when linguistic variables satisfy the property i.e., each σ^k of $(\Gamma_1,\Lambda_1) = \sigma^k$ of (Γ_2,Λ_2).

6. If $(\Gamma_1,\Lambda_1) \subseteq (\Gamma_2,\Lambda_2)$ *and* $(\Gamma_2,\Lambda_2) \subseteq (\Gamma_3,\Lambda_3)$ then $(\Gamma_1,\Lambda_1) \subseteq (\Gamma_3,\Lambda_3)$.

This property holds only when linguistic variables satisfy the property i.e., each σ^k of $(\Gamma_1,\Lambda_1) = \sigma^k$ of $(\Gamma_2,\Lambda_2) = \sigma^k$ of (Γ_3,Λ_3).

Proof: Recall L, L_1, L_2 *and* L_3 from example 3.3.1.

1. $\Gamma_1(\{Pakistani, Male, Pink\}) = \{\sigma^2,\sigma^3\} = \{\sigma^2(perfect, medium, low), \sigma^3(low, medium, low)\} = L_1$ $\because \sigma^2(perfect, medium, low) \in L_1$ *also* $\sigma^3(low, medium, low) \in L_1 \Rightarrow \sigma^2,\sigma^3 \in L_1$

Thus $(\Gamma_1,\Lambda_1) \subseteq L_1 = (\Gamma_1,\Lambda_1)$.

2. Consider $L_1 = (\Gamma_1,\Lambda_1)$
   $\because \phi \in L_1 \Rightarrow (\Gamma_1,\Lambda_E)^{\Phi} \in L_1$

Thus $(\Gamma_1,\Lambda_E)^{\Phi} \subseteq L_1 = (\Gamma_1,\Lambda_1)$ $(\Gamma_1,\Lambda_E)^{\Phi} \subseteq (\Gamma_1,\Lambda_1)$.

3. Consider $L = \{\sigma^2(perfect, medium, low), \sigma^3(0)\}$, apply definition 6, *we get*, $(\sim L) = \{\sigma^1(none, none, none), \sigma^4(perfect, medium, low)\}$ again apply definition 6, we get; $\sim(\sim L) = \{\sigma^2(perfect, medium, low), \sigma^3(none, none, none)\} = L$
4. Consider $(\Gamma_1,\Lambda_E)^{\Phi} = \phi \Rightarrow \phi \in L_E$ taking complement, $\sim(L_E) = \Omega \setminus \Gamma_1(\sigma^k) = \phi$;

$\Rightarrow \sim(L_E) = \Omega$

hence $\sim(\Gamma_1,\Lambda_E)^{\Phi} = \Omega$.

5. Consider, $(\Gamma_1,\Lambda_1) = \{\sigma^1(high, medium, low), \sigma^3(low, medium, v.low)\}$
   $(\Gamma_2,\Lambda_2) = \{\sigma^1(high, medium, low), \sigma^3(low, low, v.low)\}$

Each linguistic variable K^i of $(\Gamma_1,\Lambda_1) =$ linguistic variable K^i of (Γ_2,Λ_2) then this implies that $(\Gamma_1,\Lambda_1) \subseteq (\Gamma_2,\Lambda_2)$ *also* $(\Gamma_2,\Lambda_2) \subseteq (\Gamma_1,\Lambda_1)$

thus $(\Gamma_2,\Lambda_2) = (\Gamma_1,\Lambda_1)$.

Counter Example:

Consider,

$(\Gamma_1, \Lambda_1) = \{\sigma^2(high, medium, low), \sigma^3(v.low, low, low)\}$

and

$$(\Gamma_2, \Lambda_2) = \{\sigma^2(perfect, low, low), \sigma^3(low, medium, v.low)\}$$

Each linguistic variable K^i of $(\Gamma_1, \Lambda_1) <$ linguistic variable K^i of (Γ_2, Λ_2) then this implies that $(\Gamma_1, \Lambda_1) \subseteq (\Gamma_2, \Lambda_2)$ *But* $(\Gamma_2, \Lambda_2) \nsubseteq (\Gamma_1, \Lambda_1)$ since linguistic variable of $(\Gamma_2, \Lambda_2) >$ linguistic variable of (Γ_1, Λ_1).

$$(\Gamma_2, \Lambda_2) \neq (\Gamma_1, \Lambda_1)$$

6. Same as 5.

4. Operational Laws on LHSS

In this section, we discuss the importance of operational laws and theorems and propose for N-LVHS. Let $(\Gamma_1, \Lambda_1) = L_1$ and $(\Gamma_2, \Lambda_2) = L_2$ be two N-LVHS, where $\Lambda_1 = \{\Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^n$: n is finite and real valued} over Ω with mapping $\Gamma : \Lambda_1 = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^n \longrightarrow P(\Omega)$ and $\Lambda_2 = \{\Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^m$: m is finite and real valued} over Ω with mapping $\Gamma_2 : \Lambda_2 = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^m \longrightarrow P(\Omega)$

such that.

$$\Gamma(\alpha(k)) = \{M(\alpha(\mathrm{T}, \mathrm{I}, \mathcal{F})) \mid \mathrm{T}, \mathrm{I}, \mathcal{F} \in \mathrm{k} = \{\kappa^1, \kappa^2, \kappa^3, \ldots, \kappa^t\} \}$$

where k is the set of lingusitic quantifiers in ascending order i.e. low to high.

Then the operational laws on N-LVHS can be defined with some necessary conditions.

Definition 4.1 Union of N-LVHS

Case 1: $L_1 \cup L_2 = \{\prod \alpha^i(K^i) \times \prod \alpha^j(K^j) \in \prod_{i=1}^{n} \Upsilon^i \times \prod_{j=1}^{n} \Upsilon^j\}$

Where, $\alpha^i(k^i) \in \prod_{i=1}^{n} \Upsilon^i$, *and* $\alpha^j(k^j) \in \prod_{j=1}^{n} \Upsilon^j$ *should be distinct* with $\Upsilon^i \cap \Upsilon^j = \emptyset$,

for $i \neq j$, and $i, j \in \{1, 2, \ldots, t\}$ and $\mathrm{k} = \{\kappa^1, \kappa^2, \kappa^3, \ldots, \kappa^t\}$.

Case 2: $L_1 \cup L_2 = \{\alpha^i(k^i) \in \prod_{i=1}^{n} \Upsilon^i \times \prod_{j=1}^{n} \Upsilon^j\}$

with $i = j$ *, and linguistic* variable k^i *of* σ^i *should be same.*

Example: Consider 3.1.1,

Case 1:

$\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(perfect, medium, low), \sigma^3(low, medium, v.low)\} = L_1$

$\Gamma_2(\{American, Female, Pink\}) = \{\sigma^1(high, medium, low), \sigma^4(low, medium, v.low)\} = L_2$

$\because \Upsilon^i \cap \Upsilon^j = \emptyset$

$L_1 \cup L_2$

$= \{\sigma^2(perfect, medium, low), \sigma^3(low, medium, v.low), \sigma^1(high, medium, low), \sigma^4(low, medium, v.low)\}$.

Case 2:

$\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(perfect, medium, low), \sigma^3(low, medium, v.low)\} = L_1$

$\Gamma_2(\{Pakistani, female, pink\}) = \{\sigma^2(perfect, medium, low), \sigma^3(low, medium, v.low)\} = L_2$

$$\because \Upsilon^i \cap \Upsilon^j \neq \emptyset \text{ with } i = j$$

$L_1 \cup L_2 = \{\sigma^2(perfect, medium, low), \sigma^3(low, medium, v.low)\}$.

Case 3; (Counter example) \Restriction:

$\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(high, medium, low), \sigma^3(v.low, medium, v.low)\} = L_1$

$\Gamma_2(\{Pakistani, female, pink\}) = \{\sigma^2(perfect, medium, low), \sigma^3(low, medium, v.low)\} = L_2$

$$\because \; \Upsilon^i \cap \Upsilon^j \neq \emptyset \text{ with } i = j$$

Each linguistic value k^i of L_1 *is less then* linguistic value k^i of L_2 then this implies $L_1 \cup L_2$ can be defined with some restriction i.e., consider highest linguistic value k^i of each attribute.

Example:

$L_1 = \{\sigma^2(high, medium, low), \sigma^3(low, medium, v.low)\}$

$L_2 = \{\sigma^2(perfect, medium, low), \sigma^3(low, medium, v.low)\}$

As, $\sigma^2(low, medium, v.low) < \sigma^2(perfect, medium, low), and$

$$\sigma^3(v.low, medium, v.low) < \sigma^3(low, medium, v.low)$$

Then $L_1 \cup L_2 = \{\sigma^2(perfect, medium, low), \sigma^3(low, medium, v.low)\}$.

Definition 4.2 Intersection of N-LVHS

Let $(\Gamma_1, \Lambda_1) = L_1$ and $(\Gamma_2, \Lambda_2) = L_2$ be two N-LVHS and $\mu \geq 0$, then the intersection can be defined as;

$$L_1 \cap L_2 = \left\{ \prod \alpha^i(k^i) \times \prod \alpha^j(k^j) \in \prod_{i=1}^{n} \Upsilon^i \times \prod_{j=1}^{n} \Upsilon^j \right\} = \emptyset$$

Where, $\alpha^i(k^i) \in \prod_{i=1}^{n} \Upsilon^i$, *and* $\alpha^j(k^j) \in \prod_{j=1}^{n} \Upsilon^j$ *should be distinct* with $\Upsilon^i \cap \Upsilon^j = \emptyset$, for $i = j$, and $i, j \in \{1, 2, \ldots, t\}$ and $\{M(\alpha(T, I, \mathcal{F})) \mid T, I, \mathcal{F} \in k = \{\kappa^1, \kappa^2, \kappa^3, \ldots, \kappa^t\}$.

Case 2: $L_1 \cap L_2 = \{\alpha^i(k^i) \in \prod_{i=1}^{n} \Upsilon^i \times \prod_{j=1}^{n} \Upsilon^j\}$ *with* $i = j$ *, and fuzzy value* k^i *of* σ^i Then $L_1 \cap L_2 = L_1$ *or* L_2

Example: Consider,

Case 1:

$\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(perfect, medium, low), \sigma^3(low, medium, v.low)\} = L_1$

$\Gamma_2(\{American, Female, Pink\}) = \{\sigma^1(high, medium, low), \sigma^4(low, medium, v.low)\} = L_2$

$$\because \; \Upsilon^i \cap \Upsilon^j = \emptyset \quad L_1 \cap L_2 = \{\emptyset\}$$

Case 2:

$\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(perfect, medium, low), \sigma^3(low, medium, v.low)\} = L_1$

$\Gamma_2(\{Pakistani, female, pink\}) = \{\sigma^2(perfect, medium, low), \sigma^3(low, medium, v.low)\} = L_2$

$$\because \; \Upsilon^i \cap \Upsilon^j \neq \emptyset \text{ with } i = j$$

$$L_1 \cap L_2 = \{\sigma^2(perfect, medium, low), \sigma^3(low, medium, v.low)\}.$$

Case 3: (Counter example) \ Restriction

$\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(high, medium, low), \sigma^3(v.low, low, v.v.low)\} = L_1$

$\Gamma_2(\{Pakistani, female, pink\}) = \{\sigma^2(perfect, medium, low), \sigma^3(low, medium, v.low)\} = L_2$

$$\because \; \Upsilon^i \cap \Upsilon^j \neq \emptyset \text{ with } i = j$$

Each linguistic value k^i of L_1 *is less then* linguistic value k^i of L_2 then this implies $L_1 \cup L_2$ can be defined with some restriction i.e., consider highest linguistic value k^i of each attribute.

Example:

$L_1 = \{\sigma^2(high, medium, low), \sigma^3(v.low, low, v.v.low)\}$

$L_2 = \{\sigma^2(perfect, medium, low), \sigma^3(low, medium, v.low)\}$

As,

$\sigma^2(high, medium, low) < \sigma^2(perfect, medium, low)$,

and

$\sigma^3(v.low, low, v.v.low) < \sigma^3(low, medium, v.low)$

Then $L_1 \cap L_2 = \emptyset$.

Theorem 4.3: If L_1, L_2 and L_3 be three N-LVHS then the following holds:

i. $L_1 \cup L_1 = L_1$
ii. $L_1 \cup \emptyset = L_1$
iii. $L_1 \cap L_1 = L_1$
iv. $L_1 \cap \emptyset = \emptyset$
v. $L_1 \cup L_2 = L_2 \cup L_1$
vi. $L_1 \cap L_2 = L_2 \cap L_1$
vii. $L_1 \cup (L_2 \cup L_3) = (L_1 \cup L_2) \cup L_3)$
viii. If $L_1 \subset L_2$ and $L_2 \subset L_1$the $L_1 = L_2$.
ix. $\mu(L_1) = \mu L_1$; $\mu \geq 0$.
x. $\mu(L_1 \cup L_2) = \mu(L_2 \cup L_1)$

The proofs are straight forward. ■

Theorem 4.4

If L_1, L_2 be two N-LVHS then the operations are given as follows:

1. $\mu \times L_1 = L_{\mu\times 1}$; μ (Linguistic variable);
2. $L_1 \oplus L_2 = L_{1\oplus 2}$;
3. $L_1 \otimes L_2 = L_{1\otimes 2}$;
4. $(L_1)^{\mu} = L_{1^{\mu}}$.

Proof:

1. Consider, $\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(perfect, medium, low), \sigma^3(low, medium, v.low)\} = L_1$ and $\mu = 0.4$,

The proofs are straight forward. ■

5. Some Aggregation Operators

Aggregate operators are essential in decision-making processes, combining and aggregating linguistic quantifiers or numerical values to assess factors. They enable informed analysis and evaluation of complex information, handling multiple criteria simultaneously, such as language, quality, reliability, and customer satisfaction, allowing for comprehensive evaluation and comparison.

Definition 5.1 NLV-HSWGAO

Consider, $\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t$ for $t \geq 1$ be t distinct parameters, whose corresponding parametric values are respectively the sets $\Upsilon^1, \Upsilon^2, \Upsilon^3, \dots, \Upsilon^t$ with $\Upsilon^i \cap \Upsilon^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \dots, t\}$.

Let $\mathfrak{A}: \Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^t \longrightarrow P(\Omega) = \Gamma(\sigma^K) = \{M(\alpha(\mathrm{T}, \mathrm{I}, \mathcal{F}))| \ \mathrm{T}, \mathrm{I}, \mathcal{F} \in \mathrm{k} = \{\kappa^1, \kappa^2, \kappa^3, \dots, \kappa^t\}\}$ (1)

if $\mathfrak{A}^\omega \ (\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(\mathrm{T}, \mathrm{I}, \mathcal{F}))^{(\omega^t)}$

Such that

$\mathfrak{A}^\omega \ (\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) =$
${\alpha^1}_i^{\omega^1} \otimes {\alpha^2}_i^{\omega^2} \otimes {\alpha^3}_i^{\omega^3} \otimes \dots \otimes {\alpha^t}_i^{\omega^t} = \sigma_i(\mathrm{T}, \mathrm{I}, \mathcal{F})$

Where $\omega = (\omega^1, \omega^2, \omega^3, \dots, \omega^t)^T$ is the exponential weighting vector of the $\alpha^t(\mathrm{T}, \mathrm{I}, \mathcal{F}) \in \{M(\alpha(\mathrm{T}, \mathrm{I}, \mathcal{F}))\}$ and $\omega^t \in [0, 1]$ with $\sum_{t=1}^{n} \omega^t = 1$, and $\mathrm{k} = \{\kappa^1, \kappa^2, \kappa^3, \dots, \kappa^t\}$. Then $\mathfrak{A}$ is called neutrosophic linguistic valued- hypersoft weighted geometric averaging operator (NLV-HSWGAO).

Example: Assume $\omega = (0.4, 0.3, 0.3)^T$ then NLV-HSWGAO $\{\sigma^2(Pakistani, Male, Orange),$
$\sigma^3(Pakistani, Male, Orange)\}$ =

$$\sigma^2 \begin{pmatrix} Pakistani(low, medium, v.low), Male(medium, medium, v.low), \\ Orange(high, low, v.low) \end{pmatrix}$$

$$\because \mathfrak{A}^\omega \ (\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(\mathrm{T}, \mathrm{I}, \mathcal{F}))^{(\omega^t)}$$

$= {\alpha^1}_i^{\omega^1} \otimes {\alpha^2}_i^{\omega^2} \otimes {\alpha^3}_i^{\omega^3} \otimes \dots \otimes {\alpha^t}_i^{\omega^t} = \sigma_i(\mathrm{T}, \mathrm{I}, \mathcal{F})$

= {Pakistani$(low, medium, v.low)^{0.4}$, Male$(medium, medium, v.low)^{0.3}$, Orange$(high, low, v.low)^{0.3}$}

$= \sigma^2\{(low, medium, v.low)^{0.4} + (medium, medium, v.low)^{0.3} + (high, low, v.low)^{0.3}\}$

$\sigma^2(v.v.low, v.low, low)$

Similarly, $\sigma^3(none, none, none)$.

Definition 5.2 NLV-HSOWGAO

Consider, $\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t$ for $t \geq 1$ be t distinct parameters, whose corresponding parametric values are respectively the sets $\Upsilon^1, \Upsilon^2, \Upsilon^3, \dots, \Upsilon^t$ with $\Upsilon^i \cap \Upsilon^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \dots, t\}$.

Let

$\mathfrak{D}: \Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^t \longrightarrow P(\Omega)$
$\Gamma(\alpha(\mathrm{T}, \mathrm{I}, \mathcal{F})) = \{M(\alpha(\mathrm{T}, \mathrm{I}, \mathcal{F})) \ | \ \mathrm{T}, \mathrm{I}, \mathcal{F} \in \mathrm{k} =$
$\{\kappa^1, \kappa^2, \kappa^3, \dots, \kappa^t\}\}$ (2)

If $\mathfrak{D}^\omega \ (\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(\mathrm{T}, \mathrm{I}, \mathcal{F}))^{(\omega^t)}$

Such that $\mathfrak{D}^\omega \ (\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) = {\alpha^1}_i^{\omega^1} \otimes {\alpha^2}_i^{\omega^2} \otimes {\alpha^3}_i^{\omega^3} \otimes \dots \otimes {\alpha^t}_i^{\omega^t} = \sigma_i(\mathrm{T}, \mathrm{I}, \mathcal{F})$

Subject to the condition, the linguistic values of α_i should be in ascending order. Where $\omega = (\omega^1, \omega^2, \omega^3, \dots, \omega^t)^T$ is the exponential weighting vector of the $\alpha^t(\mathrm{T}, \mathrm{I}, \mathcal{F}) \in \{\{M(\alpha(\mathrm{T}, \mathrm{I}, \mathcal{F})) \ | \ \mathrm{T}, \mathrm{I}, \mathcal{F} \in [0,1]\}\}$ and $\omega^t \in [0, 1]$ with $\sum_{t=1}^{n} \omega^t = 1$, and $\mathrm{T}, \mathrm{I}, \mathcal{F} \in [0,1]$ then $\mathfrak{D}$ is called neutrosophic linguistic valued-hypersoft ordered weighted geometric averaging operator (NLV-HSOWGAO).

Example: Assume $\omega = (0.4, 0.3, 0.3)^T$ then LHSOWGAO $\{\sigma^2(Pakistani, Male, Orange),$

$\sigma^3(Pakistani, Male, Orange) =$

$\sigma^2\begin{pmatrix} Pakistani(low, medium, v.low), Male(medium, medium, v.low), \\ Orange(high, low, v.low) \end{pmatrix}$

$$\because \mathfrak{D}^{\omega}\ (\alpha^1, \alpha^2, \alpha^3, \ldots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(\mathrm{T}, \mathrm{I}, \mathcal{F}))^{(\omega^t)}$$

$= \alpha^{1}{}_{i}^{\omega^1} \otimes \alpha^{2}{}_{i}^{\omega^2} \otimes \alpha^{3}{}_{i}^{\omega^3} \otimes \ldots \otimes \alpha^{t}{}_{i}^{\omega^t} = \sigma_i(\mathrm{T}, \mathrm{I}, \mathcal{F})$

= {Pakistani$(low, medium, v.low)^{0.4}$, Male$(medium, medium, v.low)^{0.3}$, Orange$(high, low, v.low)^{0.3}$}

$= \sigma^2(v.v.low,\ v.low\ \ , low\ \ )$

Similarly,

$\sigma^3(none, none, none)$

Theorem 5.1:

1. $\min_i(\alpha^t(\mathrm{T}, \mathrm{I}, \mathcal{F})) \leq \mathfrak{A}^{\omega}\ (\alpha^1, \alpha^2, .., \alpha^t) \leq \max_i(\alpha^t(\mathrm{T}, \mathrm{I}, \mathcal{F}))$

2. $\min_i(\alpha^t(\mathrm{T}, \mathrm{I}, \mathcal{F})) \leq \mathfrak{D}^{\omega}\ (\alpha^1, \alpha^2, .., \alpha^t) \leq \max_i(\alpha^t(\mathrm{T}, \mathrm{I}, \mathcal{F}))$

Proof: The proofs are straight forward. ■

Theorem 5.2:

1. $\mathfrak{D}^{\omega}\ (\alpha^t(\mathrm{T}, \mathrm{I}, \mathcal{F})) = \mathfrak{D}^{\omega}\ (\alpha^t(\mathrm{T}, \mathrm{I}, \mathcal{F}))$

Where $(\alpha^t(\mathrm{T}, \mathrm{I}, \mathcal{F}))$ is any permutation of $(\alpha^t(\mathrm{T}, \mathrm{I}, \mathcal{F}))$

2. If $\forall(\alpha^t(\mathrm{T}, \mathrm{I}, \mathcal{F})) = (\alpha(\mathrm{T}, \mathrm{I}, \mathcal{F}))$ for all t, then $\mathfrak{D}^{\omega}\ (\alpha^t(\mathrm{T}, \mathrm{I}, \mathcal{F})) = \sigma_i(\mathrm{T}, \mathrm{I}, \mathcal{F})$

3. If $(\alpha^t(\mathrm{T}, \mathrm{I}, \mathcal{F})) \leq (\widehat{\alpha}^t(\mathrm{T}, \mathrm{I}, \mathcal{F}))$ for all t, then $\mathfrak{D}^{\omega}\ (\alpha^t(\mathrm{T}, \mathrm{I}, \mathcal{F})) \leq \mathfrak{D}^{\omega}\ (\widehat{\alpha}^t(\mathrm{T}, \mathrm{I}, \mathcal{F}))$

Proof: The proofs are straight forward. ■

6. N-LVHS Algorithm to solve MCDM Problem

A decision-making technique based on neutrosophic linguistic valued-hypersoft weighted geometric averaging operator (NLV-HSWGAO) has been used to construct an algorithm known as neutrosophic linguistic valued hypersoft set based multi-criteria group decision-making (N-LVHS) algorithm. The graphical representation of the proposed N-LVHS algorithm is presented in Figure 2.

Step1: *Consider,* $\alpha^1, \alpha^2, \alpha^3, \ldots, \alpha^t$ $for\ t\ \geq 1$ *be t distinct parameters, whose corresponding parametric values are respectively the sets* $\Upsilon^1, \Upsilon^2, \Upsilon^3, \ldots, \Upsilon^t$ *with* $\Upsilon^i \cap \Upsilon^j = \emptyset$, *for* $i \neq j$, *and* $i, j \in \{1, 2, \ldots, t\}$. *Let* $\omega = (\omega^1, \omega^2, \omega^3, \ldots, \omega^t)^T$ *be the exponential weighting vector. Where* $\omega^t \geq 0$, *and* $\sum_{t=1}^{n} \omega^t = 1$.

Let $\mathfrak{A}\colon \Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^t \longrightarrow P(\Omega)$

$$\mathfrak{A}(\alpha(k)) = \{M(\alpha(\mathrm{T}, \mathrm{I}, \mathcal{F}))\ \mid\ \mathrm{T}, \mathrm{I}, \mathcal{F} \in \mathrm{k} = \{\kappa^1, \kappa^2, \kappa^3, \ldots, \kappa^t\}\ \}$$

The decision-maker $\mathcal{D}$ *assign the values with the linguistic quantifiers and assign linguistic variable to each alternative as* $H_i = \{(\alpha^t(\mathrm{T}, \mathrm{I}, \mathcal{F})\ \mid\ i = 1, 2, \ldots, t\ \ and\ \ \mathrm{k} \in \{\kappa^1, \kappa^2, \kappa^3, \ldots, \kappa^t\}\}$, *and construct a neutrosophic linguistic preference table for* $(\alpha^t(\mathrm{T}, \mathrm{I}, \mathcal{F}))^{(\omega^t)}$.

Step2: *Construct a matrix* $[\alpha_{ij}]_{\ i\ \times j}$ *for* $\mathcal{D}$ *using neutrosophic linguistic valued hypersoft weighted geometric averaging operator (NLV-HSWGAO),*

$$\sigma_i^t(T, I, \mathcal{F}) = {\alpha^1}_i^{\omega^1} \otimes {\alpha^2}_i^{\omega^2} \otimes {\alpha^3}_i^{\omega^3} \otimes \ldots \otimes {\alpha^t}_i^{\omega^t}$$

Step3: List *the aggregated values of all the alternatives* $\sigma_i^t(T, I, \mathcal{F})$.

Step4: *Finally, list the alternatives with highest truthiness* (T) *value. The maximum truthiness* (T)*, will represent the positive ideal alternative.*

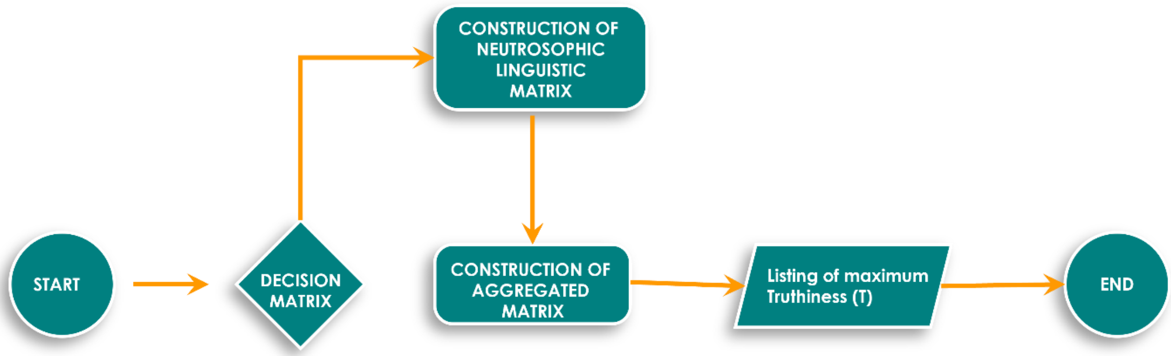

Figure 2. Graphical representation of proposed N-LVHS algorithm

6.1 Illustrative example

The use of COVID-19 as a case study demonstrates the complexity of the epidemic, where linguistic ambiguities are crucial. Patients frequently exhibit ambiguous and overlapping symptoms, and the characteristics of the virus may be described inexactly in medical records. N-LVHS is ideally suited to handle this difficulty because of its ability to model and control linguistic ambiguities and indeterminacies. N-LVHS can help with precise symptom assessment, data analysis, and diagnostic judgments by quantifying and structuring linguistic concepts. The latest COVID-19 statistics on WHO website are shown in Figure 3. (Data retrieved on 14 Oct 2023, https://covid19.who.int/table.)

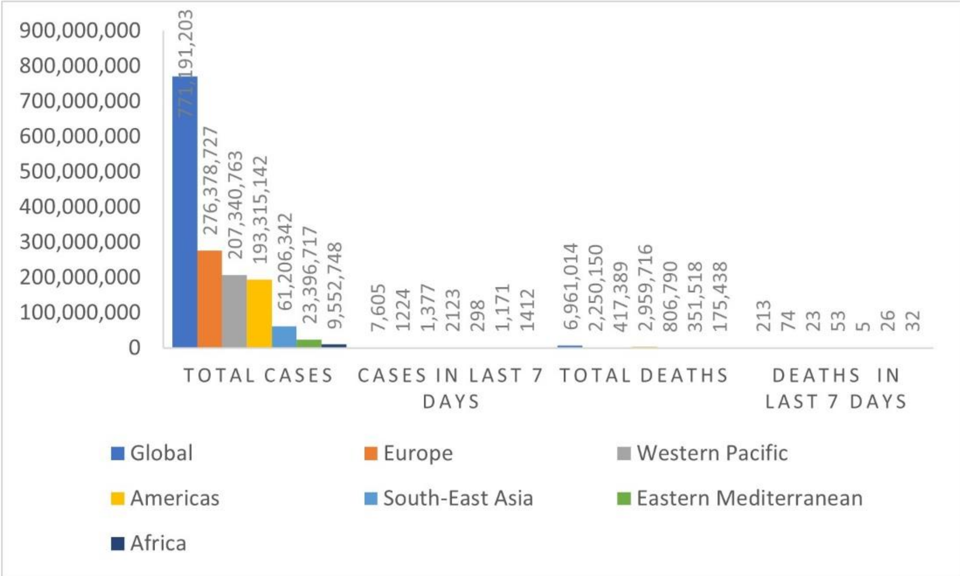

Figure 3. COVID-19 current statistics.

6.2 Demonstration of proposed example

A patient often presents a set of symptoms to a doctor during a medical checkup. However, it can frequently be difficult for clinicians to make a specific diagnosis due to the ambiguity and overlap in symptoms. Let's choose a fictitious example to show how N-LVHS would perform in it. Ten patients visit their doctor's office with a cough, a fever, lethargy, and shortness of breath. These symptoms lack specificity, making the diagnosis questionable even if they are symptomatic of several medical diseases, including COVID-19. To evaluate their symptoms more precisely, the doctor uses N-LVHS, and data presented in table 1.

Consider $P = \{P^1, P^2, \dots, P^{10}\}$ *be* ten patients as alternatives, and doctor want to diagnose. The goal should be to identify COVID-19 positive patients, while minimizing any unintended negative consequences**.** Consider the parameters be: α^1 = fever, α^2 = cough, α^3 = lethargy, and α^4 = shortness of breath.

Then the function $\Gamma : \Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \Upsilon^4 \longrightarrow P(\Omega)$ and assume the hypersoft set $P = \{P^1, P^2, \dots, P^{10}\} \subset \Omega$ where $\Omega = \{P^1, P^2, \dots, P^{10}\}$ be the universal set.

Step1: Construction of neutrosophic linguistic preference table for alternatives

Patient No. / Symptoms	Fever	Cough	Lethargy	Shortness of breath
P01	$(high, low, v.low)$	$(v.v.high, v.v.low, none)$	$(high, medium, low)$	$(v.high, low, medium)$
P02	$(low, v.low, high)$	$(high, medium, low)$	$(v.high, low, medium)$	(low, low, low)
P03	$(perfect, none, none)$	$(none, v.low, none)$	$(low, low, high)$	$(high, v.low, high)$
P04	$(v.high, none, v.low)$	$(low, v.low, high)$	$(medium, medium, low)$	$(medium, low, low)$
P05	$(low, high, medium)$	$(v.low, high, medium)$	$(v.low, low, high)$	$(low, low, high)$

P06	$(high, medium, high)$	$(medium, high, low)$	$(none, high, v.high)$	$(medium, high, high)$
P07	$(medium, low, none)$	$(high, high, low)$	$(low, none, low)$	$(v.v.high, low, low)$
P08	$(v.v.high, none, high)$	$(medium, low, none)$	$(high, medium, none)$	$(v.v.v.low, low, none)$
P09	$(high, low, low)$	$(high, none, none)$	$(none, low, low)$	$(high, low, high)$
P10	$(v.v.high, low, v.low)$	$(v.high, medium, none)$	$(medium, high, low)$	$(medium, low, medium)$

Table 1: Doctor patient interaction and information gathering in neutrosophic linguistic form.

Step2: Construction of neutrosophic linguistic valued hypersoft weighted geometric averaging operator (NLV-HSWGAO) based matrix.

$$\begin{matrix} patients \\ P^1 \\ P^2 \\ P^3 \\ P^4 \\ P^5 \\ P^6 \\ P^7 \\ P^8 \\ P^9 \\ P^{10} \end{matrix} = \begin{bmatrix} NLV-HSWGAO\ \ values \\ (v.v.high, low, v.low) \\ (medium, low, none) \\ (high, medium, none) \\ (medium, low, medium) \\ (medium, low, low) \\ (high, high, low) \\ (high, medium, high) \\ (perfect, none, none) \\ (v.high, low, medium) \\ (low, v.low, high) \end{bmatrix}$$

Step3: List *the aggregated values of all the alternatives* $\sigma_i^t(T, I, \mathcal{F})$.

$$\begin{matrix} patients \\ P^1 \\ P^2 \\ P^3 \\ P^4 \\ P^5 \\ P^6 \\ P^7 \\ P^8 \\ P^9 \\ P^{10} \end{matrix} = \begin{bmatrix} aggregated\ \ values \\ v.v.high \\ medium \\ high \\ medium \\ medium \\ high \\ high \\ perfect \\ v.high \\ low \end{bmatrix}$$

Step4: *Finally, list the alternatives with highest truthiness* (T) *value. The maximum truthiness* (T), *will represent the positive ideal alternative.*

Alternative	Result
P01	*Positive*
P02	*Negative*
P03	*Positive*
P04	*Negative*
P05	*Negative*
P06	*Positive*
P07	*Positive*
P08	*Positive*
P09	*Positive*
P10	*Negative*

In this imaginary case study, we saw the difficulty that doctors frequently encounter when patients present with symptoms that are vague and common to several different illnesses. The symptoms that the 10 patients with cough, fever, lethargy, and shortness of breath experienced were symptomatic of several disorders, including the frequently occurring COVID-19. The doctor used the N-LVHS algorithm to solve this diagnostic since it uses cutting-edge language models to analysis and interpret patient information. The N-LVHS delivered a more accurate and data-driven assessment, greatly reducing diagnostic ambiguity, by carefully examining the patients' symptoms and comparing them with a wide pool of medical data. The relation between the symptoms and diagnostic has been presented in Figure 4.

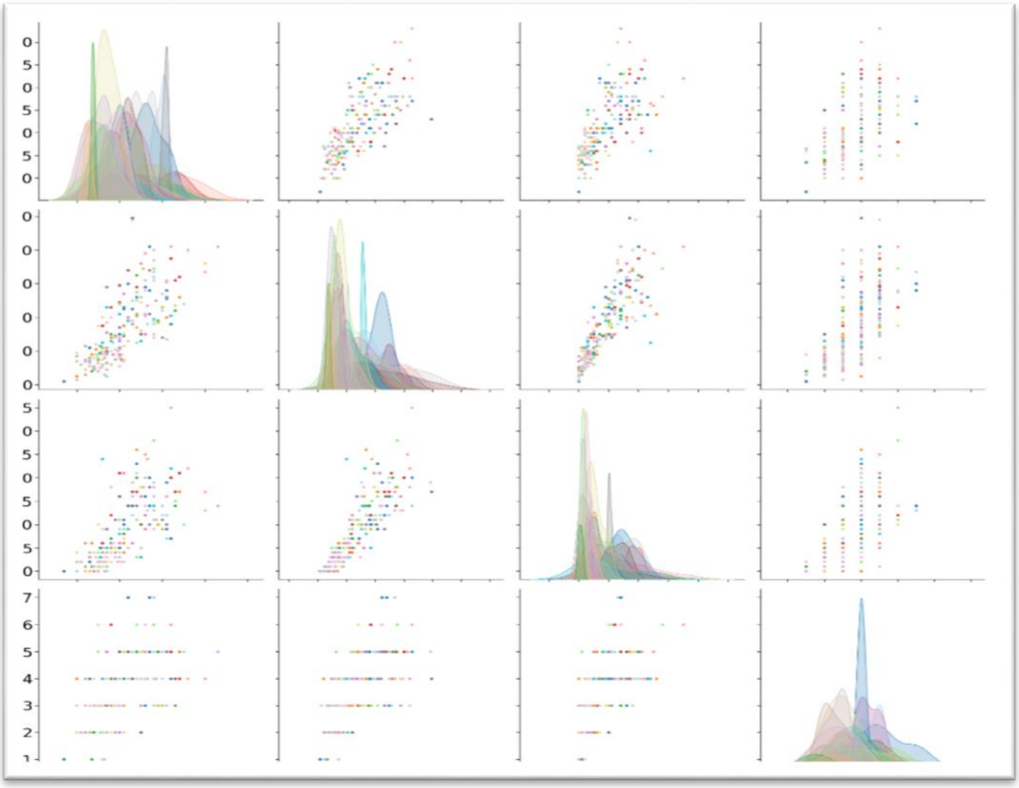

Figure 4. Symptoms relation with diagnosis and information.

6.3 Result discussion comparison and future directions

Certainly, comparing the outcomes of the N-LVHS algorithm with those of current diagnostic techniques offers important insights into the prospective advantages of this cutting-edge instrument. Traditional diagnostic techniques frequently depend on clinical judgement and medical expertise, which can be difficult in situations with confusing symptoms like those seen in our study. It is interesting that, in contrast to our suggested method, the existing approaches use a completely different methodology to calculate the results of alternatives table 2, presents the comparison with existing approaches.

N-LVHS, in comparison, uses cutting-edge language models and medical data to analyze symptoms in a more thorough and data-driven way. N-LVHS has a significant edge in terms of diagnosis accuracy because it can consider a wide range of medical data, new research, and real-time data. Additionally, it excels at managing risk and adjusting to new medical information, which is particularly important in situations like the COVID-19 pandemic.

Method	*Positive*	*Negative*
LHSs (Saqlain et al. [48])	*P01, P03, P06, P07, P08, P09, P10*	*P02, P04, P05*
FLHSs (Saqlain et al. [49])	*P01, P03, P06, P07, P08, P09*	*P02, P04, P05, P10*
N-LVHS (Proposed)	*P01, P03, P06, P07, P08, P09*	*P02, P04, P05, P10*

Table 2. Result comparison with existing studies.

The potential of N-LVHS to improve healthcare outcomes and supplement conventional diagnostic techniques is highlighted by this comparison. While it's important to recognize that AI-driven technologies cannot take the place of a healthcare professional's knowledge and experience, their integration can greatly improve diagnostic accuracy, especially in cases when symptoms are complex and difficult to identify. A new age of more precise, effective, and patient-centered healthcare is promised by further research and collaboration efforts between AI technology and the medical sector.

7. Conclusion

In conclusion, this study emphasizes the importance of language and the difficulties it presents when it comes to medical diagnosis and treatment. This study provides a possible method for enhancing healthcare decision-making by introducing Neutrosophic-Linguistic Valued Hypersoft Sets (N-LVHS), a potent tool that successfully regulates linguistic uncertainty and indeterminacy.

The necessary definitions, notions, aggregate operators, and algorithms has been proposed in this paper. The N-LVHS can be used as a crucial tool to solve the complexities of medical practice in a constantly changing healthcare environment where language-driven ambiguity and uncertainty prevail. This study contributes to the ongoing effort to provide healthcare that is more effective and patient-centered by addressing linguistic imprecision and indeterminacy. It emphasizes the significance of integrating cutting-edge linguistic and computational tools to improve healthcare practices in a complex and uncertain world. N-LVHS will need to be expanded to include a wider range of medical conditions in the future, and data scientists and healthcare professionals will need to work together to improve N-LVHS algorithms. The proposed study has the potential for a wide range of case study applications in numerous fields. It can be used in market research to understand customer attitude, environmental impact assessments to balance intricate ecological, social, and economic issues, and disaster preparedness to determine resource allocation and reaction plans.

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On S_θ-summability in neutrosophic-2-normed spaces

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Abstract. In the present paper, we aim to define S_θ-summability in neutrosophic 2-normed spaces and study some of its properties. We provide examples that shows our method of summability is stronger in these spaces. Finally we introduce S_θ-Cauchy and S_θ-completeness and prove that every neutrosophic-2-normed spaces is S_θ-complete.

Keywords: S_θ-convergence, S_θ-Cauchy, lacunary sequence, neutrosophic-2-normed spaces.

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1. **Introduction**

Statistical convergence was initially introduced by Fast [9] and later connected to summability theory by Schoenberg [12]. The concept was subsequently advanced by researchers such as Maddox [11], Connor [13], Fridy [14], Mursaleen and Edely [21], Šalát [31], and Kumar and Mursaleen [33], among numerous others.

Lacunary statistical convergence was studied by Fridy and Orhan [16] and was defined as follows: "By a lacunary sequence we mean an increasing integer sequence $\theta = (k_s)$ with $k_0 = 0$ and $h_s = k_s - k_{s-1} \to \infty$ as $s \to \infty$. The intervals determined by θ will be denoted by $I_s = (k_{s-1}, k_s]$ and the ratio $\frac{k_s}{k_{s-1}}$ will be abbreviated as q_s. For $\Re \subseteq \mathbb{N}$, the number $\delta_\theta(\Re) = \lim\limits_{s\to\infty} \dfrac{1}{h_s}|\{k \in I_s : k \in \Re\}|$ is called θ-density of $\Re$, provided the limit exists. A sequence $y = (y_k)$ is said to be lacunary statistically convergent (briefly S_θ-convergent) to y_0 if for each $\wp > 0$, $\lim\limits_s \frac{1}{h_s}|\{k \in I_s : |y_k - y_0| \geq \wp\}| = 0$ or equivalently, the set $\Re(\wp)$ has θ-density zero, where $\Re(\wp) = \{k \in \mathbb{N} : |y_k - y_0| \geq \wp\}$. In this case, we write $S_\theta - \lim\limits_{k\to\infty} y_k = y_0$." Additional noteworthy contributions to lacunary statistical convergence can be explored in references such as [7], [22], [26], and [35].

On the other hand, Zadeh [19] introduced the concept of fuzzy sets as a more suitable approach for addressing problems that cannot be adequately modeled using crisp set theory due to significant uncertainty in the data. Fuzzy set theory finds extensive applications in various scientific domains, including artificial intelligence, engineering, medicine, robotics, and numerous other fields, aiming to attain more effective solutions. Atanassov introduced intuitionistic fuzzy sets (IFS) in 1986 as an extension of fuzzy sets to better handle uncertainty. After introducing intuitionistic fuzzy sets, progressive developments were made in this field, as seen in [15], [27], etc.

Smarandache [35] proposed neutrosophic sets (NS) as another interesting generalization of fuzzy sets by introducing the indeterminacy function to intuitionistic fuzzy sets. Neutrosophic sets (NS) offer a more flexible and comprehensive way to represent uncertainty, imprecision, and indeterminacy in addressing the complexities of real-world situations. For ongoing development on neutrosophic sets (NS) and their applications, we refer to [1], [23], etc.

Kirişçi and Şimşek [20] established the concept of neutrosophic norm and investigated statistical convergence within the framework of neutrosophic normed spaces. For a comprehensive perspective in this direction, we recommend to the reader [2], [3], [4], [32], etc. Nowadays, the area of summability in these spaces is of much interest. Several summability approaches so far developed, including statistical convergence, ideal convergence, and lacunary statistical convergence in these spaces (see [5], [10], [18], [24], [25], [29], [34]). Recently in [30], the concept of neutrosophic-2-norm is introduced where the authors studied statistical convergence in neutrosophic-2-normed spaces. In the present work, we define a more general summability method, called S_θ-summability in $N-2-NS$ and develop some of its properties. We organize the paper as follows, the first and second sections are introductory and provide basic information needed in the sequel. In section 3, we define S_θ-summability in $N-2-NS$ and obtain interesting results. In section 4, we introduce S_θ-Cauchy and S_θ-completeness in $N-2-NS$. Finally, in the last section, we provide a brief conclusion regarding the whole work.

2. **Preliminaries**

This section commences with a concise overview of specific definitions and results needed in the sequel. In the course of this study, the notation $\mathbb{R}^+$ will be used to represent the open interval $(0,\infty)$, while $\mathbb{N}$ will represent the set of natural numbers.

Definition 2.1 [6] "Let $\Im=[0,1]$. A function $\circ:\Im\times\Im\to\Im$ is said to be a t-norm for all $\mu_1,\mu_2,\mu_3,\mu_4\in\Im$, we have

(i) $\mu_1\circ\mu_2=\mu_2\circ\mu_1$;

(ii)$\mu_1\circ(\mu_2\circ\mu_3)=(\mu_1\circ\mu_2)\circ\mu_3$;

(iii) $\circ$ is continuous;
(iv) $\mu_1 \circ 1 = \mu_1$ for every $\mu_1 \in \Im$ and
(v) $\mu_1 \circ \mu_2 \leq \mu_3 \circ \mu_4$ whenever $\mu_1 \leq \mu_3$ and $\mu_2 \leq \mu_4$".

Definition 2.2 [6] "Let $\Im = [0,1]$. A function $\diamond : \Im \times \Im \rightarrow \Im$ is said to be a continuous triangular conorm or t-conorm for all $\mu_1, \mu_2, \mu_3, \mu_4 \in \Im$, we have
(i) $\mu_1 \circ \mu_2 = \mu_2 \circ \mu_1$;
(ii)$\mu_1 \circ (\mu_2 \circ \mu_3) = (\mu_1 \circ \mu_2) \circ \mu_3$;
(iii) $\circ$ is continuous;
(iv) $\mu_1 \diamond 0 = \mu_1$ for every $\mu_1 \in \Im$ and
(v) $\mu_1 \circ \mu_2 \leq \mu_3 \circ \mu_4$ whenever $\mu_1 \leq \mu_3$ and $\mu_2 \leq \mu_4$".

We now recall the idea of two norm introduced in the paper [28].

Definition 2.3 [28]"Let X be a $d-$dimensional real vector space, where $2 \leq d < \infty$. A 2-norm on X is a function $\|.,.\| : X \times X \rightarrow \mathbb{R}$ fulfilling the below listed requirements: For all $\varrho_1, \varrho_2 \in X$, and scalar α, we have
(i) $||\varrho_1, \varrho_2|| = 0$ iff ϱ_1 and ϱ_2 are linearly dependent;
(ii) $||\varrho_1, \varrho_2|| = ||\varrho_2, \varrho_1||$;
(iii)$||\alpha\varrho_1, \varrho_2|| = |\alpha|||\varrho_1, \varrho_2||$ and
(iv) $||\varrho_1, \varrho_2 + \varrho_3|| \leq ||\varrho_1, \varrho_2|| + ||\varrho_1, \varrho_3||$.
The pair $(X, ||.,.||)$ is known as 2-normed space in this case.

Let $X = \mathbb{R}^2$ and for $\varrho_1 = (p_0, p'_0)$ and $\varrho_2 = (q_0, q'_0)$ we define $||\varrho_1, \varrho_2|| = |p_0 q'_0 - p'_0 q_0|$, then $||\varrho_1, \varrho_2||$ is a 2-norm on $X = \mathbb{R}^2$".

Recently, Murtaza et al. [30] defined neutrosophic 2-normed spaces as follows:

Definition 2.4 [30] "Let F is a vector space, $N_2 = (\{(\varrho_1, \varrho_2), G(\varrho_1, \varrho_2), B(\varrho_1, \varrho_2), Y(\varrho_1, \varrho_2)\} : (\varrho_1, \varrho_2) \in F \times F)$ be a 2-norm space s.t. $N_2 : F \times F \times \mathbb{R}^+ \rightarrow [0,1]$. If $\circ, \diamond$ respectively denotes t-norm and t-conorm, then the four-tuple $X = (F, N_2, \circ, \diamond)$ is known as neutrosophic 2-normed spaces (briefly $N-2-NS$) if for every $\varrho_1, \varrho_2, \omega \in X$, $\varsigma, \mu \geq 0$ and $\xi \neq 0$:
(i) $0 \leq G(\varrho_1, \varrho_2; \varsigma) \leq 1$, $0 \leq B(\varrho_1, \varrho_2; \varsigma) \leq 1$ and $0 \leq Y(\varrho_1, \varrho_2; \varsigma) \leq 1$ for every $\varsigma \in \mathbb{R}^+$;
(ii) $G(\varrho_1, \varrho_2; \varsigma) + B(\varrho_1, \varrho_2; \varsigma) + Y(\varrho_1, \varrho_2; \varsigma) \leq 3$;
(iii) $G(\varrho_1, \varrho_2; \varsigma) = 1$ iff ϱ_1, ϱ_2 are linearly dependent;
(iv) $G(\xi\varrho_1, \ \varrho_2; \ \varsigma) = G(\varrho_1, \varrho_2; \frac{\varsigma}{|\xi|})$ for each $\varsigma \neq 0$;
(v) $G(\varrho_1, \varrho_2; \varsigma) \circ G(\varrho_1, \omega; \mu) \leq G(\varrho_1, \varrho_2 + \omega; \varsigma + \mu)$;
(vi) $G(\varrho_1, \varrho_2; .) : (0, \infty) \rightarrow [0,1]$ is a non-decreasing function that runs continuously;
(vii) $\lim_{\varsigma \rightarrow \infty} G(\varrho_1, \varrho_2; \varsigma) = 1$;

(viii) $G(\varrho_1, \varrho_2; \varsigma) = G(\varrho_2, \varrho_1; \varsigma)$

(ix) $B(\varrho_1, \varrho_2; \varsigma) = 0$ iff ϱ_1, ϱ_2 are linearly dependent;

(x) $B(\xi\varrho_1, \varrho_2; \varsigma) = B(\varrho_1, \varrho_2; \frac{\varsigma}{|\xi|})$ for each $\varsigma \neq 0$;

(xi) $B(\varrho_1, \varrho_2; \varsigma) \diamond B(\varrho_1, \omega; \mu) \geq B(\varrho_1, \varrho_2 + \omega; \varsigma + \mu)$;

(xii) $B(\varrho_1, \varrho_2; .) : (0, \infty) \to [0, 1]$ is a non-increasing function that runs continuously;

(xiii) $\lim_{\varsigma\to\infty} B(\varrho_1, \varrho_2; \varsigma) = 0$;

(xiv) $B(\varrho_1, \varrho_2; \varsigma) = B(\varrho_2, \varrho_1; \varsigma)$;

(xv) $Y(\varrho_1, \varrho_2; \varsigma) = 0$ iff ϱ_1, ϱ_2 are linearly dependent;

(xvi)$Y(\xi\varrho_1,\ \varrho_2;\ \varsigma) = Y(\varrho_1,\ \varrho_2; \frac{\varsigma}{|\xi|})$ for each $\varsigma \neq 0$;

(xvii) $Y(\varrho_1, \varrho_2; \varsigma) \diamond Y(\varrho_1, \omega; \mu) \geq Y(\varrho_1, \varrho_2 + \omega; \varsigma + \mu)$;

(xviii) $Y(\varrho_1, \varrho_2; .) : (0, \infty) \to [0, 1]$ is a non-increasing function that runs continuously;

(xix) $\lim_{\lambda\to\infty} Y(\varrho_1, \varrho_2; \varsigma) = 0$;

(xx) $Y(\varrho_1, \varrho_2; \varsigma) = Y(\varrho_2, \varrho_1; \varsigma)$;

(xxi) if $\varsigma \leq 0$, then $G(\varrho_1, \varrho_2; \varsigma) = 0$, $B(\varrho_1, \varrho_2; \varsigma) = 1$, $Y(\varrho_1, \varrho_2; \varsigma) = 1$.

In this case, we call $N_2 = N_2(G, B, Y)$, a neutrosophic 2-norm on F . From now on wards, unless otherwise stated by X we shall denote the $N - 2 - NS$ $(F, N_2, \circ, \diamond)$.

A sequence (y_k) in X is said to be convergent to $y_0 \in X$ if for each $0 < \wp < 1$ and $\varsigma > 0$, $\exists$ a positive integer m s.t. $G(y_k - y_0, \omega; \varsigma) > 1 - \wp$, $B(y_k - y_0, \omega; \varsigma) < \wp$ and $Y(y_k - y_0, \omega; \varsigma) < \wp$ for all $k \geq m$ and $\omega \in X$ which is equivalently to say $\lim_{k\to\infty} G(y_k - y_0, \omega; \varsigma) = 1$, $\lim_{k\to\infty} B(y_k - y_0, \omega; \varsigma) = 0$ and $\lim_{k\to\infty} Y(y_k - y_0, \omega; \varsigma) = 0$. In this case, we write $N_2 - \lim_{k\to\infty} y_k = y_0$.

A sequence (y_k) in X is said to be Cauchy if for each $0 < \wp < 1$ and $\varsigma > 0$, $\exists$ a positive integer m s.t. $G(y_k - y_n, \omega; \varsigma) > 1 - \wp$, $B(y_k - y_n, \omega; \varsigma) < \wp$ and $Y(y_k - y_n, \omega; \varsigma) < \wp$ $\forall$ $k, n \geq m$ and $\forall$ $\omega \in X$."

3. Lacunary statistical Convergence in $N - 2 - NS$

Definition 3.1 A sequence $y = (y_k)$ in X is called lacunary statistical convergent (or S_θ-convergent) to y_0 w.r.t neutrosophic 2-norm N_2, if for each $\wp > 0$ and $\varsigma > 0$

$$\lim_{s\to\infty} \frac{1}{h_s} \left| \Big\{ k \in I_s : G(y_k - y_0, \omega; \varsigma) \leq 1 - \wp \text{ or } B(y_k - y_0, \omega; \varsigma) \geq \wp,\ Y(y_k - y_0, \omega; \varsigma) \geq \wp \Big\} \right| = 0 \text{ for every } \omega \in X;$$

or, $\delta_\theta\left(\mathfrak{A}\left(\wp, \varsigma\right)\right) = 0$, where

$$\mathfrak{A}(\wp, \varsigma) = \{k \in I_s : G(y_k - y_0, \omega; \varsigma) \leq 1 - \wp \text{ or } B(y_k - y_0, \omega; \varsigma) \geq \wp,\ Y(y_k - y_0, \omega; \varsigma) \geq \wp\}.$$

In present case, we denote $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_0$.

We now give the following Lemma and prove the uniqueness theorem.

Lemma 3.1 $y = (y_k)$ in X, the subsequent assertions are equivalent:

(i) $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_0$;

(ii) $\delta_\theta\{k \in I_s : G(y_k - y_0, \omega; \varsigma) \leq 1 - \wp\} = \delta_\theta\{k \in I_s : B(y_k - y_0, \omega; \varsigma) \geq \wp\} = \delta_\theta\{k \in I_s : Y(y_k - y_0, \omega; \varsigma) \geq \wp\} = 0$;

(iii) $\delta_\theta\{k \in I_s : G(y_k - y_0, \omega; \varsigma) > 1 - \wp$ and $B(y_k - y_0, \omega; \varsigma) < \wp$, $Y(y_k - y_0, \omega; \varsigma) < \wp\} = 1$;

(iv) $\delta_\theta\{k \in I_s : G(y_k - y_0, \omega; \varsigma) > 1 - \wp\} = \delta_\theta\{k \in I_s : B(y_k - y_0, \omega; \varsigma) < \wp\} = \delta_\theta\{k \in I_s : Y(y_k - y_0, \omega; \varsigma) < \wp\} = 1$ and

(v) $S_\theta(N_2) - \lim_{k\to\infty} G(y_k - y_0, \omega; \varsigma) = 1,\ S_\theta(N_2) - \lim_{k\to\infty} B(y_k - y_0, \omega; \varsigma) = S_\theta(N_2) - \lim_{k\to\infty} Y(y_k - y_0, \omega; \varsigma) = 0$.

Theorem 3.1 For any sequence $y = (y_k)$ in X, if $S_\theta(N_2) - \lim_{k\to\infty} y_k$ exists, then it is unique.

Proof. Suppose that $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_1$ and $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_2$. For given $\wp > 0$, choose $\nu > 0$ s.t.

$$(1 - \nu) \circ (1 - \nu) > 1 - \wp \text{ and } \nu \diamond \nu < \wp. \tag{1}$$

For any $\varsigma > 0$ and any $w \in X$. Define the following sets:

$$K_{G,1}(\nu, \varsigma) = \{k \in I_s : G(y_k - y_1, \omega; \frac{\varsigma}{2}) \leq 1 - \nu\},$$
$$K_{G,2}(\nu, \varsigma) = \{k \in I_s : G(y_k - y_2, \omega; \frac{\varsigma}{2}) \leq 1 - \nu\};$$
$$K_{B,1}(\nu, \varsigma) = \{k \in I_s : B(y_k - y_1, \omega; \frac{\varsigma}{2}) \geq \nu\},$$
$$K_{B,2}(\nu, \varsigma) = \{k \in I_s : B(y_k - y_2, \omega; \frac{\varsigma}{2}) \geq \nu\};$$
$$K_{Y,1}(\nu, \varsigma) = \{k \in I_s : \mathcal{Y}(y_k - y_1, \omega; \frac{\varsigma}{2}) \geq \nu\};$$
$$K_{Y,2}(\nu, \varsigma) = \{k \in I_s : Y(y_k - y_2, \omega; \frac{\varsigma}{2}) \geq \nu\}.$$

Since $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_1$, so by lemma 3.1, we get $\delta_\theta\{K_{G,1}(\nu, \varsigma)\} = \delta_\theta\{K_{B,1}(\nu, \varsigma)\} = \delta_\theta\{K_{Y,1}(\nu, \varsigma)\} = 0$ and therefore $\delta_\theta\{K^C_{G,1}(\nu, \varsigma)\} = \delta_\theta\{K^C_{B,1}(\nu, \varsigma)\} = \delta_\theta\{K^C_{Y,1}(\nu, \varsigma)\} = 1$. Furthermore, using $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_2$, we get, $\delta_\theta\{K_{G,2}(\nu, \varsigma)\} = \delta_\theta\{K_{B,2}(\nu, \varsigma)\} = \delta_\theta\{K_{Y,2}(\nu, \varsigma)\} = 0$ and therefore $\delta_\theta\{K^C_{G,2}(\nu, \varsigma)\} = \delta_\theta\{K^C_{B,2}(\nu, \varsigma)\} = \delta_\theta\{K^C_{Y,2}(\nu, \varsigma)\} = 1$. Now define $K_{G,B,Y}(\wp, \varsigma) = \{K_{G,1}(\nu, \varsigma) \cup K_{G,2}(\nu, \varsigma)\} \cap \{K_{B,1}(\nu, \varsigma) \cup K_{B,2}(\nu, \varsigma)\} \cap \{K_{Y,1}(\nu, \varsigma) \cup K_{Y,2}(\nu, \varsigma)\}$. Then $\delta_\theta(\{K_{G,B,Y}(\wp, \varsigma)\}) = 0$ which implies $\delta(\{K^C_{G,B,Y}(\wp, \varsigma)\}) = 1$. Let $m \in K^C_{G,B,Y}(\wp, \varsigma)$, then we have

Case 1. $m \in \{K_{G,1}(\nu, \varsigma) \cup K_{G,2}(\nu, \varsigma)\}^C$,

Case 2. $m \in \{K_{B,1}(\nu,\varsigma) \cup K_{B,2}(\nu,\varsigma)\}^C$,

Case 3. $m \in \{K_{Y,1}(\nu,\varsigma) \cup K_{Y,2}(\nu,\varsigma)\}^C$.

Case 1: Let, $m \in \{K_{G,1}(\nu,\varsigma) \cup K_{G,2}(\nu,\varsigma)\}^C$, then $m \in K^C_{G,1}(\nu,\varsigma)$ and $m \in K^C_{G,2}(\nu,\varsigma)$. Therefore, for any $\omega \in X$ we have

$$G(y_m - y_1, \omega; \frac{\varsigma}{2}) > 1-\nu \text{ and } G(y_m - y_2, \omega; \frac{\varsigma}{2}) > 1-\nu. \tag{2}$$

Now

$$\begin{aligned} G(y_1 - y_2, \omega; \varsigma) &\geq G(y_m - y_1, \omega, ; \frac{\varsigma}{2}) \circ G(y_m - y_2, \omega; \frac{\varsigma}{2}) \\ &> (1-\nu) \circ (1-\nu) \text{ by}(2) \\ &> 1 - \wp. \text{ by } (1) \end{aligned}$$

Since $\wp > 0$ is arbitrary, so we have $G(y_1 - y_2, \omega; \varsigma) = 1 \; \forall \varsigma > 0$, and therefore $y_1 - y_2 = 0$. This shows that $y_1 = y_2$.

Case 2: Let $m \in \{K_{B,1}(\nu,\varsigma) \cup K_{B,2}(l,\varsigma)\}^C$, then $m \in K^C_{B,1}(\nu,\varsigma)$ and $m \in K^C_{B,2}(\nu,\varsigma)$. Therefore, for $\omega \in X$ we have

$$B(y_m - y_1, \omega; \frac{\varsigma}{2}) < \nu \text{ and } B(y_m - y_2, \omega; \frac{\varsigma}{2}) < \nu. \tag{3}$$

Now

$$\begin{aligned} B(y_1 - y_2, \omega; \varsigma) &\leq B(y_m - y_1, \omega, ; \frac{\varsigma}{2}) \diamond B(y_m - y_2, \omega; \frac{\varsigma}{2}) \\ &< \nu \diamond \nu \text{ by } (3) \\ &< \wp. \text{ by } (1) \end{aligned}$$

Since $\wp > 0$ is arbitrary, so we have $B(y_1 - y_2, \omega; \varsigma) = 0 \; \forall \varsigma > 0$, and therefore $y_1 - y_2 = 0$. This shows that $y_1 = y_2$.

Case 3: Let $m \in \{K_{Y,1}(\nu,\varsigma) \cup K_{Y,2}(\nu,\varsigma)\}^C$, then $m \in K^C_{Y,1}(\nu,\varsigma)$ and $m \in K^C_{Y,2}(\nu,\varsigma)$. Therefore, for $\omega \in X$ we have

$$Y(y_m - y_1, \omega; \frac{\varsigma}{2}) < \nu \text{ and } Y(y_m - y_2, \omega; \frac{\varsigma}{2}) < \nu. \tag{4}$$

Now

$$\begin{aligned} Y(y_1 - y_2, \omega; \varsigma) &\leq Y(y_m - y_1, \omega; \frac{\varsigma}{2}) \diamond Y(y_m - y_2, \omega; \frac{\varsigma}{2}) \\ &< \nu \diamond \nu \text{ by } (4) \\ &< \wp. \text{ by } (1) \end{aligned}$$

Since $\wp > 0$ is arbitrary, so we have $Y(y_1 - y_2, \omega; \varsigma) = 0 \; \forall \varsigma > 0$, and therefore $y_1 - y_2 = 0$. This shows that $y_1 = y_2$.

Hence in all cases, we get $y_1 = y_2$. □

Theorem 3.2 Let $y = (y_k)$ be any sequence in X. If $N_2 - \lim_{k\to\infty} y_k = y_0$, then $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_0$.

Proof Let $N_2 - \lim_{k\to\infty} y_k = y_0$. Then for every $\wp > 0$ and $\varsigma > 0, \exists$ an integer $k_0 \in \mathbb{N}$ s.t. $G(y_k - y_0, \omega; \varsigma) > 1 - \wp$ and $B(y_k - y_0, \omega; \varsigma) < \wp$, $Y(y_k - y_0, \omega; \varsigma) < \wp$ $\forall k \geq k_0$ and every $\omega \in X$. Hence, the set $\{k \in I_s : G(y_k - y_0, \omega; \varsigma) \leq 1 - \wp$ or $B(y_k - y_0, \omega; \varsigma) \geq \wp$, $Y(y_k - y_0, \omega; \varsigma) \geq \wp\}$ has a finitely many terms whose θ-density is zero. Therefore, $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_0$. □

But the converse of the above theorem is not true in general.

Example 3.1 Let $(\mathbb{R}^2, |.|)$ be 2-normed space. For $\tau_1, \tau_2 \in [0, 1]$. Let $\tau_1 \circ \tau_2 = \tau_1\tau_2$ and $\tau_1 \diamond \tau_2 = \min\{\tau_1 + \tau_2, 1\}$. Choose $(\varrho_1, \varrho_2) \in \mathbb{R}^2$ and $\varsigma > 0$ with $\varsigma > \|\varrho_1, \varrho_2\|$. Define $G(\varrho_1, \varrho_2; \varsigma) = \frac{\varsigma}{\varsigma + \|\varrho_1, \varrho_2\|}$, $B(\varrho_1, \varrho_2; \varsigma) = \frac{\|\varrho_1, \varrho_2\|}{\varsigma + \|\varrho_1, \varrho_2\|}$ and $Y(\varrho_1, \varrho_2; \varsigma) = \frac{\|\varrho_1, \varrho_2\|}{\varsigma}$, then it is easy to see that $X = (\mathbb{R}^2, N_2, \circ, \diamond)$ is a $N - 2 - NS$. Define a sequence $y = (y_k)$ by

$$y_k = \begin{cases} (k, 0) \text{ if } k_s - [\sqrt{h_s}] + 1 \leq k \leq k_s, s \in \mathbb{N} \\ (0, 0) \text{ otherwise.} \end{cases}$$

Now, for each $\wp > 0$ and $\varsigma > 0$, let

$$\begin{aligned} \mathfrak{A}(\wp, \varsigma) = \Big\{ & k \in I_s : G(y_k - 0, \omega; \varsigma) \leq 1 - \wp \text{ or} \\ & B(y_k - 0, \omega; \varsigma) \geq \wp,\ Y(y_k - 0, \omega; \varsigma) \geq \wp \Big\} \\ = & \left\{ k \in I_s : \frac{\varsigma}{\varsigma + \|y_k, \omega\|} \leq 1 - \wp \text{ or } \frac{\|y_k, \omega\|}{\varsigma + \|y_k, \omega\|} \geq \wp,\ \frac{\|y_k, \omega\|}{\varsigma} \geq \wp \right\} \\ = & \left\{ k \in I_s : \|y_k, \omega\| \geq \frac{\varsigma\wp}{1 - \wp} \text{ or } \|y_k, \omega\| \geq \varsigma\wp \right\} \\ = & \{k \in I_s : k_s - [\sqrt{h_s}] + 1 \leq k \leq k_s; s \in \mathbb{N}\} \end{aligned}$$

and so we get

$$\frac{1}{h_s}|\mathfrak{A}(\wp, \varsigma)| \leq \frac{1}{h_s}|\{k \in I_s : k_s - [\sqrt{h_s}] + 1 \leq k \leq k_s; s \in \mathbb{N}\}| \leq \frac{[\sqrt{h_s}]}{h_s}.$$

Taking $s \to \infty$,

$$\lim_{s\to\infty} \frac{1}{h_s}|\mathfrak{A}(\wp, \varsigma)| \leq \lim_{n\to\infty} \frac{[\sqrt{h_s}]}{h_s} = 0;$$

i.e., $\delta_\theta(\mathfrak{A}(\wp, \varsigma)) = 0$. Hence, $y = (y_k)$ is S_θ-convergent to 0. But the sequence $y = (y_k)$ is not N_2-convergent to 0.

Theorem 3.3 Let $y = (y_k)$ and $z = (z_k)$ be any two sequences in X s.t $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_1$

and $S_\theta(N_2) - \lim_{k\to\infty} z_k = z_1$, then
(i) $S_\theta(N_2) - \lim_{k\to\infty} (y_k + z_k) = y_1 + z_1$ and
(ii) $S_\theta(N_2) - \lim_{k\to\infty} (cy_k) = cy_1$, where $0 \neq c \in F$.

Proof. The proof of this theorem can be derived in a manner similar to the proof of theorem 3.1 and is therefore omitted. □

We now have the following interesting implication.

Theorem 3.4 A sequence $y = (y_k)$ in X is $S_\theta(N_2)$-convergent to y_0 iff $\exists$ a subset $\Re = \{k_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ with $\delta_\theta(\Re) = 1$ and $N_2 - \lim_{n\to\infty} y_{k_n} = y_0$.

Proof. Assume that $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_0$. For any $\varsigma > 0$, $l \in \mathbb{N}$ and $\omega \in X$, define the set

$$\Re_{N_2}(l,\varsigma) = \{k \in I_s : G(y_k - y_0, \omega; \varsigma) > 1 - \frac{1}{l} \text{ and } B(y_k - y_0, \omega; \varsigma) < \frac{1}{l}, Y(y_k - y_0, \omega; \varsigma) < \frac{1}{l}\}.$$

Since $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_0$, it is clear that for $\varsigma > 0$ and $l \in \mathbb{N}$, $\Re_{N_2}(l+1,\varsigma) \subset \Re_{N_2}(l,\varsigma)$ and

$$\delta_\theta(\Re_{N_2}(l,\varsigma)) = 1. \quad (5)$$

Let r_1 be an arbitrary number in $\Re_{N_2}(1,\varsigma)$. Then, $\exists\ r_2 \in \Re_{N_2}(2,\varsigma), (r_2 > r_1)$, s.t $\forall\ n \geq r_2$, $\frac{1}{h_s}|\{k \in I_s : G(y_k - y_0, \omega; \varsigma) > 1 - \frac{1}{2}$ and $B(y_k - y_0, \omega; \varsigma) < \frac{1}{2}$, $Y(y_k - y_0, \omega; \varsigma) < \frac{1}{2}\}| > \frac{1}{2}$. Similarly, $\exists\ r_3 \in \Re_{N_2}(3,\varsigma), (r_3 > r_2)$, such that for all $n \geq r_3$, $\frac{1}{h_s}|\{k \in I_s : G(y_k - y_0, \omega; \varsigma) > 1 - \frac{1}{3}$ and $B(y_k - y_0, \omega; \varsigma) < \frac{1}{3}$, $Y(y_k - y_0, \omega; \varsigma) < \frac{1}{3}\}| > \frac{2}{3}$ and so on. So we can establish a sequence $\{r_l\}_{l\in\mathbb{N}}$ satisfying $r_l \in \Re_{N_2}(l,\varsigma)$. For all $n \geq r_l (l \in \mathbb{N})$, we have $\frac{1}{h_s}|\{k \in I_s : G(y_k - y_0, \omega; \varsigma) > 1 - \frac{1}{l}$ and $B(y_k - y_0, \omega; \varsigma) < \frac{1}{l}$, $Y(y_k - y_0, \omega; \varsigma) < \frac{1}{l}\}| > \frac{l-1}{l}$.

Define $\Re = \{n \in \mathbb{N} : 1 < n < r_1\} \cup \{\bigcup_{l\in\mathbb{N}}\{n \in \Re_{N_2}(l,\varsigma) : r_l \leq n < r_{l+1}\}\}$, Then for $r_l \leq n < r_l + 1$, we have $\frac{1}{h_s}|\{k \in I_s : k \in \Re\}| \geq \frac{1}{h_s}|\{k \in I_s : G(y_k - y_0, \omega; \varsigma) > 1 - \frac{1}{l}$ and $B(y_k - y_0, \omega; \varsigma) < \frac{1}{l}$, $Y(y_k - y_0, \omega; \varsigma) < \frac{1}{l}\}| > \frac{l-1}{l}$ and hence $\delta_\theta(\Re) = 1$ as $k \to \infty$. Now we have to demonstrate that $N_2 - \lim_{n\to\infty} \mathfrak{u}_{k_n} = \mathfrak{u}_0$. Let $\wp > 0$ and select $l \in \mathbb{N}$ with $\frac{1}{l} < \wp$. Furthermore, let $n \geq r_l$ and $n \in \Re$. Then, by definition of $\Re$, $\exists\ n_0 \geq l$ s.t, $r_{n_0} \leq n < r_{n_0+1}$ and $n \in \Re_{N_2}(l,\varsigma)$. Thus, for each $\wp > 0$, and for $\omega \in X$ we have $G(y_n - y_0, \omega; \varsigma) > 1 - \frac{1}{l} > 1 - \wp$ and $B(y_n - y_0, \omega; \varsigma) < \frac{1}{l} < \wp$, $Y(y_n - y_0, \omega; \varsigma) < \frac{1}{l} < \wp\ \forall\ n \geq r_l$ and $n \in \Re$. Hence $N_2 - \lim_{n\to\infty} y_{k_n} = y_0$.

Conversely, suppose that $\exists$ a subset $\Re = \{k_n\}_{n\in\mathbb{N}}$ of $\mathbb{N}$ with $\delta_\theta\{\Re\} = 1$ and $N_2 - \lim_{n\in\Re} y_{k_n} = y_0$. Let $\wp > 0$ and $\varsigma > 0$ $\exists\ k_{n_0} \in \mathbb{N}$ s.t $G(y_{k_n} - y_0, \omega; \varsigma) > 1 - \wp$ and $B(y_{k_n} - y_0, \omega; \varsigma) < \wp$, $Y(y_{k_n} - y_0, \omega; \varsigma) < \wp$ for each $k_n \geq k_{n_0}$ and $\omega \in X$. This implies $\mathfrak{T}_{N_2}(\wp,\varsigma) = \{k \in I_s : G(y_{k_n} - y_0, \omega; \varsigma) \leq 1 - \wp$ and $B(y_{k_n} - y_0, \omega; \varsigma) \geq \wp$, $Y(y_{k_n} - y_0, \omega; \varsigma) \geq \wp\}$

$\subseteq \mathbb{N} - \{k_{n_0}, k_{n_0+1}, k_{n_0+2}, ...\}$ and therefore $\delta_\theta\{\mathfrak{T}_{N_2}(\wp,\varsigma)\} \leq \delta_\theta(\mathbb{N}) - \delta_\theta(\{k_{n_0}, k_{n_0+1}, k_{n_0+2}, ...\})$. As $\delta_\theta\{\Re\} = 1$, so $\delta_\theta\{\mathfrak{T}_{N_2}(\wp,\varsigma)\} = 0$. This shows that $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_0$ and therefore the completes proof of the theorem. □

"For $\upsilon \in X, \varsigma > 0, \alpha \in (0,1)$ and $\omega \in X$, the ball centered at υ with radius α is denoted and defined by $H(\upsilon,\alpha,\varsigma) = \{u \in X : G(\upsilon-u,\omega,\varsigma) > 1-\alpha$ and $B(\upsilon-u,\omega,\varsigma) < \alpha,\ Y(\upsilon-u,\omega,\varsigma) < \alpha\}$."

Theorem 3.5 Let X be a $N-2-NS$. For any lacunary sequence $\theta = (k_s), S_\theta(N_2) \subseteq S(N_2)$ iff $\lim_s \sup q_s < \infty$.

Proof. If $\lim_s \sup q_s < \infty$, then $\exists\ M > 0$ s.t $q_s < M\ \forall\ s$. Suppose that $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_0$ and for $\varsigma > 0, \alpha \in (0,1), \omega \in X$, let

$$T_s = \Big|\Big\{k \in I_s : G(y_k - y_0,\omega;\varsigma) \leq 1-\alpha \text{ or } B(y_k - y_0,\omega;\varsigma) \geq \alpha,\ Y(y_k - y_0,\omega;\varsigma) \geq \alpha\Big\}\Big|.$$

Let $\wp > 0$. Since $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_0$, then $\exists\ s_0 \in \mathbb{N}$ s.t

$$\frac{T_s}{h_s} < \wp\ \forall\ s > s_0. \tag{6}$$

Now, Let $C = \max\{T_s : 1 \leq s \leq s_0\}$ and r be an integer such that $k_{s-1} < r < k_s$. Then we write

$$\begin{aligned}
&\frac{1}{r}|\{k \leq r : G(y_k - y_0,\omega;\varsigma) \leq 1-\alpha \text{ or } B(y_k - y_0,\omega;\varsigma) \geq \alpha,\ Y(y_k - y_0,\omega;\varsigma) \geq \alpha\}| \\
&\leq \frac{1}{k_{s-1}}|\{k \leq k_s : G(y_k - y_0,\omega;\varsigma) \leq 1-\alpha \text{ or } B(y_k - y_0,\omega;\varsigma) \geq \alpha,\ Y(y_k - y_0,\omega;\varsigma) \geq \alpha\}| \\
&= \frac{1}{k_{s-1}}\{T_1 + T_2 + \ldots + T_{s_0} + T_{s_0+1} + \ldots + T_s\} \\
&\leq \frac{C}{k_{s-1}}s_0 + \frac{1}{k_{s-1}}\left\{h_{s_0+1}\frac{T_{s_0+1}}{h_{s_0+1}} + \ldots + h_s\frac{T_s}{h_s}\right\} \\
&\leq \frac{s_0 C}{k_{s-1}} + \frac{1}{k_{s-1}}\left(\sup_{s>s_0}\frac{T_s}{h_s}\right)\{h_{s_0+1} + \ldots + h_s\} \\
&\leq \frac{s_0 C}{k_{s-1}} + \wp\frac{k_s - k_{s_0}}{k_{s-1}} \quad \text{by (6)} \\
&\leq \frac{s_0 C}{k_{s-1}} + \wp q_s \\
&\leq \frac{s_0 C}{k_{s-1}} + \wp M.
\end{aligned}$$

To prove the converse, assume that $\lim_s \sup q_s = \infty$. Let $\beta(\neq 0) \in X$. By applying the concept from [5], we can obtain a subsequence $(k_{s(l)})$ of $\theta = (k_s)$ s.t $q_{s(l)} > l$. Define a sequence $y = (y_k)$ by

$$y_k = \begin{cases} \beta \text{ if } k_{s(l)-1} < k \leq 2k_{s(l)-1} \text{for some } l = 1, 2, 3, \ldots \\ 0 \text{ otherwise.} \end{cases}$$

Since $\beta(\neq 0)$, so we can select $\varsigma > 0, \alpha \in (0,1)$ and $\omega \in X$ s.t $\beta \notin H(0, \alpha, \varsigma)$. Now for $l > 1$,

$$\begin{aligned} \frac{1}{h_{s(l)}} |\{k \leq k_{s(l)} : G(y_k, \omega; \varsigma) \leq 1 - \alpha \text{ or} \\ B(y_k, \omega; \varsigma) \geq \alpha,\ Y(y_k, \omega; \varsigma) \geq \alpha\}| \\ \leq \frac{1}{h_{s(l)}} \big(k_{s(l)-1}\big) \\ = \frac{1}{k_{s(l)} - k_{s(l)-1}} \big(k_{s(l)-1}\big) \\ < \frac{1}{l-1}. \end{aligned}$$

Thus, we have $y \in S_\theta(N_2)$. But $y \notin S(N_2)$. For

$$\begin{aligned} \frac{1}{2k_{s(l)-1}} |\{k \leq 2k_{s(l)-1} : G(y_k, \omega; \varsigma) \leq 1 - \alpha \text{ or} \\ B(y_k, \omega; \varsigma) \geq \alpha,\ Y(y_k, \omega; \varsigma) \geq \alpha\}| \\ \geq \frac{1}{2k_{s(l)-1}} \{k_{s(1)-1} + k_{s(2)-1} + \ldots + k_{s(l)-1}\} \\ > \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{k_{s(l)}} |\{k \leq k_{s(l)} : G(y_k - \beta, \omega; \varsigma) \leq 1 - \alpha \text{ or} \\ B(y_k - \beta, \omega; \varsigma) \geq \alpha,\ Y(y_k - \beta, \omega; \varsigma) \geq \alpha\}| \\ \geq \frac{k_{s(l)} - 2k_{s(l)-1}}{k_{s(l)}} \\ \geq 1 - \frac{2}{l}. \end{aligned}$$

This shows that $y = (y_k)$ is not S-convergent w.r.t N_2. □

Theorem 3.6 Let X be a $N-2-NS$. For any lacunary sequence $\theta = (k_s), S(N_2) \subseteq S_\theta(N_2)$ iff $\lim_s \inf q_s > 1$.

Proof. Assume that $\lim_s \inf q_s > 1$, then $\exists\ \eta > 0$ s.t $q_s \geq 1 + \eta$ for sufficiently large s, which implies that

$$\frac{h_s}{k_s} \geq \frac{\eta}{1+\eta}.$$

If $y = (y_k)$ is S-convergent to y_0 w.r.t N_2, then for each $\varsigma > 0, \alpha \in (0,1), \omega \in X$ and sufficiently large s, we have

$$\frac{1}{k_s}|\{k \leq k_s : G(y_k - y_0, \omega; \varsigma) \leq 1-\alpha \text{ or } B(y_k - y_0, \omega; \varsigma) \geq \alpha,\ Y(y_k - y_0, \omega; \varsigma) \geq \alpha\}|$$

$$\geq \frac{1}{k_s}|\{k \in I_s : G(y_k - y_0, \omega; \varsigma) \leq 1-\alpha \text{ or } B(y_k - y_0, \omega; \varsigma) \geq \alpha,\ Y(y_k - y_0, \omega; \varsigma) \geq \alpha\}|$$
$$\geq \frac{\eta}{1+\eta}\frac{1}{h_s}|\{k \in I_s : G(y_k - y_0, \omega; \varsigma) \leq 1-\alpha \text{ or } B(y_k - y_0, \omega; \varsigma) \geq \alpha,\ Y(y_k - y_0, \omega; \varsigma) \geq \alpha\}|.$$

Since $y = (y_k) \in S(N_2)$, it follows that $S_\theta(N_2) - \lim_{k\to\infty} y_k = y_0$.
To prove the converse, assume that $\liminf_s q_s = 1$. Applying the concept from [5], we can obtain a subsequence $(k_{s(l)})$ of $\theta = (k_s)$ s.t

$$\frac{k_{s(l)}}{k_{s(l)-1}} < 1 + \frac{1}{l} \text{ and } \frac{k_{s(l)} - 1}{k_{s(l-1)}} > l \text{ where } s(l) \geq s(l-1) + 2.$$

Let $\beta(\neq 0) \in X$. Define a sequence $y = (y_k)$ by

$$y_k = \begin{cases} \beta \text{ if } k \in I_{s(l)} \text{for some } l = 1,2,3,\ldots \\ 0 \text{ otherwise.} \end{cases}$$

We now show that $y = (y_k)$ is S-convergent to 0 w.r.t N_2. Let $\varsigma > 0, \alpha \in (0,1)$ and $\omega \in X$. Choose $\varsigma_1 > 0$ and $\alpha_1 \in (0,1)$ such that for previously chosen $\omega \in X$, $H(0, \alpha_1, \varsigma_1) \subset H(0, \alpha, \varsigma)$ and $\beta \notin H(0, \alpha_1, \varsigma_1)$. Also for each $r \in \mathbb{N}$, we can find $l_r > 0$ s.t $k_{s(l_r)-1} < r \leq k_{s(l_r)}$. Then for each $r \in \mathbb{N}$, we have

$$\frac{1}{r}|\{k \leq r : G(y_k, \omega; \varsigma) \leq 1 - \alpha \text{ or } B(y_k, \omega; \varsigma) \geq \alpha,\ Y(y_k, \omega; \varsigma) \geq \alpha\}|$$
$$\leq \frac{1}{k_{s(l_r)-1}}|\{k \leq r : G(y_k, \omega; \varsigma_1) \leq 1 - \alpha_1 \text{ or } B(y_k, \omega; \varsigma_1) \geq \alpha_1,\ Y(y_k, \omega; \varsigma_1) \geq \alpha_1\}|$$
$$\leq \frac{1}{k_{s(l_r)-1}}\big\{|\{k \leq k_{s(l_r)} : G(y_k, \omega; \varsigma_1) \leq 1 - \alpha_1 \text{ or } B(y_k, \omega; \varsigma_1) \geq \alpha_1,\ Y(y_k, \omega; \varsigma_1) \geq \alpha_1\}|$$
$$+|\{k_{s(l_r)-1} < k \leq r : G(y_k, \omega; \varsigma_1) \leq 1 - \alpha_1 \text{ or } B(y_k, \omega; \varsigma_1) \geq \alpha_1,\ Y(y_k, \omega; \varsigma_1) \geq \alpha_1\}|\big\}$$
$$\leq \frac{k_{s(l_r-1)}}{k_{s(l_r)-1}} + \frac{1}{k_{s(l_r)-1}}(k_{s(l_r)} - k_{s(l_r)-1})$$
$$< \frac{1}{l_r} + 1 + \frac{1}{l_r} - 1$$
$$= \frac{2}{l_r}.$$

It follows that $y = (y_k)$ is S-convergent to 0. Now we will prove that $y = (y_k)$ is not S_θ-convergent w.r.t N_2. Since $\beta \neq 0$, so we can select $\varsigma > 0, \alpha \in (0,1)$ and $\omega \in X$ s.t $\beta \notin H(0, \varsigma, \alpha)$. Thus

$$\lim_{l\to\infty} \frac{1}{h_{s(l)}}|\{k \in I_{s(l)} : G(y_k, \omega; \varsigma) \leq 1 - \alpha \text{ or } B(y_k, \omega; \varsigma) \geq \alpha,\ Y(y_k, \omega; \varsigma) \geq \alpha\}|$$
$$= \lim_{l\to\infty} \frac{1}{h_{s(l)}}(k_{s(l)} - k_{s(l)-1})$$
$$= \lim_{l\to\infty} \frac{1}{h_{s(l)}}(h_{s(l)})$$
$$= 1,$$

and for $s \neq s_l$,

$$\lim_{l\to\infty} \frac{1}{h_s}|\{k \in I_s : G(y_k - \beta, \omega; \varsigma) \leq 1 - \alpha \text{ or } B(y_k - \beta, \omega; \varsigma) \geq \alpha,\ Y(y_k - \beta, \omega; \varsigma) \geq \alpha\}| = 1 \neq 0.$$

Hence neither β nor 0 can be the S_θ-limit of the sequence $y = (y_k)$ w.r.t N_2. Furthermore, there is no other element in X that can be the S_θ-limit of y. Therefore $y \notin S_\theta(N_2)$. □

Theorems 3.5 and 3.6 together give the following result.

Theorem 3.7 Let X be a $N-2-NS$. For any lacunary sequence $\theta=(k_s), S(N_2)=S_\theta(N_2)$ iff $1<\lim_s \inf q_s \leq \lim_s \sup q_s<\infty$.

4. Lacunary statistical completeness in $N-2-NS$

Definition 4.1 A sequence $y=(y_k)$ in X is called lacunary statistically Cauchy (or S_θ-Cauchy) w.r.t neutrosophic 2-norm N_2 if for each $\wp>0$ and $\varsigma>0$, $\exists\ r\ \in\ \mathbb{N}$ s.t.

$$\lim_{s\to\infty}\frac{1}{h_s}\Bigg|\Bigg\{k\in I_s : G(y_k-y_r,\omega;\varsigma)\leq 1-\wp \text{ or } B(y_k-y_r,\omega;\varsigma)\geq\wp,\ Y(y_k-y_r,\omega;\varsigma)\geq\wp\Bigg\}\Bigg|=0\ \forall\ \omega\in X$$

or $\delta(\mathfrak{A}(\wp,\varsigma))=0$ where

$$\mathfrak{A}(\wp,\varsigma)=\{k\in I_s:\ G(y_k-y_r,\omega;\varsigma)\leq 1-\wp \text{ or } B(y_k-y_r,\omega;\varsigma)\geq\wp,\ Y\ (y_k-y_r,\omega;\varsigma)\geq\wp\}.$$

Theorem 4.1 Every $S_\theta(N_2)$-convergent sequence in X is $S_\theta(N_2)$-Cauchy.

Proof. Let $y=(y_k)$ be the S_θ-convergent sequence to y_0. Let $\wp>0$ and $\varsigma>0$. Select $\nu>0$ s.t. (1) is satisfied. Define

$$\mathfrak{A}(\nu,\varsigma)=\{k\in I_s:\ G(y_k-y_0,\omega;\ \frac{\varsigma}{2})\leq 1-\nu \text{ or } B(y_k-y_0,\omega;\frac{\varsigma}{2})\geq\nu\ \ Y(y_k-y_0,\omega;\frac{\varsigma}{2})\geq\nu\},$$

then $\delta_\theta(\mathfrak{A}(\nu,\varsigma))=0$ and $\delta_\theta(\mathfrak{A}^C(\nu,\varsigma))=1$. Let $p\in\mathfrak{A}^C(\nu,\varsigma)$ then for $\omega\in X$, we have $G(y_p-y_0,\omega;\frac{\varsigma}{2})>1-\nu$ and $B(y_p-y_0,\omega;\frac{\varsigma}{2})<\nu, Y(y_p-y_0,\omega;\frac{\varsigma}{2})<\nu$.

Now let $M(\wp,\varsigma)=\{k\in I_s : G(y_k-y_p,\omega;\varsigma)\leq 1-\wp \text{ or } B(y_k-y_p,\omega;\varsigma)\geq\wp, Y(y_k-y_p,\omega;\varsigma)\geq\wp\}$. We claim that $M(\wp,\varsigma)\subset\mathfrak{A}(l,\varsigma)$. Let $r\in M(\wp,\varsigma)$, then we have $G(y_r-y_p,\omega;\varsigma)\leq 1-\wp$ or $B(y_r-y_p,\omega;\varsigma)\geq\wp, Y(y_r-y_p,\omega;\varsigma)\geq\wp$.

Case (i): Suppose $G(y_r-y_p,\omega;\varsigma)\leq 1-\wp$, then $G(y_r-y_0,\omega;\frac{\varsigma}{2})\leq 1-\nu$ and therefore $r\in\mathfrak{A}(\nu,\varsigma)$. As otherwise, i.e, if $G(y_r-y_0,\omega;\frac{\varsigma}{2})>1-\nu$, then

$$\begin{aligned}1-\wp\geq G(y_r-y_p,\omega;\varsigma)&\geq G(y_q-y_0,\omega;\frac{\varsigma}{2})\circ G(y_p-y_0,\omega;\frac{\varsigma}{2})\\ &>(1-\nu)\circ(1-\nu)\\ &>1-\wp(\text{not possible})\end{aligned}.$$

Thus, $M(\wp,\varsigma)\subset\mathfrak{A}(\nu,\varsigma)$.

Case (ii): Suppose $B(y_r-y_p,\omega;\varsigma)\geq\wp$, then $B(y_r-y_0,\omega;\frac{\varsigma}{2})\geq\nu$ and therefore $r\in A(\nu,\varsigma)$.

As otherwise, i.e, if $B(y_r - y_0, \omega; \frac{\varsigma}{2}) < \nu$, then

$$\begin{aligned} \wp \leq B(y_r - y_p, \omega; \varsigma) &\leq B(y_r - y_0, \omega; \frac{\varsigma}{2}) \diamond B(y_p - y_0, \omega; \frac{\varsigma}{2}) \\ &< \nu \diamond \nu \\ &< \wp (\text{not possible}) \quad . \end{aligned}$$

Also, suppose $Y(y_r - y_p, \omega; \varsigma) \geq \wp$, then $Y(y_r - y_0, \omega; \frac{\varsigma}{2}) \geq \nu$ and therefore $r \in \mathfrak{A}(\nu, \varsigma)$. As otherwise, i.e, if $B(y_r - y_0, \omega; \frac{\varsigma}{2}) < \nu$, then

$$\begin{aligned} \wp \leq Y(y_r - y_p, \omega; \varsigma) &\leq Y(y_r - y_0, \omega; \frac{\varsigma}{2}) \diamond Y(y_p - y_0, \omega; \frac{\varsigma}{2}) \\ &< \nu \diamond \nu \\ &< \wp (\text{not possible}) \quad . \end{aligned}$$

Thus, $M(\wp, \varsigma) \subset \mathfrak{A}(\nu, \varsigma)$.
Hence in all cases, $M(\wp, \varsigma) \subset \mathfrak{A}(\nu, \varsigma)$. Since $\delta_\theta(\mathfrak{A}(\nu, \varsigma)) = 0$, so $\delta_\theta(M(\wp, \varsigma)) = 0$ and therefore $y = (y_k)$ is $S_\theta(N_2)$-Cauchy. □

Definition 4.2 A neutrosophic 2-normed space X is called $S_\theta(N_2)$-complete if every $S_\theta(N_2)$-Cauchy sequence in X is $S_\theta(N_2)$-convergent in X.

Theorem 4.2 Every $N-2-NS$ X is $S_\theta(N_2)$-complete.

Proof Let $y = (y_k)$ be $S_\theta(N_2)$-Cauchy sequence in X. Suppose on the contrary that $y = (y_k)$ is not $S_\theta(N_2)$-convergent. Let $\wp > 0$ and $\varsigma > 0$, then $\exists\ r \in \mathbb{N}$ such that $\omega \in X$ if we define

$$\begin{aligned} \mathfrak{A}(\wp, \varsigma) = \{k \in I_s : G(y_k - y_r, \omega; \varsigma) \leq 1 - \wp \text{ or} \\ B(y_k - y_r, \omega; \varsigma) \geq \wp, Y(y_k - y_r, \omega; \varsigma) \geq \wp\} \text{ and} \end{aligned}$$

$$\begin{aligned} \mathfrak{T}(\wp, \varsigma) = \{k \in I_s : G(y_k - y_0, \omega; \frac{\varsigma}{2}) > 1 - \wp \text{ and} \\ B(y_k - y_0, \omega; \frac{\varsigma}{2}) < \wp, Y(y_k - y_0, \omega; \frac{\varsigma}{2}) < \wp\}, \end{aligned}$$

then $\delta_\theta(\mathfrak{A}(\wp, \varsigma)) = \delta_\theta(\mathfrak{T}(\wp, \varsigma)) = 0$ and therefore we have $\delta_\theta(\mathfrak{A}^C(\wp, \varsigma)) = \delta_\theta(\mathfrak{T}^C(\wp, \varsigma)) = 1$. Since $G(y_k - y_r, \omega; \varsigma) \geq 2G(y_k - y_0, \omega; \frac{\varsigma}{2}) > 1 - \wp$ and $B(y_k - y_r, \omega; \varsigma) \leq 2B(y_k - y_0, \omega; \frac{\varsigma}{2}) < \wp$, $Y(y_k - y_r, \omega; \varsigma) \leq 2Y(y_k - y_0, \omega; \frac{\varsigma}{2}) < \wp$, if $G(y_k - y_0, \omega; \frac{\varsigma}{2}) > \frac{1-\wp}{2}$ and $B(y_k - y_0, \omega; \frac{\varsigma}{2}) < \frac{\wp}{2}$, $Y(y_k - y_0, \omega; \frac{\varsigma}{2}) < \frac{\wp}{2}$. We have $\delta_\theta(\{k \in I_s : G(y_k - y_r, \omega; \varsigma) > 1 - \wp$ and $B(y_k - y_r, \omega; \varsigma) < \wp$, $Y(y_k - y_r, \omega; \varsigma) < \wp\}) = 0$. i.e., $\delta_\theta(\mathfrak{A}^C(\wp, \varsigma)) = 0$ which contradicts the fact that $\delta_\theta(\mathfrak{A}^C(\wp, \varsigma)) = 1$. Hence, $y = (y_k)$ is S_θ-convergent w.r.t. N_2. □

Theorem 4.3 For any sequence $y = (y_k)$ in X, the subsequent assertions are equivalent.

(i) $y = (y_k)$ is a $S_\theta(N_2)$-Cauchy sequence.

(ii) $\exists$ a subset $\Re = \{k_n\}$ of $\mathbb{N}$ with $\delta_\theta(\Re) = 1$ and subsequence $(y_{k_n})_{n\in\mathbb{N}}$ is a $S_\theta(N_2)$-Cauchy sequence over $\Re$.

Proof. The proof of this theorem can be derived in a similar manner to the proof of theorem 3.4.

5. Conclusion

The fuzzy norm is a very helpful tool to analyze many situations in the real world where the crisp norm is found difficult due to huge uncertainty. In the present work, we define and study S_θ-convergence, S_θ-Cauchy and S_θ-completeness in a more general setting, i.e., in neutrosophic 2-normed spaces. The results presented in this paper will be helpful for many problems of fuzzy functional analysis in which ordinary norm can not be predictable and therefore one looks forward towards a fuzzy norm or a generalized fuzzy norm.

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Embedding Norms into Neutrosophic Multi Fuzzy Subrings

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Abstract. We have embedded the concept norm with the proposed notion Neutrosophic multifuzzy Subrings. This conception was manipulated with Neutrosophic multi fuzzy ideals and level sets. Furthermore, some propositions and theorems related to them were explored. Eventually, direct product and homomorphic properties of Neutrosophic multifuzzy Subrings were derived.

Keywords: Neutrosophic fuzzy set (NFS); Neutrosophic multisets (NMS); Neutrosophic multi fuzzy set (NMFS); T norms (T_n) and T conorms (T_c) ; Neutrosophic multifuzzy subring (NMFSR); Neutrosophic multifuzzy left(right) ideals (NMFL(R)I).

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1. Introduction

There is a lack of certainty that couldnt be manipulated by classical set. To overcome the complication, fuzzy set was enlightened by L.A.Zadeh [4]. Smarandache [5] initiated Neutrosophic set to build upon the thought of Atanassovs [11] intuitionistic fuzzy sets very convenient and effectively which is the part of philosophy. In Neutrosophic logic every hypothesis having degree of validity, neutral and non-validity is represented independently. The notion norm is a sort of dual operation tracking down numerous applications in fuzzy set, probability and statistics and other areas. A t-norm interprets intersection of fuzzy sets and conjunction in logics. There were some essential properties like Archimedean, strict and nilpotent t-norm that exist.

The Application of group theory to fuzzy set was originated by Rosenfield [10]. In view of the fuzzy set hypothesis, Multifuzzy set was initiated by Sebastian and Ramakrishnan [8]. The unified notions of Multifuzzy set and Group called as multifuzzy group was examined by

Muthuraj [1]. Also, he has discussed its Level Subgroups. The combined concepts Intuitionistic Fuzzy sets and Fuzzy Multisets together were developed as Intuitionistic Fuzzy multisets by Shinoj [9].

The thought of Intuitionistic fuzzy groups along with homomorphism and direct product had been explored by Sharma [15, 16]. Rasul Rasuli [2, 7, 18, 19] investigated his thought on Intuitionistic fuzzy subgroups and subrings regarding norms and reached out into fuzzy Multi-groups. Abu Osman [12] explored products of fuzzy subgroups. Intuitionistic fuzzy multiset was initiated by Shinoj and John [9]. Then, Wang [14] gave the comparative activities and outcomes of single esteemed neutrosophic set hypothesis. To elaborate the neutrosophic set theory, the conception neutrosophic multiset was originated by Deli [13] and Ye [21, 22] for modelling vagueness and uncertainty. VakkasUlucay [3] proposed the notion of Neutrosophic Multi Groups. Hemabala [6] gave the thought of gamma near ring applied into Anti Neutrosophic Multi fuzzy set. The extension principle was defined by Sahin[20] using neutrosophic multi-sets.

The scope of this work is predicated upon the notion of Neutrosophic set and multifuzzy set together with rings .We have characterized here a thought of Neutrosophic multifuzzy subrings along with triangular norms and made sense of certain outcomes connected with them.

2. Preliminaries

This part consists of, fundamental definitions are referred to that are essential.

Definition 2.1. [5] A NFS$\mathcal{A}$ on the space of points X is characterized by a truth membership $\mu_{\mathcal{A}}(x)$, an indeterminancy $\mathcal{N}_{\mathcal{A}}(x)$, and falsity membership $\digamma_{\mathcal{A}}(x)$ is defined as

$\mathcal{A} = \langle x, \mu_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}}(x), \digamma_{\mathcal{A}}(x) : x \in X\rangle$ where $\mu_{\mathcal{A}}, \mathcal{N}_{\mathcal{A}}, \digamma_{\mathcal{A}} : X \to [0,1]$ and

$$0 \leq \mu_{\mathcal{A}}(x) + \mathcal{N}_{\mathcal{A}}(x) + \digamma_{\mathcal{A}}(x) \leq 3$$

Definition 2.2. [13] A NMS $\mathcal{A}$ on X be defined as follows:
$\mathcal{A} = \{< x, (\mu^1_{\mathcal{A}}(x),\ \mu^2_{\mathcal{A}}(x), \ldots, \mu^n_{\mathcal{A}}(x)),\ (\mathcal{N}^1_{\mathcal{A}}(x),\ \mathcal{N}^2_{\mathcal{A}}(x), \ldots, \mathcal{N}^n_{\mathcal{A}}(x),\ (\digamma^1_{\mathcal{A}}(x),\ \digamma^2_{\mathcal{A}}(x), \ldots, \digamma^n_{\mathcal{A}}(x)) >: x \in X\}$,
where, $\mu^i_{\mathcal{A}}(x), \mathcal{N}^i_{\mathcal{A}}(x),\ \digamma^i_{\mathcal{A}}(x) : X \to [0,1]$, $0 \leq \sup \mu^i_{\mathcal{A}}(x) + \sup \mathcal{N}^i_{\mathcal{A}}(x) + \sup \digamma^i_{\mathcal{A}}(x) \leq 3$ $(i = 1, 2, \ldots, n)$ and for any x, truth membership $\mu^1_{\mathcal{A}}(x) \geq \mu^2_{\mathcal{A}}(x) \geq \cdots \geq \mu^n_{\mathcal{A}}(x)$ as decreasing order but no restrictions for indeterminacy and falsity membership. Futher more, n is called the dimension of $\mathcal{A}$, denoted $d(\mathcal{A})$.

Definition 2.3. [12] A function T_n **:** [0,1] $\times [0,1] \to [0,1]$ is a t-norm possess the following axioms.

1.$T_n\ (x, 1) = x$

2.$T_n\ (x, y) = T_n\ (x, z)$ if $y \leq z$

3.$T_n\ (x,y) = T_n\ (y,x)$

4.$T_n\ (x, T_n\,(y,z)) = T_n\,(T_n\,((x,y)\,,z))\,\forall x,y,z \in [0,1]$

Definition 2.4. [17] A function T_c **:** [0,1] $\times[0,1] \rightarrow [0,1]$ is a t-conorm possess the following axioms

1.$T_c\ (x,0) = x$

2.$T_c\ (x,y) = T_c\,(x,\text{z})$ if $y \leq z$

3.$T_c\,(x,y) = T_c\,(y,x)$

4.$T_c\,(x,T_c\,(y,z)) = T_c\,(T_c\,((x,y)\,,z))\ \forall x,y,z \in [0,1]$

Recollect if T_n is idempotent function $T_n\ (x,x) = x$. Similarly, if T_c is idempotent function $T_c\,(x,x) = x, \forall x \in [0,1]$.

3. Neutrosophic Multifuzzy Subring with respect to T_n and T_c

Definition 3.1. A NMFS $\mathcal{A} = \{< (x, \mu^i_{\mathcal{A}}(x), \mathcal{N}^i_{\mathcal{A}}, \mathcal{F}^i_{\mathcal{A}}\,(x)) >, x \in R,\ i = 1,2,\ldots,n\ \}$ of a ring R is said to be NMFSR with respect to T_n and T_c of R if

(i) $\mu^i_{\mathcal{A}}\,(x-y) \geq T_n\left(\mu^i_{\mathcal{A}}\,(x)\,,\mu^i_{\mathcal{A}}\,(y)\right);\ \mathcal{N}^i_{\mathcal{A}}\,(x-y) \leq T_c\left(\mathcal{N}^i_{\mathcal{A}}\,(x)\,,\mathcal{N}^i_{\mathcal{A}}\,(y)\right);$
$\mathcal{F}^i_{\mathcal{A}}\,(x-y) \leq T_c\left(\mathcal{F}^i_{\mathcal{A}}\,(x)\,,\mathcal{F}^i_{\mathcal{A}}\,(y)\right)$

(ii) $\mu^i_{\mathcal{A}}\,(xy) \geq T_n\left(\mu^i_{\mathcal{A}}\,(x)\,,\mu^i_{\mathcal{A}}\,(y)\right);\mathcal{N}^i_{\mathcal{A}}\,(xy) \leq T_c\left(\mathcal{N}^i_{\mathcal{A}}\,(x)\,,\mathcal{N}^i_{\mathcal{A}}\,(y)\right);$
$\mathcal{F}^i_{\mathcal{A}}\,(xy) \leq T_c\left(\mathcal{F}^i_{\mathcal{A}}\,(x)\,,\mathcal{F}^i_{\mathcal{A}}\,(y)\right)$

$\forall x,y \in R,\ i = 1,2\ldots,n.$

Example 3.2. Let ($Z_3,+,\cdot$) be a ring. For all $x \in Z_3$, we define a NMFS $\mathcal{A}$ over T_n and T_c of Z_3 as

$\mathcal{A}$= $(< 0(0.9,0.7,0.5),(0.2.0.4,0.8),(0.3,0.4,0.6) >,$
$< 1(0.9,0.5,0.4),(0.2,0.5,0.7),(0.3,0.5,0.7) >,< 2(0.8,0.5,0.4),(0.2,0.5,0.7),(0.4,0.5,0.7) >).$

Let $T_n(x,y) = xy$ and $T_c(x,y) = x+y-xy, \quad \forall x,y \in Z_3$ then $\mathcal{A}$ is a NMFSR of Z_3 over T_n and T_c

Proposition 3.3. *If $\mathcal{A}$ is a NMFSR of R with T_n and T_c, where T_n , T_c are idempotent then* $\forall \mathbf{x} \in R\ \&i = 1,2,\ldots,n$

(i) $\mu^i_{\mathcal{A}}\,(0) \geq \mu^i_{\mathcal{A}}\,(x);\ \mathcal{N}^i_{\mathcal{A}}\,(0) \leq \mathcal{N}^i_{\mathcal{A}}\,(x)\ ;\ \mathcal{F}^i_{\mathcal{A}}\,(0) \leq \mathcal{F}^i_{\mathcal{A}}\,(x)$

(ii) $\mu^i_{\mathcal{A}}\,(-x) = \mu^i_{\mathcal{A}}\,(x)\ ;\ \mathcal{N}^i_{\mathcal{A}}\,(-x) = \mathcal{N}^i_{\mathcal{A}}\,(x)\ ;\ \mathcal{F}^i_{\mathcal{A}}\,(-x) = \mathcal{F}^i_{\mathcal{A}}\,(x)$

Proof. If $x \in$ R.

(i) $\mu^i_{\mathcal{A}}\,(0) = \mu^i_{\mathcal{A}}\,(x-x) \geq T_n\left(\mu^i_{\mathcal{A}}\,(x)\,,\mu^i_{\mathcal{A}}\,(x)\right) = \mu^i_{\mathcal{A}}\,(x)$

$\mathcal{N}^i_{\mathcal{A}}\,(0) = \mathcal{N}^i_{\mathcal{A}}\,(x-x) \leq T_c\left(\mathcal{N}^i_{\mathcal{A}}\,(x)\,,\mathcal{N}^i_{\mathcal{A}}\,(x)\right) = \mathcal{N}^i_{\mathcal{A}}\,(x)$

Similarly, $\mathcal{F}^i_{\mathcal{A}}\,(0) \leq \mathcal{F}^i_{\mathcal{A}}\,(x)$

(ii) $\mu^i_{\mathcal{A}}\,(-x) = \mu^i_{\mathcal{A}}\,(0-x)$

$$\geq T_n\left(\mu^i_{\mathcal{A}}(0),\mu^i_{\mathcal{A}}(x)\right)$$
$$\geq T_n\left(\mu^i_{A}(x),\mu^i_{A}(x)\right)$$
$$=\mu^i_{A}(x)=\mu^i_{A}(0-(-\text{x}))$$
$$\geq T_n(\mu^i_{A}(0),\mu^i_{A}(-x)$$
$$\geq T_n\left(\mu^i_{A}(-x),\mu^i_{A}(-x)\right)$$
$$\geq T^i_{A}(-x))$$

So that, $\mu^i_{\mathcal{A}}(x)=\mu^i_{\mathcal{A}}(-x)$

$$\mathcal{N}^i_{\mathcal{A}}(-x)=\mathcal{N}^i_{\mathcal{A}}(0-x)$$
$$\leq T_c\left(\mathcal{N}^i_{\mathcal{A}}(0),\mathcal{N}^i_{\mathcal{A}}(x)\right)$$
$$\leq T_c\left(\mathcal{N}^i_{\mathcal{A}}(x),\mathcal{N}^i_{\mathcal{A}}(x)\right)$$
$$=\mathcal{N}^i_{\mathcal{A}}(x)=\mathcal{N}^i_{\mathcal{A}}(0-(-x))$$
$$\leq T_c\left(\mathcal{N}^i_{\mathcal{A}}(0),\mathcal{N}^i_{\mathcal{A}}(-x)\right)$$
$$\leq T_c\left(\mathcal{N}^i_{\mathcal{A}}(-x),\mathcal{N}^i_{\mathcal{A}}(-x)\right)$$
$$\leq \mathcal{N}^i_{\mathcal{A}}(-x)$$

So that, $\mathcal{N}^i_{\mathcal{A}}(x)=\mathcal{N}^i_{\mathcal{A}}(-x)$.

Similarly, $\digamma^i_{\mathcal{A}}(x)=\digamma^i_{\mathcal{A}}(-x)$. $\forall x\in R$ and $i=1,2\ldots n$ Hence the result. □

Proposition 3.4. *Let $\mathcal{A}$ be a NMFSR of R over T_n and T_c , $x\in R$ $\forall i=1,2\ldots n$ then*

$$\mu^i_{\mathcal{A}}(x-y)=1\Rightarrow\mu^i_{\mathcal{A}}(x)\geq\mu^i_{\mathcal{A}}(y)\,;\mathcal{N}^i_{\mathcal{A}}(x-y)=0\Rightarrow\mathcal{N}^i_{\mathcal{A}}(\mathbf{x})\leq\mathcal{N}^i_{\mathcal{A}}(y)$$

$$\digamma^i_{\mathcal{A}}(x-y)=0\Rightarrow\digamma^i_{\mathcal{A}}(x)\leq\digamma^i_{\mathcal{A}}(y)$$

Proof. Let $x,y\in R$ and $i=1,2\ldots n$. Then

(i) $\mu^i_{\mathcal{A}}(x)=\mu^i_{\mathcal{A}}(x-y+y)\geq T_n(\mu^i_{\mathcal{A}}(x-y),\mu^i_{\mathcal{A}}(y))=T_n(1,\mu^i_{\mathcal{A}}(y))=\mu^i_{\mathcal{A}}(y)$

(ii) $\mathcal{N}^i_{\mathcal{A}}(x)=\mathcal{N}^i_{\mathcal{A}}(x-y+y)\leq T_c\left(\mathcal{N}^i_{\mathcal{A}}(x-y),\mathcal{N}^i_{\mathcal{A}}(y)\right)=T_c\left(0,\mathcal{N}^i_{\mathcal{A}}(y)\right)=\mathcal{N}^i_{\mathcal{A}}(x)$

Similarly, $\digamma^i_{\mathcal{A}}(x)\leq\digamma^i_{\mathcal{A}}(y)$.

Hence the result. □

Proposition 3.5. *Let $\mathcal{A}$ be a NMFSR of R with respect to T_n and T_c where T_n , T_c are idempotent. Then $\mathcal{A}(x-y)=\mathcal{A}(y)$ iff $\mathcal{A}(x)=\mathcal{A}(0)$, $\forall x,y\in R$ and $i=1,2,3\ldots n$.*

Proof. Let $\mathcal{A}(x-y)=\mathcal{A}(y)$. If $y=0,\Rightarrow\mathcal{A}(x)=\mathcal{A}(0)$

Conversely, if $\mathcal{A}(x)=\mathcal{A}(0)$, Then,

(i).$\mu^i_{\mathcal{A}}(x)=\mu^i_{\mathcal{A}}(0)\geq\mu^i_{\mathcal{A}}(x-y)$

$\mu^i_{\mathcal{A}}(x)=\mu^i_{\mathcal{A}}(0)\geq\mu^i_{\mathcal{A}}(y)$ (by proposition 3.3)

$$\text{Now,} \quad \mu^i_{\mathcal{A}}(x-y) \geq T_n\left(\mu^i_{\mathcal{A}}(x), \mu^i_{\mathcal{A}}(y)\right)$$
$$\geq T_n\left(\mu^i_{\mathcal{A}}(y), \mu^i_{\mathcal{A}}(y)\right)$$
$$= \mu^i_{\mathcal{A}}(y)$$
$$= \mu^i_{\mathcal{A}}(-y)$$
$$= \mu^i_{\mathcal{A}}(x-y-x)$$
$$\geq T_n\left(\mu^i_{\mathcal{A}}(x-y), \mu^i_{\mathcal{A}}(x)\right)$$
$$\geq T_n\left(\mu^i_{\mathcal{A}}(x-y), \mu^i_{\mathcal{A}}(x-y)\right)$$
$$= \mu^i_{\mathcal{A}}(x-y)$$

So, we get $\mu^i_{\mathcal{A}}(x-y) = \mu^i_{\mathcal{A}}(y)$

(ii). $\mathcal{N}^i_{\mathcal{A}}(x) = \mathcal{N}^i_{\mathcal{A}}(0) \leq \mathcal{N}^i_{\mathcal{A}}(x-y)$

$\mathcal{N}^i_{\mathcal{A}}(x) = \mathcal{N}^i_{\mathcal{A}}(0) \leq \mathcal{N}^i_{\mathcal{A}}(y)$

Now,

$$\mathcal{N}^i_{\mathcal{A}}(x-y) \leq T_c\left(\mathcal{N}^i_{\mathcal{A}}(x), \mathcal{N}^i_{\mathcal{A}}(y)\right)$$
$$\leq T_c\left(\mathcal{N}^i_{\mathcal{A}}(y), \mathcal{N}^i_{\mathcal{A}}(y)\right)$$
$$= \mathcal{N}^i_{\mathcal{A}}(y)$$
$$= \mathcal{N}^i_{\mathcal{A}}(-y) \text{ (by theorem 3.3)}$$
$$= \mathcal{N}^i_{\mathcal{A}}(x-y-x)$$
$$\leq T_c\left(\mathcal{N}^i_{\mathcal{A}}(x-y), \mathcal{N}^i_{\mathcal{A}}(x)\right)$$
$$\leq T_c\left(\mathcal{N}^i_{\mathcal{A}}(x-y), \mathcal{N}^i_{\mathcal{A}}(x-y)\right)$$
$$= \mathcal{N}^i_{\mathcal{A}}(x-y)$$
$$\therefore \mathcal{N}^i_{\mathcal{A}}(x-y) = \mathcal{N}^i_{\mathcal{A}}(y)$$

Similarly, $\digamma^i_{\mathcal{A}}(x-y) = \digamma^i_{\mathcal{A}}(y)$

$\therefore \mathcal{A}(x-y) = \mathcal{A}(y)$ if $\mathcal{A}(x) = \mathcal{A}(0) \forall \mathbf{x}, \mathbf{y} \in R$ and $i = 1, 2 \ldots, n$. □

4. **Neutrosophic Multifuzzy ideal and level set**

Definition 4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two NMS of R. Define

$\mathcal{A} \cap \mathcal{B} = \left(\mu^i_{\mathcal{A}\cap\mathcal{B}}, \mathcal{N}^i_{\mathcal{A}\cap\mathcal{B}}, \digamma^i_{\mathcal{A}\cap\mathcal{B}}\right)$ as $\mu^i_{\mathcal{A}\cap\mathcal{B}}(x) = T_n(\mu^i_{\mathcal{A}}(x), \mu^i_{\mathcal{B}}(x))$

$\mathcal{N}^i_{\mathcal{A}\cap\mathcal{B}}(x) = T_c(\mathcal{N}^i_{\mathcal{A}}(x), \mathcal{N}^i_{\mathcal{B}}(x)); \digamma^i_{\mathcal{A}\cap\mathcal{B}}(x) = T_c(\digamma^i_{\mathcal{A}}(x), \digamma^i_{\mathcal{B}}(x));$

$\mathcal{A} \cup \mathcal{B} = (\mu^i_{\mathcal{A}\cup\mathcal{B}}, \mathcal{N}^i_{\mathcal{A}\cup\mathcal{B}}, \digamma^i_{\mathcal{A}\cup\mathcal{B}})$ as $\mu^i_{\mathcal{A}\cup\mathcal{B}}(x) = T_c\ (\ \mu^i_{\mathcal{A}}(x), \mu^i_{\mathcal{B}}(x)\ )$

$\mathcal{N}^i_{\mathcal{A}\cup\mathcal{B}}(x) = T_n\ (\ \mathcal{N}^i_{\mathcal{A}}(x), \mathcal{N}^i_{\mathcal{B}}(x)\ );\ \digamma^i_{\mathcal{A}\cup\mathcal{B}}(x\ ) = T_n\ (\ \digamma^i_{\mathcal{A}}(x), \digamma^i_{\mathcal{B}}(x)\ ), \forall\ x \in R$.

Example 4.2. Consider the ring $(Z_2, +, \cdot)$. For all $\mathbf{x} \in Z_2$, we define NMFS $\mathcal{A}$ and $\mathcal{B}$ of Z_2 as $\mathcal{A} = (< 0(0.9,0.7), (0.1.0.3), (0.4,0.6) >; < 1(0.8,0.6), (0.1,0.4), (0.4,0.7) >$
$\mathcal{B} = (< 0(0.9,0.6), (0.2.0.1), (0.5,0.4) >; < 1(0.7,0.4), (0.3,0.4), (0.6,0.7) >$
Let $T_n(x,y) = xy$ and $T_c(x,y) = x + y - xy, \quad \forall x, y \in Z_2$.Then
$\mathcal{A} \cup \mathcal{B} = \{< 0, (0.98,0.88),(0.02,0.03), (0.20,0.24) > < 1(0.94,0.76), (0.03,0.16), (0.24,0.0.49) >\}$
$\mathcal{A} \cap \mathcal{B} = (<0(0.72,0.43), (0.28.0.37), (0.7,0.76)>; <1(0.56,0.24), (0.37,0.64), (0.76,0.91) >)$.

Theorem 4.3. *If $\mathcal{A}$ and $\mathcal{B}$ are NMFSR of ring R, then $\mathcal{A} \cap \mathcal{B}$ also a NMFSR of R with respect to T_n and T_c, where T_n and T_c are idempotent.*

Proof. Let $x, y \in R$ and $i = 1, 2, 3, \ldots, n$

$$\begin{aligned}
(i)\ \mu^i_{\mathcal{A}\cap\mathcal{B}}(x-y) &= T_n(\mu^i_{\mathcal{A}}(x-y), \mu^i_{\mathcal{B}}(x-y)) \\
&\geq T_n\left\{T_n(\mu^i_{\mathcal{A}}(x), \mu^i_{\mathcal{A}}(y)), T_n(\mu^i_{\mathcal{B}}(x), \mu^i_{\mathcal{B}}(\mathbf{y}))\right\} \\
&= T_n\left\{T_n(\mu^i_{\mathcal{A}}(x), \mu^i_{\mathcal{B}}(x)), T_n(\mu^i_{A}(y), \mu^i_{\mathcal{B}}(y))\right\} \\
&= T_n(\mu^i_{\mathcal{A}\cap\mathcal{B}}(x), \mu^i_{\mathcal{A}\cap\mathcal{B}}(y)) \\
\mathcal{N}^i_{\mathcal{A}\cap\mathcal{B}}(x-y) &= T_c(\mathcal{N}^i_{\mathcal{A}}(x-y), \mathcal{N}^i_{\mathcal{B}}(x-y)) \\
&\leq T_c\left\{T_c(\mathcal{N}^i_{\mathcal{A}}(x), \mathcal{N}^i_{\mathcal{A}}(y)), T_c(\mathcal{N}^i_{\mathcal{B}}(x), \mathcal{N}^i_{\mathcal{B}}(y))\right\} \\
&= T_c\left\{T_c(\mathcal{N}^i_{\mathcal{A}}(x), \mathcal{N}^i_{\mathcal{B}}(x)), T_c(\mathcal{N}^i_{\mathcal{A}}(y), \mathcal{N}^i_{\mathcal{B}}(y))\right\} \\
&= T_c(\mathcal{N}^i_{\mathcal{A}\cap\mathcal{B}}(x), \mathcal{N}^i_{\mathcal{A}\cap\mathcal{B}}(y))
\end{aligned}$$

Similarly, $F^i_{\mathcal{A}\cap\mathcal{B}}(x-y) \leq T_c(F^i_{\mathcal{A}}(x), F^i_{\mathcal{B}}(y))$

$$\begin{aligned}
(ii)\ \mu^i_{\mathcal{A}\cap\mathcal{B}}(xy) &= T_n(\mu^i_{\mathcal{A}}(xy), \mu^i_{\mathcal{B}}(xy)) \\
&\geq T_n\left\{T_n(\mu^i_{\mathcal{A}}(x), \mu^i_{\mathcal{A}}(y)), T_n(\mu^i_{\mathcal{B}}(x), \mu^i_{\mathcal{B}}(y))\right\} \\
&= T_n\left\{T_n(\mu^i_{\mathcal{A}}(x), \mu^i_{\mathcal{B}}(x)), T_n(\mu^i_{\mathcal{A}}(y), \mu^i_{\mathcal{B}}(y))\right\} \\
&= T_n(\mu^i_{\mathcal{A}\cap\mathcal{B}}(x), \mu^i_{\mathcal{A}\cap\mathcal{B}}(y)) \\
\mathcal{N}^i_{\mathcal{A}\cap\mathcal{B}}(xy) &= T_c(\mathcal{N}^i_{\mathcal{A}}(xy), \mathcal{N}^i_{\mathcal{B}}(xy)) \\
&\leq T_c\left\{T_c(\mathcal{N}^i_{\mathcal{A}}(x), \mathcal{N}^i_{\mathcal{A}}(y)), T_c(\mathcal{N}^i_{\mathcal{B}}(x), \mathcal{N}^i_{\mathcal{B}}(y))\right\} \\
&= T_c\left\{T_c(\mathcal{N}^i_{\mathcal{A}}(x), \mathcal{N}^i_{\mathcal{B}}(x)), T_c(\mathcal{N}^i_{\mathcal{A}}(y), \mathcal{N}^i_{\mathcal{B}}(y))\right\} \\
&= T_c(\mathcal{N}^i_{\mathcal{A}\cap\mathcal{B}}(x), \mathcal{N}^i_{\mathcal{A}\cap\mathcal{B}}(y))
\end{aligned}$$

Similarly, $F^i_{\mathcal{A}\cap\mathcal{B}}(xy) \leq T_c(F^i_{A}(x), F^i_{\mathcal{B}}(y))$
Hence $\mathcal{A} \cap \mathcal{B}$ is a NMFSR of R w.r.t T_n and T_c $\forall x, y \in R$ and $i = 1, 2, \ldots, n$. □

Example 4.4. Consider the ring $(Z_2, +, \cdot)$. For all $\mathbf{x} \in Z_2$, we define NMFSR $\mathcal{A}$ and $\mathcal{B}$ of Z_2 as $\mathcal{A} = (<$ 0(0.9,0.7), (0.1.0.3), (0.4,0.6) $>; <$ 1(0.8,0.6), (0.1,0.4), (0.4,0.7) $>)$
$\mathcal{B} = (<$ 0(0.8,0.6), (0.2.0.1), (0.5,0.4) $>$; $<$ 1(0.7,0.4), (0.3,0.4), (0.6,0.7) $>)$

$\mathcal{A} \cap \mathcal{B}$=(<0(0.7,0.3), (0.3.0.4), (0.9,1)>; <1(0.5,0), (0.4,0.3), (1,1) >). Let $T_n(\mathbf{x}, \mathbf{y}) = \max(\mathbf{x} + \mathbf{y} - 1, 0)$ and $T_c(x, y) = \min(1, x + y) \forall x, y \in Z_2$ then $\mathcal{A} \cap \mathcal{B}$ is NMFSR of Z_2 over T_n & T_c.

Remark 4.5. In general, if $\mathcal{A}, \mathcal{B}$ are NMFSR of R with respect to T_n and T_c, then $\mathcal{A} \cup \mathcal{B}$ will always not be a NMFSR of R with respect to T_n and T_c. The accompanying example will show our case.

Example 4.6. Let $(Z_4, +, \cdot)$ be a ring of integers.
Let us define $\mathcal{A}$ = {(<0(0.9,0.6,0.4) (0.2,0.4,0.4) (0.3,0.5,0.6)>, <1(0.7,0.5,0.4) (0.2,0.5,0.6) (0.3,0.6,0.7)>, <2(0.6,0.5,0.4) (0.3,0.6,0.7) (0.3,0.6,0.7)>, <3(0.9,0.5,0.3) (0.2,0.5,0.7) (0.3,0.6,0.7)>}
$\mathcal{B}$ = {<(0 (0.9,0.8,0.7), (0.1,0.2,0.3), (0.2,0.4,0.6), <1 (0.8,0.4,0.3), (0.2,0.3,0.3), (0.3,0.5,0.6)>, <2 (0.9,0.5,0.4), (0.3,0.4,0.5), (0.4,0.5,0.6)> , <3 (0.5,0.2,0.1), (0.3,0.4,0.5), (0.4,0.5,0.6)> }
be two NMFSR of Z_4 under T_n and T_c.
Let us consider $T_n(x, y) = \min(x, y); T_c(x, y) = \max(x, y)$ then $\mathcal{A}$, $\mathcal{B}$ are NMFSR of Z_4.
$\mathcal{A} \cup \mathcal{B}$ = { < 0, (0.9,0.8,0.7), (0.1,0.2,0.3), (0.2,0.4,0.6) > < 1(0.8,0.5,0.4), (0.2,0.3,0.3), (0.3,0.5,0.7) >, < 2(0.9,0.5,0.4), (0.3,0.4,0.5), (0.3,0.5,0.6) >, 3(0.9,0.5,0.3), (0.2,0.4,0.5), (0.3,0.5,0.6) > }
Then for $x = 3; y = 2$. $\mu^i_{\mathcal{A}\cup\mathcal{B}}(3 - 2) = (0.8, 0.5, 0.4)$
Again, if $\mathcal{A}$ is a NMFSR with respect to T_n and T_c of R then $\forall\ x, y \in Z_4$;
$\mu^i_{\mathcal{A}\cup\mathcal{B}}(x - y) \geq T_n(\mu^i_{\mathcal{A}\cup\mathcal{B};}(x), \mu^i_{\mathcal{A}\cup\mathcal{B}}(y)\ )$
But for $x = 3; y = 2$

$$T_n \left\{\mu^i_{\mathcal{A}\cup\mathcal{B}}(x), \mu^i_{\mathcal{A}\cup\mathcal{B}}(y)\right\} = T_n(\mu^i_{\mathcal{A}\cup\mathcal{B}}(3), \mu^i_{\mathcal{A}\cup\mathcal{B}}(2)) = T_n\{(0.9, 0.5, 0.3), (0.9, 0.5, 0.4)\} = (0.9, 0.5, 0.3)$$

$\therefore \mu^i_{\mathcal{A}\cup\mathcal{B}}(3 - 2) = (0.8, 0.5, 0.4)\,; T_n\{\mu^i_{A}(3), \mu^i_{A}(2)\} = (0.9, 0.5, 0.3)$
$\mu^i_{\mathcal{A}\cup\mathcal{B}}(3 - 2) \ngeq T_n\{\mu^i_{\mathcal{A}\cup\mathcal{B}}(2), \mu^i_{\mathcal{A}\cup\mathcal{B}}(3)\}$
Hence $\mathcal{A} \cup \mathcal{B}$ is not NMFSR of Z_4 over T_n and T_c.

Corollary 4.7. *If $\mathcal{A}, \mathcal{B}$ are NMFSR of R then $\mathcal{A} \cup \mathcal{B}$ is a NMFSR of R if one is contained in other.*

Proof. Let $\mathbf{x}, \mathbf{y} \in R$ and $i = 1, 2, 3 \ldots, n$

$$\begin{aligned}(i)\ \mu^i_{\mathcal{A}\cup\mathcal{B}}(x - y) &= T_c(\mu^i_{\mathcal{A}}(x - y), \mu^i_{\mathcal{B}}(x - y)) \\ &\geq T_c\{T_n(\mu^i_{\mathcal{A}}(x), \mu^i_{\mathcal{A}}(y)), T_n(\mu^i_{\mathcal{B}}(x), \mu^i_{\mathcal{B}}(y))\}\end{aligned}$$

$$
\begin{aligned}
&= T_n\{T_c(\mu^i_{\mathcal{A}}(x), \mu^i_{\mathcal{B}}(x)), T_c(\mu^i_{\mathcal{A}}(y), \mu^i_{\mathcal{B}}(y))\}\\
&= T_n(\mu^i_{\mathcal{A}\cup\mathcal{B}}(x), \mu^i_{\mathcal{A}\cup\mathcal{B}}(x))\\
\mathcal{N}^i_{\mathcal{A}\cup\mathcal{B}}(x-y) &= T_c(\mathcal{N}^i_{\mathcal{A}}(x-y), \mathcal{N}^i_{\mathcal{B}}(x-y))\\
&\le T_c\{T_c(\mathcal{N}^i_{\mathcal{A}}(x), \mathcal{N}^i_{\mathcal{A}}(y)), T_c(\mathcal{N}^i_{\mathcal{B}}(x), \mathcal{N}^i_{\mathcal{B}}(y))\}\\
&= T_c\{T_c(\mathcal{N}^i_{\mathcal{A}}(x), \mathcal{N}^i_{\mathcal{B}}(x)), T_c(\mathcal{N}^i_{\mathcal{A}}(y), \mathcal{N}^i_{\mathcal{B}}(y))\}\\
&= T_c(\mathcal{N}^i_{\mathcal{A}\cup\mathcal{B}}(x), \mathcal{N}^i_{\mathcal{A}\cup\mathcal{B}}(y))
\end{aligned}
$$

Similarly, $F^i_{\mathcal{A}\cup\mathcal{B}}(x-y) \le T_c(F^i_{\mathcal{A}}(x), F^i_{\mathcal{B}}(y))$

$$
\begin{aligned}
(ii)\ \mu^i_{\mathcal{A}\cup\mathcal{B}}(xy) &= T_c(\mu^i_{\mathcal{A}}(xy), \mu^i_{\mathcal{B}}(xy))\\
&\ge T_c\{T_n(\mu^i_{\mathcal{A}}(x), \mu^i_{\mathcal{A}}(y)), T_n(\mu^i_{\mathcal{B}}(\mathbf{x}), \mu^i_{\mathcal{B}}(\mathbf{y}))\}\\
&= T_n\{T_c(\mu^i_{\mathcal{A}}(x), \mu^i_{\mathcal{B}}(x)), T_c(\mu^i_{\mathcal{A}}(y), \mu^i_{\mathcal{B}}(y))\}\\
&= T_n(\mu^i_{\mathcal{A}\cup\mathcal{B}}(x), \mu^i_{\mathcal{A}\cup\mathcal{B}}(y))\\
\mathcal{N}^i_{\mathcal{A}\cup\mathcal{B}}(xy) &= T_c(\mathcal{N}^i_{\mathcal{A}}(xy), \mathcal{N}^i_{\mathcal{B}}(xy))\\
&\le T_c\{T_c(\mathcal{N}^i_{\mathcal{A}}(x), \mathcal{N}^i_{\mathcal{A}}(y)), T_c(\mathcal{N}^i_{\mathcal{B}}(x), \mathcal{N}^i_{\mathcal{B}}(y))\}\\
&= T_c\{T_c(\mathcal{N}^i_{\mathcal{A}}(x), \mathcal{N}^i_{\mathcal{B}}(x)), T_c(\mathcal{N}^i_{\mathcal{A}}(y), \mathcal{N}^i_{\mathcal{B}}(y))\}\\
&= T_c(\mathcal{N}^i_{\mathcal{A}\cup\mathcal{B}}(x), \mathcal{N}^i_{\mathcal{A}\cup\mathcal{B}}(y))
\end{aligned}
$$

Similarly, $F^i_{\mathcal{A}\cup\mathcal{B}}(xy) \le T_c(F^i_{\mathcal{A}}(x), F^i_{\mathcal{B}}(y))$
Hence $\mathcal{A}\cup\mathcal{B}$ is a NMFSR of R w.r.t T_n and T_c $\forall x, y \in R$ and $i = 1, 2 \ldots, n$ □

Definition 4.8. Let $\mathcal{A}$=$\{< (x, \mu^i_{\mathcal{A}}(y), \mathcal{N}^i_{\mathcal{A}}(x), F^i_{\mathcal{A}}(x) >;\ x \in R$ and $i = 1, 2 \ldots, n\}$ be a NMFSR of R. Let $\alpha_i, \beta_i, \gamma_i \in [0, 1]$. With $0 \le \alpha_i + \beta_i + \gamma_i \le 3$. Then the set $\mathcal{A}_{\alpha,\beta,\gamma}$ is called a level set of $\mathcal{A}$, where for any $x \in \mathcal{A}_{\alpha,\beta,\gamma}$ the following inequalities hold $\mu^i_{\mathcal{A}}(x) \ge \alpha_i$; $\mathcal{N}^i_{\mathcal{A}}(x) \le \beta_i$; $F^i_{\mathcal{A}}(x) \le \gamma_i$;

Theorem 4.9. *If $\mathcal{A}$ is said to be a NMFSR of R with respect to T_n and T_c iff $\mathcal{A}_{\alpha,\beta,\gamma}$ is a subring of R with respect to T_n and T_c for all $\alpha_i, \beta_i, \gamma_i \in [0, 1]$ with $\mu_{\mathcal{A}}(x) \ge \alpha_i$; $\mathcal{N}_{\mathcal{A}}(x) \le \beta_i$; $F_{\mathcal{A}}(x) \le \gamma_i$; $i = 1, 2, \ldots, n$ and assume that T_n and T_C are idempotent.*

Proof. Since $\mu_{\mathcal{A}}(x) \ge \alpha$; $\mathcal{N}_{\mathcal{A}}(x) \le \beta$; $F_{\mathcal{A}}(x) \le \gamma$; $\forall x \in \mathcal{A}_{\alpha,\beta,\gamma}$.
(ie) $A_{\alpha,\beta,\gamma}$ is non-empty.
Then for all i, $\mu^i_{A}(x) \ge \alpha_i$; $\mathcal{N}^i_{A}(x) \le \beta_i$; $F^i_{A}(x) \le \gamma_i$;
Now, let $\mathcal{A}$ be NMFSR of R with respect to T_n and T_c and $x, y \in \mathcal{A}_{\alpha,\beta,\gamma}$
To show that, $x - y, xy \in \mathcal{A}_{\alpha,\beta,\gamma}$.
(i) $\mu^i_{\mathcal{A}}(x-y) \ge T\left(\mu^i_{\mathcal{A}}(x), \mu^i_{\mathcal{A}}(y)\right) \ge T_n\ (\ \alpha_i, \alpha_i\ ) = \alpha_i$
Again, $\mu^i_{\mathcal{A}}(xy) \ge T\left(\mu^i_{\mathcal{A}}(x), \mu^i_{\mathcal{A}}(y)\right) \ge T_n\ (\ \alpha_i, \alpha_i\ ) = \alpha_i$

(ii) $\mathcal{N}_{\mathcal{A}}^{i}(x-y) \leq T_c\left(\mathcal{N}_{\mathcal{A}}^{i}(x), \mathcal{N}_{\mathcal{A}}^{i}(y)\right) \leq T_c\ (\ \beta_i, \beta_i\ ) = \beta_i$
Again, $\mathcal{N}_{\mathcal{A}}^{i}(x) \leq T_c\left(\mathcal{N}_{\mathcal{A}}^{i}(x), \mathcal{N}_{\mathcal{A}}^{i}(y)\right) \leq T_c\ (\ \beta_i, \beta_i\ ) = \beta_i$
Similarly, $F_{\mathcal{A}}^{i}(x-y) \leq \gamma_i$; $F_{\mathcal{A}}^{i}(xy) \leq \gamma_i$
$\therefore \mu_{\mathcal{A}}(x) \geq \alpha$; $\mathcal{N}_{\mathcal{A}}(x) \leq \beta$; $F_{\mathcal{A}}(x) \leq \gamma$;
Thus $x-y, xy \in \mathcal{A}_{\alpha,\beta,\gamma}$ is a subring of R.
Conversely, let $\mathcal{A}_{\alpha,\beta,\gamma}$ be a subring of R.
To show that, $\mathcal{A}$ is a NMFSR of R with respect to T_n and T_c .
Let $x, y \in R$ then there exist $\alpha_i \in [0,1]$ such that $T_n\left(\mu_{\mathcal{A}}^{i}(x), \mu_{\mathcal{A}}^{i}(y)\right) = \alpha_i$
So, $\mu_{\mathcal{A}}^{i}(x) \geq \alpha_i$; $\mu_{\mathcal{A}}^{i}(y) \geq \alpha_i$
Also, let there exist $\beta_i, \gamma_i \in$ [0,1] such that $T_c\left(\mathcal{N}_{\mathcal{A}}^{i}(x), \mathcal{N}_{\mathcal{A}}^{i}(y)\right) = \beta_i$; $T_c\left(F_{\mathcal{A}}^{i}(x), F_{\mathcal{A}}^{i}(y)\right) = \gamma_i$.
Then $x, y \in \mathcal{A}_{\alpha,\beta,\gamma}$.
Again as $\mathcal{A}_{\alpha,\beta,\gamma}$ is a subring of R. $x-y, xy \in \mathcal{A}_{\alpha,\beta,\gamma}$
Hence,

$$\mu_{\mathcal{A}}^{i}(x-y) \geq \alpha_i = T_n\left(\mu_{\mathcal{A}}^{i}(x), \mu_{\mathcal{A}}^{i}(y)\right)$$
$$\mu_{\mathcal{A}}^{i}(xy) \geq \alpha_i = T_n\left(\mu_{\mathcal{A}}^{i}(x), \mu_{\mathcal{A}}^{i}(y)\right)$$

Similarly,

$$\mathcal{N}_{\mathcal{A}}^{i}(x-y) \leq \beta_i = T_c\left(\mathcal{N}_{\mathcal{A}}^{i}(x), \mathcal{N}_{\mathcal{A}}^{i}(y)\right); \mathcal{N}_{\mathcal{A}}^{i}(xy) \leq \beta_i = T_c\left(\mathcal{N}_{\mathcal{A}}^{i}(x), \mathcal{N}_{\mathcal{A}}^{i}(y)\right)$$
$$F_{\mathcal{A}}^{i}(x-y) \leq \gamma_i = T_c\left(F_{\mathcal{A}}^{i}(x), F_{\mathcal{A}}^{i}(y)\right); F_{\mathcal{A}}^{i}(xy) \leq \gamma_i = T_c\left(F_{\mathcal{A}}^{i}(x), F_{\mathcal{A}}^{i}(y)\right)$$

$\therefore \mathcal{A}$ is a NMFSR of R with respect to T_n and T_c . □

Proposition 4.10. *Let $\mathcal{A}$ be a NMFSR of R w.r. t. T_n and T_c where T_n, T_c are idempotent then S= $\{\ x \in R/\mu_A^i(x) = 1, \mathcal{N}_A^i(x) = 0, F_A^i(x) = 0;\ i = 1, 2 \ldots, n\}$ is a subring of R.*

Proof. Let $x, y \in$ S.Then,
(i) $\mu_{\mathcal{A}}^{i}(x-y) \geq T_n\left(\mu_{\mathcal{A}}^{i}(x), \mu_{\mathcal{A}}^{i}(y)\right) = T(1,1) = 1$
$\mathcal{N}_{\mathcal{A}}^{i}(x-y) \leq T_c\left(\mathcal{N}_{\mathcal{A}}^{i}(x), \mathcal{N}_{\mathcal{A}}^{i}(y)\right) = T_c(0,0) = 0$
Similarly, $F_{\mathcal{A}}^{i}(x-y) \leq 0$. hence $x-y \in S$.
Also,
(ii) $\mu_{\mathcal{A}}^{i}(xy) \geq T_n\left(\mu_{\mathcal{A}}^{i}(x), \mu_{\mathcal{A}}^{i}(y)\right) = T(1,1) = 1$
$\mathcal{N}_{\mathcal{A}}^{i}(xy) \leq T_c\left(\mathcal{N}_{\mathcal{A}}^{i}(x), \mathcal{N}_{\mathcal{A}}^{i}(y)\right) = T_c(0,0) = 0$
Similarly, $F_{\mathcal{A}}^{i}(xy) \leq 0$. Hence $xy \in S$.
Thus $S = \{x \in R/\mu_{\mathcal{A}}^{i}(\mathbf{x}) = 1, \mathcal{N}_{\mathcal{A}}^{i}(x) = 0, F_{\mathcal{A}}^{i}(x) = 0\}$ is a subring of R w. r. t T_n and T_c . □

Definition 4.11. Let $\mathcal{A}$ be a NMFS of R. Then $\mathcal{A}$ is Said to be NMFLI of R w.r.t, T_n and T_c if

(i) $\mu^i_{\mathcal{A}}(x-y) \geq T_n(\mu^i_{\mathcal{A}}(x), \mu^i_{\mathcal{A}}(y))$; $\mathcal{N}^i_{\mathcal{A}}(x-y) \leq T_c(\mathcal{N}^i_{\mathcal{A}}(x), \mathcal{N}^i_{\mathcal{A}}(y))$;
$F^i_{\mathcal{A}}(x-y) \leq T_c(F^i_{\mathcal{A}}(x), F^i_{\mathcal{A}}(y))$
(ii) $\mu^i_{\mathcal{A}}(xy) \geq \mu^i_{\mathcal{A}}(y)$; $\mathcal{N}^i_{\mathcal{A}}(xy) \leq \mathcal{N}^i_{\mathcal{A}}(y)$; $F^i_{\mathcal{A}}(xy) \leq F^i_{\mathcal{A}}(y)\, \forall x, y \in R, \quad i = 1, 2 \ldots, n$

Example 4.12. Let $(Z_2, +, \cdot)$ be a ring. Define
$\mathcal{A} = \{(<$ (0,(0.9,0.7),(0.1,0.5),(0.2,0.3)), (1,(0.8,0.6),(0.2,0.5),(0.3,0.6)) $> \}$
Let us consider $T_n(x, y) = xy$; $T_c(x, y) = x + y - xy$. Then $\mathcal{A}$ is NMFLI of Z_2 with T_n and T_c

Definition 4.13. Let $\mathcal{A}$ be a NMFS of R w. r. t T_n and T_c. Then $\mathcal{A}$ is NMFRI of R w. r. t T_n and T_c if
(i) $\mu^i_{\mathcal{A}}(x-y) \geq T_n\left(\mu^i_{\mathcal{A}}(x), \mu^i_{\mathcal{A}}(y)\right)$; $\mathcal{N}^i_{\mathcal{A}}(x-y) \leq T_c(\mathcal{N}^i_{\mathcal{A}}(x), \mathcal{N}^i_{\mathcal{A}}(y))$;
$F^i_{\mathcal{A}}(x-y) \leq T_c(F^i_{\mathcal{A}}(x), F^i_{\mathcal{A}}(y))$
(ii) $\mu^i_{\mathcal{A}}(xy) \geq \mu^i_{\mathcal{A}}(x)$; $\mathcal{N}^i_{\mathcal{A}}(xy) \leq \mathcal{N}^i_{\mathcal{A}}(\mathbf{x})$; $F^i_{\mathcal{A}}(xy) \leq F^i_{\mathcal{A}}(x), \quad \forall x, y \in R$, $i = 1, 2 \ldots, n$.

Example 4.14. Consider the ring $(Z_3, +, \cdot)$. For all $x \in Z_3$, we define NMFS $\mathcal{A}$ of Z_3 as $\mathcal{A} =$ ($<$ 0(0.9,0.7), (0.1.0.3), (0.4,0.6) $>$; $<$ 1(0.8,0.6), (0.1,0.4), (0.4,0.7) $>$;$<$ 2(0.7,0.4), (0.1.0.4), (0.4,0.6) $>$)
Let us consider $T_n(x, y) = \min(x, y); T_c(x, y) = \max(x, y)$ then $\mathcal{A}$ is NMFRI of Z_3 over T_n & T_c.

Definition 4.15. Let $\mathcal{A}$ be a NMFS of R with respect to T_n and T_c. Then $\mathcal{A}$ is Said to be NMFI with respect to T_n and T_c of R if
(i) $\mu^i_{\mathcal{A}}(x-y) \geq T_n(\mu^i_{\mathcal{A}}(x), \mu^i_{\mathcal{A}}(y))$; $\mathcal{N}^i_{\mathcal{A}}(x-y) \leq T_c(\mathcal{N}^i_{\mathcal{A}}(x), \mathcal{N}^i_{\mathcal{A}}(y))$
$F^i_{\mathcal{A}}(x-y) \leq T_c(F^i_{\mathcal{A}}(x), F^i_{\mathcal{A}}(y))$
(ii) $\mu^i_{\mathcal{A}}(xy) \geq T_c(\mu^i_{\mathcal{A}}(x), \mu^i_{\mathcal{A}}(y))$; $\mathcal{N}^i_{\mathcal{A}}(xy) \leq T_n(\mathcal{N}^i_{\mathcal{A}}(x), \mathcal{N}^i_{\mathcal{A}}(y))$
$F^i_{\mathcal{A}}(xy) \leq T_n(F^i_{\mathcal{A}}(x), F^i_{\mathcal{A}}(y)), \quad \forall x, y \in R.$

Example 4.16. Consider the ring $(Z_2, +, \cdot)$. For all $x \in Z_2$, we define NMFS $\mathcal{A}$ of Z_2 as $\mathcal{A} =$ ($<$ 0(0.8,0.7), (0.2.0.3), (0.1,0.4) $>$; $<$ 1(0.7,0.6), (0.2,0.3), (0.2,0.5) $>$
Let us consider $T_n(x, y) = \min(x, y); T_c(x, y) = \max(x, y)$ then $\mathcal{A}$ is NMFI of Z_2 over T_n & T_c.

Theorem 4.17. *Let $\mathcal{A}$ be a NMFS of R with respect to T_n and T_c where, T_n, T_c are idempotent. Then $\mathcal{A}$ is said to be NMFLI(NMFRI) of R with T_n and T_c iff $\mathcal{A}_{\alpha,\beta,\gamma}$ is a LI(RI) of R, $\forall$ $\alpha_i, \beta_i, \gamma_i \in$ [0,1]. with $\mu^i_{\mathcal{A}}(\mathbf{x}) \geq \alpha_i$; $\mathcal{N}^i_{\mathcal{A}}(x) \leq \beta_i$; $F^i_{\mathcal{A}}(x) \leq \gamma_i$ and $\alpha_i + \beta_i + \gamma_i \leq 3$, where $\mu^i_{\mathcal{A}}(0) \geq \alpha_i$; $\mathcal{N}^i_{\mathcal{A}}(0) \leq \beta_i$; $F^i_{\mathcal{A}}(0) \leq \gamma_i$, $i = 1, 2 \ldots, n$.*

Proof. Let $\mathcal{A}$ be a NMFLI of R with respect to T_n and T_c .
If $x, y \in \mathcal{A}_{\alpha,\beta,\gamma}$, $i = 1, 2, \ldots, n$
Then by $\mu^i_{\mathcal{A}}(x-y) \geq T_n(\mu^i_{\mathcal{A}}(x), \mu^i_{\mathcal{A}}(y)) \geq T_n(\alpha_i, \alpha_i) = \alpha_i$
$\mathcal{N}^i_{\mathcal{A}}(x-y) \leq T_C(\mathcal{N}^i_{\mathcal{A}}(x), \mathcal{N}^i_{\mathcal{A}}(x)) \leq T_C(\beta_i, \beta_i) = \beta_i$

Similarly, $F^{i}_{\mathcal{A}}(x-y) \leq \gamma_i$. $\therefore \mu_{\mathcal{A}}(x-y) \geq \alpha$; $\mathcal{N}_{\mathcal{A}}(x-y) \leq \beta$; $F_{\mathcal{A}}(x-y) \leq \gamma$;
We obtain that $x-y \in \mathcal{A}_{\alpha,\beta,\gamma}$
Now let $x \in \mathcal{A}_{\alpha,\beta,\gamma}$ and r $\in$ R. Then from $\mu^{i}_{\mathcal{A}}(rx) \geq \mu^{i}_{\mathcal{A}}(x) \geq \alpha_i$

$$\mathcal{N}^{i}_{\mathcal{A}}(rx) \leq \mathcal{N}^{i}_{\mathcal{A}}(x) \leq \beta_i$$

$$F^{i}_{\mathcal{A}}(rx) \leq F^{i}_{\mathcal{A}}(x) \leq \gamma_i$$

Therefore $rx \in \mathcal{A}_{\alpha,\beta,\gamma}$. Hence $\mathcal{A}_{\alpha,\beta,\gamma}$ is a LI of R.
Similarly, we can prove it for right ideal (ie) $xr \in \mathcal{A}_{\alpha,\beta,\gamma}$.
Conversely, let $\mathcal{A}_{\alpha,\beta,\gamma}$ be a LI of R and x, $y \in \mathcal{A}_{\alpha,\beta,\gamma}$ such that
$\mu^{i}_{\mathcal{A}}(x) = \mu^{i}_{\mathcal{A}}(y) = \alpha_i; \mathcal{N}^{i}_{\mathcal{A}}(x) = \mathcal{N}^{i}_{\mathcal{A}}(y) = \beta_i; F^{i}_{\mathcal{A}}(x) = F^{i}_{\mathcal{A}}(y) = \gamma_i$
$\because x-y \in \mathcal{A}_{\alpha,\beta,\gamma}$ so

$$\mu^{i}_{\mathcal{A}}(x-y) \geq \alpha_i = T(\alpha_i,\alpha_i) = T_n\left(\mu^{i}_{\mathcal{A}}(x), \mu^{i}_{\mathcal{A}}(y)\right)$$

$$\mathcal{N}^{i}_{\mathcal{A}}(x-y) \leq \beta_i = T_c(\beta_i,\beta_i) = T_c\left(\mathcal{N}^{i}_{\mathcal{A}}(x), \mathcal{N}^{i}_{\mathcal{A}}(y)\right)$$

Similarly, we get $F^{i}_{\mathcal{A}}(x-y) \leq T_c\left(F^{i}_{\mathcal{A}}(x), F^{i}_{\mathcal{A}}(y)\right)$. Also $\because xy \in \mathcal{A}_{\alpha,\beta,\gamma}$ then

$$\mu^{i}_{\mathcal{A}}(xy) \geq \alpha_i = \mu^{i}_{\mathcal{A}}(y)$$

$$\mathcal{N}^{i}_{\mathcal{A}}(xy) \leq \beta_i = \mathcal{N}^{i}_{\mathcal{A}}(y)$$

$$F^{i}_{\mathcal{A}}(xy) \leq \gamma_i = F^{i}_{\mathcal{A}}(y), x, y \in \mathcal{A}_{\alpha,\beta,\gamma}.$$

$\therefore \mathcal{A}$ is a NMFLR of R with T_n and T_c are idempotent. Similarly, we can prove it for RI. □

Theorem 4.18. *Let $\mathcal{A}$ be a NMFS of R with respect to T_n and T_c where T_n, T_c be idempotent. Then $\mathcal{A}$ is said to be NMFI of R with T_n and T_c iff $\mathcal{A}_{\alpha,\beta,\gamma}$ is an ideal of R $\forall$ $\alpha_i, \beta_i, \gamma_i \in$ [0,1] with $\mu^{i}_{\mathcal{A}}(x) \geq \alpha_i$; $\mathcal{N}^{i}_{\mathcal{A}}(x) \leq \beta_i$; $F^{i}_{\mathcal{A}}(x) \leq \gamma_i$ and $0 \leq \alpha_i + \beta_i + \gamma_i \leq 3$, where $\mu^{i}_{\mathcal{A}}(0) \geq \alpha_i$; $\mathcal{N}^{i}_{\mathcal{A}}(0) \leq \beta_i$; $F^{i}_{\mathcal{A}}(0) \leq \gamma_i$,i=1, 2. . . , n.*

Proof. Follows from the above theorem. □

Theorem 4.19. *If $\mathcal{A}$ and $\mathcal{B}$ are NMFLI(NMFRI) of R with respect to T_n and T_c then $\mathcal{A} \cap \mathcal{B}$ also a NMFLI(NMFRI) of R with respect to T_n and T_c where, T_n and T_c are idempotent.*

Proof. Let $x, y \in R.\mathcal{A} \cap \mathcal{B}$ is NMFSR with respect to T_n and T_c . (By theorem 4.2).
It is enough to show,

$$
\begin{aligned}
\text{(i)}\mu^{i}_{\mathcal{A}\cap\mathcal{B}}(xy) &= T_n(\mu^{i}_{\mathcal{A}}(xy), \mu^{i}_{\mathcal{B}}(xy)) \\
&\geq T_n(\mu^{i}_{\mathcal{A}}(y)), \mu^{i}_{\mathcal{B}}(y)) \\
&= T_n(\mu^{i}_{\mathcal{A}\cap\mathcal{B}}(y)) \\
\mathcal{N}^{i}_{\mathcal{A}\cap\mathcal{B}}(xy) &= T_c(\mathcal{N}^{i}_{\mathcal{A}}(\mathbf{xy}), \mathcal{N}^{i}_{\mathcal{B}}(\mathbf{xy})) \\
&\leq T_c(\mathcal{N}^{i}_{\mathcal{A}}(y), \mathcal{N}^{i}_{\mathcal{B}}(y)) \\
&= \mathcal{N}^{i}_{\mathcal{A}\cap\mathcal{B}}(\mathbf{y})
\end{aligned}
$$

Similarly, $F^{i}_{\mathcal{A}\cap\mathcal{B}}((\mathbf{xy}) \leq F^{i}_{\mathcal{A}\cap\mathcal{B}}(\mathbf{y})$. Therefore $\mathcal{A}\cap\mathcal{B}$ is a NMFLI with respect to T_n and T_c
In the similar way we can easily prove for NMFRI. □

Remark 4.20. In general, if $\mathcal{A}, \mathcal{B}$ are NMFLI(NMFRI) of R with respect to T_n and T_c, then $\mathcal{A}\cup\mathcal{B}$ will always not be a NMFLI(NMFRI) of R with respect to T_n and T_c. The accompanying example will show our case.

Example 4.21. Let $(Z_4, +, \cdot)$ be a ring of integers.
Let us define $\mathcal{A}$ = {(<0(0.9,0.6) (0.2,0.4) (0.3,0.5)>, <1(0.8,0.5) (0.3,0.6) (0.3,0.6)>, <2(0.8,0.5) (0.3,0.6) (0.3,0.6)>, <3(0.9,0.5) (0.2,0.5) (0.3,0.6)>}
$\mathcal{B}$ = {<(0 (0.9,0.8), (0.1,0.2), (0.2,0.4), <1 (0.8,0.4), (0.3,0.4), (0.4,0.5)>, <2 (0.9,0.5), (0.3,0.4), (0.4,0.5)> , <3 (0.8,0.4), (0.3,0.4), (0.4,0.5)> } be two NMFS of Z_4 under T_n and T_c.
Let us consider $T_n(\mathbf{x}, \mathbf{y}) = \min(\mathbf{x}, \mathbf{y}); T_c(\mathbf{x}, \mathbf{y}) = \max(\mathbf{x}, \mathbf{y})$ then $\mathcal{A}$,and $\mathcal{B}$ be NMFSR of Z_4.
$\mathcal{A}\cup\mathcal{B}$ = { < 0, (0.9,0.8), (0.1,0.2), (0.2,0.4) > < 1(0.8,0.5), (0.2,0.3), (0.3,0.5) >, < 2(0.9,0.5), (0.3,0.4), (0.3,0.5) >, 3(0.9,0.5), (0.2,0.4), (0.3,0.5) > }
Then for $\mathbf{x} = 3; \mathbf{y} = 2$. $\mu^{i}_{\mathcal{A}\cup\mathcal{B}}(3-2) = (0.8, 0.5)$
Again, if $\mathcal{A}$ is a NMFLI with respect to T_n and T_c of R then $\forall\ \mathbf{x}, \mathbf{y} \in Z_4$
$\mu^{i}_{\mathcal{A}\cup\mathcal{B}}(\mathbf{x}-\mathbf{y}) \geq T_n(\mu^{i}_{\mathcal{A}\cup\mathcal{B}}(\mathbf{x}), \mu^{i}_{\mathcal{A}\cup\mathcal{B}}(y))$ $\mu^{i}_{\mathcal{A}}(\mathbf{x}y) \geq \mu^{i}_{\mathcal{A}}(\mathbf{y})$; $\mathcal{N}^{i}_{\mathcal{A}}(\mathbf{x}y) \leq \mathcal{N}^{i}_{\mathcal{A}}(\mathbf{y})$; $F^{i}_{\mathcal{A}}(\mathbf{x}y) \leq F^{i}_{\mathcal{A}}(\mathbf{y}) \forall \mathbf{x}, y \in R, \quad i = 1, 2 \ldots, n$
But for $\mathbf{x} = 3; \mathbf{y} = 2$

$$T_n\left\{\mu^{i}_{\mathcal{A}\cup\mathcal{B}}(\mathbf{x}), \mu^{i}_{\mathcal{A}\cup\mathcal{B}}(y)\right\} = T_n(\mu^{i}_{\mathcal{A}\cup\mathcal{B}}(3), \mu^{i}_{\mathcal{A}\cup\mathcal{B}}(2)) = T_n\{(0.9, 0.5), (0.9, 0.5)\} = (0.9, 0.5)$$

$\therefore \mu^{i}_{\mathcal{A}\cup\mathcal{B}}(3-2) = (0.8, 0.5); T_n\{\mu^{i}_{\mathcal{A}}(3), \mu^{i}_{\mathcal{A}}(2)\} = (0.9, 0.5)$
$\mu^{i}_{\mathcal{A}\cup\mathcal{B}}(3-2) \ngeq T_n\{\mu^{i}_{\mathcal{A}\cup\mathcal{B}}(3), \mu^{i}_{\mathcal{A}\cup\mathcal{B}}(2)\}$
Hence $\mathcal{A}\cup\mathcal{B}$ is not NMFLI of Z_4 over T_n and T_c.

Theorem 4.22. *If $\mathcal{A}$ and $\mathcal{B}$ are NMFI of ring R with respect to T_n and T_c then $\mathcal{A}\cap\mathcal{B}$ also a NMFI of R w. r. t T_n and T_c where T_n and T_c are idempotent.*

Proof. Follows from above theorem. □

Corollary 4.23. *Let* $\{ \mathcal{A}_i, i = 1, 2 \ldots, n\}$ *be a NMFSR of* R *with respect to* T_n *and* T_c*. Then* $\cap \mathcal{A}_i$ *is also NMFSR of* R *.*

Definition 4.24. Let $\mathcal{A}$ and $\mathcal{B}$ be the two NMFS in R. Then $\mathcal{A} \circ \mathcal{B}$ is defined as , $\forall \mathbf{x}, \mathbf{y} \in$ R,

$$(\mathcal{A} \circ \mathcal{B})(\mathbf{x}) = \begin{cases} \underbrace{sup}_{\mathbf{x}=\mathbf{y}z} & T_n\left(\mu^i_{\mathcal{A}}(\mathbf{y}), \mu^i_{\mathcal{B}}(z)\right) \\ \underbrace{inf}_{\mathbf{x}=\mathbf{y}z} & T_c(\mathcal{N}^i_{\mathcal{A}}(\mathbf{y}), \mathcal{N}^i_{\mathcal{B}}(z)) \quad \text{if } \mathbf{x} = \mathbf{yz} \\ \underbrace{inf}_{\mathbf{x}=\mathbf{y}z} & T_c(\mathcal{F}^i_{\mathcal{A}}(\mathbf{y}), \mathcal{F}^i_{\mathcal{B}}(z)) \\ (0,1,1) & \text{if } \mathbf{x} \neq \mathbf{y}z \end{cases}$$

Theorem 4.25. *Let* $\mathcal{A}$*,* $\mathcal{B}$ *be the two NMS in R. If* $\mathcal{A}$ *and* $\mathcal{B}$ *are NMFI of R with respect to* T_n *and* T_c *then* $\mathcal{A} \circ \mathcal{B} \subset \mathcal{A} \cap \mathcal{B}$ *.*

Proof. Let $\mathbf{x} \in R$. Suppose $\mathcal{A} \circ \mathcal{B} = (0, 1, 1)$ then there is nothing to prove.

Suppose $\mathcal{A} \circ \mathcal{B} \neq (0, 1, 1)$

Then,

$$(\mathcal{A} \circ \mathcal{B})(\mathbf{x}) = \begin{cases} \underbrace{sup}_{x=yz} & T_n\left(\mu^i_{\mathcal{A}}(\mathbf{y}), \mu^i_{\mathcal{B}}(z)\right) \\ \underbrace{inf}_{x=yz} & T_c(\mathcal{N}^i_{\mathcal{A}}(\mathbf{y}), \mathcal{N}^i_{\mathcal{B}}(z) \quad \text{if } \mathbf{x} = \mathbf{yz} \\ \underbrace{inf}_{x=yz} & T_c(\mathcal{F}^i_{\mathcal{A}}(\mathbf{y}), \mathcal{F}^i_{\mathcal{B}}(z) \end{cases}$$

Since $\mathcal{A}$, $\mathcal{B}$ are NMFI of R with T_n and T_c .

(i) $\mu^i_{\mathcal{A}}(\mathbf{y}) \leq \mu^i_{\mathcal{A}}(\mathbf{y}z) = \mu^i_{\mathcal{A}}(\mathbf{x})$; $\mathcal{N}^i_{\mathcal{A}}(\mathbf{y}) \geq \mathcal{N}^i_{\mathcal{A}}(\mathbf{y}z) = \mathcal{N}_{\mathcal{A}}(\mathbf{x})$; $\mathcal{F}^i_{\mathcal{A}}(\mathbf{y}) \geq \mathcal{F}^i_{\mathcal{A}}(\mathbf{y}z) = \mathcal{F}^i_{\mathcal{A}}(\mathbf{x})$

(ii) $\mu^i_{\mathcal{B}}(z) \leq \mu^i_{\mathcal{B}}(\mathbf{y}z) = \mu^i_{\mathcal{B}}(\mathbf{x})$; $\mathcal{N}^i_{\mathcal{B}}(z) \geq \mathcal{N}^i_{\mathcal{B}}(\mathbf{y}z) = \mathcal{N}^i_{\mathcal{B}}(\mathbf{x})$; $\mathcal{F}^i_{\mathcal{B}}(z) \geq \mathcal{F}^i_{\mathcal{B}}(\mathbf{y}z) = \mathcal{F}^i_{\mathcal{B}}(\mathbf{x})$

Thus,

$$\begin{aligned} \mu^i_{\mathcal{A}\circ\mathcal{B}}(\mathbf{x}) &= \underbrace{sup}_{x=\mathbf{y}z}\{T_n(\mu^i_{\mathcal{A}}(\mathbf{y}), \mu^i_{\mathcal{B}}(z)\} \\ &\leq T_n(\mu^i_{\mathcal{A}}(\mathbf{x}), \mu^i_{\mathcal{B}}(\mathbf{x})) \\ &= \mu^i_{\mathcal{A}\cap\mathcal{B}}(\mathbf{x}) \\ \mathcal{N}^i_{\mathcal{A}\circ\mathcal{B}}(\mathbf{x}) &= \underbrace{inf}_{x=yz}\{T_c(\mathcal{N}^i_{\mathcal{A}}(\mathbf{y}), \mathcal{N}^i_{\mathcal{B}}(z)\} \\ &\geq T_c(\mathcal{N}^i_{\mathcal{A}}(\mathbf{y}), \mathcal{N}^i_{\mathcal{B}}(z)) \\ &= \mathcal{N}^i_{\mathcal{A}\cap\mathcal{B}}(\mathbf{x}) \end{aligned}$$

Similarly, $\mathcal{F}^i_{\mathcal{A}\circ\mathcal{B}}(\mathbf{x}) \geq \mathcal{F}^i_{\mathcal{A}\cap\mathcal{B}}(\mathbf{x})$. Hence $\mathcal{A} \circ \mathcal{B} \subset \mathcal{A} \cap \mathcal{B}$. □

5. Direct product and Homomorphism on Neutrosophic Multifuzzy subrings over norms

Definition 5.1. Let R_1 and R_2 be the two rings. Let $\mathcal{A}$ and $\mathcal{B}$ be the two NMFS of R_1 and R_2 respectively with T_n and T_c. Then $\mathcal{A}\times\mathcal{B}$=$\{< ((\mathbf{x},\mathbf{y}), \mu^i_{\mathcal{A}\times\mathcal{B}}(\mathbf{x},\mathbf{y}), \mathcal{N}^i_{\mathcal{A}\times\mathcal{B}}(\mathbf{x},\mathbf{y}), \digamma^i_{\mathcal{A}\times\mathcal{B}}(\mathbf{x},\mathbf{y}) > /$ $\mathbf{x}\in R_1,\ \mathbf{y}\in R_2,\ i=1,2,\dots,n\ \}$

Where $\mu^i_{\mathcal{A}\times\mathcal{B}}(\mathbf{x},\mathbf{y}) = T_n(\mu^i_{\mathcal{A}}(\mathbf{x}), \mu^i_{B}(\mathbf{y}))$

$\mathcal{N}^i_{\mathcal{A}\times\mathcal{B}}(\mathbf{x},\mathbf{y}) = T_c(\mathcal{N}^i_{\mathcal{A}}(\mathbf{x}), \mathcal{N}^i_{B}(\mathbf{y}))$, $\digamma^i_{\mathcal{A}\times\mathcal{B}}(\mathbf{x},\mathbf{y}) = T_c(\digamma^i_{\mathcal{A}}(\mathbf{x}), \digamma^i_{\mathcal{B}}(\mathbf{y}))$

Theorem 5.2. *Let R_1 and R_2 be the two rings with $\mathcal{A}$ and B are respectively NMFSR of R_1 and R_2 over T_n and T_c .Then $\mathcal{A}\times\mathcal{B}$ is also a NMFSR of $R_1\times R_2$ With respect to T_n and T_c .*

Proof. Let $\mathcal{A}$ and B are respectively NMFSR of R_1 and R_2 respectively over T_n and T_c .
Let $(\mathbf{x}_1,\mathbf{y}_1)(\mathbf{x}_2,\mathbf{y}_2)\in\mathcal{A}\times B$.

$$\begin{aligned}
\text{Then, } \mu^i_{\mathcal{A}\times\mathcal{B}}[(\mathbf{x}_1,\mathbf{y}_1)-(\mathbf{x}_2,\mathbf{y}_2)] &= \mu^i_{\mathcal{A}\times\mathcal{B}}((\mathbf{x}_1-\mathbf{x}_2),(\mathbf{y}_1-\mathbf{y}_2))\\
&= T_n\left\{\mu^i_{\mathcal{A}}((\mathbf{x}_1-\mathbf{x}_2)), \mu^i_{\mathcal{B}}(\mathbf{y}_1-\mathbf{y}_2))\right\}\\
&\geq T_n\{T_n(\mu^i_{\mathcal{A}}(\mathbf{x}_1), \mu^i_{\mathcal{A}}(\mathbf{x}_2)), T_n(\mu^i_{\mathcal{B}}(\mathbf{y}_1), \mu^i_{\mathcal{B}}(\mathbf{y}_2))\}\\
&\geq T_n\{T_n(\mu^i_{\mathcal{A}}(\mathbf{x}_1), \mu^i_{\mathcal{B}}(\mathbf{y}_1)), T_n(\mu^i_{\mathcal{A}}(\mathbf{x}_2), \mu^i_{\mathcal{B}}(\mathbf{y}_2))\}\\
&\geq T_n\left\{\mu^i_{\mathcal{A}\times\mathcal{B}}((x_1,\mathbf{y}_1)), \mu^i_{\mathcal{A}\times\mathcal{B}}((\mathbf{x},\mathbf{y}_2))\right\}.\\
\mu^i_{\mathcal{A}\times\mathcal{B}}[(\mathbf{x}_1,\mathbf{y}_1)\cdot(\mathbf{x}_2,\mathbf{y}_2)] &= \mu^i_{\mathcal{A}\times\mathcal{B}}((\mathbf{x}_1.\mathbf{x}_2),(\mathbf{y}_1.\mathbf{y}_2))\\
&= T_n\left\{\mu^i_{\mathcal{A}}((\mathbf{x}_1.\mathbf{x}_2)), \mu^i_{\mathcal{B}}(\mathbf{y}_1.\mathbf{y}_2))\right\}\\
&\geq T_n\{T_n(\mu^i_{\mathcal{A}}(\mathbf{x}), \mu^i_{\mathcal{A}}(\mathbf{x}_2)), T_n(\mu^i_{\mathcal{B}}(\mathbf{y}_1), \mu^i_{\mathcal{B}}(\mathbf{y}_2))\}\\
&\geq T_n\{T_n(\mu^i_{\mathcal{A}}(\mathbf{x}_1), \mu^i_{\mathcal{B}}(\mathbf{y}_1)), T_n(\mu^i_{\mathcal{A}}(\mathbf{x}_2), \mu^i_{\mathcal{B}}(\mathbf{y}_2))\}\\
&\geq T_n\left\{\mu^i_{\mathcal{A}\times\mathcal{B}}((\mathbf{x}_1,\mathbf{y}_1)), \mu^i_{\mathcal{A}\times\mathcal{B}}((\mathbf{x}_2,\mathbf{y}_2))\right\}.
\end{aligned}$$

$$\begin{aligned}
\text{Again, } \mathcal{N}^i_{\mathcal{A}\times\mathcal{B}}[(\mathbf{x},\mathbf{y}_1)-(\mathbf{x}_2,\mathbf{y}_2)] &= \mathcal{N}^i_{\mathcal{A}\times\mathcal{B}}((\mathbf{x}_1-\mathbf{x}_2),(\mathbf{y}_1-\mathbf{y}_2))\\
&= T_c\left\{\mathcal{N}^i_{\mathcal{A}}((\mathbf{x}_1-\mathbf{x}_2)), \mathcal{N}^i_{\mathcal{B}}(\mathbf{y}_1-\mathbf{y}_2))\right\}\\
&\leq T_c\{T_c(\mathcal{N}^i_{\mathcal{A}}(\mathbf{x}), \mathcal{N}^i_{\mathcal{A}}(\mathbf{x}_2)), T_c(\mathcal{N}^i_{\mathcal{B}}(\mathbf{y}_1), \mathcal{N}^i_{\mathcal{B}}(\mathbf{y}_2))\}\\
&\leq T_c\{T_c(\mathcal{N}^i_{\mathcal{A}}(\mathbf{x}_1), \mathcal{N}^i_{\mathcal{B}}(\mathbf{y}_1)), T_c(\mathcal{N}^i_{A}(\mathbf{x}_2), \mathcal{N}^i_{\mathcal{B}}(\mathbf{y}_2))\}\\
&\leq T_c\left\{\mathcal{N}^i_{\mathcal{A}\times\mathcal{B}}(\mathbf{x}_1,\mathbf{y}_1), \mathcal{N}^i_{\mathcal{A}\times\mathcal{B}}(\mathbf{x}_2,\mathbf{y}_2)\right\}.\\
\mathcal{N}^i_{\mathcal{A}\times\mathcal{B}}[(\mathbf{x}_1,\mathbf{y}_1).(\mathbf{x}_2,\mathbf{y}_2)] & \mathcal{N}^i_{\mathcal{A}\times\mathcal{B}}((\mathbf{x}_1.\mathbf{x}_2),(\mathbf{y}_1.\mathbf{y}_2))\\
&= T_c\left\{\mathcal{N}^i_{\mathcal{A}}((\mathbf{x}_1.\mathbf{x}_2)), \mathcal{N}^i_{\mathcal{B}}(\mathbf{y}_1.\mathbf{y}_2))\right\}\\
&\leq T_c\{T_c(\mathcal{N}^i_{\mathcal{A}}(\mathbf{x}_1), \mathcal{N}^i_{\mathcal{A}}(\mathbf{x}_2)), T_c(\mathcal{N}^i_{\mathcal{B}}(\mathbf{y}_1), \mathcal{N}^i_{\mathcal{B}}(\mathbf{y}_2))\}\\
&\leq T_c\{T_c(\mathcal{N}^i_{\mathcal{A}}(\mathbf{x}_1), \mathcal{N}^i_{\mathcal{B}}(\mathbf{y}_1)), T_c(\mathcal{N}^i_{\mathcal{A}}(\mathbf{x}_2), \mathcal{N}^i_{\mathcal{B}}(\mathbf{y}_2))\}
\end{aligned}$$

$$\leq T_c\left\{\mathcal{N}^i_{\mathcal{A}\times\mathcal{B}}((\mathbf{x}_1,\mathbf{y}_1)),\mathcal{N}^i_{\mathcal{A}\times\mathcal{B}}((\mathbf{x}_2,\mathbf{y}_2))\right\}.$$

Similarly, we get,

$$F^i_{\mathcal{A}\times\mathcal{B}}\left[(\mathbf{x}_1,\mathbf{y}_1)-(\mathbf{x}_2,\mathbf{y}_2)\right]\leq T_c\left\{F^i_{\mathcal{A}\times\mathcal{B}}((\mathbf{x}_1,\mathbf{y}_1)),F^i_{\mathcal{A}\times\mathcal{B}}((\mathbf{x}_2,\mathbf{y}_2))\right\}$$

$$F^i_{\mathcal{A}\times\mathcal{B}}\left[(\mathbf{x}_1,\mathbf{y}_1)\,.\,(\mathbf{x}_2,\mathbf{y}_2)\right]\leq T_c\left\{F^i_{\mathcal{A}\times\mathcal{B}}((\mathbf{x}_1,\mathbf{y}_1)),F^i_{\mathcal{A}\times\mathcal{B}}((\mathbf{x}_2,\mathbf{y}_2))\right\}$$

Hence $\mathcal{A}\times\mathcal{B}$ is also a NMFSR of $R_1\times R_2$ over T and T_c. □

Remark 5.3. However, $\mathcal{A}\times\mathcal{B}$ is a NMFSR of $R_1\times R_2$ over T_n and T_c. Then both $\mathcal{A}$ and $\mathcal{B}$ are need not be NMFSR of R_1 and R_2 respectively over T_n and T_c which is obvious from the accompanying case.

Example 5.4. Let $(Z_4,+,\cdot)$ and $(Z_2,+,\cdot)$ be a ring. Let $T_n\,(\mathbf{x},\mathbf{y})=\min(\mathbf{x},\mathbf{y})$ and $T_c(\mathbf{x},\mathbf{y})=\max(\mathbf{x},\mathbf{y})$. We define a NMFS $\mathcal{A}$ and $\mathcal{B}$ of Z_4 and Z_2 as

$\mathcal{A}=(\ <0$(0.9,0.8), (0.1.0.2), (0.5,0.6) $>$; <1(0.9,0.7), (0.1,0.2), (0.5,0.6) $>$ <2(0.8,0.6), (0.2,0.3), (0.6,0.7) $>$, <3(0.7,0.5), (0.3,0.2), (0.7,0.6)

$\mathcal{B}=(\ <0$(0.8,0.7), (0.2.0.3), (0.6,0.7) $>$; <1(0.7,0.7), (1,0) (0.3,0.4), (0.7,0.8) $>$).

$\mathcal{A}\times\mathcal{B}=\{\ <$ (0,0) (0.8,0.7) $>$, $<$ (0,1) (0.7,0.7) $>$, $<$ (1,0) (0.8,0.7) $>$, $<$ (1,1) (0.7,0.7) $>$, $<$ (2,0) (0.8,0.6) $>$, $<$ (2,1) (0.7,0.6) $>$, $<$ (3,0) (0.7,0.5) $>$, $<$ (3,1) (0.7,0.5) $>$)

It is clear that $\mathcal{A}\times\mathcal{B}$ a NMFSR of $Z_4\times Z_2$. But $\mathcal{A}$ is not a NMFSR of Z_2 as $\mathcal{N}^i_{\mathcal{A}}\,(1\cdot 0)=(0.1,0.3)$; $T_c\{\mathcal{N}^i_{\mathcal{A}}\,(1)\,,\mathcal{N}^i_{\mathcal{A}}\,(0)\}=(0.1,0.2)\ \mathcal{N}^i_{\mathcal{A}}\,(1\cdot 0)\nleq T_c\{\mathcal{N}^i_{\mathcal{A}}\,(1)\,,\mathcal{N}^i_{\mathcal{A}}\,(0)\ \}$

Corollary 5.5. *Let, for all $i\in\{1,\,2.\ldots\ n\}$, $(R_i,+,\cdot\ )$ are rings and $\mathcal{A}_i$ is a NMFSR of R_i over T_n and T_c .Then $\mathcal{A}_1\times\mathcal{A}_2\times\ldots.\mathcal{A}_n$ is a NMFSR of $R_1\times R_2\ldots.\times R_n$ over T_n and T_c, where $n\in N$*

Theorem 5.6. *If $\mathcal{A}$ and $\mathcal{B}$ are NMFLI(NMFRI) of R_1 and R_2 over T_n and T_c. Then $\mathcal{A}\times\mathcal{B}$ is also a NMFLI(NMFRI) of $R_1\times R_2$ With respect to T_n and T_c.*

Proof. Let $(x_1,y_1)\,(x_2,y_2)\in\mathcal{A}\times\mathcal{B}$. Assume $\mathcal{A}_1$ and $\mathcal{A}_2$ are NMFLI of R_1 and R_2 respectively over T_n and T_c.

We have to show that $\mathcal{A}\times\mathcal{B}$ is also a NMFLI of $R_1\times R_2$ over T_n and T_c

By theorem 5.2,

$\mathcal{A}\times\mathcal{B}$ is also a NMFSR of $R_1\times R_2$ over T_n and T_c .

It is enough to show

$$\mu^i_{\mathcal{A}\times\mathcal{B}}\left[(\mathbf{x}_1,\mathbf{y}_1)\,(\mathbf{x}_2,\mathbf{y}_2)\right]\geq\mu^i_{\mathcal{A}\times\mathcal{B}}((\mathbf{x}_2,\mathbf{y}_2))$$

$$\mathcal{N}^i_{\mathcal{A}\times\mathcal{B}}\left[(\mathbf{x}_1,\mathbf{y}_1)\,(\mathbf{x}_2,\mathbf{y}_2)\right]\leq\mathcal{N}^i_{\mathcal{A}\times\mathcal{B}}((\mathbf{x}_2,\mathbf{y}_2))$$

$$F^i_{\mathcal{A}\times\mathcal{B}}\left[(\mathbf{x}_1,\mathbf{y}_1)\,(\mathbf{x}_2,\mathbf{y}_2)\right]\leq F^i_{\mathcal{A}\times\mathcal{B}}((\mathbf{x}_2,\mathbf{y}_2))$$

$$
\begin{aligned}
(ie)\mu^i_{\mathcal{A}\times\mathcal{B}}\left[(\mathbf{x}_1,\mathbf{y}_1)(\mathbf{x}_2,\mathbf{y}_2)\right] &= \mu^i_{\mathcal{A}\times\mathcal{B}}\left[((\mathbf{x}_1\mathbf{x}_2,\mathbf{y}_1\mathbf{y}_2))\right]\\
&= T_n(\mu^i_{\mathcal{A}}(\mathbf{x}_1\mathbf{x}_2),\mu^i_{\mathcal{B}}(\mathbf{y}_1\mathbf{y}_2))\\
&\geq T_n(\mu^i_{\mathcal{A}}(\mathbf{x}_2),\mu^i_{\mathcal{B}}(\mathbf{y}_2))\\
&= \mu^i_{\mathcal{A}\times\mathcal{B}}((\mathbf{x}_2,\mathbf{y}_2))\\
\mathcal{N}^i_{\mathcal{A}\times\mathcal{B}}\left[(\mathbf{x}_1,\mathbf{y}_1)(\mathbf{x}_2,\mathbf{y}_2)\right] &= \mathcal{N}^i_{\mathcal{A}\times\mathcal{B}}\left[((\mathbf{x}_1\mathbf{x}_2,\mathbf{y}_1\mathbf{y}_2))\right]\\
&= T_c(\mathcal{N}^i_{\mathcal{A}}(\mathbf{x}_1\mathbf{x}_2),\mathcal{N}^i_{\mathcal{B}}(\mathbf{y}_1\mathbf{y}_2))\\
&\leq T_c(\mathcal{N}^i_{\mathcal{A}}(\mathbf{x}_2),\mathcal{N}^i_{\mathcal{B}}(\mathbf{y}_2))\\
&= \mathcal{N}^i_{\mathcal{A}\times\mathcal{B}}((\mathbf{x}_2,\mathbf{y}_2))
\end{aligned}
$$

Similarly, $F^i_{\mathcal{A}\times\mathcal{B}}\left[(\mathbf{x}_1,\mathbf{y}_1)(\mathbf{x}_2,\mathbf{y}_2)\right] \leq F^i_{\mathcal{A}\times\mathcal{B}}((\mathbf{x}_2,\mathbf{y}_2))$

Hence $\mathcal{A}\times\mathcal{B}$ is also a NMFLI of $R_1\times R_2$ over T_n and T_c

Similarly, we can show it for NMFRI. □

Theorem 5.7. *If $\mathcal{A}$ and $\mathcal{B}$ are NMFI of R_1 and R_2 over T_n and T_c .Then $\mathcal{A}\times\mathcal{B}$ is also a NMFI of $R_1\times R_2$ with respect to T_n and T_c.*

Proof. Follows from above theorem. □

Example 5.8. Let $(Z_2,+,\cdot)$ be a ring. Define

$\mathcal{A}$ = {(< (0,(0.9,0.7),(0.1,0.5),(0.2,0.3)), (1,(0.8,0.6),(0.2,0.5),(0.3,0.6)) > }

$\mathcal{B}$ ={ < (0,(0.8,0.7),(0.2,0.3),(0.1,0.4)), (1,(0.7,0.6),(0.2,0.3),(0.2,0.5)) > } be two NMFS of Z_2 under T_n and T_c. Let us consider $T_n(\mathbf{x},\mathbf{y})=\mathbf{x}\mathbf{y}$; $T_c(\mathbf{x},\mathbf{y})=\mathbf{x}+\mathbf{y}-\mathbf{x}\mathbf{y}$. Then $\mathcal{A}\times\mathcal{B}$ is NMFI with T_n and T_c of $Z_2\times Z_2$.

Corollary 5.9. *Let, for all $i\in\{1,2\ldots,n\}$, $(R_i,+,\cdot)$ are rings and A_i is a NMFI of R_i. Then $\mathcal{A}_1\times\mathcal{A}_2\times\ldots\times\mathcal{A}_n$ is a NMFI of $R_1\times R_2\ldots\times R_n$ where $n\in N$.*

Definition 5.10. If $\mathcal{A}=(\mu^i_{\mathcal{A}},\mathcal{N}^i_{\mathcal{A}},F^i_{\mathcal{A}})$ is a NMFS in R, then $\mathcal{F}(\mathcal{A})=\mathcal{B}$, is the NMFS defined by

$$\mathcal{F}\left(T^i_{\mathcal{A}}\right)(\mathbf{y}) = \begin{cases} \sup_{\mathbf{x}\in\mathcal{F}^{-1}(\mathbf{y})}\left(\mu^i_{\mathcal{A}}\right)(\mathbf{x}), & \text{if } \mathcal{F}^{-1}(\mathbf{y})\neq 0\\ 0, & \text{otherwise}\end{cases}$$

$$\mathcal{F}\left(\mathcal{N}^i_{\mathcal{A}}\right)(\mathbf{y}) = \begin{cases} \sup_{\mathbf{x}\in\mathcal{F}^{-1}(\mathbf{y})}\left(\mathcal{N}^i_{\mathcal{A}}\right)(\mathbf{x}), & \text{if } \mathcal{F}^{-1}(\mathbf{y})\neq 0\\ 1, & \text{otherwise}\end{cases}$$

$$\mathcal{F}\left(F^i_{\mathcal{A}}\right)(\mathbf{y}) = \begin{cases} \inf_{\mathbf{x}\in\mathcal{F}^{-1}(\mathbf{y})}\left(F^i_{\mathcal{A}}\right)(\mathbf{x}), & \text{if} \mathcal{F}^{-1}(\mathbf{y})\neq 0\\ 1, & \text{otherwise}\end{cases}$$

Where $\mathcal{F}$ is ring homomorphism of R onto R_1. Also $\mathcal{F}^{-1}(\mathcal{B}) = \left\{\left\langle\mathbf{x},\mathcal{F}^{-1}\left(\mu^i_{\mathcal{B}}\right)(\mathbf{x}),\mathcal{F}^{-1}\left(\mathcal{N}^i_{\mathcal{B}}\right)(\mathbf{x}),\mathcal{F}^{-1}\left(F^i_{\mathcal{B}}\right)(\mathbf{x})\right\rangle : \mathbf{x}\in\mathcal{A}\right\}$ where $\mathcal{F}^{-1}(\mathcal{B})(\mathbf{x})=(\mathcal{B})(\mathcal{F}(\mathbf{x}))$.

Theorem 5.11. *Let R and R_1 be any two rings and $\mathcal{F}$ be a homomorphism from R onto R_1. If $\mathcal{A} \in NMFSR$ of R under T_n and T_c then $\mathcal{F}(\mathcal{A}) \in NMFSR$ of R_1 over T_n and T_c.*

Proof. Let $\mathbf{x}_1, \mathbf{x}_2 \in R$ and $\mathbf{y}_1, \mathbf{y}_2 \in R_1$. If $\mathcal{A} \in$ NMFSR of R . Then

$$
\begin{aligned}
(i)\ \mathcal{F}\left(\left(\mu^i_{\mathcal{A}}\right)(\mathbf{y}_1-\mathbf{y}_2)\right) &= \sup_{\mathbf{x}\in\mathcal{F}^{-1}(\mathbf{y})} \mu^i_{\mathcal{A}}(\mathbf{x}_1-\mathbf{x}_2) \\
&\geq \sup_{\mathbf{x}\in\mathcal{F}^{-1}(\mathbf{y})} \left(T_n(\mu^i_{\mathcal{A}}(\mathbf{x}_1), \mu^i_{\mathcal{A}}(\mathbf{x}_2)\right) \\
&= T_n(\sup_{\mathbf{x}\in\mathcal{F}^{-1}(\mathbf{y})} (\mu^i_{\mathcal{A}}(\mathbf{x}_1), \sup_{\mathbf{x}\in\mathcal{F}^{-1}(\mathbf{y})} \mu^i_{\mathcal{A}}(\mathbf{x}_2)) \\
&= T_n\left(\mathcal{F}\left(\mu^i_{\mathcal{A}}(\mathbf{y}_1)\right), \mathcal{F}\left(\mu^i_{\mathcal{A}}(\mathbf{y}_2)\right)\right) \\
\text{Similarly, } \mathcal{F}\left(\mathcal{N}^i_{\mathcal{A}}(\mathbf{y}_1-\mathbf{y}_2)\right) &\leq T_c\left(\mathcal{F}\left(\mathcal{N}^i_{\mathcal{A}}(\mathbf{y}_1)\right), \mathcal{F}\left(\mathcal{N}^i_{\mathcal{A}}(\mathbf{y}_2)\right)\right) \\
\mathcal{F}\left(F^i_{\mathcal{A}}(\mathbf{y}_1-\mathbf{y}_2)\right) &\leq T_c\left(\mathcal{F}\left(F^i_{\mathcal{A}}(\mathbf{y}_1)\right), \mathcal{F}\left(F^i_{\mathcal{A}}(\mathbf{y}_2)\right)\right).
\end{aligned}
$$

$$
\begin{aligned}
(ii)\ \mathcal{F}\left(\left(\mu^i_{\mathcal{A}}\right)(\mathbf{y}_1\mathbf{y}_2)\right) &= \sup_{\mathbf{x}\in\mathcal{F}^{-1}(\mathbf{y})} \mu^i_{\mathcal{A}}(\mathbf{x}_1\mathbf{x}_2) \\
&\geq \sup_{\mathbf{x}\in\mathcal{F}^{-1}(\mathbf{y})} T_n\left(\mu^i_{\mathcal{A}}(\mathbf{x}_1), \mu^i_{\mathcal{A}}(\mathbf{x}_2)\right) \\
&= T_n(\sup_{\mathbf{x}\in\mathcal{F}^{-1}(\mathbf{y})} \mu^i_{A}(\mathbf{x}_1), \sup_{\mathbf{x}\in\mathcal{F}^{-1}(\mathbf{y})} \mu^i_{A}(\mathbf{x}_2)) \\
&= T_n\left(\mathcal{F}\left(\mu^i_{\mathcal{A}}(\mathbf{y}_1)\right), \mathcal{F}\left(\mu^i_{\mathcal{A}}(\mathbf{y}_2)\right)\right) \\
\textit{Similarly, } \mathcal{F}\left(\mathcal{N}^i_{\mathcal{A}}(\mathbf{y}_1\mathbf{y}_2)\right) &\leq T_c\left(\mathcal{F}\left(\mathcal{N}^i_{\mathcal{A}}(\mathbf{y}_1)\right), \mathcal{F}\left(\mathcal{N}^i_{\mathcal{A}}(\mathbf{y}_2)\right)\right) \\
\mathcal{F}\left(F^i_{\mathcal{A}}(\mathbf{y}_1\mathbf{y}_2)\right) &\leq T_c\left(\mathcal{F}\left(F^i_{\mathcal{A}}(\mathbf{y}_1)\right), \mathcal{F}\left(F^i_{\mathcal{A}}(\mathbf{y}_2)\right)\right)
\end{aligned}
$$

Hence then $\mathcal{F}(\mathcal{A}) \in$ NMFSR of R_1 over T_n and T_c. □

Theorem 5.12. *Let R and R_1 be any two rings and $\mathcal{F}$ be a homomorphism from R onto R_1. If $\mathcal{B} \in NMFSR$ of R_1 under T_n and T_c then $\mathcal{F}^{-1}(\mathcal{B})$ NMFSR of R under T and T_c*

Proof. Let $\mathbf{x}, \mathbf{y} \in R$. Let $B \in$ NMFSR of R_1. Then

$$
\begin{aligned}
(i)\ \ \mathcal{F}^{-1}\left(\left(\mu^i_{\mathcal{B}}\right)(\mathbf{x}-\mathbf{y})\right) &= \mu^i_{\mathcal{B}}(\mathcal{F}(\mathbf{x}-\mathbf{y})) \\
&= T^i_{\mathcal{B}}(\mathcal{F}(\mathbf{x}) - \mathcal{F}(\mathbf{y})) \\
&\geq T_n(\mu^i_{\mathcal{B}}(\mathcal{F}(\mathbf{x})), \mu^i_{\mathcal{B}}(\mathcal{F}(\mathbf{y}))) \\
&= T_n(\mathcal{F}^{-1}\left(\mu^i_{\mathcal{B}}\right)(\mathbf{x}), \mathcal{F}^{-1}\left(\mu^i_{\mathcal{B}}\right)(\mathbf{y})).
\end{aligned}
$$

Similarly, $\mathcal{F}^{-1}\left(\mathcal{N}^i_{\mathcal{B}}\right)(\mathbf{x}-\mathbf{y}) \leq T_c\left(\mathcal{F}^{-1}\left(\mathcal{N}^i_{\mathcal{B}}\right)(\mathbf{x}), \mathcal{F}^{-1}\left(\mathcal{N}^i_{\mathcal{B}}\right)(\mathbf{y})\right)$

$$\mathcal{F}^{-1}\left(F^i_{\mathcal{B}}\right)(\mathbf{x}-\mathbf{y}) \leq T_c\left(\mathcal{F}^{-1}\left(F^i_{\mathcal{B}}\right)(\mathbf{x}), \mathcal{F}^{-1}\left(F^i_{\mathcal{B}}\right)(\mathbf{y})\right)$$

$$\begin{aligned}(ii)\quad \mathcal{F}^{-1}\left(\left(\mu_{\mathcal{B}}^{i}\right)(\mathbf{xy})\right) &= \mu_{\mathcal{B}}^{i}\left(\mathcal{F}(\mathbf{xy})\right)\\ &= \mu_{\mathcal{B}}^{i}\left(\mathcal{F}(\mathbf{x})\mathcal{F}(\mathbf{y})\right)\\ &\geq T_n\left(\mu_{\mathcal{B}}^{i}\left(\mathcal{F}(\mathbf{x})\right), \mu_{\mathcal{B}}^{i}(\mathcal{F}(\mathbf{y}),)\right)\\ &= T_n\left(\mathcal{F}^{-1}\left(\mu_{\mathcal{B}}^{i}\right)(\mathbf{x}), \mathcal{F}^{-1}(\mu_{\mathcal{B}}^{i})(\mathbf{y})\right)\end{aligned}$$

Similarly, $\mathcal{F}^{-1}\left(\mathcal{N}_{\mathcal{B}}^{i}\right)(\mathbf{xy}) \leq T_c\left(\mathcal{F}^{-1}\left(\mathcal{N}_{\mathcal{B}}^{i}\right)(\mathbf{x}), \mathcal{F}^{-1}\left(\mathcal{N}_{\mathcal{B}}^{i}\right)(\mathbf{y})\right)$

$$\mathcal{F}^{-1}\left(F_{\mathcal{B}}^{i}\right)(\mathbf{xy}) \leq T_c\left(\mathcal{F}^{-1}\left(F_{\mathcal{B}}^{i}\right)(\mathbf{x}), \mathcal{F}^{-1}\left(F_{\mathcal{B}}^{i}\right)(\mathbf{y})\right)$$

Hence $\mathcal{F}^{-1}(\mathcal{B})$ is a NMFSR of R under T_n and T_c. □

6. Conclusion

We deliberated neutrosophic multifuzzy subrings and ideals along with triangular norm and made use of the concepts of direct product, image and inverse image of homomorphism. We have established some theorems and results. This study will give base for our upcoming work.

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On λ-statistical and V_λ-statistical summability in neutrosophic-2-normed spaces

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Abstract. In the present paper, we aim to define λ-statistical summability, V_λ-statistical summability in neutrosophic-2-normed spaces (briefly called $N-2-NS$) and study some relationships among these notions. We give an example that shows λ-statistical summability is stronger method in neutrosophic-2-normed spaces. Finally, we define λ-statistically Cauchy sequence and λ-statistically completeness in neutrosophic-2-normed spaces and obtain the Cauchy convergence criteria in these spaces.

Keywords: λ-statistical convergence, V_λ-summable, λ-statistical Cauchy, neutrosophic-2-normed spaces.

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1. Introduction

The idea of λ-statistical convergence was explored by Mursaleen [21] as a generalization of statistical convergence, initially introduced in [4] and [33] independently.

"Let $\lambda = (\lambda_n)\ ; \lambda_n \in \mathbb{R}^+ = (0, \infty)$ be a non decreasing sequence satisfying $\lim\limits_{n\to\infty} \lambda_n = \infty,\ \lambda_{n+1} \leq \lambda_n + 1,\ \lambda_1 = 1$ and $I_n = [n - \lambda_n + 1, n]$. For $\mathfrak{R} \subseteq \mathbb{N}$, the λ-density of $\mathfrak{R}$ is defined by $\delta_\lambda(\mathfrak{R}) = \lim\limits_n \frac{1}{\lambda_n} |\{k \in I_n : k \in \mathfrak{R}\}|$. A sequence $\mathfrak{u} = (\mathfrak{u}_k)$ of numbers is said to be λ-statistical convergent to $\mathfrak{u}_0$ if for each $\mathfrak{y} > 0$, $\lim\limits_n \frac{1}{\lambda_n} |\{k \in I_n : |\mathfrak{u}_k - \mathfrak{u}_0| \geq \mathfrak{y}\}| = 0$, i.e., $\delta_\lambda(\mathfrak{R}_\mathfrak{y}) = 0$, where $\mathfrak{R}_\mathfrak{y} = \{k \in I_n : |\mathfrak{u}_k - \mathfrak{u}_0| \geq \mathfrak{y}\}$. We write, in this case $\mathcal{S}_\lambda - \lim\limits_k \mathfrak{u}_k = \mathfrak{u}_0$."Subsequently, statistical convergence and its generalizations have been developed by numerous authors including Connor [3], Fridy [6], Hazarika et al. [8, 9], Kumar et al. [15], Maddox [20], Šalát [36] and many others.

On the other hand, many problems of real life can't be modeled via crispness due to huge uncertainty in data. In view of this Zadeh [41] defined fuzzy sets as generalization of crisp sets to deal with such problems.

One of interesting generalizations of fuzzy sets is due to Atanassov [1], called intuitionistic fuzzy sets by adding the non-membership function along with the membership function to the fuzzy sets. These sets have been applied to introduce new norms (see [5], [22]), topology [31], and metric [27] and found very useful where the crisp norms are not sufficient to work due to huge uncertainty. Intuitionistic fuzzy sets are naturally used to define intuitionistic fuzzy normed spaces [34]. Recently, statistical convergence and its generalizations have been extended and developed in these spaces (see [2], [15], [25], [26] and [38]).

Another, interesting generalization of fuzzy sets is given by Smarandache [35] by introducing the indeterminacy function to the intuitionistic fuzzy sets. For ongoing development on neutrosophic set (NS) and its applications, we refer to the reader [10], [18], [23], [29-31], etc. Kirişçi and Şimşek [13] used neutrosophic sets to define neutrosophic norm and studied statistical convergence in neutrosophic normed spaces(NNS). Nowadays, the area of summability in these spaces is of much interest. For a broad view in this direction, we recommend [28], [37-39], etc. Several summability approaches have been created, including statistical convergence, lacunary statistical convergence, and ideal convergence in these spaces (see [11], [12], [14], [16], [17], and [32]). Recently, in [24] the concept of neutrosophic-2-norm is introduced where the authors studied statistical convergence in neutrosophic-2-normed spaces. In the present study, we continue to define a more general summability method, called $\mathcal{S}_\lambda$-summability in $N-2-NS$ and develop some of its properties.

2. Prelimanaries

This section starts with a brief review on certain definitions and results needed in the sequel.

"For $\lambda = (\lambda_n)$ as defined above, the generalized de la Vallée-Poussin mean of $\mathfrak{u} = (\mathfrak{u}_k)$ is defined by $t_n(\mathfrak{u}) = \frac{1}{\lambda_n}\sum\limits_{k\in I_n} \mathfrak{u}_k$. Further, $\mathfrak{u} = (\mathfrak{u}_k)$ is called V_λ-summable to $\mathfrak{u}_0$(see[19]) if $\lim\limits_{n\to\infty} t_n(\mathfrak{u}) = \mathfrak{u}_0$. Let

$$[V_\lambda] = \left\{\mathfrak{u} = (\mathfrak{u}_n) : \exists\ \mathfrak{u}_0 \in \mathbb{R}, \lim_{n\to\infty} \frac{1}{\lambda}\sum_{k\in I_n} |\mathfrak{u}_k - \mathfrak{u}_0| = 0\right\}\text{"}$$

Definition 2.1 [40] "Let $\mathfrak{T} = [0,1]$. A binary operation $\circ : \mathfrak{T}\times\mathfrak{T} \to \mathfrak{T}$ is t-norm if $\forall\ \mathfrak{c},\ \mathfrak{e},\ \mathfrak{g},\ \mathfrak{h} \in \mathfrak{T}$ we have

1) $\circ$ is continuous, commutative and associative,

2) $\mathfrak{e} = \mathfrak{e} \circ 1$,

3) $\mathfrak{c} \circ \mathfrak{e} \leq \mathfrak{g} \circ \mathfrak{h}$ whenever $\mathfrak{c} \leq \mathfrak{g}$ and $\mathfrak{e} \leq \mathfrak{h}$."

Definition 2.2 [40]"Let $\mathfrak{T} = [0, 1]$. A binary operation $\diamond : \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$ is t-conorm if $\forall$ $\mathfrak{c},\ \mathfrak{e},\ \mathfrak{g},\ \mathfrak{h} \in \mathfrak{T}$ we have

1) $\diamond$ is continuous, commutative and associative,

2) $\mathfrak{e} = \mathfrak{e} \diamond 0$,

3) $\mathfrak{c} \diamond \mathfrak{e} \leq \mathfrak{g} \diamond \mathfrak{h}$ whenever $\mathfrak{c} \leq \mathfrak{g}$ and $\mathfrak{e} \leq \mathfrak{h}$."

Kirişci and Şimşek introduced the notion NNS in the following manner [13].

Definition 2.3 [13]"Consider $\mathfrak{F}$ to be a vector space, $N = \{\langle \mathfrak{V}, G(\mathfrak{V}), B(\mathfrak{V}), Y(\mathfrak{V})\rangle : \mathfrak{V} \in \mathfrak{F}\}$ be a normed space s.t $N : \mathfrak{F} \times \mathbb{R}^{+} \to [0,\ 1]$. Let $\circ, \diamond$ be t-norm and t-conorm, respectively. If the followings conditions hold, then the four tuple $\mathfrak{U} = (\mathfrak{F}, N,\ \circ, \diamond\ )$ is called NNS, for $\mathfrak{r}, \mathfrak{s} \in \mathfrak{F}$, $\varrho,\ \omega > 0$ and for each $\varsigma \neq 0$,

(i) $0 \leq G(\mathfrak{r},\ \varrho) \leq 1$, $0 \leq B(\mathfrak{r},\ \varrho) \leq 1$, $0 \leq Y(\mathfrak{r},\ \varrho) \leq 1$ for $\varrho \in \mathbb{R}^{+}$;

(ii) $G(\mathfrak{r},\ \varrho) +\ B(\mathfrak{r},\ \varrho) + Y(\mathfrak{r},\ \varrho) \leq 3$ for $\varrho \in \mathbb{R}^{+}$;

(iii) $G(\mathfrak{r},\ \varrho) = 1$ (for $\varrho > 0$) iff $\mathfrak{r} = 0$;

(iv) $G(\varsigma\mathfrak{r}, \varrho) = G\left(\mathfrak{r},\ \frac{\varrho}{|\varsigma|}\right)$;

(v) $G(\mathfrak{r},\ \omega) \circ G(\mathfrak{s},\ \varrho) \leq G(\mathfrak{r} + \mathfrak{s},\ \omega + \varrho)$;

(vi) $G(\mathfrak{r},\ .)$ is non-decreasing and continuous function ;

(vii) $\lim_{\varrho \to \infty} G(\mathfrak{r},\ \varrho) = 1$;

(viii) $B(\mathfrak{r}, \varrho) = 0$ (for$\varrho > 0$) iff $\mathfrak{r} = 0$;

(ix) $B(\varsigma\mathfrak{r}, \varrho) = B\left(\mathfrak{r},\ \frac{\varrho}{|\varsigma|}\right)$;

(x) $B(\mathfrak{r},\ \omega) \diamond B(\mathfrak{s},\ \varrho) \geq B(\mathfrak{r} + \mathfrak{s}, \varrho + \omega)$;

(xi) $B(\mathfrak{r},\ .)$ is non-increasing and continuous function;

(xii) $\lim_{\lambda \to \infty} B(\mathfrak{r}, \varrho) = 0$;

(xiii) $Y(\mathfrak{r}, \varrho) = 0$ (for $\varrho > 0$) iff $\mathfrak{r} = 0$;

(xiv) $Y(\varsigma\mathfrak{r}, \varrho) = Y\left(\mathfrak{r},\ \frac{\varrho}{|\varsigma|}\right)$;

(xv) $Y(\mathfrak{r},\ \omega) \diamond Y(\mathfrak{s},\ \varrho) \geq Y(\mathfrak{r} + \mathfrak{s}, \varrho + \omega)$;

(xvi) $Y(\mathfrak{r},\ .)$ is non-increasing and continuous function;

(xvii) $\lim_{\lambda \to \infty} Y(\mathfrak{r}, \varrho) = 0$;

(xviii) If $\varrho \leq 0$, then $G(\mathfrak{r}, \varrho) = 0$, $B(\mathfrak{r}, \varrho) = 1$ and $Y(\mathfrak{r}, \varrho) = 1$.

Then $N = (G, B, Y)$ is called the neutrosophic norm."

We now recall the idea of two norm introduced in the paper [7].

Definition 2.4 [7]"Let $\mathfrak{U}$ be a linear space of dimension $d > 1$. A function $\|.,.\| : \mathfrak{U} \times \mathfrak{U} \to \mathbb{R}$ satisfying the prerequisites specified below: For all $\mathfrak{s}, \mathfrak{t}, \mathfrak{l} \in \mathfrak{U}$, and scalar $\mathfrak{c}$, we have

(i) $||\mathfrak{s}, \mathfrak{t}|| = 0$ iff $\mathfrak{s}$ and $\mathfrak{t}$ are linearly dependent;

(ii) $||\mathfrak{s}, \mathfrak{t}|| = ||\mathfrak{t}, \mathfrak{s}||$;

(iii)$||\mathfrak{c}\mathfrak{s}, \mathfrak{t}|| = |\mathfrak{c}|||\mathfrak{s}, \mathfrak{t}||$ and
(iv) $||\mathfrak{s}, \mathfrak{t} + \mathfrak{l}|| \leq ||\mathfrak{s}, \mathfrak{t}|| + ||\mathfrak{s}, \mathfrak{l}||$.
The pair $(\mathfrak{U}, ||., .||)$ is then called an 2-normed space.

Let $\mathfrak{U} = \mathbb{R}^2$ and for $\mathfrak{s} = (\mathfrak{s}_1, \mathfrak{s}_2)$ and $\mathfrak{t} = (\mathfrak{t}_1, \mathfrak{t}_2)$ we define $||\mathfrak{s}, \mathfrak{t}|| = |\mathfrak{s}_1\mathfrak{t}_2 - \mathfrak{s}_2\mathfrak{t}_1|$, then $||\mathfrak{s}, \mathfrak{t}||$ is a 2-norm on $\mathfrak{U} = \mathbb{R}^2$".

Recently, Murtaza et al [24] defined neutrosophic-2- normed spaces as follows.

Definition 2.5 [24] "Consider $\mathfrak{F}$ to be a vector space, $N_2 = (\{(\mathfrak{r}, \mathfrak{s}), G(\mathfrak{r}, \mathfrak{s}), B(\mathfrak{r}, \mathfrak{s}), Y(\mathfrak{r}, \mathfrak{s})\} : (\mathfrak{r}, \mathfrak{s}) \in \mathfrak{F} \times \mathfrak{F})$ be a 2−normed space s.t $N_2 : \mathfrak{F} \times \mathfrak{F} \times \mathbb{R}^+ \to [0, 1]$. Let $\circ, \diamond$ be t-norm and t-conorm respectively. If the following conditions hold, then the four-tuple $\mathfrak{U} = (\mathfrak{F}, N_2, \circ, \diamond)$ is called a neutrosophic 2−normed spaces (briefly $N - 2 - NS$) if for each $\mathfrak{r}, \mathfrak{s}, \mathfrak{t} \in \mathfrak{U}$, $\varrho, \omega \geq 0$ and $\varsigma \neq 0$:

(i) $0 \leq G(\mathfrak{r}, \mathfrak{s}; \varrho) \leq 1$, $0 \leq B(\mathfrak{r}, \mathfrak{s}; \varrho) \leq 1$ and $0 \leq Y(\mathfrak{r}, \mathfrak{s}; \varrho) \leq 1$ for $\varrho \in \mathbb{R}^+$;
(ii) $G(\mathfrak{r}, \mathfrak{s}; \varrho) + B(\mathfrak{r}, \mathfrak{s}; \varrho) + Y(\mathfrak{r}, \mathfrak{s}; \varrho) \leq 3$;
(iii) $G(\mathfrak{r}, \mathfrak{s}; \varrho) = 1$ iff $\mathfrak{r}, \mathfrak{s}$ are linearly dependent;
(iv) $G(\varsigma\mathfrak{r},\ \mathfrak{s};\ \varrho) = G(\mathfrak{r}, \mathfrak{s}; \frac{\varrho}{|\varsigma|})$ for each $\varsigma \neq 0$;
(v) $G(\mathfrak{r}, \mathfrak{s}; \varrho) \circ G(\mathfrak{r}, \mathfrak{t}; \omega) \leq G(\mathfrak{r}, \mathfrak{s} + \mathfrak{t}; \varrho + \omega)$;
(vi) $G(\mathfrak{r}, \mathfrak{s}; .) : (0, \infty) \to [0, 1]$ is non-decreasing and continuous function;
(vii) $\lim_{\varrho \to \infty} G(\mathfrak{r}, \mathfrak{s}; \varrho) = 1$;
(viii) $G(\mathfrak{r}, \mathfrak{s}; \varrho) = G(\mathfrak{s}, \mathfrak{r}; \varrho)$
(ix) $B(\mathfrak{r}, \mathfrak{s}; \varrho) = 0$ iff $\mathfrak{r}, \mathfrak{s}$ are linearly dependent;
(x) $B(\varsigma\mathfrak{r}, \mathfrak{s}; \varrho) = B(\mathfrak{r}, \mathfrak{s}; \frac{\varrho}{|\varsigma|})$ for each $\varsigma \neq 0$;
(xi) $B(\mathfrak{r}, \mathfrak{s}; \varrho) \diamond B(\mathfrak{r}, \mathfrak{t}; \omega) \geq B(\mathfrak{r}, \mathfrak{s} + \mathfrak{t}; \varrho + \omega)$;
(xii) $B(\mathfrak{r}, \mathfrak{s}; .) : (0, \infty) \to [0, 1]$ is non-increasing and continuous function;
(xiii) $\lim_{\varrho \to \infty} B(\mathfrak{r}, \mathfrak{s}; \varrho) = 0$;
(xiv) $B(\mathfrak{r}, \mathfrak{s}; \varrho) = B(\mathfrak{s}, \mathfrak{r}; \varrho)$;
(xvi) $Y(\mathfrak{r}, \mathfrak{s}; \varrho) = 0$ iff $\mathfrak{r}, \mathfrak{s}$ are linearly dependent;
(xv)$Y(\varsigma\mathfrak{r},\ \mathfrak{s};\ \varrho) = Y(\mathfrak{r},\ \mathfrak{s}; \frac{\varrho}{|\varsigma|})$ for each $\varsigma \neq 0$;
(xvi) $Y(\mathfrak{r}, \mathfrak{s}; \varrho) \diamond Y(\mathfrak{r}, \mathfrak{t}; \omega) \geq Y(\mathfrak{r}, \mathfrak{s} + \mathfrak{t}; \varrho + \omega)$;
(xvii) $Y(\mathfrak{r}, \mathfrak{s}; .) : (0, \infty) \to [0, 1]$ is non-increasing and continuous function;
(xviii) $\lim_{\lambda \to \infty} Y(\mathfrak{r}, \mathfrak{s}; \varrho) = 0$;
(xix) $Y(\mathfrak{r}, \mathfrak{s}; \varrho) = Y(\mathfrak{s}, \mathfrak{r}; \varrho)$;
(xx) if $\varrho \leq 0$, then $G(\mathfrak{r}, \mathfrak{s}; \varrho) = 0$, $B(\mathfrak{r}, \mathfrak{s}; \varrho) = 1$, $Y(\mathfrak{r}, \mathfrak{s}; \varrho) = 1$.

In this case, $N_2 = (G, B, Y)_2$ is called a neutrosophic 2-norm. From now on wards, unless otherwise stated by $\mathfrak{U}$ we shall denote the $N - 2 - NS$ $(\mathfrak{F}, N_2, \circ, \diamond)$

A sequence $\mathfrak{u} = (\mathfrak{u}_k)$ in $\mathfrak{U}$ is called convergent to $\mathfrak{u}_0$ if for each $\mathfrak{y} > 0$ and $\varrho > 0$, $\exists\, k_0 \in \mathbb{N}$ s.t $G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) > 1 - \mathfrak{y}$, $B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}$ and $Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}\ \forall\ k \geq k_0$ and $\mathfrak{w} \in \mathfrak{U}$ which is equivalently to say $\lim_{k\to\infty} G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) = 1$, $\lim_{k\to\infty} B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) = 0$ and $\lim_{k\to\infty} Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) = 0$. In present case, we denote $N_2 - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_0$.

A sequence $\mathfrak{u} = (\mathfrak{u}_k)$ in $\mathfrak{U}$ is called Cauchy if for each $\mathfrak{y} > 0$ and $\varrho > 0$, $\exists\, k_0 \in \mathbb{N}$ s.t $G(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) > 1 - \mathfrak{y}$, $B(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) < \mathfrak{y}$ and $Y(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) < \mathfrak{y}\ \forall\ k, p \geq k_0$ and $\forall$ $\mathfrak{w} \in \mathfrak{U}$."

3. λ-Statistical Convergence in N-2-NS

In this section, we define and study λ-Statistical Convergence in $N-2-NS$ and develop some of its properties.

Definition 3.1 A sequence $\mathfrak{u} = (\mathfrak{u}_k)$ in $N-2-NS$ $\mathfrak{U}$ is called λ-statistical convergent (or $\mathcal{S}_\lambda$-convergent) to $\mathfrak{u}_0$ if for each $\mathfrak{y} > 0$ and $\varrho > 0$

$$\lim_n \frac{1}{\lambda_n}\left|\Big\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y} \text{ or } B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \geq \mathfrak{y},\ Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \geq \mathfrak{y}\Big\}\right| = 0\ \forall\ \mathfrak{w} \in \mathfrak{U};$$

or equivalently, $\delta_\lambda(\mathcal{A}(\mathfrak{y}, \varrho)) = 0$, where

$$\mathcal{A}(\mathfrak{y}, \varrho) = \{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y} \text{ or } B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \geq \mathfrak{y},\ Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \geq \mathfrak{y}\}.$$

In present case, we denote $\mathcal{S}_\lambda(N_2) - \lim\limits_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_0$.

We now give the following Lemma:

Lemma 3.1 For any sequence $\mathfrak{u} = (\mathfrak{u}_k)$ in $\mathfrak{U}$, the subsequent assertions are equivalent:

(i) $\mathcal{S}_\lambda(N_2) - \lim\limits_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_0$;

(ii) $\delta_\lambda\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y}\} = \delta_\lambda\{k \in I_n : B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \geq \mathfrak{y}\} = \delta_\lambda\{k \in I_n : Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \geq \mathfrak{y}\} = 0$;

(iii) $\delta_\lambda\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) > 1 - \mathfrak{y}$ and $B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}$, $Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}\} = 1$;

(iv) $\delta_\lambda\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) > 1 - \mathfrak{y}\} = \delta_\lambda\{k \in I_n : B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}\} = \delta_\lambda\{k \in I_n : Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}\} = 1$ and

(v) $\mathcal{S}_\lambda(N_2) - \lim\limits_{k\to\infty} G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) = 1$, $\mathcal{S}_\lambda(N_2) - \lim\limits_{k\to\infty} B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) = \mathcal{S}_\lambda(N_2) - \lim\limits_{k\to\infty} Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) = 0$.

We now have the following interesting implication.

Theorem 3.1 Let $\mathfrak{u} = (\mathfrak{u}_k)$ be any sequence in $N-2-NS$ $\mathfrak{U}$. If $N_2 - \lim\limits_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_0$, then $\mathcal{S}_\lambda(N_2) - \lim\limits_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_0$.

Proof. Let $N_2 - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_0$. Then for each $\mathfrak{y} > 0$ and $\varrho > 0, \exists$ an integer $k_0 \in \mathbb{N}$ s.t. $G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) > 1 - \mathfrak{y}$ and $B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}$, $Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}\ \forall k \geq k_0$ and every $\mathfrak{w} \in \mathfrak{U}$. Hence the set $\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y}$ or $B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}$, $Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}\}$ has a finitely many terms whose λ-density is zero. Therefore, $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_0$. □

The converse of the above theorem is not true in general.

Example 3.1 Let $(\mathbb{R}^2, |.|)$ be a 2-normed space. For $\mathfrak{e}, \mathfrak{g} \in [0, 1]$. Let $\mathfrak{e} \circ \mathfrak{g} = \mathfrak{e}\mathfrak{g}$ and $\mathfrak{e} \diamond \mathfrak{g} = \min\{\mathfrak{e} + \mathfrak{g}, 1\}$. Choose $\mathfrak{s}, \mathfrak{t} \in \mathfrak{F}$, $\varrho > 0$ and $\varrho > \|\mathfrak{s}, \mathfrak{t}\|$. Define $G(\mathfrak{s}, \mathfrak{t}; \varrho) = \frac{\varrho}{\varrho + \|\mathfrak{s}, \mathfrak{t}\|}$, $B(\mathfrak{s}, \mathfrak{t}; \varrho) = \frac{\|\mathfrak{s}, \mathfrak{t}\|}{\varrho + \|\mathfrak{s}, \mathfrak{t}\|}$ and $Y(\mathfrak{s}, \mathfrak{t}; \varrho) = \frac{\|\mathfrak{s}, \mathfrak{t}\|}{\varrho}$, then it is clear that $\mathfrak{U} = (\mathbb{R}^2, N_2, \circ, \diamond)$ is a $N - 2 - NS$. Define $\mathfrak{u} = (\mathfrak{u}_k)$ by

$$\mathfrak{u}_k = \begin{cases} (k, 0) \text{ if } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n, \\ (0, 0) \text{ otherwise.} \end{cases}$$

Now, for each $\mathfrak{y} > 0$ and $\varrho > 0$, let

$$\begin{aligned} \mathcal{A}(\mathfrak{y}, \varrho) &= \Big\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y} \text{ or} \\ &\qquad B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \geq \mathfrak{y},\ Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) \geq \mathfrak{y}\Big\} \\ &= \Big\{k \in I_n : \frac{\varrho}{\varrho + \|\mathfrak{u}_k, \mathfrak{w}\|} \leq 1 - \mathfrak{y} \text{ or} \frac{\|\mathfrak{u}_k, \mathfrak{w}\|}{\varrho + \|\mathfrak{u}_k, \mathfrak{w}\|} \geq \mathfrak{y},\ \frac{\|\mathfrak{u}_k, \mathfrak{w}\|}{\varrho} \geq \mathfrak{y}\Big\} \\ &= \Big\{k \in I_n : \|\mathfrak{u}_k, \mathfrak{w}\| \geq \frac{\varrho\mathfrak{y}}{1 - \mathfrak{y}} \text{or} \|\mathfrak{u}_k, \mathfrak{w}\| \geq \varrho\mathfrak{y}\Big\} \\ &= \{k \in I_n : n - [\sqrt{\lambda_n}] + 1 \leq k \leq n\} \end{aligned}$$

and so we get

$$\frac{1}{\lambda_n}|\mathcal{A}(\mathfrak{y}, \varrho)| = \frac{1}{\lambda_n}|\{k \in I_n : n - [\sqrt{\lambda_n}] + 1 \leq k \leq n\}| \leq \frac{[\sqrt{\lambda_n}]}{\lambda_n}.$$

Taking $n \to \infty$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n}|\mathcal{A}(\mathfrak{y}, \varrho)| \leq \lim_{n\to\infty} \frac{[\sqrt{\lambda_n}]}{\lambda_n} = 0;$$

i.e., $\delta_\lambda(\mathcal{A}(\mathfrak{y}, \varrho)) = 0$. This shows that, $\mathfrak{u}_k \to 0\big(\mathcal{S}_\lambda(N_2)\big)$ But the sequence, $\mathfrak{u} = (\mathfrak{u}_k)$ is not N_2-convergent to 0.

Theorem 3.2 Let $\mathfrak{u} = (\mathfrak{u}_k)$ be any sequence in $N - 2 - NS\ \mathfrak{U}$, if $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k$ exists, then it is unique.

Proof. Let $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_1$ and $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_2$. Let $\mathfrak{y} > 0$, select $l > 0$ s.t.

$$(1 - l) \circ (1 - l) > 1 - \mathfrak{y} \text{ and } l \diamond l < \mathfrak{y}. \tag{1}$$

For $\varrho > 0$ and $\mathfrak{w} \in \mathfrak{U}$. Define the sets:

$$K_{G,1}(l,\varrho) = \{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_1, \mathfrak{w}; \frac{\varrho}{2}) \leq 1 - l\},$$
$$K_{G,2}(l,\varrho) = \{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_2, \mathfrak{w}; \frac{\varrho}{2}) \leq 1 - l\};$$
$$K_{B,1}(l,\varrho) = \{k \in I_n : B(\mathfrak{u}_k - \mathfrak{u}_1, \mathfrak{w}; \frac{\varrho}{2}) \geq l\},$$
$$K_{B,2}(l,\varrho) = \{k \in I_n : B(\mathfrak{u}_k - \mathfrak{u}_2, \mathfrak{w}; \frac{\varrho}{2}) \geq l\};$$
$$K_{Y,1}(l,\varrho) = \{k \in I_n : Y(\mathfrak{u}_k - \mathfrak{u}_1, \mathfrak{w}; \frac{\varrho}{2}) \geq l\};$$
$$K_{Y,2}(l,\varrho) = \{k \in I_n : Y(\mathfrak{u}_k - \mathfrak{u}_2, \mathfrak{w}; \frac{\varrho}{2}) \geq l\}.$$

Since $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_1$. By Lemma 3.1, we have $\delta_\lambda\{K_{G,1}(l,\varrho)\} = \delta_\lambda\{K_{B,1}(l,\varrho)\} = \delta_\lambda\{K_{Y,1}(l,\varrho)\} = 0$ and therefore $\delta_\lambda\{K^C_{G,1}(l,\varrho)\} = \delta_\lambda\{K^C_{B,1}(l,\varrho)\} = \delta_\lambda\{K^C_{Y,1}(l,\varrho)\} = 1$. Also, using $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_2$, we get, $\delta_\lambda\{K_{G,2}(l,\varrho)\} = \delta_\lambda\{K_{B,2}(l,\varrho)\} = \delta_\lambda\{K_{Y,2}(l,\varrho)\} = 0$ and therefore $\delta_\lambda\{K^C_{G,2}(l,\varrho)\} = \delta_\lambda\{K^C_{B,2}(l,\varrho)\} = \delta_\lambda\{K^C_{Y,2}(l,\varrho)\} = 1$. Now define $K_{G,B,Y}(\mathfrak{y},\varrho) = \{K_{G,1}(l,\varrho) \cup K_{G,2}(l,\varrho)\} \cap \{K_{B,1}(l,\varrho) \cup K_{B,2}(l,\varrho)\} \cap \{K_{Y,1}(l,\varrho) \cup K_{Y,2}(l,\varrho)\}$. Then observe that $\delta_\lambda(\{K_{G,B,Y}(\mathfrak{y},\varrho)\}) = 0$ which implies $\delta(\{K^C_{G,B,Y}(\mathfrak{y},\varrho)\}) = 1$. Let $m \in K^C_{G,B,Y}(\mathfrak{y},\varrho)$, then we have

Case 1. $m \in \{K_{G,1}(l,\varrho) \cup K_{G,2}(l,\varrho)\}^C$,
Case 2. $m \in \{K_{B,1}(l,\varrho) \cup K_{B,2}(l,\varrho)\}^C$,
Case 3. $m \in \{K_{Y,1}(l,\varrho) \cup K_{Y,2}(l,\varrho)\}^C$.

Case 1: Let, $m \in \{K_{G,1}(l,\varrho) \cup K_{G,2}(l,\varrho)\}^C$, then $m \in K^C_{G,1}(l,\varrho)$ and $m \in K^C_{G,2}(l,\varrho)$. Therefore, for any $\mathfrak{w} \in \mathfrak{U}$ we have

$$G(\mathfrak{u}_m - \mathfrak{u}_1, \mathfrak{w}; \frac{\varrho}{2}) > 1 - l \text{ and } G(\mathfrak{u}_m - \mathfrak{u}_2, \mathfrak{w}; \frac{\varrho}{2}) > 1 - l. \tag{2}$$

Now

$$\begin{aligned} G(\mathfrak{u}_1 - \mathfrak{u}_2, \mathfrak{w}; \varrho) &\geq G(\mathfrak{u}_m - \mathfrak{u}_1, \mathfrak{w}, ; \frac{\varrho}{2}) \circ G(\mathfrak{u}_m - \mathfrak{u}_2, \mathfrak{w}; \frac{\varrho}{2}) \\ &> (1-l) \circ (1-l) \text{ by}(2) \\ &> 1 - \mathfrak{y}. \text{ by } (1) \qquad . \end{aligned}$$

Since $\mathfrak{y} > 0$ is arbitrary, so we have $G(\mathfrak{u}_1 - \mathfrak{u}_2, \mathfrak{w}; \varrho) = 1 \ \forall \varrho > 0$, and therefore $\mathfrak{u}_1 - \mathfrak{u}_2 = 0$. This shows that $\mathfrak{u}_1 = \mathfrak{u}_2$.

Case 2: Let, $m \in \{K_{B,1}(l,\varrho) \cup K_{B,2}(l,\varrho)\}^C$, then $m \in K^C_{B,1}(l,\varrho)$ and $m \in K^C_{B,2}(l,\varrho)$. Therefore, for $\mathfrak{w} \in \mathfrak{U}$ we have

$$B(\mathfrak{u}_m - \mathfrak{u}_1, \mathfrak{w}; \frac{\varrho}{2}) < l \text{ and } B(\mathfrak{u}_m - \mathfrak{u}_2, \mathfrak{w}; \frac{\varrho}{2}) < l. \tag{3}$$

Now

$$B(\mathfrak{u}_1-\mathfrak{u}_2,\mathfrak{w};\varrho)\leq B(\mathfrak{u}_m-\mathfrak{u}_1,\mathfrak{w};\frac{\varrho}{2})\circ B(\mathfrak{u}_m-\mathfrak{u}_2,\mathfrak{w};\frac{\varrho}{2})$$
$$<l\diamond l \text{ by (3)}$$
$$<\mathfrak{y}.\text{ by (1)}$$

Since $\mathfrak{y}>0$ is arbitrary, so we have $B(\mathfrak{u}_1-\mathfrak{u}_2,\mathfrak{w};\varrho)=0\ \forall\varrho>0$, and therefore $\mathfrak{u}_1-\mathfrak{u}_2=0$. This shows that $\mathfrak{u}_1=\mathfrak{u}_2$.

Case 3: Let, $m\in\{K_{Y,1}(l,\varrho)\cup K_{Y,2}(l,\varrho)\}^C$, then $m\in K^C_{Y,1}(l,\varrho)$ and $m\in K^C_{Y,2}(l,\varrho)$. Therefore, for $\mathfrak{w}\in\mathfrak{U}$ we have

$$Y(\mathfrak{u}_m-\mathfrak{u}_1,\mathfrak{w};\frac{\varrho}{2})<l \text{ and } Y(\mathfrak{u}_m-\mathfrak{u}_2,\mathfrak{w};\frac{\varrho}{2})<l. \tag{4}$$

Now

$$Y(\mathfrak{u}_1-\mathfrak{u}_2,\mathfrak{w};\varrho)\leq Y(\mathfrak{u}_m-\mathfrak{u}_1,\mathfrak{w};\frac{\varrho}{2})\circ Y(\mathfrak{u}_m-\mathfrak{u}_2,\mathfrak{w};\frac{\varrho}{2})$$
$$<l\diamond l \text{ by (4)}$$
$$<\mathfrak{y}.\text{ by (1)}$$

Since $\mathfrak{y}>0$ is arbitrary, so we have $Y(\mathfrak{u}_1-\mathfrak{u}_2,\mathfrak{w};\varrho)=0\ \forall\varrho>0$, and therefore $\mathfrak{u}_1-\mathfrak{u}_2=0$. This shows that $\mathfrak{u}_1=\mathfrak{u}_2$.
Hence, in all three cases, we have $\mathfrak{u}_1=\mathfrak{u}_2$, i.e., λ-statistical limit of $\mathfrak{u}=(\mathfrak{u}_k)$ is unique. □

Theorem 3.3 Let $\mathfrak{q}=(\mathfrak{q}_k)$ and $\mathfrak{u}=(\mathfrak{u}_k)$ be two sequences in $N-2-NS$ $\mathfrak{U}$ s.t. $\mathcal{S}_\lambda(N_2)-\lim_{k\to\infty}\mathfrak{q}_k=\mathfrak{q}_0$ and $\mathcal{S}_\lambda(N_2)-\lim_{k\to\infty}\mathfrak{u}_k=\mathfrak{u}_0$. Then
(i) $\mathcal{S}_\lambda(N_2)-\lim_{k\to\infty}(\mathfrak{q}_k+\mathfrak{u}_k)=\mathfrak{q}_0+\mathfrak{u}_0$.
(ii) $\mathcal{S}_\lambda(N_2)-\lim_{k\to\infty}(\mathfrak{a}\mathfrak{q}_k)=\mathfrak{a}\mathfrak{q}_0$, where $\mathfrak{a}$ is any scalar.
Proof. The proof can be obtained analogously as the proof of theorem 3.2.□

Theorem 3.4 A sequence $\mathfrak{u}=(\mathfrak{u}_k)$ in $N-2-NS$ $\mathfrak{U}$ is $\mathcal{S}_\lambda$-convergent to $\mathfrak{u}_0$, iff $\exists$ a subset $\mathfrak{R}=\{k_n:n\in\mathbb{N}\}$ of $\mathbb{N}$ with $\delta_\lambda\{\mathfrak{R}\}=1$ and $N_2-\lim_{n\to\infty}\mathfrak{u}_{k_n}=\mathfrak{u}_0$.
Proof. First assume that $\mathcal{S}_\lambda(N_2)-\lim_{k\to\infty}\mathfrak{u}_k=\mathfrak{u}_0$. For $\varrho>0$, $\mathfrak{j}\in\mathbb{N}$ and $\mathfrak{w}\in\mathfrak{U}$, define the set

$$\mathfrak{R}_{N_2}(\mathfrak{j},\varrho)=\{k\in I_n: G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)>1-\frac{1}{\mathfrak{j}} \text{ and } B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)<\frac{1}{\mathfrak{j}}, Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)<\frac{1}{\mathfrak{j}}\}.$$

Since $\mathcal{S}_\lambda(N_2)-\lim_{k\to\infty}\mathfrak{u}_k=\mathfrak{u}_0$, it is clear that for $\varrho>0$ and $\mathfrak{j}\in\mathbb{N}$, $\mathfrak{R}_{N_2}(\mathfrak{j}+1,\varrho)\subset\mathfrak{R}_{N_2}(\mathfrak{j},\varrho)$ and

$$\delta_\lambda(\mathfrak{R}_{N_2}(\mathfrak{j},\varrho))=1. \tag{5}$$

Let $m_1 \in \mathfrak{R}_{N_2}(1,\varrho)$. Then, $\exists\ m_2 \in \mathfrak{R}_{N_2}(2,\varrho),(m_2 > m_1)$, such that for all $n \geq m_2$, $\frac{1}{\lambda_n}|\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) > 1 - \frac{1}{2}$ and $B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \frac{1}{2}$, $Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \frac{1}{2}\}| > \frac{1}{2}$. Similarly, $\exists\ m_3 \in \mathfrak{R}_{N_2}(3,\varrho),(m_3 > m_2)$, such that for all $n \geq m_3$, $\frac{1}{\lambda_n}|\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) > 1 - \frac{1}{3}$ and $B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \frac{1}{3}$, $Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \frac{1}{3}\}| > \frac{2}{3}$ and so on. Thus, we can set a sequence $(m_\mathfrak{j})_{\mathfrak{j}\in\mathbb{N}}$ s.t $m_\mathfrak{j} \in \mathfrak{R}_{N_2}(\mathfrak{j},\varrho)$ and $\forall\ n \geq m_\mathfrak{j}(\mathfrak{j} \in \mathbb{N})$, $\frac{1}{\lambda_n}|\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) > 1 - \frac{1}{\mathfrak{j}}$ and $B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \frac{1}{\mathfrak{j}}$, $Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \frac{1}{\mathfrak{j}}\}| > \frac{\mathfrak{j}-1}{\mathfrak{j}}$. Now define $\mathfrak{R} = \{n \in \mathbb{N} : 1 < n < m_1\} \cup \{\bigcup_{\mathfrak{j}\in\mathbb{N}}\{n \in \mathfrak{R}_{N_2}(\mathfrak{j},\varrho) : m_\mathfrak{j} \leq n < m_{\mathfrak{j}+1}\}\}$. Then for $m_\mathfrak{j} \leq n < m_\mathfrak{j} + 1$, we have $\frac{1}{\lambda_n}|\{k \in I_n : k \in \mathfrak{R}\}| \geq \frac{1}{\lambda_n}|\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) > 1 - \frac{1}{\mathfrak{j}}$ and $B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \frac{1}{\mathfrak{j}}$, $Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \frac{1}{\mathfrak{j}}\}| > \frac{\mathfrak{j}-1}{\mathfrak{j}}$ and hence, $\delta_\lambda(\mathfrak{R}) = 1$ as $k \to \infty$. Now we have to demonstrate that $N_2 - \lim_{n\to\infty} \mathfrak{u}_{k_n} = \mathfrak{u}_0$. Let $\mathfrak{y} > 0$ and select $\mathfrak{j} \in \mathbb{N}$ s.t $\frac{1}{\mathfrak{j}} < \mathfrak{y}$. Furthermore, let $n \geq m_\mathfrak{j}$ and $n \in \mathfrak{R}$. Then, by definition of $\mathfrak{R}$, $\exists\ \mathfrak{k} \geq \mathfrak{j}$ s.t, $m_\mathfrak{k} \leq n < m_{\mathfrak{k}+1}$ and $n \in \mathfrak{R}_{N_2}(\mathfrak{j},\varrho)$. Thus, for each $\mathfrak{y} > 0$, and for $\mathfrak{w} \in \mathfrak{U}$, we have $G(\mathfrak{u}_n - \mathfrak{u}_0, \mathfrak{w}; \varrho) > 1 - \frac{1}{\mathfrak{j}} > 1 - \mathfrak{y}$ and $B(\mathfrak{u}_n - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \frac{1}{\mathfrak{j}} < \mathfrak{y}$, $Y(\mathfrak{u}_n - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \frac{1}{\mathfrak{j}} < \mathfrak{y}\ \forall\ n \geq m_\mathfrak{j}$ and $n \in \mathfrak{R}$. Hence $N_2 - \lim_{n\to\infty} \mathfrak{u}_{k_n} = \mathfrak{u}_0$.

Conversely, suppose $\exists$ a subset $\mathfrak{R} = \{k_n\}_{n\in\mathbb{N}}$ of $\mathbb{N}$ with $\delta_\lambda\{\mathfrak{R}\} = 1$ and $N_2 - \lim_{n\to\infty} \mathfrak{u}_{k_n} = \mathfrak{u}_0$. Let $\mathfrak{y} > 0$ or $\varrho > 0$, $\exists\ k_{n_0} \in \mathbb{N}$ s.t $G(\mathfrak{u}_{k_n} - \mathfrak{u}_0, \mathfrak{w}; \varrho) > 1 - \mathfrak{y}$ and $B(\mathfrak{u}_{k_n} - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}$, $Y(\mathfrak{u}_{k_n} - \mathfrak{u}_0, \mathfrak{w}; \varrho) < \mathfrak{y}\ \forall\ k_n \geq k_{n_0}$ and every $\mathfrak{w} \in \mathfrak{U}$. This implies $\mathfrak{D}_{N_2}(\mathfrak{y},\varrho) = \{k \in I_n : G(\mathfrak{u}_{k_n} - \mathfrak{u}_0, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y}$ or $B(\mathfrak{u}_{k_n} - \mathfrak{u}_0, \mathfrak{w}; \varrho) \geq \mathfrak{y}$, $Y(\mathfrak{u}_{k_n} - \mathfrak{u}_0, \mathfrak{w}; \varrho) \geq \mathfrak{y}\} \subseteq \mathbb{N} - \{k_{n_0}, k_{n_0+1}, k_{n_0+2}, ...\}$. and therefore $\delta_\lambda\{\mathfrak{D}_{N_2}(\mathfrak{y},\varrho)\} \leq\ \delta_\lambda\{\mathbb{N}\} - \delta_\lambda\{k_{n_0}, k_{n_0+1}, k_{n_0+2}, ...\} = 1 - 1 = 0$. This shows that $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_0$. $\square$

4. V_λ-summability in N-2-NS

Definition 4.1 A sequence $\mathfrak{u} = (\mathfrak{u}_k)$ in $N - 2 - NS\ \mathfrak{U}$ is called V_λ-summable to $\mathfrak{u}_0$ w.r.t the neutrosophic 2-norm N_2 if for each $\mathfrak{y} > 0, 0 < \varrho < 1$ and $\mathfrak{w} \in \mathfrak{U}$

$$G\left((\frac{1}{\lambda_n}\sum_{k\in I_n} \mathfrak{u}_k) - \mathfrak{u}_0, \mathfrak{w}; \varrho \right) > 1 - \mathfrak{y} \text{ and}$$

$$B\left((\frac{1}{\lambda_n}\sum_{k\in I_n} \mathfrak{u}_k) - \mathfrak{u}_0, \mathfrak{w}; \varrho \right) < \mathfrak{y}; Y\left((\frac{1}{\lambda_n}\sum_{k\in I_n} \mathfrak{u}_k) - \mathfrak{u}_0, \mathfrak{w}; \varrho \right) < \mathfrak{y}.$$

In present case, we denote $V_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_0$ or $\mathfrak{u}_k \to \mathfrak{u}_0(V_\lambda(N_2))$.

Theorem 4.1 Let $\lambda = (\lambda_n)$ as defined above and $\mathfrak{u} = (\mathfrak{u}_k)$ be a sequence in $N - 2 - NS\ \mathfrak{U}$ then

(I) $\mathfrak{u}_k \to \mathfrak{u}_0(V_\lambda(N_2)) \Rightarrow \mathfrak{u}_k \to \mathfrak{u}_0(\mathcal{S}_\lambda(N_2))$ and the inclusion $V_\lambda(N_2) \subseteq \mathcal{S}_\lambda(N_2)$ is proper.

(II) If $\mathfrak{u} \in l_\infty(\mathfrak{U})$ and $\mathfrak{u}_k \to \mathfrak{u}_0(\mathcal{S}_\lambda(N_2))$, then $\mathfrak{u}_k \to \mathfrak{u}_0(V_\lambda(N_2))$.

(III) $\mathcal{S}_\lambda(N_2) \cap l_\infty(\mathfrak{U}) = V_\lambda(N_2) \cap l_\infty(\mathfrak{U})$, where $l_\infty(\mathfrak{U})$ is the space of all bounded sequences in $\mathfrak{U}$.

Proof. (I) Let $\mathfrak{y} > 0$ and $\mathfrak{u}_k \to \mathfrak{u}_0(V_\lambda(N_2))$. We have,

$$\begin{aligned}
&\sum_{k\in I_n}\big(G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho) \text{ or } B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho),Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\big)\\
= &\sum_{\substack{k\in I_n\\ G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\le 1-\mathfrak{y} \text{ or}\\ B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y},Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y}}}\big(G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho) \text{ or } B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho),Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\big)\\
+ &\sum_{\substack{k\in I_n\\ G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)> 1-\mathfrak{y} \ \&\\ B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)<\mathfrak{y},Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)<\mathfrak{y}}}\big(G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho) \text{ and } B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho),Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\big)\\
\ge &\sum_{\substack{k\in I_n\\ G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\le 1-\mathfrak{y} \text{ or}\\ B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y},Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y}}}\big(G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho) \text{ or } B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho),Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\big)\\
&\ge \mathfrak{y}.\Big|\big\{k\in I_n : G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\le 1-\mathfrak{y} \text{ or } B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y},Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y}\big\}\Big|.
\end{aligned}$$

Since $\mathfrak{u}_k \to \mathfrak{u}_0(V_\lambda(N_2))$. Therefore, it follows that $\mathfrak{u}_k \to \mathfrak{u}_0\big(\mathcal{S}_\lambda(N_2)\big)$.

In order to show that the containment $V_\lambda(N_2) \subseteq \mathcal{S}_\lambda(N_2)$ is proper. We define a sequence $\mathfrak{u} = (\mathfrak{u}_k)$ by

$$\mathfrak{u}_k = \begin{cases} (k,0), & \text{for } n-[\sqrt{\lambda_n}]+1 \le k \le n,\\ (0,0), & \text{otherwise}\end{cases}$$

It is obvious that the sequence $(\mathfrak{u}_k)$ is unbounded. Then for each $\mathfrak{y} \in (0,1)$ and $\varrho > 0$ we have

$$\begin{aligned}
\frac{1}{\lambda_n}\Bigg|\Bigg\{k\in I_n : G(\mathfrak{u}_k-0,\mathfrak{w};\varrho)\le 1-\mathfrak{y} \text{ or}\\
B(\mathfrak{u}_k-0,\mathfrak{w};\varrho)\ge\mathfrak{y},Y(\mathfrak{u}_k-0,\mathfrak{w};\varrho)\ge\mathfrak{y}\Bigg\}\Bigg|\\
=\frac{[\sqrt{\lambda_n}]}{\lambda_n}\to 0 \text{ as } n\to\infty.
\end{aligned}$$

i.e., $\mathfrak{u}_k \to 0\big(\mathcal{S}_\lambda(N_2)\big)$.

Further,

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\Bigg(G(\mathfrak{u}_k-0,\mathfrak{w};\varrho) \text{ or } B(\mathfrak{u}_k-0,\mathfrak{w};\varrho),Y(\mathfrak{u}_k-0,\mathfrak{w};\varrho)\Bigg)\to\infty \text{ as } n\to\infty.$$

This implies that $\mathfrak{u}_k \nrightarrow 0(V_\lambda(N_2))$.

(II) Let $\mathfrak{u} = (\mathfrak{u}_k) \in l_\infty(\mathfrak{U})$ and $\mathfrak{u}_k \to \mathfrak{u}_0\big(\mathcal{S}_\lambda(N_2)\big)$. Then $\exists\ M > 0$ s.t $G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho) \ge 1-M$ or $B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho) \le M, Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho) \le M\ \forall\ k$. Let $\mathfrak{y} > 0$ be arbitrary selected, now as in case (I) we can write

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\big(G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho) \text{ or } B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho), Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\big)$$

$$= \frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\le 1-\mathfrak{y} \text{ or} \\ B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y}, Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y}}}\big(G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho) \text{ or } B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho), Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\big)$$

$$+\frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)> 1-\mathfrak{y} \ \& \\ B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)<\mathfrak{y}, Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)<\mathfrak{y}}}\big(G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho) \text{ and } B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho), Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\big)$$

$$\le \frac{M}{\lambda_n}\left|\left\{k\in I_n : G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\le 1-\mathfrak{y} \text{ or } B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y}, Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y}\right\}\right| + \mathfrak{y},$$

which shows that $\mathfrak{u}_k \to \mathfrak{u}_0(V_\lambda(N_2))$.

(III) Follows easily from part (I) and part (II). □

Theorem 4.2 Let $\mathfrak{u} = (\mathfrak{u}_k)$ be any sequence in $N-2-NS$ $\mathfrak{U}$. Then $\mathcal{S}(N_2) \subset \mathcal{S}_\lambda(N_2)$ if

$$\liminf_{n\to\infty}\frac{\lambda_n}{n} > 0. \tag{6}$$

Proof. Given $\mathfrak{y} > 0$ and $\varrho > 0$, we have

$$\begin{aligned}&\big\{k\le n : G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\le 1-\mathfrak{y} \text{ or} \\ &\qquad B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y},\ Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y}\big\} \\ &\supseteq \big\{k\in I_n : G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\le 1-\mathfrak{y} \text{ or} \\ &\qquad B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y},\ Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y}\big\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{n}\big|\big\{k\le n : G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\le 1-\mathfrak{y} \text{ or} \\ &\qquad B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y},\ Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y}\}\big| \\ &\ge \frac{1}{n}\big|\big\{k\in I_n : G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\le 1-\mathfrak{y} \text{ or} \\ &\qquad B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y},\ Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y}\big\}\big| \\ &\ge \frac{\lambda_n}{n}\frac{1}{\lambda_n}\big|\big\{k\in I_n : G(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\le 1-\mathfrak{y} \text{ or} \\ &\qquad B(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y},\ Y(\mathfrak{u}_k-\mathfrak{u}_0,\mathfrak{w};\varrho)\ge\mathfrak{y}\big\}\big|.\end{aligned}$$

Taking $n\to\infty$ and using (6), we get $\mathfrak{u}_k \to \mathfrak{u}_0\big(\mathcal{S}(N_2)\big) \Rightarrow \mathfrak{u}_k \to \mathfrak{u}_0\big(\mathcal{S}_\lambda(N_2)\big)$.

5. **λ-Statistical completeness in N-2-NS**

Definition 5.1 A sequence $\mathfrak{u} = (\mathfrak{u}_k)$ in $N-2-NS$ $\mathfrak{U}$ is called λ-statistically Cauchy(or $\mathcal{S}_\lambda$-Cauchy) if for each $\mathfrak{y} > 0$ and $\varrho > 0$, $\exists\, p \in \mathbb{N}$ s.t.

$$\lim_n \frac{1}{\lambda_n}|\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y} \text{ or } B(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y},\ Y(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y}\}| = 0\ \forall\ \mathfrak{w} \in \mathfrak{U}$$

or equivalently, $\delta_\lambda(\mathcal{A}(\mathfrak{y}, \varrho)) = 0$, where

$$\mathcal{A}(\mathfrak{y}, \varrho) = \{k \in I_n :\ G(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y} \text{ or } B(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y},\ Y\ (\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y}\}$$

Theorem 5.1 Every $\mathcal{S}_\lambda$-convergent sequence in $N-2-NS$ $\mathfrak{U}$ is $\mathcal{S}_\lambda$-Cauchy.

Proof. Let $\mathfrak{u} = (\mathfrak{u}_k)$ be a $\mathcal{S}_\lambda$-convergent sequence with $\mathcal{S}_\lambda(N_2) - \lim_{k\to\infty} \mathfrak{u}_k = \mathfrak{u}_0$. Let $\mathfrak{y} > 0$ and $\varrho > 0$. Choose $l > 0$ s.t. (1) is satisfied. Define a set,

$$\mathcal{A}(l,\ \varrho) = \{k \in I_n :\ G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w};\ \frac{\varrho}{2}) \leq 1 - l \text{ or } B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) \geq l\ \ Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) \geq l\},$$

then $\delta_\lambda(\mathcal{A}(l, \varrho)) = 0$ and therefore $\delta_\lambda(\mathcal{A}^C(l, \varrho)) = 1$. Let $p \in \mathcal{A}^C(l, \varrho)$ then for $\mathfrak{w} \in \mathfrak{U}$, we have $G(\mathfrak{u}_p - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) > 1 - l$ and $B(\mathfrak{u}_p - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) < l, Y(\mathfrak{u}_p - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) < l$.

Now, let $\mathcal{T}(\mathfrak{y}, \varrho) = \{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y}$ or $B(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y}, Y(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y}\}$. We claim that $\mathcal{T}(\mathfrak{y}, \varrho) \subset \mathcal{A}(l, \varrho)$. Let $m \in \mathcal{T}(\mathfrak{y}, \varrho)$, then we have $G(\mathfrak{u}_m - \mathfrak{u}_p, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y}$ or $B(\mathfrak{u}_m - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y}, Y(\mathfrak{u}_m - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y}$.

Case (i): Suppose $G(\mathfrak{u}_m - \mathfrak{u}_p, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y}$, then $G(\mathfrak{u}_m - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) \leq 1 - l$ and therefore $m \in \mathcal{A}(l, \varrho)$. As otherwise, i.e, if $G(\mathfrak{u}_m - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) > 1 - l$, then

$$\begin{aligned} 1 - \mathfrak{y} \geq G(\mathfrak{u}_m - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq G(\mathfrak{u}_m - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) &\circ G(\mathfrak{u}_p - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) \\ &> (1 - l) \circ (1 - l) \\ &> 1 - \mathfrak{y}.\text{(not possible)} \end{aligned}.$$

Thus, $\mathcal{T}(\mathfrak{y}, \varrho) \subset \mathcal{A}(l, \varrho)$.

Case (ii): Suppose $B(\mathfrak{u}_m - v_p, \mathfrak{w}; \varrho) \geq \mathfrak{y}$, then $B(\mathfrak{u}_m - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) \geq l$ and therefore $m \in \mathcal{A}(l, \varrho)$. As otherwise, i.e, if $B(\mathfrak{u}_m - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) < l$, then

$$\begin{aligned} \mathfrak{y} \leq B(\mathfrak{u}_m - \mathfrak{u}_p, \mathfrak{w}; \varrho) \leq B(\mathfrak{u}_m - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) &\diamond B(\mathfrak{u}_p - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) \\ &< l \diamond l \\ &< \mathfrak{y}.\text{(not possible)} \end{aligned}.$$

Also, suppose $Y(\mathfrak{u}_m - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y}$, then $Y(\mathfrak{u}_m - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) \geq l$ and therefore $m \in \mathcal{A}(l, \varrho)$. As otherwise, i.e, if $B(\mathfrak{u}_m - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) < l$, then

$$\begin{aligned} \mathfrak{y} \leq Y(\mathfrak{u}_m - v_p, \mathfrak{w}; \varrho) \leq Y(\mathfrak{u}_m - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) \diamond Y(\mathfrak{u}_p - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) \\ < l \diamond l \\ < \mathfrak{y}.\text{(not possible)} \end{aligned}$$

Thus, $\mathcal{T}(\mathfrak{y}, \varrho) \subset \mathcal{A}(l, \varrho)$.
Hence in all cases, $\mathcal{T}(\mathfrak{y}, \varrho) \subset \mathcal{A}(l, \varrho)$. Since $\delta_\lambda(\mathcal{A}(l, \varrho)) = 0$, so $\delta_\lambda(\mathcal{T}(\mathfrak{y}, \varrho)) = 0$ and therefore $\mathfrak{u} = (\mathfrak{u}_k)$ is λ-statistically Cauchy. □

Definition 5.2 A neutrosophic 2-normed space $\mathfrak{U}$ is called $\mathcal{S}_\lambda$-complete if every $\mathcal{S}_\lambda$-Cauchy sequence in $\mathfrak{U}$ is $\mathcal{S}_\lambda$-convergent in $\mathfrak{U}$.

Theorem 5.2 Every $N-2-NS$ $\mathfrak{U}$ is $\mathcal{S}_\lambda$-complete.
Proof. Let $\mathfrak{u} = (\mathfrak{u}_k)$ be $\mathcal{S}_\lambda$-Cauchy sequence in $\mathfrak{U}$. Suppose on the contrary that $\mathfrak{u} = (\mathfrak{u}_k)$ is not $\mathcal{S}_\lambda$-convergent. Let $\mathfrak{y} > 0$ and $\varrho > 0$, then $\exists\ p \in \mathbb{N}$ s.t $\mathfrak{w} \in \mathfrak{U}$ if we define

$$\begin{aligned} \mathcal{A}(\mathfrak{y}, \varrho) = \{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \leq 1 - \mathfrak{y} \text{ or} \\ B(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y}, Y(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq \mathfrak{y}\} \text{ and} \end{aligned}$$

$$\begin{aligned} \mathcal{T}(\mathfrak{y}, \varrho) = \{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) > 1 - \mathfrak{y} \text{ and} \\ B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) < \mathfrak{y}, Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) < \mathfrak{y}\}, \end{aligned}$$

then $\delta_\lambda(\mathcal{A}(\mathfrak{y}, \varrho)) = \delta_\lambda(\mathcal{T}(\mathfrak{y}, \varrho)) = 0$ and therefore we have $\delta_\lambda(\mathcal{A}^C(\mathfrak{y}, \varrho)) = \delta_\lambda(\mathcal{T}^C(\mathfrak{y}, \varrho)) = 1$. Since $G(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \geq 2G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) > 1 - \mathfrak{y}$ and $B(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \leq 2B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) < \mathfrak{y}$, $Y(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) \leq 2Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) < \mathfrak{y}$, if $G(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) > \frac{1-\mathfrak{y}}{2}$ and $B(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) < \frac{\mathfrak{y}}{2}$, $Y(\mathfrak{u}_k - \mathfrak{u}_0, \mathfrak{w}; \frac{\varrho}{2}) < \frac{\mathfrak{y}}{2}$. We have $\delta_\lambda(\{k \in I_n : G(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) > 1 - \mathfrak{y}$ and $B(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) < \mathfrak{y}$, $Y(\mathfrak{u}_k - \mathfrak{u}_p, \mathfrak{w}; \varrho) < \mathfrak{y}\}) = 0$. i.e., $\delta_\lambda(\mathcal{A}^C(\mathfrak{y}, \varrho)) = 0$ which contradicts the fact that $\delta_\lambda(\mathcal{A}^C(\mathfrak{y}, \varrho)) = 1$. Therefore, $\mathfrak{u} = (\mathfrak{u}_k)$ is $\mathcal{S}_\lambda$-convergent and Hence $\mathcal{S}_\lambda$-complete. □

Theorem 5.3 For any sequence $\mathfrak{u} = (\mathfrak{u}_k)$ in $N-2-NS$ $\mathfrak{U}$, the subsequent assertions are equivalent:
(i) $\mathfrak{u} = (\mathfrak{u}_k)$ is $\mathcal{S}_\lambda$-Cauchy.
(ii) $\exists$ a subset $\mathfrak{R} = \{k_n\}$ of $\mathbb{N}$ with $\delta_\lambda\{\mathfrak{R}\} = 1$ and $\{\mathfrak{u}_{k_n}\}_{n\in\mathbb{N}}$ is $\mathcal{S}_\lambda$-Cauchy sequence over $\mathfrak{R}$.
Proof. The proof of the theorem can be obtained analogously as the proof of the theorem 3.4.

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Ranking of Neutrosophic number based on values and ambiguities and its application to linear programming problem

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Abstract. The goal of this article is to establish a methodology for ordering of single-valued neutrosophic numbers (SVN-numbers) on the basis of values and ambiguities. First of all, the idea of neutrosophic numbers is discussed, and (α, β, γ)-cut and arithmetic oprations definecd over SVN-numbers are examined. Thereafter,corresponding to each components,the values and ambiguities are defined and using these definitions, the ratio ranking function is constructed. Then, for the stability of the ratio ranking function, some examples are provided for comparing this method with other approaches. Applying this ratio ranking function, neutrosophic linear programming problem(Neu-LPP) converts to the crisp linear programmning problems (CLP-Problems) and solved it by computational lingo method. At last, Neu-LPP is illustrated by two numerical real-life examples.

Keywords: Neutrosophic number, Value and ambiguity, Ranking function, Neu-LPP, C-LPP, Computational Lingo method.

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1. **Introduction**

In operation research, LP is one of the most significant and valuable optimizations methods. LP-models expand in a variety of decision problems that happen in economics, engineering, industry, and government. The practical decision problems are described not only by these models but also find applications in science. Some variation in this data must impact on optimal solution, and hence the opinion of decision maker's, that we need to investigate for a new scientific algorithm that gives us optimal solutions useful for all conditions and accepts all variations that may happen. In the work environment, we search for applications of the idea of neutrosophic science that take into consideration variations that cab happen in the work environment thorogh the indeterminacy of neutrosophic values. Hence, applying the idea of neutrosophic science, we define many practical problems.

In 1965, First of all, Zadeh [1] presented a fuzzy set(FS), which was classified through only the membership component, and then in 1986, K. Atanassov [2] presented an intuitionistic fuzzy set (IFS), which was classified by two components: membership and non-membership simultaneously. Regularly, to manage incertitude, FS and IFS perform a vital role. In 1998, Smarandache [3] presented neutrosophic set (NS) to manage some incomplete and inconsistant information in philosophical sense. The components truth, indeterminacy, and falsity independently classified on NS.Sometimes a few suitable decisions are impossible to take by IFS, and hence the indeterminacy of NS plays a vital role. Because some real-world problems such as politics, law, medicine, industry, psychology, and economics, are completely indeterminate. The ordering of SVN-number has vital role in the application of sequential problems, linear and non linear programming problems and multi-attribute selection making problems,etc. Lately, some writers [4–6, 8, 9, 14–16, 19] researched IFS models for applications and some writers [10–13, 17, 20, 22–29, 36] have researched NS models for applications. For the importance of the LP-method, we introduce the neutrosophic linear model [30]. We presented the neutrosophic linear programming method and applied it in the field of education [31]. We applied the neutrosophic linear programming method to determine optimal agricultural land use [32]. Chakraborty et al. [33] use the removal area method and apply it to time cost optimization. Jdid and Smarandache [34] used the neutrosophic method and applied it to management and corporate work. Karak et al. [21] established a ranking teachnique between SVN-numbers using the newly developed sign distance method and applied it to the transportation problem.

The struture of the paper is given step by step. Firstly, in section 2, some essential defenitions, such as NS, single valued trapezoidal and triangular neutrosophic number (SVTN-numbers, SVTrN-numbers), and arithmetic operation are given. In section 3, the value and ambiguity indexes of SVN-numbers were designed, and we presented a new ratio ranking function primarily based on expanding values and ambiguities. In this subsection, for the validity and feasibility of the ratio ranking function, we satisfied some reasonable properties. In section 4, a set of six examples is given, using these examples, the ranking results of proposed method are compared with other approaches [4,8,10,12,13,19,20]. In section 5, based on the ranking algorithm, Neu-LPP with neutrosophic constraints transfered to C-LPP with real constraints and solved by computational lingo method. In section 6, the concept of Neu-LPP is illustrated by two suitable real-life numerical examples. In the last section, the conclusion is stated briefly.

2. **Preliminaries**

Let's remind ourselves of a few fundamental definitions that are essential to reaching the main idea of this paper.

Definition 2.1. [3] Let us take ξ as an arbitary element of X, the universe of discourse. Then $\tilde{N}$ is called NS over X if it is classified through three independent components, namely $T_{\tilde{N}}$, $I_{\tilde{N}}$, and $F_{\tilde{N}}$, which were said to be truth, indeterminacy and falsity neutrosophic components, respectively. These components are maps from X to $]^{-}0, 1^{+}[$ i.e., $T_{\tilde{N}}(\xi), I_{\tilde{N}}(\xi), F_{\tilde{N}}(\xi) \in]^{-}0, 1^{+}[$ where $]^{-}0, 1^{+}[$ is called non-standard unit interval. Thus, $\tilde{N}$ is described by $\tilde{N} = \{\langle \xi; T_{\tilde{N}}(\xi), I_{\tilde{N}}(\xi), F_{\tilde{N}}(\xi)\rangle : \xi \in X\}$, with $^{-}0 \leq \sup T_{\tilde{N}}(\xi) + \sup I_{\tilde{N}}(\xi) + \sup F_{\tilde{N}}(\xi) \leq 3^{+}$.

Definition 2.2. [7] Performing non-standard analysis of neutrosophic components in real ground is too tough. So for real application, only their standard subset is taken. When three neutrosophic components take the values on $[0, 1]$, NS is said to be SVN-Set. Thus an SVN-Set $\tilde{N}$ is designed as : $\tilde{N} = \{< \xi, T_{\tilde{N}}(\xi), I_{\tilde{N}}(\xi), F_{\tilde{N}}(\xi) >: \xi \in X; T_{\tilde{N}}(\xi), I_{\tilde{N}}(\xi), F_{\tilde{N}}(\xi) \in [0, 1]$ and $0 \leq \sup T_{\tilde{N}}(\xi) + \sup I_{\tilde{N}}(\xi) + \sup F_{\tilde{N}}(\xi) \leq 3\}$.

Definition 2.3. [18] Let $\tilde{N}$ be defined as NS over $\mathbb{R}$, which is called a neutrosophic number if it fulfils three characteristics given below:

1. $T_{\tilde{N}}(\xi_0) = 1$ and $I_{\tilde{N}}(\xi_0) = F_{\tilde{N}}(\xi_0) = 0$ for some $\xi_0 \in \mathbb{R}$ i.e.,$\tilde{N}$ is normal.
2. $T_{\tilde{N}}(\nu\xi_1 + (1-\nu)\xi_2) \geq min(T_{\tilde{N}}(\xi_1), T_{\tilde{N}}(\xi_2))$, $\forall \xi_1, \xi_2 \in \mathbb{R}$, and $\nu \in [0,1]$ i.e.,N is convex for $T_{\tilde{N}}(\xi)$.
3. $I_{\tilde{N}}(\nu\xi_1+(1-\nu)\xi_2) \geq max(I_{\tilde{N}}(\xi_1), I_{\tilde{N}}(\xi_2))$, and $F_{\tilde{N}}(\nu\xi_1+(1-\nu)\xi_2) \geq max(F_{\tilde{N}}(\xi_1), F_{\tilde{N}}(\xi_2))$, $\forall \xi_1, \xi_2 \in \mathbb{R}$, and $\nu \in [0,1]$ i.e., $\tilde{N}$ is concave for $I_{\tilde{N}}(\xi)$ and $F_{\tilde{N}}(\xi)$.

Definition 2.4. [12] A NS $\tilde{m} = \langle([l,m,n]; t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}})\rangle$ defined on $\mathbb{R}$, where $t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}} \in [0,1]$ and $l,m,n \in \mathbb{R}$ satisfy the condition $l \leq m \leq n$ is called SVTrN-number whose truth,indetereminacy, and falsity component are denoted by $T_{\tilde{m}} : \mathbb{R} \mapsto [0, t_{\tilde{m}}]$, $I_{\tilde{m}} : \mathbb{R} \mapsto [i_{\tilde{m}}, 1]$, and $F_{\tilde{m}} : \mathbb{R} \mapsto [f_{\tilde{m}}, 1]$ as described below:

$$T_{\tilde{m}}(\xi) = \begin{cases} \frac{(\xi-l)t_{\tilde{m}}}{(m-l)}, & l \leq \xi \leq m, \\ \frac{(n-\xi)t_{\tilde{m}}}{(n-m)}, & m \leq \xi \leq n, \\ 0, & otherwise. \end{cases}$$

$$I_{\tilde{m}}(\xi) = \begin{cases} \frac{(m-\xi)+i_{\tilde{m}}(\xi-l)}{(m-l)}, & l \leq \xi \leq m, \\ \frac{(\xi-m)+i_{\tilde{m}}(n-\xi)}{(n-m)}, & m \leq \xi \leq n, \\ 0, & otherwise. \end{cases}$$

$$F_{\tilde{m}}(\xi) = \begin{cases} \frac{(m-\xi)+f_{\tilde{m}}(\xi-l)}{(m-l)}, & l \leq \xi \leq m, \\ \frac{(\xi-m)+f_{\tilde{m}}(n-\xi)}{(n-m)}, & m \leq \xi \leq n, \\ 0, & otherwise. \end{cases}$$

respectively.
For example, let us take SVTrN-number $\tilde{A}_1 = \langle[1,4,8]; 0.9, 0.3, 0.5\rangle$. Then the graphical representation of $\tilde{A}_1$ is given below:

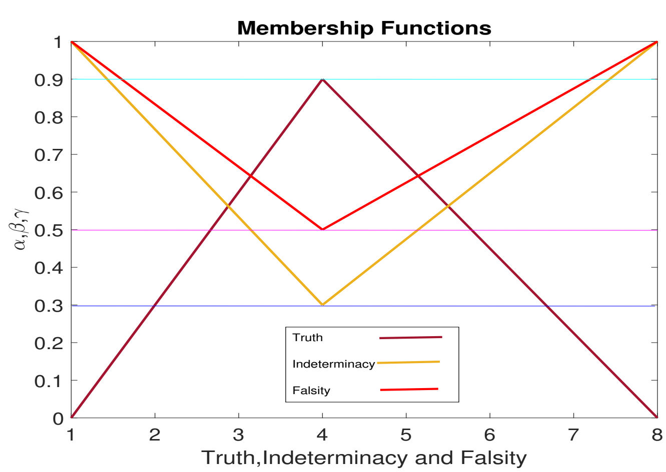

FIGURE 1. Graphical representation of single valued triangular neutrosophic number(SVTN) $\tilde{A}_1$.

Definition 2.5. [12] Let $\tilde{m} = \langle([l, m, n, p]; t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}})\rangle$ be NS on $\mathbb{R}$ where $l, m, n, p \in \mathbb{R}$, and $t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}} \in [0, 1]$ having condition $l \leq m \leq n \leq p$ is called SVTN-numbers whose truth, indetereminacy, and falsity component are denoted by $T_{\tilde{m}} : \mathbb{R} \mapsto [0, t_{\tilde{m}}]$, $I_{\tilde{m}} : \mathbb{R} \mapsto [i_{\tilde{m}}, 1]$, and $F_{\tilde{m}} : \mathbb{R} \mapsto [f_{\tilde{m}}, 1]$ as described below.

$$T_{\tilde{n}}(\xi) = \begin{cases} \frac{(\xi-l)t_{\tilde{m}}}{(m-l)}, & l \leq \xi < m, \\ t_{\tilde{m}}, & m \leq \xi \leq n, \\ \frac{(p-\xi)t_{\tilde{m}}}{(p-n)}, & n < \xi \leq p, \\ 0, & otherwise. \end{cases}$$

$$I_{\tilde{m}}(\xi) = \begin{cases} \frac{(m-\xi)+i_{\tilde{m}}(\xi-l)}{(m-l)}, & l \leq \xi < m, \\ i_{\tilde{m}}, & m \leq \xi \leq n, \\ \frac{(\xi-n)+i_{\tilde{m}}(p-\xi)}{(p-n)}, & n < \xi \leq p, \\ 0, & otherwise. \end{cases}$$

$$F_{n\tilde{m}}(\xi) = \begin{cases} \frac{(m-\xi)+f_{\tilde{m}}(\xi-l)}{(m-l)}, & l \leq \xi < m, \\ f_{\tilde{m}}, & m \leq \xi \leq n, \\ \frac{(\xi-n)+f_{\tilde{m}}(p-\xi)}{(p-n)}, & n < \xi \leq p, \\ 0, & otherwise. \end{cases}$$

respectively.
For example, let us take SVTN-number $\tilde{A}_2 = \langle[1, 3, 6, 9]; 0.7, 0.5, 0.6\rangle$. Then the graphical representation of $\tilde{A}_2$ is given below:

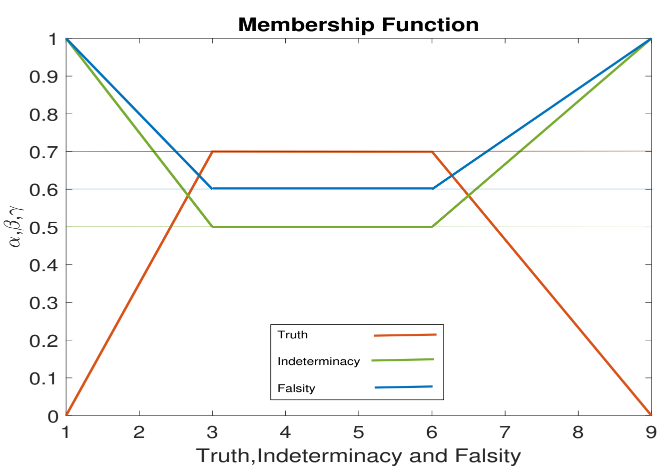

FIGURE 2. Graphical representation of single valued trapezoidal neutrosophic number(SVTN) $\tilde{A}_2$.

Definition 2.6. [12] For $\tilde{N}$ defined in 2.3, (α, β, γ)-cut is designed as : $\tilde{N}_{(\alpha,\beta,\gamma)}$=$\{\xi \in X : T_{\tilde{N}}(\xi) \geq \alpha, I_{\tilde{N}}(\xi) \leq \beta, F_{\tilde{N}}(\xi) \leq \gamma\}$ where $0 \leq \alpha, \beta, \gamma \leq 1$.
Then for SVTrN-number $\tilde{m}$ defined in 2.4, the (α, β, γ) cuts are respectively

$$\begin{aligned} \tilde{m}_\alpha = [L_{\tilde{m}}(\alpha), R_{\tilde{m}}(\alpha)] = & \left[\tfrac{(t_{\tilde{m}}-\alpha)l+\alpha m}{t_{\tilde{m}}}, \tfrac{(t_{\tilde{m}}-\alpha)n+\alpha m}{t_{\tilde{m}}}\right], \\ \tilde{m}_\beta = [L'_{\tilde{m}}(\beta), R'_{\tilde{m}}(\beta)] = & \left[\tfrac{(1-\beta)m+(\beta-i_{\tilde{m}})l}{1-i_{\tilde{m}}}, \tfrac{(1-\beta)m+(\beta-i_{\tilde{m}})n}{1-i_{\tilde{m}}}\right], \\ \text{and } \tilde{m}_\gamma = [L''_{\tilde{m}}(\gamma), R''_{\tilde{m}}(\gamma)] = & \left[\tfrac{(1-\gamma)m+(\gamma-f_{\tilde{m}})l}{1-f_{\tilde{m}}}, \tfrac{(1-\gamma)m+(\gamma-f_{\tilde{m}})n}{1-f_{\tilde{m}}}\right]. \end{aligned}$$

Here $L_{\tilde{m}}, R'_{\tilde{m}}$, and $R''_{\tilde{m}}$ are non-decreasing and continuous functions, and $R_{\tilde{m}}, L'_{\tilde{m}}$, and $L''_{\tilde{m}}$ are non-increasing continuous functions in their respectively intervals.
Similarly (α, β, γ) cut of SVTN-number $\tilde{m}$ defined in 2.5, are respectively

$$\tilde{m}_\alpha = [L_{\tilde{m}}(\alpha), R_{\tilde{m}}(\alpha)] = \left[\tfrac{(t_{\tilde{m}}-\alpha)l+\alpha m}{t_{\tilde{m}}}, \tfrac{(t_{\tilde{m}}-\alpha)p+\alpha n}{t_{\tilde{m}}}\right],$$

$$\tilde{m}_\beta = [L'_{\tilde{m}}(\beta), R'_{\tilde{m}}(\beta)] = \left[\tfrac{(1-\beta)m+(\beta-i_{\tilde{m}})l}{1-i_{\tilde{m}}}, \tfrac{(1-\beta)n+(\beta-i_{\tilde{m}})p}{1-i_{\tilde{m}}}\right],$$

$$\tilde{m}_\gamma = [L''_{\tilde{m}}(\gamma), R''_{\tilde{m}}(\gamma)] = \left[\tfrac{(1-\gamma)m+(\gamma-f_{\tilde{m}})l}{1-f_{\tilde{m}}}, \tfrac{(1-\gamma)n+(\gamma-f_{\tilde{m}})p}{1-f_{\tilde{m}}}\right].$$

Definition 2.7. [11] Let us take two SVTN-numbers $\tilde{m} = \langle([l, m, n, p]; t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}})\rangle$ and $\tilde{n} = \langle([u, v, w, x]; t_{\tilde{n}}, i_{\tilde{n}}, f_{\tilde{n}})\rangle$, and $\delta(\neq 0) \in \mathbb{R}$. Then

(i) $\tilde{m} \oplus \tilde{n} = \langle(+u, m + v, n + w, p + x); t_{\tilde{m}} \wedge t_{\tilde{n}}, i_{\tilde{m}} \vee i_{\tilde{n}}, f_{\tilde{m}} \vee f_{\tilde{n}}\rangle$.

(ii) $\delta\tilde{m} = \begin{cases} \langle(\delta l, \delta m, \delta n, \delta p); t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}}\rangle & (\delta > 0). \\ \langle(\delta p, \delta n, \delta m, \delta l); t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}}\rangle & (\delta < 0). \end{cases}$

(iii) $\tilde{m} \ominus \tilde{n} = \langle(l - x, m - w, n - v, p - u); t_{\tilde{m}} \wedge t_{\tilde{n}}, i_{\tilde{m}} \vee i_{\tilde{n}}, f_{\tilde{m}} \vee f_{\tilde{n}}\rangle$.

Definition 2.8. [11] Let us take two SVTrN-numbers $\tilde{m} = \langle([l, m, n]; t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}})\rangle$ and $\tilde{n} = \langle([u, v, w]; t_{\tilde{n}}, i_{\tilde{n}}, f_{\tilde{n}})\rangle$, and $\delta(\neq 0) \in \mathbb{R}$. Then

(i) $\tilde{m} \oplus \tilde{n} = \langle(l + u, m + v, n + w); t_{\tilde{m}} \wedge t_{\tilde{n}}, i_{\tilde{m}} \vee i_{\tilde{n}}, f_{\tilde{m}} \vee f_{\tilde{n}}\rangle$.

(ii) $\delta\tilde{m} = \begin{cases} \langle(\delta l, \delta m, \delta n); t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}}\rangle & (\delta > 0). \\ \langle(\delta n, \delta m, \delta l); t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}}\rangle & (\delta < 0). \end{cases}$

(iii) $\tilde{m} \ominus \tilde{n} = \langle(l - w, m - v, n - u); t_{\tilde{m}} \wedge t_{\tilde{n}}, i_{\tilde{m}} \vee i_{\tilde{n}}, f_{\tilde{m}} \vee f_{\tilde{n}}\rangle$.

3. Neutrosophic numbers and their ordering method

In this part, we presented an ordering method for SVN-numbers depending on values and ambiguities in a new direction.

Definition 3.1. If $\tilde{m}$ is any arbitrary SVN-number, then

1. the value and ambiguity of $\tilde{m}$ for truth component, are symbolised by $V_T(\tilde{m})$ and $A_T(\tilde{m})$ and described as follows:

(i) $V_T(\tilde{m}) = \int_0^{t_{\tilde{m}}} \{L_{\tilde{m}}(\alpha) + R_{\tilde{m}}(\alpha)\} f(\alpha) d\alpha.$

(ii) $A_T(\tilde{m}) = \int_0^{t_{\tilde{m}}} \{R_{\tilde{n}}(\alpha) - L_{\tilde{m}}(\alpha)\} f(\alpha) d\alpha.$

Where $f(\alpha) \in [0,1]$ $(\alpha \in [0, t_{\tilde{m}}])$, $f(0) = 0$, and $f(\alpha)$ is non decreasing monotonic continuous function of α.

2. the value and ambiguity of $\tilde{m}$ for indeterminacy component, are symbolised by $V_I(\tilde{m})$ and $A_I(\tilde{m})$ and described as follows:

(i) $V_I(\tilde{m}) = \int_{i_{\tilde{n}}}^1 \{L'_{\tilde{m}}(\beta) + R'_{\tilde{m}}(\beta)\} g(\beta) d\beta.$

(ii) $A_I(\tilde{m}) = \int_{i_{\tilde{n}}}^1 \{R'_{\tilde{m}}(\beta) - L'_{\tilde{m}}(\beta)\} g(\beta) d\beta.$

Where $g(\beta) \in [0,1]$ $(\beta \in [i_{\tilde{m}}, 1])$, g(1)=0 , and $g(\beta)$ is non increassing monotonic continuous function of β.

3. the value and ambiguity of $\tilde{m}$ for falsity component, are symbolised by $V_F(\tilde{m})$ and $A_F(\tilde{m})$ and described as follows:

(i) $V_F(\tilde{m}) = \int_{f_{\tilde{n}}}^1 \{L''_{\tilde{m}}(\gamma) + R''_{\tilde{m}}(\gamma)\} h(\gamma) d\gamma.$

(ii) $A_F(\tilde{m}) = \int_{f_{\tilde{m}}}^1 \{R''_{\tilde{m}}(\gamma) - L''_{\tilde{m}}(\gamma)\} h(\gamma) d\gamma.$

Where $h(\gamma) \in [0,1]$ $(\gamma \in [f_{\tilde{m}}, 1])$, h(1)=0 , and $h(\gamma)$ is non increassing monotonic continuous function of γ.

Definition 3.2. For an arbitary SVN-number $\tilde{m}$, the value and ambiguity of $\tilde{m}$ are symbolised as $V(\tilde{m})$ and $A(\tilde{m})$ and expressed as follows:

(i) $V(\tilde{m}) = \frac{1}{3}\, [V_T + V_I + V_F]$, and

(ii) $A(\tilde{m}) = \frac{1}{3}\, [A_T + A_I + A_F]$.

From now on we take $f(\alpha) = \frac{\alpha}{t_{\tilde{m}}}$, $\alpha \in [0, t_{\tilde{m}}]$ $(t_{\tilde{m}} \in (0,1])$, $g(\beta) = \frac{1-\beta}{1-i_{\tilde{m}}}$, $\beta \in [i_{\tilde{m}}, 1]$ $(i_{\tilde{m}} \in [0,1))$, $h(\gamma) = \frac{1-\gamma}{1-f_{\tilde{m}}}$, $\gamma \in [f_{\tilde{m}}, 1]$ $(f_{\tilde{m}} \in [0,1))$ for the SVN-number $\tilde{m}$, and similarly for other SVN-numbers throughout the paper.

Remark 1. It is easily derived that the value function $V(\tilde{m})$ should be maximized, whereas the ambiguity function should be minimised.

Corollary 3.1. For arbitary SVTrN-number $\tilde{m} =< [l, m, n]; t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}} >$, the value and ambiguity are given by

(i) $V(\tilde{m}) = \frac{1}{18}\, [(l + 4m + n) \times (2 + t_{\tilde{m}} - i_{\tilde{m}} - f_{\tilde{m}})]$, and

(ii) $A(\tilde{m}) = \frac{1}{18}\, [\{(n - l)\} \times (2 + t_{\tilde{m}} - i_{\tilde{m}} - f_{\tilde{m}})]$.

Corollary 3.2. for arbitary SVTN-number $\tilde{m} = \langle [l, m, n, p]; t_{\tilde{m}}, i_{\tilde{m}}, f_{\tilde{m}} \rangle$, the value and ambiguity are given by

(i) $V(\tilde{m}) = \frac{1}{18}\ [(l+2m+2n+p)\times(2+t_{\tilde{m}}-i_{\tilde{m}}-f_{\tilde{m}})]$, and

(ii) $A(\tilde{m}) = \frac{1}{18}\ [\{p-l-2(m-n)\}\times(2+t_{\tilde{m}}-i_{\tilde{m}}-f_{\tilde{m}})]$.

Property 1. For any SVN-number $\tilde{m}$ and $\delta(\neq 0)\in\mathbb{R}$,

(i) $V(\delta\tilde{m}) = \delta V(\tilde{m})$.

(ii) $A(\delta\tilde{m}) = \delta A(\tilde{m})$.

Proof: (i),(ii) are obvious (see definitions 2.7,2.8, and 3.1).

Theorem 3.1. For two SNTrN-numbers $\tilde{m}$ and $\tilde{n}$ with $t_{\tilde{n}} = t_{\tilde{m}}$, $i_{\tilde{n}} = i_{\tilde{m}}$, $f_{\tilde{n}} = f_{\tilde{m}}$,

(i) $V(\tilde{m}\oplus\tilde{n})=V(\tilde{m})+V(\tilde{n})$.

(ii) $A(\tilde{m}\oplus\tilde{n})=A(\tilde{m})+A(\tilde{n})$.

Proof:

(i) By the definition 2.8 and given condition, we get
$V(\tilde{m}\oplus\tilde{n})=\frac{1}{18}\ [\{(l+u)+2(m+v)+2(n+w)+(p+x)\}\times(2+t_{\tilde{n}}-i_{\tilde{n}}-f_{\tilde{n}})]$
$=V(\tilde{m})+V(\tilde{n})$.
Hence,the proof.

(ii) Similarly, it can be proved.

NOTE: The theorem is also true for SNTrN-numbers.

Definition 3.3. Let us consider a atio ranking function ϕ that maps from $N(\mathbb{R})$ to $\mathbb{R}$ and is described by $\phi(\tilde{m}) = \frac{V(\tilde{m})}{1+A(\tilde{m})}$ $\forall\tilde{m}\in N(\mathbb{R})$, where $N(\mathbb{R})$ indicates set of all SVN-numbers on $\mathbb{R}$ whose truth component $\in(0,1]$, indeterminacy component $\in[0,1)$, and falsity component $\in[0,1)$.

For any $\tilde{m},\tilde{n}\in N(\mathbb{R})$, we define ordering of $\tilde{m},\tilde{n}$ by

(1) $\tilde{m}\prec_{\phi}\tilde{n}$ iff $\phi(\tilde{m})<\phi(\tilde{n})$.

(2) $\tilde{m}\succ_{\phi}\tilde{n}$ iff $\phi(\tilde{m})>\phi(\tilde{n})$.

(3) $\tilde{m}\approx_{\phi}\tilde{n}$ iff $\phi(\tilde{m})=\phi(\tilde{n})$.

Then the order $\preceq_{\phi}$ is formulated as $\tilde{m}\preceq_{\phi}\tilde{n}$ iff $\tilde{m}\approx_{\phi}\tilde{n}$ or $\tilde{m}\prec_{\phi}\tilde{n}$.

Corollary 3.3. Let $\tilde{m}\in N(\mathbb{R})$ be SVTrN-number defined in definition 2.4. Then the ranking functional value of SVTrN-number $\tilde{m}$ is described by $\phi(\tilde{m}) = \frac{(l+4m+n)\times(2+t_{\tilde{m}}-i_{\tilde{m}}-f_{\tilde{m}})}{18+(n-l)\times(2+t_{\tilde{m}}-i_{\tilde{m}}-f_{\tilde{m}})}$

Corollary 3.4. Let $\tilde{m}\in N(\mathbb{R})$ be SVTN-number defined in definition 2.5. Then the ranking functional value of SVTN-number $\tilde{m}$ is describeed by $\phi(\tilde{m}) = \frac{(l+2m+2n+n)\times(2+t_{\tilde{m}}-i_{\tilde{m}}-f_{\tilde{m}})}{18+(p-l-2m+2n)\times(2+t_{\tilde{m}}-i_{\tilde{m}}-f_{\tilde{m}})}$.

Remark 2. It is easily seen that $\phi(\tilde{m})$ is not linear function of a SVN-number $\tilde{m}$ although $V(\tilde{m})$ and $A(\tilde{m})$ are linear on $\tilde{m}$. In other words, $\phi(\tilde{m}\oplus\tilde{n})\neq\phi(\tilde{m})+\phi(\tilde{n})$

Example 1. Let $\tilde{m} = \langle[1,4,7];0.6,0.1,0.4\rangle$, $\tilde{n} = \langle[3,5,6];0.7,0.1,0.2\rangle\in N(\mathbb{R})$.

Then, by definition 3.5, $\phi(\tilde{m}) = 1.1667$, and $\phi(\tilde{n}) = 2.19005$.

So, $\phi(\tilde{m})<\phi(\tilde{n})$ and hence the ranking of SVTrN-numbers $\tilde{m}$ and $\tilde{n}$ is $\tilde{m}\prec_{\phi}\tilde{n}$.

Example 2. Let $\tilde{m} = \langle[1, 2, 4, 7]; 0.7, 0.1, 0.3\rangle, \tilde{n} = \langle[1, 3, 5, 6]; 0.6, 0.2, 0.4\rangle \in N(\mathbb{R})$

Then, by definition 3.5, $\phi(\tilde{m}) = 1.1219$, and $\phi(\tilde{n}) = 1.2778$.

So, $\phi(\tilde{m}) < \phi(\tilde{n})$ and hence the ranking of SVTN-numbers $\tilde{m}$ and $\tilde{n}$ is $\tilde{m} \prec_\phi \tilde{n}$.

Property 2. The relations $\preceq_\phi$ is total ordering on $N(\mathbb{R})$.

Proof: If the relation $\preceq_\phi$ is total ordering on $N(\mathbb{R})$, then we need to prove the following:

(a) $\preceq$ is a partial order i.e., $\preceq_\phi$ is reflexive, anti symmetric, and transitive.

(b) any two element in $N(\mathbb{R})$ are comparable.

We now prove the condition (a) and (b).

(a) By definition 3.6 , it is clear that the relation $\preceq_\phi$ is reflexive i.e., $\tilde{m} \preceq_\phi \tilde{m}$, $\forall \tilde{m} \in N(\mathbb{R})$

let $\tilde{m}, \tilde{n} \in N(\mathbb{R})$ with $\tilde{m} \preceq_\phi \tilde{n}$ and $\tilde{n} \preceq_\phi \tilde{m}$

Then by definition 3.6, $\phi(\tilde{m}) - \phi(\tilde{n}) \leqslant 0$ and $\phi(\tilde{m}) - \phi(\tilde{n}) \geqslant 0$, and hence $\phi(\tilde{m}) - \phi(\tilde{n}) = 0$.

Therefore, $\tilde{m} \approx_\phi \tilde{n}$ i.e., the relation $\preceq_\phi$ is anti symmetric.

let $\tilde{m}, \tilde{n}, \tilde{p} \in N(\mathbb{R})$ with $\tilde{m} \preceq_\phi \tilde{n}$ and $\tilde{n} \preceq_\phi \tilde{p}$.

Then by definition 3.6 , $\phi(\tilde{m}) - \phi(\tilde{n}) \leqslant 0$ and $\phi(\tilde{n}) - \phi(\tilde{p}) \leqslant 0$, and hence $\phi(\tilde{m}) - \phi(\tilde{p}) \leqslant 0$.

Therefore, $\tilde{m} \preceq_\phi \tilde{p}$ i.e., the relation $\preceq_\phi$ is transitive.

Therefore, the relation $\preceq_\phi$ satisfy all the condition of partial ordering on $N(\mathbb{R})$.

(b) By the definition 3.6, we can say that any two element in $N(\mathbb{R})$ are comparable.

Therefore, the relation $\preceq_\phi$ is total ordering.

3.1. *Rationality of validation of the ratio ranking algorithm*

Seven axioms $A_1 - A_7$ proposed by Wang and Kerre [35] have reasonable properties for the validation of ratio ranking algorithm for ordering fuzzy numbers. In this article, the introduced ratio ranking method fulfils the the properties A_1, A_2, A_3, and A_5 easily. However, the properties A_4, A_6, and A_7 are not satisfied by the ratio ranking method because this method is not linear according to Remark 2. By the Remark 1, the value index $V(\tilde{m})$ should be maximized, whereas the ambiguity index $A(\tilde{m})$ should be minimised, i.e., $V(\tilde{m})$ and $A(\tilde{m})$ are in conflict. Hence, the ranking algorithm should be established depanding on the above two functions and applied it to solve Neu-LPP. Even, in general, Neu-LPP are not easily solved. Hence, the ratio ranking algorithm is used to aggregate $V(\tilde{m})$ and $A(\tilde{m})$. As a consequence, the ordering of SVN-numbers is dependent on the ratio of $V(\tilde{m})$ and $1 + A(\tilde{m})$ rather than either $V(\tilde{m})$ and $A(\tilde{m})$.

4. **Comparison Analysis**

Here the ranking of neutrosophic numbers is compared with other approaches with the proposed method by six set of examples given in the following:

Set-1: $\tilde{m} = \langle([1,5,7,8]; 0.9, 0.3, 0.4)\rangle$, $\tilde{n} = \langle([2,4,6,7]; 0.8, 0.4, 0.5)\rangle$

Then,by definition 3.5, $\phi(\tilde{m}) = 0.8601896$, $\phi(\tilde{n}) = 0.7849003$

So,$\phi(\tilde{n}) < \phi(\tilde{m})$, and hence $\tilde{n} \prec_\phi \tilde{m}$.

Set-2: $\tilde{m} = \langle([2,4,7,9]; 0.4, 0.1, 0.3\rangle$, $\tilde{n} = \langle[1,4,5,9]; 0.8, 0.2, 0.5\rangle$

Then, by definition 3.5, $\phi(\tilde{m}) = 0.75$, $\phi(\tilde{n}) = 0.7538462$

So,$\phi(\tilde{m}) < \phi(\tilde{n})$, and hence $\tilde{m} \prec_\phi \tilde{n}$.

Set-3: $\tilde{m} = \langle([1,3,6,8]; 0.7, 0.2, 0.5)\rangle$, $\tilde{n} = \langle([3,6,8,9]; 0.9, 0.1, 0.3)\rangle$

Then, by definition 3.5, $\phi(\tilde{m}) = 0.6136364$, $\phi(\tilde{n}) = 1.162791$

So,$\phi(\tilde{m}) < \phi(\tilde{n})$, and hence $\tilde{m} \prec_\phi \tilde{n}$.

Set-4: $\tilde{m} = \langle([1,2,3,4]; 0.5, 0.1, 0.2)\rangle$, $\tilde{n} = \langle([2,4,5,6]; 0.6, 0.2, 0.3)\rangle$, $\tilde{p} = \langle([3,4,6,7]; 0.7, 0.2, 0.4)\rangle$

Then, by definition 3.5, $\phi(\tilde{m}) = 0.5689655$, $\phi(\tilde{n}) = 0.8921569$, $\phi(\tilde{p}) = 0.9051724$

So, $\phi(\tilde{m}) < \phi(\tilde{n}) < \phi(\tilde{p})$, and hence $\tilde{m} \prec_\phi \tilde{n} \prec_\phi \tilde{p}$.

Set-5: $\tilde{m} = \langle([2,5,8,9]; 0.7, 0.1, 0.2\rangle$, $\tilde{n} = \langle([1,3,6,8]; 0.6, 0.2, 0.3)\rangle$, $\tilde{p} = \langle([3,4,5,7]; 0.5, 0.1, 0.3)\rangle$

Then, by definition 3.5, $\phi(\tilde{m}) = 0.9024390$, $\phi(\tilde{n}) = 0.6258278$, $\phi(\tilde{p}) = 0.9607843$

So, $\phi(\tilde{n}) < \phi(\tilde{m}) < \phi(\tilde{p})$, and hence $\tilde{n} \prec_\phi \tilde{m} \prec_\phi \tilde{p}$.

Set-6: $\tilde{m} = \langle([4,5,6,7]; 0.5, 0.1, 0.4)\rangle$, $\tilde{n} = \langle([2,4,6,8]; 0.6, 0.2, 0.3)\rangle$, $\tilde{p} = \langle([3,5,7,9]; 0.7, 0.2, 0.5)\rangle$

Then,by definition 3.5, $\phi(\tilde{m}) = 1.178571$, $\phi(\tilde{n}) = 0.8076923$, $\phi(\tilde{p}) = 0.9473684$

So, $\phi(\tilde{n}) < \phi(\tilde{p}) < \phi(\tilde{m})$, and hence $\tilde{n} \prec_\phi \tilde{p} \prec_\phi \tilde{m}$.

We now compare the ranking results of the above six set of examples with other approaches.In the articles [10,12,13,20] on NS, for ranking of these examples , we directly apply the respective approaches. But in the articles [4, 8, 19] on IFS, for the ranking of these examples, we must reject the hesitancy part and then apply the respective methods.

Table-1 : A Comparison of ordering for several approaches

Source	$Set-1$	$Set-2$	$Set-3$	$Set-4$	$Set-5$	$Set-6$
Deli et al. [12]	$\tilde{n} < \tilde{m}$	$\tilde{n} < \tilde{m}$	$\tilde{m} < \tilde{n}$	$\tilde{m} < \tilde{n} < \tilde{p}$	$\tilde{n} < \tilde{p} < \tilde{m}$	$\tilde{n} < \tilde{m} < \tilde{p}$
Peng et al. [13]	$\tilde{n} \prec \tilde{m}$	$\tilde{m} \prec \tilde{n}$	$\tilde{m} \prec \tilde{n}$	$\tilde{n} \prec \tilde{p} \prec \tilde{m}$	$\tilde{p} \prec \tilde{n} \prec \tilde{m}$	$\tilde{m} \approx \tilde{p} \prec \tilde{n}$
Ye. [10]	$\tilde{n} \prec \tilde{m}$	$\tilde{m} \prec \tilde{n}$	$\tilde{m} \prec \tilde{n}$	$\tilde{p} \prec \tilde{n} \prec \tilde{m}$	$\tilde{p} \prec \tilde{n} \prec \tilde{p}$	$\tilde{n} \prec \tilde{m} \prec \tilde{p}$
Fahad A.Alzahrani et al. [20]	$\tilde{n} < \tilde{m}$	$\tilde{n} < \tilde{m}$	$\tilde{m} < \tilde{n}$	$\tilde{m} < \tilde{n} < \tilde{p}$	$\tilde{n} < \tilde{p} < \tilde{m}$	$\tilde{n} < \tilde{m} < \tilde{p}$
Qiang and Zhong [4]	$\tilde{n} < \tilde{m}$	$\tilde{m} < \tilde{n}$	$\tilde{m} < \tilde{n}$	$\tilde{m} < \tilde{n} < \tilde{p}$	$\tilde{p} < \tilde{n} < \tilde{m}$	$\tilde{m} < \tilde{p} < \tilde{n}$
De and Das [8]	$\tilde{n} < \tilde{m}$	$\tilde{n} < \tilde{m}$	$\tilde{m} < \tilde{n}$	$\tilde{m} < \tilde{n} < \tilde{p}$	$\tilde{p} < \tilde{n} < \tilde{m}$	$\tilde{m} < \tilde{n} < \tilde{p}$
Suresh Mohan et al. [19]	$\tilde{n} < \tilde{m}$	$\tilde{n} < \tilde{m}$	$\tilde{m} < \tilde{n}$	$\tilde{m} < \tilde{n} < \tilde{p}$	$\tilde{n} < \tilde{m} < \tilde{m}$	$\tilde{n} < \tilde{m} < \tilde{p}$
proposed method	$\tilde{n} \prec \tilde{m}$	$\tilde{m} \prec \tilde{n}$	$\tilde{m} \prec \tilde{n}$	$\tilde{m} \prec \tilde{n} \prec \tilde{p}$	$\tilde{n} \prec \tilde{m} \prec \tilde{p}$	$\tilde{n} \prec \tilde{p} \prec \tilde{m}$

Here, Deli and Subas [12] applies score and accuracy to determine the ranking of SVN-numbers, and the ranking results of this method are very close to the ranking results of the introduced method. Peng et al. [13] and Ye. [10] designed score and accuracy to determine the ordering of neutrosophic numbers, and applying the score function, the ordering of six set of examples is given in Table-1, which is almost unequal to the ordering results of the introduced method. Alzahrani et al. [20] use de-neutrosophication method to determine the ordering of SVN-numbers, and the ranking results of this method are almost equal to the ranking results of the introduced method. De and Das [8] define a ranking function using value and ambiguity in IFS, and usins this ranking function, the ordering of given SVTN numbers is given in Table-1, which is very close to the ranking results of proposed method and has few difference because it has no hesitancy part. Qiang and Zhong [4] described accuracy and score functions and using this ordering of SVTN-numbers are given in Table-1, which has the same reason as De and Das for the comparison of ranking results with the proposed method. Suresh Mohan et al. [19] define magnitude to define the ordering of neutrosophic numbers and using this magnitude, the ordering of above set of examples are given in Table-1 which is almost equal to the ranking results of proposed method and has few difference because it has no hesitancy part.

5. **Neutrosophic linear programming problem and its solution**

In this section, we propose the idea of Neu-LPP in a new direction using the ranking function. First, we recall the concept of linear programming problems with crisp data, i.e., C-LPP. Usually, C-LPP is expressed as:
Maximize $Z = C\xi$
subject to $A\xi \leqslant B$, $\xi \geq 0$
Where $C \in \mathbb{R}^s$, $B^t \in \mathbb{R}^r$, $\xi \in \mathbb{R}^s$ and $A = (a_{ij})_{r\times s}$
Here, the constraints of C-LPP are crisp numbers. Next, we designed Neu-LPP.

Definition 5.1. The Neu-LPP with constraints in terms of SVN-numbers is defined in the following below:
Maximize $\tilde{Z} \approx_\phi \tilde{C}\xi$
subject to $\tilde{A}\xi \preceq_\phi \tilde{B}$, $\xi \geq 0$
Where $\tilde{A} = (\tilde{a}_{ij})_{r\times s} \in (N(\mathbb{R}))^s$, $\tilde{B} \in (N(\mathbb{R}))^r$, $\tilde{C}^t \in (N(\mathbb{R}))^s$, $\xi \in \mathbb{R}^s$.

METHODOLOGY

There are four steps to reaching the optimal solution, and the steps are given below.

Step-1: First of all, the given Neu-LPP with SVN-numbers can be wriiten in the form of a mathematically formulation.

Step-2. Using ranking function $\phi(\tilde{m}) = \frac{V(\tilde{m})}{1+A(\tilde{m})}$ convert the mentioned SVN-numbers to crisp numbers.

Step-3. Formulate the C-LPP.

Step-4. Solve the C-LPP by Computational Lingo method.

6. Numerical Example

In this section, we give two examples of Neu-LPP with constraints SVN-numbers. In the first examples, we take Neu-LPP with constraints in terms of SVTN-numbers, and in second example, we take Neu-LPP with constraints in terms of SVTrN-numbers. **Example 3.** A firm produces three products I, II, and III. The per unit profits are Rs. $\tilde{c}_1$ and Rs. $\tilde{c}_2$ and Rs. $\tilde{c}_3$ respectively, they are uncertain in nature, assuming as SVTN-numbers. The firm has two machines and each product is processed on two machines X and Y. The processing time required in hours in terms of SNTN-numbers on each product is given below the table.

Machines	$Product-I$	$Product-II$	$Product-III$
X	$\tilde{a}_1$	$\tilde{a}_2$	$\tilde{a}_3$
Y	$\tilde{a}'_1$	$\tilde{a}'_2$	$\tilde{a}'_3$

The machines X and Y have $\tilde{b}_1$ and $\tilde{b}_2$ machine hours in terms of SVTN-numbers, respectively. We have to maximize the profit of the company.

Where,

$\tilde{c}_1 = \langle([6,8,11,14];0.7,0.2,0.5)\rangle$, $\tilde{c}_2 = \langle([5,8,9,10];0.6,0.1,0.2)\rangle$, $\tilde{c}_3 = \langle([7,10,14,17];0.8,0.3,0.4)\rangle$

$\tilde{a}_1 = \langle([3,7,9,15];0.6,0.1,0.3)\rangle$, $\tilde{a}_2 = \langle([7,9,12,16];0.6,0.2,0.5)\rangle$, $\tilde{a}_3 = \langle([3,8,12,14];0.5,0.3,0.4)\rangle$,

$\tilde{a}'_1 = \langle([4,7,10,13];0.4,0.1,0.2)\rangle$, $\tilde{a}'_1 = \langle([4,7,10,13];0.4,0.1,0.2)\rangle$, $\tilde{a}'_2 = \langle([5,9,12,15];0.5,0.4,0.1)\rangle$,

$\tilde{a}'_3 = \langle([5,10,13,15];0.7,0.3,0.5)\rangle$, $\tilde{b}_1 = \langle([35,38,47,58];0.9,0.1,0.3)\rangle$, $\tilde{b}_2 = \langle([35,50,56,63];0.8,0.2,0.4)\rangle$.

Solution:

Step-1: Let the company produce the quantity ξ_1, ξ_2, ξ_3 of the products A, B, and C respectively. Then the mathematical form of the above Neu-LPP is

$$\begin{aligned} &Maximize \quad \tilde{Z} \approx_\phi && \tilde{c}_1\xi_1 \oplus \tilde{c}_2\xi_2 \oplus \tilde{c}_3\xi_3 \\ &\text{subject to,} && \tilde{a}_1\xi_1 \oplus \tilde{a}_2\xi_2 \oplus \tilde{a}_3\xi_3 \preceq_\phi \tilde{b}_1 \\ & && \tilde{a}'_1\xi_1 \oplus \tilde{a}'_2\xi_2 \oplus \tilde{a}'_3\xi_3 \preceq_\phi \tilde{b}_2 \\ &\text{and} && \xi_i \geqslant 0, i = 1,2,3. \end{aligned}$$

Step-2: In this step, we will apply ranking function to convert SVTN-numbers to real numbers. $\phi(\tilde{c}_1) = 2.521739$, $\phi(\tilde{c}_2) = 3.304985$, $\phi(\tilde{c}_3) = 2.709677$ $\phi(\tilde{a}_1) = 2.067669$, $\phi(\tilde{a}_2) = 2.655914$, $\phi(\tilde{a}_3) = 1.965517$, $\phi(\tilde{a}'_1) = 2.163636$,$\phi(\tilde{a}'_2) = 2.480000$ $\phi(\tilde{a}'_3) = 2.590909$, $\phi(\tilde{b}_1) = 5.456432$, $\phi(\tilde{b}_2) = 6.433962$.

Step-3: Thertefore, the C-LPP with constraints in terms of crisp number is

Maximize $Z = 2.521739\xi_1 + 3.304985\xi_2 + 2.709677\xi_3$

subject to

$2.067669\xi_1 + 2.655914\xi_2 + 1.965517\xi_3 \leq 5.456432$

$2.163636\xi_1 + 2.480000\xi_2 + 2.590909\xi_3 \leq 6.433962$

Step-4: By Lingo method, the optimal feasible solition is $\xi_1 = 0$, $\xi_2 = 0.7430211$, $\xi_3 = 1.772069$ and $Z_{max} = 7.257408$.

Example 4. At a cattle breeding firm it is prescribed that the food ration for one animal must contain at least $\tilde{b_1}$, $\tilde{b_2}$ and $\tilde{b_3}$ respectively, they are uncertain in nature, assuming as SVTrN-numbers. Two different kinds of fooder are available. Each unit weight of these two contains the following amounts of the three nutrients in terms of SVTrN-numbers:

	$Fodder - 1$	$Fodder - 2$
Nutrient-A	$\tilde{a}_1$	$\tilde{a}_2$
Nutrient-B	$\tilde{a}'_1$	$\tilde{a}'_2$
Nutrient-C	$\tilde{a}''_1$	$\tilde{a}''_2$

It is given that the costs of unit quantity of Fodder-1 and Fodder-2 are $\tilde{c_1}$ and $\tilde{c_2}$ monetary units respectively. Pose a linear programming problem in terms of minimizing the cost of purchasing the fodders for the above cattle breeding firm.

Where,

$\tilde{c_1} = \langle([4, 7, 8]; 0.9, 0.2, 0.5)\rangle$, $\tilde{c_2} = \langle([2, 3, 5]; 0.5, 0.3, 0.4)\rangle$, $\tilde{a_1} = \langle([1, 6, 7]; 0.6, 0.2, 0.5)\rangle$,
$\tilde{a_2} = \langle([4, 8, 9]; 0.5, 0.1, 0.4)\rangle$, $\tilde{a}'_1 = \langle([1, 2, 4]; 0.6, 0.2, 0.3)\rangle$, $\tilde{a}'_2 = \langle([2, 3, 6]; 0.4, 0.3, 0.2)\rangle$
$\tilde{a}''_1 = \langle([3, 4, 7]; 0.5, 0.4, 0.2)\rangle$, $\tilde{a}''_2 = \langle([4, 5, 6]; 0.6, 0.3, 0.4)\rangle$, $\tilde{b_1} = \langle([1, 3, 5]; 0.5, 0.3, 0.1)\rangle$,
$\tilde{b_2} = \langle([1, 2, 3]; 0.5, 0.3, 0.5)\rangle$, $\tilde{b_3} = \langle([2, 4, 6]; 0.6, 0.3, 0.4)\rangle$.

Solution:

Step-1: Let ξ_1 unit of Fodder-1 and ξ_2 unit of Fodder-2 are to be purchased to fulfil the requirement and minimizing the cost of purchasing.

Therefore, the mathematical formulation of the abpve Neu-LPP is

$$\begin{aligned}
&Minimize \quad \tilde{Z} \approx_\phi && \tilde{c}_1\xi_1 \oplus \tilde{c}_2\xi_2 \\
&\text{subject to,} && \tilde{a}_1\xi_1 \oplus \tilde{a}_2\xi_2 \succeq_\phi \tilde{b}_1 \\
& && \tilde{a}'_1\xi_1 \oplus \tilde{a}'_2\xi_2 \succeq_\phi \tilde{b}_2 \\
& && \tilde{a}''_1\xi_1 \oplus \tilde{a}''_2\xi_2 \succeq_\phi \tilde{b}_3 \\
&\text{and} && \xi_i \geqslant 0, i = 1, 2.
\end{aligned}$$

Step-2: In this step, we will apply ranking function to convert SVTrN-numbers to real numbers. $\phi(\tilde{c}_1) = 3.283582$, $\phi(\tilde{c}_2) = 1.461538$, $\phi(\tilde{a}_1) = 2.068027$, $\phi(\tilde{a}_2) = 3.214286$,

$\phi(\tilde{a}_1') = 1.123457, \phi(\tilde{a}_2') = 1.484375$, $\phi(\tilde{a}_1'') = 1.929688$, $\phi(\tilde{a}_2'') = 2.614679$, $\phi(\tilde{b}_1) = 1.431818$, $\phi(\tilde{b}_2) = 0.9532710$, $\phi(\tilde{b}_3) = 1.781250$.

Step-3: Therefore, the C-LPP with constraints in terms of crisp number is

Minimize $Z = 3.283582\xi_1 + 1.461538\xi_2$

subject to

$2.068027\xi_1 + 3.214286\xi_2 \geq 1.431818$,

$1.123457\xi_1 + 1.484375\xi_2 \geq 0.9532710$,

$1.929688\xi_1 + 2.614679\xi_2 \geq 1.781250$.

Step-4: By Computational Lingo method, the optimal feasible solition is $\xi_1 = 0$, $\xi_2 = 0.6812500$ and $Z_{min} = 0.9956727$.

7. Conclusions

In this article , we describe the ranking system of neutrosophic numbers in a new direction based on value and ambiguity. We also developed some properties and theorems about value and ambiguity. Here, we generalised C-LPP by considering the constraints in terms of SVN-numbers, and the generalised C-LPP is called Neu-LPP. Then, to solve such Neu-LPP, we proposed a simplex algirithm, and finally, this newly developed algorithm is used in real-life problems. The proposed ranking method is applied to convert the Neu-LPP with constraints in terms of SVTN-numbers to the C-LPP with constraints in terms of real numbers and solves it by the computational Lingo method. The idea has been explained by two numerical examples using both SVTN-numbers and SVTrN-numbers. For the stability and feasibility of this methodology, we also compared different existing methodologies with the proposed method. In the future, the idea of Neu-LPP may be more generalised way.

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Neutrosophic Automata and Reverse Neutrosophic Automata

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Abstract. This research is dedicated to exploring the relationships among neutrosophic automata, reverse neutrosophic automata, and double neutrosophic automata. Through the utilization of these three automata, we establish definitions for a neutrosophic subsystem, a reverse neutrosophic subsystem, and a double neutrosophic subsystem, delving into various properties associated with them. Additionally, we aim to introduce the notion of categorical aspects concerning neutrosophic automata and reverse neutrosophic automata, along with their functorial relationship.

Keywords: neutrosophic automata; reverse neutrosophic automata; neutrosophic subsystem; reverse neutrosophic subsystem; category.

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1. **Introduction**

The field of automata theory has proven instrumental in addressing computational complexity issues, finding applications across computer science and discrete mathematics. Following Zadeh's [75] introduction of fuzzy set theory, scholars such as Wee [72] and Santos [52] initiated the exploration of fuzzy automata and languages to bridge the gap between the precision of computer languages and inherent vagueness. Malik and collaborators [32, 38] introduced a simpler notion of a fuzzy finite state machine, laying the groundwork for the algebraic study of fuzzy automata and languages. Numerous researchers (cf., e.g., [5–7,14–19,25,27,30,35–37,46–48,56–63,66,68,69,76]) have contributed to the development of fuzzy automata theory, with diverse focuses. Among these works, Jin and colleagues [17] delved into the algebraic study of fuzzy automata based on po-monoids, while Kim, Kim, and Cho [25] concentrated on the algebraic aspects of fuzzy automata theory. Moćkor [35–37] explored categorical concepts in fuzzy automata theory, and Abolpour and Zahedi [5–7] applied categorical concepts to automata with membership values in various lattice structures. The work of Qiu [46–48], Tiwari and their co-authors [62, 63, 66, 68, 69] pursued algebraic, topological, and categorical studies of fuzzy automata theory based on different lattice structures. Ignjatovic and collaborators [14] investigated the notion of determinism in fuzzy automata, while Anupam and co-authors [55–61, 64, 65] explored the topological, algebraic, and categorical aspects of more generalized fuzzy automata and fuzzy languages. These collective contributions reflect the rich and diverse landscape of research in fuzzy automata theory.

Recent advancements in fuzzy automata theory are highlighted in various works, including [7, 42, 61, 67]. Fuzzy automata find practical applications in engineering contexts, particularly in areas such as information representation, pattern recognition, and machine learning systems, as discussed in [38, 43, 44, 73]. Notably, [73] proposes a non-supervised learning scheme for automatic control and pattern recognition, emphasizing the simplicity in design and computation offered by fuzzy automata as a machine learning model.

In addressing computational uncertainty, alternative mathematical tools have emerged, such as bipolar-valued fuzzy sets [31], vague sets [12], and cubic sets [20]. The generalization trend of fuzzy sets has led to the development of neutrosophy, a philosophical branch introduced and studied by Florientin Samrandache [53, 54]. Neutrosophy serves as a method for handling the computational uncertainty inherent in real-life and scientific problems. Unlike fuzzy sets, neutrosophic sets introduced by Samrandache have three independent components: the degree of membership, the degree of non-membership, and the degree of indeterminacy. Although neutrosophic sets may pose challenges in practical engineering and scientific applications, Wang et al. [70, 71] have introduced the concepts of single-valued neutrosophic sets and interval neutrosophic sets as a more manageable instance of neutrosophic sets. From a practical perspective, neutrosophic set theory has demonstrated substantial success in various fields, including topology [13,41], control theory [39,40], decision-making problems [1,3,26,50], medical diagnosis [1, 51, 74], financial management [2], and smart product-service systems [4]. Neutrosophic automata, a more recent model stemming from fuzzy automata theory, has garnered attention from numerous researchers who have extensively explored automata theory within a neutrosophic framework [21–24, 33, 34]. Neutrosophic automata offer a valuable environment for handling ambiguous computations and have demonstrated their significance in addressing substantial challenges in learning management systems [49], topology [13, 41], and algebraic structures [21–24, 33], among other applications. The concept of category theory, initially introduced by Eilenberg and Mac Lane [10], is widely recognized. Subsequent development by various researchers [11, 28, 29] has showcased its utility in advancing theoretical computer science aspects, such as the design of functional and imperative programming languages, semantic models of programming languages, algorithm development, and polymorphism [45].

1.1. *Motivation*

Various researchers have integrated neutrosophic set theory into automata theory in different ways. However, there is a notable gap in exploring the algebraic properties of automata and reverse automata within a neutrosophic environment, particularly considering t-norm and implication operators. Additionally, the application of category theory and functors between neutrosophic automata and reverse neutrosophic automata remains unexplored. This paper aims to fill these gaps by investigating and introducing the algebraic properties of neutrosophic automata, incorporating a t-norm and implication operator. Furthermore, we present fundamental properties of category theory and explore functors connecting neutrosophic automata with reverse neutrosophic automata.

The paper's structure is outlined as follows:
Section 2: Provides an introduction to the paper's content.
Section 3: Introduces and explores the concepts of neutrosophic automata, reverse neutrosophic automata, as well as subsystems (including reverse and double subsystems) for neutrosophic automata within a neutrosophic environment. This section also delves into presenting various algebraic properties associated with neutrosophic automata.
Section 4: Focuses on the introduction and examination of homomorphism and strong homomorphism between neutrosophic automata, considering specific properties as their basis. Also, proposes categorical and functorial properties of both neutrosophic automata and reverse neutrosophic automata.
Section 5: The article ends with conclusion.

2. **Preliminaries**

Within this section, we revisit fundamental notations and concepts associated with neutrosophic sets, including neutrosophic t-norms, implication operators, and category theory. The foundation for understanding neutrosophic sets is drawn from the works of [53, 54], while the principles of categories and functors are referenced from [8,9]. The discussion commences with the following points.

Definition 2.1. A **neutrosophic set (NS, in short)** A on a non-empty set X is an object having the form $A = \{< b_1, F_A(b_1), G_A(b_1), H_A(b_1) >: b_1 \in X\}$, where the functions $F_A, G_A, H_A : X \to]0^-, 1^+[$ define respectively the degree of membership (or truth), the degree of indeterminacy and the degree of non-membership (or false) of each element $b_1 \in X$ to the set A . As, the sum of $F_A(b_1), G_A(b_1), H_A(b_1)$, have no restriction. So for each $b_1 \in X, 0^- \leq F_A(b_1) + G_A(b_1) + H_A(b_1) \leq 3^+$.

Remark 2.2. A Neutrosophic Set $A = < b_1, F_A(b_1), G_A(b_1), H_A(b_1) >: b_1 \in X$ is typically denoted as an ordered triple $< F_A, G_A, H_A >$ in the non-standard unit interval $]0^-, 1^+[$ on X. The neutrosophic sets (**NSs, in short)** $\mathbf{0_N}$ and $\mathbf{1_N}$ represent constant NSs in X and are defined as $0_N =< 0, 1, 1 >$ and $1_N =< 1, 0, 0 >$, where $0, 1 : X \to]0^-, 1^+[$ are defined respectively by $\mathbf{0}(\mathbf{b_1}) = \mathbf{0}$ and $\mathbf{1}(\mathbf{b_1}) = \mathbf{1}$. The NS $\eta = (\sigma, \beta, \gamma)$ such that $\widehat{\eta} = \widehat{(\sigma, \beta, \gamma)}$ is expressed as $\widehat{\eta}(b_1) = \eta$ for all $b_1 \in X$, where σ, β, and γ are the σ-valued, β-valued, and γ-valued constant neutrosophic sets in X respectively, with the condition $0^- \leq \sigma + \beta + \gamma \leq 3^+$.

This paper opts for the interval $[0, 1]$ instead of the notation $]0^-, 1^+[$ in consideration of practical applications, as the latter might pose challenges in real-world scenarios. Also, $\mathbf{NS(X)}$ will denote the family of all neutrosophic sets in X and I^* denotes the set $\{(b_1, b_2, b_3) : ((b_1, b_2, b_3) \in [0, 1] \times [0, 1] \times [0, 1], 0 \leq b_1 + b_2 + b_3 \leq 3\}$. A neutrosophic set $A =< F_A, G_A, H_A >$ in X will frequently be viewed as a function $A : X \to I^*$, given by $A(b_1) = \{F_A(b_1), G_A(b_1), H_A(b_1) : b_1 \in X\}$.

Firstly, we recall some basic properties of NS in X.

Definition 2.3. For NSs $A =< F_A, G_A, H_A >, B =< F_B, G_B, H_B >$ and $A_i =< F_{A_i}, G_{A_i}, H_{A_i} >, i \in J$ in $b_1 \in X$. We have

(1) $A \leq B$ if $F_A(b_1) \leq F_B(b_1), G_A(b_1) \geq G_B(b_1)$ and $H_A(b_1) \geq H_B(b_1)$;
(2) $\vee_{i \in J} A_i(b_1) = (\vee_{i \in J} F_{A_i}(b_1), \wedge_{i \in J} G_{A_i}(b_1), \wedge_{i \in J} H_{A_i}(b_1))$;
(3) $\wedge_{i \in J} A_i(b_1) = (\wedge_{i \in J} F_{A_i}(b_1), \vee_{i \in J} G_{A_i}(b_1), \vee_{i \in J} H_{A_i}(b_1))$;
(4) $A^c = (1 - F_A, 1 - G_A, 1 - H_A)$;
(5) $0_N \subseteq A \subseteq 1_N; 0_N^c = 1_N$ and $1_N^c = 0_N$;
(6) $A \cup 0_N = A, A \cup 1_N = 1_N$ and $A \cap 0_N = 0_N, A \cap 1_N = A$.

Example 2.4. Let $X = \{b_1, b_2\}, A = \{< b_1, 0.2, 1, 0.3 >, < b_2, 0.4, 0.5, 0.6 >\}$ and $B = \{< b_1, 0.1, 0.3, 0.8 >, < b_2, 0, 0, 0.9 >\}$ are two NSs on X. Then $A \cup B = \{< b_1, 0.2, 0.3, 0.3 >, < b_2, 0.4, 0, 0.6 >\}, A \cap B = \{< b_1, 0.1, 1, 0.8 >, < b_2, 0, 0.5, 0.9 >\}, A^c = \{< b_1, 0.8, 0, 0.7 >, < b_2, 0.6, 0.5, 0.4 >\}, A \cup 1_N = (1, 0, 0) = 1_N, A \cap 0_N = (0, 1, 1) = 0_N$ and $B \cap 1_N = (0.1, 0.3, 0.8) = B$.

Definition 2.5. (1) A neutrosophic t-norm $\otimes : I^* \times I^* \longrightarrow I^*$ be a mapping such for all $\sigma_N = (\sigma_1, \sigma_2, \sigma_3), \beta_N = (\beta_1, \beta_2, \beta_3), \gamma_N = (\gamma_1, \gamma_2, \gamma_3), \delta_N = (\delta_1, \delta_2, \delta_3) \in I^*$ which satisfies

(i) $\sigma_N \otimes 1_N = \sigma_N$ (border condition);

(ii) $\sigma_N \otimes \beta_N = \beta_N \otimes \sigma_N$, (commutativity);

(iii) $\sigma_N \otimes (\beta_N \otimes \gamma_N) = (\sigma_N \otimes \beta_N) \otimes \gamma_N$, (associativity);

(iv) $\sigma_N \leq \beta_N$ and $\gamma_N \leq \delta_N \Rightarrow \sigma_N \otimes \gamma_N \leq \beta_N \otimes \delta_N$, (monotonicity).

(2) The neutrosophic precomplement on I^* is the mapping $\neg : I^* \longrightarrow I^*$ such that $\neg(b_1, b_2, b_3) = (b_1, b_2, b_3) \rightarrow 0_N = (b_1, b_2, b_3) \rightarrow (0, 1, 1) = (b_1 \rightarrow 0, b_2 \leftarrow 1, b_3 \leftarrow 1), \forall b_1, b_2, b_3 \in X$.

(3) The implication operator $\longrightarrow: I^* \longrightarrow I^*$ is defined as;

$$\sigma_N \rightarrow \beta_N = \vee\{\gamma_N = (\gamma_1, \gamma_2, \gamma_3) \in I^* : \sigma_N \otimes \gamma_N \leq \beta_N\}, \forall \sigma_N = (\sigma_1, \sigma_2, \sigma_3), \beta_N = (\beta_1, \beta_2, \beta_3) \in I^* \text{ with respect to } \otimes.$$

For $\sigma_N = (\sigma_1, \sigma_2, \sigma_3) \in I^*$ and $A = (F_A, G_A, H_A) \in NS(X)$, the NS $\sigma_N \rightarrow A = (\sigma_1 \rightarrow F_A, \sigma_2 \leftarrow G_A, \sigma_3 \leftarrow H_A)$ in X is defined as

$$(\sigma_1 \rightarrow F_A)(b_1) = \begin{cases} 1 & \text{if } \sigma_1(b_1) \leq F_A(b_1) \\ F_A(b_1) & \text{if } \sigma_1(b_1) > F_A(b_1) \end{cases}$$

$$(\sigma_2 \leftarrow G_A)(b_1) = \begin{cases} 0 & \text{if } \sigma_2(b_1) \geq G_A(b_1) \\ G_A(b_1) & \text{if } \sigma_2(b_1) < G_A(b_1) \end{cases}$$

and

$$(\sigma_3 \leftarrow H_A)(b_1) = \begin{cases} 0 & \text{if } \sigma_3(b_1) \geq H_A(b_1) \\ H_A(b_1) & \text{if } \sigma_3(b_1) < H_A(b_1) \end{cases}$$

$\forall b_1 \in X$.

Proposition 2.6. *Let* $A = (F_A, G_A, H_A) \in NS(X)$ *and* $\sigma_N = (\sigma_1, \sigma_2, \sigma_3), \beta_N = (\beta_1, \beta_2, \beta_3), \gamma_N = (\gamma_1, \gamma_2, \gamma_3) \in I^*$. *Then*

(i) $1_N \rightarrow A = (1, 0, 0) \rightarrow (F_A, G_A, H_A) = (F_A, G_A, H_A) = A$;

(ii) $\sigma_N \otimes \beta_N \leq \gamma_N \Leftrightarrow \sigma_N \leq \beta_N \rightarrow \gamma_N$;

(iii) $(\sigma_N \otimes \beta_N) \rightarrow \gamma_N = \sigma_N \rightarrow (\beta_N \rightarrow \gamma_N)$;

(iv) $(\sigma_N \rightarrow \beta_N) \otimes (\beta_N \rightarrow \gamma_N) \leq \sigma_N \rightarrow \gamma_N$;

(v) $\sigma_N \otimes (\vee_{i \in I} \beta_{N_i}) = \vee_{i \in I}(\sigma_N \otimes \beta_{N_i})$;

(vi) $(\sigma_N \rightarrow \beta_N) \otimes \sigma_N \leq \beta_N$;

(vii) $(\sigma_N \otimes \beta_N) \rightarrow \gamma_N = (\beta_N \otimes \sigma_N) \rightarrow \gamma_N$;

(viii) *if* $\sigma_N \leq \beta_N \Rightarrow \neg\beta_N \leq \neg\sigma_N$.

Definition 2.7. The key component of a **category thoery** T contains:

(i) a **T**- objects;

(ii) For any pair of objects X and Y within the category **T**, there exists a set denoted as $\mathbf{T}(\mathbf{X}, \mathbf{Y})$. The members of this set are referred to as **morphisms** (or **T**- morphisms), where each morphism ψ in $\mathbf{T}(\mathbf{X}, \mathbf{Y})$ is represented as $\psi : X \to Y$. These morphisms have a specified domain X and codomain Y;

(iii) For every object X within the category **T**, a morphism denoted as $id_X : X \to X$ is termed the **identity morphism** on X; and

(iv) There exists a "composition law" linked to each pair of **T**-morphisms $\psi : X \to Y$ and $\chi : Y \to Z$, a **T**-morphism denoted as $\chi \circ \psi : X \to Z$ is termed the **composition** of ψ and χ, adhering to the following properties:

(a) for any **T**-morphisms $\psi : X \to Y, \chi : Y \to Z$, and $\Phi : Z \to W$, the composition follows the associativity property: $\Phi \circ (\chi \circ \psi) = (\Phi \circ \chi) \circ \psi$.

(b) for any **T**-morphism $\psi : X \to Y$, the identity morphism id_Y satisfies the properties: $id_Y \circ \psi = \psi$ and $\psi \circ id_X = \psi$.

For simplicity, we represent the object-class of the category **T** by **T** itself.

Definition 2.8. A **functor K** : **T** $\to$ **E** is a mapping that assigns each **T**-object X to a **E**-object $K(X)$ and every **T**-morphism $\psi : X \to Y$ to a **E**-morphism $K(\psi) : K(X) \to K(Y)$ follows the conditions that:

(a) For all **T**-morphisms $\psi : X \to Y$ and $\chi : Y \to Z$, $K(\chi \circ \psi) = K(\chi) \circ K(\psi)$, and

(b) For all $X \in \mathbf{T}$, $K(id_X) = id_{K(X)}$.

3. Neutrosophic automata

In this section, we present the concept of neutrosophic automata and reverse neutrosophic automata. The introduction of neutrosophic automata naturally leads to the development of neutrosophic subsystems, including reverse neutrosophic subsystems and double neutrosophic subsystems. Throughout this exploration, we delve into various properties, such as order-preserving maps, involution and some more, associated with these neutrosophic automata and subsystems. The discussion commences with the following points.

Definition 3.1. A **neutrosophic automaton, (NA, in short)** is a triple $L = (Q, X, \delta)$, where Q and X are non-empty sets referred to as the set of states and the set of inputs (with the identity denoted as e), respectively. The neutrosophic transition function is denoted as $\delta = (F_\delta, G_\delta, H_\delta)$ and is a neutrosophic subset of $Q \times X \times Q$. In other words, δ is a mapping $\delta : Q \times X \times Q \to I^*$.

Remark 3.2. (i) Let X^* as the free monoid generated by the set X, with e being its identity. The extension of δ is denoted as $\delta^* = (F_{\delta^*}, G_{\delta^*}, H_{\delta^*}) : Q \times X^* \times Q \to I^*$. This extension is characterized by the property that for any $q_1, q_2 \in Q, u \in X^*$, and $b_1 \in X$, the following holds:

$$F_{\delta^*}(q_1, e, q_2) = \begin{cases} 1 & \text{if } q_1 = q_2 \\ 0 & \text{if } q_1 \neq q_2, \end{cases} \quad G_{\delta^*}(q_1, e, q_2) = H_{\delta^*}(q_1, e, q_2) = \begin{cases} 0 & \text{if } q_1 = q_2 \\ 1 & \text{if } q_1 \neq q_2 \end{cases}$$

$F_{\delta^*}(q_1, ub_1, q_2) = \vee\{F_{\delta^*}(q_1, u, q_3) \otimes F_\delta(q_3, b_1, q_2) : q_3 \in Q\}, G_{\delta^*}(q_1, ub_1, q_2) = \wedge\{G_{\delta^*}(q_1, u, q_3) \otimes G_\delta(q_3, b_1, q_2) : q_3 \in Q\}$, and $H_{\delta^*}(q_1, ub_1, q_2) = \wedge\{H_{\delta^*}(q_1, u, q_3) \otimes H_\delta(q_3, b_1, q_2) : q_3 \in Q\}$.

(ii) For $u \in X^*$, we can establish a mapping $\delta_u = (F_{\delta_u}, G_{\delta_u}, H_{\delta_u}) : Q \times Q \to I^*$ such that $\forall q_1, q_2 \in Q, F_{\delta_u}(q_1, q_2) = F_{\delta^*}(q_1, u, q_2), G_{\delta_u}(q_1, q_2) = G_{\delta^*}(q_1, u, q_2)$ and $H_{\delta_u}(q_1, q_2) = H_{\delta^*}(q_1, u, q_2)$.

Definition 3.3. A **reverse neutrosophic automaton (RNA, in short)** of a NA $L = (Q, X, \delta)$ is a NA $\overline{L} = (Q, X, \overline{\delta})$, where $\overline{\delta} : Q \times X \times Q \to I^*$ is a mapping such that $\overline{\delta}(q_1, b_1, q_2) = \delta(q_2, b_1, q_1), \forall q_1, q_2 \in Q$ and $\forall b_1 \in X$.

Definition 3.4. Let $L = (Q, X, \delta)$ be a NA. Then $A = (F_A, G_A, H_A) \in NS(S)$ is called

(i) **neutrosophic subsystem, (NSS, in short)** of L if $F_A(q_1) \otimes F_\delta(q_1, b_1, q_2) \leq F_A(q_2), G_A(q_1) \otimes G_\delta(q_1, b_1, q_2) \geq G_A(q_2)$ and $H_A(q_1) \otimes H_\delta(q_1, b_1, q_2) \geq H_A(q_2), \forall q_1, q_2 \in Q$ and $\forall b_1 \in X$.

(ii) **reverse neutrosophic subsystem, (RNSS, in short)** of L if $F_A(q_2) \otimes F_\delta(q_1, b_1, q_2) \leq F_A(q_1), G_A(q_2) \otimes G_\delta(q_1, b_1, q_2) \geq G_A(q_1)$ and $H_A(q_2) \otimes H_\delta(q_1, b_1, q_2) \geq H_A(q_1), \forall q_1, q_2 \in Q$ and $\forall b_1 \in X$.

(iii) **double neutrosophic subsystem, (DNSS, in short)** of L if it is both NSS and RNSS of L.

Proposition 3.5. *If A is a NSS in a NA $L = (Q, X, \delta)$, then A is a RNSS in a RNA $\overline{L} = (Q, X, \overline{\delta})$.*

Proof: Let $A = (F_A, G_A, H_A)$ be a NSS in L. Then $\forall q_1, q_2 \in Q$ and $\forall b_1 \in X, F_A(q_1) \otimes F_\delta(q_1, b_1, q_2) \leq F_A(q_2), G_A(q_1) \otimes G_\delta(q_1, b_1, q_2) \geq G_A(q_2)$ and $H_A(q_1) \otimes H_\delta(q_1, b_1, q_2) \geq H_A(q_2) \Rightarrow F_A(q_1) \otimes F_{\overline{\delta}}(q_2, b_1, q_1) \leq F_A(q_2), G_A(q_1) \otimes G_{\overline{\delta}}((q_2, b_1, q_1) \geq G_A(q_2)$ and $H_A(q_1) \otimes H_{\overline{\delta}}((q_2, b_1, q_1) \geq H_A(q_2)$. Hence A is a RNSS in a RNA $\overline{L}$.

Proposition 3.6. *Let $L = (Q, X, \delta)$ be a NA and $A \in NS(S)$. Then*

(i) *$A = (F_A, G_A, H_A)$ is a NSS of L if and only if $A : (Q, X, \delta) \longrightarrow (I^*, \to)$ is an order preserving map.*

(ii) *$A = (F_A, G_A, H_A)$ is a RNSS of L if and only if $A : (Q, X, \overline{\delta}) \longrightarrow (I^*, \to)$ is an order preserving map.*

(iii) *$A=(F_A,G_A,H_A)$ is a DNSS of L if and only if $A:(Q,X,\delta)\longrightarrow(I^*,\rightarrow)$ is an order preserving map.*

Proof: (i) Let $A=(F_A,G_A,H_A)\in NS(S)$ be a NSS of L. Then $\forall q_1,q_2\in Q$ and $b_1\in X, F_A(q_1)\otimes F_\delta(q_1,b_1,q_2)\leq F_A(q_2), G_A(q_1)\otimes G_\delta(q_1,b_1,q_2)\geq G_A(q_2)$ and $H_A(q_1)\otimes H_\delta(q_1,b_1,q_2)\geq H_A(q_2)$, then $F_\delta(q_1,b_1,q_2)\leq F_A(q_1)\rightarrow F_A(q_2), G_\delta(q_1,b_1,q_2)\geq G_A(q_1)\leftarrow G_A(q_2)$ and $H_\delta(q_1,b_1,q_2)\geq H_A(q_1)\leftarrow H_A(q_2)$ (cf., Proposition 2.6). Hence $A:(Q,X,\delta)\longrightarrow(I^*,\rightarrow)$ preserve order. Converse follows similarly.
(ii) Similar to (i).
(iii) Derives from (i) and (ii).

Proposition 3.7. *Let $L=(Q,X,\delta)$ be a NA and $q_1,q_3\in Q,b_1\in X$. Then*

(i) *$[q_3]^{\delta_{b_1}}=(F_{[q_3]^{\delta_{b_1}}},G_{[q_3]^{\delta_{b_1}}},H_{[q_3]^{\delta_{b_1}}})\in NS(Q)$ such that $F_{[q_3]^{\delta_{b_1}}}(q_1)=F_{\delta_{b_1}}(q_3,q_1), G_{[q_3]^{\delta_{b_1}}}(q_1)=G_{\delta_{b_1}}(q_3,q_1)$ and $H_{[q_3]^{\delta_{b_1}}}(q_1)=H_{\delta_{b_1}}(q_3,q_1)$ is a NSS of L,*

(ii) *$[q_3]_{\delta_{b_1}}=(F_{[q_3]_{\delta_{b_1}}},G_{[q_3]_{\delta_{b_1}}},H_{[q_3]_{\delta_{b_1}}})\in NS(Q)$ such that $F_{[q_3]_{\delta_{b_1}}}(q_1)=F_{\delta_{b_1}}(q_1,q_3), G_{[q_3]_{\delta_{b_1}}}(q_1)=G_{\delta_{b_1}}(q_1,q_3)$ and $H_{[q_3]_{\delta_{b_1}}}(q_1)=H_{\delta_{b_1}}(q_1,q_3)$ is a RNSS of L* , and

(iii) *$[q_3]^{\delta_{b_1}}$ and $[q_3]_{\delta_{b_1}}$ is a DNSS of L.*

Proof: (i) Let $F_{[q_3]^{\delta_{b_1}}}(q_1)=F_{\delta_{b_1}}(q_3,q_1), G_{[q_3]^{\delta_{b_1}}}(q_1)=G_{\delta_{b_1}}(q_3,q_1)$ and $H_{[q_3]^{\delta_{b_1}}}(q_1)=H_{\delta_{b_1}}(q_3,q_1)$. Then $F_{[q_3]^{\delta_{b_1}}}(q_1)\otimes F_\delta(q_1,b_1,q_2)=F_{\delta_{b_1}}(q_3,q_1)\otimes F_\delta(q_1,b_1,q_2)=F_\delta(q_3,b_1,q_1)\otimes F_\delta(q_1,b_1,q_2)\leq F_\delta(q_3,b_1,q_2)=F_{\delta_{b_1}}(q_3,q_2)=F_{[q_3]^{\delta_{b_1}}}(q_2), G_{[q_3]^{\delta_{b_1}}}(q_1)\otimes G_\delta(q_1,b_1,q_2)=G_{\delta_{b_1}}(q_3,q_1)\otimes G_\delta(q_1,b_1,q_2)=G_\delta(q_3,b_1,q_1)\otimes G_\delta(q_1,b_1,q_2)\geq G_\delta(q_3,b_1,q_2)=G_{\delta_{b_1}}(q_3,q_2)=G_{[q_3]^{\delta_{b_1}}}(q_2)$ and $H_{[q_3]^{\delta_{b_1}}}(q_1)\otimes H_\delta(q_1,b_1,q_2)=H_{\delta_{b_1}}(q_3,q_1)\otimes H_\delta(q_1,b_1,q_2)=H_\delta(q_3,a,q_1)\otimes H_\delta(q_1,b_1,q_2)\geq H_\delta(q_3,b_1,q_2)=H_{\delta_{b_1}}(q_3,q_2)=H_{[q_3]^{\delta_{b_1}}}(q_2)$, as δ_{b_1} is transitive. Hence $F_{[q_3]^{\delta_{b_1}}}(q_1)\otimes F_\delta(q_1,b_1,q_2)\leq F_{[q_3]^{\delta_{b_1}}}(q_2), G_{[q_3]^{\delta_{b_1}}}(q_1)\otimes G_\delta(q_1,b_1,q_2)\geq G_{[q_3]^{\delta_{b_1}}}(q_2)$ and $H_{[q_3]^{\delta_{b_1}}}(q_1)\otimes H_\delta(q_1,b_1,q_2)\geq H_{[q_3]^{\delta_{b_1}}}(q_2)$. Thus $[q_3]^{\delta_{b_1}}$ is a NSS of L.
(ii) Derives from (i) and the transitivity of δ_{b_1}.
(iii) Derives from (i) and (ii).

Proposition 3.8. *Let $L=(Q,X,\delta)$ be a NA and $A\in NS(Q)$. Then*

(i) *if $A=(F_A,G_A,H_A)$ is a NSS of a NA L, then for each $\eta\in I^*, A\rightarrow\widehat{\eta}$ is a RNSS of L.*

(ii) *if $A=(F_A,G_A,H_A)$ is a RNSS of a NA L, then for each $\eta\in I^*, A\rightarrow\widehat{\eta}$ is a NSS of L.*

Proof: Let $A=(F_A,G_A,H_A)$ is a NSS of a NA L , i.e., $\forall q_1,q_2\in Q$ and $b_1\in X, F_A(q_1)\otimes F_\delta(q_1,b_1,q_2)\leq F_A(q_2), G_A(q_1)\otimes G_\delta(q_1,b_1,q_2)\geq G_A(q_2)$ and $H_A(q_1)\otimes H_\delta(q_1,b_1,q_2)\geq$

$H_A(q_2)$. Then, we have to show that $A \to \widehat{\eta}$ is a RNSS of L, or that $\forall q_1, q_2 \in Q$ and $b_1 \in X, (F_A(q_2) \to \sigma) \otimes F_\delta(q_1, b_1, q_2) \le (F_A(q_1) \to \sigma), (G_A(q_2) \leftarrow \beta) \otimes G_\delta(q_1, b_1, q_2) \ge (G_A(q_1) \leftarrow \beta)$ and $(H_A(q_2) \leftarrow \gamma) \otimes H_\delta(q_1, b_1, q_2) \ge (H_A(q_1) \leftarrow \gamma)$ which implies that $(F_A(q_2) \to \sigma) \otimes F_\delta(q_1, b_1, q_2) \otimes F_A(q_1) \le \sigma, (G_A(q_2) \leftarrow \beta) \otimes G_\delta(q_1, b_1, q_2) \otimes G_A(q_1) \ge \beta$ and $(H_A(q_2) \leftarrow \gamma) \otimes H_\delta(q_1, b_1, q_2) \otimes (H_A(q_1) \ge \gamma$. So $(F_A(q_2) \to \sigma) \otimes F_\delta(q_1, b_1, q_2) \otimes F_A(q_1) \le (F_A(q_2) \to \sigma) \otimes F_A(q_2) \le \sigma, (G_A(q_2) \leftarrow \beta) \otimes G_\delta(q_1, b_1, q_2) \otimes G_A(q_1) \ge (G_A(q_2) \leftarrow \beta) \otimes G_A(q_2) \ge \beta$ and $(H_A(q_2) \leftarrow \gamma) \otimes H_\delta(q_1, b_1, q_2) \otimes (H_A(q_1) \ge (H_A(q_2) \leftarrow \gamma) \otimes (H_A(q_2) \ge \gamma$ (cf., Proposition 2.6). Hence $A \to \widehat{\eta}$ is a RNSS of L.

(ii) In a similar manner, it can be prove that if $A = (F_A, G_A, H_A)$ is a RNSS of a NA L, then for each $\eta \in I^*, A \to \widehat{\eta}$ is a NSS of L.

Proposition 3.9. *Let $L = (Q, X, \delta)$ be a NA and $A \in NS(Q)$. Then*

- (i) *if $A = (F_A, G_A, H_A)$ is a NSS of a NA L, then for each $\eta \in I^*, \widehat{\eta} \otimes A$ is a NSS of L.*
- (ii) *if $A = (F_A, G_A, H_A)$ is a RNSS of a NA L, then for each $\eta \in I^*, \widehat{\eta} \otimes A$ is a RNSS of L.*

Proof: (i) Let $A = (F_A, G_A, H_A)$ is a NSS of a NA L and $\eta \in I^*$. Then $\forall q_1, q_2 \in Q$ and $b_1 \in X, F_A(q_1) \otimes F_\delta(q_1, b_1, q_2) \le F_A(q_2), G_A(q_1) \otimes G_\delta(q_1, b_1, q_2) \ge G_A(q_2)$ and $H_A(q_1) \otimes H_\delta(q_1, b_1, q_2) \ge H_A(q_2)$ which implies that $\forall q_1, q_2 \in Q$ and $b_1 \in X, (\sigma \otimes F_A(q_1)) \otimes F_\delta(q_1, b_1, q_2) \le (\sigma \otimes F_A(q_2)), (\beta \otimes G_A(q_1)) \otimes G_\delta(q_1, b_1, q_2) \ge (\beta \otimes G_A(q_2))$ and $(\gamma \otimes H_A(q_1)) \otimes H_\delta(q_1, b_1, q_2) \ge (\gamma \otimes H_A(q_2))$. Hence $\widehat{\eta} \otimes A$ is a NSS of L.

(ii) In a similar manner, one can demonstrate that if $A = (F_A, G_A, H_A)$ is a RNSS of a NA L, then for each $\eta \in I^*, \widehat{\eta} \otimes A$ is a RNSS of L.

The following provides a characterization of the neutrosophic transition function of a NA based on its NSS.

Proposition 3.10. *For given a NA $L = (Q, X, \delta)$. We have*

- (1) *let **E** be the family of all NSS. Then $\forall q_1, q_2 \in Q$ and $b_1 \in X, F_{\delta_{b_1}}(q_1, q_2) = \wedge\{F_A(q_1) \to F_A(q_2) : F_A \in \boldsymbol{E}\}; G_{\delta_{b_1}}(q_1, q_2) = \vee\{G_A(q_1) \leftarrow G_A(q_2) : G_A \in \boldsymbol{E}\}; H_{\delta_{b_1}}(q_1, q_2) = \vee\{H_A(q_1) \leftarrow H_A(q_2) : H_A \in \boldsymbol{E}\}$.*
- (2) *let **E'** be the family of all RNSS. Then $\forall q_1, q_2 \in Q$ and $b_1 \in X, F_{\delta_{b_1}}(q_1, q_2) = \wedge\{F_A(q_2) \to F_A(q_1) : F_A \in \boldsymbol{E'}\}; G_{\delta_{b_1}}(q_1, q_2) = \vee\{G_A(q_2) \leftarrow G_A(q_1) : G_A \in \boldsymbol{E'}\}; H_{\delta_{b_1}}(q_1, q_2) = \vee\{H_A(q_2) \leftarrow H_A(q_1) : H_A \in \boldsymbol{E'}\}$.*

Proof: We only prove here for NSS of L. The RNSS of L can be proved in a similar way.

(i) Let A be a NSS of a NA L. Then $\forall q_1, q_2 \in Q, b_1 \in X, F_A(q_1) \otimes F_\delta(q_1, b_1, q_2) \le$

$F_A(q_2), G_A(q_1) \otimes G_\delta(q_1, b_1, q_2) \geq G_A(q_2)$ and $H_A(q_1) \otimes H_\delta(q_1, b_1, q_2) \geq H_A(q_2)$, i.e. $F_A(q_1) \otimes F_{\delta_{b_1}}(q_1, q_2) \leq F_A(q_2), G_A(q_1) \otimes G_{\delta_{b_1}}(q_1, q_2) \geq G_A(q_2)$ and $H_A(q_1) \otimes H_{\delta_{b_1}}(q_1, q_2) \geq H_A(q_2)$, or that $F_{\delta_{b_1}}(q_1, q_2) \leq F_A(q_1) \to F_A(q_2), G_{\delta_{b_1}}(q_1, q_2) \geq G_A(q_1) \leftarrow G_A(q_2)$ and $H_{\delta_{b_1}}(q_1, q_2) \geq H_A(q_1) \leftarrow H_A(q_2) \Rightarrow F_{\delta_{b_1}}(q_1, q_2) \leq \wedge\{F_A(q_1) \to F_A(q_2) : F_A \in \mathbf{E}\}, G_{\delta_{b_1}}(q_1, q_2) \geq \vee\{G_A(q_1) \leftarrow G_A(q_2) : G_A \in \mathbf{E}\}$ and $H_{\delta_{b_1}}(q_1, q_2) \geq \vee\{H_A(q_1) \leftarrow H_A(q_2) : H_A \in \mathbf{E}\}$. Next for $q_3 \in Q, b_1 \in X$, as $[q_3]^{\delta_{b_1}}(q_1) = (F_{[q_3]^{\delta_{b_1}}}(q_1), G_{[q_3]^{\delta_{b_1}}}(q_1), H_{[q_3]^{\delta_{b_1}}}(q_1))$ is a NSS of M. Then $\wedge\{F_{[q_3]^{\delta_{b_1}}}(q_1) \to F_{[q_3]^{\delta_{b_1}}}(q_2) : q_3 \in Q\} \leq \{F_{\delta_e}(q_1, q_1) \to F_{\delta_{b_1}}(q_1, q_2)\} = 1 \to F_{\delta_{b_1}}(q_1, q_2) = F_{\delta_{b_1}}(q_1, q_2), \vee\{G_{[q_3]^{\delta_{b_1}}}(q_1) \leftarrow G_{[q_3]^{\delta_{b_1}}}(q_2) : q_3 \in Q\} \geq \{G_{\delta_e}(q_1, q_1) \leftarrow G_{\delta_{b_1}}(q_1, q_2)\} = 0 \leftarrow G_{\delta_{b_1}}(q_1, q_2) = G_{\delta_{b_1}}(q_1, q_2)$ and $\vee\{H_{[q_3]^{\delta_{b_1}}}(q_1) \leftarrow H_{[q_3]^{\delta_{b_1}}}(q_2) : q_3 \in Q\} \geq \{H_{\delta_e}(q_1, q_1) \leftarrow H_{\delta_{b_1}}(q_1, q_2)\} = 0 \leftarrow H_{\delta_{b_1}}(q_1, q_2) = H_{\delta_{b_1}}(q_1, q_2)$ (cf., Proposition 2.6). Thus $\forall q_1, q_2 \in Q$ and $b_1 \in X, F_{\delta_{b_1}}(q_1, q_2) = \wedge\{F_A(q_1) \to F_A(q_2) : F_A \in \mathbf{E}\}; G_{\delta_{b_1}}(q_1, q_2) = \vee\{G_A(q_1) \leftarrow G_A(q_2) : G_A \in \mathbf{E}\}; H_{\delta_{b_1}}(q_1, q_2) = \vee\{H_A(q_1) \leftarrow H_A(q_2) : H_A \in \mathbf{E}\}$.

Proposition 3.11. *Let $L = (Q, X, \delta)$ be a NA and $A \in NS(Q)$. Then*

(1) *if $A = (F_A, G_A, H_A)$ is a NSS of L, so for each $\eta \in I^*, \widehat{\eta} \to A$ is a NSS of L.*

(2) *if $A = (F_A, G_A, H_A)$ is a RNSS of L, so for each $\eta \in I^*, \widehat{\eta} \to A$ is a RNSS of L.*

Proof: We only prove here for NSS of L. The RNSS of L can be proved in a similar way. (i) Let $A = (F_A, G_A, H_A)$ be a NSS of a NA L and $\eta \in I^*$. Then $\forall q_1, q_2 \in Q$ and $b_1 \in X, (\sigma \to F_A(q_1)) \otimes (F_A(q_1) \to F_A(q_2)) \leq (\sigma \to F_A(q_2)), (\beta \leftarrow G_A(q_1)) \otimes (G_A(q_1) \leftarrow G_A(q_2)) \geq (\beta \leftarrow G_A(q_2))$ and $(\gamma \leftarrow H_A(q_1)) \otimes (H_A(q_1) \leftarrow H_A(q_2)) \geq (\gamma \leftarrow H_A(q_2))$ (cf., Proposition 2.6). So that $(F_A(q_1) \to F_A(q_2)) \leq (\sigma \to F_A(q_1)) \to (\sigma \to F_A(q_2)), (G_A(q_1) \leftarrow G_A(q_2)) \geq (\beta \leftarrow G_A(q_1)) \leftarrow (\beta \leftarrow G_A(q_2))$ and $(H_A(q_1) \leftarrow H_A(q_2)) \geq (\gamma \leftarrow H_A(q_1)) \leftarrow (\gamma \leftarrow H_A(q_2))$, or that $F_\delta(q_1, q_2) \leq (\sigma \to F_A(q_1)) \to (\sigma \to F_A(q_2)), G_\delta(q_1, q_2) \geq (\beta \leftarrow G_A(q_1)) \leftarrow (\beta \leftarrow G_A(q_2))$ and $H_\delta(q_1, q_2) \geq (\gamma \leftarrow H_A(q_1)) \leftarrow (\gamma \leftarrow H_A(q_2))$ (cf., Proposition 2.6), which implies that $(\sigma \to F_A(q_1)) \otimes F_\delta(q_1, q_2) \leq (\sigma \to F_A(q_2)), (\beta \leftarrow G_A(q_1)) \otimes G_\delta(q_1, q_2) \geq (\beta \leftarrow G_A(q_2))$ and $(\gamma \leftarrow H_A(q_1)) \otimes H_\delta(q_1, q_2) \geq (\gamma \leftarrow H_A(q_2))$. Thus $\widehat{\eta} \to A$ is a NSS of L.

Proposition 3.12. *Let $L = (Q, X, \delta)$ be a NA and $A \in NS(Q)$ is a RNSS of L if and only if it is a NSS of the RNA $\overline{L} = (Q, X, \overline{\delta})$.*

Proof: Let A is a NSS of the RNA $\overline{L} = (Q, X, \overline{\delta})$, then $\forall q_1, q_2 \in Q$ and $b_1 \in X, F_A(q_1) \otimes F_{\overline{\delta}}(q_1, b_1, q_2) \leq F_A(q_2); G_A(q_1) \otimes G_{\overline{\delta}}(q_1, b_1, q_2) \geq G_A(q_2)$ and $H_A(q_1) \otimes H_{\overline{\delta}}(q_1, b_1, q_2) \geq H_A(q_2)$ if and only if $F_A(q_1) \otimes F_\delta(q_2, b_1, q_1) \leq F_A(q_2); G_A(q_1) \otimes G_\delta(q_2, b_1, q_1) \geq G_A(q_2)$ and $H_A(q_1) \otimes H_\delta(q_2, b_1, q_1) \geq H_A(q_2)$. Thus A is a RNSS of L. Converse is trivial.

Proposition 3.13. *Let $L = (Q, X, \delta)$ be a NA with $A \in NS(Q)$ and let $\neg$ be involutive. Then*

(i) *If A is a NSS, then $\neg A = (\neg F_A, \neg G_A, \neg H_A)$ is a RNSS,* and

(ii) *if A is a RNSS, then $\neg A = (\neg F_A, \neg G_A, \neg H_A)$ is a NSS.*

(iii) *if A is a DNSS, then $\neg A = (\neg F_A, \neg G_A, \neg H_A)$ is also a DNSS.*

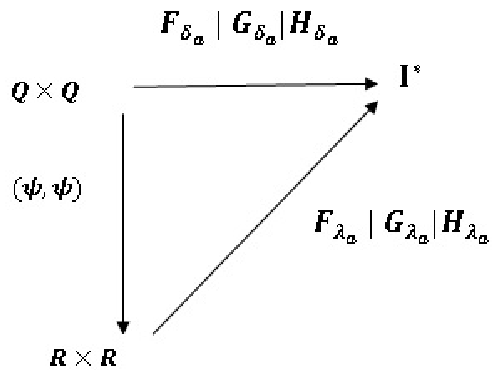
Proof: (i) Let $A = (F_A, G_A, H_A)$ is a NSS of L, then $\forall q_1, q_2 \in Q$ and $b_1 \in X, F_A(q_1) \otimes F_\delta(q_1, b_1, q_2) \leq F_A(q_2), G_A(q_1) \otimes G_\delta(q_1, b_1, q_2) \geq G_A(q_2)$ and $H_A(q_1) \otimes H_\delta(q_1, b_1, q_2) \geq H_A(q_2)$, or that $\neg(F_A(q_1) \otimes F_\delta(q_1, b_1, q_2)) \geq \neg F_A(q_2); \neg(G_A(q_1) \otimes G_\delta(q_1, b_1, q_2)) \leq \neg G_A(q_2)$ and $\neg(H_A(q_1) \otimes H_\delta(q_1, b_1, q_2)) \leq \neg H_A(q_2)$ which implies that $(F_A(q_1) \otimes F_\delta(q_1, b_1, q_2)) \rightarrow 0 \geq \neg F_A(q_2); (G_A(q_1) \otimes G_\delta(q_1, b_1, q_2)) \leftarrow 1 \leq \neg G_A(q_2)$ and $(H_A(q_1) \otimes H_\delta(q_1, b_1, q_2)) \leftarrow 1 \leq \neg H_A(q_2) \Rightarrow (F_\delta(q_1, b_1, q_2) \otimes F_A(q_1)) \rightarrow 0 \geq \neg F_A(q_2); (G_\delta(q_1, b_1, q_2) \otimes G_A(q_1)) \leftarrow 1 \leq \neg G_A(q_2)$ and $(H_\delta(q_1, b_1, q_2) \otimes H_A(q_1)) \leftarrow 1 \leq \neg H_A(q_2) \Rightarrow F_\delta(q_1, b_1, q_2) \rightarrow (F_A(q_1) \rightarrow 0) \geq \neg F_A(q_2); G_\delta(q_1, b_1, q_2) \leftarrow (G_A(1_1) \leftarrow 1) \leq \neg G_A(q_2)$ and $H_\delta(q_1, b_1, q_2) \leftarrow (H_A(q_1) \leftarrow 1) \leq \neg H_A(q_2) \Rightarrow F_\delta(q_1, b_1, q_2) \rightarrow \neg F_A(q_1) \geq \neg F_A(q_2); G_\delta(q_1, b_1, q_2) \leftarrow \neg G_A(q_1) \leq \neg G_A(q_2)$ and $H_\delta(q_1, b_1, q_2) \leftarrow \neg H_A(q_1) \leq \neg H_A(q_2) \Rightarrow \neg F_A(q_2) \otimes F_\delta(q_1, b_1, q_2) \leq \neg F_A(q_1); \neg G_A(q_2) \otimes G_\delta(q_1, b_1, q_2) \geq \neg G_A(q_1)$ and $\neg H_A(q_2) \otimes H_\delta(q_1, b_1, q_2) \geq \neg H_A(q_1)$ (cf., Proposition 2.6). Hence $\neg A$ is a RNSS of L.

(ii) Let $A = (F_A, G_A, H_A)$ is a RNSS of L, then $\forall q_1, q_2 \in Q$ and $b_1 \in X, F_A(q_2) \otimes F_\delta(q_1, b_1, q_2) \leq F_A(q_1); G_A(q_2) \otimes G_\delta(q_1, b_1, q_2) \geq G_A(q_1)$ and $H_A(q_2) \otimes H_\delta(q_1, b_1, q_2) \geq H_A(q_1)$, or that $\neg(F_A(q_2) \otimes F_\delta(q_1, b_1, q_2)) \geq \neg F_A(q_1); \neg(G_A(q_2) \otimes G_\delta(q_1, b_1, q_2)) \leq \neg G_A(q_1)$ and $\neg(H_A(q_2) \otimes H_\delta(q_1, b_1, q_2)) \leq \neg H_A(q_1)$ which implies that $(F_A(q_2) \otimes F_\delta(q_1, b_1, q_2)) \rightarrow 0 \geq \neg F_A(q_1); (G_A(q_2) \otimes G_\delta(q_1, b_1, q_2)) \leftarrow 1 \leq \neg G_A(q_1)$ and $(H_A(q_2) \otimes H_\delta(q_1, b_1, q_2)) \leftarrow 1 \leq \neg H_A(q_1) \Rightarrow (F_\delta(q_1, b_1, q_2) \otimes F_A(q_2)) \rightarrow 0 \geq \neg F_A(q_1); (G_\delta(q_1, b_1, q_2) \otimes G_A(q_2)) \leftarrow 1 \leq \neg G_A(q_1)$ and $(H_\delta(q_1, b_1, q_2) \otimes H_A(q_2)) \leftarrow 1 \leq \neg H_A(q_1) \Rightarrow F_\delta(q_1, b_1, q_2) \rightarrow (F_A(q_2) \rightarrow 0) \geq \neg F_A(q_1); G_\delta(q_1, b_1, q_2) \leftarrow (G_A(q_2) \leftarrow 1) \leq \neg G_A(q_1)$ and $H_\delta(q_1, b_1, q_2) \leftarrow (H_A(q_2) \leftarrow 1) \leq \neg H_A(q_1) \Rightarrow F_\delta(q_1, b_1, q_2) \rightarrow \neg F_A(q_2) \geq \neg F_A(q_1); G_\delta(q_1, b_1, q_2) \leftarrow \neg G_A(q_2) \leq \neg G_A(q_1)$ and $H_\delta(q_1, b_1, q_2) \leftarrow \neg H_A(q_2) \leq \neg H_A(q_1) \Rightarrow \neg F_A(q_1) \otimes F_\delta(q_1, b_1, q_2) \leq \neg F_A(q_2); \neg G_A(q_1) \otimes G_\delta(q_1, b_1, q_2) \geq \neg G_A(q_2)$ and $\neg H_A(q_1) \otimes H_\delta(q_1, b_1, q_2) \geq \neg H_A(q_2)$ (cf., Proposition 2.6). Hence $\neg A$ is a NSS of L.

(iii) Derives from (i) and (ii).

4. Neutrosophic automata and reverse neutrosophic automata: a categorical approach

In this section, we initially demonstrate that an isomorphism among neutrosophic automata (NA) establishes an equivalence relation. Additionally, we present the categorical characteristics of both neutrosophic automata and reverse neutrosophic automata. Furthermore, we identify the functorial relationship that exists between the categories of neutrosophic automata and reverse neutrosophic automata. The discussion begins with the following points.



(Fig.1)Homomorphism between L and N

Definition 4.1. Let $L = (Q, \delta)$ and $N = (R, \lambda)$ are two NA over X. A homomorphism from L to N is a function $\psi : Q \to R$ such that, for each element $b_1 \in X$, the diagram depicted in Figure 1 remains consistent.

Remark 4.2. (i) In Figure 1, the commutativity of a diagram signifies $(F_{\lambda_{b_1}} o(\psi, \psi))(q_1, q_2) = F_{\delta_{b_1}}(q_1, q_2); (G_{\lambda_{b_1}} o(\psi, \psi))(q_1, q_2) = G_{\delta_{b_1}}(q_1, q_2)$ and $(H_{\lambda_{b_1}} o(\psi, \psi))(q_1, q_2) = H_{\delta_{b_1}}(q_1, q_2), \forall q_1, q_2 \in Q$.

(ii) Throughout, we will use the notation $F_A|G_A|H_A$ diagrams to denote a neutrosophic set A. Furthermore, the commutativity of these diagrams remains consistent with the discussion in part (i).

Remark 4.3. (i). The pair (ψ_1, ψ_2) is known as a strong homomorphism if, $\forall(q_1, b_1, q_2) \in Q \times X \times Q, F_\lambda(\psi_1(q_1), \psi_2(b_1), \psi_1(q_2)) = \vee\{F_\delta(q_1, b_1, q_3) : q_3 \in Q, \psi_1(q_3) = \psi_1(q_2)\}, G_\lambda(\psi_1(q_3), \psi_2(b_1), \psi_1(q_2)) = \wedge\{G_\delta(q_1, b_1, q_3) : q_3 \in Q, \psi_1(q_3) = \psi_1(q_2)\}$ and $H_\lambda(\psi_1(q_1), \psi_2(b_1), \psi_1(q_2)) = \wedge\{H_\delta(q_1, b_1, q_3) : q_3 \in Q, \psi_1(q_3) = \psi_1(q_2)\}$.

(ii). A bijective homomorphism (strong homomorphism) with the property $\lambda(\psi_1(q_1), \psi_2(b_1), \psi_1(q_2)) = \delta(q_1, b_1, q_2)$ is called an isomorphism (strong isomorphism).

Definition 4.4. Let $L = (Q, X, \delta)$ and $N = (R, X, \lambda)$ be two NA and $\psi : L \longrightarrow N$ be a homomorphism. Then for $A \in NS(Q)$, the neutrosophic subset $\psi(A) \in NS(R)$ can be defined as

$$F_{\psi(A)}(q_3) = \begin{cases} \vee(F_A(q_1) : q_1 \in Q, \psi(q_1) = q_3) & \text{if } \psi^{-1}(q_3) \neq \phi \\ 0 & \text{if } \psi^{-1}(q_3) = \phi \end{cases}$$

$$G_{\psi(A)}(q_3) = \begin{cases} \wedge(G_A(q_1) : q_1 \in Q, \psi(q_1) = q_3) & \text{if } \psi^{-1}(q_3) \neq \phi \\ 1 & \text{if } \psi^{-1}(q_3) = \phi \ and \end{cases}$$

$$H_{\psi(A)}(q_3) = \begin{cases} \wedge(H_A(q_1) : q_1 \in Q, \psi(q_1) = q_3) & \text{if } \psi^{-1}(q_3) \neq \phi \\ 1 & \text{if } \psi^{-1}(q_3) = \phi, \end{cases}$$

In this context, we explore the properties of NSS under strong homomorphism.

Proposition 4.5. *Let $L = (Q, X, \delta)$ and $N = (R, X, \lambda)$ be two NA and $\psi : L \longrightarrow N$ be an onto strong homomorphism. Then for a NSS A of $L, \psi(A)$ is a NSS of N.*

Proof: Let $q_1, q_2 \in Q$ and $r_1, r_2 \in R$ such that $f(q_1) = r_1$ and $f(q_2) = r_2$. If A is a NSS of L, then $\forall r_1, r_2 \in R$ and $b_1 \in X$, we have $F_{\psi(A)}(r_1) \otimes F_\lambda(r_1, b_1, r_2) = F_A(r_1) \otimes F_\lambda(r_1, b_1, r_2) = F_A(q_1) \otimes F_\lambda(f(q_1), b_1, f(q_2))$ (where $f(q_1) = r_1, \forall q_1 \in Q$) $= F_A(q_1) \otimes \vee\{F_\delta(q_1, b_1, q_3) : q_3 \in Q, \psi(q_3) = \psi(q_2) = r_2\} = \vee\{F_A(q_1) \otimes F_\delta(q_1, b_1, q_3) : q_3 \in Q, \psi(q_3) = \psi(q_2) = r_2\} \leq \vee\{F_A(q_3) : q_3 \in Q, \psi(q_3) = \psi(q_2) = r_2\} = F_{\psi(A)}(r_2)$. Similarly, we can show that $G_{\psi(A)}(r_1) \otimes G_\lambda(r_1, b_1, r_2) \geq G_{\psi(A)}(r_2)$ and $H_{\psi(A)}(r_1) \otimes H_\lambda(r_1, b_1, r_2) \geq H_{\psi(A)}(r_2)$. Hence $\psi(A)$ is a NSS of N.

The proposition mentioned above holds true solely for NSS and does not apply to RNSS.

Proposition 4.6. *An isomorphism among NA establishes an equivalence relation.*

Proof:-The reflexivity and symmetry are evident. To establish transitivity, we let $(\psi_1, \psi_2) : L_1 \longrightarrow L_2$ and $(\chi_1, \chi_2) : L_2 \longrightarrow L_3$ where $\psi_1 : Q_1 \longrightarrow Q_2$, $\chi_1 : Q_2 \longrightarrow Q_3$ and $\psi_2, \chi_2 : X \longrightarrow X$ be the isomorphism of L_1 onto L_2 and L_2 onto L_3 respectively. Then $(\chi_1, \chi_2)o(\psi_1, \psi_2) : L_1 \longrightarrow L_3$ is bijective map from L_1 to L_3, where$((\chi_1, \chi_2)o(\psi_1, \psi_2))(q_1, b_1, q_1') = (\chi_1, \chi_2)((\psi_1, \psi_2)(q_1, b_1, q_1')), \forall (q_1, b_1, q_1') \in Q_1 \times X \times Q_1$.

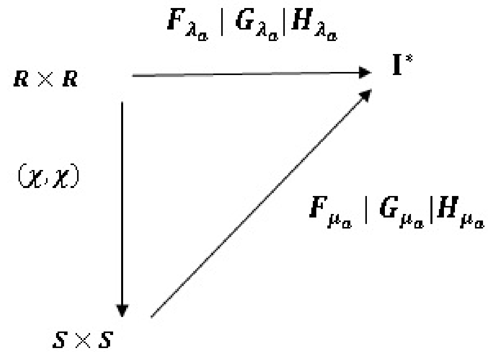
Since a map $(\psi_1, \psi_2) : L_1 \to L_2$ defined as $\psi_1(q_1) = q_2, \psi_1(q_1') = q_2', \psi_2(b_1) = b_1$ is an isomorphism. So, we have $F_{\delta_1}(q_1, b_1, q_1') = F_{\delta_2}(\psi_1(q_1), \psi_2(b_1), \psi_1(q_1')) = F_{\delta_2}(q_2, b_1, q_2')$. Similarly,$G_{\delta_1}(q_1, b_1, q_1') = G_{\delta_2}(q_2, b_1, q_2')$ and $H_{\delta_1}(q_1, b_1, q_1') = H_{\delta_2}(q_2, b_1, q_2'), \forall (q_1, b_1, q_1') \in Q_1 \times X \times Q_1$ and $\forall (q_2, b_1, q_2') \in Q_2 \times X \times Q_2$.

.....(1)

Next, since a map $(\chi_1, \chi_2) : L_2 \to L_3$ defined as $\chi_1(q_2) = q_3, \chi_1(q_2') = q_3'$ and $\chi_2(b_1) = b_1$ is an isomorphism. So, we have $F_{\delta_2}(q_2, b_1, q_2') = F_{\delta_3}(\chi_1(q_2), \chi_2(b_1), \chi_1(q_2')) = F_{\delta_3}(q_3, b_1, q_3')$. Similarly $G_{\delta_2}(q_2, b_1, q_2') = G_{\delta_3}(q_3, b_1, q_3')$ and $H_{\delta_2}(q_2, b_1, q_2') = H_{\delta_3}(q_3, b_1, q_3'), \forall (q_2, b_1, q_2') \in Q_2 \times X \times Q_2$ and $(q_3, b_1, q_3') \in Q_3 \times X \times Q_3$.

.........(2)

Thus from expressions $(1), (2)$ and$\psi_1(q_1) = q_2, \psi_1(q_1') = q_2', \psi_2(b_1) = b_1, \forall (q_1, b_1, q_1') \in Q_1 \times X \times Q_1$, we have $F_{\delta_1}(q_1, b_1, q_1') = F_{\delta_2}(\psi_1(q_1), \psi_2(b_1), \psi_1(q_1')) = F_{\delta_2}(q_2, b_1, q_2') = F_{\delta_3}(\chi_1(q_2), \chi_2(b_1), \chi_1(q_2')) = F_{\delta_3}((\chi_1, \chi_2)(q_2, b_1, q_2')) = F_{\delta_3}((\chi_1, \chi_2)((\psi_1, \psi_2)(q_1, b_1, q_1')) =$



(Fig.2)Homomorphism between N and P

$F_{\delta_3}((\chi_1,\chi_2)o(\psi_1,\psi_2))(q_1,b_1,q_1')$. Similarly $G_{\delta_1}(q_1,b_1,q_1') = G_{\delta_3}((\chi_1,\chi_2)o(\psi_1,\psi_2))(q_1,b_1,q_1')$ and $H_{\delta_1}(q_1,b_1,q_1') = H_{\delta_3}((\chi_1,\chi_2)o(\psi_1,\psi_2))(q_1,b_1,q_1'), \forall (q_1,b_1,q_1') \in Q_1 \times X \times Q_1$. Hence $(\chi_1,\chi_2)o(\psi_1,\psi_2)$ is an isomorphism between L_1 and L_3.

Proposition 4.7. *An isomorphism among RNA establishes an equivalence relation.*

Proof:- A direct consequence of the proposition 4.6.

Proposition 4.8. *An isomorphism among DNA establishes an equivalence relation.*

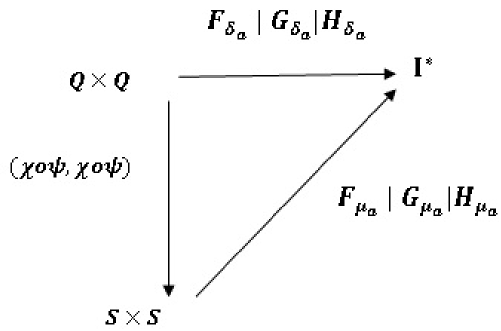
Proof:- This is a direct consequence of the propositions 4.6 and 4.7.

We will represent the category of NA over X as $\mathbf{NeA}(X)$ and the category of NA over X^* as $\mathbf{NeA}(X^*)$. Additionally, the object-class of the categories NeA(X) and NeA(X^*) will be denoted as NeA(X) and NeA(X^*), respectively. Now, we proceed with the following.

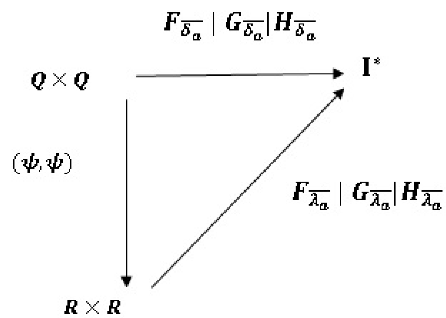
Proposition 4.9. *The class of NA over X and their homomorphisms constitute a category.*

Proof: We demonstrate solely that the composition of two homomorphisms is again a homomorphism, as follows, let $L = (Q,\delta), N = (R,\lambda)$ and $P = (S,\mu)$ be NA over X and $\psi : L \to N, \chi : N \to P$ be homomorphisms, i.e., $\psi : Q \to R, \chi : R \to S$ are the maps such that for all $b_1 \in X$, the diagrams in Fig.1 and Fig. 2 holds. Then the following shows that for all $b_1 \in X$, the diagram in Fig. 3 also hold. So, let $q_1, q_2 \in Q$. Then $(F_{\mu_{b_1}} o(\chi o\psi, \chi o\psi))(q_1,q_2) = F_{\mu_{b_1}}(\chi(\psi(q_1)), \chi(\psi(q_2))) = (F_{\mu_{b_1}} o(\chi,\chi))(\psi(q_1),\psi(q_2)) = F_{\lambda_{b_1}}(\psi(q_1),\psi(q_2)) = (F_{\lambda_{b_1}} o(\psi,\psi))(q_1,q_2) = F_{\delta_{b_1}}(q_1,q_2)$. Hence $F_{\delta_{b_1}} = F_{\mu_{b_1}} o(\chi o\psi, \chi o\psi)$. Similarly, we can show that $G_{\delta_{b_1}} = G_{\mu_{b_1}} o(\chi o\psi, \chi o\psi)$ and $H_{\delta_{b_1}} = H_{\mu_{b_1}} o(\chi o\psi, \chi o\psi)$. Thus $\chi o\psi : L \to P$ is a homomorphism.

We will represent the category of RNA over X as $\mathbf{RNeA}(X)$ and the category of RNA over



(Fig.3) Homomorphism between L and P



(Fig.4) Homomorphism between $\overline{L}$ and $\overline{N}$

X^* as **RNeA**(X^*). Additionally, the object-class of the categories $RNeA(X)$ and $RNeA(X^*)$ will be denoted as $RNeA(X)$ and $RNeA(X^*)$, respectively.

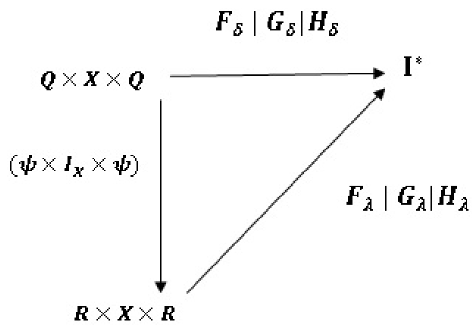
Definition 4.10. Let $\overline{L}$=(Q, $\overline{\delta}$) and $\overline{N}$=(R, $\overline{\lambda}$) be RNA over X. A homomorphism from $\overline{L}$ to $\overline{N}$ is a map $\psi : Q \to R$ such that for all $b_1 \in X$, the diagram in Fig.4 hold.

Now, we present the introduction of functors between the categories of NA as described earlier.

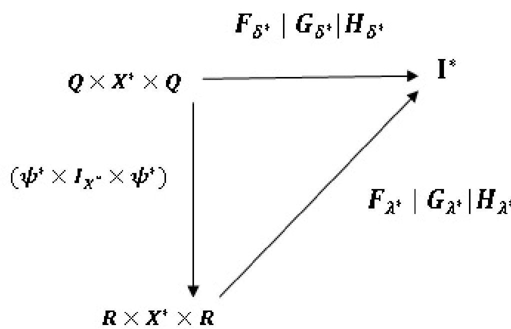
Proposition 4.11. *From $NeA(X)$ to $NeA(X^*)$, there exists a functor.*

Proof:- Let $L = (Q, X, \delta) \in NeA(X)$. We establish a mapping $K : NeA(X) \to NeA(X^*)$ such that $K(L) = (Q, X^*, \delta^*)$, then $K(L) \in NeA(X^*)$. Also, for a $NeA(X)$-morphism $\psi : L = (Q, X, \delta) \to N = (R, X, \lambda)$, let $K(\psi) : K(L) \to K(N)$,i.e., $K(\psi) = \psi^*$. Subsequently, it can be demonstrated that ψ^* is a $NeA(X^*)$-morphism from $K(L)$ to $K(N)$, i.e., the depicted diagram in Figure 5 is valid, indicating that the diagram in Figure 6 also holds. Consequently, based on Figure 5, we obtain

$F_\delta = F_\lambda o(\psi \times I_X \times \psi), G_\delta = G_\lambda o(\psi \times I_X \times \psi)$ and $H_\delta = H_\lambda o(\psi \times I_X \times \psi)$



(Fig.5) Morphism between L and N



(Fig.6) Morphism between $K(L)$and $K(N)$

Now, $K(F_\delta) = F_{\delta^*} = K[F_\lambda o(\psi \times I_X \times \psi)] = F_{\lambda^*} o(\psi^* \times I_{X^*} \times \psi^*)$. In a similar manner $K(G_\delta) = G_{\delta^*} = K[G_\lambda o(\psi \times I_X \times \psi)] = G_{\lambda^*} o(\psi^* \times I_{X^*} \times \psi^*)$ and $K(H_\delta) = H_{\delta^*} = K[H_\lambda o(\psi \times I_X \times \psi)] = H_{\lambda^*} o(\psi^* \times I_{X^*} \times \psi^*)$. This implies the validity of Figure 6. Additionally, the identity and composition properties of maps K are evident. Therefore, the mapping $K : NeA(X) \to NeA(X^*)$ is a functor.

Proposition 4.12. *From $NeA(X^*)$ to $NeA(X)$, there exists a functor.*

Proof:- Define a mapping $\beta : NeA(X^*) \to NeA(X)$ such that $\beta(L) = (Q, X, \delta), \forall L \in NeA(X^*)$. Then $\beta(L) \in NeA(X)$. Therefore, based on proposition 4.11, we demonstrate that β operates as a functor.

In this context, we present the functor between the category of RNA, as defined earlier.

Proposition 4.13. *From $RNeA(X)$ to $RNeA(X^*)$, there exists a functor.*

Proof:- This is a direct consequence of the proposition 4.11.

Proposition 4.14. *From $RNeA(X^*)$ to $RNeA(X)$, there exists a functor.*

Proof:-This is a direct consequence of the proposition 4.12.

5. Conclusions

This paper has introduced the novel concepts of neutrosophic automata and reverse neutrosophic automata, extending the groundwork laid by fuzzy automata. The exploration includes the introduction of neutrosophic subsystems, reverse neutrosophic subsystems, and double neutrosophic subsystems linked to these automata, with an investigation into algebraic results derived from these concepts. Additionally, the categorical properties of neutrosophic automata and their functorial relationships have been examined. In future work, the focus will extend to exploring the topological properties of neutrosophic automata based on the aforementioned concepts.

Conflicts of Interest: The authors assert that there are no conflicts of interest..

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New algebraic structure for Diophantine neutrosophic subbisemirings of bisemirings

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Abstract. In this paper, we define the concept of Diophantine neutrosophic subbisemiring (DioNSBS) of bisemirings (BSs). The DioNSBS is the new approach for fuzzy subbisemiring (FSBS) over a BS. Let Ξ be the Diophantine neutrosophic subset (DioNSS) in $\mathcal{T}$, we show that $\Xi = \langle(\mho^{\mathcal{T}}_{\Xi}, \mho^{\mathcal{I}}_{\Xi}, \mho^{\mathcal{F}}_{\Xi}), (\Gamma_{\Xi}, \Lambda_{\Xi}, \Theta_{\Xi})\rangle$ is a DioNSBS of $\mathcal{T}$ if and only if all non-empty level set $\Xi^{(\beta,\gamma)}$ is a subbisemiring (SBS) of $\mathcal{T}$, $\forall \beta, \gamma \in [0, 1]$. Let Ξ be the DioNSBS of a BS $\mathcal{T}$ and Z be the strongest Diophantine neutrosophic relation of $\mathcal{T}$. Then Ξ is a DioNSBS of $\mathcal{T}$ if and only if Z is a DioNSBS of $\mathcal{T} \times \mathcal{T}$. Let $\Xi_1, \Xi_2, \ldots, \Xi_n$ be the family of DioNSBSs of $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n$, respectively. We show that $\Xi_1 \times \Xi_2 \times \ldots \times \Xi_n$ is a DioNSBS of $\mathcal{T}_1 \times \mathcal{T}_2 \times \ldots \times \mathcal{T}_n$. The homomorphic image of every DioNSBS is a DioNSBS. Let Ξ be any DioNSBS of $\mathcal{T}$, then pseudo Diophantine neutrosophic coset $(a\Xi)^p$ is a DioNSBS of $\mathcal{T}$, for every $a \in \mathcal{T}$. Let $(\mathcal{T}_1, \circledast_1, \circledast_2, \circledast_3)$ and $(\mathcal{T}_2, \otimes_1, \otimes_2, \otimes_3)$ be any two BSs. The homomorphic preimage of every DioNSBS of $\mathcal{T}_2$ is a DioNSBS of $\mathcal{T}_1$. Let $(\mathcal{T}_1, \circledast_1, \circledast_2, \circledast_3)$ and $(\mathcal{T}_2, \otimes_1, \otimes_2, \otimes_3)$ be any two BSs. Let Ξ and Δ be any two DioNSBSs of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, then $\Xi \times \Delta$ is a DioNSBS of $\mathcal{T}_1 \times \mathcal{T}_2$. If $\mathcal{L} : \mathcal{T}_1 \to \mathcal{T}_2$ is a homomorphism, then $\mathcal{L}(\Xi_{(\beta,\gamma)})$ is a level SBS of DioNSBS Z of $\mathcal{T}_2$. Examples are given to demonstrate our findings.

Keywords: BS; FSBS; NSBS; DioNSBS.

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1. **Introduction**

Most real-world problems are characterized by uncertainty. Numerous uncertain theories, such as the fuzzy set (FS) [31], intuitionistic fuzzy set (IFS) [5], Pythagorean fuzzy set (PFS)

[28] and neutrosophic set (NSS) [26] are proposed to deal with the uncertainties. An FS is one in which every element in the universe is a member, but only to a degree of belongingness that ranges from zero to one. In the set of elements, these grades are known as membership values. Clustering techniques [29] are used in applications of FSs like regression prediction for fuzzy time series [27]and fuzzy c-numbersIn applications that require precise data. Atanassov [5] introduced the idea of an IFS. An organization whose membership degree and non-membership degree values are less than or equal to one. Occasionally, we have difficulty making decisions when the combined value of the membership degree and the non-membership degree is greater than one. As part of a generalization of IFS, Yager [28] introduced the concept of PFS as defined by the sum of membership degrees with non-membership degrees having a value less than or equal to one. The numerous applications based on PFSs were addressed by Akram et al. [2–4]. The study of semirings resulted from Dedekind's engagement with commutative ring theories. Vandiver [30] introduces semirings as part of his generalization of rings. In the 1880s, the German mathematician Dedekind began to investigate semirings and commutative rings as ideals. Vandiver developed a fundamental algebraic structure in 1934 due to his later research on semirings. A distributive lattice was essentially a generalization of rings. On the other hand, semiring theory has advanced since 1950. Rings and distributive lattices were essentially generalized. The theory of semirings has nevertheless been developing since 1950. Iseki [8, 9] was introduced by the semiring concept that is not always commutative under either operation. Without zero, Iseki [10] demonstrated numerous significant results based on semirings by using this abstraction for semirings. Many authors and academics have described the various ideals based on semirings [7]. Semigroups, semirings, and hypersemigroups are a few examples of ordered algebraic structures that many writers have researched. Zadeh invented the concept of FS [31]. A function described by a membership value is what this definition refers to as an FS. In real unit intervals, degrees are taken. A combination of membership and non-membership has been considered, and an insufficient definition has been reached. NSS extend FS and IFS by delineating truth and indeterminacy memberships separately. To manage the uncertainty presented, Atanassov [5] described a set referred to as an IFS. Several application-related problems are present in this information set, and Smarandache [26] proposed neutrosophy to address these issues. Reference parameters were included in the discussion of the linear Diophantine fuzzy set (LDFS) by Riaz et al. [23]. Because reference parameters are used, the LDFS is more effective and adaptable than other methods. By modifying the reference parameter's physical sense, the LDFS classifies the data in MADM difficulties. A fundamental difference between FS and IFS can be found in neutrosophy, which focuses on neutral cognition. Smarandache [26] invented neurosophic logic. Each proposition is given an estimated degree of truth, degree of ambiguity and degree of falsity according to this logic. Every component of

the cosmos in NSS has a degree of truth, indeterminacy and falsity that ranges from $[0,1]$. The FS, interval-valued FS and classical sets can be generalized to an NSS from a philosophical perspective.

A semiring $(S,+,\cdot)$ is a non-empty set in which $(S,+)$ and $(S,\cdot)$ are semigroups such that "$\cdot$" is distributive over "$+$" [7]. In 1993, Ahsan et al. [1] introduced the notion of fuzzy semirings. In 2001, Sen and Ghosh were introduced in BSs. A bisemiring (BS) $(\mathcal{T},+,\circ,\times)$ is an algebraic structure in which $(\mathcal{T},+,\circ)$ and $(\mathcal{T},\circ,\times)$ are semirings in which $(\mathcal{T},+),(\mathcal{T},\circ)$ and $(\mathcal{T},\times)$ are semigroups such that (i) $\tau_a\circ(\tau_b+\tau_c)=(\tau_a\circ\tau_b)+(\tau_a\circ\tau_c)$, (ii) $(\tau_b+\tau_c)\circ\tau_a=(\tau_b\circ\tau_a)+(\tau_c\circ\tau_a)$ (iii) $\tau_a\times(\tau_b\circ\tau_c)=(\tau_a\times\tau_b)\circ(\tau_a\times\tau_c)$ and (iv) $(\tau_b\circ\tau_c)\times\tau_a=(\tau_b\times\tau_a)\circ(\tau_c\times\tau_a),\forall\tau_a,\tau_c\in\mathcal{T}$ [25]. A non-empty subset Ξ of a BS $(\mathcal{T},+,\circ,\times)$ is an SBS if and only if $\tau_a+\tau_b\in\Xi,\tau_a\circ\tau_b\in\Xi$ and $\tau_a\times\tau_b\in\Xi,\forall\tau_a,\tau_b,\tau_c\in\Xi$ [6]. Palanikumar et al. discussed the various ideal structures of SBS theory and its applications [11]- [20]. The concept of DioNSBSs is introduced in this study. This paper is focused on the following: The introduction is in Section 1. The preliminary definitions and results are found in Section 2. Section 3 introduces the notion of DioNSBS and its several illustrative examples.

2. Basic Concepts

Definition 2.1. [26] An NSS Ξ in the universe $\mathcal{U}$ is $\Xi=\{\epsilon,\mho^{\mathcal{T}}_{\Xi}(\epsilon),\mho^{\mathcal{I}}_{\Xi}(\epsilon),\mho^{\mathcal{F}}_{\Xi}(\epsilon)\mid\epsilon\in\mathcal{U}\}$, where $\mho^{\mathcal{T}}_{\Xi}(\epsilon)$, $\mho^{\mathcal{I}}_{\Xi}(\epsilon)$ $\mho^{\mathcal{F}}_{\Xi}(\epsilon)$ represents the degree of truth-membership, indeterminacy membership and falsity-membership of Ξ, respectively. The mapping $\mho^{\mathcal{T}}_{\Xi},\mho^{\mathcal{I}}_{\Xi},\mho^{\mathcal{F}}_{\Xi}:\mathcal{U}\to[0,1]$ and $0\le\mho^{\mathcal{T}}_{\Xi}(\epsilon)+\mho^{\mathcal{I}}_{\Xi}(\epsilon)+\mho^{\mathcal{F}}_{\Xi}(\epsilon)\le 3$.

Definition 2.2. [26] Let $\Xi_1=\langle\mho^{\mathcal{T}}_{\Xi_1},\mho^{\mathcal{I}}_{\Xi_1},\mho^{\mathcal{F}}_{\Xi_1}\rangle$, $\Xi_2=\langle\mho^{\mathcal{T}}_{\Xi_2},\mho^{\mathcal{I}}_{\Xi_2},\mho^{\mathcal{F}}_{\Xi_2}\rangle$ and $\Xi_3=\langle\mho^{\mathcal{T}}_{\Xi_3},\mho^{\mathcal{I}}_{\Xi_3},\mho^{\mathcal{F}}_{\Xi_3}\rangle$ be the three neutrosophic numbers over $\mathcal{U}$. Then
(i) ${\Xi_1}^c=\langle\mho^{\mathcal{F}}_{\Xi_1},\mho^{\mathcal{I}}_{\Xi_1},\mho^{\mathcal{T}}_{\Xi_1}\rangle$
(ii) $\Xi_2\veebar\Xi_3=\left\langle\max(\mho^{\mathcal{T}}_{\Xi_2},\mho^{\mathcal{T}}_{\Xi_3}),\min(\mho^{\mathcal{I}}_{\Xi_2},\mho^{\mathcal{I}}_{\Xi_3}),\min(\mho^{\mathcal{F}}_{\Xi_2},\mho^{\mathcal{F}}_{\Xi_3})\right\rangle$
(iii) $\Xi_2\barwedge\Xi_3=\left\langle\min(\mho^{\mathcal{T}}_{\Xi_2},\mho^{\mathcal{T}}_{\Xi_3}),\max(\mho^{\mathcal{I}}_{\Xi_2},\mho^{\mathcal{I}}_{\Xi_3}),\max(\mho^{\mathcal{F}}_{\Xi_2},\mho^{\mathcal{F}}_{\Xi_3})\right\rangle$
(iv) $\Xi_2\ge\Xi_3$ iff $\mho^{\mathcal{T}}_{\Xi_2}\ge\mho^{\mathcal{T}}_{\Xi_3}$ and $\mho^{\mathcal{I}}_{\Xi_2}\le\mho^{\mathcal{I}}_{\Xi_3}$ and $\mho^{\mathcal{F}}_{\Xi_2}\le\mho^{\mathcal{F}}_{\Xi_3}$
(v) $\Xi_2=\Xi_3$ iff $\mho^{\mathcal{T}}_{\Xi_2}=\mho^{\mathcal{T}}_{\Xi_3}$ and $\mho^{\mathcal{I}}_{\Xi_2}=\mho^{\mathcal{I}}_{\Xi_3}$ and $\mho^{\mathcal{F}}_{\Xi_2}=\mho^{\mathcal{F}}_{\Xi_3}$.

Definition 2.3. [26] For any NSS $\Xi=\{\xi_a,\mho^{\mathcal{T}}_{\Xi}(\xi_a),\mho^{\mathcal{I}}_{\Xi}(\xi_a),\mho^{\mathcal{F}}_{\Xi}(\xi_a)\}$ of $\mathcal{U}$, we defined a (τ,σ)-cut of as the crisp subset $\{\xi_a\in U\mid\mho^{\mathcal{T}}_{\Xi}(\xi_a)\ge\tau,\mho^{\mathcal{I}}_{\Xi}(\xi_a)\ge\tau,\mho^{\mathcal{F}}_{\Xi}(\xi_a)\le\sigma\}$.

Definition 2.4. [26] Let Ξ and Δ be be two NSSs of $\mathcal{T}$. The Cartesian product of Ξ and Δ is defined as $\Xi\times\Delta=\{\mho^{\mathcal{T}}_{\Xi\times\Delta}(\xi_a,\xi_b),\mho^{\mathcal{I}}_{\Xi\times\Delta}(\xi_a,\xi_b),\mho^{\mathcal{F}}_{\Xi\times\Delta}(\xi_a,\xi_b)\mid\ for\ all\ \xi_a,\xi_b\in\mathcal{T}\}$, where $\mho^{\mathcal{T}}_{\Xi\times\Delta}(\xi_a,\xi_b)=\min\{\mho^{\mathcal{T}}_{\Xi}(\xi_a),\mho^{\mathcal{T}}_{\Delta}(\xi_b)\},\mho^{\mathcal{I}}_{\Xi\times\Delta}(\xi_a,\xi_b)=\frac{\mho^{\mathcal{I}}_{\Xi}(\xi_a)+\mho^{\mathcal{I}}_{\Delta}(\xi_b)}{2},\mho^{\mathcal{F}}_{\Xi\times\Delta}(\xi_a,\xi_b)=\max\{\mho^{\mathcal{F}}_{\Xi}(\xi_a),\mho^{\mathcal{F}}_{\Delta}(\xi_b)\}$.

Definition 2.5. [?] An FS Ξ of a BS $(\mathcal{T}, \oslash_1, \oslash_2, \oslash_3)$ is said to be a fuzzy subbisemiring (FSBS) of $\mathcal{T}$ if $\mho_\Xi(\xi_a \oslash_1 \xi_b) \geq \min\{\mho_\Xi(\xi_a), \mho_\Xi(\xi_b)\}, \mho_\Xi(\xi_a \oslash_2 \xi_b) \geq \min\{\mho_\Xi(\xi_a), \mho_\Xi(\xi_b)\}, \mho_\Xi(\xi_a \oslash_3 \xi_b) \geq \min\{\mho_\Xi(\xi_a), \mho_\Xi(\xi_b)\}$, $\forall \xi_a, \xi_b \in \mathcal{T}$.

Definition 2.6. [?] An FS Ξ of a BS $(\mathcal{T}, \oslash_1, \oslash_2, \oslash_3)$ is said to be a fuzzy normal subbisemiring (FNSBS) of $\mathcal{T}$ if $\mho_\Xi(\xi_a \oslash_1 \xi_b) = \mho_\Xi(\xi_b \oslash_1 \xi_a), \mho_\Xi(\xi_a \oslash_2 \xi_b) = \mho_\Xi(\xi_b \oslash_2 \xi_a), \mho_\Xi(\xi_a \oslash_3 \xi_b) = \mho_\Xi(\xi_b \oslash_3 \xi_a)$, $\forall \xi_a, \xi_b \in \mathcal{T}$.

Definition 2.7. [6] Let $(\mathcal{T}, +, \cdot, \times)$ and $(\mathcal{T}_1, \circledast, \circ, \otimes)$ be two BSs. A mapping $\kappa : \mathcal{T} \to \mathcal{T}_1$ is said to be a homomorphism if $\kappa(\xi_a + \xi_b) = \kappa(\xi_a) \circledast \kappa(\xi_b), \kappa(\xi_a \cdot \xi_b) = \kappa(\xi_a) \circ \kappa(\xi_b), \kappa(\xi_a \times \xi_b) = \kappa(\xi_a) \otimes \kappa(\xi_b)$, $\forall \xi_a, \xi_b \in \mathcal{T}$.

3. Diophantine Neutrosophic Subbisemirings

In the following, let $\mathcal{T}$ denote a BS unless otherwise stated.

Definition 3.1. A DioNSS Ξ in $\mathcal{U}$ is $\Xi = \Big\{\epsilon, \Big(\mho^{\mathcal{T}}_\Xi(\epsilon), \mho^{\mathcal{I}}_\Xi(\epsilon), \mho^{\mathcal{F}}_\Xi(\epsilon)\Big), \Big(\Gamma_\Xi(\epsilon), \Lambda_\Xi(\epsilon), \Theta_\Xi(\epsilon)\Big) \mid \epsilon \in \mathcal{U}\Big\}$, where $\mho^{\mathcal{T}}_\Xi(\epsilon)$, $\mho^{\mathcal{I}}_\Xi(\epsilon)$ $\mho^{\mathcal{F}}_\Xi(\epsilon)$ represents the degree of truth-membership, degree of indeterminacy membership and degree of falsity-membership of Ξ, respectively, and $\Gamma_\Xi(\epsilon) + \Lambda_\Xi(\epsilon) + \Theta_\Xi(\epsilon) \leq 1$. The mapping $\mho^{\mathcal{T}}_\Xi, \mho^{\mathcal{I}}_\Xi, \mho^{\mathcal{F}}_\Xi : \mathcal{U} \to [0, 1]$ and $0 \leq (\Gamma_\Xi(\epsilon) \cdot \mho^{\mathcal{T}}_\Xi(\epsilon)) + (\Lambda_\Xi(\epsilon) \cdot \mho^{\mathcal{I}}_\Xi(\epsilon)) + (\Theta_\Xi(\epsilon) \cdot \mho^{\mathcal{F}}_\Xi(\epsilon)) \leq 2$.

Definition 3.2. A DioNSS Ξ of $\mathcal{T}$ is said to be a DioNSBS of $\mathcal{T}$ if $(\forall \zeta, \eta \in \mathcal{T})$

$$\begin{Bmatrix} \mho^{\mathcal{T}}_\Xi(\zeta \oslash_1 \eta) \geq \min\{\mho^{\mathcal{T}}_\Xi(\zeta), \mho^{\mathcal{T}}_\Xi(\eta)\} \\ \mho^{\mathcal{T}}_\Xi(\zeta \oslash_2 \eta) \geq \min\{\mho^{\mathcal{T}}_\Xi(\zeta), \mho^{\mathcal{T}}_\Xi(\eta)\} \\ \mho^{\mathcal{T}}_\Xi(\zeta \oslash_3 \eta) \geq \min\{\mho^{\mathcal{T}}_\Xi(\zeta), \mho^{\mathcal{T}}_\Xi(\eta)\} \end{Bmatrix} \quad \begin{Bmatrix} \mho^{\mathcal{I}}_\Xi(\zeta \oslash_1 \eta) \geq \frac{\mho^{\mathcal{I}}_\Xi(\zeta) + \mho^{\mathcal{I}}_\Xi(\eta)}{2} \\ \text{OR} \\ \mho^{\mathcal{I}}_\Xi(\zeta \oslash_2 \eta) \geq \frac{\mho^{\mathcal{I}}_\Xi(\zeta) + \mho^{\mathcal{I}}_\Xi(\eta)}{2} \\ \text{OR} \\ \mho^{\mathcal{I}}_\Xi(\zeta \oslash_3 \eta) \geq \frac{\mho^{\mathcal{I}}_\Xi(\zeta) + \mho^{\mathcal{I}}_\Xi(\eta)}{2} \end{Bmatrix}$$

$$\begin{Bmatrix} \mho^{\mathcal{F}}_\Xi(\zeta \oslash_1 \eta) \leq \max\{\mho^{\mathcal{F}}_\Xi(\zeta), \mho^{\mathcal{F}}_\Xi(\eta)\} \\ \mho^{\mathcal{F}}_\Xi(\zeta \oslash_2 \eta) \leq \max\{\mho^{\mathcal{F}}_\Xi(\zeta), \mho^{\mathcal{F}}_\Xi(\eta)\} \\ \mho^{\mathcal{F}}_\Xi(\zeta \oslash_3 \eta) \leq \max\{\mho^{\mathcal{F}}_\Xi(\zeta), \mho^{\mathcal{F}}_\Xi(\eta)\} \end{Bmatrix}$$

$$\begin{Bmatrix} \Gamma_\Xi(\zeta \oslash_1 \eta) \geq \min\{\Gamma_\Xi(\zeta), \Gamma_\Xi(\eta)\} \\ \Gamma_\Xi(\zeta \oslash_2 \eta) \geq \min\{\Gamma_\Xi(\zeta), \Gamma_\Xi(\eta)\} \\ \Gamma_\Xi(\zeta \oslash_3 \eta) \geq \min\{\Gamma_\Xi(\zeta), \Gamma_\Xi(\eta)\} \end{Bmatrix} \quad \begin{Bmatrix} \Lambda_\Xi(\zeta \oslash_1 \eta) \geq \frac{\Lambda_\Xi(\zeta) + \Lambda_\Xi(\eta)}{2} \\ \text{OR} \\ \Lambda_\Xi(\zeta \oslash_2 \eta) \geq \frac{\Lambda_\Xi(\zeta) + \Lambda_\Xi(\eta)}{2} \\ \text{OR} \\ \Lambda_\Xi(\zeta \oslash_3 \eta) \geq \frac{\Lambda_\Xi(\zeta) + \Lambda_\Xi(\eta)}{2} \end{Bmatrix}$$

$$\begin{Bmatrix} \Theta_\Xi(\zeta \oslash_1 \eta) \leq \max\{\Theta_\Xi(\zeta), \Theta_\Xi(\eta)\} \\ \Theta_\Xi(\zeta \oslash_2 \eta) \leq \max\{\Theta_\Xi(\zeta), \Theta_\Xi(\eta)\} \\ \Theta_\Xi(\zeta \oslash_3 \eta) \leq \max\{\Theta_\Xi(\zeta), \Theta_\Xi(\eta)\} \end{Bmatrix}.$$

Example 3.3. Let $\mathcal{T} = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ be the BS with the tables:

$\oslash_1$	θ_a	θ_b	θ_c	θ_d
θ_a	θ_a	θ_a	θ_a	θ_a
θ_b	θ_a	θ_b	θ_a	θ_b
θ_c	θ_a	θ_a	θ_c	θ_c
θ_d	θ_a	θ_b	θ_c	θ_d

$\oslash_2$	θ_a	θ_b	θ_c	θ_d
θ_a	θ_a	θ_b	θ_c	θ_d
θ_b	θ_b	θ_b	θ_d	θ_d
θ_c	θ_c	θ_d	θ_c	θ_d
θ_d	θ_d	θ_d	θ_d	θ_d

$\oslash_3$	θ_a	θ_b	θ_c	θ_d
θ_a	θ_a	θ_a	θ_a	θ_a
θ_b	θ_a	θ_b	θ_c	θ_d
θ_c	θ_d	θ_d	θ_d	θ_d
θ_d	θ_d	θ_d	θ_d	θ_d

	$\theta = \theta_a$	$\theta = \theta_b$	$\theta = \theta_c$	$\theta = \theta_d$
$\left(\mho^{\mathcal{T}}_\Xi(\theta), \Gamma_\Xi(\theta)\right)$	$(0.97, 0.40)$	$(0.95, 0.35)$	$(0.92, 0.25)$	$(0.94, 0.30)$
$\left(\mho^{\mathcal{I}}_\Xi(\theta), \Lambda_\Xi(\theta)\right)$	$(0.80, 0.25)$	$(0.78, 0.20)$	$(0.73, 0.10)$	$(0.75, 0.15)$
$\left(\mho^{\mathcal{F}}_\Xi(\theta), \Theta_\Xi(\theta)\right)$	$(0.85, 0.30)$	$(0.89, 0.35)$	$(0.91, 0.45)$	$(0.90, 0.40)$

Clearly, Ξ is a DioNSBS of $\mathcal{T}$.

Theorem 3.4. *The intersection of a family of DioNSBSs of $\mathcal{T}$ is a DioNSBS of $\mathcal{T}$.*

Proof. Let $\{Z_i : i \in \mathcal{I}\}$ be a family of DioNSBSs of $\mathcal{T}$ and $\Xi = \bigcap_{i\in\mathcal{I}} Z_i$.
Let ζ and η in $\mathcal{T}$. Then

$$\begin{aligned} \mho^{\mathcal{T}}_\Xi(\zeta \oslash_1 \eta) &= \inf_{i\in\mathcal{I}} \mho^{\mathcal{T}}_{Z_i}(\zeta \oslash_1 \eta) \\ &\geq \inf_{i\in\mathcal{I}} \min\{\mho^{\mathcal{T}}_{Z_i}(\zeta), \mho^{\mathcal{T}}_{Z_i}(\eta)\} \\ &= \min\left\{\inf_{i\in\mathcal{I}} \mho^{\mathcal{T}}_{Z_i}(\zeta), \inf_{i\in\mathcal{I}} \mho^{\mathcal{T}}_{Z_i}(\eta)\right\} \\ &= \min\{\mho^{\mathcal{T}}_\Xi(\zeta), \mho^{\mathcal{T}}_\Xi(\eta)\}. \end{aligned}$$

Similarly, $\mho^{\mathcal{T}}_\Xi(\zeta \oslash_2 \eta) \geq \min\{\mho^{\mathcal{T}}_\Xi(\zeta), \mho^{\mathcal{T}}_\Xi(\eta)\}$, $\mho^{\mathcal{T}}_\Xi(\zeta \oslash_3 \eta) \geq \min\{\mho^{\mathcal{T}}_\Xi(\zeta), \mho^{\mathcal{T}}_\Xi(\eta)\}$. Now,

$$\begin{aligned} \mho^{\mathcal{I}}_\Xi(\zeta \oslash_1 \eta) &= \inf_{i\in\mathcal{I}} \mho^{\mathcal{I}}_{Z_i}(\zeta \oslash_1 \eta) \\ &\geq \inf_{i\in\mathcal{I}} \frac{\mho^{\mathcal{I}}_{Z_i}(\zeta) + \mho^{\mathcal{I}}_{Z_i}(\eta)}{2} \\ &= \frac{\inf_{i\in\mathcal{I}} \mho^{\mathcal{I}}_{Z_i}(\zeta) + \inf_{i\in\mathcal{I}} \mho^{\mathcal{I}}_{Z_i}(\eta)}{2} \\ &= \frac{\mho^{\mathcal{I}}_\Xi(\zeta) + \mho^{\mathcal{I}}_\Xi(\eta)}{2}. \end{aligned}$$

Similarly, $\mho_{\Xi}^{\mathcal{I}}(\zeta \oslash_2 \eta) \geq \frac{\mho_{\Xi}^{\mathcal{I}}(\zeta)+\mho_{\Xi}^{\mathcal{I}}(\eta)}{2}$ and $\mho_{\Xi}^{\mathcal{I}}(\zeta \oslash_3 \eta) \geq \frac{\mho_{\Xi}^{\mathcal{I}}(\zeta)+\mho_{\Xi}^{\mathcal{I}}(\eta)}{2}$. Now,

$$\begin{aligned}
\mho_{\Xi}^{\mathcal{F}}(\zeta \oslash_1 \eta) &= \sup_{i\in\mathcal{I}} \mho_{Z_i}(\zeta \oslash_1 \eta) \\
&\leq \sup_{i\in\mathcal{I}} \max\{\mho_{Z_i}(\zeta), \mho_{Z_i}(\eta)\} \\
&= \max\left\{\sup_{i\in\mathcal{I}} \mho_{Z_i}(\zeta), \sup_{i\in\mathcal{I}} \mho_{Z_i}(\eta)\right\} \\
&= \max\{\mho_{\Xi}^{\mathcal{F}}(\zeta), \mho_{\Xi}^{\mathcal{F}}(\eta)\}.
\end{aligned}$$

Similarly, $\mho_{\Xi}^{\mathcal{F}}(\zeta \oslash_2 \eta) \leq \max\{\mho_{\Xi}^{\mathcal{F}}(\zeta), \mho_{\Xi}^{\mathcal{F}}(\eta)\}$, $\mho_{\Xi}^{\mathcal{F}}(\zeta \oslash_3 \eta) \leq \max\{\mho_{\Xi}^{\mathcal{F}}(\zeta), \mho_{\Xi}^{\mathcal{F}}(\eta)\}$.

$$\begin{aligned}
\Gamma_{\Xi}(\zeta \oslash_1 \eta) &= \inf_{i\in\mathcal{I}} \Gamma_{Z_i}(\zeta \oslash_1 \eta) \\
&\geq \inf_{i\in\mathcal{I}} \min\{\Gamma_{Z_i}(\zeta), \Gamma_{Z_i}(\eta)\} \\
&= \min\left\{\inf_{i\in\mathcal{I}} \Gamma_{Z_i}(\zeta), \inf_{i\in\mathcal{I}} \Gamma_{Z_i}(\eta)\right\} \\
&= \min\{\Gamma_{\Xi}(\zeta), \Gamma_{\Xi}(\eta)\}.
\end{aligned}$$

Similarly, $\Gamma_{\Xi}(\zeta \oslash_2 \eta) \geq \min\{\Gamma_{\Xi}(\zeta), \Gamma_{\Xi}(\eta)\}$, $\Gamma_{\Xi}(\zeta \oslash_3 \eta) \geq \min\{\Gamma_{\Xi}(\zeta), \Gamma_{\Xi}(\eta)\}$. Now,

$$\begin{aligned}
\Lambda_{\Xi}(\zeta \oslash_1 \eta) &= \inf_{i\in\mathcal{I}} \Lambda_{Z_i}(\zeta \oslash_1 \eta) \\
&\geq \inf_{i\in\mathcal{I}} \frac{\Lambda_{Z_i}(\zeta) + \Lambda_{Z_i}(\eta)}{2} \\
&= \frac{\inf_{i\in\mathcal{I}} \Lambda_{Z_i}(\zeta) + \inf_{i\in\mathcal{I}} \Lambda_{Z_i}(\eta)}{2} \\
&= \frac{\Lambda_{\Xi}(\zeta) + \Lambda_{\Xi}(\eta)}{2}.
\end{aligned}$$

Similarly, $\Lambda_{\Xi}(\zeta \oslash_2 \eta) \geq \frac{\Lambda_{\Xi}(\zeta)+\Lambda_{\Xi}(\eta)}{2}$ and $\Lambda_{\Xi}(\zeta \oslash_3 \eta) \geq \frac{\Lambda_{\Xi}(\zeta)+\Lambda_{\Xi}(\eta)}{2}$. Now,

$$\begin{aligned}
\Theta_{\Xi}(\zeta \oslash_1 \eta) &= \sup_{i\in\mathcal{I}} \Theta_{Z_i}(\zeta \oslash_1 \eta) \\
&\leq \sup_{i\in\mathcal{I}} \max\{\Theta_{Z_i}(\zeta), \Theta_{Z_i}(\eta)\} \\
&= \max\left\{\sup_{i\in\mathcal{I}} \Theta_{Z_i}(\zeta), \sup_{i\in\mathcal{I}} \Theta_{Z_i}(\eta)\right\} \\
&= \max\{\Theta_{\Xi}(\zeta), \Theta_{\Xi}(\eta)\}.
\end{aligned}$$

Similarly, $\Theta_{\Xi}(\zeta \oslash_2 \eta) \leq \max\{\Theta_{\Xi}(\zeta), \Theta_{\Xi}(\eta)\}$, $\Theta_{\Xi}(\zeta \oslash_3 \eta) \leq \max\{\Theta_{\Xi}(\zeta), \Theta_{\Xi}(\eta)\}$. Hence Ξ is a DioNSBS of $\mathcal{T}$.

Theorem 3.5. *If Ξ and Δ are two DioNSBSs of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, then $\Xi \times \Delta$ is a DioNSBS of $\mathcal{T}_1 \times \mathcal{T}_2$.*

Proof. Let Ξ and Δ be two DioNSBSs of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. Let $\zeta_1, \zeta_2 \in \mathcal{T}_1$ and $\eta_1, \eta_2 \in \mathcal{T}_2$. Then (ζ_1, η_1) and (ζ_2, η_2) are in $\mathcal{T}_1 \times \mathcal{T}_2$. Now,

$$\begin{aligned}
\mho^{\mathcal{T}}_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_1(\zeta_2,\eta_2)] &= \mho^{\mathcal{T}}_{\Xi\times\Delta}(\zeta_1\oslash_1\zeta_2, \eta_1\oslash_1\eta_2)\\
&= \min\{\mho^{\mathcal{T}}_{\Xi}(\zeta_1\oslash_1\zeta_2), \mho^{\mathcal{T}}_{\Delta}(\eta_1\oslash_1\eta_2)\}\\
&\geq \min\{\min\{\mho^{\mathcal{T}}_{\Xi}(\zeta_1), \mho^{\mathcal{T}}_{\Xi}(\zeta_2)\}, \min\{\mho^{\mathcal{T}}_{\Delta}(\eta_1), \mho^{\mathcal{T}}_{\Delta}(\eta_2)\}\}\\
&= \min\{\min\{\mho^{\mathcal{T}}_{\Xi}(\zeta_1), \mho^{\mathcal{T}}_{\Delta}(\eta_1)\}, \min\{\mho^{\mathcal{T}}_{\Xi}(\zeta_2), \mho^{\mathcal{T}}_{\Delta}(\eta_2)\}\}\\
&= \min\{\mho^{\mathcal{T}}_{\Xi\times\Delta}(\zeta_1,\eta_1), \mho^{\mathcal{T}}_{\Xi\times\Delta}(\zeta_2,\eta_2)\}.
\end{aligned}$$

Also, $\mho^{\mathcal{T}}_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_2(\zeta_2,\eta_2)] \geq \min\{\mho^{\mathcal{T}}_{\Xi\times\Delta}(\zeta_1,\eta_1), \mho^{\mathcal{T}}_{\Xi\times\Delta}(\zeta_2,\eta_2)\}$ and $\mho^{\mathcal{T}}_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_3(\zeta_2,\eta_2)] \geq \min\{\mho^{\mathcal{T}}_{\Xi\times\Delta}(\zeta_1,\eta_1), \mho^{\mathcal{T}}_{\Xi\times\Delta}(\zeta_2,\eta_2)\}$. Now,

$$\begin{aligned}
\mho^{\mathcal{I}}_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_1(\zeta_2,\eta_2)] &= \mho^{\mathcal{I}}_{\Xi\times\Delta}(\zeta_1\oslash_1\zeta_2, \eta_1\oslash_1\eta_2)\\
&= \frac{\mho^{\mathcal{I}}_{\Xi}(\zeta_1\oslash_1\zeta_2)+\mho^{\mathcal{I}}_{\Delta}(\eta_1\oslash_1\eta_2)}{2}\\
&\geq \frac{1}{2}\left[\frac{\mho^{\mathcal{I}}_{\Xi}(\zeta_1)+\mho^{\mathcal{I}}_{\Xi}(\zeta_2)}{2}+\frac{\mho^{\mathcal{I}}_{\Delta}(\eta_1)+\mho^{\mathcal{I}}_{\Delta}(\eta_2)}{2}\right]\\
&= \frac{1}{2}\left[\frac{\mho^{\mathcal{I}}_{\Xi}(\zeta_1)+\mho^{\mathcal{I}}_{\Delta}(\eta_1)}{2}+\frac{\mho^{\mathcal{I}}_{\Xi}(\zeta_2)+\mho^{\mathcal{I}}_{\Delta}(\eta_2)}{2}\right]\\
&= \frac{1}{2}\Big[\mho^{\mathcal{I}}_{\Xi\times\Delta}(\zeta_1,\eta_1)+\mho^{\mathcal{I}}_{\Xi\times\Delta}(\zeta_2,\eta_2)\Big].
\end{aligned}$$

Also, $\mho^{\mathcal{I}}_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_2(\zeta_2,\eta_2)] \geq \frac{1}{2}\Big[\mho^{\mathcal{I}}_{\Xi\times\Delta}(\zeta_1,\eta_1)+\mho^{\mathcal{I}}_{\Xi\times\Delta}(\zeta_2,\eta_2)\Big]$ and $\mho^{\mathcal{I}}_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_3(\zeta_2,\eta_2)] \geq \frac{1}{2}\Big[\mho^{\mathcal{I}}_{\Xi\times\Delta}(\zeta_1,\eta_1)+\mho^{\mathcal{I}}_{\Xi\times\Delta}(\zeta_2,\eta_2)\Big]$. Now,

$$\begin{aligned}
\mho^{\mathcal{F}}_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_1(\zeta_2,\eta_2)] &= \mho^{\mathcal{F}}_{\Xi\times\Delta}(\zeta_1\oslash_1\zeta_2, \eta_1\oslash_1\eta_2)\\
&= \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_1\oslash_1\zeta_2), \mho^{\mathcal{F}}_{\Delta}(\eta_1\oslash_1\eta_2)\}\\
&\leq \max\{\max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_1), \mho^{\mathcal{F}}_{\Xi}(\zeta_2)\}, \max\{\mho^{\mathcal{F}}_{\Delta}(\eta_1), \mho^{\mathcal{F}}_{\Delta}(\eta_2)\}\}\\
&= \max\{\max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_1), \mho^{\mathcal{F}}_{\Delta}(\eta_1)\}, \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_2), \mho^{\mathcal{F}}_{\Delta}(\eta_2)\}\}\\
&= \max\{\mho^{\mathcal{F}}_{\Xi\times\Delta}(\zeta_1,\eta_1), \mho^{\mathcal{F}}_{\Xi\times\Delta}(\zeta_2,\eta_2)\}.
\end{aligned}$$

Also, $\mho^{\mathcal{F}}_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_2(\zeta_2,\eta_2)] \leq \max\{\mho^{\mathcal{F}}_{\Xi\times\Delta}(\zeta_1,\eta_1), \mho^{\mathcal{F}}_{\Xi\times\Delta}(\zeta_2,\eta_2)\}$ and $\mho^{\mathcal{F}}_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_3(\zeta_2,\eta_2)] \leq \max\{\mho^{\mathcal{F}}_{\Xi\times\Delta}(\zeta_1,\eta_1), \mho^{\mathcal{F}}_{\Xi\times\Delta}(\zeta_2,\eta_2)\}$.

$$\begin{aligned}
\Gamma_{\Xi\times\Delta}[(\zeta_1,\eta_1)\oslash_1(\zeta_2,\eta_2)] &= \Gamma_{\Xi\times\Delta}(\zeta_1\oslash_1\zeta_2, \eta_1\oslash_1\eta_2)\\
&= \min\{\Gamma_{\Xi}(\zeta_1\oslash_1\zeta_2), \Gamma_{\Delta}(\eta_1\oslash_1\eta_2)\}\\
&\geq \min\{\min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Xi}(\zeta_2)\}, \min\{\Gamma_{\Delta}(\eta_1), \Gamma_{\Delta}(\eta_2)\}\}\\
&= \min\{\min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Delta}(\eta_1)\}, \min\{\Gamma_{\Xi}(\zeta_2), \Gamma_{\Delta}(\eta_2)\}\}\\
&= \min\{\Gamma_{\Xi\times\Delta}(\zeta_1,\eta_1), \Gamma_{\Xi\times\Delta}(\zeta_2,\eta_2)\}.
\end{aligned}$$

Also, $\Gamma_{\Xi\times\Delta}[(\zeta_1,\eta_1) \oslash_2 (\zeta_2,\eta_2)] \geq \min\{\Gamma_{\Xi\times\Delta}(\zeta_1,\eta_1),\Gamma_{\Xi\times\Delta}(\zeta_2,\eta_2)\}$ and $\Gamma_{\Xi\times\Delta}[(\zeta_1,\eta_1) \oslash_3 (\zeta_2,\eta_2)] \geq \min\{\Gamma_{\Xi\times\Delta}(\zeta_1,\eta_1),\Gamma_{\Xi\times\Delta}(\zeta_2,\eta_2)\}$. Now,

$$\begin{aligned}
\Lambda_{\Xi\times\Delta}[(\zeta_1,\eta_1) \oslash_1 (\zeta_2,\eta_2)] &= \Lambda_{\Xi\times\Delta}(\zeta_1 \oslash_1 \zeta_2, \eta_1 \oslash_1 \eta_2)\\
&= \frac{\Lambda_{\Xi}(\zeta_1 \oslash_1 \zeta_2) + \Lambda_{\Delta}(\eta_1 \oslash_1 \eta_2)}{2}\\
&\geq \frac{1}{2}\Big[\frac{\Lambda_{\Xi}(\zeta_1)+\Lambda_{\Xi}(\zeta_2)}{2} + \frac{\Lambda_{\Delta}(\eta_1)+\Lambda_{\Delta}(\eta_2)}{2}\Big]\\
&= \frac{1}{2}\Big[\frac{\Lambda_{\Xi}(\zeta_1)+\Lambda_{\Delta}(\eta_1)}{2} + \frac{\Lambda_{\Xi}(\zeta_2)+\Lambda_{\Delta}(\eta_2)}{2}\Big]\\
&= \frac{1}{2}\Big[\Lambda_{\Xi\times\Delta}(\zeta_1,\eta_1) + \Lambda_{\Xi\times\Delta}(\zeta_2,\eta_2)\Big].
\end{aligned}$$

Also, $\Lambda_{\Xi\times\Delta}[(\zeta_1,\eta_1) \oslash_2 (\zeta_2,\eta_2)] \geq \frac{1}{2}\Big[\Lambda_{\Xi\times\Delta}(\zeta_1,\eta_1) + \Lambda_{\Xi\times\Delta}(\zeta_2,\eta_2)\Big]$ and $\Lambda_{\Xi\times\Delta}[(\zeta_1,\eta_1) \oslash_3 (\zeta_2,\eta_2)] \geq \frac{1}{2}\Big[\Lambda_{\Xi\times\Delta}(\zeta_1,\eta_1) + \Lambda_{\Xi\times\Delta}(\zeta_2,\eta_2)\Big]$. Now,

$$\begin{aligned}
\Theta_{\Xi\times\Delta}[(\zeta_1,\eta_1) \oslash_1 (\zeta_2,\eta_2)] &= \Theta_{\Xi\times\Delta}(\zeta_1 \oslash_1 \zeta_2, \eta_1 \oslash_1 \eta_2)\\
&= \max\{\Theta_{\Xi}(\zeta_1 \oslash_1 \zeta_2), \Theta_{\Delta}(\eta_1 \oslash_1 \eta_2)\}\\
&\leq \max\{\max\{\Theta_{\Xi}(\zeta_1),\Theta_{\Xi}(\zeta_2)\}, \max\{\Theta_{\Delta}(\eta_1),\Theta_{\Delta}(\eta_2)\}\}\\
&= \max\{\max\{\Theta_{\Xi}(\zeta_1),\Theta_{\Delta}(\eta_1)\}, \max\{\Theta_{\Xi}(\zeta_2),\Theta_{\Delta}(\eta_2)\}\}\\
&= \max\{\Theta_{\Xi\times\Delta}(\zeta_1,\eta_1), \Theta_{\Xi\times\Delta}(\zeta_2,\eta_2)\}.
\end{aligned}$$

Also, $\Theta_{\Xi\times\Delta}[(\zeta_1,\eta_1) \oslash_2 (\zeta_2,\eta_2)] \leq \max\{\Theta_{\Xi\times\Delta}(\zeta_1,\eta_1),\Theta_{\Xi\times\Delta}(\zeta_2,\eta_2)\}$ and $\Theta_{\Xi\times\Delta}[(\zeta_1,\eta_1) \oslash_3 (\zeta_2,\eta_2)] \leq \max\{\Theta_{\Xi\times\Delta}(\zeta_1,\eta_1),\Theta_{\Xi\times\Delta}(\zeta_2,\eta_2)\}$. Hence $\Xi\times\Delta$ is a DioNSBS of $\mathcal{T}$.

Corollary 3.6. *If $\Xi_1,\Xi_2,\ldots,\Xi_n$ are DioNSBSs of $\mathcal{T}_1,\mathcal{T}_2,\ldots,\mathcal{T}_n$, respectively, then $\Xi_1\times\Xi_2\times\ldots\times\Xi_n$ is a DioNSBS of $\mathcal{T}_1\times\mathcal{T}_2\times\ldots\times\mathcal{T}_n$.*

Definition 3.7. Let Ξ be a DioNSS in $\mathcal{T}$, the strongest Diophantine neutrosophic relation on $\mathcal{T}$. That is a Diophantine neutrosophic relation on Ξ is Z given by

$$\left\{\begin{array}{l}
\mho^{\mathcal{T}}_Z(\zeta,\eta) = \min\{\mho^{\mathcal{T}}_Z(\zeta),\mho^{\mathcal{T}}_Z(\eta)\}\\
\mho^{\mathcal{I}}_Z(\zeta,\eta) = \frac{\mho^{\mathcal{I}}_Z(\zeta)+\mho^{\mathcal{I}}_Z(\eta)}{2}\\
\mho^{\mathcal{F}}_Z(\zeta,\eta) = \max\{\mho^{\mathcal{F}}_Z(\zeta),\mho^{\mathcal{F}}_Z(\eta)\}
\end{array}\right\}
\quad
\left\{\begin{array}{l}
\Gamma_Z(\zeta,\eta) = \min\{\Gamma_Z(\zeta),\Gamma_Z(\eta)\}\\
\Lambda_Z(\zeta,\eta) = \frac{\Lambda_Z(\zeta)+\Lambda_Z(\eta)}{2}\\
\Theta_Z(\zeta,\eta) = \max\{\Theta_Z(\zeta),\Theta_Z(\eta)\}
\end{array}\right\}.$$

Theorem 3.8. *Let Ξ be the DioNSBS of $\mathcal{T}$ and Z be the strongest Diophantine neutrosophic relation of $\mathcal{T}$. Then Ξ is a DioNSBS of $\mathcal{T}$ if and only if Z is a DioNSBS of $\mathcal{T}\times\mathcal{T}$.*

Proof. Let Ξ be the DioNSBS of $\mathcal{T}$ and Z be the strongest Diophantine neutrosophic relation of $\mathcal{T}$. Then for any $\zeta = (\zeta_1, \zeta_2)$ and $\eta = (\eta_1, \eta_2)$ are in $\mathcal{T} \times \mathcal{T}$. We have

$$\begin{aligned}
\mho_Z^{\mathcal{T}}(\zeta \oslash_1 \eta) &= \mho_Z^{\mathcal{T}}[((\zeta_1, \zeta_2) \oslash_1 (\eta_1, \eta_2)] \\
&= \mho_Z^{\mathcal{T}}(\zeta_1 \oslash_1 \eta_1, \zeta_2 \oslash_1 \eta_2) \\
&= \min\{\mho_\Xi^{\mathcal{T}}(\zeta_1 \oslash_1 \eta_1), \mho_\Xi^{\mathcal{T}}(\zeta_2 \oslash_1 \eta_2)\} \\
&\geq \min\{\min\{\mho_\Xi^{\mathcal{T}}(\zeta_1), \mho_\Xi^{\mathcal{T}}(\eta_1)\}, \min\{\mho_\Xi^{\mathcal{T}}(\zeta_2), \mho_\Xi^{\mathcal{T}}(\eta_2)\}\} \\
&= \min\{\min\{\mho_\Xi^{\mathcal{T}}(\zeta_1), \mho_\Xi^{\mathcal{T}}(\zeta_2)\}, \min\{\mho_\Xi^{\mathcal{T}}(\eta_1), \mho_\Xi^{\mathcal{T}}(\eta_2)\}\} \\
&= \min\{\mho_Z^{\mathcal{T}}(\zeta_1, \zeta_2), \mho_Z^{\mathcal{T}}(\eta_1, \eta_2)\} \\
&= \min\{\mho_Z^{\mathcal{T}}(\zeta), \mho_Z^{\mathcal{T}}(\eta)\}.
\end{aligned}$$

Also, $\mho_Z^{\mathcal{T}}(\zeta \oslash_2 \eta) \geq \min\{\mho_Z^{\mathcal{T}}(\zeta), \mho_Z^{\mathcal{T}}(\eta)\}, \mho_Z^{\mathcal{T}}(\zeta \oslash_3 \eta) \geq \min\{\mho_Z^{\mathcal{T}}(\zeta), \mho_Z^{\mathcal{T}}(\eta)\}$. Now,

$$\begin{aligned}
\mho_Z^{\mathcal{I}}(\zeta \oslash_1 \eta) &= \mho_Z^{\mathcal{I}}[((\zeta_1, \zeta_2) \oslash_1 (\eta_1, \eta_2)] \\
&= \mho_Z^{\mathcal{I}}(\zeta_1 \oslash_1 \eta_1, \zeta_2 \oslash_1 \eta_2) \\
&= \frac{\mho_\Xi^{\mathcal{I}}(\zeta_1 \oslash_1 \eta_1) + \mho_\Xi^{\mathcal{I}}(\zeta_2 \oslash_1 \eta_2)}{2} \\
&\geq \frac{1}{2}\left[\frac{\mho_\Xi^{\mathcal{I}}(\zeta_1) + \mho_\Xi^{\mathcal{I}}(\eta_1)}{2} + \frac{\mho_\Xi^{\mathcal{I}}(\zeta_2) + \mho_\Xi^{\mathcal{I}}(\eta_2)}{2}\right] \\
&= \frac{1}{2}\left[\frac{\mho_\Xi^{\mathcal{I}}(\zeta_1) + \mho_\Xi^{\mathcal{I}}(\zeta_2)}{2} + \frac{\mho_\Xi^{\mathcal{I}}(\eta_1) + \mho_\Xi^{\mathcal{I}}(\eta_2)}{2}\right] \\
&= \frac{\mho_Z^{\mathcal{I}}(\zeta_1, \zeta_2) + \mho_Z^{\mathcal{I}}(\eta_1, \eta_2)}{2} \\
&= \frac{\mho_Z^{\mathcal{I}}(\zeta) + \mho_Z^{\mathcal{I}}(\eta)}{2}.
\end{aligned}$$

Also, $\mho_Z^{\mathcal{I}}(\zeta \oslash_2 \eta) \geq \frac{\mho_Z^{\mathcal{I}}(\zeta)+\mho_Z^{\mathcal{I}}(\eta)}{2}$ and $\mho_Z^{\mathcal{I}}(\zeta \oslash_3 \eta) \geq \frac{\mho_Z^{\mathcal{I}}(\zeta)+\mho_Z^{\mathcal{I}}(\eta)}{2}$. Similarly, $\mho_Z^{\mathcal{F}}(\zeta \oslash_1 \eta) \leq \max\{\mho_Z^{\mathcal{F}}(\zeta), \mho_Z^{\mathcal{F}}(\eta)\}, \mho_Z^{\mathcal{F}}(\zeta \oslash_2 \eta) \leq \max\{\mho_Z^{\mathcal{F}}(\zeta), \mho_Z^{\mathcal{F}}(\eta)\}$ and $\mho_Z^{\mathcal{F}}(\zeta \oslash_3 \eta) \leq \max\{\mho_Z^{\mathcal{F}}(\zeta), \mho_Z^{\mathcal{F}}(\eta)\}$. Now,

$$\begin{aligned}
\Gamma_Z(\zeta \oslash_1 \eta) &= \Gamma_{\Xi Z}[((\zeta_1, \zeta_2) \oslash_1 (\eta_1, \eta_2)] \\
&= \Gamma_Z(\zeta_1 \oslash_1 \eta_1, \zeta_2 \oslash_1 \eta_2) \\
&= \min\{\Gamma_\Xi(\zeta_1 \oslash_1 \eta_1), \Gamma_\Xi(\zeta_2 \oslash_1 \eta_2)\} \\
&\geq \min\{\min\{\Gamma_\Xi(\zeta_1), \Gamma_\Xi(\eta_1)\}, \min\{\Gamma_\Xi(\zeta_2), \Gamma_\Xi(\eta_2)\}\} \\
&= \min\{\min\{\Gamma_\Xi(\zeta_1), \Gamma_\Xi(\zeta_2)\}, \min\{\Gamma_\Xi(\eta_1), \Gamma_\Xi(\eta_2)\}\} \\
&= \min\{\Gamma_Z(\zeta_1, \zeta_2), \Gamma_Z(\eta_1, \eta_2)\} \\
&= \min\{\Gamma_Z(\zeta), \Gamma_Z(\eta)\}.
\end{aligned}$$

Also, $\Gamma_Z(\zeta \oslash_2 \eta) \geq \min\{\Gamma_Z(\zeta), \Gamma_Z(\eta)\}$ and $\Gamma_Z(\zeta \oslash_3 \eta) \geq \min\{\Gamma_Z(\zeta), \Gamma_Z(\eta)\}$. Now,

$$\begin{aligned}
\Lambda_Z(\zeta \oslash_1 \eta) &= \Lambda_Z[((\zeta_1, \zeta_2) \oslash_1 (\eta_1, \eta_2)] \\
&= \Lambda_Z(\zeta_1 \oslash_1 \eta_1, \zeta_2 \oslash_1 \eta_2) \\
&= \frac{\Lambda_\Xi(\zeta_1 \oslash_1 \eta_1) + \Lambda_\Xi(\zeta_2 \oslash_1 \eta_2)}{2} \\
&\geq \frac{1}{2}\left[\frac{\Lambda_\Xi(\zeta_1) + \Lambda_\Xi(\eta_1)}{2} + \frac{\Lambda_\Xi(\zeta_2) + \Lambda_\Xi(\eta_2)}{2}\right] \\
&= \frac{1}{2}\left[\frac{\Lambda_\Xi(\zeta_1) + \Lambda_\Xi(\zeta_2)}{2} + \frac{\Lambda_\Xi(\eta_1) + \Lambda_\Xi(\eta_2)}{2}\right] \\
&= \frac{\Lambda_Z(\zeta_1, \zeta_2) + \Lambda_Z(\eta_1, \eta_2)}{2} \\
&= \frac{\Lambda_Z(\zeta) + \Lambda_Z(\eta)}{2}.
\end{aligned}$$

Also, $\Lambda_Z(\zeta \oslash_2 \eta) \geq \frac{\Lambda_Z(\zeta)+\Lambda_Z(\eta)}{2}$ and $\Lambda_Z(\zeta \oslash_3 \eta) \geq \frac{\Lambda_Z(\zeta)+\Lambda_Z(\eta)}{2}$. Similarly, $\Theta_Z(\zeta \oslash_1 \eta) \leq \max\{\Theta_Z(\zeta), \Theta_Z(\eta)\}, \Theta_Z(\zeta \oslash_2 \eta) \leq \max\{\Theta_Z(\zeta), \Theta_Z(\eta)\}$ and $\Theta_Z(\zeta \oslash_3 \eta) \leq \max\{\Theta_Z(\zeta), \Theta_Z(\eta)\}$. Hence Z is a DioNSBS of $\mathcal{T} \times \mathcal{T}$.

Conversely, assume that Z is a DioNSBS of $\mathcal{T} \times \mathcal{T}$, then for any $\zeta = (\zeta_1, \zeta_2)$ and $\eta = (\eta_1, \eta_2)$ are in $\mathcal{T} \times \mathcal{T}$. We have

$$\begin{aligned}
\min\{\mho^{\mathcal{T}}_\Xi(\zeta_1 \oslash_1 \eta_1), \mho^{\mathcal{T}}_\Xi(\zeta_2 \oslash_1 \eta_2)\} &= \mho^{\mathcal{T}}_Z(\zeta_1 \oslash_1 \eta_1, \zeta_2 \oslash_1 \eta_2) \\
&= \mho^{\mathcal{T}}_Z[(\zeta_1, \zeta_2) \oslash_1 (\eta_1, \eta_2)] \\
&= \mho^{\mathcal{T}}_Z(\zeta \oslash_1 \eta) \\
&\geq \min\{\mho^{\mathcal{T}}_Z(\zeta), \mho^{\mathcal{T}}_Z(\eta)\} \\
&= \min\{\mho^{\mathcal{T}}_Z(\zeta_1, \zeta_2)\}, \mho^{\mathcal{T}}_Z(\eta_1, \eta_2)\} \\
&= \min\{\min\{\mho^{\mathcal{T}}_\Xi(\zeta_1), \mho^{\mathcal{T}}_\Xi(\zeta_2)\}, \min\{\mho^{\mathcal{T}}_\Xi(\eta_1), \mho^{\mathcal{T}}_\Xi(\eta_2)\}\}.
\end{aligned}$$

If $\mho^{\mathcal{T}}_\Xi(\zeta_1 \oslash_1 \eta_1) \leq \mho^{\mathcal{T}}_\Xi(\zeta_2 \oslash_1 \eta_2)$, then $\mho^{\mathcal{T}}_\Xi(\zeta_1) \leq \mho^{\mathcal{T}}_\Xi(\zeta_2)$ and $\mho^{\mathcal{T}}_\Xi(\eta_1) \leq \mho^{\mathcal{T}}_\Xi(\eta_2)$. We get $\mho^{\mathcal{T}}_\Xi(\zeta_1 \oslash_1 \eta_1) \geq \min\{\mho^{\mathcal{T}}_\Xi(\zeta_1), \mho^{\mathcal{T}}_\Xi(\eta_1)\}$ $\forall \zeta_1, \eta_1 \in \mathcal{T}$ and $\min\{\mho^{\mathcal{T}}_\Xi(\zeta_1 \oslash_2 \eta_1), \mho^{\mathcal{T}}_\Xi(\zeta_2 \oslash_2 \eta_2)\} \geq \min\{\min\{\mho^{\mathcal{T}}_\Xi(\zeta_1), \mho^{\mathcal{T}}_\Xi(\zeta_2)\}, \min\{\mho^{\mathcal{T}}_\Xi(\eta_1), \mho^{\mathcal{T}}_\Xi(\eta_2)\}\}$. If $\mho^{\mathcal{T}}_\Xi(\zeta_1 \oslash_2 \eta_1) \leq \mho^{\mathcal{T}}_\Xi(\zeta_2 \oslash_2 \eta_2)$, then $\mho^{\mathcal{T}}_\Xi(\zeta_1 \oslash_2 \eta_1) \geq \min\{\mho^{\mathcal{T}}_\Xi(\zeta_1), \mho^{\mathcal{T}}_\Xi(\eta_1)\}$. So, $\min\{\mho^{\mathcal{T}}_\Xi(\zeta_1 \oslash_3 \eta_1), \mho^{\mathcal{T}}_\Xi(\zeta_2 \oslash_3 \eta_2)\} \geq \min\{\min\{\mho^{\mathcal{T}}_\Xi(\zeta_1), \mho^{\mathcal{T}}_\Xi(\zeta_2)\}, \min\{\mho^{\mathcal{T}}_\Xi(\eta_1), \mho^{\mathcal{T}}_\Xi(\eta_2)\}\}$. If $\mho^{\mathcal{T}}_\Xi(\zeta_1 \oslash_3 \eta_1) \leq \mho^{\mathcal{T}}_\Xi(\zeta_2 \oslash_3 \eta_2)$, then

$\mho_{\Xi}^{\mathcal{I}}(\zeta_1 \oslash_3 \eta_1) \geq \min\{\mho_{\Xi}^{\mathcal{I}}(\zeta_1), \mho_{\Xi}^{\mathcal{I}}(\eta_1)\}$. Now,

$$\begin{aligned}\frac{1}{2}\Big[\mho_{\Xi}^{\mathcal{I}}(\zeta_1 \oslash_1 \eta_1) + \mho_{\Xi}^{\mathcal{I}}(\zeta_2 \oslash_1 \eta_2)\Big] &= \mho_{Z}^{\mathcal{I}}(\zeta_1 \oslash_1 \eta_1, \zeta_2 \oslash_1 \eta_2)\\ &= \mho_{Z}^{\mathcal{I}}[(\zeta_1, \zeta_2) \oslash_1 (\eta_1, \eta_2)]\\ &= \mho_{Z}^{\mathcal{I}}(\zeta \oslash_1 \eta)\\ &\geq \frac{\mho_{Z}^{\mathcal{I}}(\zeta) + \mho_{Z}^{\mathcal{I}}(\eta)}{2}\\ &= \frac{\mho_{Z}^{\mathcal{I}}(\zeta_1, \zeta_2) + \mho_{Z}^{\mathcal{I}}(\eta_1, \eta_2)}{2}\\ &= \frac{1}{2}\left[\frac{\mho_{\Xi}^{\mathcal{I}}(\zeta_1) + \mho_{\Xi}^{\mathcal{I}}(\zeta_2)}{2} + \frac{\mho_{\Xi}^{\mathcal{I}}(\eta_1) + \mho_{\Xi}^{\mathcal{I}}(\eta_2)}{2}\right].\end{aligned}$$

If $\mho_{\Xi}^{\mathcal{I}}(\zeta_1 \oslash_1 \eta_1) \leq \mho_{\Xi}^{\mathcal{I}}(\zeta_2 \oslash_1 \eta_2)$, then $\mho_{\Xi}^{\mathcal{I}}(\zeta_1) \leq \mho_{\Xi}^{\mathcal{I}}(\zeta_2)$ and $\mho_{\Xi}^{\mathcal{I}}(\eta_1) \leq \mho_{\Xi}^{\mathcal{I}}(\eta_2)$. We get, $\mho_{\Xi}^{\mathcal{I}}(\zeta_1 \oslash_1 \eta_1) \geq \frac{\mho_{\Xi}^{\mathcal{I}}(\zeta_1)+\mho_{\Xi}^{\mathcal{I}}(\eta_1)}{2}$. Similarly, $\mho_{\Xi}^{\mathcal{I}}(\zeta_1 \oslash_2 \eta_1) \geq \frac{\mho_{\Xi}^{\mathcal{I}}(\zeta_1)+\mho_{\Xi}^{\mathcal{I}}(\eta_1)}{2}$ and $\mho_{\Xi}^{\mathcal{I}}(\zeta_1 \oslash_3 \eta_1) \geq \frac{\mho_{\Xi}^{\mathcal{I}}(\zeta_1)+\mho_{\Xi}^{\mathcal{I}}(\eta_1)}{2}$. Similarly to prove that $\max\{\mho_{\Xi}^{\mathcal{F}}(\zeta_1 \oslash_1 \eta_1), \mho_{\Xi}^{\mathcal{F}}(\zeta_2 \oslash_1 \eta_2)\} \leq \max\{\max\{\mho_{\Xi}^{\mathcal{F}}(\zeta_1), \mho_{\Xi}^{\mathcal{F}}(\zeta_2)\}, \max\{\mho_{\Xi}^{\mathcal{F}}(\eta_1), \mho_{\Xi}^{\mathcal{F}}(\eta_2)\}\}$. If $\mho_{\Xi}^{\mathcal{F}}(\zeta_1 \oslash_1 \eta_1) \geq \mho_{\Xi}^{\mathcal{F}}(\zeta_2 \oslash_1 \eta_2)$, then $\mho_{\Xi}^{\mathcal{F}}(\zeta_1) \geq \mho_{\Xi}^{\mathcal{F}}(\zeta_2)$ and $\mho_{\Xi}^{\mathcal{F}}(\eta_1) \geq \mho_{\Xi}^{\mathcal{F}}(\eta_2)$. We get, $\mho_{\Xi}^{\mathcal{F}}(\zeta_1 \oslash_1 \eta_1) \leq \max\{\mho_{\Xi}^{\mathcal{F}}(\zeta_1), \mho_{\Xi}^{\mathcal{F}}(\eta_1)\}$. So, $\max\{\mho_{\Xi}^{\mathcal{F}}(\zeta_1 \oslash_2 \eta_1), \mho_{\Xi}^{\mathcal{F}}(\zeta_2 \oslash_2 \eta_2)\} \leq \max\{\max\{\mho_{\Xi}^{\mathcal{F}}(\zeta_1), \mho_{\Xi}^{\mathcal{F}}(\zeta_2)\}, \max\{\mho_{\Xi}^{\mathcal{F}}(\eta_1), \mho_{\Xi}^{\mathcal{F}}(\eta_2)\}\}$. If $\mho_{\Xi}^{\mathcal{F}}(\zeta_1 \oslash_2 \eta_1) \geq \mho_{\Xi}^{\mathcal{F}}(\zeta_2 \oslash_2 \eta_2)$, then $\mho_{\Xi}^{\mathcal{F}}(\zeta_1 \oslash_2 \eta_1) \leq \max\{\mho_{\Xi}^{\mathcal{F}}(\zeta_1), \mho_{\Xi}^{\mathcal{F}}(\eta_1)\}$. So, $\max\{\mho_{\Xi}^{\mathcal{F}}(\zeta_1 \oslash_3 \eta_1), \mho_{\Xi}^{\mathcal{F}}(\zeta_2 \oslash_3 \eta_2)\} \leq \max\{\max\{\mho_{\Xi}^{\mathcal{F}}(\zeta_1), \mho_{\Xi}^{\mathcal{F}}(\zeta_2)\}, \max\{\mho_{\Xi}^{\mathcal{F}}(\eta_1), \mho_{\Xi}^{\mathcal{F}}(\eta_2)\}\}$. If $\mho_{\Xi}^{\mathcal{F}}(\zeta_1 \oslash_3 \eta_1) \geq \mho_{\Xi}^{\mathcal{F}}(\zeta_2 \oslash_3 \eta_2)$, then $\mho_{\Xi}^{\mathcal{F}}(\zeta_1 \oslash_3 \eta_1) \leq \max\{\mho_{\Xi}^{\mathcal{F}}(\zeta_1), \mho_{\Xi}^{\mathcal{F}}(\eta_1)\}$. Now,

$$\begin{aligned}\min\{\Gamma_{\Xi}(\zeta_1 \oslash_1 \eta_1), \Gamma_{\Xi}(\zeta_2 \oslash_1 \eta_2)\} &= \Gamma_Z(\zeta_1 \oslash_1 \eta_1, \zeta_2 \oslash_1 \eta_2)\\ &= \Gamma_Z[(\zeta_1, \zeta_2) \oslash_1 (\eta_1, \eta_2)]\\ &= \Gamma_Z(\zeta \oslash_1 \eta)\\ &\geq \min\{\Gamma_Z(\zeta), \Gamma_Z(\eta)\}\\ &= \min\{\Gamma_Z(\zeta_1, \zeta_2)\}, \Gamma_Z(\eta_1, \eta_2)\}\\ &= \min\{\min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Xi}(\zeta_2)\}, \min\{\Gamma_{\Xi}(\eta_1), \Gamma_{\Xi}(\eta_2)\}\}.\end{aligned}$$

If $\Gamma_{\Xi}(\zeta_1 \oslash_1 \eta_1) \leq \Gamma_{\Xi}(\zeta_2 \oslash_1 \eta_2)$, then $\Gamma_{\Xi}(\zeta_1) \leq \Gamma_{\Xi}(\zeta_2)$ and $\Gamma_{\Xi}(\eta_1) \leq \Gamma_{\Xi}(\eta_2)$. We get $\Gamma_{\Xi}(\zeta_1 \oslash_1 \eta_1) \geq \min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Xi}(\eta_1)\}$ $\forall \zeta_1, \eta_1 \in \mathcal{T}$ and $\min\{\Gamma_{\Xi}(\zeta_1 \oslash_2 \eta_1), \Gamma_{\Xi}(\zeta_2 \oslash_2 \eta_2)\} \geq \min\{\min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Xi}(\zeta_2)\}, \min\{\Gamma_{\Xi}(\eta_1), \Gamma_{\Xi}(\eta_2)\}\}$. If $\Gamma_{\Xi}(\zeta_1 \oslash_2 \eta_1) \leq \Gamma_{\Xi}(\zeta_2 \oslash_2 \eta_2)$, then $\Gamma_{\Xi}(\zeta_1 \oslash_2 \eta_1) \geq \min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Xi}(\eta_1)\}$. So, $\min\{\Gamma_{\Xi}(\zeta_1 \oslash_3 \eta_1), \Gamma_{\Xi}(\zeta_2 \oslash_3 \eta_2)\} \geq \min\{\min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Xi}(\zeta_2)\}, \min\{\Gamma_{\Xi}(\eta_1), \Gamma_{\Xi}(\eta_2)\}\}$. If $\Gamma_{\Xi}(\zeta_1 \oslash_3 \eta_1) \leq \Gamma_{\Xi}(\zeta_2 \oslash_3 \eta_2)$, then

$\Gamma_{\Xi}(\zeta_1 \oslash_3 \eta_1) \geq \min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Xi}(\eta_1)\}$. Now,

$$\begin{aligned}\frac{1}{2}\Big[\Lambda_{\Xi}(\zeta_1 \oslash_1 \eta_1) + \Lambda_{\Xi}(\zeta_2 \oslash_1 \eta_2)\Big] &= \Lambda_Z(\zeta_1 \oslash_1 \eta_1, \zeta_2 \oslash_1 \eta_2)\\ &= \Lambda_Z[(\zeta_1, \zeta_2) \oslash_1 (\eta_1, \eta_2)]\\ &= \Lambda_Z(\zeta \oslash_1 \eta)\\ &\geq \frac{\Lambda_Z(\zeta) + \Lambda_Z(\eta)}{2}\\ &= \frac{\Lambda_Z(\zeta_1, \zeta_2) + \Lambda_Z(\eta_1, \eta_2)}{2}\\ &= \frac{1}{2}\Bigg[\frac{\Lambda_{\Xi}(\zeta_1) + \Lambda_{\Xi}(\zeta_2)}{2} + \frac{\Lambda_{\Xi}(\eta_1) + \Lambda_{\Xi}(\eta_2)}{2}\Bigg].\end{aligned}$$

If $\Lambda_{\Xi}(\zeta_1 \oslash_1 \eta_1) \leq \Lambda_{\Xi}(\zeta_2 \oslash_1 \eta_2)$, then $\Lambda_{\Xi}(\zeta_1) \leq \Lambda_{\Xi}(\zeta_2)$ and $\Lambda_{\Xi}(\eta_1) \leq \Lambda_{\Xi}(\eta_2)$. We get, $\Lambda_{\Xi}(\zeta_1 \oslash_1 \eta_1) \geq \frac{\Lambda_{\Xi}(\zeta_1)+\Lambda_{\Xi}(\eta_1)}{2}$. Similarly, $\Lambda_{\Xi}(\zeta_1 \oslash_2 \eta_1) \geq \frac{\Lambda_{\Xi}(\zeta_1)+\Lambda_{\Xi}(\eta_1)}{2}$ and $\Lambda_{\Xi}(\zeta_1 \oslash_3 \eta_1) \geq \frac{\Lambda_{\Xi}(\zeta_1)+\Lambda_{\Xi}(\eta_1)}{2}$. Similarly to prove that $\max\{\Theta_{\Xi}(\zeta_1 \oslash_1 \eta_1), \Theta_{\Xi}(\zeta_2 \oslash_1 \eta_2)\} \leq \max\{\max\{\Theta_{\Xi}(\zeta_1), \Theta_{\Xi}(\zeta_2)\}, \max\{\Theta_{\Xi}(\eta_1), \Theta_{\Xi}(\eta_2)\}\}$. If $\Theta_{\Xi}(\zeta_1 \oslash_1 \eta_1) \geq \Theta_{\Xi}(\zeta_2 \oslash_1 \eta_2)$, then $\Theta_{\Xi}(\zeta_1) \geq \Theta_{\Xi}(\zeta_2)$ and $\Theta_{\Xi}(\eta_1) \geq \Theta_{\Xi}(\eta_2)$. We get, $\Theta_{\Xi}(\zeta_1 \oslash_1 \eta_1) \leq \max\{\Theta_{\Xi}(\zeta_1), \Theta_{\Xi}(\eta_1)\}$. So, $\max\{\Theta_{\Xi}(\zeta_1 \oslash_2 \eta_1), \Theta_{\Xi}(\zeta_2 \oslash_2 \eta_2)\} \leq \max\{\max\{\Theta_{\Xi}(\zeta_1), \Theta_{\Xi}(\zeta_2)\}, \max\{\Theta_{\Xi}(\eta_1), \Theta_{\Xi}(\eta_2)\}\}$. If $\Theta_{\Xi}(\zeta_1 \oslash_2 \eta_1) \geq \Theta_{\Xi}(\zeta_2 \oslash_2 \eta_2)$, then $\Theta_{\Xi}(\zeta_1 \oslash_2 \eta_1) \leq \max\{\Theta_{\Xi}(\zeta_1), \Theta_{\Xi}(\eta_1)\}$. So, $\max\{\Theta_{\Xi}(\zeta_1 \oslash_3 \eta_1), \Theta_{\Xi}(\zeta_2 \oslash_3 \eta_2)\} \leq \max\{\max\{\Theta_{\Xi}(\zeta_1), \Theta_{\Xi}(\zeta_2)\}, \max\{\Theta_{\Xi}(\eta_1), \Theta_{\Xi}(\eta_2)\}\}$. If $\Theta_{\Xi}(\zeta_1 \oslash_3 \eta_1) \geq \Theta_{\Xi}(\zeta_2 \oslash_3 \eta_2)$, then $\Theta_{\Xi}(\zeta_1 \oslash_3 \eta_1) \leq \max\{\Theta_{\Xi}(\zeta_1), \Theta_{\Xi}(\eta_1)\}$. Hence Ξ is a DioNSBS of $\mathcal{T}$.

Theorem 3.9. *Let Ξ be a DioNSS in $\mathcal{T}$. Then $\Xi = \langle(\mho^{\mathcal{T}}_{\Xi}, \mho^{\mathcal{I}}_{\Xi}, \mho^{\mathcal{F}}_{\Xi}), (\Gamma_{\Xi}, \Lambda_{\Xi}, \Theta_{\Xi})\rangle$ is a DioNSBS of $\mathcal{T}$ if and only if all non-empty level set $\Xi^{(\beta,\gamma)}$ is an SBS of $\mathcal{T}$ for $\beta, \gamma \in [0, 1]$.*

Proof. Assume that Ξ is a DioNSBS of $\mathcal{T}$. For each $\beta, \gamma \in [0, 1]$ and $\zeta_1, \zeta_2 \in \Xi^{(\beta,\gamma)}$. We have $\mho^{\mathcal{T}}_{\Xi}(\zeta_1) \geq \beta, \mho^{\mathcal{T}}_{\Xi}(\zeta_2) \geq \beta$, $\mho^{\mathcal{I}}_{\Xi}(\zeta_1) \geq \beta, \mho^{\mathcal{I}}_{\Xi}(\zeta_2) \geq \beta$, $\mho^{\mathcal{F}}_{\Xi}(\zeta_1) \leq \gamma, \mho^{\mathcal{F}}_{\Xi}(\zeta_2) \leq \gamma$ and $\Gamma_{\Xi}(\zeta_1) \geq \beta, \Gamma_{\Xi}(\zeta_2) \geq \beta$, $\Lambda_{\Xi}(\zeta_1) \geq \beta, \Lambda_{\Xi}(\zeta_2) \geq \beta$ and $\Theta_{\Xi}(\zeta_1) \leq \gamma, \Theta_{\Xi}(\zeta_2) \leq \gamma$. Now, $\mho^{\mathcal{T}}_{\Xi}(\zeta_1 \oslash_1 \zeta_2) \geq \min\{\mho^{\mathcal{T}}_{\Xi}(\zeta_1), \mho^{\mathcal{T}}_{\Xi}(\zeta_2)\} \geq \beta$ and $\mho^{\mathcal{I}}_{\Xi}(\zeta_1 \oslash_1 \zeta_2) \geq \frac{\mho^{\mathcal{I}}_{\Xi}(\zeta_1)+\mho^{\mathcal{I}}_{\Xi}(\zeta_2)}{2} \geq \frac{t+t}{2} = t$ and $\mho^{\mathcal{F}}_{\Xi}(\zeta_1 \oslash_1 \zeta_2) \leq \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_1), \mho^{\mathcal{F}}_{\Xi}(\zeta_2)\} \leq \gamma$. Similarly, $\Gamma_{\Xi}(\zeta_1 \oslash_1 \zeta_2) \geq \min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Xi}(\zeta_2)\} \geq \beta$ and $\Lambda_{\Xi}(\zeta_1 \oslash_1 \zeta_2) \geq \frac{\Lambda_{\Xi}(\zeta_1)+\Lambda_{\Xi}(\zeta_2)}{2} \geq \frac{t+t}{2} = t$ and $\Theta_{\Xi}(\zeta_1 \oslash_1 \zeta_2) \leq \max\{\Theta_{\Xi}(\zeta_1), \Theta_{\Xi}(\zeta_2)\} \leq \gamma$. This implies that $\zeta_1 \oslash_1 \zeta_2 \in \Xi^{(\beta,\gamma)}$. Similarly, $\zeta_1 \oslash_2 \zeta_2 \in \Xi^{(\beta,\gamma)}$ and $\zeta_1 \oslash_3 \zeta_2 \in \Xi^{(\beta,\gamma)}$. Therefore $\Xi^{(\beta,\gamma)}$ is a SBS of $\mathcal{T}$ for each $\beta, \gamma \in [0, 1]$.

Conversely, assume that $\Xi^{(\beta,\gamma)}$ is an SBS of $\mathcal{T}$ for each $\beta, \gamma \in [0, 1]$. Suppose if there exist $\zeta_1, \zeta_2 \in \mathcal{T}$ such that $\mho^{\mathcal{T}}_{\Xi}(\zeta_1 \oslash_1 \zeta_2) < \min\{\mho^{\mathcal{T}}_{\Xi}(\zeta_1), \mho^{\mathcal{T}}_{\Xi}(\zeta_2)\}$, $\mho^{\mathcal{I}}_{\Xi}(\zeta_1 \oslash_1 \zeta_2) < \frac{\mho^{\mathcal{I}}_{\Xi}(\zeta_1)+\mho^{\mathcal{I}}_{\Xi}(\zeta_2)}{2}$, $\mho^{\mathcal{F}}_{\Xi}(\zeta_1 \oslash_1 \zeta_2) > \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_1), \mho^{\mathcal{F}}_{\Xi}(\zeta_2)\}$ and $\Gamma_{\Xi}(\zeta_1 \oslash_1 \zeta_2) < \min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Xi}(\zeta_2)\}$, $\Lambda_{\Xi}(\zeta_1 \oslash_1 \zeta_2) < \frac{\Lambda_{\Xi}(\zeta_1)+\Lambda_{\Xi}(\zeta_2)}{2}$ and $\Theta_{\Xi}(\zeta_1 \oslash_1 \zeta_2) > \max\{\Theta_{\Xi}(\zeta_1), \Theta_{\Xi}(\zeta_2)\}$. Select $\beta, \gamma \in [0, 1]$ such that $\mho^{\mathcal{T}}_{\Xi}(\zeta_1 \oslash_1 \zeta_2) < \beta \leq \min\{\mho^{\mathcal{T}}_{\Xi}(\zeta_1), \mho^{\mathcal{T}}_{\Xi}(\zeta_2)\}$ and $\mho^{\mathcal{I}}_{\Xi}(\zeta_1 \oslash_1 \zeta_2) < \beta \leq \frac{\mho^{\mathcal{I}}_{\Xi}(\zeta_1)+\mho^{\mathcal{I}}_{\Xi}(\zeta_2)}{2}$ and $\mho^{\mathcal{F}}_{\Xi}(\zeta_1 \oslash_1 \zeta_2) > \gamma \geq \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_1), \mho^{\mathcal{F}}_{\Xi}(\zeta_2)\}$. Then $\zeta_1, \zeta_2 \in \Xi^{(\beta,\gamma)}$, but $\zeta_1 \oslash_1 \zeta_2 \notin \Xi^{(\beta,\gamma)}$. This contradicts to that $\Xi^{(\beta,\gamma)}$ is an SBS of $\mathcal{T}$. Hence $\mho^{\mathcal{T}}_{\Xi}(\zeta_1 \oslash_1 \zeta_2) \geq \min\{\mho^{\mathcal{T}}_{\Xi}(\zeta_1), \mho^{\mathcal{T}}_{\Xi}(\zeta_2)\}$, $\mho^{\mathcal{I}}_{\Xi}(\zeta_1 \oslash_1 \zeta_2) \geq \frac{\mho^{\mathcal{I}}_{\Xi}(\zeta_1)+\mho^{\mathcal{I}}_{\Xi}(\zeta_2)}{2}$ and $\mho^{\mathcal{F}}_{\Xi}(\zeta_1 \oslash_1 \zeta_2) \leq \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta_1), \mho^{\mathcal{F}}_{\Xi}(\zeta_2)\}$. Select $\beta, \gamma \in [0,1]$ such that $\Gamma_{\Xi}(\zeta_1 \oslash_1 \zeta_2) < \beta \leq \min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Xi}(\zeta_2)\}$ and $\Lambda_{\Xi}(\zeta_1 \oslash_1 \zeta_2) < \beta \leq \frac{\Lambda_{\Xi}(\zeta_1)+\Lambda_{\Xi}(\zeta_2)}{2}$ and $\Theta_{\Xi}(\zeta_1 \oslash_1 \zeta_2) > \gamma \geq \max\{\Theta_{\Xi}(\zeta_1), \Theta_{\Xi}(\zeta_2)\}$. Then $\zeta_1, \zeta_2 \in \Xi^{(\beta,\gamma)}$, but $\zeta_1 \oslash_1 \zeta_2 \notin \Xi^{(\beta,\gamma)}$. This contradicts to that $\Xi^{(\beta,\gamma)}$ is an SBS of $\mathcal{T}$. Hence $\Gamma_{\Xi}(\zeta_1 \oslash_1 \zeta_2) \geq \min\{\Gamma_{\Xi}(\zeta_1), \Gamma_{\Xi}(\zeta_2)\}$, $\Lambda_{\Xi}(\zeta_1 \oslash_1 \zeta_2) \geq \frac{\Lambda_{\Xi}(\zeta_1)+\Lambda_{\Xi}(\zeta_2)}{2}$ and $\Theta_{\Xi}(\zeta_1 \oslash_1 \zeta_2) \leq \max\{\Theta_{\Xi}(\zeta_1), \Theta_{\Xi}(\zeta_2)\}$. Similarly, $\oslash_2$ and $\oslash_3$ cases. Hence $\Xi = \langle(\mho^{\mathcal{T}}_{\Xi}, \mho^{\mathcal{I}}_{\Xi}, \mho^{\mathcal{F}}_{\Xi}), (\Gamma_{\Xi}, \Lambda_{\Xi}, \Theta_{\Xi})\rangle$ is a DioNSBS of $\mathcal{T}$.

Definition 3.10. Let Ξ be any DioNSBS of $\mathcal{T}$, $a \in \mathcal{T}$ and P is any set. Then the pseudo Diophantine neutrosophic coset $(a\Xi)^p$ is defined by

$$\begin{Bmatrix} ((a\mho^{\mathcal{T}}_{\Xi})^p)(\zeta) = p(a)\mho^{\mathcal{T}}_{\Xi}(\zeta) \\ ((a\mho^{\mathcal{I}}_{\Xi})^p)(\zeta) = p(a)\mho^{\mathcal{I}}_{\Xi}(\zeta) \\ ((a\mho^{\mathcal{F}}_{\Xi})^p)(\zeta) = p(a)\mho^{\mathcal{F}}_{\Xi}(\zeta) \end{Bmatrix} \quad \begin{Bmatrix} ((a\Gamma_{\Xi})^p)(\zeta) = p(a)\Gamma_{\Xi}(\zeta) \\ ((a\Lambda_{\Xi})^p)(\zeta) = p(a)\Lambda_{\Xi}(\zeta) \\ ((a\Theta_{\Xi})^p)(\zeta) = p(a)\Theta_{\Xi}(\zeta) \end{Bmatrix}$$

for every $\zeta \in \mathcal{T}$ and for some $p \in P$.

Theorem 3.11. *Let Ξ be any DioNSBS of $\mathcal{T}$, then the pseudo Diophantine neutrosophic coset $(a\Xi)^p$ is a DioNSBS of $\mathcal{T}$, for every $a \in \mathcal{T}$.*

Proof. Let Ξ be any DioNSBS of $\mathcal{T}$ and for every $\zeta, \eta \in \mathcal{T}$. Now, $((a\mho^{\mathcal{T}}_{\Xi})^p)(\zeta \oslash_1 \eta) = p(a)\ \mho^{\mathcal{T}}_{\Xi}(\zeta \oslash_1 \eta) \geq p(a)\ \min\{\mho^{\mathcal{T}}_{\Xi}(\zeta), \mho^{\mathcal{T}}_{\Xi}(\eta)\} = \min\{p(a)\ \mho^{\mathcal{T}}_{\Xi}(\zeta), p(a)\ \mho^{\mathcal{T}}_{\Xi}(\eta)\} = \min\{((a\mho^{\mathcal{T}}_{\Xi})^p)(\zeta), ((a\mho^{\mathcal{T}}_{\Xi})^p)(\eta)\}$. Thus, $((a\mho^{\mathcal{T}}_{\Xi})^p)(\zeta \oslash_1 \eta) \geq \min\{((a\mho^{\mathcal{T}}_{\Xi})^p)(\zeta), ((a\mho^{\mathcal{T}}_{\Xi})^p)(\eta)\}$. Now, $((a\mho^{\mathcal{I}}_{\Xi})^p)(\zeta \oslash_1 \eta) = p(a)\ \mho^{\mathcal{I}}_{\Xi}(\zeta \oslash_1 \eta) \geq p(a)\left[\frac{\mho^{\mathcal{I}}_{\Xi}(\zeta)+\mho^{\mathcal{I}}_{\Xi}(\eta)}{2}\right] = \frac{p(a)\ \mho^{\mathcal{I}}_{\Xi}(\zeta)+p(a)\ \mho^{\mathcal{I}}_{\Xi}(\eta)}{2} = \frac{((a\mho^{\mathcal{I}}_{\Xi})^p)(\zeta)+((a\mho^{\mathcal{I}}_{\Xi})^p)(\eta)}{2}$. Thus, $((a\mho^{\mathcal{I}}_{\Xi})^p)(\zeta \oslash_1 \eta) \geq \frac{((a\mho^{\mathcal{I}}_{\Xi})^p)(\zeta)+((a\mho^{\mathcal{I}}_{\Xi})^p)(\eta)}{2}$. Now, $((a\mho^{\mathcal{F}}_{\Xi})^p)(\zeta \oslash_1 \eta) = p(a)\ \mho^{\mathcal{F}}_{\Xi}(\zeta \oslash_1 \eta) \leq p(a)\ \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta), \mho^{\mathcal{F}}_{\Xi}(\eta)\} = \max\{p(a)\ \mho^{\mathcal{F}}_{\Xi}(\zeta), p(a)\ \mho^{\mathcal{F}}_{\Xi}(\eta)\} = \max\{((a\mho^{\mathcal{F}}_{\Xi})^p)(\zeta), ((a\mho^{\mathcal{F}}_{\Xi})^p)(\eta)\}$. Thus, $((a\mho^{\mathcal{F}}_{\Xi})^p)(\zeta \oslash_1 \eta) \leq \max\{((a\mho^{\mathcal{F}}_{\Xi})^p)(\zeta), ((a\mho^{\mathcal{F}}_{\Xi})^p)(\eta)\}$. Now,

$$\begin{aligned} ((a\Gamma_{\Xi})^p)(\zeta \oslash_1 \eta) &= p(a)\ \Gamma_{\Xi}(\zeta \oslash_1 \eta) \\ &\geq p(a)\ \min\{\Gamma_{\Xi}(\zeta), \Gamma_{\Xi}(\eta)\} \\ &= \min\{p(a)\ \Gamma_{\Xi}(\zeta), p(a)\ \Gamma_{\Xi}(\eta)\} \\ &= \min\{((a\Gamma_{\Xi})^p)(\zeta), ((a\Gamma_{\Xi})^p)(\eta)\}. \end{aligned}$$

Thus, $((a\Gamma_\Xi)^p)(\zeta \oslash_1 \eta) \geq \min\{((a\Gamma_\Xi)^p)(\zeta), ((a\Gamma_\Xi)^p)(\eta)\}$. Now,

$$\begin{aligned} ((a\Lambda_\Xi)^p)(\zeta \oslash_1 \eta) &= p(a)\ \Lambda_\Xi(\zeta \oslash_1 \eta) \\ &\geq p(a)\left[\frac{\Lambda_\Xi(\zeta) + \Lambda_\Xi(\eta)}{2}\right] \\ &= \frac{p(a)\ \Lambda_\Xi(\zeta) + p(a)\ \Lambda_\Xi(\eta)}{2} \\ &= \frac{((a\Lambda_\Xi)^p)(\zeta) + ((a\Lambda_\Xi)^p)(\eta)}{2}. \end{aligned}$$

Thus, $((a\Lambda_\Xi)^p)(\zeta \oslash_1 \eta) \geq \frac{((a\Lambda_\Xi)^p)(\zeta)+((a\Lambda_\Xi)^p)(\eta)}{2}$. Now,

$$\begin{aligned} ((a\Theta_\Xi)^p)(\zeta \oslash_1 \eta) &= p(a)\ \Theta_\Xi(\zeta \oslash_1 \eta) \\ &\leq p(a)\ \max\{\Theta_\Xi(\zeta), \Theta_\Xi(\eta)\} \\ &= \max\{p(a)\ \Theta_\Xi(\zeta), p(a)\ \Theta_\Xi(\eta)\} \\ &= \max\{((a\Theta_\Xi)^p)(\zeta), ((a\Theta_\Xi)^p)(\eta)\}. \end{aligned}$$

Thus, $((a\Theta_\Xi)^p)(\zeta \oslash_1 \eta) \leq \max\{((a\Theta_\Xi)^p)(\zeta), ((a\Theta_\Xi)^p)(\eta)\}$. Similarly, $\oslash_2$ and $\oslash_3$ cases. Hence $(a\Xi)^p$ is a DioNSBS of $\mathcal{T}$.

Definition 3.12. Let $(\mathcal{T}_1, \circledast_1, \circledast_2, \circledast_3)$ and $(\mathcal{T}_2, \otimes_1, \otimes_2, \otimes_3)$ be any two BSs. Let $\mathcal{L} : \mathcal{T}_1 \to \mathcal{T}_2$ be any function and Ξ be any DioNSBS in $\mathcal{T}_1$, Z be any DioNSBS in $\mathcal{L}(\mathcal{T}_1) = \mathcal{T}_2$. If $\mho_\Xi = \langle(\mho_\Xi^{\mathcal{T}}, \mho_\Xi^{\mathcal{I}}, \mho_\Xi^{\mathcal{F}}), (\Gamma_\Xi, \Lambda_\Xi, \Theta_\Xi)\rangle$ is a DioNSS in $\mathcal{T}_1$, then $\mho_Z$ is a DioNSS in $\mathcal{T}_2$, defined by $\forall \zeta \in \mathcal{T}_1$ and $\eta \in \mathcal{T}_2$,

$$\mho_Z^{\mathcal{T}}(\eta) = \begin{cases} \sup \mho_\Xi^{\mathcal{T}}(\zeta) & if\ \ \zeta \in \mathcal{L}^{-1}\eta \\ 0 & \text{otherwise} \end{cases} \qquad \mho_Z^{\mathcal{I}}(\eta) = \begin{cases} \sup \mho_\Xi^{\mathcal{I}}(\zeta) & if\ \ \zeta \in \mathcal{L}^{-1}\eta \\ 0 & \text{otherwise} \end{cases}$$

$$\mho_Z^{\mathcal{F}}(\eta) = \begin{cases} \inf \mho_\Xi^{\mathcal{F}}(\zeta) & if\ \ \zeta \in \mathcal{L}^{-1}\eta \\ 1 & \text{otherwise} \end{cases}$$

$$\Gamma_Z(\eta) = \begin{cases} \sup \Gamma_\Xi(\zeta) & if\ \ \zeta \in \mathcal{L}^{-1}\eta \\ 0 & \text{otherwise} \end{cases} \qquad \Lambda_Z(\eta) = \begin{cases} \sup \Lambda_\Xi(\zeta) & if\ \ \zeta \in \mathcal{L}^{-1}\eta \\ 0 & \text{otherwise} \end{cases}$$

$$\Theta_Z(\eta) = \begin{cases} \inf \Theta_\Xi(\zeta) & if\ \ \zeta \in \mathcal{L}^{-1}\eta \\ 1 & \text{otherwise} \end{cases}$$

which is called the image of $\mho_\Xi$ under $\mathcal{L}$.
Similarly, If $\mho_Z = \langle(\mho_Z^{\mathcal{T}}, \mho_Z^{\mathcal{I}}, \mho_Z^{\mathcal{F}}), (\Gamma_Z, \Lambda_Z, \Theta_Z)\rangle$ is a DioNSS in $\mathcal{T}_2$, then DioNSS $\mho_\Xi = \mathcal{L} \circ \mho_Z$ in $\mathcal{T}_1$ [i.e., the DioNSS defined by $\mho_\Xi(\zeta) = \mho_Z(\mathcal{L}(\zeta))$] is called the preimage of $\mho_Z$ under $\mathcal{L}$.

Theorem 3.13. *Let $(\mathcal{T}_1, \circledast_1, \circledast_2, \circledast_3)$ and $(\mathcal{T}_2, \otimes_1, \otimes_2, \otimes_3)$ be any two BSs. The homomorphic image of every DioNSBS of $\mathcal{T}_1$ is a DioNSBS of $\mathcal{T}_2$.*

Proof. Let $\mathcal{L} : \mathcal{T}_1 \to \mathcal{T}_2$ be any homomorphism. Then $\mathcal{L}(\zeta \circledast_1 \eta) = \mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta), \mathcal{L}(\zeta \circledast_2 \eta) = \mathcal{L}(\zeta) \otimes_2 \mathcal{L}(\eta)$ and $\mathcal{L}(\zeta \circledast_3 \eta) = \mathcal{L}(\zeta) \otimes_3 \mathcal{L}(\eta)$ $\forall \zeta, \eta \in \mathcal{T}_1$. Let $Z = \mathcal{L}(\Xi)$, Ξ is any DioNSBS of $\mathcal{T}_1$. Let $\mathcal{L}(\zeta), \mathcal{L}(\eta) \in \mathcal{T}_2$. Let $\zeta \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))$ and $\eta \in \mathcal{L}^{-1}(\mathcal{L}(\eta))$ be such that $\mho^{\mathcal{T}}_{\Xi}(\zeta) = \sup_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))} \mho^{\mathcal{T}}_{\Xi}(\zeta')$ and $\mho^{\mathcal{T}}_{\Xi}(\eta) = \sup_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\eta))} \mho^{\mathcal{T}}_{\Xi}(\zeta')$. Now,

$$\begin{aligned}
\mho^{\mathcal{T}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) &= \sup_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta)\otimes_1\mathcal{L}(\eta))} \mho^{\mathcal{T}}_{\Xi}(\zeta'') \\
&= \sup_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta \circledast_1 \eta)} \mho^{\mathcal{T}}_{\Xi}(\zeta'') \\
&= \mho^{\mathcal{T}}_{\Xi}(\zeta \circledast_1 \eta) \\
&\geq \min\{\mho^{\mathcal{T}}_{\Xi}(\zeta), \mho^{\mathcal{T}}_{\Xi}(\eta)\} \\
&= \min\{\mho^{\mathcal{T}}_{Z}\mathcal{L}(\zeta), \mho^{\mathcal{T}}_{Z}\mathcal{L}(\eta)\}.
\end{aligned}$$

Thus, $\mho^{\mathcal{T}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \min\{\mho^{\mathcal{T}}_{Z}\mathcal{L}(\zeta), \mho^{\mathcal{T}}_{Z}\mathcal{L}(\eta)\}$. Similarly, $\mho^{\mathcal{T}}_{Z}(\mathcal{L}(\zeta) \otimes_2 \mathcal{L}(\eta)) \geq \min\{\mho^{\mathcal{T}}_{Z}\mathcal{L}(\zeta), \mho^{\mathcal{T}}_{Z}\mathcal{L}(\eta)\}$ and $\mho^{\mathcal{T}}_{Z}(\mathcal{L}(\zeta) \otimes_3 \mathcal{L}(\eta)) \geq \min\{\mho^{\mathcal{T}}_{Z}\mathcal{L}(\zeta), \mho^{\mathcal{T}}_{Z}\mathcal{L}(\eta)\}$. Let $\zeta \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))$ and $\eta \in \mathcal{L}^{-1}(\mathcal{L}(\eta))$ be such that $\mho^{\mathcal{I}}_{\Xi}(\zeta) = \sup_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))} \mho^{\mathcal{I}}_{\Xi}(\zeta')$ and $\mho^{\mathcal{I}}_{\Xi}(\eta) = \sup_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\eta))} \mho^{\mathcal{I}}_{\Xi}(\zeta')$. Now,

$$\begin{aligned}
\mho^{\mathcal{I}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) &= \sup_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta)\otimes_1\mathcal{L}(\eta))} \mho^{\mathcal{I}}_{\Xi}(\zeta'') \\
&= \sup_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta \circledast_1 \eta)} \mho^{\mathcal{I}}_{\Xi}(\zeta'') \\
&= \mho^{\mathcal{I}}_{\Xi}(\zeta \circledast_1 \eta) \\
&\geq \frac{\mho^{\mathcal{I}}_{\Xi}(\zeta) + \mho^{\mathcal{I}}_{\Xi}(\eta)}{2} \\
&= \frac{\mho^{\mathcal{I}}_{Z}\mathcal{L}(\zeta) + \mho^{\mathcal{I}}_{Z}\mathcal{L}(\eta)}{2}.
\end{aligned}$$

Thus, $\mho^{\mathcal{I}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \frac{\mho^{\mathcal{I}}_{Z}\mathcal{L}(\zeta)+\mho^{\mathcal{I}}_{Z}\mathcal{L}(\eta)}{2}$. Similarly, $\mho^{\mathcal{I}}_{Z}(\mathcal{L}(\zeta) \otimes_2 \mathcal{L}(\eta)) \geq \frac{\mho^{\mathcal{I}}_{Z}\mathcal{L}(\zeta)+\mho^{\mathcal{I}}_{Z}\mathcal{L}(\eta)}{2}$ and $\mho^{\mathcal{I}}_{Z}(\mathcal{L}(\zeta)\otimes_3\mathcal{L}(\eta)) \geq \frac{\mho^{\mathcal{I}}_{Z}\mathcal{L}(\zeta)+\mho^{\mathcal{I}}_{Z}\mathcal{L}(\eta)}{2}$. Let $\mathcal{L}(\zeta), \mathcal{L}(\eta) \in \mathcal{T}_2$. Let $\zeta \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))$ and $\eta \in \mathcal{L}^{-1}(\mathcal{L}(\eta))$ be such that $\mho^{\mathcal{F}}_{\Xi}(\zeta) = \inf_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))} \mho^{\mathcal{F}}_{\Xi}(\zeta')$ and $\mho^{\mathcal{F}}_{\Xi}(\eta) = \inf_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\eta))} \mho^{\mathcal{F}}_{\Xi}(\zeta')$. Now,

$$\begin{aligned}
\mho^{\mathcal{F}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) &= \inf_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta)\otimes_1\mathcal{L}(\eta))} \mho^{\mathcal{F}}_{\Xi}(\zeta'') \\
&= \inf_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta \circledast_1 \eta)} \mho^{\mathcal{F}}_{\Xi}(\zeta'') \\
&= \mho^{\mathcal{F}}_{\Xi}(\zeta \circledast_1 \eta) \\
&\leq \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta), \mho^{\mathcal{F}}_{\Xi}(\eta)\} \\
&= \max\{\mho^{\mathcal{F}}_{Z}\mathcal{L}(\zeta), \mho^{\mathcal{F}}_{Z}\mathcal{L}(\eta)\}.
\end{aligned}$$

Thus, $\mho^{\mathcal{F}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \leq \max\{\mho^{\mathcal{F}}_{Z}\mathcal{L}(\zeta), \mho^{\mathcal{F}}_{Z}\mathcal{L}(\eta)\}$. Similarly, $\mho^{\mathcal{F}}_{Z}(\mathcal{L}(\zeta) \otimes_2 \mathcal{L}(\eta)) \leq \max\{\mho^{\mathcal{F}}_{Z}\mathcal{L}(\zeta), \mho^{\mathcal{F}}_{Z}\mathcal{L}(\eta)\}$ and $\mho^{\mathcal{F}}_{Z}(\mathcal{L}(\zeta) \otimes_3 \mathcal{L}(\eta)) \leq \max\{\mho^{\mathcal{F}}_{Z}\mathcal{L}(\zeta), \mho^{\mathcal{F}}_{Z}\mathcal{L}(\eta)\}$. Let $\zeta \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))$

and $\eta \in \mathcal{L}^{-1}(\mathcal{L}(\eta))$ be such that $\Gamma_{\Xi}(\zeta) = \sup_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))} \Gamma_{\Xi}(\zeta')$ and $\Gamma_{\Xi}(\eta) = \sup_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\eta))} \Gamma_{\Xi}(\zeta')$. Now,

$$\begin{aligned}\Gamma_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) &= \sup_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta))} \Gamma_{\Xi}(\zeta'') \\ &= \sup_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta \circledast_1 \eta)} \Gamma_{\Xi}(\zeta'') \\ &= \Gamma_{\Xi}(\zeta \circledast_1 \eta) \\ &\geq \min\{\Gamma_{\Xi}(\zeta), \Gamma_{\Xi}(\eta)\} \\ &= \min\{\Gamma_Z\mathcal{L}(\zeta), \Gamma_Z\mathcal{L}(\eta)\}.\end{aligned}$$

Thus, $\Gamma_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \min\{\Gamma_Z\mathcal{L}(\zeta), \Gamma_Z\mathcal{L}(\eta)\}$. Similarly, $\Gamma_Z(\mathcal{L}(\zeta) \otimes_2 \mathcal{L}(\eta)) \geq \min\{\Gamma_Z\mathcal{L}(\zeta), \Gamma_Z\mathcal{L}(\eta)\}$ and $\Gamma_Z(\mathcal{L}(\zeta) \otimes_3 \mathcal{L}(\eta)) \geq \min\{\Gamma_Z\mathcal{L}(\zeta), \Gamma_Z\mathcal{L}(\eta)\}$. Let $\zeta \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))$ and $\eta \in \mathcal{L}^{-1}(\mathcal{L}(\eta))$ be such that $\Lambda_{\Xi}(\zeta) = \sup_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))} \Lambda_{\Xi}(\zeta')$ and $\Lambda_{\Xi}(\eta) = \sup_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\eta))} \Lambda_{\Xi}(\zeta')$. Now,

$$\begin{aligned}\Lambda_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) &= \sup_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta))} \Lambda_{\Xi}(\zeta'') \\ &= \sup_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta \circledast_1 \eta)} \Lambda_{\Xi}(\zeta'') \\ &= \Lambda_{\Xi}(\zeta \circledast_1 \eta) \\ &\geq \frac{\Lambda_{\Xi}(\zeta) + \Lambda_{\Xi}(\eta)}{2} \\ &= \frac{\Lambda_Z\mathcal{L}(\zeta) + \Lambda_Z\mathcal{L}(\eta)}{2}.\end{aligned}$$

Thus, $\Lambda_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \frac{\Lambda_Z\mathcal{L}(\zeta)+\Lambda_Z\mathcal{L}(\eta)}{2}$. Similarly, $\Lambda_Z(\mathcal{L}(\zeta) \otimes_2 \mathcal{L}(\eta)) \geq \frac{\Lambda_Z\mathcal{L}(\zeta)+\Lambda_Z\mathcal{L}(\eta)}{2}$ and $\Lambda_Z(\mathcal{L}(\zeta) \otimes_3 \mathcal{L}(\eta)) \geq \frac{\Lambda_Z\mathcal{L}(\zeta)+\Lambda_Z\mathcal{L}(\eta)}{2}$. Let $\mathcal{L}(\zeta), \mathcal{L}(\eta) \in \mathcal{T}_2$. Let $\zeta \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))$ and $\eta \in \mathcal{L}^{-1}(\mathcal{L}(\eta))$ be such that $\Theta_{\Xi}(\zeta) = \inf_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta))} \Theta_{\Xi}(\zeta')$ and $\Theta_{\Xi}(\eta) = \inf_{\zeta' \in \mathcal{L}^{-1}(\mathcal{L}(\eta))} \Theta_{\Xi}(\zeta')$. Now,

$$\begin{aligned}\Theta_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) &= \inf_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta))} \Theta_{\Xi}(\zeta'') \\ &= \inf_{\zeta'' \in \mathcal{L}^{-1}(\mathcal{L}(\zeta \circledast_1 \eta)} \Theta_{\Xi}(\zeta'') \\ &= \Theta_{\Xi}(\zeta \circledast_1 \eta) \\ &\leq \max\{\Theta_{\Xi}(\zeta), \Theta_{\Xi}(\eta)\} \\ &= \max\{\Theta_Z\mathcal{L}(\zeta), \Theta_Z\mathcal{L}(\eta)\}.\end{aligned}$$

Thus, $\Theta_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \leq \max\{\Theta_Z\mathcal{L}(\zeta), \Theta_Z\mathcal{L}(\eta)\}$. Similarly, $\Theta_Z(\mathcal{L}(\zeta) \otimes_2 \mathcal{L}(\eta)) \leq \max\{\Theta_Z\mathcal{L}(\zeta), \Theta_Z\mathcal{L}(\eta)\}$ and $\Theta_Z(\mathcal{L}(\zeta) \otimes_3 \mathcal{L}(\eta)) \leq \max\{\Theta_Z\mathcal{L}(\zeta), \Theta_Z\mathcal{L}(\eta)\}$. Hence Z is a DioNSBS of $\mathcal{T}_2$.

Theorem 3.14. *Let $(\mathcal{T}_1, \circledast_1, \circledast_2, \circledast_3)$ and $(\mathcal{T}_2, \otimes_1, \otimes_2, \otimes_3)$ be any two BSs. The homomorphic preimage of DioNSBS of $\mathcal{T}_2$ is a DioNSBS of $\mathcal{T}_1$.*

Proof. Let $\mathcal{L} : \mathcal{T}_1 \to \mathcal{T}_2$ be any homomorphism. Then $\mathcal{L}(\zeta \circledast_1 \eta) = \mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta), \mathcal{L}(\zeta \circledast_2 \eta) = \mathcal{L}(\zeta) \otimes_2 \mathcal{L}(\eta)$ and $\mathcal{L}(\zeta \circledast_3 \eta) = \mathcal{L}(\zeta) \otimes_3 \mathcal{L}(\eta)\ \forall \zeta, \eta \in \mathcal{T}_1$. Let $Z = \mathcal{L}(\Xi)$, where Z is any DioNSBS of $\mathcal{T}_2$. Let $\zeta, \eta \in \mathcal{T}_1$. Now, $\mho^{\mathcal{T}}_{\Xi}(\zeta \circledast_1 \eta) = \mho^{\mathcal{T}}_{Z}(\mathcal{L}(\zeta \circledast_1 \eta)) = \mho^{\mathcal{T}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \min\{\mho^{\mathcal{T}}_{Z}\mathcal{L}(\zeta), \mho^{\mathcal{T}}_{Z}\mathcal{L}(\eta)\} = \min\{\mho^{\mathcal{T}}_{\Xi}(\zeta), \mho^{\mathcal{T}}_{\Xi}(\eta)\}$. Thus, $\mho^{\mathcal{T}}_{\Xi}(\zeta \circledast_1 \eta) \geq \min\{\mho^{\mathcal{T}}_{\Xi}(\zeta), \mho^{\mathcal{T}}_{\Xi}(\eta)\}$. Now, $\mho^{\mathcal{I}}_{\Xi}(\zeta \circledast_1 \eta) = \mho^{\mathcal{I}}_{Z}(\mathcal{L}(\zeta \circledast_1 \eta)) = \mho^{\mathcal{I}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \frac{\mho^{\mathcal{I}}_{Z}\mathcal{L}(\zeta) + \mho^{\mathcal{I}}_{Z}\mathcal{L}(\eta)}{2} = \frac{\mho^{\mathcal{I}}_{\Xi}(\zeta) + \mho^{\mathcal{I}}_{\Xi}(\eta)}{2}$. Thus, $\mho^{\mathcal{I}}_{\Xi}(\zeta \circledast_1 \eta) \geq \frac{\mho^{\mathcal{I}}_{\Xi}(\zeta) + \mho^{\mathcal{I}}_{\Xi}(\eta)}{2}$. Now, $\mho^{\mathcal{F}}_{\Xi}(\zeta \circledast_1 \eta) = \mho^{\mathcal{F}}_{Z}(\mathcal{L}(\zeta \circledast_1 \eta)) = \mho^{\mathcal{F}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \leq \max\{\mho^{\mathcal{F}}_{Z}\mathcal{L}(\zeta), \mho^{\mathcal{F}}_{Z}\mathcal{L}(\eta)\} = \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta), \mho^{\mathcal{F}}_{\Xi}(\eta)\}$. Thus, $\mho^{\mathcal{F}}_{\Xi}(\zeta \circledast_1 \eta) \leq \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta), \mho^{\mathcal{F}}_{\Xi}(\eta)\}$. Now, $\Gamma_{\Xi}(\zeta \circledast_1 \eta) = \Gamma_Z(\mathcal{L}(\zeta \circledast_1 \eta)) = \Gamma_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \min\{\Gamma_Z\mathcal{L}(\zeta), \Gamma_Z\mathcal{L}(\eta)\} = \min\{\Gamma_{\Xi}(\zeta), \Gamma_{\Xi}(\eta)\}$. Thus, $\Gamma_{\Xi}(\zeta \circledast_1 \eta) \geq \min\{\Gamma_{\Xi}(\zeta), \Gamma_{\Xi}(\eta)\}$. Now, $\Lambda_{\Xi}(\zeta \circledast_1 \eta) = \Lambda_Z(\mathcal{L}(\zeta \circledast_1 \eta)) = \Lambda_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \frac{\Lambda_Z\mathcal{L}(\zeta) + \Lambda_Z\mathcal{L}(\eta)}{2} = \frac{\Lambda_{\Xi}(\zeta) + \Lambda_{\Xi}(\eta)}{2}$. Thus, $\Lambda_{\Xi}(\zeta \circledast_1 \eta) \geq \frac{\Lambda_{\Xi}(\zeta) + \Lambda_{\Xi}(\eta)}{2}$. Now, $\Theta_{\Xi}(\zeta \circledast_1 \eta) = \Theta_Z(\mathcal{L}(\zeta \circledast_1 \eta)) = \Theta_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \leq \max\{\Theta_Z\mathcal{L}(\zeta), \Theta_Z\mathcal{L}(\eta)\} = \max\{\Theta_{\Xi}(\zeta), \Theta_{\Xi}(\eta)\}$. Thus, $\Theta_{\Xi}(\zeta \circledast_1 \eta) \leq \max\{\Theta_{\Xi}(\zeta), \Theta_{\Xi}(\eta)\}$. Similarly to prove two other operations, Ξ is a DioNSBS of $\mathcal{T}_1$.

Theorem 3.15. *Let $(\mathcal{T}_1, \circledast_1, \circledast_2, \circledast_3)$ and $(\mathcal{T}_2, \otimes_1, \otimes_2, \otimes_3)$ be any two BSs. If $\mathcal{L} : \mathcal{T}_1 \to \mathcal{T}_2$ is a homomorphism, then $\mathcal{L}(\Xi_{(\beta,\gamma)})$ is a level SBS of DioNSBS Z of $\mathcal{T}_2$.*

Proof. Let $\mathcal{L} : \mathcal{T}_1 \to \mathcal{T}_2$ be any homomorphism. Then $\mathcal{L}(\zeta \circledast_1 \eta) = \mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta), \mathcal{L}(\zeta \circledast_2 \eta) = \mathcal{L}(\zeta) \otimes_2 \mathcal{L}(\eta)$ and $\mathcal{L}(\zeta \circledast_3 \eta) = \mathcal{L}(\zeta) \otimes_3 \mathcal{L}(\eta)\ \forall \zeta, \eta \in \mathcal{T}_1$. Let $Z = \mathcal{L}(\Xi)$, Ξ is a DioNSBS of $\mathcal{T}_1$. By Theorem 3.13, Z is a DioNSBS of $\mathcal{T}_2$. Let $\Xi_{(\beta,\gamma)}$ be any level SBS of Ξ. Suppose that $\zeta, \eta \in \Xi_{(\beta,\gamma)}$. Then $\mathcal{L}(\zeta \circledast_1 \eta), \mathcal{L}(\zeta \circledast_2 \eta)$ and $\mathcal{L}(\zeta \circledast_3 \eta) \in \Xi_{(\beta,\gamma)}$. Now, $\mho^{\mathcal{T}}_{Z}(\mathcal{L}(\zeta)) = \mho^{\mathcal{T}}_{\Xi}(\zeta) \geq \beta, \mho^{\mathcal{T}}_{Z}(\mathcal{L}(\eta)) = \mho^{\mathcal{T}}_{\Xi}(\eta) \geq \beta$. Thus, $\mho^{\mathcal{T}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \mho^{\mathcal{T}}_{\Xi}(\zeta \circledast_1 \eta) \geq \beta$. Now, $\mho^{\mathcal{I}}_{Z}(\mathcal{L}(\zeta)) = \mho^{\mathcal{I}}_{\Xi}(\zeta) \geq \beta, \mho^{\mathcal{I}}_{Z}(\mathcal{L}(\eta)) = \mho^{\mathcal{I}}_{\Xi}(\eta) \geq \beta$. Thus, $\mho^{\mathcal{I}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \mho^{\mathcal{I}}_{\Xi}(\zeta \circledast_1 \eta) \geq \beta$. Now, $\mho^{\mathcal{F}}_{Z}(\mathcal{L}(\zeta)) = \mho^{\mathcal{F}}_{\Xi}(\zeta) \leq \gamma, \mho^{\mathcal{F}}_{Z}(\mathcal{L}(\eta)) = \mho^{\mathcal{F}}_{\Xi}(\eta) \leq \gamma$. Thus, $\mho^{\mathcal{F}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \leq \mho^{\mathcal{F}}_{\Xi}(\zeta \circledast_1 \eta) \leq \gamma$, $\forall \mathcal{L}(\zeta), \mathcal{L}(\eta) \in \mathcal{T}_2$. Now, $\Gamma_{\Xi Z}(\mathcal{L}(\zeta)) = \Gamma_{\Xi}(\zeta) \geq \beta, \Gamma_Z(\mathcal{L}(\eta)) = \Gamma(\eta) \geq \beta$. Thus, $\Gamma_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \Gamma_{\Xi}(\zeta \circledast_1 \eta) \geq \beta$. Now, $\Lambda_Z(\mathcal{L}(\zeta)) = \Lambda_{\Xi}(\zeta) \geq \beta, \Lambda_Z(\mathcal{L}(\eta)) = \Lambda_{\Xi}(\eta) \geq \beta$. Thus, $\Lambda_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \geq \Lambda_{\Xi}(\zeta \circledast_1 \eta) \geq \beta$. Now, $\Theta_Z(\mathcal{L}(\zeta)) = \Theta_{\Xi}(\zeta) \leq \gamma, \Theta_Z(\mathcal{L}(\eta)) = \Theta_{\Xi}(\eta) \leq \gamma$. Thus, $\Theta_Z(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \leq \Theta_{\Xi}(\zeta \circledast_1 \eta) \leq \gamma$, $\forall \mathcal{L}(\zeta), \mathcal{L}(\eta) \in \mathcal{T}_2$. Similarly to prove other operations, hence $\mathcal{L}(\Xi_{(\beta,\gamma)})$ is a level SBS of DioNSBS Z of $\mathcal{T}_2$.

Theorem 3.16. *Let $(\mathcal{T}_1, \circledast_1, \circledast_2, \circledast_3)$ and $(\mathcal{T}_2, \otimes_1, \otimes_2, \otimes_3)$ be any two BSs. If $\mathcal{L} : \mathcal{T}_1 \to \mathcal{T}_2$ is any homomorphism, then $\Xi_{(\beta,\gamma)}$ is a level SBS of DioNSBS Ξ of $\mathcal{T}_1$.*

Proof. Let $\mathcal{L} : \mathcal{T}_1 \to \mathcal{T}_2$ be any homomorphism. Then $\mathcal{L}(\zeta \circledast_1 \eta) = \mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta), \mathcal{L}(\zeta \circledast_2 \eta) = \mathcal{L}(\zeta) \otimes_2 \mathcal{L}(\eta)$ and $\mathcal{L}(\zeta \circledast_3 \eta) = \mathcal{L}(\zeta) \otimes_3 \mathcal{L}(\eta)\ \forall \zeta, \eta \in \mathcal{T}_1$. Let $Z = \mathcal{L}(\Xi)$, Z is a DioNSBS of $\mathcal{T}_2$. By Theorem 3.14, Ξ is a DioNSBS of $\mathcal{T}_1$. Let $\mathcal{L}(\Xi_{(\beta,\gamma)})$ be a level SBS of Z. Suppose that $\mathcal{L}(\zeta), \mathcal{L}(\eta) \in \mathcal{L}(\Xi_{(\beta,\gamma)})$. Then $\mathcal{L}(\zeta \circledast_1 \eta), \mathcal{L}(\zeta \circledast_2 \eta)$ and $\mathcal{L}(\zeta \circledast_3 \eta) \in \mathcal{L}(\Xi_{(\beta,\gamma)})$. Now, $\mho^{\mathcal{T}}_{\Xi}(\zeta) = \mho^{\mathcal{T}}_{Z}(\mathcal{L}(\zeta)) \geq t, \mho^{\mathcal{T}}_{\Xi}(\eta) = \mho^{\mathcal{T}}_{Z}(\mathcal{L}(\eta)) \geq \beta$. Thus, $\mho^{\mathcal{T}}_{\Xi}(\zeta \circledast_1 \eta) \geq \min\{\mho^{\mathcal{T}}_{\Xi}(\zeta), \mho^{\mathcal{T}}_{\Xi}(\eta)\} \geq \beta$. Now, $\mho^{\mathcal{I}}_{\Xi}(\zeta) = \mho^{\mathcal{I}}_{Z}(\mathcal{L}(\zeta)) \geq t, \mho^{\mathcal{I}}_{\Xi}(\eta) = \mho^{\mathcal{I}}_{Z}(\mathcal{L}(\eta)) \geq \beta$. Thus, $\mho^{\mathcal{I}}_{\Xi}(\zeta \circledast_1 \eta) \geq \frac{\mho^{\mathcal{I}}_{\Xi}(\zeta)+\mho^{\mathcal{I}}_{\Xi}(\eta)}{2} \geq \beta$. Now, $\mho^{\mathcal{F}}_{\Xi}(\zeta) = \mho^{\mathcal{F}}_{Z}(\mathcal{L}(\zeta)) \leq \gamma, \mho^{\mathcal{F}}_{\Xi}(\eta) = \mho^{\mathcal{F}}_{Z}(\mathcal{L}(\eta)) \leq \gamma$. Thus, $\mho^{\mathcal{F}}_{\Xi}(\zeta \circledast_1 \eta) = \mho^{\mathcal{F}}_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \leq \max\{\mho^{\mathcal{F}}_{\Xi}(\zeta), \mho^{\mathcal{F}}_{\Xi}(\eta)\} \leq \gamma$, $\forall \zeta, \eta \in \mathcal{T}_1$. Now, $\Gamma_{\Xi}(\zeta) = \Gamma_{\Xi Z}(\mathcal{L}(\zeta)) \geq t, \Gamma_{\Xi}(\eta) = \Gamma_{Z}(\mathcal{L}(\eta)) \geq \beta$. Thus, $\Gamma_{\Xi}(\zeta \circledast_1 \eta) \geq \min\{\Gamma_{\Xi}(\zeta), \Gamma_{\Xi}(\eta)\} \geq \beta$. Now, $\Lambda_{\Xi}(\zeta) = \Lambda_{Z}(\mathcal{L}(\zeta)) \geq t, \Lambda_{\Xi}(\eta) = \Lambda_{Z}(\mathcal{L}(\eta)) \geq \beta$. Thus, $\Lambda_{\Xi}(\zeta \circledast_1 \eta) \geq \frac{\Lambda_{\Xi}(\zeta)+\Lambda_{\Xi}(\eta)}{2} \geq \beta$. Now, $\Theta_{\Xi}(\zeta) = \Theta_{Z}(\mathcal{L}(\zeta)) \leq \gamma, \Theta_{\Xi}(\eta) = \Theta_{Z}(\mathcal{L}(\eta)) \leq \gamma$. Thus, $\Theta_{\Xi}(\zeta \circledast_1 \eta) = \Theta_{Z}(\mathcal{L}(\zeta) \otimes_1 \mathcal{L}(\eta)) \leq \max\{\Theta_{\Xi}(\zeta), \Theta_{\Xi}(\eta)\} \leq \gamma$, $\forall \zeta, \eta \in \mathcal{T}_1$. Similarly to prove other two operations, hence $\Xi_{(\beta,\gamma)}$ is a level SBS of DioNSBS Ξ of $\mathcal{T}_1$.

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The Minimum Spanning Tree Problem on networks with Neutrosophic numbers

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Abstract. The minimum spanning tree problem (MSTP) revolves around creating a spanning tree (ST) within a graph/network that incurs the least cost compared to all other potential STs. This represents a vital and fundamental issue in the realm of combinational optimization problems (COP). Supply chain management, communication, transportation, and routing are a few examples of real-world issues that have been represented using the MSTP. Uncertainties exist in almost every real life application of MSTP due to inconsistency, improperness, incompleteness, vagueness and indeterminacy of the information and It generates really challenging scenarios to determine the arc length precisely. The main motivation behind this research work is to design a method for MST which will be simple enough and effective in real world scenarios. Neutrosophic set (NS) is a well known renowned theory, which one can this type of uncertainty in the edge weights of the ST. In this article, we review trapezoid neutrosophic set/number to describe the arc weight of a neutrosophic network for MSTP. Here, we introduce an algorithm for solving MSTP in neutrosophic environment. In our proposed method, we describe the uncertainties in Prim's algorithm for MSTP using trapezoid neutrosophic set as edge cost. Here examples of numerical sets are used to explain the proposed algorithm. **Keywords:** Neutrosophic set; MSTP; neutrosophic network/graph; Prim's algorithm.

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1. **Introduction**

The MSTP, a renowned and extensively applied constraint optimization problem, finds its applications in both operation research and graph theory. It has numerous practical applications [1–3], including communications problem, transportation problem, logistics problem, supply chain management problem, image processing, wireless telecommunication networks and cluster analysis.

Consider $G = (X, Y)$ as a connected, undirected, weighted graph, where X represents a collection items/nodes/vertices and Y denotes a finite set of arcs characterized by integral cost or weight. Tree is a connected graph without circuits. Consider t to represent as a ST of connected graph if and only if t is a sub graph of the graph G and t must consist all vertices of graph G. Because it has the most edges surrounded by all feasible trees in graph G, a ST T is alternatively referred to as the largest possible subtree within graph G. The MST is a ST where the aggregate weights of edges is minimized.

The traditional MSTP is built on a graph or network. As a result, the G's nodes are used to represent the things, points, and objects, while the arcs are used to indicate the relationship or specific link between the nodes (for instance, highways connecting communities). The information connected to objects/items and the relationship between two things are assumed to be fully known in the traditional networks/graphs descriptions of simple deterministic scenarios. In practical situations, achieving this may prove challenging due to presence of uncertainties that can exist in any conceivable description of an object or the relationship between two objects, or even both cases. Due to this reason, the crisp graph model is not useful to model those problem.

Zadeh [4] proposed the concept of a fuzzy set (FS), which may deal with the occurrence of uncertainty, ambiguity, and imprecision in everyday life. The main characteristic of a FS, described by a membership grade/function/degree, is a grade/function/degree whose interval is [0,1]. The idea of FSs has been applied to model several COPs in many fields. The FS (type-1/classical FS), whose membership grade is an actual value, is incapable of managing many different kinds of uncertainty that are present in problems in the real world. In [5], Turksen described the concept of FS with interval membership vale membership [6] and they developed the idea of intuitionistic FSs to capture the problem of non membership grade of classical FSs. It has applied in several problems, e.g., decision making, COPs, artificial neural network, medical analysis, and so forth. It is a modified version of classical FS that can consider not only one a membership grade for each element and but also it considers a non-membership grade. It helps to capture more flexibility to work with uncertainties of real problem [7] than the simple FS. It has three different types of membership grade: membership, non-membership and hesitation of all elements in this set. In [8], the author modified the idea of FSs to the interval valued intuitionistic FSs to capture more uncertainties than intuitionistic FSs.

However, the intuitionistic FS and intuitionistic fuzzy logic have been used to find the solutions of many COPs, but it cannot be captured several type of uncertainty properly.

For e.g., the FS cannot work with the uncertainties due to indeterminate information and inconsistent information. When seeking the opinion of an expert regarding a decision, then he might state that the likelihood of the decision being true is 0.5, likelihood of it being false is 0.9, and the chances of uncertainty is 0.4. This kind of real-world issue cannot be modeled with FS. To solve this issue, new thinking is therefore necessary.

In [9], The idea of NS was established by the author to explain information and facts that might be insufficient, unsure, hazy, imprecise, indeterminate, and inconsistent in various real-world settings. Three membership grades are used to define it: a true membership degree/grade/function, an indeterminate membership degree/grade/function, and a false membership degree/grade/function independently. They fall inside the nonstandard or standard unit interval in terms of value. Because it can adeptly deal with information that lacks consistency and completeness , NS is frequently employed by researchers to handle challenges that arise in real-life situations [10], [11], [12], [13], [**?**], [**?**], [14].

MSTP is an COP [15] in graph theory [16], [17] which can determine the minimum cost ST of a graph. The classical MSTP has several real life applications, including cluster analysis, wireless communication, computer networks, speech recognition, social network, etc. In the classical MSTP, the edge lengths are considered to be precise and expert assumes some crisp values (real number) to describe the edge lengths of the graph. However, in our day to day life [18], [19], [20], the edge length may represent a criterion such as cost, time, demand, capacity, etc. that shouldn't have a fixed parameter. We can consider a real life scenarios in a road networks of city. The edge length describing the time it takes for the car journey could vary due to changing weather conditions, strong traffic flow, or other unanticipated factors, however the geometric distance/road distance between two cities is fixed. [21]. For this reason, it is very confusing for an expert to assume a proper edge length in real number, i.e., crisp values. Experts may consider a range of feasible values of edge lengths in form of approximate intervals, linguistic terms, etc. In this road network problem, the edge lengths can represent as, "around 30 to 90 minute", "about 1 hour", "between 5 and 10 hour" and "nearly 2 to 3 hours", etc. Many researchers used classical FS to describe those uncertainties in edge weights. But simple FS is not properly model those vagueness/incomplete information because their membership grades are fully crisp. The idea of neutrosophic network/graph can be considered as a modified version of fuzzy graph to deal this types of uncertain situations.

The idea of MSTP and its several applications have taken lot of attention of researchers throughout the prior decade and numerous successful approaches have been created for finding the MSTP solution in classical graphs. Refs. [22], [23], [24], [25], [26] can be found related to this MSTP. Prim's algorithm [23] is an effective and well-known algorithmic technique to solve the MST of a crisp graph. Expert can determine the MST using Prim's algorithm [23] if and only if the edge weight/length of the graph are real number/crisp number. Almost all MSTP applications in real life include a certain amount of uncertainty. Two different causes of parameter uncertainty are randomness and fuzzy or insufficient information. The probability theory uses to handle the uncertainties due randomness. Due to this reason, many scientist assume the link/edge of a MSTP as a random variables. In [27] presents this problem with random edge weights/costs. In [28], the author introduced a method that can efficiently solve this problem within a polynomial time frame. In this method, the arc length is calculated based on the several parameters of the probability distributions and those parameters are determined by considering a confidence interval from a set of stochastic data. However in real life scenarios, those parameter are unknown and the parameter values are uncertain (fuzzy, vague or incomplete) in nature. In [29], the authors has described first time the MSTP in fuzzy environment. They used the idea of chance constrained programming and necessity measurement to solve this problem. Then Chang [30] presented the fuzzy MSTP whose fuzzy arc length are fuzzy number. They applied three different techniques using the ranking index method [31] for comparing the several fuzzy arc lengths. Combing the idea of probability theory and FS theory, the authors have developed algorithmic technique to solve this problem. They have also described a genetic algorithm for this problem. In [32], the author has described the fuzzy ST problem in which the lengths are denoted by interval fuzzy number. They have used the principles of possibility theory to compare and select the edge of the ST for the fuzzy graph. In [33], the author considered different intuitionistic FS/number to denote the edge weight of a fuzzy graph. They have described a new algorithmic approach to find the solution of this COP. In [34], the authors presented the fuzzy MSTP with hesitant FSs as fuzzy edge weight and introduce an algorithmic approach to find the solution of this problem.

Recently, few scientists have worked MSTP in neutrosophic environment. This MSTP is defined as neutrosophic MSTP (NMSTP) problem. In several real life application of MSTP, a NMSTP may be more logical, reliable and reasonable. Ye [35] developed an algorithmic approach to solve the MSTP of a neutrosophic network/graph where objects/nodes/vertices are described in NSs and link between two different nodes describes the dissimilarity between objects. Kandasamy [36] described a double-valued NMSTP and present a clustering method to classify the cluster of data/information. A novel approach to solving optimum ST issues by assuming inconsistent, inappropriate, partial, ambiguous, and indeterminate data was presented by Mandal and Basu [37]. They represented the arc length with NSs. Neutosophic numbers and fuzzy numbers have essentially identical notations, but their representations couldn't be further apart. No algorithm for MST with interval neutrosophic arc lengths exists as far as we are aware.

Due to its extensive uses in real-world situations, the MSTP is a well known COPs in the field of operation research. It is particularly difficult to calculate the edge weights correctly since the information used in the real world application of MSTP is inconsistent, inappropriate, partial, unclear, and indeterminable. The NS/logic theory is well known for its ability to explain the inconsistent, inappropriate, incomplete, nebulous, and undetermined arc lengths of the ST. Numerous scientists believe that neutrosophicness should be used instead of fuzziness to represent doubt because it is inconsistent, incorrect, incomplete, ambiguous, and indefinite.

Although some research works have been done to develop for MSTP using neutrosophic set and its generalizations, still there are some scope of research works in this field. The key driving force behind this scientific study is to identify an algorithmic method for the MSTP of a neutrosophic network which will be able to efficiently handle the MSTP. In the past few years, few researchers [38–40] developed some algorithms to determine the MST of a neutrosophic network/graph. In those algorithms, they consider the simple NS to describe the MST of the neutrosophic network/graph. We consider the interval neutrosophic number to represent the arc length. The objective of this scientific study is to introduce an algorithm that can determine the MST. In this research paper, a MSTP is considered whose edge weights are described by neutrosophic number. We have described a modified Prim's algorithm to determine the NMSTP of a graph and its weight as a score value.Our focus is on a neutrosophic network [41–44] or graph, with its edge weights expressed through the use of neutrosophic numbers. Opt for the arc with the most minimal score, a ranking mechanism based on scores is utilized.

The structure of the paper is as follows. A few essential definitions and concepts related to neutrosophic graphs, single valued trapezoidal neutrosophic number, ranking and addition operation are reviewed in brief in Section 2. We provide a mathematical model for Neutrosophic Minimal Spanning Tree in Section 3. In Section 4 we provide our proposed algorithm for this problem. Section 5 presents the outcomes of the suggested method and draws comparisons with binary programming. In Section 6, we finally come to an end.

2. Preliminary

Definition 2.1. Let $\mathcal{U}$ and Z represent an universal set and NS (NS). The NS [9] Z consists of 3 membership degree. There are true membership grade $\mathcal{T}_Z(k)$, indeterminate membership grade $\mathcal{I}_Z(k)$ and false membership grade $\mathcal{F}_Z(k)$ respectively.

$$^{-}0 \leq \sup\ \mathcal{T}_Z(k) + \sup\ \mathcal{I}_Z(k) + \sup\ \mathcal{F}_Z(k) \leq 3^{+} \tag{1}$$

Definition 2.2. The single valued NS [45] D on the $\mathcal{U}$ is presented as following

$$A = \{\langle k : \mathcal{T}_Z(k), \mathcal{I}_Z(k), \mathcal{F}_Z(k) | k \in \mathcal{U}\rangle\} \tag{2}$$

The true function $\mathcal{T}_Z(k)$ lies in the interval $[0,1]$, indeterminate membership grade $\mathcal{I}_A(k)$ lies between $[0,1]$ and false membership grade $\mathcal{F}_A(k)$ is in the interval $[0,1]$, satisfy the following condition:

$$^{-}0 \leq \sup\ \mathcal{T}_Z(k) + \sup\ \mathcal{I}_Z(k) + \sup\ \mathcal{F}_Z(k) \leq 3^{+} \tag{3}$$

Definition 2.3. Let $\tilde{D}$ describes a single valued trapezoidal neutrosophic number (SVTNN) [?] where $\tilde{D} = \left\langle (d_r, d_l, d_p, d_s); w_{\tilde{D}}, u_{\tilde{D}}, y_{\tilde{D}} \right\rangle$. The membership values can be calculated as follows

$$\mu_{\tilde{D}}(q) = \begin{cases} \frac{(q-d_r)w_{\tilde{D}}}{(d_l-d_r)} & (d_r \leq q < d_l) \\ w_{\tilde{D}} & (d_l \leq q \leq d_p) \\ \frac{(d_s-p)w_{\tilde{D}}}{(d_s-d_p)} & (d_p < p \leq d_s) \\ 0 & otherwise \end{cases}$$

$$v_{\tilde{D}}(q) = \begin{cases} \frac{\left(d_l-p+u_{\tilde{D}}(q-d_r)\right)}{(d_l-d_r)} & (d_r \leq x < d_l) \\ u_{\tilde{D}} & (d_l \leq q \leq d_p) \\ \frac{\left(q-d_p+u_{\tilde{D}}(d_s-p)\right)}{(d_s-d_p)} & (d_p < x \leq d_s) \\ 0 & otherwise \end{cases}$$

and

$$\lambda_{\tilde{D}}(q) = \begin{cases} \frac{\left(d_l-p+y_{\tilde{D}}(q-d_r)\right)}{(d_l-d_r)} & (d_r \leq q < d_l) \\ y_{\tilde{D}} & (d_l \leq q \leq d_p) \\ \frac{\left(x-d_p+y_{\tilde{D}}(d_s-p)\right)}{(d_s-d_p)} & (d_p < p \leq d_s) \\ 0 & otherwise \end{cases}$$

respectively.

Definition 2.4. Let $\tilde{D}$ is a single valued triangular neutrosophic number (SVTrN-number) [?] where $\tilde{D} = \left\langle (d_r, d_l, d_p,); w_{\tilde{D}}, u_{\tilde{D}}, y_{\tilde{D}} \right\rangle$. We can calculate membership values in the following manner:

$$\mu_{\tilde{D}}(q) = \begin{cases} \frac{(q-d_r)w_{\tilde{D}}}{(d_l-d_r)}, & (d_r \leq q < d_l) \\ \frac{(d_p-p)w_{\tilde{D}}}{(d_p-d_l)}, & (d_l \leq q < d_p) \\ 0, & otherwise \end{cases}$$

$$v_{\tilde{D}}(q) = \begin{cases} \frac{\left(d_l-p+u_{\tilde{D}}(q-d_r)\right)}{(d_l-d_r)}, & (d_r \leq q < d_l) \\ \frac{\left(q-d_l+u_{\tilde{D}}(d_p-p)\right)}{(d_p-d_l)}, & (d_l \leq q \leq d_p) \\ 0, & otherwise \end{cases}$$

$$\lambda_{\tilde{D}}(q) = \begin{cases} \frac{\left(d_l-p+y_{\tilde{D}}(q-d_r)\right)}{(d_l-d_r)}, & (d_r \leq q < d_l) \\ \frac{\left(q-d_l+y_{\tilde{D}}(d_p-p)\right)}{(d_p-d_l)}, & (d_l \leq x \leq d_p) \\ 0, & otherwise \end{cases}$$

If $d_r \geq 0$and at least $c > 0$ then $\tilde{D} = \left\langle (d_r, d_l, d_p, d_s); w_{\tilde{D}}, u_{\tilde{D}}, y_{\tilde{D}} \right\rangle$ it is affirmed to be positive SVTrN-number, denoted by $\tilde{D} > 0$.

Definition 2.5.
Let $\tilde{D} = \langle (d_{r1}, d_{l1}, d_{p1}, d_{s1}) ; w_{\tilde{D}}, u_{\tilde{D}}, y_{\tilde{D}} \rangle$ and $\tilde{B} = \langle (d_{r2}, d_{l2}, d_{p2}, d_{s2}) ; w_{\tilde{D}}, u_{\tilde{D}}, y_{\tilde{D}} \rangle$ be two SVTN-number and $d_s \neq 0$ be any real number. Then, the addition operation between $\tilde{D}$ and $\tilde{B}$

$$\tilde{D} + \tilde{B} = \langle (d_{r1} + d_{r2}, d_{l1} + d_{l2}, d_{p1} + d_{p2}, d_{s1} + d_{s2}) ; w_{\tilde{D}} \wedge w_{\tilde{B}}, u_{\tilde{D}} \vee u_{\tilde{B}}, y_{\tilde{D}} \vee y_{\tilde{B}} \rangle \tag{4}$$

Definition 2.6. The score functions is defined as follows:

$$s(\tilde{D}) = \frac{1}{12} (d_{r1} + d_{r2}\, d_{r3} + d_{r4}) * (2 + w_{\tilde{D}} - u_{\tilde{D}} - y_{\tilde{D}}) \tag{5}$$

Definition 2.7. Let D_1 and D_2 are two SVNs. Then

$D_1 \succ D_2$ if and only if $s(D_1) > s(D_2)$.

$D_1 \prec D_2$ if and only if $s(D_1) < s(D_2)$.

$D_1 \sim D_2$ if and only if $s(D_1) = s(D_2)$.

Here, $D_1 \succ D_2$ expresses that the cost of the arc/edge represented by D_1 is larger than the cost of the arc/edge represented by D_2. Similarly, $D_1 \prec D_2$ expresses that the cost of the edge described by D_1 is lower than the cost of the arc/edge descried by $D2$. $D_1 \sim D_2$ expresses that the cost of the edge presented by D_1 is equal to the cost of the arc/edge described by D_2.

3. Neutrosophic Minimal Spanning Tree (NMSTP)

When considering a connected graph G, an ST, defined as connected, acyclic and maximum sub graph, comprises all nodes within G. Each ST contains precisely $n - 1$ arcs, where n represents the number of nodes in the graph G. A MSTP is to identify a ST such that the total length of its arcs is minimum. The precise weights connected to the graph's arcs are taken into account by the traditional MSTP. However, due to insufficient or absent evidence in real-world settings, the arc lengths might not be exact. Neutosophic graphs are the best solution for dealing with this imprecision.

3.1. *Problem formulation for NMSTP*

Let G represents a neutrosophic network/graph. The graph G consists of p number of vertices $V = \{v_1, v_2, ..., v_p\}$ and q number of arcs $A \subseteq V \times V$. Each edge of G is denoted by r, which is a pair of vertices (n, m), where $n, m \in V$ and $n \neq m$. If the edge e is present in the NMSTP then $x_r = 1$, otherwise $x_r = 0$. The NMSTP is expressed as the following linear programming problem.

$$\min \sum_{r \in A} D_r y_r \tag{6}$$

Subject to

$$\sum_{r\in A} y_r = p-1 \tag{7}$$

$$\sum_{r\in d_l(s)} y_r \geq 1 \; \forall s \subset V, \; \emptyset \neq s \neq V \tag{8}$$

$$y_r \in \{0,1\} \;\; \forall r \in A \tag{9}$$

Here, D_r is a NS that describes the edge weight $r \in A$ and $\sum$ in (6) is the addition of all NSs. Next Eq. (7) describes that the total number of arc in the ST is $p-1$. In (8), $d_l(s) = \{(n, m|n \in s, m \notin s)\}$ is applied for the cut of a subset of node s, i.e., the edges that consists of only one node s and the different node outside the s.

4. **Proposed Algorithm for the NMSTP and its cost**

In this document, we acquaint you with a neutrosophic version of Prim's algorithm for finding the MST in an uncertain environment [46, 47]. For managing the uncertainty in the existences world scenarios, we employ NS. In a neutrosophic environment, we present the MSTP on a neutrosophic network or graph where each edge is given a trapezoid neutrosophic number as its edge weight. In this optimization problem MSTP, since it needs ordering and summation between trapezoid neutrosophic number, is not same from the strand MSTP, which can only consider real numbers/value. The neutrosophic number's scoring function is utilized for comparison, and neutrosophic numbers are combined by applying their designated addition formula. Based on this two concept of neutrosophic number, we propose a neutrosophic edition of the conventional Prim's method to solve the NMSTP. We take into account numerous variables that are crucial to describing in our proposed approach. An undirected connected weighted neutrosophic network/graph $G = (X, Y)$ with neutrosophic edge weight, where X contains a set of nodes and E contains a set of arcs. Let $number = |X|$ and $number_1 = |Y|$, so we have a finite set of nodes $X = \{x_1, x_2, ..., x_{number}\}$ and edges $Y = \{y_1, y_2, ..., y_{number_1}\}$. X_n, Y_n and $C_{\tilde{M}}$ describe the finite set of node, arc and weight of the corresponding neutrosophic MST (NMST).

A random vertex t is selected from G. We calculate the score value for all the arcs in graph G using Eq. (7). Start from the node t and add the node t to its nearest neighbour node, say r. To select the nearest neighbour vertex, first we have to determine all the adjacent edge with p. Then, select the arc, i.e., (p, r) with lowest score value among all the adjacent edges of p. Using the same concept, we have to find the nearest neighbour vertex for all other nodes of the graph. Now, we assume p and r as one simple sub-graph and add this sub graph to its nearest neighbour node. Let us consider, the new node is Q. Next time, the neutrosophic tree with nodes p, q and r as one another sub graph and repeat this method until all $number$ nodes have joined by $number - 1$ edges. Our proposed neutrosophic Prim algorithm is presented in Algorithm 1.

Algorithm 1 Algorithm of the modified neutrosophic Prim's algorithm for NMSTP.

Input: An undirected connected weighted neutrosophic network/graph $G = (X, Y)$ with neutrosophic edge weight.

Output: NMST $\tilde{M} = (X_n, Y_n)$ of G and its cost.

```
X_n ← {t};          ▷ M̃ is a randomly selected source node and S is the vertex set of M̃
Y_n ← ∅;
C_M̃ ← 0;
Calculate the score value for each arc in G using (7);
while X \ X_n ≠ ∅ do
    Select an arc (p, r) with minimum score value such that p is in X_n and r is not;
    C_M ← C_M ⊕ Score(p, r);
    Y_n ← Y_n ∪ (p, r);
    X_n ← X_n ∪ ({p, r} \ X_n);
end while
```

5. Numerical illustrations

To give an idea, put forward a suggested algorithm where we have included an example of the NMSTP in this section. A network/graph with undirected connections, weighted edges, and neutrosophic nature where network/graph $G = (X, Y)$, neutrosophic edge weight is considered here. This graph has 6 nodes and 9 arcs. Our propose algorithm can solve this NMSTP and it finds the NMST of a neutrosophic network/graph, whose arc length are expressed by trapezoid neutrosophic set/number. The eight trapezoid neutrosophic number, presented in Table 1 are considered as edge weight of neutrosophic network/graph. For this graph, presented in Figure 1, those trapezoid neutrosophic number are given to the arcs as arc length of this graph randomly.

The source vertex 1 is selected randomly from the set of vertices of the neutrosophic network/graph G. The Prim's algorithm will start from the node 1. Initially, $X_n = \{1\}$, Y_n=$\{\emptyset\}$ and $C_{\tilde{M}}$=0.

Step 1. Find all the connected edges with vertex 1 in the first step. The three edges, $(1, 2)$, $(1, 5)$ and $(1, 3)$, are joined with vertex 1. We employ a ranking approach to determine the numeric value associated with three edges. Among them, the smallest one $(1, 2)$ is picked out along with minimum score value. Now, $X_n = 1, 2$ and Y_n=$(1, 2)$.

Step 2. In this second stage, all the arcs must be chosen so that one end vertex is either in $1 or 2$ while the other is in 5,3, or 6. Three edges, $(1, 5)$, $(2, 3)$ and $(2, 6)$ are determined.

Table 1. Arc length of MSTP

Index	Edge	STVN number
1	(1,2)	$\langle(4.6, 5.5, 7.6, 8.9), (0.4, 0.7, 0.2)\rangle$
2	(1,5)	$\langle(4.2, 6.9, 7.5, 8.7), (0.7, 0.2, 0.6)\rangle$
3	(1,3)	$\langle(6.0, 7.6, 8.2, 8.4), (0.4, 0.1, 0.3)\rangle$
4	(2,3)	$\langle(6.1, 6.7, 8.3, 8.7), (0.5, 0.2, 0.4)\rangle$
5	(4,5)	$\langle(4.7, 5.9, 7.2, 7.4), (0.7, 0.2, 0.3)\rangle$
6	(2,6)	$\langle(6.6, 8.8, 10, 12), (0.6, 0.2, 0.2)\rangle$
7	(4,6)	$\langle(6.3, 6.5, 8.9, 8.99), (0.7, 0.4, 0.6)\rangle$
8	(3,4)	$\langle(5.2, 7.9, 9.1, 9.4), (0.6, 0.3, 0.5)\rangle$

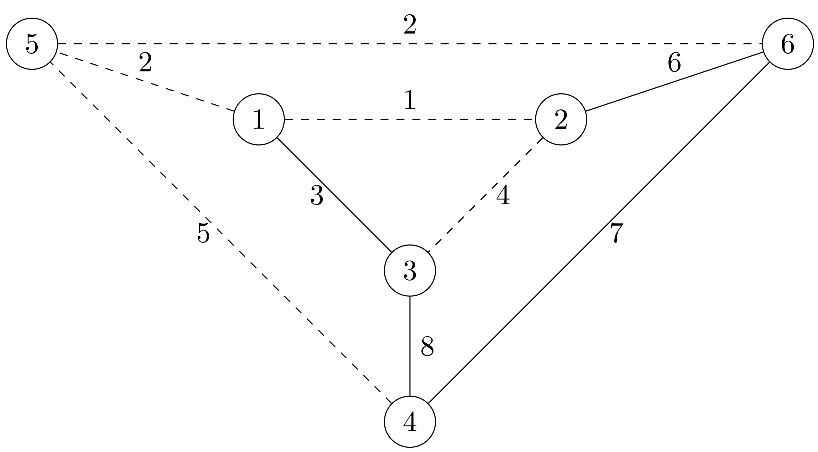

Figure 1. A neutrosophic network/graph

Among this three, the lightest one $(1, 5)$ is selected with their value of score. Now, $X_n = 1, 2, 5$, Y_n=$(1, 2), (1, 5)$.

Step 3. Similarly, we add all other edges of the neutrosophic network/graph. Now, $X_n = 1, 2, 3, 4, 5, 6, D$, Y_n=$(1, 2), (1, 5), (5, 6), (2, 3), (4, 5)$.

A LPP model is also considered to solve NMSTP. To find the solution, LINGO software is employed. We describe our obtained result in Table 2 which is determined by software LINGO. We use a variable $x_{i,j}$=1, if any edge i, j is in the MST. Table 2 also provides a description of the Prim's algorithm solution. We get an identical solution of LINGO and our proposed algorithm.

Table 2. Result of NMSTP

LINGO using Solution	**Prim's algorithm using Solution**
Min Z= 75.06	Cost = 75.06
$x_{23} = 1$, $x_{12} = 1$, $x_{56} = 1$	MSTP=(23)(12)(51)(45)(56)
$x_{15} = 1$,$x_{45} = 1$	

6. Conclusion

The main objective of this paper is to consider MSTP with neutrosophic set and its generalizations. In this study, we investigate the NMSTP, whose arc lengths are characterized by trapezoid neutrosophic numbers. In addition, we discuss the necessity of using the trapezoid neutrosophic number in MSTP. Using trapezoid neutrosophic number for NMSTP, the classical Prim's algorithm is updated to incorporate uncertainty. In order to demonstrate the efficacy of our algorithm, we have included an illustrative numeric instance for clarification. The propose method is practical and easy to use in scenarios found in the real world. The supply chain management, routing, commutation, and other significant fields will be among the next areas to which we attempt to apply our proposed algorithm. It is important to note that there is more uncertainty in the arc length of a neutrosophic graph in NMSTP than just the geometric distance. For instance, even if the geometric distance is set, the travel time between two cities may be represented as a neutrosophic number because of weather and other unforeseen circumstances.

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Neutrosophic interval-valued anti fuzzy linear space

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Abstract. In this research paper, we have introduced the concept of Neutrosophic interval-valued anti fuzzy linear space **(NIVAFLS)** and have also examined its various distinct characteristics. A counter example has demonstrated that the intersection of two Neutrosophic interval-valued anti fuzzy linear spaces **(NIVAFLSs)** does not possess the capability to be a Neutrosophic interval-valued anti fuzzy linear space **(NIVAFLS)**. Conversely, the union of two Neutrosophic interval-valued anti fuzzy linear spaces **(NIVAFLSs)** does form a Neutrosophic interval-valued anti fuzzy linear space **(NIVAFLS)**. Additionally, we have defined and provided an explanation for the cartesian product of two (NIVAFLSs). Furthermore, we have performed a study on the homomorphic image and inverse image of Neutrosophic Anti-Fuzzy Linear Space **(NIVAFLS)**, along with investigating some related properties.

Keywords: Fuzzy linear space; Anti fuzzy linear space; Neutrosophic fuzzy set; Neutrosophic interval-valued anti fuzzy linear space.

1. Introduction

In 1965, Zadeh [1] came up with the concept of "fuzzy set". His pioneering research on coping with uncertainty culminated in a magnificent notion. A membership value has been assigned to each member of a fuzzy set, representing the degree of membership or the degree to which the element belonging to the set. Those membership values range from 0 to 1, with 0 indicating that the element has no membership in the set and 1 indicating full membership. Values between 0 and 1 signify different levels of partial membership. This framework paved the route for an extensive variety of mathematical applications as well as real-world challenges.Later, in 1986, Atanassov [2] developed fuzzy sets theory by introducing the idea of intuitionistic fuzzy sets. Each element in a universe is given a membership value in classic fuzzy sets, ranging from 0 to 1, which indicates how much the element belongs to a specific set. But by introducing the idea of non-membership functions, Atanassov [2] offered a more adaptable framework.. Intuitionistic fuzzy sets give away an expanded framework for decision-making and knowledge portrayal by providing a more complex and powerful tool for resolving ambiguity and uncertainty in real-world scenarios, and Smarandache [3] enacted an entirely novel idea known as Neutrosophic set **(NS)** through the addition of an intermediate membership in 2005. In their research, Arockiarani and Martina [4] delve into the concept of the Neutrosophic set, which is a mathematical paradigm that encompasses values ranging from the subset of $[0, 1]$. This set allows for the representation of uncertain, indeterminate and contradictory information, making it a valuable tool in various fields such as data analysis, pattern recognition and decision-making . Vijayabalaji and Sivaramakrishnan [5, 6] set the standard for the concept of cubic linear space, as well as cubic intuitionistic linear space. The work of Sivaramakrishnann and Vijayabalaji [7] has contributed significantly to the development and understanding of interval-valued anti fuzzy linear space.

Anti fuzzy sets have emerged as a promising alternative to classical fuzzy sets when it comes to dealing with uncertain and imperfect data. In scenarios where the available information is not fully reliable or the data exhibits ambiguity, fuzzy sets may fall short in accurately representing the underlying uncertainty. This is where anti fuzzy sets come into play, providing a different and more robust approach. In the context of interval-valued anti fuzzy sets **(IVAFS)**, the extent of non-membership is expressed through intervals rather than individual values. In this sense, considering a number of alternatives for degrees of non-membership allows for a more adaptive portrayal of uncertainty. **IVAFS** is beneficial in decision-making, risk assessment, pattern detection and other activities that require the successful management of uncertainty and imprecision. Union, intersection and complement operations can be stated for **(IVAFS)** in the same manner as they can for standard fuzzy sets. Because the data is interval-valued, these processes become more complex and may require the inclusion of extra variables.

The amalgamation of Fuzzy Set **(FS)** and Intuitionistic Fuzzy Set **(IFS)** is referred to as Neutrosophic Fuzzy Set (NFS). This mathematical structure that makes it possible to describe inconsistent, ambiguous and incomplete data. Mathematical applications of **NFS** can be found in many fields, including as natural language processing, production planning and scheduling, pattern classification, data mining, data analysis, optimization and decision making. Neutrosophy is concerned with indeterminacy and is made up of three components: truth (T), falsity (F) and indeterminacy (I). Different levels of truth, falsehood and unknowns in a given statement or proposition are indicated by these components: the evidence, the context and the logical reasoning. Neutrosophic fuzzy sets **(NFS)** combine neutrosophy and fuzzy sets to provide a more comprehensive strategy to dealing with unclear and imprecise data. In **(NFS)** theory, In the extension of classical set theory, each element of a set is assigned a membership degree, as well as non-membership and indeterminacy degrees. This enables a more refined depiction of uncertainty and ambiguity in the inclusion of fragments within a set. In **(NIVAFS)**, each element in a set is connected with an interval that indicates the possibility of non-membership (uncertainty) within the neutrosophic context of truth, falsity, and indeterminacy. The **(NIVAFS)** membership function ties universe elements to intervals while taking neutrosophic traits and anti-fuzzy degrees into account. Neutrosophic interval-valued anti fuzzy settings provide an effective way to cope with uncertainty, indeterminacy and imprecision in each of these decision-making challenges, allowing decision-makers to make more complete and informed decisions in difficult real-world scenarios. Because of its adaptability and strength, numerous decision-analysis and problem-solving tasks are ideally suited to this paradigm.

In the realm of algebraic structures, a homomorphism is a function that preserves the operations of two algebraic structures of the same kind. For example, if two groups were homomorphized, the group operation would be maintained. The inverse image of a subset in the codomain is the set of all elements in the domain that map to elements in the given subset, based on a function or mapping between the two sets.

This study presents a methodology for determining the framework of linear space for single-valued Neutrosophic sets. The concepts of Neutrosophic set **(NS)** and interval-valued anti fuzzy setting of linear space **(IVAFLS)** are utilized to define Neutrosophic interval-valued anti fuzzy linear space **(NIVAFLS)**. **(NIVAFLS)** is defined as the union of two Neutrosophic interval-valued anti fuzzy linear spaces **(NIVAFLSs)**. However, the intersection of two **(NIVAFLSs)** may not necessarily be a NIVAFLS. Additionally, the definition and theory for the cartesian product of two **(NIVAFLSs)**, as well as the image and inverse image of a Neutrosophic Anti-Fuzzy Linear Space **(NIVAFLS)**, are established.

2. Preliminaries

Definition 2.1 (3)**.** Assume $\mathbf{X_U}$ is universe of discourse. A **NS** is Ω=$\{v,\ \xi_\Omega(v), \Psi_\Omega(v), \zeta_\Omega(v) | v \in \mathbf{X_U}\}$ or simply denoted by $\Omega = (\xi_\Omega(v), \Psi_\Omega(v), \zeta_\Omega(v))$, where $\xi : \mathbf{X_U} \to [0,1]$, $\Psi : \mathbf{X_U} \to [0,1]$ $\zeta : \mathbf{X_U} \to [0,1]$ the object's degree of truth can be represented through its membership, indeterminacy, and false membership $v \in \mathbf{X_U}$ and $0 \leq \xi_\Omega(v) + \Psi_\Omega(v) + \zeta_\Omega(v) \leq 3$.

Definition 2.2 (6)**.** Let $\mathbf{V_S}$ represent a crisp linear space over a field $\mathbf{F}$, which is symbolized by $(\mathbf{V_S F})$, a mapping $\delta : \mathbf{V_S} \to [0,1]$ is called as an anti fuzzy linear space (**AFLS**) if $\delta(av_1 * bv_2) \leq \mathbf{max}\{\delta(v_1), \delta(v_2)\}, \forall v_1, v_2 \in \mathbf{V_S}$ and $a, b \in \mathbf{F}$ and $*$ is any binary operation on $\mathbf{F}$.

3. Neutrosophic interval-valued anti fuzzy linear space

Definition 3.1. A (**NS**) $\overline{\Omega} = (\overline{\xi}_{\overline{\Omega}}, \overline{\Psi}_{\overline{\Omega}}, \overline{\zeta}_{\overline{\Omega}})$ is known to be a **NIVAFLS** of $\mathbf{V_S}$, if for all $v_1, v_2 \in \mathbf{V_S}$ and $a, b \in \mathbf{F}$, the following holds.

$$(i)\ \overline{\xi}_{\overline{\Omega}}(av_1 * bv_2) \leq \max\{\overline{\xi}_{\overline{\Omega}}(v_1), \overline{\xi}_{\overline{\Omega}}(v_2)\},$$

$$(ii)\ \overline{\Psi}_{\overline{\Omega}}(av_1 * bv_2) \leq \max\{\overline{\Psi}_{\overline{\Omega}}(v_1), \overline{\Psi}_{\overline{\Omega}}(v_2)\},$$

$$(iii)\ \overline{\zeta}_{\overline{\Omega}}(av_1 * bv_2) \geq \min\{\overline{\zeta}_{\overline{\Omega}}(v_1), \overline{\zeta}_{\overline{\Omega}}(v_2)\}$$

Example 3.2. Let $\mathbf{V_S} = \mathbf{R}^2$ be a crisp linear space over a field $\mathbf{R}$ and let **NS** $\overline{\Omega}$= $(\overline{\xi}_{\overline{\Omega}}, \overline{\Psi}_{\overline{\Omega}}, \overline{\zeta}_{\overline{\Omega}})$ be a **NIVAFLS**. For each $v = (v_1, v_2) \in \mathbf{R}^2$, mappings $\overline{\xi}_{\overline{\Omega}} : \mathbf{V_S} \to \mathbf{D}[0,1]$, $\overline{\Psi}_{\overline{\Omega}} : \mathbf{V_S} \to \mathbf{D}[0,1]$ and $\overline{\zeta}_{\overline{\Omega}} : \mathbf{V_S} \to \mathbf{D}[0,1]$ are defined by

$$\overline{\xi}_{\overline{\Omega}}(v) = \begin{cases} [0.8,\ 0.9], & if\ \ v_1 = 0\ or\ v_2 = 0, \\ [0.28,\ 0.31], & otherwise. \end{cases}$$

$$\overline{\Psi}_{\overline{\Omega}}(v) = \begin{cases} [0.73,\ 0.82], & if\ \ v_1 = 0\ or\ v_2 = 0, \\ [0.32,\ 0.41], & otherwise. \end{cases}$$

and

$$\overline{\zeta}_{\overline{\Omega}}(v) = \begin{cases} [0.51,\ 0.6], & if\ \ v_1 = 0\ or\ v_2 = 0, \\ [0.93,\ 1], & otherwise. \end{cases}$$

Clearly, $\overline{\Omega} = (\overline{\xi}_{\overline{\Omega}}, \overline{\Psi}_{\overline{\Omega}}, \overline{\zeta}_{\overline{\Omega}})$ is a **NIVAFLS** in $\mathbf{V_S}$.

Example 3.3. Consider a Klein 4-group $\mathbf{V_S} = \{\epsilon s_1, \epsilon s_2, \epsilon s_3, \epsilon s_4\}$ with the binary operation $*$.

$*$	ϵs_1	ϵs_2	ϵs_3	ϵs_4
ϵs_1	ϵs_1	ϵs_2	ϵs_3	ϵs_4
ϵs_2	ϵs_2	ϵs_1	ϵs_4	ϵs_3
ϵs_3	ϵs_3	ϵs_4	ϵs_1	ϵs_2
ϵs_4	ϵs_4	ϵs_3	ϵs_2	ϵs_1

Assume $\mathbf{F}$ to be a GF(2). Suppose that $(0)w = e$, $(1)w = w$ for all $w \in \mathbf{V_S}$.

Define the mappings $\overline{\xi}_{\overline{\Omega}} : \mathbf{V_S} \to \mathbf{D}[0,1]$, $\overline{\Psi}_{\overline{\Omega}} : \mathbf{V_S} \to \mathbf{D}[0,1]$ and $\overline{\zeta}_{\overline{\Omega}} : \mathbf{V_S} \to \mathbf{D}[0,1]$ by

$$\overline{\xi}_{\overline{\Omega}}(v) = \begin{cases} [0.4,\ 0.5], & if\ \ v = \epsilon s_1, \\ [0.91,\ 1], & otherwise. \end{cases}$$

$$\overline{\Psi}_{\overline{\Omega}}(v) = \begin{cases} [0.22,\ 0.31], & if\ \ v = \epsilon s_1, \\ [0.72,\ 0.9], & otherwise. \end{cases}$$

and

$$\overline{\zeta}_{\overline{\Omega}}(v) = \begin{cases} [0.8,\ 0.9], & if\ \ v = \epsilon s_1, \\ [0.5,\ 0.42], & otherwise. \end{cases}$$

Note that $\overline{\Omega} = (\overline{\xi}_{\overline{\Omega}}, \overline{\Psi}_{\overline{\Omega}}, \overline{\zeta}_{\overline{\Omega}})$ is a **NIVAFLS** in $\mathbf{V_S}$.

Theorem 3.4. *If $\overline{\Omega}_1$ and $\overline{\Omega}_2$ are **NIVAFLSs** of $\mathbf{V_S}$, then the union $\overline{\Omega}_1 \cup \overline{\Omega}_2$ so is.*

Proof. Let v_1 and $v_2 \in \mathbf{V_S}$ and $a, b \in \mathbf{F}$.

Define $\overline{\Omega}_1 \cup \overline{\Omega}_2 = \{\langle \mathrm{v}, \overline{\xi}_{\overline{\Omega}_1 \cup \overline{\Omega}_2}(v), \Psi_{\overline{\Omega}_1 \cup \overline{\Omega}_2}(v), \zeta_{\overline{\Omega}_1 \cup \overline{\Omega}_2}(v) \rangle : v \in \mathbf{V_S}\}$.

Now $(\overline{\xi}_{\overline{\Omega}_1 \cup \overline{\Omega}_2})(av_1 * bv_2) = \max\{\overline{\xi}_{\overline{\Omega}_1}(av_1 * bv_2), \overline{\xi}_{\overline{\Omega}_2}(av_1 * bv_2)\}$

$$\leq \max\{\max[\overline{\xi}_{\overline{\Omega}_1}(v_1), \overline{\xi}_{\overline{\Omega}_1}(v_2)], \max[\overline{\xi}_{\overline{\Omega}_2}(v_1), \overline{\xi}_{\overline{\Omega}_2}(v_2)]\}$$

$$= \max\{\max[\overline{\xi}_{\overline{\Omega}_1}(v_1), \overline{\xi}_{\overline{\Omega}_2}(v_1)], \max[\overline{\xi}_{\overline{\Omega}_1}(v_2), \overline{\xi}_{\overline{\Omega}_2}(v_2)]\}$$

$$\Rightarrow (\overline{\xi}_{\overline{\Omega}_1 \cup \overline{\Omega}_2})(av_1 * bv_2) \leq \max\{(\overline{\xi}_{\overline{\Omega}_1 \cup \overline{\Omega}_2})(v_1), (\overline{\xi}_{\overline{\Omega}_1 \cup \overline{\Omega}_2})(v_2)\}$$

$$(\overline{\Psi}_{\overline{\Omega}_1 \cup \overline{\Omega}_2})(av_1 * bv_2) = \max\{\overline{\Psi}_{\overline{\Omega}_1}(av_1 * bv_2), \overline{\Psi}_{\overline{\Omega}_2}(av_1 * bv_2)\}$$

$$\leq \max\{\max[\overline{\Psi}_{\overline{\Omega}_1}(v_1), \overline{\Psi}_{\overline{\Omega}_1}(v_2)], \max[\overline{\Psi}_{\overline{\Omega}_2}(v_1), \overline{\Psi}_{\overline{\Omega}_2}(v_2)]\}$$

$$= \max\{\max[\overline{\Psi}_{\overline{\Omega}_1}(v_1), \overline{\Psi}_{\overline{\Omega}_2}(v_1)], \max[\overline{\Psi}_{\overline{\Omega}_1}(v_2), \overline{\Psi}_{\overline{\Omega}_2}(v_2)]\}$$

$$\Rightarrow (\overline{\Psi}_{\overline{\Omega}_1 \cup \overline{\Omega}_2})(av_1 * bv_2) \leq \max\{(\overline{\Psi}_{\overline{\Omega}_1 \cup \overline{\Omega}_2})(v_1), (\overline{\Psi}_{\overline{\Omega}_1 \cup \overline{\Omega}_2})(v_2)\}$$

$$(\overline{\zeta}_{\overline{\Omega}_1 \cup \overline{\Omega}_2})(av_1 * bv_2) = \min\{\overline{\zeta}_{\overline{\Omega}_1}(av_1 * bv_2), \overline{\zeta}_{\overline{\Omega}_2}(av_1 * bv_2)\}$$

$$\geq \text{mim}\{\min[\overline{\zeta}_{\overline{\Omega}_1}(v_1), \overline{\zeta}_{\overline{\Omega}_1}(v_2)], \min[\overline{\zeta}_{\overline{\Omega}_2}(v_1), \overline{\zeta}_{\overline{\Omega}_2}(v_2)]\}$$

$$= \min\{\min[\overline{\zeta}_{\overline{\Omega}_1}(v_1), \overline{\zeta}_{\overline{\Omega}_2}(v_1)], \min[\overline{\zeta}_{\overline{\Omega}_1}(v_2), \overline{\zeta}_{\overline{\Omega}_2}(v_2)]\}$$

$$\Rightarrow (\overline{\zeta}_{\overline{\Omega}_1 \cup \overline{\Omega}_2})(av_1 * bv_2) \geq \min\{(\overline{\zeta}_{\overline{\Omega}_1 \cup \overline{\Omega}_2})(v_1), (\overline{\zeta}_{\overline{\Omega}_1 \cup \overline{\Omega}_2})(v_2)\}$$

Thus $(\overline{\Omega}_1 \cup \overline{\Omega}_2)$ is a **NIVAFLS** of $\mathbf{V_S}$.□

Remark 3.5. The intersection of two **(NIVAFLSs)** of $\mathbf{V_S}$ need not be a **(NIVAFLS)** of $\mathbf{V_S}$.

Proof. An example will be used to demonstrate the aforementioned claim.

Let $\mathbf{V_S} = \{\epsilon s_1, \epsilon s_2, \epsilon s_3, \epsilon s_4\}$ be the Klein 4-group as in Example 3.3.

Let $\mathbf{F}$ be the field GF(2). Let $(0)w = e$, $(1)w = w$ for all $w \in \mathbf{V_S}$. Then $\mathbf{V_S}$ is a linear space over $\mathbf{F}$.

Define $\overline{\xi}_{\overline{\Omega}_1}$ and $\overline{\xi}_{\overline{\Omega}_2}$ as follows:

$\overline{\xi}_{\overline{\Omega}_1}(\epsilon s_1) = [0.1, 0.2]$, $\overline{\xi}_{\overline{\Omega}_1}(\epsilon s_2) = [0.6, 0.7] = \overline{\xi}_{\overline{\Omega}_1}(\epsilon s_3)$, $\overline{\xi}_{\overline{\Omega}_1}(\epsilon s_4) = [0.4, 0.5]$ and

$\overline{\xi}_{\overline{\Omega}_2}(\epsilon s_1) = [0.2, 0.3]$, $\overline{\xi}_{\overline{\Omega}_2}(\epsilon s_2) = [0.3, 0.4]$, $\overline{\xi}_{\overline{\Omega}_2}(\epsilon s_3) = [0.5, 0.6] = \overline{\xi}_{\overline{\Omega}_2}(\epsilon s_4)$.

Define $\overline{\xi}_{\overline{\Omega}_1\cap\overline{\Omega}_2}$ by $(\overline{\xi}_{\overline{\Omega}_1\cap\overline{\Omega}_2})(v) = \min\{\overline{\xi}_{\overline{\Omega}_1}(v), \overline{\xi}_{\overline{\Omega}_2}(v)\}$ for all $v \in \mathbf{V_S}$.

So, $(\overline{\xi}_{\overline{\Omega}_1\cap\overline{\Omega}_2})(\epsilon s_1) = [0.1, 0.2]$, $(\overline{\xi}_{\overline{\Omega}_1\cap\overline{\Omega}_2})(\epsilon s_2) = [0.3, 0.4]$,

$(\overline{\xi}_{\overline{\Omega}_1\cap\overline{\Omega}_2})(\epsilon s_3) = [0.5, 0.6]$, $(\overline{\xi}_{\overline{\Omega}_1\cap\overline{\Omega}_2})(\epsilon s_4) = [0.4, 0.5]$.

When $a = b = 1$, then the Definition 3.1 in (i) becomes

$(\overline{\xi}_{\overline{\Omega}_1\cap\overline{\Omega}_2})(\epsilon s_2 * \epsilon s_4) \leq \max\{(\overline{\xi}_{\overline{\Omega}_1\cap\overline{\Omega}_2})(\epsilon s_2), (\overline{\xi}_{\overline{\Omega}_1\cap\overline{\Omega}_2})(\epsilon s_4)\}$

$\Rightarrow (\overline{\xi}_{\overline{\Omega}_1\cap\overline{\Omega}_2})(\epsilon s_3) \leq \max\{[0.3, 0.4], [0.4, 0.5]\}$

But $(\overline{\xi}_{\overline{\Omega}_1\cap\overline{\Omega}_2})(\epsilon s_3) = [0.5, 0.6] \leq [0.4, 0.5]$

This is absurd.

The other inequalities are similarly proved.

So, the intersection of two **NIVAFLSs** need not be a **NIVAFLS**. □

Definition 3.6. The complement of a Neutrosophic fuzzy subset $\overline{\Omega}$ is denoted by $\overline{\Omega}^c$ and is defined by $\overline{\Omega}^c = \{\langle \text{v}, \overline{\xi}_{\overline{\Omega}^c}(v), \overline{\Psi}_{\overline{\Omega}^c}(v), \overline{\zeta}_{\overline{\Omega}^c}(v)\rangle : v \in \mathbf{V_S}\}$, where $\overline{\xi}_{\overline{\Omega}^c}(v) = \overline{\zeta}_{\overline{\Omega}}(v)$, $\overline{\Psi}_{\overline{\Omega}^c}(v) = 1 - \overline{\Psi}_{\overline{\Omega}}(v)$, $\overline{\zeta}_{\overline{\Omega}^c}(v) = \overline{\xi}_{\overline{\Omega}}(v)$, for all $v \in \mathbf{V_S}$.

Theorem 3.7. *If $\overline{\Omega}$ be a Neutrosophic fuzzy linear space of $\boldsymbol{V_S}$ then its complement $\overline{\Omega}^c$ is a **NIVAFLS** of $\boldsymbol{V_S}$.*

Proof. Let v_1 and $v_2 \in \mathbf{V_S}$ and $a, b \in \mathbf{F}$.

$$\overline{\xi}_{\overline{\Omega}^c}(av_1 * bv_2) = \overline{\zeta}_{\overline{\Omega}}(av_1 * bv_2)$$

$$\leq \max\{\overline{\zeta}_{\overline{\Omega}}(v_1), \overline{\zeta}_{\overline{\Omega}}(v_2)\}$$

$$= \max\{\overline{\xi}_{\overline{\Omega}^c}(v_1), \xi_{\overline{\Omega}^c}(v_2)\}$$

$$\overline{\Psi}_{\overline{\Omega}^c}(av_1 * bv_2) = 1 - \overline{\Psi}_{\overline{\Omega}}(av_1 * bv_2)$$

$$= 1- \leq \min\{\overline{\Psi}_{\overline{\Omega}}(v_1), \overline{\Psi}_{\overline{\Omega}}(v_2)\}$$

$$= \max\{1 - \overline{\Psi}_{\overline{\Omega}}(v_1), 1 - \overline{\Psi}_{\overline{\Omega}}(v_2)\}$$

$$= \max\{\overline{\Psi}_{\overline{\Omega}^c}(v_1), \overline{\Psi}_{\overline{\Omega}^c}(v_2)\}$$

$$\overline{\zeta}_{\overline{\Omega}^c}(av_1 * bv_2) = \overline{\xi}_{\overline{\Omega}}(av_1 * bv_2)$$

$$\geq \min\{\overline{\xi}_{\overline{\Omega}}(v_1), \overline{\xi}_{\overline{\Omega}}(v_2)\}$$

$$\geq \min\{\overline{\zeta}_{\overline{\Omega}^c}(v_1), \overline{\zeta}_{\overline{\Omega}^c}(v_2)\}$$

So, $\overline{\Omega}^c$ is a **NIVAFLS** of $\mathbf{V_S}$.□

Definition 3.8. Let $\overline{\Omega}_1$, $\overline{\Omega}_2$ be Neutrosophic anti fuzzy subsets of $\mathbf{V_{S1}}$ and $\mathbf{V_{S2}}$ respectively. Then the cartesian product of $\overline{\Omega}_1$ and $\overline{\Omega}_2$ denoted by $\overline{\Omega}_1 \times \overline{\Omega}_2$ is defined by
$\overline{\Omega}_1 \times \overline{\Omega}_2 = \{\langle(v_1 \times v_2), \xi_{\overline{\Omega}_1\times\overline{\Omega}_2}(v_1, v_2), \overline{\Psi}_{\overline{\Omega}_1\times\overline{\Omega}_2}(v_1, v_2), \overline{\zeta}_{\overline{\Omega}_1\times\overline{\Omega}_2}(v_1, v_2)\rangle : v_1 \in \mathbf{V_{S1}}, v_2 \in \mathbf{V_{S2}}\}$, where $\xi_{\overline{\Omega}_1\times\overline{\Omega}_2}(v_1, v_2) = \max\{\overline{\xi}_{\overline{\Omega}_1}(v_1), \overline{\xi}_{\overline{\Omega}_2}(v_2)\}$, $\overline{\Psi}_{\overline{\Omega}_1\times\overline{\Omega}_2}(v_1, v_2) = \max\{\overline{\Psi}_{\overline{\Omega}_1}(v_1), \overline{\Psi}_{\overline{\Omega}_2}(v_2)\}$ and $\overline{\zeta}_{\overline{\Omega}_1\times\overline{\Omega}_2}(v_1, v_2) = \min\{\overline{\zeta}_{\overline{\Omega}_1}(v_1), \overline{\zeta}_{\overline{\Omega}_2}(v_2)\}$.

Theorem 3.9. *If $\overline{\Omega}_1$ and $\overline{\Omega}_2$ are **NIVAFLSs** of $\boldsymbol{V_S}$, then $(\overline{\Omega}_1 \times \overline{\Omega}_2)$ is a **NIVAFLS** of $\boldsymbol{V_{S1}} \times \boldsymbol{V_{S2}}$.*

Proof. Let $v = (v_1, v_2)$, $w = (w_1, w_2) \in \mathbf{V_{S1}} \times \mathbf{V_{S2}}$. Then

$$\overline{\xi}_{\overline{\Omega}_1\times\overline{\Omega}_2}(av * bw) = \overline{\xi}_{\overline{\Omega}_1\times\overline{\Omega}_2}(a(v_1, v_2) * b(w_1, w_2))$$

$$= \overline{\xi}_{\overline{\Omega}_1\times\overline{\Omega}_2}((av_1 * bw_1), (av_2 * bw_2))$$

$$= \max\{\overline{\xi}_{\overline{\Omega}_1}(av_1 * bw_1), \overline{\xi}_{\overline{\Omega}_2}(av_2 * bw_2)\}$$

$$\leq \max\{\max[\overline{\xi}_{\overline{\Omega}_1}(v_1), \overline{\xi}_{\overline{\Omega}_1}(w_1)], \max[\overline{\xi}_{\overline{\Omega}_2}(v_2), \overline{\xi}_{\overline{\Omega}_2}(w_2)]\}$$

$$= \max\{\max[\overline{\xi}_{\overline{\Omega}_1}(v_1), \overline{\xi}_{\overline{\Omega}_2}(v_2)], \max[\overline{\xi}_{\overline{\Omega}_1}(w_1), \overline{\xi}_{\overline{\Omega}_2}(w_2)]\}$$

$$= \max\{\overline{\xi}_{\overline{\Omega}_1\times\overline{\Omega}_2}(v_1, v_2), \overline{\xi}_{\overline{\Omega}_1\times\overline{\Omega}_2}(w_1, w_2)\}$$

$$= \max\{\overline{\xi}_{\overline{\Omega}_1 \times \overline{\Omega}_2}(v), \overline{\xi}_{\overline{\Omega}_1 \times \overline{\Omega}_2}(w)\}$$

$$\overline{\Psi}_{\overline{\Omega}_1 \times \overline{\Omega}_2}(av * bw) = \overline{\Psi}_{\overline{\Omega}_1 \times \overline{\Omega}_2}(a(v_1, v_2) * b(w_1, w_2))$$

$$= \overline{\Psi}_{\overline{\Omega}_1 \times \overline{\Omega}_2}((av_1 * bw_1), (av_2 * bw_2))$$

$$= \max\{\overline{\Psi}_{\overline{\Omega}_1}(av_1 * bw_1), \overline{\Psi}_{\overline{\Omega}_2}(av_2 * bw_2)\}$$

$$\leq \max\{\max[\overline{\Psi}_{\overline{\Omega}_1}(v_1), \overline{\Psi}_{\overline{\Omega}_1}(w_1)], \max[\overline{\Psi}_{\overline{\Omega}_2}(v_2), \overline{\Psi}_{\overline{\Omega}_2}(w_2)]\}$$

$$= \max\{\max[\overline{\Psi}_{\overline{\Omega}_1}(v_1), \overline{\Psi}_{\overline{\Omega}_2}(v_2)], \max[\overline{\Psi}_{\overline{\Omega}_1}(w_1), \overline{\Psi}_{\overline{\Omega}_2}(w_2)]\}$$

$$= \max\{\overline{\Psi}_{\overline{\Omega}_1 \times \overline{\Omega}_2}(v_1, v_2), \overline{\Psi}_{\overline{\Omega}_1 \times \overline{\Omega}_2}(w_1, w_2)\}$$

$$= \max\{\overline{\Psi}_{\overline{\Omega}_1 \times \overline{\Omega}_2}(v), \overline{\Psi}_{\overline{\Omega}_1 \times \overline{\Omega}_2}(w)\}$$

$$\overline{\zeta}_{\overline{\Omega}_1 \times \overline{\Omega}_2}(av * bw) = \overline{\zeta}_{\overline{\Omega}_1 \times \overline{\Omega}_2}(a(v_1, v_2) * b(w_1, w_2))$$

$$= \overline{\zeta}_{\overline{\Omega}_1 \times \overline{\Omega}_2}((av_1 * bw_1), (av_2 * bw_2))$$

$$= \min\{\overline{\zeta}_{\overline{\Omega}_1}(av_1 * bw_1), \overline{\zeta}_{\overline{\Omega}_2}(av_2 * bw_2)\}$$

$$\geq \min\{\min[\overline{\zeta}_{\overline{\Omega}_1}(v_1), \overline{\zeta}_{\overline{\Omega}_1}(w_1)], \min[\overline{\zeta}_{\overline{\Omega}_2}(v_2), \overline{\zeta}_{\overline{\Omega}_2}(w_2)]\}$$

$$= \min\{\min[\overline{\zeta}_{\overline{\Omega}_1}(v_1), \overline{\zeta}_{\overline{\Omega}_2}(v_2)], \min[\overline{\zeta}_{\overline{\Omega}_1}(w_1), \overline{\zeta}_{\overline{\Omega}_2}(w_2)]\}$$

$$= \min\{\overline{\zeta}_{\overline{\Omega}_1 \times \overline{\Omega}_2}(v_1, v_2), \overline{\zeta}_{\overline{\Omega}_1 \times \overline{\Omega}_2}(w_1, w_2)\}$$

$$= \min\{\overline{\zeta}_{\overline{\Omega}_1 \times \overline{\Omega}_2}(v), \overline{\zeta}_{\overline{\Omega}_1 \times \overline{\Omega}_2}(w)\}$$

So, $(\overline{\Omega}_1 \times \overline{\Omega}_2)$ is a **NIVAFLS** of $\mathbf{V_{S1}} \times \mathbf{V_{S2}}$. □

Definition 3.10. Let $\varpi : \mathbf{V_{S_1}} \to \mathbf{V_{S_2}}$ be a mapping of linear spaces of $\mathbf{V_S}$ over $\mathbf{F}$. If $\overline{\Omega} = (\overline{\xi}_{\overline{\Omega}}, \overline{\Psi}_{\overline{\Omega}}, \overline{\zeta}_{\overline{\Omega}})$ is a **NIVAFLS** of $\mathbf{V_{S2}}$ over $\mathbf{F}$, then the inverse image of $\overline{\Omega} = (\overline{\xi}_{\overline{\Omega}}, \overline{\Psi}_{\overline{\Omega}}, \overline{\zeta}_{\overline{\Omega}})$ under ϖ, denoted by $\varpi^{-1}(\overline{\Omega}) = (\varpi^{-1}(\overline{\xi}_{\overline{\Omega}}), \varpi^{-1}(\overline{\Psi}_{\overline{\Omega}}), \varpi^{-1}(\overline{\zeta}_{\overline{\Omega}}))$, is a **NIVAFLS** of $\mathbf{V_{S1}}$, defined by $\varpi^{-1}(\overline{\Omega})(x) = \overline{\Omega}(\varpi(x)) = (\overline{\xi}_{\overline{\Omega}}(\varpi(x)), \overline{\Psi}_{\overline{\Omega}}(\varpi(x)), \overline{\zeta}_{\overline{\Omega}}(\varpi(x)))$ for all $x \in \mathbf{V_{S1}}$.

Theorem 3.11. *Consider the homomorphism $\varpi : \boldsymbol{V_S}_1 \to \boldsymbol{V_S}_2$, which represents the mapping between linear spaces $\boldsymbol{V_S F}$. If $\overline{\Omega} = (\overline{\xi}_{\overline{\Omega}}, \overline{\Psi}_{\overline{\Omega}}, \overline{\zeta}_{\overline{\Omega}})$ is a* ***NIVAFLS*** *of $\boldsymbol{V_S}_2$, then $\varpi^{-1}(\overline{\Omega})(x) = \overline{\Omega}(\varpi(x)) = (\overline{\xi}_{\overline{\Omega}}(\varpi(x)), \overline{\Psi}_{\overline{\Omega}}(\varpi(x)), \overline{\zeta}_{\overline{\Omega}}(\varpi(x)))$ for every x that belongs to $\boldsymbol{V_S}_1$.*

Proof. Suppose that $\overline{\Omega} = (\overline{\xi}_{\overline{\Omega}}, \overline{\Psi}_{\overline{\Omega}}, \overline{\zeta}_{\overline{\Omega}})$ is a **NIVAFLS** of $\mathbf{V_{S2}}$ and x and y belong to $\mathbf{V_{S1}}$ and a and b belong to $\mathbf{F}$.
Next, we have

$$\begin{aligned}
\text{(i)}\varpi^{-1}(\overline{\xi}_{\overline{\Omega}})(ax*by) &= \overline{\xi}_{\overline{\Omega}}(\varpi(ax*by)) \\
&= \overline{\xi}_{\overline{\Omega}}(\varpi(x)\varpi(y)) \text{ (since } \varpi \text{ is homomorphism)} \\
&\leq \max\{\overline{\xi}_{\overline{\Omega}}(\varpi(x)), \overline{\xi}_{\overline{\Omega}}(\varpi(y))\} \\
&= \max\{\varpi^{-1}(\overline{\xi}_{\overline{\Omega}}(x)),\ \varpi^{-1}(\overline{\xi}_{\overline{\Omega}}(y))\} \\
\Rightarrow \varpi^{-1}(\overline{\xi}_{\overline{\Omega}})(ax*by) &\leq \max\{\varpi^{-1}(\overline{\xi}_{\overline{\Omega}}(x)),\ \varpi^{-1}(\overline{\xi}_{\overline{\Omega}}(y))\}
\end{aligned}$$

Therefore $\varpi^{-1}(\overline{\xi}_{\overline{\Omega}})$ is a **NIVAFLS** of $\mathbf{V_{S1}}$.

$$\begin{aligned}
\text{(ii)}\varpi^{-1}(\overline{\Psi}_{\overline{\Omega}})(ax*by) &= \overline{\Psi}_{\overline{\Omega}}(\varpi(ax*by)) \\
&= \overline{\Psi}_{\overline{\Omega}}(\varpi(x)\varpi(y)) \text{ (since } \varpi \text{ is homomorphism)} \\
&\leq \max\{\overline{\Psi}_{\overline{\Omega}}(\varpi(x)), \overline{\Psi}_{\overline{\Omega}}(\varpi(y))\} \\
&= \max\{\varpi^{-1}(\overline{\Psi}_{\overline{\Omega}}(x)),\ \varpi^{-1}(\overline{\Psi}_{\overline{\Omega}}(y))\} \\
\Rightarrow \varpi^{-1}(\overline{\Psi}_{\overline{\Omega}})(ax*by) &\leq \max\{\varpi^{-1}(\overline{\Psi}_{\overline{\Omega}}(x)),\ \varpi^{-1}(\overline{\Psi}_{\overline{\Omega}}(y))\}
\end{aligned}$$

Therefore $\varpi^{-1}(\overline{\Psi}_{\overline{\Omega}})$ is a **NIVAFLS** of $\mathbf{V_{S1}}$.

$$\begin{aligned}
\text{(iii)}\varpi^{-1}(\overline{\zeta}_{\overline{\Omega}})(ax*by) &= \overline{\zeta}_{\overline{\Omega}}(\varpi(ax*by)) \\
&= \overline{\zeta}_{\overline{\Omega}}(\varpi(x)\varpi(y)) \text{ (since } \varpi \text{ is homomorphism)} \\
&\geq \min\{\overline{\zeta}_{\overline{\Omega}}(\varpi(x)), \overline{\zeta}_{\overline{\Omega}}(\varpi(y))\} \\
&= \min\{\varpi^{-1}(\overline{\zeta}_{\overline{\Omega}}(x)),\ \varpi^{-1}(\overline{\zeta}_{\overline{\Omega}}(y))\} \\
\Rightarrow \varpi^{-1}(\overline{\zeta}_{\overline{\Omega}})(ax*by) &\geq \min\{\varpi^{-1}(\overline{\zeta}_{\overline{\Omega}}(x)),\ \varpi^{-1}(\overline{\zeta}_{\overline{\Omega}}(y))\}
\end{aligned}$$

Therefore $\varpi^{-1}(\overline{\zeta}_{\overline{\Omega}})$ is a **NIVAFLS** of $\mathbf{V_{S1}}$. □

Theorem 3.12. *Let $\overline{\Omega} = (\overline{\xi}_{\overline{\Omega}}, \overline{\Psi}_{\overline{\Omega}}, \overline{\zeta}_{\overline{\Omega}})$ be a* ***NIVAFLS*** *of $\boldsymbol{V_S}$ and an onto homomorphism $\varpi : \boldsymbol{V_S} \to \boldsymbol{V_S}$. Subsequently, the mapping $\overline{\Omega}^{\varpi} : \boldsymbol{V_S} \to \boldsymbol{D}[0,1]$ is defined as follows: for every $x \in \overline{\Omega}^{\varpi}(x) = \overline{\Omega}(\varpi(x))$ for all $x \in \boldsymbol{V_S}$ is a* ***NIVAFLS*** *of $\boldsymbol{V_S}$.*

Proof. For any $x, y \in \mathbf{V_S}$ and $a, b \in \mathbf{F}$.

$$\begin{aligned}
(\text{i})\overline{\xi}^{\varpi}_{\overline{\Omega}}(ax*by) &= \overline{\xi}_{\overline{\Omega}}(\varpi(ax*by)) \\
&= \overline{\xi}_{\overline{\Omega}}(\varpi(x)\varpi(y)) \text{ (since } \varpi \text{ is homomorphism)} \\
&\leq \max\{\overline{\xi}_{\overline{\Omega}}(\varpi(x)), \overline{\xi}_{\overline{\Omega}}(\varpi(y))\} \\
&= \max\{\overline{\xi}^{\varpi}_{\overline{\Omega}}(x), \overline{\xi}^{\varpi}_{\overline{\Omega}}(y)\} \\
\Rightarrow \overline{\xi}^{\varpi}_{\overline{\Omega}}(ax*by) &\leq \max\{\overline{\xi}^{\varpi}_{\overline{\Omega}}(x), \overline{\xi}^{\varpi}_{\overline{\Omega}}(y)\}.
\end{aligned}$$

$$\begin{aligned}
(\text{ii})\overline{\Psi}^{\varpi}_{\overline{\Omega}}(ax*by) &= \overline{\Psi}_{\overline{\Omega}}(\varpi(ax*by)) \\
&= \overline{\Psi}_{\overline{\Omega}}(\varpi(x)\varpi(y)) \text{ (since } \varpi \text{ is homomorphism)} \\
&\leq \max\{\overline{\Psi}_{\overline{\Omega}}(\varpi(x)), \overline{\Psi}_{\overline{\Omega}}(\varpi(y))\} \\
&= \max\{\overline{\Psi}^{\varpi}_{\overline{\Omega}}(x), \overline{\Psi}^{\varpi}_{\overline{\Omega}}(y)\} \\
\Rightarrow \overline{\Psi}^{\varpi}_{\overline{\Omega}}(ax*by) &\leq \max\{\overline{\Psi}^{\varpi}_{\overline{\Omega}}(x), \overline{\Psi}^{\varpi}_{\overline{\Omega}}(y)\}.
\end{aligned}$$

$$\begin{aligned}
(\text{iii})\overline{\zeta}^{\varpi}_{\overline{\Omega}}(ax*by) &= \overline{\zeta}_{\overline{\Omega}}(\varpi(ax*by)) \\
&= \overline{\zeta}_{\overline{\Omega}}(\varpi(x)\varpi(y)) \text{ (since } \varpi \text{ is homomorphism)} \\
&\leq \max\{\overline{\zeta}_{\overline{\Omega}}(\varpi(x)), \overline{\zeta}_{\overline{\Omega}}(\varpi(y))\} \\
&= \max\{\overline{\zeta}^{\varpi}_{\overline{\Omega}}(x), \overline{\zeta}^{\varpi}_{\overline{\Omega}}(y)\} \\
\Rightarrow \overline{\zeta}^{\varpi}_{\overline{\Omega}}(ax*by) &\leq \max\{\overline{\zeta}^{\varpi}_{\overline{\Omega}}(x), \overline{\zeta}^{\varpi}_{\overline{\Omega}}(y)\}.
\end{aligned}$$

So, $\overline{\Omega}^{\varpi}$ is a **NIVAFLS** of $\mathbf{V_S}$. □

Theorem 3.13. *Consider an epimorphism $\varpi : \boldsymbol{V_{S1}} \to \boldsymbol{V_{S2}}$ that maps linear spaces $\boldsymbol{V_{S1}}$ and $\boldsymbol{V_{S2}}$ over $\boldsymbol{F}$. Let's assume that$\overline{\Omega} = (\overline{\xi}_{\overline{\Omega}}, \overline{\Psi}_{\overline{\Omega}}, \overline{\zeta}_{\overline{\Omega}})$ be a ϖ-invariant **NIVAFLS** of $\boldsymbol{V_{S1}}$. Consequently, $\varpi(\overline{\Omega})$ is a **NIVAFLS** of $\boldsymbol{V_{S2}}$.*

Proof. For any elements $x^{'}$ and $y^{'}$ belonging to $\mathbf{V_{S2}}$ and a and b belonging to $\mathbf{F}$, there exists x and y belonging to $\mathbf{V_{S1}}$ such that $\varpi(x)$ is equals $x^{'}$ and $\varpi(y)$ equals $y^{'}$.

Also $ax^{'} * by^{'} = \varpi(ax*by)$. Since $\overline{\Omega}$ is ϖ-invariant,

$$\begin{aligned}
(\text{i})\varpi(\overline{\xi}_{\overline{\Omega}})(ax*by) &= \overline{\xi}_{\overline{\Omega}}(ax^{'} * by^{'}) \leq \max\{\overline{\xi}_{\overline{\Omega}}(x^{'}), \overline{\xi}_{\overline{\Omega}}(y^{'})\} \\
&= \max\{\varpi(\overline{\xi}_{\overline{\Omega}})(x),\ \varpi(\overline{\xi}_{\overline{\Omega}})(y)\} \\
\Rightarrow \varpi(\overline{\xi}_{\overline{\Omega}})(ax*by) &\leq \max\{\varpi(\overline{\xi}_{\overline{\Omega}})(x), \varpi(\overline{\xi}_{\overline{\Omega}})(y)\}
\end{aligned}$$

Therefore $\varpi(\bar{\xi}_{\overline{\Omega}})$ is a **NIVAFLS** of $\mathbf{V_{S2}}$.

$$\begin{aligned}(\text{ii})\varpi(\overline{\Psi}_{\overline{\Omega}})(ax*by) &= \overline{\Psi}_{\overline{\Omega}}(ax^{'}*by^{'}) \leq \max\{\overline{\Psi}_{\overline{\Omega}}(x^{'}), \overline{\Psi}_{\overline{\Omega}}(y^{'})\} \\ &= \max\{\varpi(\overline{\Psi}_{\overline{\Omega}})(x),\ \varpi(\overline{\Psi}_{\overline{\Omega}})(y)\} \\ \Rightarrow \varpi(\overline{\Psi}_{\overline{\Omega}})(ax*by) &\leq \max\{\varpi(\overline{\Psi}_{\overline{\Omega}})(x), \varpi(\overline{\Psi}_{\overline{\Omega}})(y)\}\end{aligned}$$

Therefore $\varpi(\overline{\Psi}_{\overline{\Omega}})$ is a **NIVAFLS** of $\mathbf{V_{S2}}$.

$$\begin{aligned}(iii)\varpi(\bar{\zeta}_{\overline{\Omega}})(ax*by) &= \bar{\zeta}_{\overline{\Omega}}(ax^{'}*by^{'}) \geq \min\{\bar{\zeta}_{\overline{\Omega}}(x^{'}), \bar{\zeta}_{\overline{\Omega}}(y^{'})\} \\ &= \min\{\varpi(\bar{\zeta}_{\overline{\Omega}})(x),\ \varpi(\bar{\zeta}_{\overline{\Omega}})(y)\} \\ \Rightarrow \varpi(\bar{\zeta}_{\overline{\Omega}})(ax*by) &\geq \min\{\varpi(\bar{\zeta}_{\overline{\Omega}})(x), \varpi(\bar{\zeta}_{\overline{\Omega}})(y)\}\end{aligned}$$

Therefore $\varpi(\bar{\zeta}_{\overline{\Omega}})$ is a **NIVAFLS** of $\mathbf{V_{S2}}$. □

4. Conclusion

The present paper introduces a novel concept referred to as a **NIVAFLS**. A counterexample is employed to demonstrate that the intersection of two **NIVAFLSs** need not be a **NIVAFLS**, and we examine into some of the aspects of **NIVAFLS** to show that the union of two **NIVAFLSs** is likewise a **NIVAFLS**.

In the future, we will apply this idea to different algebraic structures like

- semigroup,
- M-semigroup,
- ring,
- rough set,
- soft set together with problems based on decision-makings.

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Sine Trigonometric Aggregation Operators with Single-Valued Neutrosophic Z-Numbers: Application in Business Site Selection

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Abstract. Business has a massive effect on both people and society as a whole in today's culture. Businesses make significant contributions to humanity's general well-being and progress by promoting economic growth, supporting innovation, producing wealth, and addressing social challenges. However, choosing the right location for a business is a complex task, involving multiple criteria and qualitative and quantitative factors that heavily rely on expert judgement. The researchers propose a unique approach called the SVNZN multi-attributed decision-making method to aid decision-makers in this process. They introduce new operating laws for SVNZNs based on the sine trigonometric (ST) function, known for its periodicity and symmetry over the origin, making it favorable for decision-makers over multi-time phase parameters. Additionally, novel AOs such as SVNZN weighted averaging and geometric operators are defined to combine SVNZNs effectively. Based on these AOs, a decision-making technique for MADM problems is presented, and its applicability is demonstrated through a numerical example of selecting the best location for a business. To validate and enhance the understanding of these proposed techniques, the researchers conducted a comprehensive comparison analysis, including a sensitivity analysis, considering existing literature on MADM difficulties. Figure 1 provides a thorough graphical summary by visually representing all of the contributions and outcomes.

Keywords: Single-valued neutrosophic set, Decision Making, Sine Trigonometric aggregation operator, Z-number.

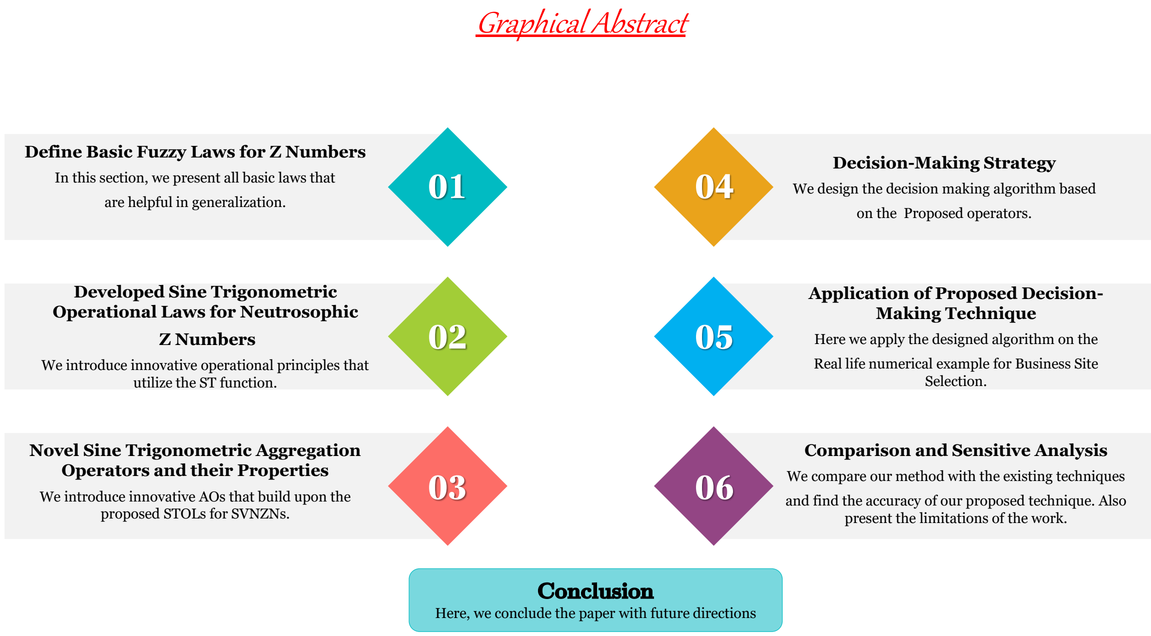

FIGURE 1. Graphical Abstract

We express specific symbols in Table 1, that will be used throughout this work.

TABLE 1. Abbreviations and Descriptions of this Manuscript

Abbreviation	Description	Abbreviation	Description
FS	fuzzy set	ZN	Z-number
FZN	fuzzy ZN	SFZN	spherical FZN
SVNZN	single-valued neutrosophic ZN	AO	aggregation operator
MADM	multi-attribute decision-making	PFZN	picture FZN
ST	sine trigonometric	STAO	ST Aggregation Operator
SVN	single-valued neutrosophic	STOL	ST operational law
DMP	decision-making problem	DMk	decision-maker
WV	weight vector	DM	decision-making

1. Introduction

Research is crucial before deciding on a site for a company, since this choice may affect the organization's long-term performance in significant ways. Before making a final selection, it's important to thoroughly analyze many factors to make sure the selected site can match the organization's goals and operational needs. Numerous important factors must be considered while making decisions; they include, but are not limited to, demographics, infrastructure, and market availability. For instance, several significant factors directly impact a corporation's cost-effectiveness and operational efficiency [2, 3]. The accessibility of trained workers, public transit systems, suppliers, and buyers are all examples of such factors. To guarantee compliance and avoid hazards, more research on legal and regulatory aspects, including zoning limits and tax incentives, is required. Gathering and analyzing as much data as possible is crucial for making decisions; this data must account for market trends, economic patterns, and any hazards that may be present. To gain a variety of opinions and support from everyone involved, it is also important to solicit feedback and participation from stakeholder. According to [4, 5], companies may improve their operational efficiency, promote sustainability, and secure long-term success by thoroughly analyzing these complex challenges and making wise choices.

An additional layer of complexity is added to the decision-making (DM) process when fuzzy set (FS) theory is applied to the process of selecting a location for a company. In the field of FS theory, it is acknowledged that several location criteria may include inherent ambiguities and imprecisions [6]. The application of fuzzy logic assists decision-makers (DMks) in taking into account the complexity of the situation, which is a result of the fact that the variables that exist in the real world are often not binary but rather exist on a spectrum. When it comes to matters like the availability of competent people or a market, for example, there may not be any clear regulations in place. Decision-makers are able to define these features using degrees thanks to FS theory, which enables a representation that is both more realistic and flexible [7] than traditional methods. The provision of a framework for coping with ambiguity is one of the ways in which this strategy improves DM, particularly in the setting of fast-paced commercial environments. The use of FS theory allows companies to make better and more context-aware judgments throughout the site selection process. This helps to guarantee that the selected location is the most suitable for the achievement of their objectives and the fulfillment of their operational requirements [8–10].

In 1965, Zadeh was the first person to present the idea of an FS, which was a generalization of crisp sets. Through the process of providing a membership value between 0 and 1 to each element, FSs provide a unique viewpoint. This value indicates the degree to which the element is associated with the set. The amendment was made in order to provide an achievable solution that could effectively address the inherent ambiguity and uncertainty that is present in reasoning and DM processes [11]. When it comes to processing complicated and imprecise information, embracing FSs enables a more adaptive and intelligent approach. This, in turn, enables a more exact depiction of real-world occurrences and facilitates well-informed decision making in contexts that are unpredictable. The versatility of FSs becomes clear via their wide-ranging applications in many domains, including artificial intelligence [12], control systems [13], pattern recognition [14], decision analysis [15], and decision modeling [16]. Decision modeling is exactly what it sounds like: a strategy. It is a method that is deliberate and planned, and it is used to organize and visualize judgments as well as the qualities that are associated with them. In order to do this, it is required to develop models that include the fundamental aspects of a DM problem (DMP), such as objectives, alternatives, constraints, unpredictability, and interests [17, 18]. The first step in decision modeling is called issue identification, and it involves expressing the choice problem in a clear and simple way while also outlining particular objectives [19]. This phase is when the decision modeling process begins. During this particular phase, it is important to possess a thorough comprehension of the context, to identify the most significant challenges, and to describe the objectives that the process of selecting is intended to achieve [20]. The term "distinctive decision simulation," on the other hand, refers to the use of novel strategies, approaches, or methods within the realm of decision modeling. The process involves experimenting with unique and creative methods of expressing, evaluating, and addressing decision difficulties, often via the use of technological, data analytical, and computational advances. Some examples of inventive document management methods are shown here [21, 22]. Using intelligence-based techniques and predictive algorithms, you may develop decision models that are able to learn from data, make predictions, and optimize the consequences of decisions. Over time, these models are able to autonomously modify and improve themselves based on fresh information and input being received. Leverage the power of big data to provide DM with information. It is important to include vast and complicated data sets from a variety of sources into decision models. Some examples of these sources are social media, sensors, and transaction logs. Obtaining insights and providing support for DM processes may be accomplished via the use of sophisticated data analytic methods like as data mining, predictive modeling, and pattern recognition.

There are two components that make up the representation of a Z-number (ZN), which is represented as $Z = (X, Y)$. These components are described in more detail below. In the case of a real-valued uncertain variable, the first component, which is represented by the letter X, serves as a constraint that determines the range of values that are acceptable with regard to the variable. The second component, which is represented by the letter Y, is accountable for measuring the degree of dependability or confidence that is associated with the information that was provided by the first component. This is the responsible party. Kang et al. [24] were the ones who did the first presentation of fuzzy ZNs (FZNs), while Sari and Kahraman [25] were the ones who developed intuitionistic FZNs. Pythagorean FZNs were presented in [26], and [27] detailed a number of operations via their description.

Each element in an SVN set is given a truth-membership degree denoted by $(\mathfrak{T})$, an indeterminacy-membership degree denoted by $(\mathfrak{I})$, and a falsity-membership degree denoted by ($\mathfrak{F}$), which adds up to 3. The depiction of uncertain and ambiguous features of an element is made easier by these degrees, which also provide a means of accommodating the many degrees of truth and falsehood that are associated with the element. Within the frames of ZNs, FZNs, intuitionistic FZNs, and Pythagorean FZNs, we provide single-valued neutrosophic ZN (SVNZN), a novel notion that enhances the capabilities of these existing frameworks. The formation of SVNZNs involves the combination of two well-known sets, namely SVN sets and ZNs. By combining a number of different frameworks, SVNZNs provide a strong instrument for dealing with the uncertainty and indeterminacy that are present in DM situations. As a result of the introduction of SVNZN, there are now more choices available for dealing with complex DM scenarios that include a variety of different types of uncertainty. This unique idea expands the scope of previously established approaches while simultaneously enhancing the capability to accurately and totally imitate and comprehend occurrences that occur in the actual world. The symbol Z is used to indicate an SVNZN, which is composed of two components: $Z = (X, Y)$ respectively. The first component is made up of three different elements: the truth-membership ($\mathfrak{T}_{\mathfrak{Y}}$), the indeterminacy-membership ($\mathfrak{I}_{\mathfrak{Y}}$), and the falsity-membership ($\mathfrak{F}_{\mathfrak{Y}}$) degree, with the sum of these three components being equal to three. As an alternative, the second component, which is denoted by the symbol Y, is responsible for assessing the level of trustworthiness or dependability that is connected to the information provided by the first component. Additionally, it is composed of three elements, namely ($\mathfrak{T}_{\mathfrak{R}}$), ($\mathfrak{I}_{\mathfrak{R}}$), and ($\mathfrak{F}_{\mathfrak{R}}$), with the sum of these three factors being equal to 3. When used in DM environments where uncertainty and indeterminacy coexist, SVNZNs provide a method that is both comprehensive and rigorous, designed to successfully resolve choice difficulties that occur in the real world. As a consequence of SVNZs, decision makers have access to a more flexible way of presenting and handling complicated data, which ultimately leads to more informed judgments. The procedures and computations that are performed by ZN and SVN sets, which include arithmetic, comparison, and aggregation operations, are comparable to those that are performed by SVNZ. The last point is that SVNZNs provide a flexible framework for dealing with uncertainty and indeterminacy in DM. This framework enables DMks to take into consideration a variety of views and arrive at more robust conclusions.

Motivation and Novelty: It is standard practice to use ST Aggregation Operators (STAOs) in multi-criteria decision making and analysis. As a result of their reliance on the mathematical concept of the sine function, these operators are especially adept at dealing with input that is both perplexing and wrong. STAOs provide a single overall value or preference rating by aggregating individual assessments or criteria values [28]. STAOs have the following important characteristics: STAOs perform a sine change on the input data prior to aggregation. The sine function normalizes values between -1 and 1, allowing for the depiction of imprecise and uncertain data in a continuous and smooth manner. Weighting techniques are used in STAOs to provide relative importance or priority to certain assessments or criteria. This enables DMks to evaluate the relative value of each criterion during the DM process. The weights specify how much each evaluation or criterion effects the overall value or ranking. STAOs offer various AOs to combine the transformed values. These operators include the sine weighted average, sine weighted geometric mean, sine weighted harmonic mean, sine weighted quadratic mean, and other variations etc. Each operator has its own mathematical formula for aggregating the transformed values. The output of STAOs is typically a single aggregated value or a preference ranking of alternatives based on their aggregated scores. The interpretation of the aggregated result depends on the context of the decision problem and the specific requirements of the DMk. STAOs may also handle uncertainty and sensitivity analysis by considering several scenarios or modifications in the given data. Sensitivity analysis assesses the robustness of aggregated data and gives information on the influence of numerous factors on decision making. STAOs are utilized in a wide range of disciplines, such as DM under uncertainty, multi-criteria decision analysis, group DM, and consensus-building techniques. When dealing with ambiguous, vague, or volatile information, STAOs, specifically ST weighted average AOs and ST weighted geometric AOs, provide significant benefits since they effectively capture and represent this complexity. It is important to remember, however, that STAOs are simply one of several AO families used in decision modeling. The most appropriate AO is decided by the specific criteria and features of the decision issue, and various families of operators may be more suited in other cases.

The following justifies the use of ST weighted average AOs and ST weighted geometric AOs on SVNZNs:

- We develop a technique that assigns different criteria different degrees of significance, reflecting their relative relevance in the decision making process, by integrating ST weights. This sensitivity makes sure that important elements are given greater weight, providing a more sophisticated assessment of possible business locations.
- The selected operators are particularly good at managing non-linear connections and interactions between the site selection parameters. The curvature that is created by the ST function has the ability to capture complex and non-linear features that are present in the original data. It is vital to have this insight in order to comprehend how certain factors could interact in non-linearly additive ways, which would result in a more accurate picture of the intricate interactions that influence site suitability.
- By using specified operators, it is possible to efficiently reduce the impact of the extreme values that are included inside the dataset. This is very important since outliers

have the potential to distort the outcomes of choices made when selecting a site for a company. These operators naturally lessen the influence of extreme values, so avoiding outliers from unnecessarily skewing the DM process and encouraging outputs that are reliable and trustworthy.

- The newly implemented aggregation processes are more responsive to even the most minute changes in the variables that are received as input. This is a crucial feature in dynamic business scenarios, where the acceptability of a location may be significantly influenced by even modest changes in the criteria. It is guaranteed that DMks will be able to be more proactive and intelligent in their site selection process if the operators are able to quickly change their judgments in reaction to changing conditions.
- Within the context of site selection, it is usually essential to make compromises between opposing requirements. The operators that were selected provide an approach that is equitable for integrating a large number of criteria, which makes it simpler to evaluate these trade-offs. For instance, the weighted geometric mean takes into consideration a mix of weights and values by default. This encourages a fair compromise between the criteria, as opposed to giving priority to one of the criterion over the others. When it comes to selecting a site for a firm, this makes it feasible for DMks to effectively handle the inherent difficulties of trade-off circumstances.

In light of this, the following are the conclusions of the research:

- The weighted geometric mean and the ST weighted average are two innovative aggregation strategies that we propose. Both of these approaches were built exclusively for statistically significant SVNZNs.
- The incorporation of these additional operators results in the development of a complex decision algorithm that enhances the process of selecting places for commercial ventures.
- Our strategy focuses on SVNZNs in particular, recognizing their special qualities and tackling the difficulties in choosing a site by using designed aggregation operators (AOs).
- We prove the decision algorithm's usefulness in a real-world scenario of business site selection, putting its efficacy in a concrete, applied setting. The system is not merely theoretical.
- Our research leads to the development of a comprehensive framework for DM that makes use of the AOs that we developed. This framework opens the door to the development of more sophisticated and complex business site selection techniques.

This article uses the structure indicated below. In Section 2, fundamental ideas that underpin FS, NS, SVNZNs, and several fundamental operational laws are discussed. We introduce novel

AOs, including Novel Sine Trigonometric Operational Laws, in Section 3. Section 4 introduces new aggregating operators. Key Properties of the Suggested AOs for SVNZNs were also established by Novel Sine Trigonometric Aggregation Operators. In Section 5, a numerical issue solution, numerical illustrations, and a DM method based on the proposed AOs are developed. In Section 6, we compare and contrast a few current practices with suggested ones. In Section 7, we come to a conclusion.

2. Preliminaries and Basic Concepts

In this section, we define some fundamental properties related to our work.

Definition 2.1. [34] In the context of a predetermined set $\curlyvee$, a picture FZN (PFZN) set $\eth$ in $\curlyvee$ is viewed as

$$\eth = \left\{\left\langle \mathfrak{Z}, \left(\mathfrak{T}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right), \left(\mathfrak{I}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right), \left(\mathfrak{F}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right)\right\rangle \mid \mathfrak{Z} \in \curlyvee\right\},$$

for each element $\mathfrak{Z}$ in the set $\curlyvee$, the memberships of $\mathfrak{Z}$ beneath the PFZN $\eth$ have been classified into positive $\left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right) : \curlyvee \to \phi$, neutral $\left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right) : \curlyvee \to \phi$, and negative $\left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right) : \curlyvee \to \phi$ categories, each associated with a specific degree. These degrees are represented by the unit interval $\phi = [0, 1]$. Furthermore, it is essential to guarantee that the sum of $\left(\mathfrak{T}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right), \left(\mathfrak{I}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right)$, and $\left(\mathfrak{F}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right)$ for each $\mathfrak{Z}$ in $\curlyvee$ remains within the range of 0 to 1.

Definition 2.2. [30] In the context of a predetermined set $\curlyvee$, a spherical FZN (SFZN) set $\eth$ in $\curlyvee$ is viewed as

$$\eth = \left\{\left\langle \mathfrak{Z}, \left(\mathfrak{T}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right), \left(\mathfrak{I}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right), \left(\mathfrak{F}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right)\right\rangle \mid \mathfrak{Z} \in \curlyvee\right\},$$

for each element $\mathfrak{Z}$ in the set $\curlyvee$, the memberships of $\mathfrak{Z}$ beneath the SFZN $\eth$ have been classified into positive $\left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right) : \curlyvee \to \phi$, neutral $\left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right) : \curlyvee \to \phi$, and negative $\left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right) : \curlyvee \to \phi$ categories, each associated with a specific degree. These degrees are represented by the unit interval $\phi = [0, 1]$. Furthermore, it is essential to guarantee that the sum of $\left(\mathfrak{T}^2_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{T}^2_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right), \left(\mathfrak{I}^2_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{I}^2_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right)$, and $\left(\mathfrak{F}^2_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{F}^2_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right)$ for each $\mathfrak{Z}$ in $\curlyvee$ remains within the range of 0 to 1.

Definition 2.3. [35] In the context of a predetermined set $\curlyvee$, a SVNZN set $\eth$ in $\curlyvee$ is viewed as

$$\eth = \left\{\left\langle \mathfrak{Z}, \left(\mathfrak{T}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right), \left(\mathfrak{I}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right), \left(\mathfrak{F}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right)\right\rangle \mid \mathfrak{Z} \in \curlyvee\right\},$$

for each element $\mathfrak{Z}$ in the set $\curlyvee$, the memberships of $\mathfrak{Z}$ beneath the SVNZN $\eth$ have been classified into truth $\left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right) : \curlyvee \to \phi$, indeterminacy $\left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right) : \curlyvee \to \phi$, and falsity $\left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right) : \curlyvee \to \phi$ categories, each associated with a specific degree. These degrees are

represented by the unit interval $\phi = [0,1]$. Furthermore, it is essential to guarantee that the sum of $\left(\mathfrak{T}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right), \left(\mathfrak{I}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right)$, and $\left(\mathfrak{F}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right)$ for each $\mathfrak{Z}$ in $\curlyvee$ remains within the range of 0 to 3.
In brief, the triplet $\eth = \left\{\left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right), \left(\mathfrak{I}_{Vji}, \mathfrak{I}_{Rji}\right), \left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right)\right\}$ SVNZN in the entirety of the attention and the ensemble of SVNZNs signified by SVNZN($\curlyvee$).

Definition 2.4. [35] In the context of a predetermined set $\curlyvee$, $SVNZN(\curlyvee)$ set in a universe set $\curlyvee$ is viewed as:

$$\eth_Z = \{\langle \mathfrak{Z},\ \mathfrak{T}(\mathfrak{V},\ \mathfrak{R})(\mathfrak{Z}),\ \mathfrak{I}(\mathfrak{V},\ \mathfrak{R})(\mathfrak{Z}),\ \mathfrak{F}(\mathfrak{V},\ \mathfrak{R})(\mathfrak{Z})\rangle \mid \mathfrak{Z} \in \curlyvee\},$$

where $\mathfrak{T}(\mathfrak{V},\ \mathfrak{R})(\mathfrak{Z}) = (\mathfrak{T}_{\mathfrak{V}}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}}(\mathfrak{Z}))$, $\mathfrak{I}(\mathfrak{V},\ \mathfrak{R})(\mathfrak{Z}) = (\mathfrak{I}_{\mathfrak{V}}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}}(\mathfrak{Z}))$, $\mathfrak{F}(\mathfrak{V},\ \mathfrak{R})(\mathfrak{Z}) = (\mathfrak{F}_{\mathfrak{V}}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}}(\mathfrak{Z})) : \curlyvee \to [0,1]^2$ are the order pairs of truth, indeterminacy and falsity membership, then the component $\mathfrak{R}$ is neutrosophic measures of reliability for $\mathfrak{V}$, along with the sum of $\mathfrak{T}_{\mathfrak{V}}(\mathfrak{Z})$, $\mathfrak{I}_{\mathfrak{V}}(\mathfrak{Z})$, $\mathfrak{F}_{\mathfrak{V}}(\mathfrak{Z})$ remains within the range of 0 to 3, and also sum of $\mathfrak{T}_{\mathfrak{R}}(\mathfrak{Z})$, $\mathfrak{I}_{\mathfrak{R}}(\mathfrak{Z})$, $\mathfrak{F}_{\mathfrak{R}}(\mathfrak{Z})$ remains within the range of 0 to 3. The element $\langle \mathfrak{Z},\ \mathfrak{T}(\mathfrak{V},\ \mathfrak{R})(\mathfrak{Z}),\ \mathfrak{I}(\mathfrak{V},\ \mathfrak{R})(\mathfrak{Z}),\ \mathfrak{F}(\mathfrak{V},\ \mathfrak{R})(\mathfrak{Z})\rangle$ in $\eth_Z$ has been simplified for ease of depiction referred as $\eth_Z = \langle \mathfrak{T}(\mathfrak{V},\ \mathfrak{R}), \mathfrak{I}(\mathfrak{V},\ \mathfrak{R}), \mathfrak{F}(\mathfrak{V},\ \mathfrak{R})\rangle = \langle(\mathfrak{T}_{\mathfrak{V}},\ \mathfrak{T}_{\mathfrak{R}}),\ (\mathfrak{I}_{\mathfrak{V}},\ \mathfrak{I}_{\mathfrak{R}}),\ (\mathfrak{F}_{\mathfrak{V}},\ \mathfrak{F}_{\mathfrak{R}})\rangle$, which is designated as $SVNZN(\curlyvee)$.

Definition 2.5. [35] Let $\eth_1 = \{(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1}), (\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}), (\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1})\}$ and $\eth_2 = \{(\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2}), (\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2}), (\mathfrak{F}_{\mathfrak{V}_2}, \mathfrak{F}_{\mathfrak{R}_2})\} \in SVNZN(\curlyvee)$. then,

- **(1):** $\eth_1 \subseteq \eth_2$ if and only if $(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1}) \leq (\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2})$,$(\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}) \geq (\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2})$ and $(\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1}) \geq (\mathfrak{F}_{\mathfrak{V}_2}, \mathfrak{F}_{\mathfrak{R}_2})$ for each $\mathfrak{Z} \in \curlyvee$.
- **(2):** $\eth_1 = \eth_2$ if and only if $\eth_1 \subseteq \eth_2$ and $\eth_2 \subseteq \eth_1$.
- **(3):** $\eth_1 \cap \eth_2 = \left\{\begin{array}{c} \inf\left((\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1}), (\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2})\right), \sup\left((\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}), (\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2})\right), \\ \sup\left((\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1}), (\mathfrak{F}_{\mathfrak{V}_2}, \mathfrak{F}_{\mathfrak{R}_2})\right) \end{array}\right\}$,
- **(4):** $\eth_1 \cup \eth_2 = \left\{\begin{array}{c} \sup\left((\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1}), (\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2})\right), \inf\left((\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}), (\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2})\right), \\ \inf\left((\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1}), (\mathfrak{F}_{\mathfrak{V}_2}, \mathfrak{F}_{\mathfrak{R}_2})\right) \end{array}\right\}$,
- **(5):** $\eth_1^c = \{(\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1}), (\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}), (\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1})\}$.

Definition 2.6. [35] Let $\eth_1 = \{(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1}), (\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}), (\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1})\}$ and $\eth_2 = \{(\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2}), (\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2}), (\mathfrak{F}_{\mathfrak{V}_2}, \mathfrak{F}_{\mathfrak{R}_2})\} \in SVNZN(\curlyvee)$ with $\xi > 0$. then,

(1):

$$\eth_1 \otimes \eth_2 = \left\{\begin{array}{c} (\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1})(\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2}), (\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}) + (\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2}) - (\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}) \cdot (\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2}), \\ (\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1}) + (\mathfrak{F}_{\mathfrak{V}_2}, \mathfrak{F}_{\mathfrak{R}_2}) - (\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1}) \cdot (\mathfrak{F}_{\mathfrak{V}_2}, \mathfrak{F}_{\mathfrak{R}_2}) \end{array}\right\};$$

(2):

$$\eth_1 \oplus \eth_2 = \left\{\begin{array}{c} (\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1}) + (\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2}) - (\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1})(\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2}), (\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1})(\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2}), \\ (\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1})(\mathfrak{F}_{\mathfrak{V}_2}, \mathfrak{F}_{\mathfrak{R}_2}) \end{array}\right\};$$

(3):

$$(\eth_1)^{\xi} = \left\{ (\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1})^{\xi}, 1 - (1 - (\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}))^{\xi}, 1 - (1 - (\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1}))^{\xi} \right\};$$

(4):

$$\xi \cdot \eth_1 = \left\{ 1 - (1 - (\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1}))^{\xi}, (\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1})^{\xi}, (\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1})^{\xi} \right\};$$

(5):

$$\xi^{\eth_1} = \left\{ \begin{array}{c} \left(\xi^{1-\left(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1}\right)}, 1 - \xi^{\left(\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}\right)}, 1 - \xi^{\left(\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1}\right)} \right) \quad \text{if } \xi \in (0,1) \\ \left(\left(\frac{1}{\xi}\right)^{1-\left(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1}\right)}, 1 - \left(\frac{1}{\xi}\right)^{\left(\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}\right)}, 1 - \left(\frac{1}{\xi}\right)^{\left(\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1}\right)} \right) \quad \text{if } \xi \geq 1 \end{array} \right\}$$

Definition 2.7. [35] Let $\eth_{\Xi} = \{(\mathfrak{T}_{\mathfrak{V}_{\Xi}}, \mathfrak{T}_{\mathfrak{R}_{\Xi}}), (\mathfrak{I}_{\mathfrak{V}_{\Xi}}, \mathfrak{I}_{\mathfrak{R}_{\Xi}}), (\mathfrak{F}_{\mathfrak{V}_{\Xi}}, \mathfrak{F}_{\mathfrak{R}_{\Xi}})\} \in SVNZN(\Upsilon)$ $(\Xi = 1, 2, 3, ..., n)$. Subsequently, the algebraic averaging AO associated with the set $SVNZN(\Upsilon)$ is identified as $SVNZNWA$ and outlined in the ensuing manner:

$$\begin{aligned} SVNZNWA(\eth_1, \eth_2, \eth_3, ..., \eth_n) &= \sum_{\Xi=1}^{n} \xi_{\Xi} \eth_{\Xi}, \\ &= \left\{ \begin{array}{c} 1 - \prod_{\Xi=1}^{n} (1 - (\mathfrak{T}_{\mathfrak{V}_{\Xi}}, \mathfrak{T}_{\mathfrak{R}_{\Xi}}))^{\xi_{\Xi}}, \prod_{\Xi=1}^{n} ((\mathfrak{I}_{\mathfrak{V}_{\Xi}}, \mathfrak{I}_{\mathfrak{R}_{\Xi}}))^{\xi_{\Xi}}, \\ \prod_{\Xi=1}^{n} ((\mathfrak{F}_{\mathfrak{V}_{\Xi}}, \mathfrak{F}_{\mathfrak{R}_{\Xi}}))^{\xi_{\Xi}} \end{array} \right\}. \end{aligned}$$

Here, ξ_{Ξ} (where Ξ ranges from 1 to n) signifies the weights assigned to $\eth_{\Xi}$ (where Ξ ranges from 1 to n), with the stipulation that ξ_{Ξ} is non-negative and the summation of all ξ_{Ξ} values equals 1.

Definition 2.8. [35] Let $\eth_{\Xi} = \{(\mathfrak{T}_{\mathfrak{V}_{\Xi}}, \mathfrak{T}_{\mathfrak{R}_{\Xi}}), (\mathfrak{I}_{\mathfrak{V}_{\Xi}}, \mathfrak{I}_{\mathfrak{R}_{\Xi}}), (\mathfrak{F}_{\mathfrak{V}_{\Xi}}, \mathfrak{F}_{\mathfrak{R}_{\Xi}})\} \in SVNZN(\Upsilon)$ $(\Xi = 1, 2, 3, ..., n)$. Subsequently, the algebraic geometric AO associated with the set $SVNZN(\Upsilon)$ is identified as $SVNZNWG$ and outlined in the ensuing manner:

$$\begin{aligned} SVNZNWG(\eth_1, \eth_2, \eth_3, ..., \eth_n) &= \prod_{\Xi=1}^{n} (\eth_{\Xi})^{\xi_{\Xi}}, \\ &= \left\{ \begin{array}{c} \prod_{\Xi=1}^{n} ((\mathfrak{T}_{\mathfrak{V}_{\Xi}}, \mathfrak{T}_{\mathfrak{R}_{\Xi}}))^{\xi_{\Xi}}, 1 - \prod_{\Xi=1}^{n} (1 - (\mathfrak{I}_{\mathfrak{V}_{\Xi}}, \mathfrak{I}_{\mathfrak{R}_{\Xi}}))^{\xi_{\Xi}}, \\ 1 - \prod_{\Xi=1}^{n} (1 - (\mathfrak{F}_{\mathfrak{V}_{\Xi}}, \mathfrak{F}_{\mathfrak{R}_{\Xi}}))^{\xi_{\Xi}} \end{array} \right\} \end{aligned}$$

Here, ξ_{Ξ} (where Ξ ranges from 1 to n) signifies the weights assigned to $\eth_{\Xi}$ (where Ξ ranges from 1 to n), with the stipulation that ξ_{Ξ} is non-negative and the summation of all ξ_{Ξ} values equals 1.

3. Novel Sine Trigonometric Operational Laws For SVNZNs

Within this portion, we introduce innovative principles that utilize the ST function within SVNZN settings.

Definition 3.1. Let $\eth = \left\{\left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right), \left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right), \left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right)\right\} \in SVNZN\left(\Upsilon\right)$. Subsequently,ST operational laws (STOLs) for SVNZN $\eth$ are outlined below:

$$\sin\left(\eth\right) = \left\{\begin{pmatrix} \mathfrak{Z}, \sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_{ji}}\left(\mathfrak{Z}\right), \mathfrak{T}_{\mathfrak{R}_{ji}}\left(\mathfrak{Z}\right)\right)\right), 1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_{ji}}\left(\mathfrak{Z}\right), \mathfrak{I}_{\mathfrak{R}_{ji}}\left(\mathfrak{Z}\right)\right)\right)\right), \\ 1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_{ji}}\left(\mathfrak{Z}\right), \mathfrak{F}_{\mathfrak{R}_{ji}}\left(\mathfrak{Z}\right)\right)\right)\right) \end{pmatrix} | \mathfrak{Z} \in \Upsilon\right\}$$

The fact that $\sin\left(\eth\right)$ also exhibits the SVNZN property is evident. It is evident that, for every element $\mathfrak{Z}$ within the set Υ, the values representing truth, indeterminacy, and falsity, denoted as $\left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right) : \Upsilon \to \phi$, $\left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right) : \Upsilon \to \phi$, and $\left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right) : \Upsilon \to \phi$ respectively, pertain to the SVNZN set $\eth$. Here, $\phi = [0, 1]$ designates the unit interval. Furthermore, it is essential to guarantee that the sum of $\left(\mathfrak{T}_{\mathfrak{V}_{ji}}\left(\mathfrak{Z}\right), \mathfrak{T}_{\mathfrak{R}_{ji}}\left(\mathfrak{Z}\right)\right), \left(\mathfrak{I}_{\mathfrak{V}_{ji}}\left(\mathfrak{Z}\right), \mathfrak{I}_{\mathfrak{R}_{ji}}\left(\mathfrak{Z}\right)\right)$, and $\left(\mathfrak{F}_{\mathfrak{V}_{ji}}\left(\mathfrak{Z}\right), \mathfrak{F}_{\mathfrak{R}_{ji}}\left(\mathfrak{Z}\right)\right)$ for each $\mathfrak{Z}$ in Υ remains within the range of 0 to 3. Moreover, the membership degree of truth

$$\begin{aligned} \sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right)\right) &: \Upsilon \to \phi, \\ \text{for each } \mathfrak{Z} &\in \Upsilon \to \sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_{ji}}\left(\mathfrak{Z}\right), \mathfrak{T}_{\mathfrak{R}_{ji}}\left(\mathfrak{Z}\right)\right)\right) \in [0, 1], \end{aligned}$$

membership degree of indeterminacy

$$\begin{aligned} 1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right)\right)\right) &: \Upsilon \to \phi, \\ \text{for each } \mathfrak{Z} &\in \Upsilon \to 1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_{ji}}\left(\mathfrak{Z}\right), \mathfrak{I}_{\mathfrak{R}_{ji}}\left(\mathfrak{Z}\right)\right)\right)\right) \in [0, 1], \end{aligned}$$

and membership degree of falsity

$$\begin{aligned} 1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right)\right)\right) &: \Upsilon \to \phi, \\ \text{for each } \mathfrak{Z} &\in \Upsilon \to 1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_{ji}}\left(\mathfrak{Z}\right), \mathfrak{F}_{\mathfrak{R}_{ji}}\left(\mathfrak{Z}\right)\right)\right)\right) \in [0, 1]. \end{aligned}$$

As such,

$$\sin\left(\eth\right) = \left\{\begin{pmatrix} \mathfrak{Z}, \sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_{ji}}\left(\mathfrak{Z}\right), \mathfrak{T}_{\mathfrak{R}_{ji}}\left(\mathfrak{Z}\right)\right)\right), 1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_{ji}}\left(\mathfrak{Z}\right), \mathfrak{I}_{\mathfrak{R}_{ji}}\left(\mathfrak{Z}\right)\right)\right)\right), \\ 1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_{ji}}\left(\mathfrak{Z}\right), \mathfrak{F}_{\mathfrak{R}_{ji}}\left(\mathfrak{Z}\right)\right)\right)\right) \end{pmatrix} | \mathfrak{Z} \in \Upsilon\right\}$$

is SVNZN.

Definition 3.2. Let $\eth = \left\{\left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right), \left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right), \left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right)\right\} \in SVNZN\left(\Upsilon\right)$. If

$$\sin\left(\eth\right) = \left\{\begin{pmatrix} \mathfrak{Z}, \sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_{ji}}\left(\mathfrak{Z}\right), \mathfrak{T}_{\mathfrak{R}_{ji}}\left(\mathfrak{Z}\right)\right)\right), 1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_{ji}}\left(\mathfrak{Z}\right), \mathfrak{I}_{\mathfrak{R}_{ji}}\left(\mathfrak{Z}\right)\right)\right)\right), \\ 1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_{ji}}\left(\mathfrak{Z}\right), \mathfrak{F}_{\mathfrak{R}_{ji}}\left(\mathfrak{Z}\right)\right)\right)\right) \end{pmatrix} | \mathfrak{Z} \in \Upsilon\right\}$$

Subsequently, the function $\sin(\eth)$ is referred to as the ST operator, and the outcome of $\sin(\eth)$ is termed the ST-SVNZN (STSVNZN).

Theorem 3.3. *Let $\eth = \left\{\left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right), \left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right), \left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right)\right\} \in SVNZN\left(\Upsilon\right)$. Then, the result yielded by the operator $\sin(\eth)$ possesses the SVNZN property.*

Proof. As $\eth = \left\{\left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right), \left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right), \left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right)\right\}$, that is, $0 \leq \left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right) \leq 1$, $0 \leq \left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right) \leq 1$ and $0 \leq \left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right) \leq 1$. Moreover, $\left(\mathfrak{T}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right) + \left(\mathfrak{I}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right) + \left(\mathfrak{F}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right) \leq 3$, for every $\mathfrak{Z} \in \Upsilon$. In order to demonstrate that sin($\eth$) holds the SVNZN characteristic, two essential conditions are considered:

- **(1):** $\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right)\right)$,$1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right)\right)\right)$ and $1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right)\right)\right) \in [0,1]$
- **(2):** $\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right)\right)+1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right)\right)\right)+1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right)\right)\right) \leq 3$.

As $0 \leq \left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right) \leq 1$ this leads to the inference that $0 \leq \frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right) \leq \frac{\Pi}{2}$. Additionally, it's important to note that the function "sin" is monotonically increasing within the first quadrant; therefore, we have $0 \leq \sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right)\right) \leq 1$.
As $0 \leq \left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right) \leq 1$ this leads to the inference that $0 \leq \frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right)\right) \leq \frac{\Pi}{2}$,
$\Rightarrow 0 \leq \sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right)\right)\right) \leq 1$. Consequently, we obtain $0 \leq 1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right)\right)\right) \leq 1$.
Likewise, we acquire $0 \leq 1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right)\right)\right) \leq 1$. Hence, part (1) is established.
As $\eth \in SVNZN(\Upsilon) \Rightarrow 0 \leq \left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right), \left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right), \left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right) \leq 1$, and
$\left(\mathfrak{T}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right) + \left(\mathfrak{I}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right) + \left(\mathfrak{F}_{\mathfrak{V}_{ji}}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right) \leq 3$, for every $\mathfrak{Z} \in \Upsilon$.
Subsequently, (1) indicates that $0 \leq \sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right)\right), 1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right)\right)\right), 1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right)\right)\right) \leq 1$. Furthermore, as per Definition 3.1, we possess $0 \leq \sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right)\right)+1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right)\right)\right)+1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right)\right)\right) \leq 3$. Consequently, it can be concluded that sin($\eth$) exhibits the SVNZN property. □

Definition 3.4. Let $\sin(\eth_1) = \left\{\begin{pmatrix} \sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1}\right)\right), \\ 1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}\right)\right)\right), \\ 1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1}\right)\right)\right) \end{pmatrix}\right\}$, and $\sin(\eth_2) = \left\{\begin{pmatrix} \sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2}\right)\right), \\ 1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2}\right)\right)\right), \\ 1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_2}, \mathfrak{F}_{\mathfrak{R}_2}\right)\right)\right) \end{pmatrix}\right\}$ be two STSVNZNs. Then the operational laws are as follows

(1):

$$\sin(\eth_1) \bigoplus \sin(\eth_2) = \begin{pmatrix} 1-\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1}\right)\right)\right)\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2}\right)\right)\right), \\ \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}\right)\right)\right)\right)\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2}\right)\right)\right)\right), \\ \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1}\right)\right)\right)\right)\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_2}, \mathfrak{F}_{\mathfrak{R}_2}\right)\right)\right)\right) \end{pmatrix},$$

(2):

$$\psi \cdot \sin(\eth_1) = \begin{pmatrix} 1-\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1}\right)\right)\right)^{\psi}, \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}\right)\right)\right)\right)^{\psi}, \\ \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1}\right)\right)\right)\right)^{\psi} \end{pmatrix},$$

(3):

$$\sin(\eth_1)\bigotimes\sin(\eth_2)=\begin{pmatrix}\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_1},\mathfrak{T}_{\mathfrak{R}_1}\right)\right)\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_2},\mathfrak{T}_{\mathfrak{R}_2}\right)\right),\\ 1-\left(\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_1},\mathfrak{I}_{\mathfrak{R}_1}\right)\right)\right)\right)\left(\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_2},\mathfrak{I}_{\mathfrak{R}_2}\right)\right)\right)\right),\\ 1-\left(\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_1},\mathfrak{F}_{\mathfrak{R}_1}\right)\right)\right)\right)\left(\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_2},\mathfrak{F}_{\mathfrak{R}_2}\right)\right)\right)\right)\end{pmatrix},$$

(4):

$$(\sin(\eth_1))^{\psi}=\begin{pmatrix}\left(\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_1},\mathfrak{T}_{\mathfrak{R}_1}\right)\right)\right)^{\psi},\\ 1-\left(\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_1},\mathfrak{I}_{\mathfrak{R}_1}\right)\right)\right)\right)^{\psi},\\ 1-\left(\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_1},\mathfrak{F}_{\mathfrak{R}_1}\right)\right)\right)\right)^{\psi}\end{pmatrix}.$$

In order to make comparisons between the STSVNZNs, we have introduced the subsequent definitions.

Definition 3.5. Consider $\eth=\left\{\left(\mathfrak{T}_{\mathfrak{V}_{ji}},\mathfrak{T}_{\mathfrak{R}_{ji}}\right),\left(\mathfrak{I}_{\mathfrak{V}_{ji}},\mathfrak{I}_{\mathfrak{R}_{ji}}\right),\left(\mathfrak{F}_{\mathfrak{V}_{ji}},\mathfrak{F}_{\mathfrak{R}_{ji}}\right)\right\}\in SVNZN(\Upsilon)$. In this context, we symbolize and establish the *partial* score and accuracy this way:

(1): $\overline{sc}(\eth)=\left(\mathfrak{T}_{\mathfrak{V}_{ji}},\mathfrak{T}_{\mathfrak{R}_{ji}}\right)-\left(\mathfrak{I}_{\mathfrak{V}_{ji}},\mathfrak{I}_{\mathfrak{R}_{ji}}\right)-\left(\mathfrak{F}_{\mathfrak{V}_{ji}},\mathfrak{F}_{\mathfrak{R}_{ji}}\right)$, and

(2): $\underline{ac}(\eth)=\left(\mathfrak{T}_{\mathfrak{V}_{ji}},\mathfrak{T}_{\mathfrak{R}_{ji}}\right)+\left(\mathfrak{I}_{\mathfrak{V}_{ji}},\mathfrak{I}_{\mathfrak{R}_{ji}}\right)+\left(\mathfrak{F}_{\mathfrak{V}_{ji}},\mathfrak{F}_{\mathfrak{R}_{ji}}\right)$.

Definition 3.6. Let $\eth_1=\{(\mathfrak{T}_{\mathfrak{V}_1},\mathfrak{T}_{\mathfrak{R}_1}),(\mathfrak{I}_{\mathfrak{V}_1},\mathfrak{I}_{\mathfrak{R}_1}),(\mathfrak{F}_{\mathfrak{V}_1},\mathfrak{F}_{\mathfrak{R}_1})\}$ and $\eth_2=\{(\mathfrak{T}_{\mathfrak{V}_2},\mathfrak{T}_{\mathfrak{R}_2}),(\mathfrak{I}_{\mathfrak{V}_2},\mathfrak{I}_{\mathfrak{R}_2}),(\mathfrak{F}_{\mathfrak{V}_2},\mathfrak{F}_{\mathfrak{R}_2})\}\in SVNZN(\Upsilon)$. Then,

(1): If $\overline{sc}(\eth_1)<\overline{sc}(\eth_2)$, it follows that $\eth_1<\eth_2$.

(2): If $\overline{sc}(\eth_1)>\overline{sc}(\eth_2)$, it follows that $\eth_1>\eth_2$.

(3): If $\overline{sc}(\eth_1)=\overline{sc}(\eth_2)$, then

(a): $\underline{ac}(\eth_1)<\underline{ac}(\eth_2)$, it follows that $\eth_1<\eth_2$,

(b): $\underline{ac}(\eth_1)>\underline{ac}(\eth_2)$, it follows that $\eth_1>\eth_2$,

(c): $\underline{ac}(\eth_1)=\underline{ac}(\eth_2)$, it follows that $\eth_1=\eth_2$.

To compare $SVNZNs$ $\mathfrak{W}_{Z_i}=\langle\mathfrak{T}_i(\mathfrak{V},\ \mathfrak{R}),\ \mathfrak{I}_i(\mathfrak{V},\ \mathfrak{R}),\ \mathfrak{F}_i(\mathfrak{V},\ \mathfrak{R})\rangle=\langle(\mathfrak{T}_{\mathfrak{V}_i},\mathfrak{T}_{\mathfrak{R}_i}),(\mathfrak{I}_{\mathfrak{V}_i},\mathfrak{I}_{\mathfrak{R}_i}),(\mathfrak{F}_{\mathfrak{V}_i},\mathfrak{F}_{\mathfrak{R}_i})\rangle$ $(i=1,2)$, we introduce a score function:

$$Y(\mathfrak{W}_{Z_i})=\frac{2+\mathfrak{T}_{Vi}\mathfrak{T}_{Ri}-\mathfrak{I}_{Vi}\mathfrak{I}_{Ri}-\mathfrak{F}_{Vi}\mathfrak{F}_{Ri}}{3}$$

for $Y(\mathfrak{W}_{Z_i})\in[0,1]$, when $Y(\mathfrak{W}_{Z_1})\geq Y(\mathfrak{W}_{Z_2})$, this leads to the conclusion that the ranking is $\mathfrak{W}_{Z_1}\geq\mathfrak{W}_{Z_2}$.

Example 3.7. Set two $SVNZNs$ as $\mathfrak{W}_{Z_1}=\langle(0.9,\ 0.6),\ \ (0.6,\ 0.8),\ \ (0.7,\ 0.9)\rangle$ and $\mathfrak{W}_{Z_2}=\langle(0.8,\ 0.5),\ \ (0.1,\ 0.4),\ \ (0.3,\ 0.6)\rangle$. By using score function, we have

$$Y(\mathfrak{W}_{Z_1})=\frac{(2+0.9\times0.6-0.6\times0.8-0.7\times0.9)}{3}=0.477$$

and

$$Y(\mathfrak{W}_{Z_2})=\frac{(2+0.8\times0.5-0.1\times0.4-0.3\times0.6)}{3}=0.727$$

As $Y(\mathfrak{W}_{Z_1}) < Y(\mathfrak{W}_{Z_2})$, it follows that their ranking is $\mathfrak{W}_{Z_1} < \mathfrak{W}_{Z_2}$. Subsequently, we delve into a discussion of fundamental properties of STSVNZNs built upon the introduced STOLs.

Theorem 3.8. *Let* $\eth_1 = \{(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1}), (\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}), (\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1})\}$ *and* $\eth_2 = \{(\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2}), (\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2}), (\mathfrak{F}_{\mathfrak{V}_2}, \mathfrak{F}_{\mathfrak{R}_2})\} \in SVNZN(\curlyvee)$. *Then,*

(1): $\sin(\eth_1) \bigoplus \sin(\eth_2) = \sin(\eth_2) \bigoplus \sin(\eth_1)$,

(2): $\sin(\eth_1) \bigotimes \sin(\eth_2) = \sin(\eth_2) \bigotimes \sin(\eth_1)$.

Proof. This is evident directly from Definition 3.2. □

Theorem 3.9. *Let* $\eth_\Xi = \{(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}), (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}), (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})\} \in SVNZN(\curlyvee)$ $(\Xi = 1, 2, 3)$. *Then,*

(1): $(\sin(\eth_1) \bigoplus \sin(\eth_2)) \bigoplus \sin(\eth_3) = \sin(\eth_1) \bigoplus (\sin(\eth_2) \bigoplus \sin(\eth_3))$,

(2): $(\sin(\eth_1) \bigotimes \sin(\eth_2)) \bigotimes \sin(\eth_3) = \sin(\eth_1) \bigotimes (\sin(\eth_2) \bigotimes \sin(\eth_3))$.

Proof. This is evident directly from Definition 3.2. □

Theorem 3.10. *Let* $\eth_\Xi = \{(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}), (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}), (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})\} \in SVNZN(\curlyvee)$ $(\Xi = 1, 2)$ *and* $\psi, \psi_1, \psi_2 > 0$. *Then,*

(1): $\psi(\sin(\eth_1) \bigoplus \sin(\eth_2)) = \psi \sin(\eth_1) \bigoplus \psi \sin(\eth_2)$,

(2): $(\sin(\eth_1) \bigotimes \sin(\eth_2))^\psi = (\sin(\eth_1))^\psi \bigotimes (\sin(\eth_2))^\psi$,

(3): $\psi_1 \sin(\eth_1) \bigoplus \psi_2 \sin(\eth_1) = (\psi_1 + \psi_2)\sin(\eth_1)$,

(4): $(\sin(\eth_1))^{\psi_1} \bigotimes (\sin(\eth_1))^{\psi_2} = (\sin(\eth_1))^{\psi_1+\psi_2}$,

(5): $\left((\sin(\eth_1))^{\psi_1}\right)^{\psi_2} = (\sin(\eth_1))^{\psi_1 . \psi_2}$.

Proof.
Let $\eth_\Xi = \{(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}), (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}), (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})\} \in SVNZN(\curlyvee)$ $(\Xi = 1, 2)$ and $\psi, \psi_1, \psi_2 > 0$. Then, by the Definition 3.2, we have $\sin(\eth_1) = \left\{\begin{pmatrix} \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1})\right), \\ 1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}))\right), \\ 1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1}))\right) \end{pmatrix}\right\}$ and $\sin(\eth_2) = \left\{\begin{pmatrix} \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2})\right), \\ 1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2}))\right), \\ 1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{F}_{\mathfrak{V}_2}, \mathfrak{F}_{\mathfrak{R}_2}))\right) \end{pmatrix}\right\}$ be two STSVNZNs. Therefore, using the STOLs for SVNZNs, we obtain

$$\sin(\eth_1) \bigoplus \sin(\eth_2) = \begin{pmatrix} 1 - \left(1 - \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1})\right)\right)\left(1 - \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2})\right)\right), \\ \left(1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}))\right)\right)\left(1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2}))\right)\right), \\ \left(1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1}))\right)\right)\left(1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{F}_{\mathfrak{V}_2}, \mathfrak{F}_{\mathfrak{R}_2}))\right)\right) \end{pmatrix}.$$

(1): For any $\psi > 0$, the following holds

$$\psi\left(\sin(\eth_1)\bigoplus\sin(\eth_2)\right) = \begin{pmatrix} 1-\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_1},\mathfrak{T}_{\mathfrak{R}_1}\right)\right)\right)^{\psi}\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_2},\mathfrak{T}_{\mathfrak{R}_2}\right)\right)\right)^{\psi}, \\ \left(\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_1},\mathfrak{I}_{\mathfrak{R}_1}\right)\right)\right)\right)\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_2},\mathfrak{I}_{\mathfrak{R}_2}\right)\right)\right)\right)\right)^{\psi}, \\ \left(\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_1},\mathfrak{F}_{\mathfrak{R}_1}\right)\right)\right)\right)\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_2},\mathfrak{F}_{\mathfrak{R}_2}\right)\right)\right)\right)\right)^{\psi} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_1},\mathfrak{T}_{\mathfrak{R}_1}\right)\right)\right)^{\psi}, \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_1},\mathfrak{I}_{\mathfrak{R}_1}\right)\right)\right)\right)^{\psi}, \\ \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_1},\mathfrak{F}_{\mathfrak{R}_1}\right)\right)\right)\right)^{\psi} \end{pmatrix}$$

$$\bigoplus \begin{pmatrix} 1-\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_2},\mathfrak{T}_{\mathfrak{R}_2}\right)\right)\right)^{\psi}, \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_2},\mathfrak{I}_{\mathfrak{R}_2}\right)\right)\right)\right)^{\psi}, \\ \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_2},\mathfrak{F}_{\mathfrak{R}_2}\right)\right)\right)\right)^{\psi} \end{pmatrix}$$

$$= \psi\sin(\eth_1)\bigoplus\psi\sin(\eth_2).$$

(2): The proof follows a similar pattern as (1).

(3): For any $\psi_1, \psi_2 > 0$, we have

$$\psi_1\sin(\eth_1) = \begin{pmatrix} 1-\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_1},\mathfrak{T}_{\mathfrak{R}_1}\right)\right)\right)^{\psi_1}, \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_1},\mathfrak{I}_{\mathfrak{R}_1}\right)\right)\right)\right)^{\psi_1}, \\ \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_1},\mathfrak{F}_{\mathfrak{R}_1}\right)\right)\right)\right)^{\psi_1} \end{pmatrix}$$

and

$$\psi_2\sin(\eth_1) = \begin{pmatrix} 1-\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_1},\mathfrak{T}_{\mathfrak{R}_1}\right)\right)\right)^{\psi_2}, \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_1},\mathfrak{I}_{\mathfrak{R}_1}\right)\right)\right)\right)^{\psi_2}, \\ \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_1},\mathfrak{F}_{\mathfrak{R}_1}\right)\right)\right)\right)^{\psi_2} \end{pmatrix}.$$

Thus, by STOLs for SVNZNs, we get

$$\psi_1\sin(\eth_1)\bigoplus\psi_2\sin(\eth_1) = \begin{pmatrix} 1-\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_1},\mathfrak{T}_{\mathfrak{R}_1}\right)\right)\right)^{\psi_1}, \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_1},\mathfrak{I}_{\mathfrak{R}_1}\right)\right)\right)\right)^{\psi_1}, \\ \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_1},\mathfrak{F}_{\mathfrak{R}_1}\right)\right)\right)\right)^{\psi_1} \end{pmatrix}$$

$$\bigoplus \begin{pmatrix} 1-\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_1},\mathfrak{T}_{\mathfrak{R}_1}\right)\right)\right)^{\psi_2}, \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_1},\mathfrak{I}_{\mathfrak{R}_1}\right)\right)\right)\right)^{\psi_2}, \\ \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_1},\mathfrak{F}_{\mathfrak{R}_1}\right)\right)\right)\right)^{\psi_2} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_1},\mathfrak{T}_{\mathfrak{R}_1}\right)\right)\right)^{\psi_1+\psi_2}, \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_1},\mathfrak{I}_{\mathfrak{R}_1}\right)\right)\right)\right)^{\psi_1+\psi_2}, \\ \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_1},\mathfrak{F}_{\mathfrak{R}_1}\right)\right)\right)\right)^{\psi_1+\psi_2} \end{pmatrix}$$

$$= (\psi_1+\psi_2)\sin(\eth_1)$$

The proof of (4), and (5) is similarly as (3). □

Theorem 3.11. *Let* $\eth_\Xi = \{(\mathfrak{T}_{\mathfrak{V}_\Xi},\mathfrak{T}_{\mathfrak{R}_\Xi}), (\mathfrak{I}_{\mathfrak{V}_\Xi},\mathfrak{I}_{\mathfrak{R}_\Xi}), (\mathfrak{F}_{\mathfrak{V}_\Xi},\mathfrak{F}_{\mathfrak{R}_\Xi})\} \in SVNZN(\Upsilon)$ $(\Xi = 1, 2)$ *such that* $(\mathfrak{T}_{\mathfrak{V}_1},\mathfrak{T}_{\mathfrak{R}_1}) \geq (\mathfrak{T}_{\mathfrak{V}_2},\mathfrak{T}_{\mathfrak{R}_2})$, $(\mathfrak{I}_{\mathfrak{V}_1},\mathfrak{I}_{\mathfrak{R}_1}) \leq (\mathfrak{I}_{\mathfrak{V}_2},\mathfrak{I}_{\mathfrak{R}_2})$ *and* $(\mathfrak{F}_{\mathfrak{V}_1},\mathfrak{F}_{\mathfrak{R}_1}) \leq (\mathfrak{F}_{\mathfrak{V}_2},\mathfrak{F}_{\mathfrak{R}_2})$. *Then* $\sin(\eth_1) \geq \sin(\eth_2)$.

Proof. For $\eth_\Xi = \{(\mathfrak{T}_{\mathfrak{V}_\Xi},\mathfrak{T}_{\mathfrak{R}_\Xi}), (\mathfrak{I}_{\mathfrak{V}_\Xi},\mathfrak{I}_{\mathfrak{R}_\Xi}), (\mathfrak{F}_{\mathfrak{V}_\Xi},\mathfrak{F}_{\mathfrak{R}_\Xi})\} \in SVNZN(\Upsilon)$ $(\Xi = 1, 2)$, we have $(\mathfrak{T}_{\mathfrak{V}_1},\mathfrak{T}_{\mathfrak{R}_1}) \geq (\mathfrak{T}_{\mathfrak{V}_2},\mathfrak{T}_{\mathfrak{R}_2})$. As "sin" is an increasing function in $\left[0, \frac{\Pi}{2}\right]$, thus we have $\sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_1},\mathfrak{T}_{\mathfrak{R}_1})\right) \geq \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_2},\mathfrak{T}_{\mathfrak{R}_2})\right)$. Similarly, we have $(\mathfrak{I}_{\mathfrak{V}_1},\mathfrak{I}_{\mathfrak{R}_1}) \leq (\mathfrak{I}_{\mathfrak{V}_2},\mathfrak{I}_{\mathfrak{R}_2})$, which implies that $1 - (\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}) \geq 1 - (\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2})$. Thus, $\sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}))\right) \geq \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2}))\right)$, which further implies that

$$1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}))\right) \leq 1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2}))\right)$$

and similarly we get

$$1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1}))\right) \leq 1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{F}_{\mathfrak{V}_2}, \mathfrak{F}_{\mathfrak{R}_2}))\right).$$

Therefore we get

$$\left\{\begin{pmatrix} \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1})\right), \\ 1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}))\right), \\ 1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1}))\right) \end{pmatrix}\right\} \geq \left\{\begin{pmatrix} \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2})\right), \\ 1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2}))\right), \\ 1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{F}_{\mathfrak{V}_2}, \mathfrak{F}_{\mathfrak{R}_2}))\right) \end{pmatrix}\right\}.$$

Hence, according to Definition 3.2, it follows that $\sin(\eth_1) \geq \sin(\eth_2)$. □

4. **Novel Sine Trigonometric Aggregation Operators for SVNZNs**

In this part, we present novel AOs that extend the suggested STOLs for SVNZNs. We define the geometric AOs and weighted averaging that follow.

4.1. *Sine Trigonometric Weighted Averaging AOs for SVNZNs*

Definition 4.1. Let $\eth_\Xi = \{(\mathfrak{T}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_\Xi}(\mathfrak{Z})), (\mathfrak{I}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_\Xi}(\mathfrak{Z})), (\mathfrak{F}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_\Xi}(\mathfrak{Z}))\} \in SVNZN(\Upsilon)$ $(\Xi = 1, 2, 3, ..., n)$. Subsequently, the ST weighted averaging AO for $SVNZN(\Upsilon)$ known as ST-SVNZNWA is symbolized and outlined in the subsequent manner:

$$\begin{aligned} ST\text{-}SVNZNWA(\eth_1, \eth_2, ...\eth_n) &= \xi_1 \sin(\eth_1) \bigoplus \xi_2 \sin(\eth_2) \bigoplus ... \bigoplus \xi_n \sin(\eth_n) \\ &= \sum_{\Xi=1}^{n} \xi_\Xi \sin(\eth_\Xi). \end{aligned}$$

Here, $\xi_\Xi (\Xi = 1, 2, ..., n)$ signifies the weights associated with $\eth_\Xi (\Xi = 1, 2, 3, ..., n)$, where $\xi_\Xi \geq 0$ and $\sum_{\Xi=1}^{n} \xi_\Xi = 1$.

Theorem 4.2. *Let* $\eth_\Xi = \{(\mathfrak{T}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_\Xi}(\mathfrak{Z})), (\mathfrak{I}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_\Xi}(\mathfrak{Z})), (\mathfrak{F}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_\Xi}(\mathfrak{Z}))\} \in SVNZN(\Upsilon)$ $(\Xi = 1, 2, 3, ..., n)$ *and the WV of* $\eth\Xi (\Xi = 1, 2, 3, ..., n)$ *is represented by* $\xi = (\xi 1, \xi_2, ..., \xi_n)^{\mathfrak{T}}$ *with the constraint* $\sum_{\Xi=1}^{n} \xi_\Xi = 1$. *The ST-SVNZNWA operator is a mapping*

$G^n \longrightarrow G$ *that satisfies:*

$$ST\text{-}SVNZNWA(\eth_1, \eth_2, ...\eth_n) = \sum_{\Xi=1}^{n} \xi_\Xi \sin(\eth_\Xi) = \begin{pmatrix} 1 - \prod_{\Xi=1}^{n} \left(1 - \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi})\right)\right)^{\xi_\Xi}, \\ \prod_{\Xi=1}^{n} \left(1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}))\right)\right)^{\xi_\Xi}, \\ \prod_{\Xi=1}^{n} \left(1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi}))\right)\right)^{\xi_\Xi} \end{pmatrix}$$

Proof. We establish the proof of Theorem 4.2 by utilizing mathematical induction on n. For each Ξ,
$\eth_\Xi = \{(\mathfrak{T}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_\Xi}(\mathfrak{Z})), (\mathfrak{I}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_\Xi}(\mathfrak{Z})), (\mathfrak{F}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_\Xi}(\mathfrak{Z}))\} \in SVNZN(\Upsilon)$, which signifies that
$(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}), (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}), (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi}) \in [0,1]$ and $(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}) + (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}) + (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi}) \leq 3$. Subsequently, the subsequent stages of the mathematical induction process have been carried out.

Step-1: For $n = 2$, we get $ST\text{-}SVNZNWA(\eth_1, \eth_2) = \xi_1 \sin(\eth_1) \bigoplus \xi_2 \sin(\eth_2)$.
Since, by Definition 3.2, $\sin(\eth_1)$ and $\sin(\eth 2)$ are SVNZNs, it follows that $\xi 1 \sin(\eth 1) \bigoplus \xi 2 \sin(\eth_2)$ is also an SVNZN. Furthermore, in the case of $\eth_1$ and $\eth_2$, we have

$$\begin{aligned} ST\text{-}SVNZNWA(\eth_1, \eth_2) &= \xi_1 \sin(\eth_1) \bigoplus \xi_2 \sin(\eth_2) \\ &= \begin{pmatrix} 1 - \left(1 - \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1})\right)\right)^{\xi_1}, \\ \left(1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}))\right)\right)^{\xi_1}, \\ \left(1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1}))\right)\right)^{\xi_1} \end{pmatrix} \\ &\bigoplus \begin{pmatrix} 1 - \left(1 - \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2})\right)\right)^{\xi_2}, \\ \left(1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2}))\right)\right)^{\xi_2}, \\ \left(1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{F}_{\mathfrak{V}_2}, \mathfrak{F}_{\mathfrak{R}_2}))\right)\right)^{\xi_2} \end{pmatrix} \\ &= \begin{pmatrix} 1 - \prod_{\Xi=1}^{2} \left(1 - \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi})\right)\right)^{\xi_\Xi}, \\ \prod_{\Xi=1}^{2} \left(1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}))\right)\right)^{\xi_\Xi}, \\ \prod_{\Xi=1}^{2} \left(1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi}))\right)\right)^{\xi_\Xi} \end{pmatrix} \end{aligned} \tag{1}$$

Step-2: Assume that Equation (1) holds for $n = \kappa$. Consequently, we have:

$$ST\text{-}SVNZNWA\left(\eth_1,\eth_2,...\eth_\kappa\right)=\begin{pmatrix}1-\prod_{\Xi=1}^{\kappa}\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_\Xi},\mathfrak{T}_{\mathfrak{R}_\Xi}\right)\right)\right)^{\xi_\Xi},\\ \prod_{\Xi=1}^{\kappa}\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_\Xi},\mathfrak{I}_{\mathfrak{R}_\Xi}\right)\right)\right)\right)^{\xi_\Xi},\\ \prod_{\Xi=1}^{\kappa}\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_\Xi},\mathfrak{F}_{\mathfrak{R}_\Xi}\right)\right)\right)\right)^{\xi_\Xi}\end{pmatrix}$$

Step-3: Our next objective is to demonstrate that Equation (1) holds for $n=\kappa+1$.

$$\begin{aligned}ST\text{-}SVNZNWA\left(\eth_1,\eth_2,...\eth_{\kappa+1}\right) &= \sum_{\Xi=1}^{\kappa}\xi_\Xi\sin\left(\eth_\Xi\right)\bigoplus\xi_{\kappa+1}\sin\left(\eth_{\kappa+1}\right)\\ &= \begin{pmatrix}1-\prod_{\Xi=1}^{\kappa}\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_\Xi},\mathfrak{T}_{\mathfrak{R}_\Xi}\right)\right)\right)^{\xi_\Xi},\\ \prod_{\Xi=1}^{\kappa}\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_\Xi},\mathfrak{I}_{\mathfrak{R}_\Xi}\right)\right)\right)\right)^{\xi_\Xi},\\ \prod_{\Xi=1}^{\kappa}\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_\Xi},\mathfrak{F}_{\mathfrak{R}_\Xi}\right)\right)\right)\right)^{\xi_\Xi}\end{pmatrix}\\ &\bigoplus\begin{pmatrix}1-\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_{\kappa+1}},\mathfrak{T}_{\mathfrak{R}_{\kappa+1}}\right)\right)\right)^{\xi_{\kappa+1}},\\ \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_{\kappa+1}},\mathfrak{I}_{\mathfrak{R}_{\kappa+1}}\right)\right)\right)\right)^{\xi_{\kappa+1}},\\ \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_{\kappa+1}},\mathfrak{F}_{\mathfrak{R}_{\kappa+1}}\right)\right)\right)\right)^{\xi_{\kappa+1}}\end{pmatrix}\\ &= \begin{pmatrix}1-\prod_{\Xi=1}^{\kappa+1}\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_\Xi},\mathfrak{T}_{\mathfrak{R}_\Xi}\right)\right)\right)^{\xi_\Xi},\\ \prod_{\Xi=1}^{\kappa+1}\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_\Xi},\mathfrak{I}_{\mathfrak{R}_\Xi}\right)\right)\right)\right)^{\xi_\Xi},\\ \prod_{\Xi=1}^{\kappa+1}\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_\Xi},\mathfrak{F}_{\mathfrak{R}_\Xi}\right)\right)\right)\right)^{\xi_\Xi}\end{pmatrix}\end{aligned}$$

Specifically, for $n=\kappa+1$, we need to establish that Equation (1) remains valid. Hence, it can be concluded that Equation (1) holds for all values of n. □

Example 4.3. Suppose

$$\eth_1=\left\{\left(0.22,0.34\right),\left(0.15,0.57\right),\left(0.66,0.18\right)\right\},$$

$$\eth_2=\left\{\left(0.17,0.63\right),\left(0.52,0.31\right),\left(0.37,0.28\right)\right\},$$

$$\eth_3=\left\{\left(0.71,0.38\right),\left(0.25,0.42\right),\left(0.25,0.67\right)\right\},$$

and

$$\eth_4=\left\{\left(0.32,0.56\right),\left(0.47,0.23\right),\left(0.35,0.41\right)\right\}$$

are the SVNZNs with $\xi=\left(0.245,0.239,0.254,0.262\right)^{\mathfrak{T}}$ is the WV. Initially, we determine the $\xi_\Xi=\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_\Xi},\mathfrak{T}_{\mathfrak{R}_\Xi}\right)\right)$ we get

$$\begin{aligned}\xi_1 &= \left(0.3387,0.5090\right);\xi_2=\left(0.2639,0.8358\right)\\ \xi_3 &= \left(0.8980,0.5621\right);\xi_4=\left(0.4818,0.7705\right)\end{aligned}$$

As a result, we obtain

$$\begin{aligned}\prod_{\Xi=1}^{4}\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_\Xi},\mathfrak{T}_{\mathfrak{R}_\Xi}\right)\right)\right)^{\xi_\Xi} &= (1-\xi_1)^{0.245}\times(1-\xi_2)^{0.239}\times\\ &\quad (1-\xi_3)^{0.254}\times(1-\xi_4)^{0.262}\\ &= (0.9036,0.8401)\times(0.9294,0.6493)\times\\ &\quad (0.5600,0.8108)\times(0.8418,0.6800)\\ &= (0.3959,0.3007)\end{aligned}$$

Similarly, if $m_\Xi=\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_\Xi},\mathfrak{I}_{\mathfrak{R}_\Xi}\right)\right)\right)$, we get

$$\begin{aligned} m_1 &= (0.9724,0.6252)\,;m_2=(0.6845,0.8838)\\ m_3 &= (0.9239,0.7902)\,;m_4=(0.7396,0.9354)\end{aligned}$$

As a result, we obtain

$$\begin{aligned}\prod_{\Xi=1}^{4}\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_\Xi},\mathfrak{I}_{\mathfrak{R}_\Xi}\right)\right)\right)\right)^{\xi_\Xi} &= (1-m_1)^{0.245}\times(1-m_2)^{0.239}\times\\ &\quad (1-m_3)^{0.254}\times(1-m_4)^{0.262}\\ &= (0.4150,0.7863)\times(0.7590,0.5978)\times\\ &\quad (0.5198,0.6726)\times(0.7029,0.4878)\\ &= (0.1151,0.1542)\end{aligned}$$

Similarly, if $n_\Xi=\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_\Xi},\mathfrak{F}_{\mathfrak{R}_\Xi}\right)\right)\right)$, we get

$$\begin{aligned} n_1 &= (0.5090,0.9603)\,;n_2=(0.8358,0.9048)\\ n_3 &= (0.9239,0.4955)\,;n_4=(0.8526,0.7997)\end{aligned}$$

As a result, we obtain

$$\begin{aligned}\prod_{\Xi=1}^{4}\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_\Xi},\mathfrak{F}_{\mathfrak{R}_\Xi}\right)\right)\right)\right)^{\xi_\Xi} &= (1-n_1)^{0.245}\times(1-n_2)^{0.239}\times\\ &\quad (1-n_3)^{0.254}\times(1-n_4)^{0.262}\\ &= (0.8401,0.4536)\times(0.6493,0.5700)\times\\ &\quad (0.5198,0.8405)\times(0.6055,0.6562)\\ &= (0.1717,0.1426)\end{aligned}$$

Therefore,

$$\begin{aligned} ST\text{-}SVNZNWA\left(\eth_1,\eth_2,\eth_3,\eth_4\right) &= \begin{pmatrix} 1-\prod_{\Xi=1}^{4}\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_\Xi},\mathfrak{T}_{\mathfrak{R}_\Xi}\right)\right)\right)^{\xi_\Xi}, \\ \prod_{\Xi=1}^{4}\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_\Xi},\mathfrak{I}_{\mathfrak{R}_\Xi}\right)\right)\right)\right)^{\xi_\Xi}, \\ \prod_{\Xi=1}^{4}\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_\Xi},\mathfrak{F}_{\mathfrak{R}_\Xi}\right)\right)\right)\right)^{\xi_\Xi} \end{pmatrix} \\ &= \left(1-\left(0.3959,0.3007\right),\left(0.1151,0.1542\right),\left(0.1717,0.1426\right)\right) \\ &= \left(\left(0.6041,0.6993\right),\left(0.1151,0.1542\right),\left(0.1717,0.1426\right)\right) \end{aligned}$$

Moving forward, we outline a series of properties associated with the proposed ST-SVNZNWA AO. Given that these AOs are rooted in the ST function, they exhibit attributes such as idempotency, boundedness, monotonicity, and symmetry.

Theorem 4.4. *(idempotency)*
Let $\eth_\Xi = \{(\mathfrak{T}_{\mathfrak{V}_\Xi}(\mathfrak{Z}),\mathfrak{T}_{\mathfrak{R}_\Xi}(\mathfrak{Z})),(\mathfrak{I}_{\mathfrak{V}_\Xi}(\mathfrak{Z}),\mathfrak{I}_{\mathfrak{R}_\Xi}(\mathfrak{Z})),(\mathfrak{F}_{\mathfrak{V}_\Xi}(\mathfrak{Z}),\mathfrak{F}_{\mathfrak{R}_\Xi}(\mathfrak{Z}))\}$ *belong to* $SVNZN(\Upsilon)\,(\Xi=1,2,3,...,n)$ *where* $\eth_\Xi=\eth$. *Then* $ST\text{-}SVNZNWA(\eth_1,\eth_2,...,\eth_n)=\sin(\eth)$.

Proof. Since $\eth_\Xi=\eth$ $(\Xi=1,2,3,...,n)$, we can apply Theorem 4.2 to deduce:

$$\begin{aligned} ST\text{-}SVNZNWA\left(\eth_1,\eth_2,...,\eth_n\right) &= \begin{pmatrix} 1-\prod_{\Xi=1}^{n}\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_\Xi},\mathfrak{T}_{\mathfrak{R}_\Xi}\right)\right)\right)^{\xi_\Xi}, \\ \prod_{\Xi=1}^{n}\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_\Xi},\mathfrak{I}_{\mathfrak{R}_\Xi}\right)\right)\right)\right)^{\xi_\Xi}, \\ \prod_{\Xi=1}^{n}\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_\Xi},\mathfrak{F}_{\mathfrak{R}_\Xi}\right)\right)\right)\right)^{\xi_\Xi} \end{pmatrix} \\ &= \begin{pmatrix} 1-\prod_{\Xi=1}^{n}\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_{ji}},\mathfrak{T}_{\mathfrak{R}_{ji}}\right)\right)\right)^{\xi_\Xi}, \\ \prod_{\Xi=1}^{n}\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_{ji}},\mathfrak{I}_{\mathfrak{R}_{ji}}\right)\right)\right)\right)^{\xi_\Xi}, \\ \prod_{\Xi=1}^{n}\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_{ji}},\mathfrak{F}_{\mathfrak{R}_{ji}}\right)\right)\right)\right)^{\xi_\Xi} \end{pmatrix} \\ &= \begin{pmatrix} 1-\left(1-\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_{ji}},\mathfrak{T}_{\mathfrak{R}_{ji}}\right)\right)\right)^{\sum_{\Xi=1}^{n}\xi_\Xi}, \\ \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_{ji}},\mathfrak{I}_{\mathfrak{R}_{ji}}\right)\right)\right)\right)^{\sum_{\Xi=1}^{n}\xi_\Xi}, \\ \left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_{ji}},\mathfrak{F}_{\mathfrak{R}_{ji}}\right)\right)\right)\right)^{\sum_{\Xi=1}^{n}\xi_\Xi} \end{pmatrix} \\ &= \begin{pmatrix} \sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_{ji}},\mathfrak{T}_{\mathfrak{R}_{ji}}\right)\right),1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_{ji}},\mathfrak{I}_{\mathfrak{R}_{ji}}\right)\right)\right), \\ 1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_{ji}},\mathfrak{F}_{\mathfrak{R}_{ji}}\right)\right)\right) \end{pmatrix} \\ &= \sin\left(\eth\right) \end{aligned}$$

□

Theorem 4.5. *(Boundedness)*
Let $\eth_\Xi = \{(\mathfrak{T}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_\Xi}(\mathfrak{Z})), (\mathfrak{I}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_\Xi}(\mathfrak{Z})), (\mathfrak{F}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_\Xi}(\mathfrak{Z}))\}$,
$\eth_\Xi^- = \{\min((\mathfrak{T}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_\Xi}(\mathfrak{Z}))), \max((\mathfrak{I}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_\Xi}(\mathfrak{Z}))), \max((\mathfrak{F}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_\Xi}(\mathfrak{Z})))\}$ *and* $\eth_\Xi^+ = \{\max((\mathfrak{T}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_\Xi}(\mathfrak{Z}))), \min((\mathfrak{I}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_\Xi}(\mathfrak{Z}))), \min((\mathfrak{F}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_\Xi}(\mathfrak{Z})))\} \in SVNZN(\Upsilon)$ $(\Xi = 1, 2, 3, ..., n)$. *Then,* $\sin(\eth_\Xi^-) \leq ST\text{-}SVNZNWA(\eth_1, \eth_2, ..., \eth_n) \leq \sin(\eth_\Xi^+)$.

Proof. For any value of Ξ, $\min_\Xi((\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi})) \leq (\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}) \leq \min_\Xi((\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}))$, $\min_\Xi((\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi})) \leq (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}) \leq \min_\Xi((\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}))$ and $\min_\Xi((\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})) \leq (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi}) \leq \min_\Xi((\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi}))$. This implies that $\eth_\Xi^- \leq \eth_\Xi \leq \eth_\Xi^+$. Suppose that $ST\text{-}SVNZNWA(\eth_1, \eth_2, ...\eth_n) = \sin(\eth_\Xi) = \{(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}), (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}), (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})\}$, $\sin(\eth_\Xi^-) = \{(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi})^-, (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi})^-, (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})^-\}$ and $\sin(\eth_\Xi^+) = \{(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi})^+, (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi})^+, (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})^+\}$. Then, leveraging the monotonic nature of the sine function, we observe that

$$\begin{aligned}
(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}) &= 1 - \prod_{\Xi=1}^{n} \left(1 - \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi})\right)\right)^{\xi_\Xi} \\
&\geq 1 - \prod_{\Xi=1}^{n} \left(1 - \sin\left(\frac{\Pi}{2}\min_\Xi((\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}))\right)\right)^{\xi_\Xi} \\
&= \sin\left(\frac{\Pi}{2}\min_\Xi((\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}))\right) = (\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi})^-
\end{aligned}$$

and,

$$\begin{aligned}
(\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}) &= \prod_{\Xi=1}^{n} \left(1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}))\right)\right)^{\xi_\Xi} \\
&\geq \prod_{\Xi=1}^{n} \left(1 - \sin\left(\frac{\Pi}{2}(1 - (\min_\Xi(\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi})))\right)\right)^{\xi_\Xi} \\
&= 1 - \sin\left(\frac{\Pi}{2}(1 - (\min_\Xi(\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi})))\right) = (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi})^-
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi}) &= \prod_{\Xi=1}^{n} \left(1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi}))\right)\right)^{\xi_\Xi} \\
&\geq \prod_{\Xi=1}^{n} \left(1 - \sin\left(\frac{\Pi}{2}(1 - (\min_\Xi(\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})))\right)\right)^{\xi_\Xi} \\
&= 1 - \sin\left(\frac{\Pi}{2}(1 - (\min_\Xi(\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})))\right) = (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})^-
\end{aligned}$$

Also, we have

$$\begin{aligned}(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}) &= 1-\prod_{\Xi=1}^{n}\left(1-\sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi})\right)\right)^{\xi_\Xi} \\ &\leq 1-\prod_{\Xi=1}^{n}\left(1-\sin\left(\frac{\Pi}{2}\max_\Xi\left((\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi})\right)\right)\right)^{\xi_\Xi} \\ &= \sin\left(\frac{\Pi}{2}\max_\Xi\left((\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi})\right)\right) = (\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi})^{+}\end{aligned}$$

and

$$\begin{aligned}(\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}) &= \prod_{\Xi=1}^{n}\left(1-\sin\left(\frac{\Pi}{2}(1-(\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}))\right)\right)^{\xi_\Xi} \\ &\leq \prod_{\Xi=1}^{n}\left(1-\sin\left(\frac{\Pi}{2}(1-(\max_\Xi(\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi})))\right)\right)^{\xi_\Xi} \\ &= 1-\sin\left(\frac{\Pi}{2}(1-(\max_\Xi(\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi})))\right) = (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi})^{+}\end{aligned}$$

Similarly,

$$\begin{aligned}(\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi}) &= \prod_{\Xi=1}^{n}\left(1-\sin\left(\frac{\Pi}{2}(1-(\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi}))\right)\right)^{\xi_\Xi} \\ &\leq \prod_{\Xi=1}^{n}\left(1-\sin\left(\frac{\Pi}{2}(1-(\max_\Xi(\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})))\right)\right)^{\xi_\Xi} \\ &= 1-\sin\left(\frac{\Pi}{2}(1-(\max_\Xi(\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})))\right) = (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})^{+}\end{aligned}$$

Based on the score function, we get

$$\begin{aligned}\overline{sc}(\sin(\eth_\Xi)) &= (\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}) - (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}) - (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi}) \\ &\leq (\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi})^{+} - (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi})^{-} - (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})^{-} = \overline{sc}\left(\sin\left(\eth_\Xi^{+}\right)\right)\end{aligned}$$

and

$$\begin{aligned}\overline{sc}(\sin(\eth_\Xi)) &= (\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}) - (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}) - (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi}) \\ &\geq (\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi})^{-} - (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi})^{+} - (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})^{+} = \overline{sc}\left(\sin\left(\eth_\Xi^{-}\right)\right)\end{aligned}$$

□

Thus, we have $\overline{sc}\left(\sin\left(\eth_\Xi^{-}\right)\right) \leq \overline{sc}(\sin(\eth_\Xi)) \leq \overline{sc}\left(\sin\left(\eth_\Xi^{+}\right)\right)$. We will now delve into a discussion of the three cases:

(Case-1): If $\overline{sc}\left(\sin\left(\eth_\Xi^{-}\right)\right) < \overline{sc}(\sin(\eth_\Xi)) < \overline{sc}\left(\sin\left(\eth_\Xi^{+}\right)\right)$, then the conclusion remains valid.

(Case-2): If $\overline{sc}\left(\sin\left(\eth_{\Xi}^{+}\right)\right) = \overline{sc}\left(\sin\left(\eth_{\Xi}\right)\right)$, then we have: $(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi})^{+} - (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi})^{+} - (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})^{+} = (\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}) - (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}) - (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})$. This implies that $(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi})^{+} = (\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi})$, $(\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi})^{+} = (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi})$, and $(\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})^{+} = (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})$. Consequently, $\underline{ac}\left(\sin\left(\eth_{\Xi}\right)\right) = \underline{ac}\left(\sin\left(\eth_{\Xi}^{+}\right)\right)$.

(Case-3): If $\overline{sc}\left(\sin\left(\eth_{\Xi}\right)\right) = \overline{sc}\left(\sin\left(\eth_{\Xi}^{-}\right)\right)$, then we have: $(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}) - (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}) - (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi}) = (\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi})^{-} - (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi})^{-} - (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})^{-}$. This implies that $(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}) = (\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi})^{-}$, $(\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}) = (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi})^{-}$, and $(\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi}) = (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})^{-}$. Consequently, $\underline{ac}\left(\sin\left(\eth_{\Xi}\right)\right) = \underline{ac}\left(\sin\left(\eth_{\Xi}^{-}\right)\right)$. Therefore, we ultimately establish $\sin\left(\eth_{\Xi}^{-}\right) \leq ST\text{-}SVNZNWA\left(\eth_1, \eth_2, ..., \eth_n\right) \leq \sin\left(\eth_{\Xi}^{+}\right)$.

Theorem 4.6. *(Monotonically)*
Let $\eth\Xi = \{(\mathfrak{T}\mathfrak{V}_{\Xi}(\mathfrak{z}), \mathfrak{T}_{\mathfrak{R}_\Xi}(\mathfrak{z})), (\mathfrak{I}_{\mathfrak{V}_\Xi}(\mathfrak{z}), \mathfrak{I}_{\mathfrak{R}_\Xi}(\mathfrak{z})), (\mathfrak{F}_{\mathfrak{V}_\Xi}(\mathfrak{z}), \mathfrak{F}_{\mathfrak{R}_\Xi}(\mathfrak{z}))\}$ *and* $\eth\Xi^{*} = \{(\mathfrak{T}\mathfrak{V}_{\Xi}(\mathfrak{z}), \mathfrak{T}_{\mathfrak{R}_\Xi}(\mathfrak{z}))^{*}, (\mathfrak{I}_{\mathfrak{V}_\Xi}(\mathfrak{z}), \mathfrak{I}_{\mathfrak{R}_\Xi}(\mathfrak{z}))^{*}, (\mathfrak{F}_{\mathfrak{V}_\Xi}(\mathfrak{z}), \mathfrak{F}_{\mathfrak{R}_\Xi}(\mathfrak{z}))^{*}\} \in SVNZN(\curlyvee)$ $(\Xi = 1, 2, 3, ..., n)$. *If* $(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}) \leq (\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi})^{*}$, $(\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}) \leq (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi})^{*}$, *and* $(\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi}) \leq (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})^{*}$, *then:*
$ST\text{-}SVNZNWA\left(\eth_1, \eth_2, ..., \eth_n\right) \leq ST\text{-}SVNZNWA\left(\eth_1^{*}, \eth_2^{*}, ..., \eth_n^{*}\right)$.

Proof. Indeed, this conclusion is a direct consequence of Theorem 4.5, and as such, it is not necessary to elaborate further on this point. □

Theorem 4.7. *(Symmetric)*
Let $\eth\Xi = \{(\mathfrak{T}\mathfrak{V}_{\Xi}(\mathfrak{z}), \mathfrak{T}_{\mathfrak{R}_\Xi}(\mathfrak{z})), (\mathfrak{I}_{\mathfrak{V}_\Xi}(\mathfrak{z}), \mathfrak{I}_{\mathfrak{R}_\Xi}(\mathfrak{z})), (\mathfrak{F}_{\mathfrak{V}_\Xi}(\mathfrak{z}), \mathfrak{F}_{\mathfrak{R}_\Xi}(\mathfrak{z}))\}$ *and* $\eth\Xi^{*} = \{(\mathfrak{T}\mathfrak{V}_{\Xi}(\mathfrak{z}), \mathfrak{T}_{\mathfrak{R}_\Xi}(\mathfrak{z}))^{*}, (\mathfrak{I}_{\mathfrak{V}_\Xi}(\mathfrak{z}), \mathfrak{I}_{\mathfrak{R}_\Xi}(\mathfrak{z}))^{*}, (\mathfrak{F}_{\mathfrak{V}_\Xi}(\mathfrak{z}), \mathfrak{F}_{\mathfrak{R}_\Xi}(\mathfrak{z}))^{*}\} \in SVNZN(\curlyvee)$ $(\Xi = 1, 2, 3, ..., n)$.
Then, we have: $ST\text{-}SVNZNWA\left(\eth_1, \eth_2, ..., \eth_n\right) = ST - SVNZNWA\left(\eth_1^{*}, \eth_2^{*}, ..., \eth_n^{*}\right)$, *whenever* $\eth_{\Xi}^{*}$ $(\Xi = 1, 2, 3, ..., n)$ *is any version of* $\eth_{\Xi}$ $(\Xi = 1, 2, 3, ..., n)$.

Proof. Indeed, this conclusion is a direct consequence of Theorem 4.5, and as such, it is not necessary to elaborate further on this point. □

4.2. *Sine Trigonometric Weighted Geometric AOs for SVNZNs*

Definition 4.8. Let $\eth_{\Xi} = \{(\mathfrak{T}_{\mathfrak{V}_\Xi}(\mathfrak{z}), \mathfrak{T}_{\mathfrak{R}_\Xi}(\mathfrak{z})), (\mathfrak{I}_{\mathfrak{V}_\Xi}(\mathfrak{z}), \mathfrak{I}_{\mathfrak{R}_\Xi}(\mathfrak{z})), (\mathfrak{F}_{\mathfrak{V}_\Xi}(\mathfrak{z}), \mathfrak{F}_{\mathfrak{R}_\Xi}(\mathfrak{z}))\} \in SVNZN(\curlyvee)$ $(\Xi = 1, 2, 3, ..., n)$. Then, the ST-SVNZNWG operator for $SVNZN(\curlyvee)$ is designated as the ST weighted geometric AO, and defined in the following manner:

$$\begin{aligned} ST\text{-}SVNZNWG\left(\eth_1, \eth_2, ..., \eth_n\right) &= \left(\sin\left(\eth_1\right)\right)^{\xi_1} \bigotimes \left(\sin\left(\eth_2\right)\right)^{\xi_2} \bigotimes ... \bigotimes \left(\sin\left(\eth_n\right)\right)^{\xi_n} \\ &= \prod_{\Xi=1}^{n} \left(\sin\left(\eth_{\Xi}\right)\right)^{\xi_{\Xi}}. \end{aligned}$$

Here, $\xi_{\Xi}\,(\Xi = 1, 2, ..., n)$ denotes the weights assigned to $\eth\Xi\,(\Xi = 1, 2, 3, ..., n)$, where $\xi\Xi \geq 0$ and the summation over all Ξ values is constrained to be equal to 1.

Theorem

4.9. *Assume that* $\eth\Xi = \{(\mathfrak{T}\mathfrak{V}_{\Xi}\,(\mathfrak{Z})\,, \mathfrak{T}_{\mathfrak{R}_{\Xi}}\,(\mathfrak{Z}))\,, (\mathfrak{I}_{\mathfrak{V}_{\Xi}}\,(\mathfrak{Z})\,, \mathfrak{I}_{\mathfrak{R}_{\Xi}}\,(\mathfrak{Z})), (\mathfrak{F}_{\mathfrak{V}_{\Xi}}\,(\mathfrak{Z})\,, \mathfrak{F}_{\mathfrak{R}_{\Xi}}\,(\mathfrak{Z}))\} \in SVNZN\,(\Upsilon)$ *for* $(\Xi = 1, 2, 3, ..., n)$*. Additionally, the weight vector (WV) associated with each* $\eth\Xi\,(\Xi = 1, 2, 3, ..., n)$ *is denoted by* $\xi = (\xi 1, \xi_2, ..., \xi_n)^{\mathfrak{T}}$*, with the requirement that* $\sum\limits_{\Xi=1}^{n} \xi_{\Xi} = 1$*. The ST-SVNZNWG operator is then a function mapping* G^n *to* G*, defined as follows:*

$$
\begin{aligned}
ST\text{-}SVNZNWG\,(\eth_1, \eth_2, ..., \eth_n) &= \prod_{\Xi=1}^{n} \left(\sin\left(\eth_{\Xi}\right)\right)^{\xi_{\Xi}} \\
&= \begin{pmatrix} \prod_{\Xi=1}^{n} \left(\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_{\Xi}}, \mathfrak{T}_{\mathfrak{R}_{\Xi}}\right)\right)\right)^{\xi_{\Xi}}, \\ 1 - \prod_{\Xi=1}^{n} \left(\sin\left(\frac{\Pi}{2}\left(1 - \left(\mathfrak{I}_{\mathfrak{V}_{\Xi}}, \mathfrak{I}_{\mathfrak{R}_{\Xi}}\right)\right)\right)\right)^{\xi_{\Xi}}, \\ 1 - \prod_{\Xi=1}^{n} \left(\sin\left(\frac{\Pi}{2}\left(1 - \left(\mathfrak{F}_{\mathfrak{V}_{\Xi}}, \mathfrak{F}_{\mathfrak{R}_{\Xi}}\right)\right)\right)\right)^{\xi_{\Xi}} \end{pmatrix} \qquad (2)
\end{aligned}
$$

Proof. We establish the proof for Theorem 4.9 through mathematical induction based on n. For each value of Ξ, $\eth\Xi = \{(\mathfrak{T}\mathfrak{V}_{\Xi}\,(\mathfrak{Z})\,, \mathfrak{T}_{\mathfrak{R}_{\Xi}}\,(\mathfrak{Z}))\,, (\mathfrak{I}_{\mathfrak{V}_{\Xi}}\,(\mathfrak{Z})\,, \mathfrak{I}_{\mathfrak{R}_{\Xi}}\,(\mathfrak{Z})), (\mathfrak{F}_{\mathfrak{V}_{\Xi}}\,(\mathfrak{Z})\,, \mathfrak{F}_{\mathfrak{R}_{\Xi}}\,(\mathfrak{Z}))\} \in SVNZN\,(\Upsilon)$. This implies that $(\mathfrak{T}_{\mathfrak{V}_{\Xi}}, \mathfrak{T}_{\mathfrak{R}_{\Xi}})\,, (\mathfrak{I}_{\mathfrak{V}_{\Xi}}, \mathfrak{I}_{\mathfrak{R}_{\Xi}}), (\mathfrak{F}_{\mathfrak{V}_{\Xi}}, \mathfrak{F}_{\mathfrak{R}_{\Xi}}) \in [0, 1]$ and $(\mathfrak{T}_{\mathfrak{V}_{\Xi}}, \mathfrak{T}_{\mathfrak{R}_{\Xi}}) + (\mathfrak{I}_{\mathfrak{V}_{\Xi}}, \mathfrak{I}_{\mathfrak{R}_{\Xi}}) + (\mathfrak{F}_{\mathfrak{V}_{\Xi}}, \mathfrak{F}_{\mathfrak{R}_{\Xi}}) \leq 3$. Following this, we proceed with the steps of mathematical induction.

Step-1: When $n = 2$, then the equation becomes as: $ST\text{-}SVNZNWG\,(\eth_1, \eth_2) = (\sin(\eth_1))^{\xi_1} \bigotimes (\sin(\eth_2))^{\xi_2}$. Since according to Definition 3.2, we know that $\sin(\eth_1)$ and $\sin(\eth_2)$ are both SVNZNs, hence, it follows that $(\sin(\eth 1))^{\xi 1} \bigotimes (\sin(\eth 2))^{\xi 2}$ also exhibits the properties of SVNZN. Moving forward, when considering $\eth_1$ and $\eth_2$, we observe that

$$
\begin{aligned}
ST\text{-}SVNZNWG\,(\eth_1, \eth_2) &= (\sin(\eth_1))^{\xi_1} \bigotimes (\sin(\eth_2))^{\xi_2} \\
&= \begin{pmatrix} \left(\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1}\right)\right)\right)^{\xi_1}, \\ 1 - \left(\sin\left(\frac{\Pi}{2}\left(1 - \left(\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}\right)\right)\right)\right)^{\xi_1}, \\ 1 - \left(\sin\left(\frac{\Pi}{2}\left(1 - \left(\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1}\right)\right)\right)\right)^{\xi_1} \end{pmatrix} \\
&\quad \bigotimes \begin{pmatrix} \left(\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2}\right)\right)\right)^{\xi_2}, \\ 1 - \left(\sin\left(\frac{\Pi}{2}\left(1 - \left(\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2}\right)\right)\right)\right)^{\xi_2}, \\ 1 - \left(\sin\left(\frac{\Pi}{2}\left(1 - \left(\mathfrak{F}_{\mathfrak{V}_2}, \mathfrak{F}_{\mathfrak{R}_2}\right)\right)\right)\right)^{\xi_2} \end{pmatrix} \\
&= \begin{pmatrix} \prod_{\Xi=1}^{2} \left(\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_{\Xi}}, \mathfrak{T}_{\mathfrak{R}_{\Xi}}\right)\right)\right)^{\xi_{\Xi}}, \\ 1 - \prod_{\Xi=1}^{2} \left(\sin\left(\frac{\Pi}{2}\left(1 - \left(\mathfrak{I}_{\mathfrak{V}_{\Xi}}, \mathfrak{I}_{\mathfrak{R}_{\Xi}}\right)\right)\right)\right)^{\xi_{\Xi}}, \\ 1 - \prod_{\Xi=1}^{2} \left(\sin\left(\frac{\Pi}{2}\left(1 - \left(\mathfrak{F}_{\mathfrak{V}_{\Xi}}, \mathfrak{F}_{\mathfrak{R}_{\Xi}}\right)\right)\right)\right)^{\xi_{\Xi}} \end{pmatrix}
\end{aligned}
$$

Step-2: Assuming that Equation (2) holds for $n = \kappa$, we can then conclude:

$$ST\text{-}SVNZNWG\left(\eth_1, \eth_2, ..., \eth_\kappa\right) = \begin{pmatrix} \prod_{\Xi=1}^{\kappa}\left(\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}\right)\right)\right)^{\xi_\Xi}, \\ 1-\prod_{\Xi=1}^{\kappa}\left(\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}\right)\right)\right)\right)^{\xi_\Xi}, \\ 1-\prod_{\Xi=1}^{\kappa}\left(\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi}\right)\right)\right)\right)^{\xi_\Xi} \end{pmatrix}$$

Step-3: Our next step is to demonstrate the validity of Equation (2) for $n = \kappa + 1$.

$$\begin{aligned} ST\text{-}SVNZNWG\left(\eth_1, \eth_2, ..., \eth_{\kappa+1}\right) &= \prod_{\Xi=1}^{\kappa}\left(\sin\left(\eth_\Xi\right)\right)^{\xi_\Xi} \bigotimes \left(\sin\left(\eth_{\kappa+1}\right)\right)^{\xi_{\kappa+1}} \\ &= \begin{pmatrix} \prod_{\Xi=1}^{\kappa}\left(\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}\right)\right)\right)^{\xi_\Xi}, \\ 1-\prod_{\Xi=1}^{\kappa}\left(\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}\right)\right)\right)\right)^{\xi_\Xi}, \\ 1-\prod_{\Xi=1}^{\kappa}\left(\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi}\right)\right)\right)\right)^{\xi_\Xi} \end{pmatrix} \\ &\quad \bigotimes \begin{pmatrix} \left(\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_{\kappa+1}}, \mathfrak{T}_{\mathfrak{R}_{\kappa+1}}\right)\right)\right)^{\xi_{\kappa+1}}, \\ 1-\left(\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_{\kappa+1}}, \mathfrak{I}_{\mathfrak{R}_{\kappa+1}}\right)\right)\right)\right)^{\xi_{\kappa+1}}, \\ 1-\left(\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_{\kappa+1}}, \mathfrak{F}_{\mathfrak{R}_{\kappa+1}}\right)\right)\right)\right)^{\xi_{\kappa+1}} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \prod_{\Xi=1}^{\kappa+1}\left(\sin\left(\frac{\Pi}{2}\left(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}\right)\right)\right)^{\xi_\Xi}, \\ 1-\prod_{\Xi=1}^{\kappa+1}\left(\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}\right)\right)\right)\right)^{\xi_\Xi}, \\ 1-\prod_{\Xi=1}^{\kappa+1}\left(\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi}\right)\right)\right)\right)^{\xi_\Xi} \end{pmatrix}$$

In other words, Equation (2) holds true when $n = \kappa + 1$.

Consequently, we can conclude that Equation (2) holds for all values of n. □

Theorem 4.10. *(idempotency)*
Let $\eth_\Xi = \{(\mathfrak{T}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_\Xi}(\mathfrak{Z})), (\mathfrak{I}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_\Xi}(\mathfrak{Z})), (\mathfrak{F}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_\Xi}(\mathfrak{Z}))\} \in SVNZN(\Upsilon)$ $(\Xi = 1, 2, 3, ..., n)$ *such that* $\eth_\Xi = \eth$. *Then,* $ST\text{-}SVNZNWG(\eth_1, \eth_2, ..., \eth_n) = \sin(\eth)$.

Theorem 4.11. *(Boundedness)*
Let $\eth_\Xi = \{(\mathfrak{T}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_\Xi}(\mathfrak{Z})), (\mathfrak{I}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_\Xi}(\mathfrak{Z})), (\mathfrak{F}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_\Xi}(\mathfrak{Z}))\}$,
$\eth_\Xi^- = \{\min((\mathfrak{T}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_\Xi}(\mathfrak{Z}))), \max((\mathfrak{I}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_\Xi}(\mathfrak{Z}))), \max((\mathfrak{F}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_\Xi}(\mathfrak{Z})))\}$ *and*
$\eth_\Xi^+ = \{\max((\mathfrak{T}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_\Xi}(\mathfrak{Z}))), \min((\mathfrak{I}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_\Xi}(\mathfrak{Z}))), \min((\mathfrak{F}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_\Xi}(\mathfrak{Z})))\} \in SVNZN(\Upsilon)$
$(\Xi = 1, 2, 3, ..., n)$. *Then,* $\sin(\eth_\Xi^-) \leq ST\text{-}SVNZNWG(\eth_1, \eth_2, ..., \eth_n) \leq \sin(\eth_\Xi^+)$.

Theorem 4.12. *(Monotonically)*
Let $\eth_\Xi = \{(\mathfrak{T}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_\Xi}(\mathfrak{Z})), (\mathfrak{I}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_\Xi}(\mathfrak{Z})), (\mathfrak{F}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_\Xi}(\mathfrak{Z}))\}$,
$\eth_\Xi^* = \{(\mathfrak{T}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_\Xi}(\mathfrak{Z}))^*, (\mathfrak{I}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_\Xi}(\mathfrak{Z}))^*, (\mathfrak{F}_{\mathfrak{V}_\Xi}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_\Xi}(\mathfrak{Z}))^*\} \in SVNZN(\Upsilon)$ $(\Xi = 1, 2, 3, ..., n)$.
If $(\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi}) \leq (\mathfrak{T}_{\mathfrak{V}_\Xi}, \mathfrak{T}_{\mathfrak{R}_\Xi})^*$, $(\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi}) \leq (\mathfrak{I}_{\mathfrak{V}_\Xi}, \mathfrak{I}_{\mathfrak{R}_\Xi})^*$ *and* $(\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi}) \leq (\mathfrak{F}_{\mathfrak{V}_\Xi}, \mathfrak{F}_{\mathfrak{R}_\Xi})^*$, *then* $ST\text{-}SVNZNWG(\eth_1, \eth_2, ..., \eth_n) \leq ST\text{-}SVNZNWG(\eth_1^*, \eth_2^*, ..., \eth_n^*)$.

Theorem 4.13. *(Symmetric)*
Let $\eth_{\Xi} = \{(\mathfrak{T}_{\mathfrak{V}_{\Xi}}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_{\Xi}}(\mathfrak{Z})), (\mathfrak{I}_{\mathfrak{V}_{\Xi}}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_{\Xi}}(\mathfrak{Z})), (\mathfrak{F}_{\mathfrak{V}_{\Xi}}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_{\Xi}}(\mathfrak{Z}))\}$, $\eth_{\Xi}^{*} = \{(\mathfrak{T}_{\mathfrak{V}_{\Xi}}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_{\Xi}}(\mathfrak{Z}))^{*}, (\mathfrak{I}_{\mathfrak{V}_{\Xi}}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_{\Xi}}(\mathfrak{Z}))^{*}, (\mathfrak{F}_{\mathfrak{V}_{\Xi}}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_{\Xi}}(\mathfrak{Z}))^{*}\} \in SVNZN(\Upsilon)$ $(\Xi = 1, 2, 3, ..., n)$. *Then* $ST\text{-}SVNZNWG(\eth_1, \eth_2, ..., \eth_n) = ST\text{-}SVNZNWG(\eth_1^{*}, \eth_2^{*}, ..., \eth_n^{*})$, *whenever* $\eth_{\Xi}^{*}$ $(\Xi = 1, 2, 3, ..., n)$ *is any of* $\eth_{\Xi}$ $(\Xi = 1, 2, 3, ..., n)$.

Proof. Proofs of the above theorems, Theorem 4.10–4.13 follow from Theorems 4.4–4.7 likewise.

□

4.3. *Fundamental Properties of the Proposed AOs for SVNZNs*

In this section, we have delved into different connections among the suggested AOs and analyzed some of their essential characteristics.

Theorem 4.14. *Let* $\eth_{\Xi} = \{(\mathfrak{T}_{\mathfrak{V}_{\Xi}}(\mathfrak{Z}), \mathfrak{T}_{\mathfrak{R}_{\Xi}}(\mathfrak{Z})), (\mathfrak{I}_{\mathfrak{V}_{\Xi}}(\mathfrak{Z}), \mathfrak{I}_{\mathfrak{R}_{\Xi}}(\mathfrak{Z})), (\mathfrak{F}_{\mathfrak{V}_{\Xi}}(\mathfrak{Z}), \mathfrak{F}_{\mathfrak{R}_{\Xi}}(\mathfrak{Z}))\} \in SVNZN(\Upsilon)$ $(\Xi = 1, 2)$. *Then,*

$$\sin(\eth_1) \bigoplus \sin(\eth_2) \geq \sin(\eth_1) \bigotimes \sin(\eth_2).$$

Proof. Given that $\eth_{\Xi} \in SVNZN(\Upsilon)$ $(\Xi = 1, 2)$, we can apply Definition 3.4 to obtain:

$$\sin(\eth_1) \bigoplus \sin(\eth_2) = \begin{pmatrix} 1 - \left(1 - \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1})\right)\right)\left(1 - \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2})\right)\right), \\ \left(1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}))\right)\right)\left(1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2}))\right)\right), \\ \left(1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1}))\right)\right)\left(1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{F}_{\mathfrak{V}_2}, \mathfrak{F}_{\mathfrak{R}_2}))\right)\right) \end{pmatrix}$$

and

$$\sin(\eth_1) \bigotimes \sin(\eth_2) = \begin{pmatrix} \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1})\right)\sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2})\right), \\ 1 - \left(\sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}))\right)\right)\left(\sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2}))\right)\right), \\ 1 - \left(\sin\left(\frac{\Pi}{2}(1 - (\mathfrak{F}_{\mathfrak{V}_1}, \mathfrak{F}_{\mathfrak{R}_1}))\right)\right)\left(\sin\left(\frac{\Pi}{2}(1 - (\mathfrak{F}_{\mathfrak{V}_2}, \mathfrak{F}_{\mathfrak{R}_2}))\right)\right) \end{pmatrix}$$

For any pair of non-negative real numbers ξ and m, we know that their arithmetic mean is greater than or equal to their geometric mean, expressed as $\frac{\xi+m}{2} \geq \sqrt{lm}$. This inequality can be rearranged as $\xi + m - 2\sqrt{lm} \geq 0$, which further simplifies to $1 - \sqrt{1-\xi}\sqrt{1-m} \geq lm$.
Hence, by considering $\xi = \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1})\right)$ and $m = \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2})\right)$, we obtain
$1 - \left(1 - \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1})\right)\right)\left(1 - \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2})\right)\right) \geq \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1})\right)\sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2})\right)$, which leads to
$1 - \left(1 - \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1})\right)\right)\left(1 - \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2})\right)\right) \geq \sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_1}, \mathfrak{T}_{\mathfrak{R}_1})\right)\sin\left(\frac{\Pi}{2}(\mathfrak{T}_{\mathfrak{V}_2}, \mathfrak{T}_{\mathfrak{R}_2})\right)$.
Likewise, we obtain
$\left(1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}))\right)\right)\left(1 - \sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2}))\right)\right) \leq 1 - \left(\sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_1}, \mathfrak{I}_{\mathfrak{R}_1}))\right)\right)\left(\sin\left(\frac{\Pi}{2}(1 - (\mathfrak{I}_{\mathfrak{V}_2}, \mathfrak{I}_{\mathfrak{R}_2}))\right)\right)$
and

$\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_1},\mathfrak{F}_{\mathfrak{R}_1}\right)\right)\right)\right)\left(1-\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_2},\mathfrak{F}_{\mathfrak{R}_2}\right)\right)\right)\right) \leq$
$1-\left(\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_1},\mathfrak{F}_{\mathfrak{R}_1}\right)\right)\right)\right)\left(\sin\left(\frac{\Pi}{2}\left(1-\left(\mathfrak{F}_{\mathfrak{V}_2},\mathfrak{F}_{\mathfrak{R}_2}\right)\right)\right)\right)$. Therefore,

$$\sin(\eth_1)\bigoplus\sin(\eth_2)\geq\sin(\eth_1)\bigotimes\sin(\eth_2).$$

□

Theorem 4.15. *Let* $\eth=\left\{\left(\mathfrak{T}_{\mathfrak{V}_{ji}}(\mathfrak{Z}),\mathfrak{T}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right),\left(\mathfrak{I}_{\mathfrak{V}_{ji}}(\mathfrak{Z}),\mathfrak{I}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right),\left(\mathfrak{F}_{\mathfrak{V}_{ji}}(\mathfrak{Z}),\mathfrak{F}_{\mathfrak{R}_{ji}}(\mathfrak{Z})\right)\right\}\in SVNZN(\Upsilon)$ *and* $\psi\geq 0$ *be any real number, then*

- **(1):** $\psi\sin(\eth)\geq(\sin(\eth))^{\psi}$ *if and only if* $\psi\geq 1$,
- **(2):** $\psi\sin(\eth)\leq(\sin(\eth))^{\psi}$ *if and only if* $0<\psi\leq 1$.

Proof. This can be deduced from Theorem 4.14 in a similar manner. □

Lemma 4.16. *For* $\xi_\Xi\geq 0$ *and* $m_\Xi\geq 0$, *then we have* $\prod_{\Xi=1}^{n}(\xi_\Xi)^{m_\Xi}\leq\sum_{\Xi=1}^{n}m_\Xi\xi_\Xi$ *and if* $\xi_1=\xi_2=...=\xi_n$ *then equality holds.*

Lemma 4.17. *Let* $0\leq\xi,\ m\leq 1$, *and* $0\leq x\leq 1$, *then* $0\leq lx+m(1-x)\leq 1$.

Lemma 4.18. *Let* $0\leq\xi,\ m\leq 1$, *then* $\sqrt{1-(1-\xi^2)(1-m^2)}\geq lm$.

Theorem 4.19. *Let* $\eth_\Xi=\{(\mathfrak{T}_{\mathfrak{V}_\Xi}(\mathfrak{Z}),\mathfrak{T}_{\mathfrak{R}_\Xi}(\mathfrak{Z})),(\mathfrak{I}_{\mathfrak{V}_\Xi}(\mathfrak{Z}),\mathfrak{I}_{\mathfrak{R}_\Xi}(\mathfrak{Z})),(\mathfrak{F}_{\mathfrak{V}_\Xi}(\mathfrak{Z}),\mathfrak{F}_{\mathfrak{R}_\Xi}(\mathfrak{Z}))\}\in SVNZN(\Upsilon)\ (\Xi=1,2,3,...,n)$. *Then,*

$$ST\text{-}SVNZNWA(\eth_1,\eth_2,...,\eth_n)\geq ST\text{-}SVNZNWG(\eth_1,\eth_2,...,\eth_n),$$

where equality holds if and only if $\eth_1=\eth_2=...=\eth_n$.

Proof. Likewise, it derives from Theorem 4.14. □

5. Decision-Making Strategy

This section presents a DM methodology, along with an illustrative example, designed to address DMPs in the context of the SVNZN framework. Aspects related to multi-attribute DM (MADM) can be effectively showcased through the utilization of a decision matrix structure, where columns signify attributes and rows pertain to alternatives. For a given decision matrix $D_{n\times m}$, we consider a set of n alternatives: $\{\Upsilon_1,\Upsilon_2,\Upsilon_3,...,\Upsilon_n\}$, and m attributes: $\{t_1,t_2,t_3,...,t_m\}$. The undetermined WV associated with the m attributes is signified as $W=\{\kappa_1,\kappa_2,\kappa_3,...,\kappa_m\}$, subject to the constraint that $\xi_\Xi\in[0,1]$ and $\sum_{\Xi=1}^{m}\xi_\Xi=1$. Let's designate the SVN decision matrix as $D=(\eth_{ji})_{n\times m}=\left\langle\left(\mathfrak{T}_{\mathfrak{V}_{ji}},\mathfrak{T}_{\mathfrak{R}_{ji}}\right),\left(\mathfrak{I}_{\mathfrak{V}_{ji}},\mathfrak{I}_{\mathfrak{R}_{ji}}\right),\left(\mathfrak{F}_{\mathfrak{V}_{ji}},\mathfrak{F}_{\mathfrak{R}_{ji}}\right)\right\rangle_{n\times m}$, where $\left(\mathfrak{T}_{\mathfrak{V}_{ji}},\mathfrak{T}_{\mathfrak{R}_{ji}}\right)$ signifies the truth degree of the alternative satisfying the criteria t_j assessed by DMk, $\left(\mathfrak{I}_{\mathfrak{V}_{ji}},\mathfrak{I}_{\mathfrak{R}_{ji}}\right)$ represents the degree of the alternative's indeterminacy with respect to the criteria t_j evaluated by DMk, and $\left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right)$ denotes the degree of the alternative not meeting the criteria t_j considered by DMk. The algorithm encompasses the following steps:

Step-1: Compile the assessments of each alternative into the decision matrix $D^{(k)} = \left(\eth_{ji}^{(k)}\right)_{n\times m}$ using the SVNZN information.

Step-2: Form the normalized decision matrix $P = (p_{ji})$ from $D = (\eth_{ji})$, where p_{ji} is computed as follows:

$$p_{ji} = \left\{ \begin{array}{l} \left(\left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right), \left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right), \left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right)\right) \text{ in case if the criteria are of the benefit type} \\ \left(\left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right), \left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right), \left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right)\right) \text{ in case if the criteria are of the cost type} \end{array} \right\} \tag{3}$$

Step-3: Compute the collective information derived from DMk's input using either the SVNZNWA/SVNZNWG operator:

$$SVNZNWA\left(\eth_1, \eth_2, ..., \eth_n\right) = \left\{ \begin{array}{c} 1 - \prod_{\Xi=1}^{n}\left(1 - \left(\mathfrak{T}_{\mathfrak{V}_{\Xi}}, \mathfrak{T}_{\mathfrak{R}_{\Xi}}\right)\right)^{\xi_\Xi}, \prod_{\Xi=1}^{n}\left(\left(\mathfrak{I}_{\mathfrak{V}_{\Xi}}, \mathfrak{I}_{\mathfrak{R}_{\Xi}}\right)\right)^{\xi_\Xi}, \\ \prod_{\Xi=1}^{n}\left(\left(\mathfrak{F}_{\mathfrak{V}_{\Xi}}, \mathfrak{F}_{\mathfrak{R}_{\Xi}}\right)\right)^{\xi_\Xi} \end{array} \right\}$$

or

$$SVNZNWG\left(\eth_1, \eth_2, ..., \eth_n\right) = \left\{ \begin{array}{c} \prod_{\Xi=1}^{n}\left(\left(\mathfrak{T}_{\mathfrak{V}_{\Xi}}, \mathfrak{T}_{\mathfrak{R}_{\Xi}}\right)\right)^{\xi_\Xi}, 1 - \prod_{\Xi=1}^{n}\left(1 - \left(\mathfrak{I}_{\mathfrak{V}_{\Xi}}, \mathfrak{I}_{\mathfrak{R}_{\Xi}}\right)\right)^{\xi_\Xi}, \\ 1 - \prod_{\Xi=1}^{n}\left(1 - \left(\mathfrak{F}_{\mathfrak{V}_{\Xi}}, \mathfrak{F}_{\mathfrak{R}_{\Xi}}\right)\right)^{\xi_\Xi} \end{array} \right\}$$

Step-4: If the attribute weights are pre-determined, they should be employed. However, if they are not known, they can be calculated using the entropy measure concept. In this context, the entropy-based information for criteria t_j is determined as follows:

$$E_j\left(\eth\right) = \frac{1}{\left(\sqrt{2}-1\right)m} \sum_{i=1}^{m} \left[\begin{array}{c} \sin\left(\frac{\Pi}{4}\left(1 + \left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right) - \left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right) - \left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right)\right)\right) + \\ \sin\left(\frac{\Pi}{4}\left(1 - \left(\mathfrak{T}_{\mathfrak{V}_{ji}}, \mathfrak{T}_{\mathfrak{R}_{ji}}\right) + \left(\mathfrak{I}_{\mathfrak{V}_{ji}}, \mathfrak{I}_{\mathfrak{R}_{ji}}\right) + \left(\mathfrak{F}_{\mathfrak{V}_{ji}}, \mathfrak{F}_{\mathfrak{R}_{ji}}\right)\right)\right) - 1 \end{array} \right].$$

Here, the term $\frac{1}{\left(\sqrt{2}-1\right)m}$ serves as a constant to ensure that $0 \leq E_j\left(\eth\right) \leq 1$.

Step-5: By employing the suggested STAOs and attribute WV, the combined SVN information for each alternative within the set $\{\curlyvee_1, \curlyvee_2, \curlyvee_3, ..., \curlyvee_n\}$ is acquired.

Step-6: Compute the score values $\overline{sc}\left(\eth\right)$ for the aggregated SVN numbers and arrange them in descending order of their score values. If two sets $\eth_1$ and $\eth_2$ yield identical score values, proceed to determine the accuracy degrees $\underline{ac}\left(\eth_1\right)$ and $\underline{ac}\left(\eth_2\right)$ for each set respectively. Subsequently, rank $\eth_1$ and $\eth_2$ based on the highest accuracy degree.

Step-7: Choose the optimal alternative based on either the highest score value or the maximum accuracy degree.

5.1. *Application of Proposed Decision-Making Technique*

Forecasting in business is an activity that is very valuable to both the strategic planning and DM processes that are carried out inside firms. It has a tremendous impact, both in terms of the day-to-day operations of the business as well as the results that it produces. It is crucial to have the ability to successfully anticipate market trends, the requirements of clients, and the achievement of success by a company. It is of the utmost importance for the process of DM due to the fact that it enables administration to make informed judgements about production, stock management, resource allocation, promotion, and development. An instance of the approach for making judgements that has been presented is first demonstrated with the help of a numerical application concerning the forecasting of a firm's selection problem in this section of the article. The first example that will be given is this one. In order to emphasize the characteristics and benefits offered by the given AOs, a comparison is made between the STAOs that have been delivered and the SVNZN AOs that are already in use. This is done in order to showcase the attributes and benefits of the AOs that are currently being provided.

5.1.1. *Practical Case Study*

- f_1 **Efficient Resource Allocation, Cost Reduction, and Risk Mitigation:** When businesses are able to accurately forecast future demand, they are in a better position to utilize their available resources in an effective way, which not only helps them save costs but also helps them avoid risks. One further advantage is that this assists cut down on expenditures. They are able to adapt the levels of production, inventory, and people needs to fit the predicted demand, which lowers the risk of either overstocking or running out of supplies. This is because they are able to alter the levels of production, inventory, and manpower requirements. When businesses have precise estimates, they are better able to optimize their supply chains and manufacturing processes, which, in turn, results in less waste and less expenses that aren't necessary. This can be a win-win situation for everyone involved. This has the potential to be a win-win circumstance for all parties concerned. It helps to avoid having an excess inventory, which may lead to charges associated with storage and holding, and it reduces the frequency of urgent orders, which may contribute to higher production costs. Both of these factors may lead to higher overall costs. Both of these considerations have the potential to drive up total expenditures. The practise of forecasting future business activity may be of assistance to firms in spotting prospective hazards and ambiguities, which, in turn, enables these companies to design strategies to cope with unanticipated results. If companies are aware of the many difficulties that might be thrown their way, they may be better prepared to cope with unanticipated occurrences such as swings in the economy, disruptions in the market, and other unexpected occurrences.
- f_2 **Strategic Planning, Enhanced Budgeting, and Competitive Advantage:** The process of forecasting must act as the foundation for the planning process in order for it to be effective when it comes to long-term strategy planning. Not only is it possible to enhance one's finances via accurate forecasting, but it may also provide one

an advantage over their competitors. It offers aid in the process of setting goals that are attainable, defining objectives, and devising strategies that can be implemented in order to accomplish targets for growth and profitability. Due to the fact that it gives estimates of both revenue and expenditures, realistic forecasting makes it feasible to construct budgets that are more realistic. This makes it possible to distribute resources in a more effective way and helps firms to more effectively integrate the financial procedures they utilize with the larger corporate goals. In addition, this makes it possible to deploy resources in a more effective manner. If a company is able to precisely forecast what will occur in the future and react swiftly to changing circumstances in the market, they will have a considerable advantage over their rivals and will be able to more effectively compete. When companies have the ability to anticipate the needs of their customers and the trends in the market, they are in a better position to tailor the goods and services they provide in order to fulfil the particular demands of their customers.

f_3 **Customer Satisfaction and Investor Confidence:** An growth in both the amount of confidence maintained by investors and the degree to which consumers are happy with the product or service offered. An accurate forecast provides a regular supply of goods or services, which in turn leads in improved levels of customer satisfaction. Predictions may be made using historical data or by using predictive models. Customers who are pleased with the products or services they acquire are more likely to continue their patronage of the business and to suggest it to their friends and family, all of which contribute to the sustained prosperity of the enterprize. The capacity to create accurate forecasts inspires higher confidence among investors because it demonstrates an acute awareness and understanding of the mechanics of the market. In other words, it demonstrates that the investor is well-informed. This is due to the fact that it reveals to everyone that the investor has a solid grasp of the dynamics involved. This may be successful in persuading a greater number of investors and other stakeholders to support the firm's aims of development and expansion, which may be advantageous to the company.

The practise of business forecasting, in general, is advantageous to companies because it assists them in adapting to the ever-changing circumstances of the market, in making choices based on credible information, and in maximizing their operations in order to achieve long-term development and success. It makes it possible for organizations to construct their future in a way that is proactive and to effectively react to both opportunities and challenges in a positive manner.

When it comes to dealing with the placement of all renewable resources for the purpose of developing the most accurate business projections, the subject of making the appropriate choice is always one that is crucial for business. In order to find a solution to this issue, industry experts and those in charge of making decisions need to take into account as many qualitative and quantitative factors as they possibly can. A DM procedure that takes into consideration a wide range of factors is often used when one is tasked with selecting the most suitable location at which to build a new location of an existing firm. This is because the task is considered to be of especially high importance. The business industry is one of the most productive and ecologically friendly kinds of business. It also provides a significant contribution to the development of a nation.

The areas under examination must have been selected by the knowledgeable specialists after engaging in professional discussion with one another. The viewpoint of the person responsible for making the choice as well as the available research were used to compile a list of all of the factors that had a role in the selection of the location. The individuals in charge of making decisions need to collect and consider all of the available information in order to choose the most suitable place or location. We pick a case study for this selection issue and place it in a typical frame. In this example, there are four possible sites, which we will refer to as $\mathfrak{W}_1$, $\mathfrak{W}_2$, $\mathfrak{W}_3$, and $\mathfrak{W}_4$, and all of them will be taken into account while attempting to solve the problem. These websites have been scrutinized in a methodical manner with regard to the three primary characteristics, which are referred to above as f_1, f_2, and f_3 respectively. When the number of characteristics is raised, it is reasonable to anticipate an improvement in the solution. The issue of picking the best feasible location for a company from the set of possibilities that are now accessible is being mathematically and critically addressed within the context of the SVNZN environment, taking into account the expert's or DMk's viewpoint as well as the weights of the criteria. They are unable to supply the whole choice information because of the fuzziness and doubt that exists inside their brain, and the information on the assessment can be found in Table 2, which can be seen below. During this assessment, the expert was requested to utilize SVN information, with attribute weights set as $(0.33, 0.35, 0.32)^{\mathfrak{T}}$.

Step-1: Table 2 reveals the information handed over by the expert.

Step-2: As per the expert's input, attributes f_1 and f_3 are categorized as benefit types, while f_2 is a cost attribute. The normalized matrix computed using equation 3 yields the following results, which are displayed in Table 3.

Step-3: There is no need to estimate the aggregation decision matrix in this real-world case study because just one analyst DMk is involved.

Step-4: WV is a well-known criterion:

$$\kappa = \{\kappa_1 = 0.33, \kappa_2 = 0.35, \kappa_3 = 0.32\}$$

Table 2. Information result of the expert

	f_1	f_2	f_3
$\mathfrak{W}_1$	$\left\langle \begin{array}{c} (0.6, 0.8), \\ (0.2, 0.3), \\ (0.1, 0.5) \end{array} \right\rangle$	$\left\langle \begin{array}{c} (0.1, 0.3), \\ (0.2, 0.6), \\ (0.7, 0.8) \end{array} \right\rangle$	$\left\langle \begin{array}{c} (0.1, 0.3), \\ (0.5, 0.6), \\ (0.2, 0.9) \end{array} \right\rangle$
$\mathfrak{W}_2$	$\left\langle \begin{array}{c} (0.8, 0.7), \\ (0.1, 0.8), \\ (0.2, 0.6) \end{array} \right\rangle$	$\left\langle \begin{array}{c} (0.5, 0.4), \\ (0.3, 0.2), \\ (0.7, 0.6) \end{array} \right\rangle$	$\left\langle \begin{array}{c} (0.4, 0.2), \\ (0.1, 0.5), \\ (0.7, 0.6) \end{array} \right\rangle$
$\mathfrak{W}_3$	$\left\langle \begin{array}{c} (0.4, 0.6), \\ (0.2, 0.5), \\ (0.9, 0.4) \end{array} \right\rangle$	$\left\langle \begin{array}{c} (0.4, 0.1), \\ (0.5, 0.6), \\ (0.2, 0.4) \end{array} \right\rangle$	$\left\langle \begin{array}{c} (0.4, 0.3), \\ (0.5, 0.1), \\ (0.2, 0.1) \end{array} \right\rangle$
$\mathfrak{W}_4$	$\left\langle \begin{array}{c} (0.1, 0.5), \\ (0.2, 0.3), \\ (0.7, 0.5) \end{array} \right\rangle$	$\left\langle \begin{array}{c} (0.3, 0.1), \\ (0.2, 0.9), \\ (0.6, 0.2) \end{array} \right\rangle$	$\left\langle \begin{array}{c} (0.1, 0.5), \\ (0.4, 0.1), \\ (0.1, 0.2) \end{array} \right\rangle$

Table 3. Normalized matrix

	f_1	f_2	f_3
$\mathfrak{W}_1$	$\left\langle \begin{array}{c} (0.6, 0.8), \\ (0.2, 0.3), \\ (0.1, 0.5) \end{array} \right\rangle$	$\left\langle \begin{array}{c} (0.7, 0.8), \\ (0.2, 0.6), \\ (0.1, 0.3) \end{array} \right\rangle$	$\left\langle \begin{array}{c} (0.1, 0.3), \\ (0.5, 0.6), \\ (0.2, 0.9) \end{array} \right\rangle$
$\mathfrak{W}_2$	$\left\langle \begin{array}{c} (0.8, 0.7), \\ (0.1, 0.8), \\ (0.2, 0.6) \end{array} \right\rangle$	$\left\langle \begin{array}{c} (0.7, 0.6), \\ (0.3, 0.2), \\ (0.5, 0.4) \end{array} \right\rangle$	$\left\langle \begin{array}{c} (0.4, 0.2), \\ (0.1, 0.5), \\ (0.7, 0.6) \end{array} \right\rangle$
$\mathfrak{W}_3$	$\left\langle \begin{array}{c} (0.4, 0.6), \\ (0.2, 0.5), \\ (0.9, 0.4) \end{array} \right\rangle$	$\left\langle \begin{array}{c} (0.2, 0.4), \\ (0.5, 0.6), \\ (0.4, 0.1) \end{array} \right\rangle$	$\left\langle \begin{array}{c} (0.4, 0.3), \\ (0.5, 0.1), \\ (0.2, 0.1) \end{array} \right\rangle$
$\mathfrak{W}_4$	$\left\langle \begin{array}{c} (0.1, 0.5), \\ (0.2, 0.3), \\ (0.7, 0.5) \end{array} \right\rangle$	$\left\langle \begin{array}{c} (0.6, 0.2), \\ (0.2, 0.9), \\ (0.3, 0.1) \end{array} \right\rangle$	$\left\langle \begin{array}{c} (0.1, 0.5), \\ (0.4, 0.1), \\ (0.1, 0.2) \end{array} \right\rangle$

Step-5: By utilizing the proposed STAOs and the provided WV, the collective SVNZN information for each alternative is obtained and presented in Table 4.

Step-6: Calculate the score values for each aggregated SVNZN information of every alternative, as demonstrated in Table 5.

Step-7: As shown in Table 6, select the best option based on the greatest score value. Our goal in our case study is to use three factors to help us choose the best location for the company. Following the application of the planned algorithm stages, the aggregate

Table 4. aggregated SVNZN information of each alternative

	$ST\text{-}SVNZNWA$	$ST\text{-}SVNZNWG$
$\mathfrak{W}_1$	$\left\langle \begin{array}{c} (0.748, 0.894), \\ (0.086, 0.266), \\ (0.019, 0.291) \end{array} \right\rangle$	$\left\langle \begin{array}{c} (0.495, 0.751), \\ (0.135, 0.326), \\ (0.024, 0.527) \end{array} \right\rangle$
$\mathfrak{W}_2$	$\left\langle \begin{array}{c} (0.872, 0.760), \\ (0.026, 0.208), \\ (0.198, 0.315) \end{array} \right\rangle$	$\left\langle \begin{array}{c} (0.797, 0.614), \\ (0.047, 0.403), \\ (0.323, 0.343) \end{array} \right\rangle$
$\mathfrak{W}_3$	$\left\langle \begin{array}{c} (0.506, 0.650), \\ (0.162, 0.120), \\ (0.202, 0.030) \end{array} \right\rangle$	$\left\langle \begin{array}{c} (0.469, 0.601), \\ (0.220, 0.262), \\ (0.505, 0.075) \end{array} \right\rangle$
$\mathfrak{W}_4$	$\left\langle \begin{array}{c} (0.498, 0.604), \\ (0.075, 0.111), \\ (0.092, 0.054) \end{array} \right\rangle$	$\left\langle \begin{array}{c} (0.278, 0.529), \\ (0.096, 0.499), \\ (0.263, 0.126) \end{array} \right\rangle$

Table 5. The aggregated SVNZN information of each alternative's score value

	$Y(\mathfrak{W}_1)$	$Y(\mathfrak{W}_2)$	$Y(\mathfrak{W}_3)$	$Y(\mathfrak{W}_4)$
$ST\text{-}SVNZNWA$	0.880	0.865	0.768	0.762
$ST\text{-}SVNZNWG$	0.787	0.772	0.729	0.689

Table 6. Optimal alternative based on the highest score value

	Score Ranking	Best Alternatives
$ST\text{-}SVNZNWA$	$Y(\mathfrak{W}_1) > Y(\mathfrak{W}_2) > Y(\mathfrak{W}_3) > Y(\mathfrak{W}_4)$	$\mathfrak{W}_1$
$ST\text{-}SVNZNWG$	$Y(\mathfrak{W}_2) > Y(\mathfrak{W}_1) > Y(\mathfrak{W}_3) > Y(\mathfrak{W}_4)$	$\mathfrak{W}_1$

data based on the innovative ST operational principles is presented as an SVNZN set. Drawing conclusions from the computational method described above, we find that $\mathfrak{W}_2$ is the optimal choice among the alternatives; as such, it is strongly advised to use it for the necessary work or plan.

6. Comparison Analysis

The feasibility of the proposed process, its aggregation's adaptability to specific inputs and outcomes, the influence of scoring functions, analysis of sensitivity, supremacy, and, lastly, a comparison of the proposed technique with current methods are all covered in this part. The recommended approach was accurate and appropriate for a wide range of input data types. The approach that was developed worked well for managing uncertainty. It included

Z numbers and STAO-based SVNS spaces. We may effectively use our approach in a wide range of circumstances by expanding the distance among the pleasure and displeasure classes by altering the real-world importance of particular parameters. We came across a variety of elements and parameters for input in multiple MADM problems that were appropriate for the given situation. The suggested SVNZNs were easy to grasp, basic, and versatile enough to fit a wide range of needs. We saw in Table 7, that every one of our suggested aggregating operators generated the same outcomes, demonstrating accuracy and strength. This essay's goal was to show, through a comparative analysis with a few current approaches, the superiority and reliability of our original study. We compared our results with neutrosophic ZNs (NZN) weighted arithmetic averaging (NZNWAA) and NZN weighted geometric averaging (NZN-WGA) operators [35], NZN AczelAlsina weighted arithmetic averaging (NZNAAWAA) and NZN AczelAlsina weighted geometric averaging (NZNAAWGA) operators [32] and linguistic neutrosophic ZN (LNZN) weighted arithmetic mean (LNZNWAM) and LNZN weighted geometric mean (LNZNWGM) operators [33], and the work that is connected to decision making difficulties in [34–36] as well as the great work that is important to SVN structure in [37] is really remarkable.

Table 7. Comparison Analysis

	Score Ranking	Best Alternatives
$ST\text{-}NZNWAA$	$Y(\mathfrak{W}_1) > Y(\mathfrak{W}_2) > Y(\mathfrak{W}_4) > Y(\mathfrak{W}_3)$	$\mathfrak{W}_1$
$ST\text{-}NZNWGA$	$Y(\mathfrak{W}_1) > Y(\mathfrak{W}_2) > Y(\mathfrak{W}_3) > Y(\mathfrak{W}_4)$	$\mathfrak{W}_1$
$ST\text{-}NZNAAWAA$	$Y(\mathfrak{W}_2) > Y(\mathfrak{W}_1) > Y(\mathfrak{W}_3) > Y(\mathfrak{W}_4)$	$\mathfrak{W}_2$
$ST\text{-}NZNAAWGA$	$Y(\mathfrak{W}_1) > Y(\mathfrak{W}_2) > Y(\mathfrak{W}_3) > Y(\mathfrak{W}_4)$	$\mathfrak{W}_1$
$ST\text{-}LNZNWAM$	$Y(\mathfrak{W}_1) > Y(\mathfrak{W}_2) > Y(\mathfrak{W}_3) > Y(\mathfrak{W}_4)$	$\mathfrak{W}_1$
$ST\text{-}LNZNWGM$	$Y(\mathfrak{W}_2) > Y(\mathfrak{W}_1) > Y(\mathfrak{W}_3) > Y(\mathfrak{W}_4)$	$\mathfrak{W}_2$
$ST\text{-}SVNZNWA$	$Y(\mathfrak{W}_1) > Y(\mathfrak{W}_2) > Y(\mathfrak{W}_3) > Y(\mathfrak{W}_4)$	$\mathfrak{W}_1$
$ST\text{-}SVNZNWG$	$Y(\mathfrak{W}_1) > Y(\mathfrak{W}_2) > Y(\mathfrak{W}_3) > Y(\mathfrak{W}_4)$	$\mathfrak{W}_1$

7. Conclusion

The rapid process of industrialization has led to a significant expansion of the global business landscape. In this research manuscript, our goal is to introduce a novel method for selecting business locations. To achieve this, we propose the utilization of fresh operational laws based on the ST function under SVNZNs, which we refer to as Z-STOLs. Below is a more detailed discussion of these strategies' benefits.

- The important features of the SVNZNs and their operational characteristics, such as boundedness, monotonicity, commutativity, and idempotency, are covered first.
- Next, the ideas for developing specialized AOs such ST ZN SVNZN-weighted AOs and ST ZN SVNZN-ordered weighted averaging/geometric AOs are formed. We investigate the underlying links between these newly introduced aggregation operations in depth.
- Then we developed a new MADM algorithm for dealing with DM situations in which preferences are evaluated using SVNZNs. This enables us to apply the proposed legislation to Decision-Making Problems correctly.
- Our research findings highlight the high efficacy of using SVNZ information measures in handling ambiguity in DM issues. We employ a real-world case to assess the efficacy of our suggested strategy of site selection for a business, subjecting it to thorough scrutiny to ascertain its superiority and viability.
- At the end, we conduct a comparative analysis with several previously published studies to further validate its efficiency. The approach presented in this study holds significant promise for application in various domains, including medical diagnostics, green supplier selection, and more.
- Future studies on two-sided combining making choices with multi-granular and unfinished criteria weight information, widespread agreement accomplishing with uncooperative behavioral DMPs, and personalized individual uniformity control consensus problems could make use of the suggested AOs. This examination of the constraints imposed by proposed AOs is independent of the levels of involvement, abstention, and non-membership. On this side of the intended AOs, an innovative hybrid structure of interactive, prioritized AOs is being implemented.
- In future studies, we aim to extend this approach to address other ambiguous domains, such as interval-value SVNZNs and probabilistic linguistic term sets. The versatility of our proposed approach makes it a valuable tool for DMks across different industries.

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Extension for neutrosophic vague subbisemirings of bisemirings

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Abstract. This paper introduces the idea of a neutrosophic vague subbisemiring (NSVSBS), level sets of NSVSBS, and (ρ, σ)-neutrosophic vague subbisemiring ((ρ, σ)-NSVSBS) of a bisemiring. NSVSBSs are generalizations of neutrosophic subbisemirings and SBS based on bisemirings. Let Λ be a neutrosophic vague subset in $\mathcal{B}$, we show that $\mathcal{V} = ([\mathcal{T}_\Lambda^-, \mathcal{T}_\Lambda^+], [\mathcal{I}_\Lambda^-, \mathcal{I}_\Lambda^+], [\mathcal{F}_\Lambda^-, \mathcal{F}_\Lambda^+])$ is a NSVSBS of $\mathcal{B}$ if and only if all non empty level set $\mathcal{V}^{(t_1,t_2,s)}$ is a SBS of $\mathcal{B}$ for $t_1, t_2, s \in [0, 1]$. In the case that Λ is a NSVSBS of a bisemiring $\mathcal{B}$ and V is the strongest neutrosophic vague relation of $\mathcal{B}$, we prove that Λ is a NSVSBS of $\mathcal{B} \times \mathcal{B}$. Let Λ be any NSVSBS of $\mathcal{B}$, prove that pseudo neutrosophic vague coset $(\tau\Lambda)^p$ is a NSVSBS of $\mathcal{B}$, for every $\tau \in \mathcal{B}$. Let $\Lambda_1, \Lambda_2, ..., \Lambda_n$ be the family of $NSVSBS^s$ of $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n$ respectively. We show that $\Lambda_1 \times \Lambda_2 \times ... \times \Lambda_n$ is a NSVSBS of $\mathcal{B}_1 \times \mathcal{B}_2 \times ... \times \mathcal{B}_n$. The homomorphic image of every NSVSBS is a NSVSBS. The homomorphic pre-image of every NSVSBS is a NSVSBS. Examples are provided to strengthen our results.

Keywords: subbisemiring; neutrosophic subbisemiring; neutrosophic vague bisemiring; homomorphism

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1. Introduction

Due to the limitations of classical mathematics, such as fuzzy set (FS) [1] and vague set (VS) [2], mathematical theories have been developed to address uncertainty and fuzziness. In the case of uncertain or vague situations, FS introduced by Zadeh [1] is the most appropriate technique. In recent years, many hybrid fuzzy models have been developed based on FS. A generalization of FS, intuitionistic fuzzy set (IFS) incorporate hesitation levels into the notion of FS, which were first proposed by Attanasov [3] in 1983. The neutrosophic set (NSS) was proposed in 1999 by Smarandache [4]. In NSS, each proposition is estimated to have a degree of truth, an indeterminacy degree, and a falsity degree. As a result of Smarandache [5], he further generalised and expanded the theory of IFSs to include the neutrosophic model as well. A study of fuzzy semirings was initiated by Ahsan et al. [6]. Palanikumar et al. [?, ?] discussed tri-quasi-ideals and bi-quasi-ideals are natural generalizations of rings such that they constitute a natural generalization of ternary semirings, semirings and ordered semirings. In 2004, Sen et al. [17] extended the study of semirings and proposed the concept of bisemiring to further develop them. The study of vague algebra was initiated by Biswas [18] through the introduction of vague groups, vague cuts and vague normal groups. In their work, Arulmozhi et al. [19] focus on the interaction between semirings, ternary semirings and other algebraic structures. A semiring $(S, +, \cdot)$ is a non-empty set in which $(S, +)$ and $(S, \cdot)$ are semigroups such that "$\cdot$" is distributive over "+" [20]. In 1993, Ahsan et al. [6] introduced the notion of fuzzy semirings.

An introduction to bisemirings was made in 2001 by Sen et al. [21]. A bisemiring $(\mathcal{B}, \partial, \circledcirc, \boxtimes)$ is an algebraic structure in which $(\mathcal{B}, \partial, \circledcirc)$ and $(\mathcal{B}, \circledcirc, \boxtimes)$ are semirings in which $(\mathcal{B}, \partial), (\mathcal{B}, \circledcirc)$ and $(\mathcal{B}, \boxtimes)$ are semigroups such that (a) $\zeta \circledcirc (\Im \partial \tau) = (\Re \circledcirc \Im) \partial (\Re \circledcirc \tau)$, (b) $(\Im \partial \tau) \circledcirc \Re = (\Im \circledcirc \Re) \partial (\tau \circledcirc \Re)$, (c) $\Re \boxtimes (\Im \circledcirc \tau) = (\Re \boxtimes \Im) \circledcirc (\Re \boxtimes \tau)$ and (d) $(\Im \circledcirc \tau) \boxtimes \Re = (\Im \boxtimes \Re) \circledcirc (\tau \boxtimes \Re)$ for all $\Re, \Im, \tau \in \mathcal{B}$ [17]. A non-empty subset Λ of a bisemiring $(\mathcal{B}, \partial, \circledcirc, \boxtimes)$ is a subbisemiring (SBS) if and only if $\Re \partial \Im \in \Lambda, \Re \circledcirc \Im \in \Lambda$ and $\Re \boxtimes \Im \in \Lambda$ for all $\Re, \Im \in \Lambda$ [21]. Palanikumar et al. discussed the various ideal structures of SBS theory and its applications [7]- [16]. However, numerous algebraic concepts had been generalized using FS theory. Fuzzy algebraic structures of semirings have been extensively investigated by Vandiver [22]. These are generalizations of rings requiring only a monoid, rather than a group, to achieve a particular additive structure and have been shown to be useful for a wide range of problems. Golan [20] and Glazek [23] have both extensively studied the application of semirings.

Bipolar fuzzy information has been applied to various algebraic structures over the past few years, like semigroups [?, 14, 15] and BCK/BCI algebras [24–27]. An application of bipolar fuzzy metric spaces was discussed by Zararsz et al. [28]. A vague soft hyperring and a vague soft hyper ideal were introduced by Selvachandran [29]. The bipolar fuzzy translation was introduced by Jun et al. [30] and BCK/BCI-algebra and its properties were investigated. A bipolar fuzzy regularity, bipolar fuzzy regular sub-algebra, a bipolar fuzzy filter, and a bipolar fuzzy closed quasi filter have been introduced into BCH algebras in [31]. In 2004, Sen et al. [17] contributed to the field of semirings by proposing bisemiring as a concept. Hussain et al. [32] defined the congruence relation between bisemirings and bisemiring homomorphisms. In addition to bisemiring, Hussain et al. [21, 32] described an algebraic structure called semiring and congruence relations between homomorphisms and n-semirings based on this algebraic structure.

Neutosophic vague subbisemirings (NSVSBS) are discussed here, as well as their level sets. Subbisemirings are a generalization of bisemirings, and NSVSBSs are a generalization of subbisemirings. A number of illustrative examples are provided to illustrate the theory for (ξ, τ)-NSVSBS over bisemiring theory. Following is an outline of the preliminary definitions and results presented in Section 2. The concept of a NSVSBS is introduced in Section 3. There is more information about (ξ, τ)-NSVSBS in Section 4.

2. Basic concepts

For our future studies, we will quickly review some fundamental terms in this section.

Definition 2.1. [4] A neutrosophic set (NSS) Λ in a universal set $\mathcal{U}$ is $\Lambda = \{(\Re, \mathcal{T}_\Lambda(\Re), \mathcal{I}_\Lambda(\Re), \mathcal{F}_\Lambda(\Re)) : \Re \in \mathcal{U}\}$, where $\mathcal{T}_\Lambda, \mathcal{I}_\Lambda, \mathcal{F}_\Lambda : \mathcal{U} \to [0,1]$ denotes the truth, indeterminacy and the falsity membership function, respectively. For $\langle \mathcal{T}_\Lambda, \mathcal{I}_\Lambda, \mathcal{F}_\Lambda \rangle$ is used for the NSS $\Lambda = \{(\Re, \mathcal{T}_\Lambda(\Re), \mathcal{I}_\Lambda(\Re), \mathcal{F}_\Lambda(\Re)) : \Re \in \mathcal{U}\}$.

Definition 2.2. [4] Let $\Lambda = \langle \mathcal{T}_\Lambda, \mathcal{I}_\Lambda, \mathcal{F}_\Lambda \rangle$ and $\Psi = \langle \mathcal{T}_\Psi, \mathcal{I}_\Psi, \mathcal{F}_\Psi \rangle$ be the two NSS of $\mathcal{U}$. Then

(1) $\Lambda \cap \Psi = \{(\Re, \min\{\mathcal{T}_\Lambda(\Re), \mathcal{T}_\Psi(\Re)\}, \min\{\mathcal{I}_\Lambda(\Re), \mathcal{I}_\Psi(\Re)\}, \max\{\mathcal{F}_\Lambda(\Re), \mathcal{F}_\Psi(\Re)\}) : \Re \in \mathcal{U}\}$,

(2) $\Lambda \cup \Psi = \{(\Re, \max\{\mathcal{T}_\Lambda(\Re), \mathcal{T}_\Psi(\Re)\}, \max\{\mathcal{I}_\Lambda(\Re), \mathcal{I}_\Psi(\Re)\}, \min\{\mathcal{F}_\Lambda(\Re), \mathcal{F}_\Psi(\Re)\}) : \Re \in \mathcal{U}\}$.

Definition 2.3. [4] For any NSS $\Lambda = \langle \mathcal{T}_\Lambda, \mathcal{I}_\Lambda, \mathcal{F}_\Lambda \rangle$ of $\mathcal{U}$, we defined a (ρ, σ)-cut of as the crisp subset $\{\Re \in \mathcal{U} : \mathcal{T}_\Lambda(\Re) \geq \rho, \mathcal{I}_\Lambda(\Re) \geq \rho, \mathcal{F}_\Lambda(\Re) \leq \sigma\}$ of $\mathcal{U}$.

Definition 2.4. [4] Let Λ and Ψ be two neutrosophic subsets of S. The Cartesian product of Λ and Ψ is defined as $\Lambda \times \Psi = \{((\Re, \Im), \mathcal{T}_{\Lambda\times\Psi}(\Re, \Im), \mathcal{I}_{\Lambda\times\Psi}(\Re, \Im), \mathcal{F}_{\Lambda\times\Psi}(\Re, \Im)) : \Re, \Im \in S\}$, where $\mathcal{T}_{\Lambda\times\Psi}(\Re, \Im) = \min\{\mathcal{T}_\Lambda(\Re), \mathcal{T}_\Psi(\Im)\}, \mathcal{I}_{\Lambda\times\Psi}(\Re, \Im) = \frac{\mathcal{I}_\Lambda(\Re)+\mathcal{I}_\Psi(\Im)}{2}$ and $\mathcal{F}_{\Lambda\times\Psi}(\Re, \Im) = \max\{\mathcal{F}_\Lambda(\Re), \mathcal{F}_\Psi(\Im)\}$.

Definition 2.5. [18] A vague set (VS) $\Lambda = (\mathcal{T}_\Lambda,\ \mathcal{F}_\Lambda)$ of $\mathcal{B}$ is said to be vague semiring if

$$\begin{Bmatrix} \mathcal{T}_\Lambda(\ell_1 + \ell_2) \geq \min\{\mathcal{T}_\Lambda(\ell_1), \mathcal{T}_\Lambda(\ell_2)\} \\ \mathcal{T}_\Lambda(\ell_1 \cdot \ell_2) \geq \min\{\mathcal{T}_\Lambda(\ell_1), \mathcal{T}_\Lambda(\ell_2)\} \end{Bmatrix}$$

and

$$\begin{Bmatrix} 1 - \mathcal{F}_\Lambda(\ell_1 + \ell_2) \geq \min\{1 - \mathcal{F}_\Lambda(\ell_1), 1 - \mathcal{F}_\Lambda(\ell_2)\} \\ 1 - \mathcal{F}_\Lambda(\ell_1 \cdot \ell_2) \geq \min\{1 - \mathcal{F}_\Lambda(\ell_1), 1 - \mathcal{F}_\Lambda(\ell_2)\} \end{Bmatrix}.$$

for all $\ell_1, \ell_2 \in \mathcal{B}$.

Definition 2.6. [18] A VS Λ in $\mathcal{U}$. Then

(1) A VS $\Lambda = (\mathcal{T}_\Lambda, \mathcal{F}_\Lambda)$, where $\mathcal{T}_\Lambda : \mathcal{U} \to [0,1], \mathcal{F}_\Lambda : \mathcal{U} \to [0,1]$ are mappings such that $\mathcal{T}_\Lambda(\Re)+\mathcal{F}_\Lambda(\Re) \leq 1$, for all $\Re \in \mathcal{U}$ where $\mathcal{T}_\Lambda$ and $\mathcal{F}_\Lambda$ are called true and false membership function, respectively.

(2) The interval $[\mathcal{T}_\Lambda(\Re), 1 - \mathcal{F}_\Lambda(\Re)]$ is called the vague value of $\Re$ in Λ and it is denoted by $V_\Lambda(\Re)$, i.e., $V_\Lambda(\Re) = [\mathcal{T}_\Lambda(\Re), 1 - \mathcal{F}_\Lambda(\Re)]$.

Definition 2.7. [18] Let Λ and Ψ be the two VSs of $\mathcal{U}$. Then

(1) Λ is contained in Ψ as $\Lambda \subseteq \Psi$ if and only if $V_\Lambda(\Re) \leq V_\Psi(\Re)$, i.e. $\mathcal{T}_\Lambda(\Re) \leq \mathcal{T}_\Psi(\Re)$ and $1 - \mathcal{F}_\Lambda(\Re) \leq 1 - \mathcal{F}_\Psi(\Re)$ for all $\Re \in \mathcal{U}$,

(2) the union of Λ and Ψ as $\Delta = \Lambda \cup \Psi, \mathcal{T}_\Delta = \max\{\mathcal{T}_\Lambda, \mathcal{T}_\Psi\}$ and $1 - \mathcal{F}_\Delta = \max\{1 - \mathcal{F}_\Lambda, 1 - \mathcal{F}_\Psi\} = 1 - \min\{\mathcal{F}_\Lambda, \mathcal{F}_\Psi\}$,

(3) the intersection of Λ and Ψ as $\Delta = \Lambda \cap \Psi, \mathcal{T}_\Delta = \min\{\mathcal{T}_\Lambda, \mathcal{T}_\Psi\}$ and $1 - \mathcal{F}_\Delta = \min\{1 - \mathcal{F}_\Lambda, 1 - \mathcal{F}_\Psi\} = 1 - \max\{\mathcal{F}_\Lambda, \mathcal{F}_\Psi\}$.

Definition 2.8. [18] Let Λ and Ψ be any two VSs in $\mathcal{U}$. Then

(1) $\Lambda \cap \Psi = \big\{(\Re, \min\{\mathcal{T}_\Lambda(\Re), \mathcal{T}_\Psi(\Re)\}, \min\{1 - \mathcal{F}_\Lambda(\Re), 1 - \mathcal{F}_\Psi(\Re)\}) : \Re \in \mathcal{U}\big\}$,

(2) $\Lambda \cup \Psi = \big\{(\Re, \max\{\mathcal{T}_\Lambda(\Re), \mathcal{T}_\Psi(\Re)\}, \max\{1 - \mathcal{F}_\Lambda(\Re), 1 - \mathcal{F}_\Psi(\Re)\}) : \Re \in \mathcal{U}\big\}$,

(3) $\Box\Lambda = \big\{(\Re, \mathcal{T}_\Lambda(\Re), 1 - \mathcal{T}_\Lambda(\Re)) : \Re \in \mathcal{U}\big\}$,

(4) $\Diamond\Lambda = \big\{(\Re, 1 - \mathcal{F}_\Lambda(\Re), \mathcal{F}_\Lambda(\Re)) : \Re \in U\big\}$.

3. Neutrosophic vague subbisemirings

In all cases, assume that $\mathcal{B}$ represents a bisemiring.

Definition 3.1. A neutrosophic VS Λ of $\mathcal{B}$ is represent a NSVSBS of $\mathcal{B}$ if

$$\begin{Bmatrix} \mathcal{V}^{\mathcal{T}}_\Lambda(\Re\Diamond_1\Im) \geq \min\{\mathcal{V}^{\mathcal{T}}_\Lambda(\Re), \mathcal{V}^{\mathcal{T}}_\Lambda(\Im)\} \\ \mathcal{V}^{\mathcal{T}}_\Lambda(\Re\Diamond_2\Im) \geq \min\{\mathcal{V}^{\mathcal{T}}_\Lambda(\Re), \mathcal{V}^{\mathcal{T}}_\Lambda(\Im)\} \\ \mathcal{V}^{\mathcal{T}}_\Lambda(\Re\Diamond_3\Im) \geq \min\{\mathcal{V}^{\mathcal{T}}_\Lambda(\Re), \mathcal{V}^{\mathcal{T}}_\Lambda(\Im)\} \end{Bmatrix} \quad \begin{Bmatrix} \mathcal{V}^{\mathcal{I}}_\Lambda(\Re\Diamond_1\Im) \geq \frac{\mathcal{V}^{\mathcal{I}}_\Lambda(\Re)+\mathcal{V}^{\mathcal{I}}_\Lambda(\Im)}{2} \\ OR \\ \mathcal{V}^{\mathcal{I}}_\Lambda(\Re\Diamond_2\Im) \geq \frac{\mathcal{V}^{\mathcal{I}}_\Lambda(\Re)+\mathcal{V}^{\mathcal{I}}_\Lambda(\Im)}{2} \\ OR \\ \mathcal{V}^{\mathcal{I}}_\Lambda(\Re\Diamond_3\Im) \geq \frac{\mathcal{V}^{\mathcal{I}}_\Lambda(\Re)+\mathcal{V}^{\mathcal{I}}_\Lambda(\Im)}{2} \end{Bmatrix}$$

$$\begin{Bmatrix} \mathcal{V}^{\mathcal{F}}_\Lambda(\Re\Diamond_1\Im) \leq \max\{\mathcal{V}^{\mathcal{F}}_\Lambda(\Re), \mathcal{V}^{\mathcal{F}}_\Lambda(\Im)\} \\ \mathcal{V}^{\mathcal{F}}_\Lambda(\Re\Diamond_2\Im) \leq \max\{\mathcal{V}^{\mathcal{F}}_\Lambda(\Re), \mathcal{V}^{\mathcal{F}}_\Lambda(\Im)\} \\ \mathcal{V}^{\mathcal{F}}_\Lambda(\Re\Diamond_3\Im) \leq \max\{\mathcal{V}^{\mathcal{F}}_\Lambda(\Re), \mathcal{V}^{\mathcal{F}}_\Lambda(\Im)\} \end{Bmatrix}.$$

That is,

$$\left\{\begin{array}{c}\left(\begin{array}{c}\mathcal{T}_\Lambda^-(\Re\lozenge_1\Im)\geq\min\{\mathcal{T}_\Lambda^-(\Re),\mathcal{T}_\Lambda^-(\Im)\},\\1-\mathcal{F}_\Lambda^-(\Re\lozenge_1\Im)\geq\min\{1-\mathcal{F}_\Lambda^-(\Re),1-\mathcal{F}_\Lambda^-(\Im)\}\end{array}\right)\\\left(\begin{array}{c}\mathcal{T}_\Lambda^-(\Re\lozenge_2\Im)\geq\min\{\mathcal{T}_\Lambda^-(\Re),\mathcal{T}_\Lambda^-(\Im)\},\\1-\mathcal{F}_\Lambda^-(\Re\lozenge_2\Im)\geq\min\{1-\mathcal{F}_\Lambda^-(\Re),1-\mathcal{F}_\Lambda^-(\Im)\}\end{array}\right)\\\left(\begin{array}{c}\mathcal{T}_\Lambda^-(\Re\lozenge_3\Im)\geq\min\{\mathcal{T}_\Lambda^-(\Re),\mathcal{T}_\Lambda^-(\Im)\},\\1-\mathcal{F}_\Lambda^-(\Re\lozenge_3\Im)\geq\min\{1-\mathcal{F}_\Lambda^-(\Re),1-\mathcal{F}_\Lambda^-(\Im)\}\end{array}\right)\end{array}\right\}\quad\left\{\begin{array}{c}\left(\begin{array}{c}\mathcal{I}_\Lambda^+(\Re\lozenge_1\Im)\geq\frac{\mathcal{I}_\Lambda^+(\Re)+\mathcal{I}_\Lambda^+(\Im)}{2},\\\mathcal{I}_\Lambda^-(\Re\lozenge_1\Im)\geq\frac{\mathcal{I}_\Lambda^-(\Re)-\mathcal{I}_\Lambda^-(\Im)}{2}\end{array}\right)\\OR\\\left(\begin{array}{c}\mathcal{I}_\Lambda^+(\Re\lozenge_2\Im)\geq\frac{\mathcal{I}_\Lambda^+(\Re)+\mathcal{I}_\Lambda^+(\Im)}{2},\\\mathcal{I}_\Lambda^-(\Re\lozenge_2\Im)\geq\frac{\mathcal{I}_\Lambda^-(\Re)-\mathcal{I}_\Lambda^-(\Im)}{2}\end{array}\right)\\OR\\\left(\begin{array}{c}\mathcal{I}_\Lambda^+(\Re\lozenge_3\Im)\geq\frac{\mathcal{I}_\Lambda^+(\Re)+\mathcal{I}_\Lambda^+(\Im)}{2},\\\mathcal{I}_\Lambda^-(\Re\lozenge_3\Im)\geq\frac{\mathcal{I}_\Lambda^-(\Re)-\mathcal{I}_\Lambda^-(\Im)}{2}\end{array}\right)\end{array}\right\}$$

$$\left\{\begin{array}{c}\left(\begin{array}{c}\mathcal{F}_\Lambda^-(\Re\lozenge_1\Im)\leq\max\{\mathcal{F}_\Lambda^-(\Re),\mathcal{F}_\Lambda^-(\Im)\},\\1-\mathcal{T}_\Lambda^-(\Re\lozenge_1\Im)\leq\max\{1-\mathcal{T}_\Lambda^-(\Re),1-\mathcal{T}_\Lambda^-(\Im)\}\end{array}\right)\\\left(\begin{array}{c}\mathcal{F}_\Lambda^-(\Re\lozenge_2\Im)\leq\max\{\mathcal{F}_\Lambda^-(\Re),\mathcal{F}_\Lambda^-(\Im)\},\\1-\mathcal{T}_\Lambda^-(\Re\lozenge_2\Im)\leq\max\{1-\mathcal{T}_\Lambda^-(\Re),1-\mathcal{T}_\Lambda^-(\Im)\}\end{array}\right)\\\left(\begin{array}{c}\mathcal{F}_\Lambda^-(\Re\lozenge_3\Im)\leq\max\{\mathcal{F}_\Lambda^-(\Re),\mathcal{F}_\Lambda^-(\Im)\},\\1-\mathcal{T}_\Lambda^-(\Re\lozenge_3\Im)\leq\max\{1-\mathcal{T}_\Lambda^-(\Re),1-\mathcal{T}_\Lambda^-(\Im)\}\end{array}\right)\end{array}\right\}$$

for all $\Re,\Im\in\mathcal{B}$.

Example 3.2. Let $\mathcal{B}=\{\dot{a},\ddot{a},\tilde{a},\vec{a}\}$ be the bisemiring.

$\lozenge_1$	$\dot{a}$	$\ddot{a}$	$\tilde{a}$	$\vec{a}$
$\dot{a}$	$\dot{a}$	$\dot{a}$	$\dot{a}$	$\dot{a}$
$\ddot{a}$	$\dot{a}$	$\ddot{a}$	$\dot{a}$	$\ddot{a}$
$\tilde{a}$	$\dot{a}$	$\dot{a}$	$\tilde{a}$	$\tilde{a}$
$\vec{a}$	$\dot{a}$	$\ddot{a}$	$\tilde{a}$	$\vec{a}$

$\lozenge_2$	$\dot{a}$	$\ddot{a}$	$\tilde{a}$	$\vec{a}$
$\dot{a}$	$\dot{a}$	$\ddot{a}$	$\tilde{a}$	$\vec{a}$
$\ddot{a}$	$\ddot{a}$	$\ddot{a}$	$\vec{a}$	$\vec{a}$
$\tilde{a}$	$\tilde{a}$	$\vec{a}$	$\tilde{a}$	$\vec{a}$
$\vec{a}$	$\vec{a}$	$\vec{a}$	$\vec{a}$	$\vec{a}$

$\lozenge_3$	$\dot{a}$	$\ddot{a}$	$\tilde{a}$	$\vec{a}$
$\dot{a}$	$\dot{a}$	$\dot{a}$	$\dot{a}$	$\dot{a}$
$\ddot{a}$	$\dot{a}$	$\ddot{a}$	$\tilde{a}$	$\vec{a}$
$\tilde{a}$	$\vec{a}$	$\vec{a}$	$\vec{a}$	$\vec{a}$
$\vec{a}$	$\vec{a}$	$\vec{a}$	$\vec{a}$	$\vec{a}$

	$[\mathcal{T}_\Lambda^-(\varphi),\mathcal{T}_\Lambda^+(\varphi)]$	$[\mathcal{I}_\Lambda^-(\varphi),\mathcal{I}_\Lambda^+(\varphi)]$	$[\mathcal{F}_\Lambda^-(\varphi),\mathcal{F}_\Lambda^+(\varphi)]$
$\varphi=\dot{a}$	$[0.75,0.8]$	$[0.85,0.9]$	$[0.2,0.25]$
$\varphi=\ddot{a}$	$[0.65,0.75]$	$[0.8,0.85]$	$[0.25,0.35]$
$\varphi=\tilde{a}$	$[0.50,0.55]$	$[0.65,0.70]$	$[0.45,0.50]$
$\varphi=\vec{a}$	$[0.55,0.65]$	$[0.75,0.80]$	$[0.35,0.45]$

Clearly, Λ is a NSVSBS of $\mathcal{B}$.

Theorem 3.3. *The intersection of a family of every $NSVSBS^s$ of $\mathcal{B}$ is a NSVSBS of $\mathcal{B}$.*

Proof. Let $\{\mathcal{V}_i : i \in I\}$ be a collection of $NSVSBS^s$ of $\mathcal{B}$ and $\Lambda = \bigcap_{i\in I} \mathcal{V}_i$.
Let $\Re, \Im$ in $\mathcal{B}$. Then

$$\begin{aligned}
\mathcal{T}^-_\Lambda(\Re\lozenge_1\Im) &= \inf_{i\in I}\ \mathcal{T}^-_{\mathcal{V}_i}(\Re\lozenge_1\Im)\\
&\geq \inf_{i\in I}\ \min\{\mathcal{T}^-_{\mathcal{V}_i}(\Re), \mathcal{T}^-_{\mathcal{V}_i}(\Im)\}\\
&= \min\left\{\inf_{i\in I}\ \mathcal{T}^-_{\mathcal{V}_i}(\Re), \inf_{i\in I}\ \mathcal{T}^-_{\mathcal{V}_i}(\Im)\right\}\\
&= \min\{\mathcal{T}^-_\Lambda(\Re), \mathcal{T}^-_\Lambda(\Im)\}.
\end{aligned}$$

$$\begin{aligned}
1-\mathcal{F}^-_\Lambda(\Re\lozenge_1\Im) &= \inf_{i\in I}\ 1-\mathcal{F}^-_{\mathcal{V}_i}(\Re\lozenge_1\Im)\\
&\geq \inf_{i\in I}\ \min\{1-\mathcal{F}^-_{\mathcal{V}_i}(\Re), 1-\mathcal{F}^-_{\mathcal{V}_i}(\Im)\}\\
&= \min\left\{\inf_{i\in I}\ 1-\mathcal{F}^-_{\mathcal{V}_i}(\Re), \inf_{i\in I}\ 1-\mathcal{F}^-_{\mathcal{V}_i}(\Im)\right\}\\
&= \min\{1-\mathcal{F}^-_\Lambda(\Re), 1-\mathcal{F}^-_\Lambda(\Im)\}.
\end{aligned}$$

Thus $\mathcal{V}^\mathcal{T}_\Lambda(\Re\lozenge_1\Im) \geq \min\{\mathcal{V}_\Lambda(\Re), \mathcal{V}_\Lambda(\Im)\}$. Similarly, $\mathcal{V}^\mathcal{T}_\Lambda(\Re\lozenge_2\Im) \geq \min\{\mathcal{V}_\Lambda(\Re), \mathcal{V}_\Lambda(\Im)\}$ and $\mathcal{V}^\mathcal{T}_\Lambda(\Re\lozenge_3\Im) \geq \min\{\mathcal{V}_\Lambda(\Re), \mathcal{V}_\Lambda(\Im)\}$. Now,

$$\begin{aligned}
\mathcal{I}^-_\Lambda(\Re\lozenge_1\Im) &= \inf_{i\in I-}\ \mathcal{I}^-_{\mathcal{V}_i}(\Re\lozenge_1\Im)\\
&\geq \inf_{i\in I-}\ \frac{\mathcal{I}^-_{\mathcal{V}_i}(\Re)+\mathcal{I}^-_{\mathcal{V}_i}(\Im)}{2}\\
&= \frac{\inf_{i\in I-}\ \mathcal{I}^-_{\mathcal{V}_i}(\Re)+\inf_{i\in I-}\ \mathcal{I}^-_{\mathcal{V}_i}(\Im)}{2}\\
&= \frac{\mathcal{I}^-_\Lambda(\Re)+\mathcal{I}^-_\Lambda(\Im)}{2}.
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}^+_\Lambda(\Re\lozenge_1\Im) &= \inf_{i\in I+}\ \mathcal{I}^+_{\mathcal{V}_i}(\Re\lozenge_1\Im)\\
&\geq \inf_{i\in I+}\ \frac{\mathcal{I}^+_{\mathcal{V}_i}(\Re)+\mathcal{I}^+_{\mathcal{V}_i}(\Im)}{2}\\
&= \frac{\inf_{i\in I+}\ \mathcal{I}^+_{\mathcal{V}_i}(\Re)+\inf_{i\in I+}\ \mathcal{I}^+_{\mathcal{V}_i}(\Im)}{2}\\
&= \frac{\mathcal{I}^+_\Lambda(\Re)+\mathcal{I}^+_\Lambda(\Im)}{2}.
\end{aligned}$$

Thus $\mathcal{V}^\mathcal{I}_\Lambda(\Re\lozenge_1\Im) \geq \min\{\mathcal{V}_\Lambda(\Re), \mathcal{V}_\Lambda(\Im)\}$. Similarly, $\mathcal{V}^\mathcal{I}_\Lambda(\Re\lozenge_2\Im) \geq \min\{\mathcal{V}_\Lambda(\Re), \mathcal{V}_\Lambda(\Im)\}$ and $\mathcal{V}^\mathcal{I}_\Lambda(\Re\lozenge_3\Im) \geq \min\{\mathcal{V}_\Lambda(\Re), \mathcal{V}_\Lambda(\Im)\}$.

Now,

$$\begin{aligned}\mathcal{F}^-_\Lambda(\Re\Diamond_1\Im) &= \sup_{i\in I} \mathcal{F}^-_{\mathcal{V}_i}(\Re\Diamond_1\Im)\\ &\leq \sup_{i\in I} \max\{\mathcal{F}^-_{\mathcal{V}_i}(\Re), \mathcal{F}^-_{\mathcal{V}_i}(\Im)\}\\ &= \max\left\{\sup_{i\in I} \mathcal{F}^-_{\mathcal{V}_i}(\Re), \sup_{i\in I} \mathcal{F}^-_{\mathcal{V}_i}(\Im)\right\}\\ &= \max\{\mathcal{F}^-_\Lambda(\Re), \mathcal{F}^-_\Lambda(\Im)\}.\end{aligned}$$

$$\begin{aligned}1-\mathcal{T}^-_\Lambda(\Re\Diamond_1\Im) &= \sup_{i\in I} 1-\mathcal{T}^-_{\mathcal{V}_i}(\Re\Diamond_1\Im)\\ &\leq \sup_{i\in I} \max\{1-\mathcal{T}^-_{\mathcal{V}_i}(\Re), 1-\mathcal{T}^-_{\mathcal{V}_i}(\Im)\}\\ &= \max\left\{\sup_{i\in I} 1-\mathcal{T}^-_{\mathcal{V}_i}(\Re), \sup_{i\in I} 1-\mathcal{T}^-_{\mathcal{V}_i}(\Im)\right\}\\ &= \max\{1-\mathcal{T}^-_\Lambda(\Re), 1-\mathcal{T}^-_\Lambda(\Im)\}.\end{aligned}$$

Thus $\mathcal{V}^{\mathcal{F}}_\Lambda(\Re\Diamond_1\Im) \leq \max\{\mathcal{V}_\Lambda(\Re), \mathcal{V}_\Lambda(\Im)\}$. Similarly, $\mathcal{V}^{\mathcal{F}}_\Lambda(\Re\Diamond_2\Im) \leq \max\{\mathcal{V}_\Lambda(\Re), \mathcal{V}_\Lambda(\Im)\}$ and $\mathcal{V}^{\mathcal{F}}_\Lambda(\Re\Diamond_3\Im) \leq \max\{\mathcal{V}_\Lambda(\Re), \mathcal{V}_\Lambda(\Im)\}$. Hence, Λ is a NSVSBS of $\mathcal{B}$.

Theorem 3.4. *If Λ and Ψ are the $NSVSBS^s$ of $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, then $\Lambda\times\Psi$ is a NSVSBS of $\mathcal{B}_1\times\mathcal{B}_2$.*

Proof. Let Λ and Ψ be the $NSVSBS^s$ of $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Let $\Re_1, \Re_2 \in \mathcal{B}_1$ and $\Im_1, \Im_2 \in \mathcal{B}_2$. Then $(\Re_1, \Im_1), (\Re_2, \Im_2)$ belong to $\mathcal{B}_1\times\mathcal{B}_2$. Now

$$\begin{aligned}\mathcal{T}^-_{\Lambda\times\Psi}[(\Re_1,\Im_1)\Diamond_1(\Re_2,\Im_2)] &= \mathcal{T}^-_{\Lambda\times\Psi}(\Re_1\Diamond_1\Re_2, \Im_1\Diamond_1\Im_2)\\ &= \min\{\mathcal{T}^-_\Lambda(\Re_1\Diamond_1\Re_2), \mathcal{T}^-_\Psi(\Im_1\Diamond_1\Im_2)\}\\ &\geq \min\{\min\{\mathcal{T}^-_\Lambda(\Re_1), \mathcal{T}^-_\Lambda(\Re_2)\}, \min\{\mathcal{T}^-_\Psi(\Im_1), \mathcal{T}^-_\Psi(\Im_2)\}\}\\ &= \min\{\min\{\mathcal{T}^-_\Lambda(\Re_1), \mathcal{T}^-_\Psi(\Im_1)\}, \min\{\mathcal{T}^-_\Lambda(\Re_2), \mathcal{T}^-_\Psi(\Im_2)\}\}\\ &= \min\{\mathcal{T}^-_{\Lambda\times\Psi}(\Re_1,\Im_1), \mathcal{T}^-_{\Lambda\times\Psi}(\Re_2,\Im_2)\}.\end{aligned}$$

$$\begin{aligned}1-\mathcal{F}^-_{\Lambda\times\Psi}[(\Re_1,\Im_1)\Diamond_1(\Re_2,\Im_2)] &= 1-\mathcal{F}^-_{\Lambda\times\Psi}(\Re_1\Diamond_1\Re_2, \Im_1\Diamond_1\Im_2)\\ &= \min\{1-\mathcal{F}^-_\Lambda(\Re_1\Diamond_1\Re_2), 1-\mathcal{F}^-_\Psi(\Im_1\Diamond_1\Im_2)\}\\ &\geq \min\{\min\{1-\mathcal{F}^-_\Lambda(\Re_1), 1-\mathcal{F}^-_\Lambda(\Re_2)\}, \min\{1-\mathcal{F}^-_\Psi(\Im_1), 1-\mathcal{F}^-_\Psi(\Im_2)\}\}\\ &= \min\{\min\{1-\mathcal{F}^-_\Lambda(\Re_1), 1-\mathcal{F}^-_\Psi(\Im_1)\}, \min\{1-\mathcal{F}^-_\Lambda(\Re_2), 1-\mathcal{F}^-_\Psi(\Im_2)\}\}\\ &= \min\{1-\mathcal{F}^-_{\Lambda\times\Psi}(\Re_1,\Im_1), 1-\mathcal{F}^-_{\Lambda\times\Psi}(\Re_2,\Im_2)\}.\end{aligned}$$

Thus $\mathcal{V}^{\mathcal{T}}_{\Lambda\times\Psi}(\Re\Diamond_1\Im) \geq \min\{\mathcal{V}^{\mathcal{T}}_{\Lambda\times\Psi}(\Re), \mathcal{V}^{\mathcal{T}}_{\Lambda\times\Psi}(\Im)\}$. Similarly, $\mathcal{V}^{\mathcal{T}}_{\Lambda\times\Psi}(\Re\Diamond_2\Im) \geq \min\{\mathcal{V}^{\mathcal{T}}_{\Lambda\times\Psi}(\Re), \mathcal{V}^{\mathcal{T}}_{\Lambda\times\Psi}(\Im)\}$ and $\mathcal{V}^{\mathcal{T}}_{\Lambda\times\Psi}(\Re\Diamond_3\Im) \geq \min\{\mathcal{V}^{\mathcal{T}}_{\Lambda\times\Psi}(\Re), \mathcal{V}^{\mathcal{T}}_{\Lambda\times\Psi}(\Im)\}$.

Now,

$$\begin{aligned}
\mathcal{I}^-_{\Lambda\times\Psi}[(\Re_1,\Im_1)\Diamond_1(\Re_2,\Im_2)] &= \mathcal{I}^-_{\Lambda\times\Psi}(\Re_1\Diamond_1\Re_2,\Im_1\Diamond_1\Im_2)\\
&= \frac{\mathcal{I}^-_{\Lambda}(\Re_1\Diamond_1\Re_2)+\mathcal{I}^-_{\Psi}(\Im_1\Diamond_1\Im_2)}{2}\\
&\geq \frac{1}{2}\left[\frac{\mathcal{I}^-_{\Lambda}(\Re_1)+\mathcal{I}^-_{\Lambda}(\Re_2)}{2}+\frac{\mathcal{I}^-_{\Psi}(\Im_1)+\mathcal{I}^-_{\Psi}(\Im_2)}{2}\right]\\
&= \frac{1}{2}\left[\frac{\mathcal{I}^-_{\Lambda}(\Re_1)+\mathcal{I}^-_{\Psi}(\Im_1)}{2}+\frac{\mathcal{I}^-_{\Lambda}(\Re_2)+\mathcal{I}^-_{\Psi}(\Im_2)}{2}\right]\\
&= \frac{1}{2}\left[\mathcal{I}^-_{\Lambda\times\Psi}(\Re_1,\Im_1)+\mathcal{I}^-_{\Lambda\times\Psi}(\Re_2,\Im_2)\right].
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}^+_{\Lambda\times\Psi}[(\Re_1,\Im_1)\Diamond_1(\Re_2,\Im_2)] &= \mathcal{I}^+_{\Lambda\times\Psi}(\Re_1\Diamond_1\Re_2,\Im_1\Diamond_1\Im_2)\\
&= \frac{\mathcal{I}^+_{\Lambda}(\Re_1\Diamond_1\Re_2)+\mathcal{I}^+_{\Psi}(\Im_1\Diamond_1\Im_2)}{2}\\
&\geq \frac{1}{2}\left[\frac{\mathcal{I}^+_{\Lambda}(\Re_1)+\mathcal{I}^+_{\Lambda}(\Re_2)}{2}+\frac{\mathcal{I}^+_{\Psi}(\Im_1)+\mathcal{I}^+_{\Psi}(\Im_2)}{2}\right]\\
&= \frac{1}{2}\left[\frac{\mathcal{I}^+_{\Lambda}(\Re_1)+\mathcal{I}^+_{\Psi}(\Im_1)}{2}+\frac{\mathcal{I}^+_{\Lambda}(\Re_2)+\mathcal{I}^+_{\Psi}(\Im_2)}{2}\right]\\
&= \frac{1}{2}\left[\mathcal{I}^+_{\Lambda\times\Psi}(\Re_1,\Im_1)+\mathcal{I}^+_{\Lambda\times\Psi}(\Re_2,\Im_2)\right].
\end{aligned}$$

Thus $\mathcal{V}^{\mathcal{I}}_{\Lambda\times\Psi}(\Re\Diamond_1\Im) \geq \frac{1}{2}\left[\mathcal{V}^{\mathcal{I}}_{\Lambda\times\Psi}(\Re_1,\Im_1)+\mathcal{V}^{\mathcal{I}}_{\Lambda\times\Psi}(\Re_2,\Im_2)\right]$. Similarly, $\mathcal{V}^{\mathcal{I}}_{\Lambda\times\Psi}(\Re\Diamond_2\Im) \geq \frac{1}{2}\left[\mathcal{V}^{\mathcal{I}}_{\Lambda\times\Psi}(\Re_1,\Im_1)+\mathcal{V}^{\mathcal{I}}_{\Lambda\times\Psi}(\Re_2,\Im_2)\right]$ and $\mathcal{V}^{\mathcal{I}}_{\Lambda\times\Psi}(\Re\Diamond_3\Im) \geq \frac{1}{2}\left[\mathcal{V}^{\mathcal{I}}_{\Lambda\times\Psi}(\Re_1,\Im_1)+\mathcal{V}^{\mathcal{I}}_{\Lambda\times\Psi}(\Re_2,\Im_2)\right]$. Now

$$\begin{aligned}
\mathcal{F}^-_{\Lambda\times\Psi}[(\Re_1,\Im_1)\Diamond_1(\Re_2,\Im_2)] &= \mathcal{F}^-_{\Lambda\times\Psi}(\Re_1\Diamond_1\Re_2,\Im_1\Diamond_1\Im_2)\\
&= \max\{\mathcal{F}^-_{\Lambda}(\Re_1\Diamond_1\Re_2),\mathcal{F}^-_{\Psi}(\Im_1\Diamond_1\Im_2)\}\\
&\leq \max\{\max\{\mathcal{F}^-_{\Lambda}(\Re_1),\mathcal{F}^-_{\Lambda}(\Re_2)\},\max\{\mathcal{F}^-_{\Psi}(\Im_1),\mathcal{F}^-_{\Psi}(\Im_2)\}\}\\
&= \max\{\max\{\mathcal{F}^-_{\Lambda}(\Re_1),\mathcal{F}^-_{\Psi}(\Im_1)\},\max\{\mathcal{F}^-_{\Lambda}(\Re_2),\mathcal{F}^-_{\Psi}(\Im_2)\}\}\\
&= \max\{\mathcal{F}^-_{\Lambda\times\Psi}(\Re_1,\Im_1),\mathcal{F}^-_{\Lambda\times\Psi}(\Re_2,\Im_2)\}.
\end{aligned}$$

$$\begin{aligned}
1-\mathcal{T}^-_{\Lambda\times\Psi}[(\Re_1,\Im_1)\Diamond_1(\Re_2,\Im_2)] &= 1-\mathcal{T}^-_{\Lambda\times\Psi}(\Re_1\Diamond_1\Re_2,\Im_1\Diamond_1\Im_2)\\
&= \max\{1-\mathcal{T}^-_{\Lambda}(\Re_1\Diamond_1\Re_2),1-\mathcal{T}^-_{\Psi}(\Im_1\Diamond_1\Im_2)\}\\
&\leq \max\{\max\{1-\mathcal{T}^-_{\Lambda}(\Re_1),1-\mathcal{T}^-_{\Lambda}(\Re_2)\},\max\{1-\mathcal{T}^-_{\Psi}(\Im_1),1-\mathcal{T}^-_{\Psi}(\Im_2)\}\}\\
&= \max\{\max\{1-\mathcal{T}^-_{\Lambda}(\Re_1),1-\mathcal{T}^-_{\Psi}(\Im_1)\},\max\{1-\mathcal{T}^-_{\Lambda}(\Re_2),1-\mathcal{T}^-_{\Psi}(\Im_2)\}\}\\
&= \max\{1-\mathcal{T}^-_{\Lambda\times\Psi}(\Re_1,\Im_1),1-\mathcal{T}^-_{\Lambda\times\Psi}(\Re_2,\Im_2)\}.
\end{aligned}$$

Thus $\mathcal{V}^{\mathcal{F}}_{\Lambda\times\Psi}(\Re\Diamond_1\Im) \leq \max\{\mathcal{V}^{\mathcal{F}}_{\Lambda\times\Psi}(\Re),\mathcal{V}^{\mathcal{F}}_{\Lambda\times\Psi}(\Im)\}$. Similarly, $\mathcal{V}^{\mathcal{F}}_{\Lambda\times\Psi}(\Re\Diamond_2\Im) \leq \max\{\mathcal{V}^{\mathcal{F}}_{\Lambda\times\Psi}(\Re),\mathcal{V}^{\mathcal{F}}_{\Lambda\times\Psi}(\Im)\}$ and $\mathcal{V}^{\mathcal{F}}_{\Lambda\times\Psi}(\Re\Diamond_3\Im) \leq \max\{\mathcal{V}^{\mathcal{F}}_{\Lambda\times\Psi}(\Re),\mathcal{V}^{\mathcal{F}}_{\Lambda\times\Psi}(\Im)\}$.

Corollary 3.5. *If $\Lambda_1, \Lambda_2, ..., \Lambda_n$ are the families of $NSVSBS^s$ of $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n$ respectively, then $\Lambda_1 \times \Lambda_2 \times ... \times \Lambda_n$ is a NSVSBS of $\mathcal{B}_1 \times \mathcal{B}_2 \times ... \times \mathcal{B}_n$.*

Definition 3.6. Let Λ be a neutrosophic VS in $\mathcal{B}$, the strongest neutrosophic vague relation (SNSVR) on $\mathcal{B}$, that is a NSVR on Λ is defined as

$$\left\{\begin{array}{l} \mathcal{V}_V^{\mathcal{T}}(\Re, \Im) = \min\{\mathcal{V}_\Lambda^{\mathcal{T}}(\Re), \mathcal{V}_\Lambda^{\mathcal{T}}(\Im)\} \\ \mathcal{V}_V^{\mathcal{I}}(\Re, \Im) = \frac{\mathcal{V}_\Lambda^{\mathcal{I}}(\Re) + \mathcal{V}_\Lambda^{\mathcal{I}}(\Im)}{2} \\ \mathcal{V}_V^{\mathcal{F}}(\Re, \Im) = \max\{\mathcal{V}_\Lambda^{\mathcal{F}}(\Re), \mathcal{V}_\Lambda^{\mathcal{F}}(\Im)\} \end{array}\right\}.$$

Theorem 3.7. *Let Λ be the NSVSBS of $\mathcal{B}$ and V be the SNSVR of $\mathcal{B}$. Then Λ is a NSVSBS of $\mathcal{B}$ if and only if V is a NSVSBS of $\mathcal{B} \times \mathcal{B}$.*

Proof. Let Λ be the NSVSBS of $\mathcal{B}$ and V be the SNSVR of $\mathcal{B}$. Then for any $\Re = (\Re_1, \Re_2)$ and $\Im = (\Im_1, \Im_2)$ are in $\mathcal{B} \times \mathcal{B}$. Now,

$$\begin{aligned} \mathcal{T}_V^-(\Re \Diamond_1 \Im) &= \mathcal{T}_V^-[((\Re_1, \Re_2) \Diamond_1 (\Im_1, \Im_2)] \\ &= \mathcal{T}_V^-(\Re_1 \Diamond_1 \Im_1, \Re_2 \Diamond_1 \Im_2) \\ &= \min\{\mathcal{T}_\Lambda^-(\Re_1 \Diamond_1 \Im_1), \mathcal{T}_\Lambda^-(\Re_2 \Diamond_1 \Im_2)\} \\ &\geq \min\{\min\{\mathcal{T}_\Lambda^-(\Re_1), \mathcal{T}_\Lambda^-(\Im_1)\}, \min\{\mathcal{T}_\Lambda^-(\Re_2), \mathcal{T}_\Lambda^-(\Im_2)\}\} \\ &= \min\{\min\{\mathcal{T}_\Lambda^-(\Re_1), \mathcal{T}_\Lambda^-(\Re_2)\}, \min\{\mathcal{T}_\Lambda^-(\Im_1), \mathcal{T}_\Lambda^-(\Im_2)\}\} \\ &= \min\{\mathcal{T}_V^-(\Re_1, \Re_2), \mathcal{T}_V^-(\Im_1, \Im_2)\} \\ &= \min\{\mathcal{T}_V^-(\Re), \mathcal{T}_V^-(\Im)\}. \end{aligned}$$

$$\begin{aligned} 1 - \mathcal{F}_V^-(\Re \Diamond_1 \Im) &= 1 - \mathcal{F}_V^-[((\Re_1, \Re_2) \Diamond_1 (\Im_1, \Im_2)] \\ &= 1 - \mathcal{F}_V^-(\Re_1 \Diamond_1 \Im_1, \Re_2 \Diamond_1 \Im_2) \\ &= \min\{1 - \mathcal{F}_\Lambda^-(\Re_1 \Diamond_1 \Im_1), 1 - \mathcal{F}_\Lambda^-(\Re_2 \Diamond_1 \Im_2)\} \\ &\geq \min\{\min\{1 - \mathcal{F}_\Lambda^-(\Re_1), 1 - \mathcal{F}_\Lambda^-(\Im_1)\}, \min\{1 - \mathcal{F}_\Lambda^-(\Re_2), 1 - \mathcal{F}_\Lambda^-(\Im_2)\}\} \\ &= \min\{\min\{1 - \mathcal{F}_\Lambda^-(\Re_1), 1 - \mathcal{F}_\Lambda^-(\Re_2)\}, \min\{1 - \mathcal{F}_\Lambda^-(\Im_1), 1 - \mathcal{F}_\Lambda^-(\Im_2)\}\} \\ &= \min\{1 - \mathcal{F}_V^-(\Re_1, \Re_2), 1 - \mathcal{F}_V^-(\Im_1, \Im_2)\} \\ &= \min\{1 - \mathcal{F}_V^-(\Re), 1 - \mathcal{F}_V^-(\Im)\}. \end{aligned}$$

Thus $\mathcal{V}_V^{\mathcal{T}}(\Re\Diamond_1\Im) \geq \min\{\mathcal{V}_V^{\mathcal{T}}(\Re), \mathcal{V}_V^{\mathcal{T}}(\Im)\}$. Similarly, $\mathcal{V}_V^{\mathcal{T}}(\Re\Diamond_2\Im) \geq \min\{\mathcal{V}_V^{\mathcal{T}}(\Re), \mathcal{V}_V^{\mathcal{T}}(\Im)\}$ and $\mathcal{V}_V^{\mathcal{T}}(\Re\Diamond_3\Im) \geq \min\{\mathcal{V}_V^{\mathcal{T}}(\Re), \mathcal{V}_V^{\mathcal{T}}(\Im)\}$. Now,

$$\begin{aligned}
\mathcal{I}_V^-(\Re\Diamond_1\Im) &= \mathcal{I}_V^-[((\Re_1,\Re_2)\Diamond_1(\Im_1,\Im_2)] \\
&= \mathcal{I}_V^-(\Re_1\Diamond_1\Im_1, \Re_2\Diamond_1\Im_2) \\
&= \frac{\mathcal{I}_\Lambda^-(\Re_1\Diamond_1\Im_1) + \mathcal{I}_\Lambda^-(\Re_2\Diamond_1\Im_2)}{2} \\
&\geq \frac{1}{2}\left[\frac{\mathcal{I}_\Lambda^-(\Re_1) + \mathcal{I}_\Lambda^-(\Im_1)}{2} + \frac{\mathcal{I}_\Lambda^-(\Re_2) + \mathcal{I}_\Lambda^-(\Im_2)}{2}\right] \\
&= \frac{1}{2}\left[\frac{\mathcal{I}_\Lambda^-(\Re_1) + \mathcal{I}_\Lambda^-(\Re_2)}{2} + \frac{\mathcal{I}_\Lambda^-(\Im_1) + \mathcal{I}_\Lambda^-(\Im_2)}{2}\right] \\
&= \frac{\mathcal{I}_V^-(\Re_1,\Re_2) + \mathcal{I}_V^-(\Im_1,\Im_2)}{2} \\
&= \frac{\mathcal{I}_V^-(\Re) + \mathcal{I}_V^-(\Im)}{2}.
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_V^+(\Re\Diamond_1\Im) &= \mathcal{I}_V^+[((\Re_1,\Re_2)\Diamond_1(\Im_1,\Im_2)] \\
&= \mathcal{I}_V^+(\Re_1\Diamond_1\Im_1, \Re_2\Diamond_1\Im_2) \\
&= \frac{\mathcal{I}_\Lambda^+(\Re_1\Diamond_1\Im_1) + \mathcal{I}_\Lambda^+(\Re_2\Diamond_1\Im_2)}{2} \\
&\geq \frac{1}{2}\left[\frac{\mathcal{I}_\Lambda^+(\Re_1) + \mathcal{I}_\Lambda^+(\Im_1)}{2} + \frac{\mathcal{I}_\Lambda^+(\Re_2) + \mathcal{I}_\Lambda^+(\Im_2)}{2}\right] \\
&= \frac{1}{2}\left[\frac{\mathcal{I}_\Lambda^+(\Re_1) + \mathcal{I}_\Lambda^+(\Re_2)}{2} + \frac{\mathcal{I}_\Lambda^+(\Im_1) + \mathcal{I}_\Lambda^+(\Im_2)}{2}\right] \\
&= \frac{\mathcal{I}_V^+(\Re_1,\Re_2) + \mathcal{I}_V^+(\Im_1,\Im_2)}{2} \\
&= \frac{\mathcal{I}_V^+(\Re) + \mathcal{I}_V^+(\Im)}{2}.
\end{aligned}$$

Thus $\mathcal{V}_V^{\mathcal{I}}(\Re\Diamond_1\Im) \geq \frac{\mathcal{V}_V(\Re)+\mathcal{V}_V(\Im)}{2}$. Similarly, $\mathcal{V}_V^{\mathcal{I}}(\Re\Diamond_2\Im) \geq \frac{\mathcal{V}_V(\Re)+\mathcal{V}_V(\Im)}{2}$ and $\mathcal{V}_V^{\mathcal{I}}(\Re\Diamond_3\Im) \geq \frac{\mathcal{V}_V(\Re)+\mathcal{V}_V(\Im)}{2}$.

Similarly, $\mathcal{V}_V^{\mathcal{F}}(\Re\Diamond_1\Im) \leq \max\{\mathcal{V}_V^{\mathcal{F}}(\Re), \mathcal{V}_V^{\mathcal{F}}(\Im)\}$, $\mathcal{V}_V^{\mathcal{F}}(\Re\Diamond_2\Im) \leq \max\{\mathcal{V}_V^{\mathcal{F}}(\Re), \mathcal{V}_V^{\mathcal{F}}(\Im)\}$ and $\mathcal{V}_V^{\mathcal{F}}(\Re\Diamond_3\Im) \leq \max\{\mathcal{V}_V^{\mathcal{F}}(\Re), \mathcal{V}_V^{\mathcal{F}}(\Im)\}$.

Conversely let us assume that V is a NSVSBS of $\mathcal{B} \times \mathcal{B}$, then for any $\Re = (\Re_1, \Re_2)$ and $\Im = (\Im_1, \Im_2)$ are in $\mathcal{B} \times \mathcal{B}$. Now,

$$\begin{aligned}\min\{\mathcal{T}_\Lambda^-(\Re_1\Diamond_1\Im_1), \mathcal{T}_\Lambda^-(\Re_2\Diamond_1\Im_2)\} &= \mathcal{T}_V^-(\Re_1\Diamond_1\Im_1, \Re_2\Diamond_1\Im_2)\\ &= \mathcal{T}_V^-[(\Re_1, \Re_2)\Diamond_1(\Im_1, \Im_2)]\\ &= \mathcal{T}_V^-(\Re\Diamond_1\Im)\\ &\geq \min\{\mathcal{T}_V^-(\Re), \mathcal{T}_V^-(\Im)\}\\ &= \min\{\mathcal{T}_V^-(\Re_1, \Re_2)\}, \mathcal{T}_V^-(\Im_1, \Im_2)\}\\ &= \min\{\min\{\mathcal{T}_\Lambda^-(\Re_1), \mathcal{T}_\Lambda^-(\Re_2)\}, \min\{\mathcal{T}_\Lambda^-(\Im_1), \mathcal{T}_\Lambda^-(\Im_2)\}\}.\end{aligned}$$

If $\mathcal{T}_\Lambda^-(\Re_1\Diamond_1\Im_1) \leq \mathcal{T}_\Lambda^-(\Re_2\Diamond_1\Im_2)$, then $\mathcal{T}_\Lambda^-(\Re_1) \leq \mathcal{T}_\Lambda^-(\Re_2)$ and $\mathcal{T}_\Lambda^-(\Im_1) \leq \mathcal{T}_\Lambda^-(\Im_2)$. We get $\mathcal{T}_\Lambda^-(\Re_1\Diamond_1\Im_1) \geq \min\{\mathcal{T}_\Lambda^-(\Re_1), \mathcal{T}_\Lambda^-(\Im_1)\}$ for all $\Re_1, \Im_1 \in \mathcal{B}$, and
$\min\{\mathcal{T}_\Lambda^-(\Re_1\Diamond_2\Im_1), \mathcal{T}_\Lambda^-(\Re_2\Diamond_2\Im_2)\} \geq \min\{\min\{\mathcal{T}_\Lambda^-(\Re_1), \mathcal{T}_\Lambda^-(\Re_2)\}, \min\{\mathcal{T}_\Lambda^-(\Im_1), \mathcal{T}_\Lambda^-(\Im_2)\}\}$
If $\mathcal{T}_\Lambda^-(\Re_1\Diamond_2\Im_1) \leq \mathcal{T}_\Lambda^-(\Re_2\Diamond_2\Im_2)$, then $\mathcal{T}_\Lambda^-(\Re_1\Diamond_2\Im_1) \geq \min\{\mathcal{T}_\Lambda^-(\Re_1), \mathcal{T}_\Lambda^-(\Im_1)\}$.
$\min\{\mathcal{T}_\Lambda^-(\Re_1\Diamond_3\Im_1), \mathcal{T}_\Lambda^-(\Re_2\Diamond_3\Im_2)\} \geq \min\{\min\{\mathcal{T}_\Lambda^-(\Re_1), \mathcal{T}_\Lambda^-(\Re_2)\}, \min\{\mathcal{T}_\Lambda^-(\Im_1), \mathcal{T}_\Lambda^-(\Im_2)\}\}$.
If $\mathcal{T}_\Lambda^-(\Re_1\Diamond_3\Im_1) \leq \mathcal{T}_\Lambda^-(\Re_2\Diamond_3\Im_2)$, then $\mathcal{T}_\Lambda^-(\Re_1\Diamond_3\Im_1) \geq \min\{\mathcal{T}_\Lambda^-(\Re_1), \mathcal{T}_\Lambda^-(\Im_1)\}$.

$$\begin{aligned}&\min\{1 - \mathcal{F}_\Lambda^-(\Re_1\Diamond_1\Im_1), 1 - \mathcal{F}_\Lambda^-(\Re_2\Diamond_1\Im_2)\}\\ &= 1 - \mathcal{F}_V^-(\Re_1\Diamond_1\Im_1, \Re_2\Diamond_1\Im_2)\\ &= 1 - \mathcal{F}_V^-[(\Re_1, \Re_2)\Diamond_1(\Im_1, \Im_2)]\\ &= 1 - \mathcal{F}_V^-(\Re\Diamond_1\Im)\\ &\geq \min\{1 - \mathcal{F}_V^-(\Re), 1 - \mathcal{F}_V^-(\Im)\}\\ &= \min\{1 - \mathcal{F}_V^-(\Re_1, \Re_2)\}, 1 - \mathcal{F}_V^-(\Im_1, \Im_2)\}\\ &= \min\{\min\{1 - \mathcal{F}_\Lambda^-(\Re_1), 1 - \mathcal{F}_\Lambda^-(\Re_2)\}, \min\{1 - \mathcal{F}_\Lambda^-(\Im_1), 1 - \mathcal{F}_\Lambda^-(\Im_2)\}\}.\end{aligned}$$

If $1 - \mathcal{F}_\Lambda^-(\Re_1\Diamond_1\Im_1) \leq 1 - \mathcal{F}_\Lambda^-(\Re_2\Diamond_1\Im_2)$, then $1 - \mathcal{F}_\Lambda^-(\Re_1) \leq 1 - \mathcal{F}_\Lambda^-(\Re_2)$ and $1 - \mathcal{F}_\Lambda^-(\Im_1) \leq 1 - \mathcal{F}_\Lambda^-(\Im_2)$. We get $1 - \mathcal{F}_\Lambda^-(\Re_1\Diamond_1\Im_1) \geq \min\{1 - \mathcal{F}_\Lambda^-(\Re_1), 1 - \mathcal{F}_\Lambda^-(\Im_1)\}$ for all $\Re_1, \Im_1 \in \mathcal{B}$, and $\min\{1 - \mathcal{F}_\Lambda^-(\Re_1\Diamond_2\Im_1), 1 - \mathcal{F}_\Lambda^-(\Re_2\Diamond_2\Im_2)\} \geq \min\{\min\{1 - \mathcal{F}_\Lambda^-(\Re_1), 1 - \mathcal{F}_\Lambda^-(\Re_2)\}, \min\{1 - \mathcal{F}_\Lambda^-(\Im_1), 1 - \mathcal{F}_\Lambda^-(\Im_2)\}\}$.
If $1 - \mathcal{F}_\Lambda^-(\Re_1\Diamond_2\Im_1) \leq 1 - \mathcal{F}_\Lambda^-(\Re_2\Diamond_2\Im_2)$, then $1 - \mathcal{F}_\Lambda^-(\Re_1\Diamond_2\Im_1) \geq \min\{1 - \mathcal{F}_\Lambda^-(\Re_1), 1 - \mathcal{F}_\Lambda^-(\Im_1)\}$.
$\min\{1 - \mathcal{F}_\Lambda^-(\Re_1\Diamond_3\Im_1), 1 - \mathcal{F}_\Lambda^-(\Re_2\Diamond_3\Im_2)\} \geq \min\{\min\{1 - \mathcal{F}_\Lambda^-(\Re_1), 1 - \mathcal{F}_\Lambda^-(\Re_2)\}, \min\{1 - \mathcal{F}_\Lambda^-(\Im_1), 1 - \mathcal{F}_\Lambda^-(\Im_2)\}\}$. If $1 - \mathcal{F}_\Lambda^-(\Re_1\Diamond_3\Im_1) \leq 1 - \mathcal{F}_\Lambda^-(\Re_2\Diamond_3\Im_2)$, then $1 - \mathcal{F}_\Lambda^-(\Re_1\Diamond_3\Im_1) \geq \min\{1 - \mathcal{F}_\Lambda^-(\Re_1), 1 - \mathcal{F}_\Lambda^-(\Im_1)\}$.
Thus $\mathcal{V}_V^{\mathcal{T}}(\Re\Diamond_1\Im) \geq \min\{\mathcal{V}_V^{\mathcal{T}}(\Re), \mathcal{V}_V^{\mathcal{T}}(\Im)\}$. Similarly, $\mathcal{V}_V^{\mathcal{T}}(\Re\Diamond_2\Im) \geq \min\{\mathcal{V}_V^{\mathcal{T}}(\Re), \mathcal{V}_V^{\mathcal{T}}(\Im)\}$ and

$\mathcal{V}_V^{\mathcal{T}}(\Re\lozenge_3\Im) \geq \min\{\mathcal{V}_V^{\mathcal{T}}(\Re), \mathcal{V}_V^{\mathcal{T}}(\Im)\}$. Now,

$$\begin{aligned}\frac{1}{2}\Big[\mathcal{I}_\Lambda^-(\Re_1\lozenge_1\Im_1) + \mathcal{I}_\Lambda^-(\Re_2\lozenge_1\Im_2)\Big] &= \mathcal{I}_V^-(\Re_1\lozenge_1\Im_1, \Re_2\lozenge_1\Im_2) \\ &= \mathcal{I}_V^-[(\Re_1,\Re_2)\lozenge_1(\Im_1,\Im_2)] \\ &= \mathcal{I}_V^-(\Re\lozenge_1\Im) \\ &\geq \frac{\mathcal{I}_V^-(\Re) + \mathcal{I}_V^-(\Im)}{2} \\ &= \frac{\mathcal{I}_V^-(\Re_1,\Re_2) + \mathcal{I}_V^-(\Im_1,\Im_2)}{2} \\ &= \frac{1}{2}\Bigg[\frac{\mathcal{I}_\Lambda^-(\Re_1) + \mathcal{I}_\Lambda^-(\Re_2)}{2} + \frac{\mathcal{I}_\Lambda^-(\Im_1) + \mathcal{I}_\Lambda^-(\Im_2)}{2}\Bigg].\end{aligned}$$

If $\mathcal{I}_\Lambda^-(\Re_1\lozenge_1\Im_1) \leq \mathcal{I}_\Lambda^-(\Re_2\lozenge_1\Im_2)$, then $\mathcal{I}_\Lambda^-(\Re_1) \leq \mathcal{I}_\Lambda^-(\Re_2)$ and $\mathcal{I}_\Lambda^-(\Im_1) \leq \mathcal{I}_\Lambda^-(\Im_2)$.
We get $\mathcal{I}_\Lambda^-(\Re_1\lozenge_1\Im_1) \geq \frac{\mathcal{I}_\Lambda^-(\Re_1)+\mathcal{I}_\Lambda^-(\Im_1)}{2}$. Similarly, $\mathcal{I}_\Lambda^-(\Re_1\lozenge_2\Im_1) \geq \frac{\mathcal{I}_\Lambda^-(\Re_1)+\mathcal{I}_\Lambda^-(\Im_1)}{2}$ and $\mathcal{I}_\Lambda^-(\Re_1\lozenge_3\Im_1) \geq \frac{\mathcal{I}_\Lambda^-(\Re_1)+\mathcal{I}_\Lambda^-(\Im_1)}{2}$.
Also, $\frac{1}{2}\Big[\mathcal{I}_\Lambda^+(\Re_1\lozenge_1\Im_1) + \mathcal{I}_\Lambda^+(\Re_2\lozenge_1\Im_2)\Big] \geq \frac{1}{2}\Bigg[\frac{\mathcal{I}_\Lambda^+(\Re_1)+\mathcal{I}_\Lambda^+(\Re_2)}{2} + \frac{\mathcal{I}_\Lambda^+(\Im_1)+\mathcal{I}_\Lambda^+(\Im_2)}{2}\Bigg]$.
If $\mathcal{I}_\Lambda^+(\Re_1\lozenge_1\Im_1) \leq \mathcal{I}_\Lambda^+(\Re_2\lozenge_1\Im_2)$, then $\mathcal{I}_\Lambda^+(\Re_1) \leq \mathcal{I}_\Lambda^+(\Re_2)$ and $\mathcal{I}_\Lambda^+(\Im_1) \leq \mathcal{I}_\Lambda^+(\Im_2)$.
We get $\mathcal{I}_\Lambda^+(\Re_1\lozenge_1\Im_1) \geq \frac{\mathcal{I}_\Lambda^+(\Re_1)+\mathcal{I}_\Lambda^+(\Im_1)}{2}$ and $\mathcal{I}_\Lambda^+(\Re_1\lozenge_2\Im_1) \geq \frac{\mathcal{I}_\Lambda^+(\Re_1)+\mathcal{I}_\Lambda^+(\Im_1)}{2}$ and $\mathcal{I}_\Lambda^+(\Re_1\lozenge_3\Im_1) \geq \frac{\mathcal{I}_\Lambda^+(\Re_1)+\mathcal{I}_\Lambda^+(\Im_1)}{2}$.
Thus $\mathcal{V}_V^{\mathcal{I}}(\Re\lozenge_1\Im) \geq \frac{\mathcal{V}_V(\Re)+\mathcal{V}_V(\Im)}{2}$. Similarly, $\mathcal{V}_V^{\mathcal{I}}(\Re\lozenge_2\Im) \geq \frac{\mathcal{V}_V(\Re)+\mathcal{V}_V(\Im)}{2}$ and $\mathcal{V}_V^{\mathcal{I}}(\Re\lozenge_3\Im) \geq \frac{\mathcal{V}_V(\Re)+\mathcal{V}_V(\Im)}{2}$. Similarly, $\max\{\mathcal{F}_\Lambda^-(\Re_1\lozenge_1\Im_1), \mathcal{F}_\Lambda^-(\Re_2\lozenge_1\Im_2)\} \leq \max\{\max\{\mathcal{F}_\Lambda^-(\Re_1), \mathcal{F}_\Lambda^-(\Re_2)\}, \max\{\mathcal{F}_\Lambda^-(\Im_1), \mathcal{F}_\Lambda^-(\Im_2)\}\}$.
If $\mathcal{F}_\Lambda^-(\Re_1\lozenge_1\Im_1) \geq \mathcal{F}_\Lambda^-(\Re_2\lozenge_1\Im_2)$, then $\mathcal{F}_\Lambda^-(\Re_1) \geq \mathcal{F}_\Lambda^-(\Re_2)$ and $\mathcal{F}_\Lambda^-(\Im_1) \geq \mathcal{F}_\Lambda^-(\Im_2)$.
We get $\mathcal{F}_\Lambda^-(\Re_1\lozenge_1\Im_1) \leq \max\{\mathcal{F}_\Lambda^-(\Re_1), \mathcal{F}_\Lambda^-(\Im_1)\}$.
$\max\{\mathcal{F}_\Lambda^-(\Re_1\lozenge_2\Im_1), \mathcal{F}_\Lambda^-(\Re_2\lozenge_2\Im_2)\} \leq \max\{\max\{\mathcal{F}_\Lambda^-(\Re_1), \mathcal{F}_\Lambda^-(\Re_2)\}, \max\{\mathcal{F}_\Lambda^-(\Im_1), \mathcal{F}_\Lambda^-(\Im_2)\}\}$.
If $\mathcal{F}_\Lambda^-(\Re_1\lozenge_2\Im_1) \geq \mathcal{F}_\Lambda^-(\Re_2\lozenge_2\Im_2)$, then $\mathcal{F}_\Lambda^-(\Re_1\lozenge_2\Im_1) \leq \max\{\mathcal{F}_\Lambda^-(\Re_1), \mathcal{F}_\Lambda^-(\Im_1)\}$.
$\max\{\mathcal{F}_\Lambda^-(\Re_1\lozenge_3\Im_1), \mathcal{F}_\Lambda^-(\Re_2\lozenge_3\Im_2)\} \leq \max\{\max\{\mathcal{F}_\Lambda^-(\Re_1), \mathcal{F}_\Lambda^-(\Re_2)\}, \max\{\mathcal{F}_\Lambda^-(\Im_1), \mathcal{F}_\Lambda^-(\Im_2)\}\}$
If $\mathcal{F}_\Lambda^-(\Re_1\lozenge_3\Im_1) \geq \mathcal{F}_\Lambda^-(\Re_2\lozenge_3\Im_2)$, then $\mathcal{F}_\Lambda^-(\Re_1\lozenge_3\Im_1) \leq \max\{\mathcal{F}_\Lambda^-(\Re_1), \mathcal{F}_\Lambda^-(\Im_1)\}$.
Also, Similarly to prove that $\max\{1-\mathcal{T}_\Lambda^-(\Re_1\lozenge_1\Im_1), 1-\mathcal{T}_\Lambda^-(\Re_2\lozenge_1\Im_2)\} \leq \max\{\max\{1-\mathcal{T}_\Lambda^-(\Re_1), 1-\mathcal{T}_\Lambda^-(\Re_2)\}, \max\{1-\mathcal{T}_\Lambda^-(\Im_1), 1-\mathcal{T}_\Lambda^-(\Im_2)\}\}$.
If $1-\mathcal{T}_\Lambda^-(\Re_1\lozenge_1\Im_1) \geq 1-\mathcal{T}_\Lambda^-(\Re_2\lozenge_1\Im_2)$, then $1-\mathcal{T}_\Lambda^-(\Re_1) \geq 1-\mathcal{T}_\Lambda^-(\Re_2)$ and $1-\mathcal{T}_\Lambda^-(\Im_1) \geq 1-\mathcal{T}_\Lambda^-(\Im_2)$.
We get $1-\mathcal{T}_\Lambda^-(\Re_1\lozenge_1\Im_1) \leq \max\{1-\mathcal{T}_\Lambda^-(\Re_1), 1-\mathcal{T}_\Lambda^-(\Im_1)\}$.
$\max\{1-\mathcal{T}_\Lambda^-(\Re_1\lozenge_2\Im_1), 1-\mathcal{T}_\Lambda^-(\Re_2\lozenge_2\Im_2)\} \leq \max\{\max\{1-\mathcal{T}_\Lambda^-(\Re_1), 1-\mathcal{T}_\Lambda^-(\Re_2)\}, \max\{1-\mathcal{T}_\Lambda^-(\Im_1), 1-\mathcal{T}_\Lambda^-(\Im_2)\}\}$.
If $1-\mathcal{T}_\Lambda^-(\Re_1\lozenge_2\Im_1) \geq 1-\mathcal{T}_\Lambda^-(\Re_2\lozenge_2\Im_2)$, then $1-\mathcal{T}_\Lambda^-(\Re_1\lozenge_2\Im_1) \leq \max\{1-\mathcal{T}_\Lambda^-(\Re_1), 1-\mathcal{T}_\Lambda^-(\Im_1)\}$.
$\max\{1-\mathcal{T}_\Lambda^-(\Re_1\lozenge_3\Im_1), 1-\mathcal{T}_\Lambda^-(\Re_2\lozenge_3\Im_2)\} \leq \max\{\max\{1-\mathcal{T}_\Lambda^-(\Re_1), 1-\mathcal{T}_\Lambda^-(\Re_2)\}, \max\{1-$

$\mathcal{T}_{\Lambda}^{-}(\Im_1), 1-\mathcal{T}_{\Lambda}^{-}(\Im_2)\}\}$.

If $1-\mathcal{T}_{\Lambda}^{-}(\Re_1\diamondsuit_3\Im_1) \geq 1-\mathcal{T}_{\Lambda}^{-}(\Re_2\diamondsuit_3\Im_2)$, then $1-\mathcal{T}_{\Lambda}^{-}(\Re_1\diamondsuit_3\Im_1) \leq \max\{1-\mathcal{T}_{\Lambda}^{-}(\Re_1), 1-\mathcal{T}_{\Lambda}^{-}(\Im_1)\}$. Hence, $\mathcal{V}_V^{\mathcal{F}}(\Re\diamondsuit_1\Im) \leq \max\{\mathcal{V}_V^{\mathcal{F}}(\Re), \mathcal{V}_V^{\mathcal{F}}(\Im)\}$, $\mathcal{V}_V^{\mathcal{F}}(\Re\diamondsuit_2\Im) \leq \max\{\mathcal{V}_V^{\mathcal{F}}(\Re), \mathcal{V}_V^{\mathcal{F}}(\Im)\}$ and $\mathcal{V}_V^{\mathcal{F}}(\Re\diamondsuit_3\Im) \leq \max\{\mathcal{V}_V^{\mathcal{F}}(\Re), \mathcal{V}_V^{\mathcal{F}}(\Im)\}$. Hence, Λ is a NSVSBS of $\mathcal{B}$.

Theorem 3.8. *Let Λ be a NSV subset in $\mathcal{B}$. Then $\mathcal{V} = ([\mathcal{T}_{\Lambda}^{-}, \mathcal{T}_{\Lambda}^{+}], [\mathcal{I}_{\Lambda}^{-}, \mathcal{I}_{\Lambda}^{+}], [\mathcal{F}_{\Lambda}^{-}, \mathcal{F}_{\Lambda}^{+}])$ is a NSVSBS of $\mathcal{B}$ if and only if all non empty level set $\mathcal{V}^{(t_1,t_2,s)}$ is a SBS of $\mathcal{B}$ for $t_1, t_2, s \in [0,1]$.*

Proof. Assume that $\mathcal{V}$ is a NSVSBS of $\mathcal{B}$. For $t_1, t_2, s \in [0,1]$ and $\xi_1, \xi_2 \in \mathcal{V}^{(t_1,t_2,s)}$. We have $\mathcal{T}_{\Lambda}^{-}(\xi_1) \geq t_1, \mathcal{T}_{\Lambda}^{-}(\xi_2) \geq t_1$ and $1-\mathcal{F}_{\Lambda}^{-}(\xi_1) \geq s, 1-\mathcal{F}_{\Lambda}^{-}(\xi_2) \geq s$ and $\mathcal{I}_{\Lambda}^{-}(\xi_1) \geq t_2, \mathcal{I}_{\Lambda}^{-}(\xi_2) \geq t_2$ and $\mathcal{I}_{\Lambda}^{+}(\xi_1) \geq t_2, \mathcal{I}_{\Lambda}^{+}(\xi_2) \geq t_2$, $1-\mathcal{T}_{\Lambda}^{-}(\xi_1) \leq t_1, 1-\mathcal{T}_{\Lambda}^{-}(\xi_2) \leq t_1$ and $\mathcal{F}_{\Lambda}^{-}(\xi_1) \leq s, \mathcal{F}_{\Lambda}^{-}(\xi_2) \leq s$. Now, $\mathcal{T}_{\Lambda}^{-}(\xi_1\diamondsuit_1\xi_2) \geq \min\{\mathcal{T}_{\Lambda}^{-}(\xi_1), \mathcal{T}_{\Lambda}^{-}(\xi_2)\} \geq t_1$, $1-\mathcal{F}_{\Lambda}^{-}(\xi_1\diamondsuit_1\xi_2) \geq \min\{1-\mathcal{F}_{\Lambda}^{-}(\xi_1), 1-\mathcal{F}_{\Lambda}^{-}(\xi_2)\} \geq s$ and $\mathcal{I}_{\Lambda}^{-}(\xi_1\diamondsuit_1\xi_2) \geq \frac{\mathcal{I}_{\Lambda}^{-}(\xi_1)+\mathcal{I}_{\Lambda}^{-}(\xi_2)}{2} \geq t_2$, $\mathcal{I}_{\Lambda}^{+}(\xi_1\diamondsuit_1\xi_2) \geq \frac{\mathcal{I}_{\Lambda}^{+}(\xi_1)+\mathcal{I}_{\Lambda}^{+}(\xi_2)}{2} \geq t_2$ and $\mathcal{F}_{\Lambda}^{-}(\xi_1\diamondsuit_1\xi_2) \leq \max\{\mathcal{F}_{\Lambda}^{-}(\xi_1), \mathcal{F}_{\Lambda}^{-}(\xi_2)\} \leq s$ and $1-\mathcal{T}_{\Lambda}^{-}(\xi_1\diamondsuit_1\xi_2) \leq \max\{1-\mathcal{T}_{\Lambda}^{-}(\xi_1), 1-\mathcal{T}_{\Lambda}^{-}(\xi_2)\} \leq t_1$. This implies that $\xi_1\diamondsuit_1\xi_2 \in \mathcal{V}^{(t_1,t_2,s)}$. Similarly, $\xi_1\diamondsuit_2\xi_2 \in \mathcal{V}^{(t_1,t_2,s)}$ and $\xi_1\diamondsuit_3\xi_2 \in \mathcal{V}^{(t_1,t_2,s)}$. Therefore $\mathcal{V}^{(t_1,t_2,s)}$ is a SBS of $\mathcal{B}$, where $t_1, t_2, s \in [0,1]$.

Conversely, assume that $\mathcal{V}^{(t_1,t_2,s)}$ is a SBS of $\mathcal{B}$, where $t_1, t_2, s \in [0,1]$. Suppose if there exist $\xi_1, \xi_2 \in \mathcal{B}$ such that $\mathcal{T}_{\Lambda}^{-}(\xi_1\diamondsuit_1\xi_2) < \min\{\mathcal{T}_{\Lambda}^{-}(\xi_1), \mathcal{T}_{\Lambda}^{-}(\xi_2)\}$, $1-\mathcal{F}_{\Lambda}^{-}(\xi_1\diamondsuit_1\xi_2) < \min\{1-\mathcal{F}_{\Lambda}^{-}(\xi_1), 1-\mathcal{F}_{\Lambda}^{-}(\xi_2)\}$, $\mathcal{I}_{\Lambda}^{-}(\xi_1\diamondsuit_1\xi_2) < \frac{\mathcal{I}_{\Lambda}^{-}(\xi_1)+\mathcal{I}_{\Lambda}^{-}(\xi_2)}{2}$, $\mathcal{I}_{\Lambda}^{+}(\xi_1\diamondsuit_1\xi_2) < \frac{\mathcal{I}_{\Lambda}^{+}(\xi_1)+\mathcal{I}_{\Lambda}^{+}(\xi_2)}{2}$ and $\mathcal{F}_{\Lambda}^{-}(\xi_1\diamondsuit_1\xi_2) > \max\{\mathcal{F}_{\Lambda}^{-}(\xi_1), \mathcal{F}_{\Lambda}^{-}(\xi_2)\}$. $1-\mathcal{T}_{\Lambda}^{-}(\xi_1\diamondsuit_1\xi_2) > \max\{1-\mathcal{T}_{\Lambda}^{-}(\xi_1), 1-\mathcal{T}_{\Lambda}^{-}(\xi_2)\}$. Select $t_1, t_2, s \in [0,1]$ such that $\mathcal{T}_{\Lambda}^{-}(\xi_1\diamondsuit_1\xi_2) < t_1 \leq \min\{\mathcal{T}_{\Lambda}^{-}(\xi_1), \mathcal{T}_{\Lambda}^{-}(\xi_2)\}$ and $1-\mathcal{F}_{\Lambda}^{-}(\xi_1\diamondsuit_1\xi_2) < t_1 \leq \min\{1-\mathcal{F}_{\Lambda}^{-}(\xi_1), 1-\mathcal{F}_{\Lambda}^{-}(\xi_2)\}$ and $\mathcal{I}_{\Lambda}^{-}(\xi_1\diamondsuit_1\xi_2) < t_2 \leq \frac{\mathcal{I}_{\Lambda}^{-}(\xi_1)+\mathcal{I}_{\Lambda}^{-}(\xi_2)}{2}$ and $\mathcal{I}_{\Lambda}^{+}(\xi_1\diamondsuit_1\xi_2) < t_2 \leq \frac{\mathcal{I}_{\Lambda}^{+}(\xi_1)+\mathcal{I}_{\Lambda}^{+}(\xi_2)}{2}$ and $\mathcal{F}_{\Lambda}^{-}(\xi_1\diamondsuit_1\xi_2) > s \geq \max\{\mathcal{F}_{\Lambda}^{-}(\xi_1), \mathcal{F}_{\Lambda}^{-}(\xi_2)\}$, $1-\mathcal{T}_{\Lambda}^{-}(\xi_1\diamondsuit_1\xi_2) > s \geq \max\{1-\mathcal{T}_{\Lambda}^{-}(\xi_1), 1-\mathcal{T}_{\Lambda}^{-}(\xi_2)\}$. Then $\xi_1, \xi_2 \in \mathcal{V}^{(t_1,t_2,s)}$, but $\xi_1\diamondsuit_1\xi_2 \notin \mathcal{V}^{(t_1,t_2,s)}$. This contradicts to that $\mathcal{V}^{(t_1,t_2,s)}$ is a SBS of $\mathcal{B}$. Hence, $\mathcal{T}_{\Lambda}^{-}(\xi_1\diamondsuit_1\xi_2) \geq \min\{\mathcal{T}_{\Lambda}^{-}(\xi_1), \mathcal{T}_{\Lambda}^{-}(\xi_2)\}$, $1-\mathcal{F}_{\Lambda}^{-}(\xi_1\diamondsuit_1\xi_2) \geq \min\{1-\mathcal{F}_{\Lambda}^{-}(\xi_1), 1-\mathcal{F}_{\Lambda}^{-}(\xi_2)\}$, $\mathcal{I}_{\Lambda}^{-}(\xi_1\diamondsuit_1\xi_2) \geq \frac{\mathcal{I}_{\Lambda}^{-}(\xi_1)+\mathcal{I}_{\Lambda}^{-}(\xi_2)}{2}$, $\mathcal{I}_{\Lambda}^{+}(\xi_1\diamondsuit_1\xi_2) \geq \frac{\mathcal{I}_{\Lambda}^{+}(\xi_1)+\mathcal{I}_{\Lambda}^{+}(\xi_2)}{2}$ and $\mathcal{F}_{\Lambda}^{-}(\xi_1\diamondsuit_1\xi_2) \leq \max\{\mathcal{F}_{\Lambda}^{-}(\xi_1), \mathcal{F}_{\Lambda}^{-}(\xi_2)\}$ and $1-\mathcal{T}_{\Lambda}^{-}(\xi_1\diamondsuit_1\xi_2) \leq \max\{1-\mathcal{T}_{\Lambda}^{-}(\xi_1), 1-\mathcal{T}_{\Lambda}^{-}(\xi_2)\}$. Similarly, $\diamondsuit_2$ and $\diamondsuit_3$ cases. Hence, $\mathcal{V} = ([\mathcal{T}_{\Lambda}^{-}, \mathcal{T}_{\Lambda}^{+}], [\mathcal{I}_{\Lambda}^{-}, \mathcal{I}_{\Lambda}^{+}], [\mathcal{F}_{\Lambda}^{-}, \mathcal{F}_{\Lambda}^{+}])$ is a NSVSBS of $\mathcal{B}$.

Definition 3.9. Let Λ be any NSVSBS of $\mathcal{B}$ and $\tau \in \mathcal{B}$. Then the pseudo NSV coset $(\tau\Lambda)^p$ is defined by

$$\left\{\begin{array}{l} (\tau\mathcal{V}_{\Lambda}^{\mathcal{T}})^p(\Re) = p(\tau)\mathcal{V}_{\Lambda}^{\mathcal{T}}(\Re), \\ (\tau\mathcal{V}_{\Lambda}^{\mathcal{I}})^p(\Re) = p(\tau)\mathcal{V}_{\Lambda}^{\mathcal{I}}(\Re), \\ (\tau\mathcal{V}_{\Lambda}^{\mathcal{F}})^p(\Re) = p(\tau)\mathcal{V}_{\Lambda}^{\mathcal{F}}(\Re) \end{array}\right\}.$$

That is,

$$\left\{\begin{array}{l} (\tau\mathcal{T}_{\Lambda}^{-})^p(\Re) = p(\tau)\mathcal{T}_{\Lambda}^{-}(\Re),\ 1-(\tau\mathcal{F}_{\Lambda}^{-})^p(\Re) = p(\tau)(1-\mathcal{F}_{\Lambda}^{-})(\Re), \\ (\tau\mathcal{I}_{\Lambda}^{-})^p(\Re) = p(\tau)\mathcal{I}_{\Lambda}^{-}(\Re),\ (\tau\mathcal{I}_{\Lambda}^{+})^p(\Re) = p(\tau)\mathcal{I}_{\Lambda}^{+}(\Re), \\ (\tau\mathcal{F}_{\Lambda}^{-})^p(\Re) = p(\tau)\mathcal{F}_{\Lambda}^{-}(\Re),\ 1-(\tau\mathcal{T}_{\Lambda}^{-})^p(\Re) = p(\tau)(1-\mathcal{T}_{\Lambda}^{-})(\Re) \end{array}\right\}$$

each $\Re \in \mathcal{B}$ and for any non-empty set $p \in P$.

Theorem 3.10. *Let Λ be any NSVSBS of $\mathcal{B}$, then the pseudo NSV coset $(\tau\Lambda)^p$ is a NSVSBS of $\mathcal{B}$.*

Proof. Let Λ be any NSVSBS of $\mathcal{B}$ and for each $\Re, \Im \in \mathcal{B}$. Now, $(\tau\mathcal{T}^-_\Lambda)^p(\Re\Diamond_1\Im) = p(\tau)\ \mathcal{T}^-_\Lambda(\Re\Diamond_1\Im) \geq p(\tau)\ \min\{\mathcal{T}^-_\Lambda(\Re), \mathcal{T}^-_\Lambda(\Im)\} = \min\{p(\tau)\ \mathcal{T}^-_\Lambda(\Re), p(\tau)\ \mathcal{T}^-_\Lambda(\Im)\} = \min\{(\tau\mathcal{T}^-_\Lambda)^p(\Re), (\tau\mathcal{T}^-_\Lambda)^p(\Im)\}$. Thus $(\tau\mathcal{T}^-_\Lambda)^p(\Re\Diamond_1\Im) \geq \min\{(\tau\mathcal{T}^-_\Lambda)^p(\Re), (\tau\mathcal{T}^-_\Lambda)^p(\Im)\}$ and $1 - (\tau\mathcal{F}^-_\Lambda)^p(\Re\Diamond_1\Im) = p(\tau)\ (1 - \mathcal{F}^-_\Lambda(\Re\Diamond_1\Im)) \geq p(\tau)\ \min\{1 - \mathcal{F}^-_\Lambda(\Re), 1 - \mathcal{F}^-_\Lambda(\Im)\} = \min\{p(\tau)\ (1 - \mathcal{F}^-_\Lambda(\Re)), p(\tau)\ (1 - \mathcal{F}^-_\Lambda(\Im))\} = \min\{1 - (\tau\mathcal{F}^-_\Lambda)^p(\Re), 1 - (\tau\mathcal{F}^-_\Lambda)^p(\Im)\}$. Thus $1 - (\tau\mathcal{F}^-_\Lambda)^p(\Re\Diamond_1\Im) \geq \min\{1 - (\tau\mathcal{F}^-_\Lambda)^p(\Re), 1 - (\tau\mathcal{F}^-_\Lambda)^p(\Im)\}$. Now, $(\tau\mathcal{I}^-_\Lambda)^p(\Re\Diamond_1\Im) = p(\tau)\ \mathcal{I}^-_\Lambda(\Re\Diamond_1\Im) \geq p(\tau)\left[\frac{\mathcal{I}^-_\Lambda(\Re)+\mathcal{I}^-_\Lambda(\Im)}{2}\right] = \frac{p(\tau)\ \mathcal{I}^-_\Lambda(\Re)+p(\tau)\ \mathcal{I}^-_\Lambda(\Im)}{2} = \frac{(\tau\mathcal{I}^-_\Lambda)^p(\Re)+(\tau\mathcal{I}^-_\Lambda)^p(\Im)}{2}$. Thus $(\tau\mathcal{I}^-_\Lambda)^p(\Re\Diamond_1\Im) \geq \frac{(\tau\mathcal{I}^-_\Lambda)^p(\Re)+(\tau\mathcal{I}^-_\Lambda)^p(\Im)}{2}$ and $(\tau\mathcal{I}^+_\Lambda)^p(\Re\Diamond_1\Im) = p(\tau)\ \mathcal{I}^+_\Lambda(\Re\Diamond_1\Im) \geq p(\tau)\left[\frac{\mathcal{I}^+_\Lambda(\Re)+\mathcal{I}^+_\Lambda(\Im)}{2}\right] = \frac{p(\tau)\ \mathcal{I}^+_\Lambda(\Re)+p(\tau)\ \mathcal{I}^+_\Lambda(\Im)}{2} = \frac{(\tau\mathcal{I}^+_\Lambda)^p(\Re)+(\tau\mathcal{I}^+_\Lambda)^p(\Im)}{2}$. Thus $(\tau\mathcal{I}^+_\Lambda)^p(\Re\Diamond_1\Im) \geq \frac{(\tau\mathcal{I}^+_\Lambda)^p(\Re)+(\tau\mathcal{I}^+_\Lambda)^p(\Im)}{2}$. Now, $(\tau\mathcal{F}^-_\Lambda)^p(\Re\Diamond_1\Im) = p(\tau)\ \mathcal{F}^-_\Lambda(\Re\Diamond_1\Im) \leq p(\tau)\ \max\{\mathcal{F}^-_\Lambda(\Re), \mathcal{F}^-_\Lambda(\Im)\} = \max\{p(\tau)\ \mathcal{F}^-_\Lambda(\Re), p(\tau)\ \mathcal{F}^-_\Lambda(\Im)\} = \max\{(\tau\mathcal{F}^-_\Lambda)^p(\Re), (\tau\mathcal{F}^-_\Lambda)^p(\Im)\}$. Thus $(\tau\mathcal{F}^-_\Lambda)^p(\Re\Diamond_1\Im) \leq \max\{(\tau\mathcal{F}^-_\Lambda)^p(\Re), (\tau\mathcal{F}^-_\Lambda)^p(\Im)\}$ and $1-(\tau\mathcal{T}^-_\Lambda)^p(\Re\Diamond_1\Im) = p(\tau)\ (1-\mathcal{T}^-_\Lambda(\Re\Diamond_1\Im)) \leq p(\tau)\ \max\{1-\mathcal{T}^-_\Lambda(\Re), 1-\mathcal{T}^-_\Lambda(\Im)\} = \max\{p(\tau)\ (1-\mathcal{T}^-_\Lambda(\Re)), p(\tau)\ (1-\mathcal{T}^-_\Lambda(\Im))\} = \max\{1-(\tau\mathcal{T}^-_\Lambda)^p(\Re), 1-(\tau\mathcal{T}^-_\Lambda)^p(\Im)\}$. Thus $1-(\tau\mathcal{T}^-_\Lambda)^p(\Re\Diamond_1\Im) \leq \max\{1-(\tau\mathcal{T}^-_\Lambda)^p(\Re), 1-(\tau\mathcal{T}^-_\Lambda)^p(\Im)\}$. Similarly, $\Diamond_2$ and $\Diamond_3$ cases. Hence, $(\tau\Lambda)^p$ is a NSVSBS of $\mathcal{B}$.

Definition 3.11. Let $(\mathcal{B}_1, \varnothing_1, \varnothing_2, \varnothing_3)$ and $(\mathcal{B}_2, \eth_1, \eth_2, \eth_3)$ be the bisemirings. Let $\Upsilon : \mathcal{B}_1 \to \mathcal{B}_2$ and Λ be an NSVSBS in $\mathcal{B}_1$, V be an NSVSBS in $\Upsilon(\mathcal{B}_1) = \mathcal{B}_2$, the image of VS is defined as $\mathcal{V}_{\mho(V)}(\ell_2) = [T^-_{\mho(V)}(\ell_2), 1 - F^-_{\mho(V)}(\ell_2)], [I^-_{\mho(V)}(\ell_2), I^+_{\mho(V)}(\ell_2)], [F^-_{\mho(V)}(\ell_2), 1 - T^-_{\mho(V)}(\ell_2)]$ where $T^-_{\mho(V)}(\ell_2) = T^-_V\mho(\ell_2)$, $I^-_{\mho(V)}(\ell_2) = I^-_V\mho(\ell_2)$, $I^+_{\mho(V)}(\ell_2) = I^+_V\mho(\ell_2)$ and $F^-_{\mho(V)}(\ell_2) = F^-_V\mho(\ell_2)$.

Definition 3.12. Let $(\mathcal{B}_1, \varnothing_1, \varnothing_2, \varnothing_3)$ and $(\mathcal{B}_2, \eth_1, \eth_2, \eth_3)$ be the bisemirings. Let $\mho : \mathcal{B}_1 \to \mathcal{B}_2$ be any function. Let V be a VS in $\mho(\mathcal{B}_1) = \mathcal{B}_2$. Then the inverse image of V, $\mho^{-1}$ is the VS in $\mathcal{B}_1$ by $\mathcal{V}_{\mho^{-1}(V)}(\ell_1) = [T^-_{\mho^{-1}(V)}(\ell_1), 1 - F^-_{\mho^{-1}(V)}(\ell_1)], [I^-_{\mho^{-1}(V)}(\ell_1), I^+_{\mho^{-1}(V)}(\ell_1)], [F^-_{\mho^{-1}(V)}(\ell_1), 1 - T^-_{\mho^{-1}(V)}(\ell_1)]$, where $T^-_{\mho^{-1}(V)}(\ell_1) = T^-_V(\mho^{-1}(\ell_1))$, $I^-_{\mho^{-1}(V)}(\ell_1) = I^-_V(\mho^{-1}(\ell_1))$, $I^+_{\mho^{-1}(V)}(\ell_1) = I^+_V(\mho^{-1}(\ell_1))$, $F^-_{\mho^{-1}(V)}(\ell_1) = F^-_V(\mho^{-1}(\ell_1))$.

Theorem 3.13. *Every homomorphic image of NSVSBS of $\mathcal{B}_1$ is a NSVSBS of $\mathcal{B}_2$.*

Proof. Let $\mho : \mathcal{B}_1 \to \mathcal{B}_2$ be a homomorphism. Now, $\mho(\ell_1\varnothing_1\ell_2) = \mho(\ell_1)\eth_1\mho(\ell_2), \mho(\ell_1\varnothing_2\ell_2) = \mho(\ell_1)\eth_2\mho(\ell_2)$ and $\mho(\ell_1\varnothing_3\ell_2) = \mho(\ell_1)\eth_3\mho(\ell_2)$ for all $\ell_1, \ell_2 \in \mathcal{B}_1$. Let $V = \mho(\Lambda)$, Λ is a NSVSBS of $\mathcal{B}_1$. Let $\mho(\ell_1), \mho(\ell_2) \in \mathcal{B}_2$, $T^-_V(\mho(\ell_1)\eth_1\mho(\ell_2)) \geq T^-_\Lambda(\ell_1\varnothing_1\ell_2) \geq \min\{T^-_\Lambda(\ell_1), T^-_\Lambda(\ell_2)\} = \min\{T^-_V\mho(\ell_1), T^-_V\mho(\ell_2)\}$ and $1 - F^-_V(\mho(\ell_1)\eth_1\mho(\ell_2)) \geq 1 - F^-_\Lambda(\ell_1\varnothing_1\ell_2) \geq \min\{1 - F^-_\Lambda(\ell_1), 1 - F^-_\Lambda(\ell_2)\} =$

$\min\{1 - F_V^{-}\mho(\ell_1), 1 - F_V^{-}\mho(\ell_2)\}$. Thus $\mathcal{V}_V^{\mathcal{T}}(\mho(\ell_1)\eth_1\mho(\ell_2)) \geq \min\{\mathcal{V}_V^{\mathcal{T}}\mho(\ell_1), \mathcal{V}_V^{\mathcal{T}}\mho(\ell_2)\}$. Similarly, $\mathcal{V}_V^{\mathcal{T}}(\mho(\ell_1)\eth_2\mho(\ell_2)) \geq \min\{\mathcal{V}_V^{\mathcal{T}}\mho(\ell_1), \mathcal{V}_V^{\mathcal{T}}\mho(\ell_2)\}$ and $\mathcal{V}_V^{\mathcal{T}}(\mho(\ell_1)\eth_3\mho(\ell_2)) \geq \min\{\mathcal{V}_V^{\mathcal{T}}\mho(\ell_1), \mathcal{V}_V^{\mathcal{T}}\mho(\ell_2)\}$. Now, $I_V^{-}(\mho(\ell_1)\eth_1\mho(\ell_2)) \geq I_\Lambda^{-}(\ell_1\varnothing_1\ell_2) \geq \frac{I_\Lambda^{-}(\ell_1)+I_\Lambda^{-}(\ell_2)}{2} = \frac{I_V^{-}\mho(\ell_1)+I_V^{-}\mho(\ell_2)}{2}$ and $I_V^{+}(\mho(\ell_1)\eth_1\mho(\ell_2)) \geq I_\Lambda^{+}(\ell_1\varnothing_1\ell_2) \geq \frac{I_\Lambda^{+}(\ell_1)+I_\Lambda^{+}(\ell_2)}{2} = \frac{I_V^{+}\mho(\ell_1)+I_V^{+}\mho(\ell_2)}{2}$. Thus $\mathcal{V}_V^{\mathcal{I}}(\mho(\ell_1)\eth_1\mho(\ell_2)) \geq \frac{\mathcal{V}_V^{\mathcal{I}}\mho(\ell_1)+\mathcal{V}_V^{\mathcal{I}}\mho(\ell_2)}{2}$. Similarly, $\mathcal{V}_V^{\mathcal{I}}(\mho(\ell_1)\eth_2\mho(\ell_2)) \geq \min\{\mathcal{V}_V^{\mathcal{I}}\mho(\ell_1), \mathcal{V}_V^{\mathcal{I}}\mho(\ell_2)\}$ and $\mathcal{V}_V^{\mathcal{I}}(\mho(\ell_1)\eth_3\mho(\ell_2)) \geq \min\{\mathcal{V}_V^{\mathcal{I}}\mho(\ell_1), \mathcal{V}_V^{\mathcal{I}}\mho(\ell_2)\}$. Now, $F_V^{-}(\mho(\ell_1)\eth_1\mho(\ell_2)) \leq F_\Lambda^{-}(\ell_1\varnothing_1\ell_2) \leq \max\{F_\Lambda^{-}(\ell_1), F_\Lambda^{-}(\ell_2)\} = \max\{F_V^{-}\mho(\ell_1), F_V^{-}\mho(\ell_2)\}$ and $1 - T_V^{-}(\mho(\ell_1)\eth_1\mho(\ell_2)) \leq 1 - T_\Lambda^{-}(\ell_1\varnothing_1\ell_2) \leq \max\{1 - T_\Lambda^{-}(\ell_1), 1 - T_\Lambda^{-}(\ell_2)\} = \max\{1 - T_V^{-}\mho(\ell_1), 1 - T_V^{-}\mho(\ell_2)\}$. Thus $\mathcal{V}_V^{\mathcal{F}}(\mho(\ell_1)\eth_1\mho(\ell_2)) \leq \max\{\mathcal{V}_V^{\mathcal{F}}\mho(\ell_1), \mathcal{V}_V^{\mathcal{F}}\mho(\ell_2)\}$. Similarly, $\mathcal{V}_V^{\mathcal{F}}(\mho(\ell_1)\eth_2\mho(\ell_2)) \leq \max\{\mathcal{V}_V^{\mathcal{F}}\mho(\ell_1), \mathcal{V}_V^{\mathcal{F}}\mho(\ell_2)\}$ and $\mathcal{V}_V^{\mathcal{F}}(\mho(\ell_1)\eth_3\mho(\ell_2)) \leq \max\{\mathcal{V}_V^{\mathcal{F}}\mho(\ell_1), \mathcal{V}_V^{\mathcal{F}}\mho(\ell_2)\}$. Hence, V is a NSVSBS of $\mathcal{B}_2$.

Theorem 3.14. *Every homomorphic pre-image of NSVSBS of $\mathcal{B}_2$ is a NSVSBS of $\mathcal{B}_1$.*

Proof. Let $\mho : \mathcal{B}_1 \to \mathcal{B}_2$ and $\mho(\Re\varnothing_1\Im) = \mho(\Re)\eth_1\mho(\Im), \mho(\Re\varnothing_2\Im) = \mho(\Re)\eth_2\mho(\Im)$ and $\mho(\Re\varnothing_3\Im) = \mho(\Re)\eth_3\mho(\Im)$ for all $\Re, \Im \in \mathcal{B}_1$. Let $V = \mho(\Lambda)$, where V is any NSVSBS of $\mathcal{B}_2$. Let $\Re, \Im \in \mathcal{B}_1$. Now, $T_\Lambda^{-}(\Re\varnothing_1\Im) = T_V^{-}(\mho(\Re\varnothing_1\Im)) = T_V^{-}(\mho(\Re)\eth_1\mho(\Im)) \geq \min\{T_V^{-}\mho(\Re), T_V^{-}\mho(\Im)\} = \min\{T_\Lambda^{-}(\Re), T_\Lambda^{-}(\Im)\}$. Thus $T_\Lambda^{-}(\Re\varnothing_1\Im) \geq \min\{T_\Lambda^{-}(\Re), T_\Lambda^{-}(\Im)\}$ and $1 - F_\Lambda^{-}(\Re\varnothing_1\Im) = 1 - F_V^{-}(\mho(\Re\varnothing_1\Im)) = 1 - F_V^{-}(\mho(\Re)\eth_1\mho(\Im)) \geq \min\{1 - F_V^{-}\mho(\Re), 1 - F_V^{-}\mho(\Im)\} = \min\{1 - F_\Lambda^{-}(\Re), 1 - F_\Lambda^{-}(\Im)\}$. Thus $1 - F_\Lambda^{-}(\Re\varnothing_1\Im) \geq \min\{1 - F_\Lambda^{-}(\Re), 1 - F_\Lambda^{-}(\Im)\}$. Hence, $\mathcal{V}_V^{\mathcal{T}}(\Re\varnothing_1\Im) \geq \min\{\mathcal{V}_V^{\mathcal{T}}(\Re), \mathcal{V}_V^{\mathcal{T}}(\Im)\}$. Similarly, $\mathcal{V}_V^{\mathcal{T}}(\Re\varnothing_2\Im) \geq \min\{\mathcal{V}_V^{\mathcal{T}}(\Re), \mathcal{V}_V^{\mathcal{T}}(\Im)\}$ and $\mathcal{V}_V^{\mathcal{T}}(\Re\varnothing_3\Im) \geq \min\{\mathcal{V}_V^{\mathcal{T}}(\Re), \mathcal{V}_V^{\mathcal{T}}(\Im)\}$. Now, $I_\Lambda^{-}(\Re\varnothing_1\Im) = I_V^{-}(\mho(\Re\varnothing_1\Im)) = I_V^{-}(\mho(\Re)\eth_1\mho(\Im)) \geq \frac{I_V^{-}\mho(\Re)+I_V^{-}\mho(\Im)}{2} = \frac{I_\Lambda^{-}(\Re)+I_\Lambda^{-}(\Im)}{2}$. Thus $I_\Lambda^{-}(\Re\varnothing_1\Im) \geq \frac{I_\Lambda^{-}(\Re)+I_\Lambda^{-}(\Im)}{2}$ and $I_\Lambda^{+}(\Re\varnothing_1\Im) = I_V^{+}(\mho(\Re\varnothing_1\Im)) = I_V^{+}(\mho(\Re)\eth_1\mho(\Im)) \geq \frac{I_V^{+}\mho(\Re)+I_V^{+}\mho(\Im)}{2} = \frac{I_\Lambda^{+}(\Re)+I_\Lambda^{+}(\Im)}{2}$. Thus $I_\Lambda^{+}(\Re\varnothing_1\Im) \geq \frac{I_\Lambda^{+}(\Re)+I_\Lambda^{+}(\Im)}{2}$. Hence, $\mathcal{V}_V^{\mathcal{I}}(\Re\varnothing_1\Im) \geq \frac{\mathcal{V}_V^{\mathcal{I}}(\Re)+\mathcal{V}_V^{\mathcal{I}}(\Im)}{2}$. Similarly, $\mathcal{V}_V^{\mathcal{I}}(\Re\varnothing_2\Im) \geq \frac{\mathcal{V}_V^{\mathcal{I}}(\Re)+\mathcal{V}_V^{\mathcal{I}}(\Im)}{2}$ and $\mathcal{V}_V^{\mathcal{I}}(\Re\varnothing_3\Im) \geq \frac{\mathcal{V}_V^{\mathcal{I}}(\Re)+\mathcal{V}_V^{\mathcal{I}}(\Im)}{2}$. Now, $F_\Lambda^{-}(\Re\varnothing_1\Im) = F_V^{-}(\mho(\Re\varnothing_1\Im)) = F_V^{-}(\mho(\Re)\eth_1\mho(\Im)) \leq \max\{F_V^{-}\mho(\Re), F_V^{-}\mho(\Im)\} = \max\{F_\Lambda^{-}(\Re), F_\Lambda^{-}(\Im)\}$. Thus $F_\Lambda^{-}(\Re\varnothing_1\Im) \leq \max\{F_\Lambda^{-}(\Re), F_\Lambda^{-}(\Im)\}$ and $1 - T_\Lambda^{-}(\Re\varnothing_1\Im) = 1 - T_V^{-}(\mho(\Re\varnothing_1\Im)) = 1 - T_V^{-}(\mho(\Re)\eth_1\mho(\Im)) \leq \max\{1 - T_V^{-}\mho(\Re), 1 - T_V^{-}\mho(\Im)\} = \max\{1 - T_\Lambda^{-}(\Re), 1 - T_\Lambda^{-}(\Im)\}$. Thus $1 - T_\Lambda^{-}(\Re\varnothing_1\Im) \leq \max\{1 - T_\Lambda^{-}(\Re), 1 - T_\Lambda^{-}(\Im)\}$. Hence, $\mathcal{V}_V^{\mathcal{F}}(\Re\varnothing_1\Im) \leq \max\{\mathcal{V}_V^{\mathcal{F}}(\Re), \mathcal{V}_V^{\mathcal{F}}(\Im)\}$. Similarly, $\mathcal{V}_V^{\mathcal{F}}(\Re\varnothing_2\Im) \leq \max\{\mathcal{V}_V^{\mathcal{F}}(\Re), \mathcal{V}_V^{\mathcal{F}}(\Im)\}$ and $\mathcal{V}_V^{\mathcal{F}}(\Re\varnothing_3\Im) \leq \max\{\mathcal{V}_V^{\mathcal{F}}(\Re), \mathcal{V}_V^{\mathcal{F}}(\Im)\}$. Hence, Λ is a NSVSBS of $\mathcal{B}_1$.

Theorem 3.15. *If $\mho : \mathcal{B}_1 \to \mathcal{B}_2$ is a homomorphism, then $\mho(\Lambda_{(t_1,t_2,s)})$ is a level SBS of NSVSBS V of $\mathcal{B}_2$.*

Proof. Let $\mho : \mathcal{B}_1 \to \mathcal{B}_2$ be a homomorphism and $\mho(\Re\varnothing_1\Im) = \mho(\Re)\eth_1\mho(\Im), \mho(\Re\varnothing_2\Im) = \mho(\Re)\eth_2\mho(\Im)$ and $\mho(\Re\varnothing_3\Im) = \mho(\Re)\eth_3\mho(\Im)$ for all $\Re, \Im \in \mathcal{B}_1$. Let $V = \mho(\Lambda)$, Λ is a NSVSBS of $\mathcal{B}_1$. By Theorem 3.13, V is a NSVSBS of $\mathcal{B}_2$. Let $\Lambda_{(t_1,t_2,s)}$ be any level SBS of Λ. Suppose that $\Re, \Im \in \Lambda_{(t_1,t_2,s)}$. Then $\mho(\Re\varnothing_1\Im), \mho(\Re\varnothing_2\Im)$ and $\mho(\Re\varnothing_3\Im) \in \Lambda_{(t_1,t_2,s)}$. Now, $T_V^{-}(\mho(\Re)) =$

$T_\Lambda^-(\Re) \geq t_1, T_V^-(\mho(\Im)) = T_\Lambda^-(\Im) \geq t_1$. Thus $T_V^-(\mho(\Re)\partial_1\mho(\Im)) \geq T_\Lambda^-(\Re\oslash_1\Im) \geq t_1$ and $1-F_V^-(\mho(\Re)) = 1-F_\Lambda^-(\Re) \geq s, 1-F_V^-(\mho(\Im)) = 1-F_\Lambda^-(\Im) \geq s$. Thus $1-F_V^-(\mho(\Re)\partial\mho(\Im)) \geq 1-F_\Lambda^-(\Re\oslash_1\Im) \geq s$. Now, $I_V^-(\mho(\Re)) = I_\Lambda^-(\Re) \geq t_2, I_V^-(\mho(\Im)) = I_\Lambda^-(\Im) \geq t_2$. Thus $I_V^-(\mho(\Re)\partial_1\mho(\Im)) \geq I_\Lambda^-(\Re\oslash_1\Im) \geq t_2$ and $I_V^+(\mho(\Re)) = I_\Lambda^+(\Re) \geq t_2, I_V^+(\mho(\Im)) = I_\Lambda^+(\Im) \geq t_2$. Thus $I_V^+(\mho(\Re)\partial_1\mho(\Im)) \geq I_\Lambda^+(\Re\oslash_1\Im) \geq t_2$. Now, $F_V^-(\mho(\Re)) = F_\Lambda^-(\Re) \leq s, F_V^-(\mho(\Im)) = F_\Lambda^-(\Im) \leq s$. Thus $F_V^-(\mho(\Re)\partial_1\mho(\Im)) \leq F_\Lambda^-(\Re\oslash_1\Im) \leq s$ and $1-T_V^-(\mho(\Re)) = 1-T_\Lambda^-(\Re) \leq t_1, 1-T_V^-(\mho(\Im)) = 1-T_\Lambda^-(\Im) \leq t_1$. Thus $1-T_V^-(\mho(\Re)\partial\mho(\Im)) \leq 1-T_\Lambda^-(\Re\oslash_1\Im) \leq t_1$, for all $\mho(\Re), \mho(\Im) \in \mathcal{B}_2$. Similarly to prove other operations. Hence proved.

Theorem 3.16. *If $\mho : \mathcal{B}_1 \to \mathcal{B}_2$ is any homomorphism, then $\Lambda_{(t_1,t_2,s)}$ is a level SBS of NSVSBS Λ of $\mathcal{B}_1$.*

Proof. Let $\mho : \mathcal{B}_1 \to \mathcal{B}_2$ be a homomorphism and $\mho(\Re\oslash_1\Im) = \mho(\Re)\partial_1\mho(\Im), \mho(\Re\oslash_2\Im) = \mho(\Re)\partial_2\mho(\Im)$ and $\mho(\Re\oslash_3\Im) = \mho(\Re)\partial_3\mho(\Im)$ for all $\Re, \Im \in \mathcal{B}_1$. Let $V = \mho(\Lambda)$, V is a NSVSBS of $\mathcal{B}_2$. By Theorem 3.14, Λ is a NSVSBS of $\mathcal{B}_1$. Let $\mho(\Lambda_{(t_1,t_2,s)})$ be a level SBS of V. Suppose that $\mho(\Re), \mho(\Im) \in \mho(\Lambda_{(t_1,t_2,s)})$. Then $\mho(\Re\oslash_1\Im), \mho(\Re\oslash_2\Im)$ and $\mho(\Re\oslash_3\Im) \in \mho(\Lambda_{(t_1,t_2,s)})$. Now, $T_\Lambda^-(\Re) = T_V^-(\mho(\Re)) \geq t_1, T_\Lambda^-(\Im) = T_V^-(\mho(\Im)) \geq t_1$. Thus $T_\Lambda^-(\Re\oslash_1\Im) \geq \min\{T_\Lambda^-(\Re), T_\Lambda^-(\Im)\} \geq t_1$ and $1-F_\Lambda^-(\Re) = 1-F_V^-(\mho(\Re)) \geq s, 1-F_\Lambda^-(\Im) = 1-F_V^-(\mho(\Im)) \geq s$. Thus $1-F_\Lambda^-(\Re\oslash_1\Im) \geq \min\{1-F_\Lambda^-(\Re), 1-F_\Lambda^-(\Im)\} \geq s$. Now, $I_\Lambda^-(\Re) = I_V^-(\mho(\Re)) \geq t_2, I_\Lambda^-(\Im) = I_V^-(\mho(\Im)) \geq t_2$. Thus $I_\Lambda^-(\Re\oslash_1\Im) \geq \frac{I_\Lambda^-(\Re)+I_\Lambda^-(\Im)}{2} \geq t_2$ and $I_\Lambda^+(\Re) = I_V^+(\mho(\Re)) \geq t_2, I_\Lambda^+(\Im) = I_V^+(\mho(\Im)) \geq t_2$. Thus $I_\Lambda^+(\Re\oslash_1\Im) \geq \frac{I_\Lambda^+(\Re)+I_\Lambda^+(\Im)}{2} \geq t_2$. Now, $F_\Lambda^-(\Re) = F_V^-(\mho(\Re)) \leq s, F_\Lambda^-(\Im) = F_V^-(\mho(\Im)) \leq s$. Thus $F_\Lambda^-(\Re\oslash_1\Im) = F_V^-(\mho(\Re)\partial_1\mho(\Im)) \leq \max\{F_\Lambda^-(\Re), F_\Lambda^-(\Im)\} \leq s$ and $1-T_\Lambda^-(\Re) = 1-T_V^-(\mho(\Re)) \leq t_1, 1-T_\Lambda^-(\Im) = 1-T_V^-(\mho(\Im)) \leq t_1$. Thus $1-T_\Lambda^-(\Re\oslash_1\Im) = 1-T_V^-(\mho(\Re)\partial_1\mho(\Im)) \leq \max\{1-T_\Lambda^-(\Re), 1-T_\Lambda^-(\Im)\} \leq t_1$, for all $\Re, \Im \in \mathcal{B}_1$. Similarly to prove other two operations. Hence proved.

4. (ρ,σ)-Neutrosophic vague SBSs

We discuss about (ρ,σ)-NSVSBS and $(\rho,\sigma) \in [0,1]$ be such that $0 \leq \rho < \sigma \leq 1$.

Definition 4.1. Let Λ be any NSVS of $\mathcal{B}$ is called a (ρ,σ)-NSVSBS of $\mathcal{B}$ if

$$\left\{\begin{array}{l} \max\{\mathcal{V}_\Lambda^{\mathcal{T}}(\Re\Diamond_1\Im), \rho\} \geq \min\{\mathcal{V}_\Lambda^{\mathcal{T}}(\Re), \mathcal{V}_\Lambda^{\mathcal{T}}(\Im), \sigma\} \\ \max\{\mathcal{V}_\Lambda^{\mathcal{T}}(\Re\Diamond_2\Im), \rho\} \geq \min\{\mathcal{V}_\Lambda^{\mathcal{T}}(\Re), \mathcal{V}_\Lambda^{\mathcal{T}}(\Im), \sigma\} \\ \max\{\mathcal{V}_\Lambda^{\mathcal{T}}(\Re\Diamond_3\Im), \rho\} \geq \min\{\mathcal{V}_\Lambda^{\mathcal{T}}(\Re), \mathcal{V}_\Lambda^{\mathcal{T}}(\Im), \sigma\} \end{array}\right\} \quad \left\{\begin{array}{l} \max\{\mathcal{V}_\Lambda^{\mathcal{I}}(\Re\Diamond_1\Im), \rho\} \geq \min\left\{\frac{\mathcal{V}_\Lambda^{\mathcal{I}}(\Re)+\mathcal{V}_\Lambda^{\mathcal{I}}(\Im)}{2}, \sigma\right\} \\ \qquad OR \\ \max\{\mathcal{V}_\Lambda^{\mathcal{I}}(\Re\Diamond_2\Im), \rho\} \geq \min\left\{\frac{\mathcal{V}_\Lambda^{\mathcal{I}}(\Re)+\mathcal{V}_\Lambda^{\mathcal{I}}(\Im)}{2}, \sigma\right\} \\ \qquad OR \\ \max\{\mathcal{V}_\Lambda^{\mathcal{I}}(\Re\Diamond_3\Im), \rho\} \geq \min\left\{\frac{\mathcal{V}_\Lambda^{\mathcal{I}}(\Re)+\mathcal{V}_\Lambda^{\mathcal{I}}(\Im)}{2}, \sigma\right\} \end{array}\right\}$$

$$\begin{Bmatrix} \min\{\mathcal{V}_\Lambda^{\mathcal{F}}(\Re\Diamond_1\Im),\rho\} \le \max\{\mathcal{V}_\Lambda^{\mathcal{F}}(\Re),\mathcal{V}_\Lambda^{\mathcal{F}}(\Im),\sigma\} \\ \min\{\mathcal{V}_\Lambda^{\mathcal{F}}(\Re\Diamond_2\Im),\rho\} \le \max\{\mathcal{V}_\Lambda^{\mathcal{F}}(\Re),\mathcal{V}_\Lambda^{\mathcal{F}}(\Im),\sigma\} \\ \min\{\mathcal{V}_\Lambda^{\mathcal{F}}(\Re\Diamond_3\Im),\rho\} \le \max\{\mathcal{V}_\Lambda^{\mathcal{F}}(\Re),\mathcal{V}_\Lambda^{\mathcal{F}}(\Im),\sigma\} \end{Bmatrix}.$$

That is,

$$\begin{Bmatrix} \begin{pmatrix} \max\{\mathcal{T}_\Lambda^-(\Re\Diamond_1\Im),\rho\} \ge \min\{\mathcal{T}_\Lambda^-(\Re),\mathcal{T}_\Lambda^-(\Im),\sigma\}, \\ \max\{1-\mathcal{F}_\Lambda^-(\Re\Diamond_1\Im),\rho\} \ge \min\{1-\mathcal{F}_\Lambda^-(\Re),1-\mathcal{F}_\Lambda^-(\Im),\sigma\} \end{pmatrix} \\ \begin{pmatrix} \max\{\mathcal{T}_\Lambda^-(\Re\Diamond_2\Im),\rho\} \ge \min\{\mathcal{T}_\Lambda^-(\Re),\mathcal{T}_\Lambda^-(\Im),\sigma\}, \\ \max\{1-\mathcal{F}_\Lambda^-(\Re\Diamond_2\Im),\rho\} \ge \min\{1-\mathcal{F}_\Lambda^-(\Re),1-\mathcal{F}_\Lambda^-(\Im),\sigma\} \end{pmatrix} \\ \begin{pmatrix} \max\{\mathcal{T}_\Lambda^-(\Re\Diamond_3\Im),\rho\} \ge \min\{\mathcal{T}_\Lambda^-(\Re),\mathcal{T}_\Lambda^-(\Im),\sigma\}, \\ \max\{1-\mathcal{F}_\Lambda^-(\Re\Diamond_3\Im),\rho\} \ge \min\{1-\mathcal{F}_\Lambda^-(\Re),1-\mathcal{F}_\Lambda^-(\Im),\sigma\} \end{pmatrix} \end{Bmatrix}$$

$$\begin{Bmatrix} \begin{pmatrix} \max\{\mathcal{I}_\Lambda^+(\Re\Diamond_1\Im),\rho\} \ge \min\left\{\frac{\mathcal{I}_\Lambda^+(\Re)+\mathcal{I}_\Lambda^+(\Im)}{2},\sigma\right\} \\ \max\{\mathcal{I}_\Lambda^-(\Re\Diamond_1\Im),\rho\} \ge \min\left\{\frac{\mathcal{I}_\Lambda^-(\Re)-\mathcal{I}_\Lambda^-(\Im)}{2},\sigma\right\} \end{pmatrix} \\ OR \\ \begin{pmatrix} \max\{\mathcal{I}_\Lambda^+(\Re\Diamond_2\Im),\rho\} \ge \min\left\{\frac{\mathcal{I}_\Lambda^+(\Re)+\mathcal{I}_\Lambda^+(\Im)}{2},\sigma\right\} \\ \max\{\mathcal{I}_\Lambda^-(\Re\Diamond_2\Im),\rho\} \ge \min\left\{\frac{\mathcal{I}_\Lambda^-(\Re)-\mathcal{I}_\Lambda^-(\Im)}{2},\sigma\right\} \end{pmatrix} \\ OR \\ \begin{pmatrix} \max\{\mathcal{I}_\Lambda^+(\Re\Diamond_3\Im),\rho\} \ge \min\left\{\frac{\mathcal{I}_\Lambda^+(\Re)+\mathcal{I}_\Lambda^+(\Im)}{2},\sigma\right\} \\ \max\{\mathcal{I}_\Lambda^-(\Re\Diamond_3\Im),\rho\} \ge \min\left\{\frac{\mathcal{I}_\Lambda^-(\Re)-\mathcal{I}_\Lambda^-(\Im)}{2},\sigma\right\} \end{pmatrix} \end{Bmatrix}$$

$$\begin{Bmatrix} \begin{pmatrix} \min\{\mathcal{F}_\Lambda^-(\Re\Diamond_1\Im),\rho\} \le \max\{\mathcal{F}_\Lambda^-(\Re),\mathcal{F}_\Lambda^-(\Im),\sigma\}, \\ \min\{1-\mathcal{T}_\Lambda^-(\Re\Diamond_1\Im),\rho\} \le \max\{1-\mathcal{T}_\Lambda^-(\Re),1-\mathcal{T}_\Lambda^-(\Im),\sigma\} \end{pmatrix} \\ \begin{pmatrix} \min\{\mathcal{F}_\Lambda^-(\Re\Diamond_2\Im),\rho\} \le \max\{\mathcal{F}_\Lambda^-(\Re),\mathcal{F}_\Lambda^-(\Im),\sigma\}, \\ \min\{1-\mathcal{T}_\Lambda^-(\Re\Diamond_2\Im),\rho\} \le \max\{1-\mathcal{T}_\Lambda^-(\Re),1-\mathcal{T}_\Lambda^-(\Im),\sigma\} \end{pmatrix} \\ \begin{pmatrix} \min\{\mathcal{F}_\Lambda^-(\Re\Diamond_3\Im),\rho\} \le \max\{\mathcal{F}_\Lambda^-(\Re),\mathcal{F}_\Lambda^-(\Im),\sigma\}, \\ \min\{1-\mathcal{T}_\Lambda^-(\Re\Diamond_3\Im),\rho\} \le \max\{1-\mathcal{T}_\Lambda^-(\Re),1-\mathcal{T}_\Lambda^-(\Im),\sigma\} \end{pmatrix} \end{Bmatrix}$$

for all $\Re,\Im \in \mathcal{B}$.

Example 4.2. By Example 3.2,

	$[\mathcal{T}_\Lambda^-(\varphi),\mathcal{T}_\Lambda^+(\varphi)]$	$[\mathcal{I}_\Lambda^-(\varphi),\mathcal{I}_\Lambda^+(\varphi)]$	$[\mathcal{F}_\Lambda^-(\varphi),\mathcal{F}_\Lambda^+(\varphi)]$
$\varphi=\dot{a}$	$[0.65,0.70]$	$[0.55,0.65]$	$[0.3,0.35]$
$\varphi=\ddot{a}$	$[0.6,0.65]$	$[0.50,0.60]$	$[0.35,0.40]$
$\varphi=\tilde{a}$	$[0.35,0.40]$	$[0.25,0.30]$	$[0.60,0.65]$
$\varphi=\vec{a}$	$[0.45,0.55]$	$[0.40,0.50]$	$[0.45,0.55]$

Clearly, Λ is a $(0.25,0.85)$ NSVSBS of $\mathcal{B}$.

Theorem 4.3. *The intersection of a family of every (ρ,σ)- $NSVSBS^s$ is a (ρ,σ)-NSVSBS.*

Proof. The proof is similar to Theorem 3.3.

Theorem 4.4. *If Λ and Ψ are any two (ρ, σ)- $NSVSBS^s$ of $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, then $\Lambda \times \Psi$ is a (ρ, σ)-NSVSBS of $\mathcal{B}_1 \times \mathcal{B}_2$.*

Proof. The proof is similar to Theorem 3.4.

Corollary 4.5. *If $\Lambda_1, \Lambda_2, ..., \Lambda_n$ are the families of (ρ, σ)-$NSVSBS^s$ of $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n$ respectively, then $\Lambda_1 \times \Lambda_2 \times ... \times \Lambda_n$ is a (ρ, σ)-NSVSBS of $\mathcal{B}_1 \times \mathcal{B}_2 \times ... \times \mathcal{B}_n$.*

Definition 4.6. Let Λ be a (ρ, σ)- NSVS in $\mathcal{B}$, the (ρ, σ)-SNSVR on $\mathcal{B}$. ie) (ρ, σ)- NSVR on Λ is V given by

$$\left\{ \begin{array}{l} \max\{\mathcal{V}_{\Lambda}^{\mathcal{T}}(\Re, \Im), \rho\} = \min\{\mathcal{V}_{\Lambda}^{\mathcal{T}}(\Re), \mathcal{V}_{\Lambda}^{\mathcal{T}}(\Im), \sigma\} \\ \max\{\mathcal{V}_{\Lambda}^{\mathcal{I}}(\Re, \Im), \rho\} = \min\left\{\frac{\mathcal{V}_{\Lambda}^{\mathcal{I}}(\Re) + \mathcal{V}_{\Lambda}^{\mathcal{I}}(\Im)}{2}, \sigma\right\} \\ \min\{\mathcal{V}_{\Lambda}^{\mathcal{F}}(\Re, \Im), \rho\} = \max\{\mathcal{V}_{\Lambda}^{\mathcal{F}}(\Re), \mathcal{V}_{\Lambda}^{\mathcal{F}}(\Im), \sigma\} \end{array} \right\}.$$

Theorem 4.7. *Let Λ be a (ρ, σ)-NSVSBS of $\mathcal{B}$ and V be the (ρ, σ)-SNSVR of $\mathcal{B}$. Then Λ is a (ρ, σ)-NSVSBS of $\mathcal{B}$ if and only if V is a (ρ, σ)-NSVSBS of $\mathcal{B} \times \mathcal{B}$.*

Proof. A similar proof is given in Theorem 3.7.

Theorem 4.8. *A homomorphic image of (ρ, σ)-NSVSBS of $\mathcal{B}_1$ is a (ρ, σ)-NSVSBS of $\mathcal{B}_2$.*

Proof. A similar proof is given in Theorem 3.13.

Theorem 4.9. *A homomorphic pre-image of (ρ, σ)-NSVSBS of $\mathcal{B}_2$ is a (ρ, σ)-NSVSBS of $\mathcal{B}_1$.*

Proof. A similar proof is given in Theorem 3.14.

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Comprehensive Review MEREC weighting method for Smart Building Selection for New Capital using neutrosophic theory

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Abstract

Population growth has become a serious problem in many countries, especially Egypt. Which leads to an increase in the population area and an increase in buildings, which then leads to several problems, including large energy consumption, increased pollution, traffic congestion, and others. Therefore, many governments have resorted to using technology and applying it to build smart buildings to help save energy by using renewable energy to improve its impact on the environment, improve the quality of life of citizens, provide security and safety, and so on. The selection of smart buildings depends on many criteria. Since this problem is described as a multi-criteria decision-making (MCDM) problem, MCDM methods will be used in this paper. A hybrid method is presented to evaluate smart buildings. The first method, MEREC, was used to calculate the weights of criteria, and the VIKOR model was used for ranking alternatives. Then applying those weights to the CoCoSo, COPRAS, and TOPSIS methods for making comparisons using Spearman`s correlation coefficients for ranking these four methods. All methods used are applied in the T2NN environment.

Keywords: Multi-Criteria Decision Making; Smart Building; Neutrosophic Theory; MEREC, VIKOR, TOPSIS, COPRAS, CoCoSo.

1. Introduction

The smart city has gained a lot of attention in recent years because it promises benefits such as high quality of life, economic prosperity, and environmental sustainability through advanced technologies [1]. Smart cities are a dynamic, integrated ecosystem that uses advanced technology such as integration of information and communication technologies (ICT), internet of things (IoT) devices, software solutions, user interfaces (UI), AI, data analysis and communication networks.

Smart cities use data analysis to collect and analyze data from various sources, and this data is processed to enable decision makers to take the necessary measures to create a sustainable environment, facilitate citizens' lives, improve energy efficiency, and improve quality of life in general, such as transportation, energy, public safety, water resources, etc. Smart cities use this technology to be applied in different parts of the city, such as the smart traffic system, to improve traffic, avoid congestion, save time, and maintain citizen safety. Smart lighting system to save energy and reduce costs. Waste management and recycling systems, and water management systems to preserve materials. These applications result in improving energy consumption, generating clean energy, and enhancing the efficiency of its use.

Smart cities are characterized by several features like, connectivity, data collection and analysis, infrastructure, sustainability, public services, citizen engagement, security, innovation, ecosystem, efficient transportation, see Figure 1.

With the rapid development of artificial intelligence, the concept of smart building has been proposed to improve the performance and efficiency in the life cycle of a building [2]. the whole world is starting to realize the important of data and technology to improve citizens' quality of life, enhance sustainability, and optimize urban infrastructure. The whole world is seeking to build smart cities to help leaders make decisions that contribute to improving the quality of life for citizens, enhance efficiency, safety, and overall performance. Smart Buildings play a crucial role in transforming urban landscapes by incorporating advanced technologies and intelligent systems contributes to sustainability goals by reducing energy consumption and environmental impact, optimize resource utilization, and enhance overall building performance.

In smart buildings, technology and some advanced algorithms are used to monitor air quality, temperature, energy levels, humidity, the extent of pollution produced, and water consumption to enable officials to take the necessary measures to maintain a sustainable environment. Systems have also been built that can evaluate risk and emergency situations and respond quickly to them, such as natural disasters, accidents, and terrorist attacks, and provide means of safety and preservation of citizens, such as providing immediate evacuation methods or providing first aid.

Here are key features and components of smart buildings:

- Energy Efficiency: Smart buildings using advanced sensors, automation systems, and real-time data analytics to detect energy consumption and adjust lighting and HVAC systems to improve efficiency, reduce costs, reduce dependence on non-renewable energies, and use renewable energies.

- Building Automation Systems (BAS): BAS It is an integrated system that includes the building's various systems, such as heating, ventilation, air conditioning (HVAC), lighting, security, and so on, to facilitate the process of managing and controlling the building and improving its maintenance.
- Smart Infrastructure: Smart Buildings are part of an interconnected ecosystem within Smart Cities. Using networked devices, cloud-based platforms, and interoperable technologies we can share data with other systems, such as transportation, water management, and public safety, enabling better coordination and resource allocation for improved management of building ensuring optimal building performance and reducing maintenance costs.
- Enhanced Connectivity: Smart buildings are equipped with sensors that monitor ambient conditions, occupancy, and other parameters, allowing monitoring and control of building performance, occupancy patterns, and energy use.
- Advanced Security Systems: Smart buildings are equipped with security systems such as surveillance cameras and intrusion detection devices to detect and prevent threats and risks such as fires and burglaries.
- Resilience and Adaptability: Technology is used in smart buildings to monitor environmental conditions to make buildings have the ability to adapt to conditions as well as respond to them dynamically, adjust energy usage during peak demand, and integrate with renewable energy sources for increased resilience.
- Economic Benefits: Smart Buildings attract businesses and stimulate economic growth in smart cities. Their energy-efficient features and advanced infrastructure make them attractive to companies seeking sustainable and technologically advanced spaces.
- Predictive Maintenance: Sensors and data analytics enable predictive maintenance by monitoring the condition of building equipment. This helps identify potential issues before they lead to failures, reducing downtime and maintenance costs.

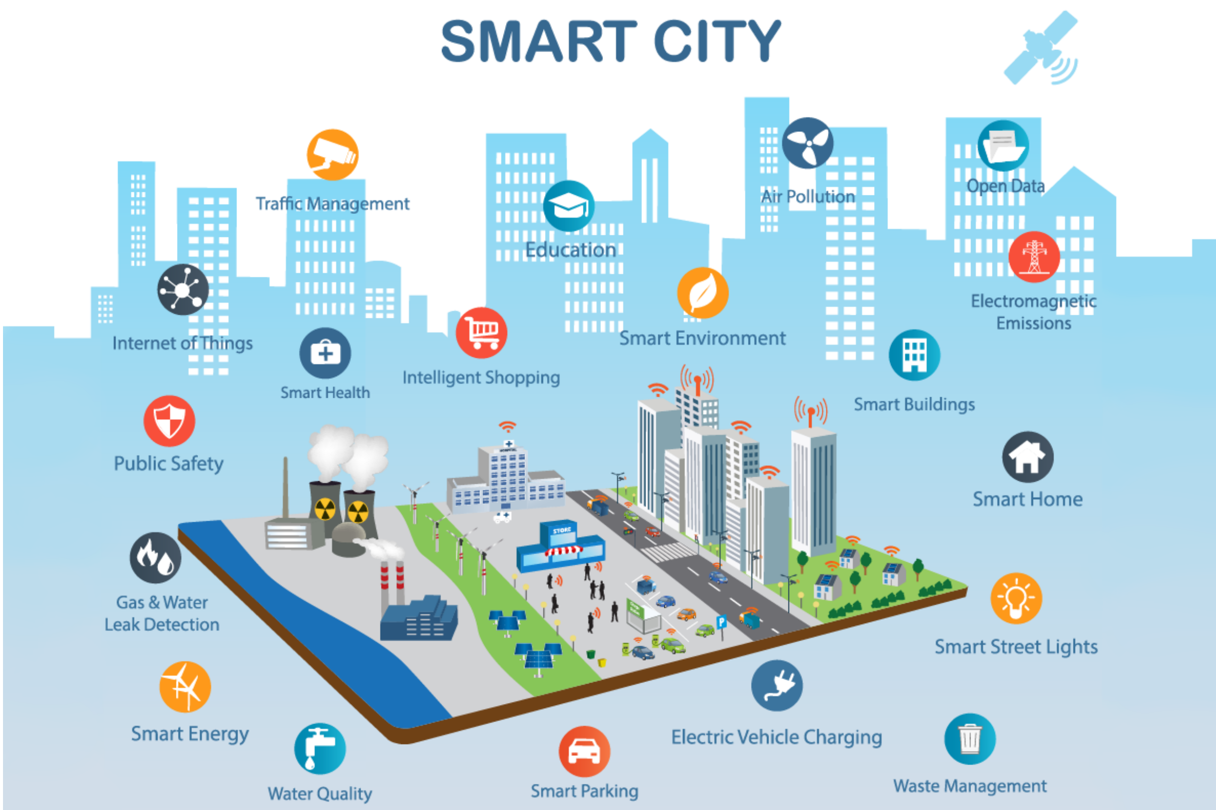

Figure 1. Smart city.

Egypt aim to build new administrative capital and intend to make it smart city. The Egyptian government has planned to build many smart cities recently, and they are currently being built, with the new administrative capital on top of these cities. The New Administrative Capital is one of the most important smart cities that Egypt is building according to the standards of fourth generation cities, as it was designed to become one of the largest capitals in the world. Its total area is about 170 thousand acres, which is larger than the area of Singapore, to accommodate 18 to 40 million people by 2025. President Abdel Fattah El-Sisi decided to build the Administrative Capital to relieve congestion in Cairo, so that the new capital will be the new headquarters for Egypt's government. Among the most important features of the city are the elegance of architectural design, the electricity generation system, ease of transportation, a smart waste collection system, and a security command and control center in the capital. One of the most important features of smart cities is smart buildings, so the government seeks to make the capital's buildings smart buildings.

Multi-criteria decision-making (MCDM) problem has many methods that can assess each criterion. To choose the most suitable smart building solution, MCDM approach are applied.

MCDM is a problem-solving technique that incorporates decision-makers' preferences to identify the best alternative. By assigning weights to each criterion based on the decision-makers' preferences and using suitable evaluation methods, such as the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) or the Simple Additive Weighting (SAW) method, you can identify the smart building solution that best aligns with your organization's needs and priorities. The weights of criteria are very crucial and imperative to the problem as they influence the outcome of the decision- making process and may lead to unpredictable results [3]. weights show the importance for each criteria in the problem. Weighting methods can be divided into three categories: subjective, objective and combinative. Subjective methods require DMs to take responsibility for assigning weights to the criteria depending on their preference, subjective methods like (Direct Ranking, Point allocation, Pairwise Comparisons, SMART) but this type of method is not efficient enough when the number of criteria increase. In contrast, objective methods do not involve DMs in determining the relative importance of the criteria but instead use mathematical algorithms based on initial data or decision matrix like Entropy, CRITI. The combinative approach involves a blend of both subjective and objective methods [4]. This paper aim to show a new hybrid method to help decision makers to select best city. First MEREC method (Method based on the Removal Effects of Criteria), for determining criteria weights [5], The VIKOR method used to solve various decision-making issues based on multi-criteria. Additionally, the proposed approach is presented in the type-2 neutrosophic number (T2NN). Hence, the T2NN-MEREC method is used to calculate the weight of each criterion then T2NN-VIKOR method is used to evaluate and rank alternatives.

Finally, this paper including a comparative between four methods VIKOR, COPRAS, TOPSIS and COCOSO and rank disagreements are expressed using spearman`s correlation coefficients.

1.1 Contributions of this study

The primary contribution of this study are summarized below: This paper development of a new approach MEREC method with VIKOR method based on T2NN. The proposed approach T2NN-MEREC-VIKOR improve performance of decision making problems. This study provides a suggestion for the government for selecting best smart city and proposed a comparative analysis between MCDM methods for evaluating alternatives. Finally, sensitivity analysis and a comparative analysis are presented to prove the robustness, and stability of the proposed approach.

1.2 Organization of the paper

This paper is organized as follows: In section 2, a literature review the studies used in this paper. Section 3, introduce the concept and methodology for the suggested approach T2NN-MEREC-VIKOR. Section4, introduce case study for this method. Finally, Section5, proposed the sensitivity and comparative analysis between some of MCDM methods using spearman`s correlation coefficients. Section 6 display the conclusion of this study.

2. Literature Review

In this section, simple explanation will be given contain literature associated with this study. This part consists of three sub-parts. the first one present studies related to smart building. second part introduce the studies that explain the neutrosophic numbers T2NN. Third part present some literature about MEREC, VIKOR, COCOSO, COPRAS, and TOPSIS methods.

2.1 Smart Building

Building performance optimization is a multidisciplinary field that encompasses various aspects, including building rating systems, energy simulation algorithms, AI and ML technologies, and project delivery methodologies. As the global energy crisis continues to exert pressure on the construction industry, there is an increasing need for innovative solutions to ensure the efficient and sustainable operation of buildings. Given the importance of building performance, developing an innovative MCDM method is necessary to promote the Efficiency and effectiveness in building performance-based design [6].

The goal of optimization in building performance design is to identify the best design solution for a specific building application, considering factors such as energy efficiency, indoor environmental quality, cost, and other criteria specified by the client or the regulatory requirements. MCDM can be applied to evaluate and select the final optimization solution among several alternative solutions. This process involves assigning weights to the criteria, forming decision matrices, and calculating the normalized decision matrix to determine the relative preference for each solution. Building performance optimization is a critical issue in the AEC field, which requires the development and implementation of innovative algorithms and methodologies. By applying MCDM and other evaluation techniques, the optimal solution for a specific building performance optimization problem can be determined, leading to the design and construction of more efficient, sustainable, and cost-effective buildings. MCDM or Multi-Criteria Decision

Analysis (MCDA), is one of the most accurate methods of decision-making, and it can be known as a revolution in this field [7].

2.2 T2NN Environment

Type-1 neutrosophic number (T1NNS) is a mathematical concept introduced by Florentin Smarandache in early 1990s as a generalization of fuzzy numbers to capture the nature of human judgments and beliefs, which can be expressed as true, false, or indeterminate. It has been successfully applied in various fields, including building performance optimization, to improve the accuracy and robustness of decision-making processes. The concept of T1NNS is based on three levels of truth: True, False, and Indeterminate. The True level represents beliefs that are confirmed, the false level represents beliefs that are refuted, and the indeterminate level represents beliefs that are uncertain or have not been evaluated yet. Smarandache proposed the neutrosophic sets in [8-9, 23]. Type-2 neutrosophic number (T2NNS) is an extension of the concept of a T1NN to a higher level of indeterminacy [24]. This extension enables a more comprehensive representation of the beliefs of decision-makers and their degree of confidence in the beliefs. The neutrosophic sets proved to be a valid workspace in describing incompatible and indefinite information. z(T, I, F) is a Type-1 Neutrosophic Number. But z((T_t, T_i,T_f), (I_t, I_i,I_f), (F_t, F_i, F_f)) is a Type-2 Neutrosophic Number, which means that each neutrosophic component T, I, and F is split into its truth, indeterminacy, and falsehood subparts [10]. Then, T2NN has become a preferred tool by scholar and researchers in recent time.

2.3 MCDM Methods

MCDM methods are used in many fields [11, 12]. These methods help to compare alternatives and find the best one [13]. There are various MCDM technique that have been employed to deal with several real-world decision making issues. Keshavarz-Ghorabaee et al. [5] a new Method based on the Removal Effects of Criteria (MEREC). This method used for determining criteria weights. Saidin et al. [14] mention that MEREC can solve fuzzy MCDM problems. Shanmugasundar et al. [15] introduce application of MEREC in multi-criteria selection. MEREC focuses on the change in the total criteria weight by disabling that criterion when determining the weight of a criterion.

Also, VIKOR method has been utilizes in several literatures. VIKOR used to prioritize and rank different alternatives. It is based on the concept of stochastic dominance, which considers both the strength and the number of attributes that exceed a particular threshold. VIKOR (Vise

Kriterijumska Optimizajica I Kompromisno Resenje) was first introduced by Serafim Opricovic in 1998. VIKOR aims to complete decision-making on existing alternatives by ranking and choosing sample sets with conflicting criteria [16]. Sayadi et al. [17] introduce extension of VIKOR method for decision making problem with interval numbers. Shumaiza et al. [18] present VIKOR method with trapezoidal bipolar fuzzy information. Yazdani et al. [19] proposed a technique called the combined compromise solution (CoCoSo) for an MCDM problem which is based on integrated simple additive weighing and an exponentially weighted product model. TOPSIS (Technique for Order Performance by Similarity to Ideal Solution) method, which is one of the most widely used MCDM methods [20]. TOPSIS is one of the fundamental methods in MADM domain and has been immensely popular in applications and as foundation to numerous method development [21, 22].

3. Methodology

This section introduces the methodology for each study in this paper. this section also divided into three parts. First, some basic concept and definitions about T2NN. Second MEREC method to determine the weights for each criterion, then the MCDM methods proposed for ranking best alternatives form smart buildings.

Four steps to evaluate process using MCDM approaches:

- Defining alternatives and criteria related to problem.
- Determine weights of each criteria using one of the MCDM methods.
- Assigning individual performance to each option.
- Evaluate alternatives based on the aggregate performance of them on all criteria.

3.1 Preliminaries

In this part definitions and some concepts and operations associated with T2NN are given below:

Definition 1 [10]. We consider that Z is limited universe of discourse and F[0,1] is the set of all triangular neutrosophic numbers on F[0,1] .

A Type 2 neutrosophic number set (T2NNS) $\widetilde{U}$ in Z is represented by:

$$\widetilde{U} = \left\langle \left(T_{T_{\widetilde{U}}}(z), T_{I_{\widetilde{U}}}(z), T_{F_{\widetilde{U}}}(z)\right), \left(I_{T_{\widetilde{U}}}(z), I_{I_{\widetilde{U}}}(z), I_{F_{\widetilde{U}}}(z)\right), \left(F_{T_{\widetilde{U}}}(z), F_{I_{\widetilde{U}}}(z), F_{F_{\widetilde{U}}}(z)\right) \right\rangle \tag{1}$$

Where $\check{T}_{\check{U}}(z): Z \rightarrow F[0,1]$, $\tilde{I}_{\check{U}}(z): Z \rightarrow F[0,1]$, $\check{F}_{\check{U}}(z): Z \rightarrow F[0,1]$. The type -2 neutrosophic number set $\check{T}_{\check{U}}(z) = \left(T_{T_{\check{U}}}(z), T_{I_{\check{U}}}(z), T_{F_{\check{U}}}(z)\right)$, $\tilde{I}_{\check{U}}(z) = \left(I_{T_{\check{U}}}(z), I_{I_{\check{U}}}(z), I_{F_{\check{U}}}(z)\right)$, $\check{F}_{\check{U}}(z) = \left(F_{T_{\check{U}}}(z), F_{I_{\check{U}}}(z), F_{F_{\check{U}}}(z)\right)$ defined as the truth, indeterminacy and falsity member-ships of z in $\widetilde{U}$.

Definition 2 [10]. Suppose that

$\widetilde{U}_1 = \left\langle \left(T_{T_{\check{U}1}}(z), T_{I_{\widetilde{U1}}}(z), T_{F_{\widetilde{U1}}}(z)\right), \left(I_{T_{\widetilde{U1}}}(z), I_{I_{\widetilde{U1}}}(z), I_{F_{\widetilde{U1}}}(z)\right), \left(F_{T_{\widetilde{U1}}}(z), F_{I_{\widetilde{U1}}}(z), F_{F_{\widetilde{U1}}}(z)\right)\right\rangle$ and

$\widetilde{U}_2 = \left\langle \left(T_{T_{\check{U}2}}(z), T_{I_{\check{U}2}}(z), T_{F_{\widetilde{U2}}}(z)\right), \left(I_{T_{\widetilde{U2}}}(z), I_{I_{\widetilde{U2}}}(z), I_{F_{\widetilde{U2}}}(z)\right), \left(F_{T_{\widetilde{U2}}}(z), F_{I_{\widetilde{U2}}}(z), F_{F_{\check{U}2}}(z)\right)\right\rangle$

Are two T2NNs then the following equations describe some of T2NN operators.

- $$\widetilde{U}_1 \oplus \widetilde{U}_2 = \left\langle \begin{matrix} \begin{pmatrix} T_{T_{\check{U}_1}}(z) + T_{T_{\check{U}_2}}(z) - T_{T_{\check{U}_1}}(z).T_{T_{\check{U}2}}(z),\ T_{I_{\widetilde{U1}}}(z) + T_{I_{\widetilde{U2}}}(z) - T_{I_{\widetilde{U1}}}(z).T_{I_{\widetilde{U2}}}(z), \\ T_{F_{\widetilde{U1}}}(z) + T_{F_{\widetilde{U2}}}(z) - T_{F_{\widetilde{U1}}}(z).T_{F_{\widetilde{U2}}}(z) \end{pmatrix}, \\ \left(I_{T_{\widetilde{U1}}}(z).I_{T_{\widetilde{U2}}}(z), I_{I_{\widetilde{U1}}}(z).I_{I_{\widetilde{U2}}}(z), I_{F_{\widetilde{U1}}}(z).I_{F_{\widetilde{U2}}}(z)\right), \\ \left(F_{T_{\widetilde{U1}}}(z).F_{T_{\widetilde{U2}}}(z), F_{I_{\widetilde{U1}}}(z).F_{I_{\widetilde{U2}}}(z), F_{F_{\widetilde{U1}}}(z).F_{F_2}(z)\right) \end{matrix} \right\rangle \quad (2)$$
- $$\widetilde{U}_1 \otimes \widetilde{U}_2 = \left\langle \begin{matrix} \left(\left(T_{T_{\check{U}_1}}(z).T_{T_{\check{U}_2}}(z), T_{I_{\widetilde{U1}}}(z).T_{I_{\widetilde{U2}}}(z), T_{F_{\widetilde{U1}}}(z).T_{F_{\widetilde{U2}}}(z)\right)\right), \\ \left(\left(I_{T_{\widetilde{U1}}}(z) + I_{T_{\widetilde{U2}}}(z) - I_{T_{\widetilde{U1}}}(z).I_{T_{\widetilde{U2}}}(z)\right), \left(I_{I_{\widetilde{U1}}}(z) + I_{I_{\widetilde{U2}}}(z) - I_{I_{\widetilde{U1}}}(z).I_{I_{\widetilde{U2}}}(z)\right), \begin{pmatrix} I_{F_{\widetilde{U1}}}(z) + I_{F_{\widetilde{U2}}}(z) - \\ I_{F_{\widetilde{U1}}}(z).I_{F_{\widetilde{U2}}}(z) \end{pmatrix}\right) \\ \left(\left(F_{T_{\widetilde{U1}}}(z) + F_{T_{\widetilde{U2}}}(z) - F_{T_{\widetilde{U1}}}(z).F_{T_{\widetilde{U2}}}(z)\right), \left(F_{I_{\widetilde{U1}}}(z) + F_{I_{\widetilde{U2}}}(z) - F_{I_{\widetilde{U1}}}(z).F_{I_{\widetilde{U2}}}(z)\right), \begin{pmatrix} F_{F_{\widetilde{U1}}}(z) + F_{F_2}(z) - \\ F_{F_{\widetilde{U1}}}(z).F_{F_2}(z) \end{pmatrix}\right) \end{matrix} \right\rangle \quad (3)$$
- Score function

$$S(\widetilde{U}) = \frac{1}{12}\left\langle 8 + \left(T_{T_{\widetilde{U1}}}(Z) + 2\left(T_{I_{\widetilde{U1}}}(Z)\right) + T_{F_{\widetilde{U1}}}(Z)\right) - \left(I_{T_{\widetilde{U1}}}(Z) + 2\left(I_{I_{\widetilde{U1}}}(Z)\right) + I_{F_{\widetilde{U1}}}(Z)\right) - \left(F_{T_{\widetilde{U1}}}(Z) + 2\left(F_{I_{\widetilde{U1}}}(Z)\right) + F_{F_{\widetilde{U1}}}(Z)\right)\right\rangle \quad (4)$$

Definition 3 [10]. To Build the evaluation matrix $A_i \times Œ_{ip}$ to assess the classification of alternatives with respect to each criterion.

$$\check{R} = \begin{matrix} & Œ_{ip} & \cdots & Œ_{in} \\ Alt_1 & \check{Z}_{11} & \cdots & \check{z}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ Alt_m & \check{Z}_{m1} & \cdots & \check{Z}_{mn} \end{matrix} \quad (5)$$

3.2 MEREC method

In this section, the following steps present the MEREC method that used to evaluate the weights of criteria in MCDM problems as mentioned if the Figure 2.

Step 1. Build the decision matrix which element will be x_{ij}, and matrix consist of n x m where n numbers of criteria and m numbers of alternatives , then matrix form is :

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1j} & \cdots & x_{im} \\ x_{21} & x_{22} & \cdots & x_{2j} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{i1} & x_{i2} & \cdots & x_{ij} & \cdots & x_{im} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nj} & \cdots & x_{nm} \end{bmatrix} \quad (6)$$

Step 2. Normalize this matrix using the following Eq. (7).

$$n_{ij}^{x} = \begin{cases} \frac{\min_k x_{kj}}{x_{ij}}, \ if\ j\ \in B \\ \frac{x_{ij}}{\max_k x_{kj}}, \ if\ j\ \in H \end{cases} \quad (7)$$

where B is the set of beneficial criteria and H is the set of non-beneficial criteria.

Step 3. The overall efficiency of the alternatives (S_i) is calculated using Eq. (8).

$$S_i = \ln\left(1 + \frac{1}{m}\ \sum_j \left|\ln(n_{ij}^{x})\right|\right) \quad (8)$$

Step 4. Based on method idea, calculate the performance of the alternatives by removing each criterion. So, $\grave{S_{ij}}$ denotes as the overall performance of ith alternative concerning the removal of jth criterion.

$$\grave{S_{ij}} = \ln\left(1 + \left(\frac{1}{m}\ \sum_{k,k\neq j} \left|\ln(n_{ik}^{x})\right|\right)\right) \quad (9)$$

Step 5. Calculating the absolute value of the deviations using Eq. (10), E_j the difference between Step 3 and Step 4.

$$E_j = \sum_i \left|\ \grave{S_{ij}} - S_i\right| \quad (10)$$

Step 6: The weights of criteria is computed as follow using Eq. (11).

$$W_j = \frac{E_j}{\sum_k E_k} \quad (11)$$

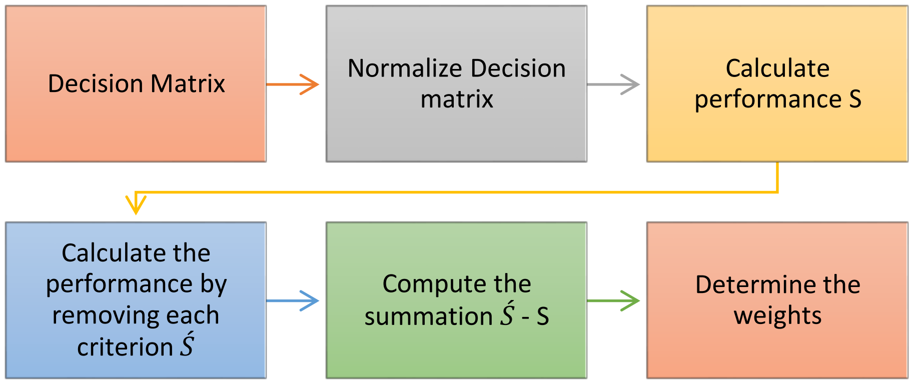

Figure 2. Steps for MEREC method.

3.3 VIKOR method

In this part VIKOR steps are introduced to rank alternatives based on weights given from MEREC method as mentioned in Figure 3.

Step 1. Define the decision matrix. This matrix is defined as follow:

$$F= \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccc} C_{x1} & C_{x2} & \cdots & C_{xn} \end{array} \\ \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix} \end{array} \quad (12)$$

Where A_i denote alternatives as i = 1,2, 3, …., n and C_{xn} denote criteria as j = 1,2, 3, …., m

Step 2. Determining best (f_j^*) and worst (f_j^-) performance values as the ideal solution for all criteria, to normalize decision matrix as the following equations:

$$f_j^* = max_j\, f_{ij} \text{ and } f_j^- = min_j\, f_{ij} \quad (13)$$

$$f_j^* = min_j\, f_{ij} \text{ and } f_j^- = max_j\, f_{ij} \quad (14)$$

Step 3. The utility measure (S_i) and regret measures (R_i) are calculated as follow:

$$S_i = \sum_{j=1}^{n} W_j \frac{(f^* - f_{ij})}{(f_j^* - f_j^-)} \quad (15)$$

$$R_i = max_j \left[W_j \frac{(f^* - f_{ij})}{(f_j^* - f_j^-)} \right] \quad (16)$$

where W_j is the weight of each criterion with the MEREC.

Step 4. Finally, the value of Q_i is calculated known as VIKOR index using Eq. (17).

$$Q_i = v\left[\frac{S_i - S^*}{S^- - S^*}\right] + (1-v)\left[\frac{R_i - R^*}{R^- - R^*}\right] \quad (17)$$

Where $S^* = min_i\, S_i \;\; and\; S^- = max_i\, S_i$

$R^* = min_i\, R_i \;\; and\; R^- = max_i\, R_i$

Step 5. Ranking is applied on Q_i by ascending order.

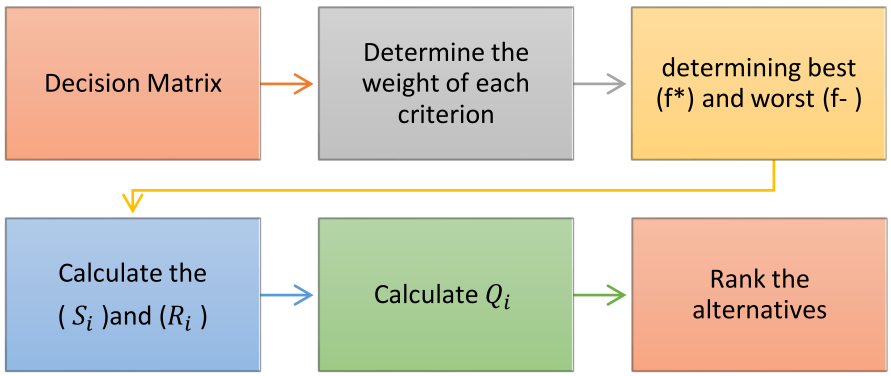

Figure 3. Steps for VIKOR method.

3.4 TOPSIS method

Here are the steps for TOPSIS method.

Step 1. Construct decision matrix same as the following above.

Step 2. Calculating the normalized matrix based on this equation:

$$\alpha_{ij} = \frac{x_{ij}}{\sqrt{\sum (X_{ij})^2}}$$

Step 3. Assigning weights to decision matrix as follow:

$$X_{ij} = \alpha_{ij} * W_j$$

Step 4. Define best and worst solution

$$X_i^b = \max x_{ij} \quad \text{as best value}$$

$$X_i^w = \min x_{ij} \quad \text{as worst value}$$

Step 5. Calculating Euclidean distance for best and worst values.

$$d_i^b = \sqrt{\sum (x_{ij} - X_j^b)^2}$$

$$d_i^w = \sqrt{\sum (x_{ij} - X_j^w)^2}$$

Step 6. Calculating Value of D_i by

$$D_i = \frac{d_i^w}{d_i^w + d_i^b}$$

Step 7. Ranking based on D_i values while the largest D_i is best alternatives.

3.5 COCOSO method

Here are the steps for COCOSO method.

Step 1. Construct decision matrix same as the following above.

Step 2. Determine the normalized matrix by the following equation:

$$r_{ij} = \frac{x_{ij} - minx_{ij}}{maxx_{ij} - minx_{ij}} \quad \text{for benefit criterion}$$

$$r_{ij} = \frac{maxx_{ij} - x_{ij}}{maxx_{ij} - minx_{ij}} \quad \text{for cost criterion}$$

Step 3. As, CoCoSo method consists of the integration of methods such as the WASPAS, SAW and EWP. So, based on WASPAS method S_i ,P_i are computed as follow:

$$S_i = \sum w_i \,.\, r_{ij}$$

$$P_i = \sum (\ r_{ij})^{w_i}$$

Step 4. Three appraisal score strategies are calculated

$$k_{ia} = \frac{P_i + S_i}{\sum_{i=1}^{m} \ P_i + S_i}$$

$$k_{ib} = \frac{S_i}{min\ S_i} + \frac{P_i}{min\ \ P_i}$$

$$k_{ic} = \frac{\lambda.S_i \ + (1-\lambda)P_i}{\lambda.\max S_i + (1-\lambda)\max P_i}$$

Where λ usually =0.5 but its range from 0 to1.

Step 5. Final step final ranking for all alternatives based on performance k_i

$$k_i = (k_{ic} + k_{ib} + k_{ic})^{\frac{1}{3}} + \frac{1}{3}\ (k_{ic} + k_{ib} + k_{ic})$$

3.6 COPRAS method [26]

Step 1. Same as all MCDM method first step is to construct decision matrix.

Step 2. Normalize matrix using these formula.

$$r_{ij} = \frac{x_{ij}}{\sum x_{ij}}$$

Step 3. Obtain weighted normalized matrix by:

$$\breve{r_{ij}} = r_{ij} . w_i$$

Step 4. Determine maximize and minimize for each alternative

$$S^+ = \sum_{j=1}^{k} \breve{r_{ij}}$$

$$S^- = \sum_{j=k+1}^{k} \breve{r_{ij}}$$

Step 5. Calculate the relative weight for each alternative

$$Q_i = S^+ + \frac{min\, S^- \; \sum_{i=1}^{m} S^-}{S^- \quad \sum_{i=1}^{m} \frac{min\, S^-}{S^-}}$$

Step 6: The priority order of the alternatives is ranked using the value of Qi in descending order. The highest relative weight is the most acceptable alternative.

4. Case Study

4.1 problem definition

The problem definition of smart buildings revolves around addressing challenges and inefficiencies in traditional building systems by integrating advanced technologies to enhance efficiency, sustainability, safety, and occupant comfort.

With the dense population increase, there are several problems facing traditional buildings that affect the environment and the quality of life of citizens, as well as the consumption of energy and resources in general. Traditional buildings face several problems, including overlapping buildings and an increase in shared spaces, thus increasing the risk of theft, harm to citizens, and lack of security. Therefore, smart buildings are designed to provide more privacy and security through sensors, surveillance cameras, and a security system to prevent unauthorized persons from accessing the buildings.

Among the most important problems that traditional buildings suffer from is the increase in energy costs and their increased impact on the environment resulting from increased heat. Therefore, smart buildings contain an energy management system that can track energy consumption and adjust the system settings to adapt to the results after obtaining them after collecting data and adjusting the control of heating and ventilation mechanisms. And air conditioning. There are many other problems that must be overcome, therefore, by addressing these challenges by using technology in smart buildings to improve citizens' lives, reduce energy consumption, provide safety, improve the quality of buildings, optimize the use of resources, and improve economic growth.

Smart cities seek to build a better future through advanced technology and modern technologies. One of the most important smart cities is OSLA the first smart city in the world. There are several smart cities like Barcelona, Spain, Columbus, Ohio, USA, Dubai, United Arab Emirates, Hong Kong, China, Kansas City, Missouri, USA, London, England, Melbourne, Australia. Egypt also aims to become an ideal model for cultural environmental development, and in order to choose the best solutions, the Egyptian government can use different evaluation methods. In this paper, a method is used to obtain the weights of the criteria. Smart buildings use wide rang technology and its intelligence to design building and collect data from citizen, systems and sensors and analyze these data and optimize smart building.

To get a comprehensive and balanced ranking, the government can use the T2NN algorithm, which is an Artificial Neural Network approach that considers both qualitative and quantitative factors. This approach can provide a reliable ranking of potential smart city candidates based on their overall suitability and ability to contribute to Egypt's goal of developing smart, sustainable, and environmentally friendly cities.

4.2 Description of alternatives and criteria

Several cities around the world have been implementing smart technologies, including smart buildings, to enhance urban living. So, we choose of cities that have made strides in adopting smart building technologies:

- Alt1: Singapore: has been a pioneer in the development of smart city technologies. As it depends on the use of sensors and data analytics in buildings for energy efficiency, waste management, and urban planning.
- Alt2: Barcelona, Spain: has implemented the "Smart City Barcelona" initiative, leveraging technologies for smart lighting, waste management, and transportation. Smart building solutions are integrated into the city's infrastructure to enhance energy efficiency and sustainability.

- Alt3: Seoul, South Kore: has focused on creating a smart city infrastructure with an emphasis on smart buildings. The city has implemented energy-efficient technologies, smart grids, and advanced transportation systems to improve overall urban sustainability.
- Alt4: Dubai, United Arab Emirates: has been working towards becoming a smart city with initiatives like the Smart Dubai project. The city has incorporated smart building technologies for energy management, smart lighting, and integrated data systems.

The criteria for defining a smart building can vary, but generally, they encompass the integration of intelligent systems and data-driven solutions. Here are key criteria for smart buildings:

- C1: Energy Efficiency: Integration of energy-efficient technologies, such as smart lighting systems, occupancy sensors, and energy management systems, to optimize energy consumption and reduce environmental impact.
- C2: Building Advanced Security Systems: Implementation of security systems, including access control, and intrusion detection, often integrated with other building systems.
- C3: Data Analytics and Predictive Maintenance: Use of data analytics to gain insights into building performance, enabling predictive maintenance to address potential issues before they become critical.
- C4: Resilience and Disaster preparedness: Conduct a comprehensive risk assessment to identify potential hazards and vulnerabilities specific to the building's location, such as earthquakes, floods, hurricanes, or other natural disasters.

4.3 Applying MEREC to get weights then using VIKOR method to rank alternatives

Step 1. Organize alternative and criteria based on our expert's opinion in Table 1, according to Eq. (5).

- Experts use the linguistic terms presented in Table 7 [10].
- Aggregate the finial evaluation matrix using Eq. (4) to form the decision matrix in Table 2.

Table 1. Classification of alternatives by experts.

Expert	Alt n	C1	C2	C3	C4
$Expert_1$	Alt1	MG	G	VG	MG
$Expert_2$	Alt2	VB	VG	G	B
$Expert_3$	Alt3	B	MG	MG	M
$Expert_4$	Alt4	G	MB	VG	MG

Table 2. Decision matrix.

	Criteria	
	$C_1 \in B$	$C_2 \in B$
Alt_1	⟨(0.617,0.599,0.623); (0.013,0.001,0.014); (0.003,0.005,0.005)⟩	⟨(0.476,0.440,0.453); (0.013,0.018,0.030); (0.008,0.011,0.026)⟩
Alt_2	⟨(0.353, 0.308, 0.358); (0.043, 0.042, 0.064); (0.024, 0.032, 0.063)⟩	⟨(0.640, 0.613, 0.637); (0.002, 0.003, 0.007); (0.004, 0.007, 0.006)⟩
Alt_3	⟨(0.245, 0.245, 0.099); (0.070, 0.160, 0.193); (0.034, 0.160, 0.106)⟩	⟨(0.475, 0.393, 0.358); (0.006, 0.006, 0.017); (0.002, 0.016, 0.005)⟩
Alt_4	⟨(0.589, 0.588, 0.591); (0.006, 0.005, 0.003); (0.008, 0.005, 0.002)⟩	⟨(0.383, 0.261, 0.358); (0.054, 0.003, 0.057); (0.030, 0.024, 0.084)⟩
	Criteria	
	$C_3 \in H$	$C_4 \in H$
Alt_1	⟨(0.650,0.619,0.650); (0.003,0.004,0.004); (0.001,0.008,0.005)⟩	⟨(0.653,0.629,0.650); (0.006, 0.005,0.006); (0.008, 0.006,0.001)⟩
Alt_2	⟨(0.552, 0.544, 0.593); (0.004, 0.005, 0.009); (0.002, 0.002, 0.008)⟩	⟨(0.393, 0.351, 0.358); (0.033, 0.039, 0.060); (0.026, 0.030, 0.059)⟩
Alt_3	⟨(0.603, 0.539, 0.571); (0.001, 0.008, 0.001); (0.001, 0.001, 0.004)⟩	⟨(0.485, 0.420, 0.458); (0.005, 0.003, 0.011); (0.001, 0.008, 0.003)⟩
Alt_4	⟨(0.664, 0.657, 0.664); (0.003, 0.003, 0.004); (0.004, 0.004, 0.004)⟩	⟨(0.588, 0.510, 0.571); (0.003, 0.001, 0.002); (0.008, 0.001, 0.002)⟩

Here we have two beneficial criteria C1, C2 and two non-beneficial criteria C3, C4.

Step 2. Use Eq. (4) to modify type 2 neutrosophic numbers to the crisp represented in Table 3.

Table 3. Crisp numbers.

Alternatives	$C1$	$C2$	$C3$	$C4$
Alt1	0.8659	0.8061	0.8765	0.8598
Alt2	0.7488	0.8720	0.8497	0.7614
Alt3	0.6493	0.7954	0.8523	0.8118
Alt4	0.8597	0.7487	0.8844	0.8467

Step 3. Applying MEREC method to get weight, use Eq. (7) to get normalized decision matrix as shown in Table 4.

Table 4. Normalized decision matrix

Alternatives	$C1$	$C2$	$C3$	$C4$
Alt1	0.75	0.93	0.99	1

Alt2	0.87	0.86	0.96	0.89
Alt3	1	0.94	0.96	0.94
Alt4	0.75	1	1	0.98

Step 4. Obtain overall efficiency of the alternatives (S_i), using Eq. (8).

$$S_1 = ln\left(1+\frac{1}{4}\left(|\ln(0.75)| + |\ln(0.93)| + |\ln(0.99)| + |\ln(1)|\right)\right) = 0.088$$

$$S_2 = ln\left(1+\frac{1}{4}\left(|\ln(0.87)| + |\ln(0.86)| + |\ln(0.96)| + |\ln(0.89)|\right)\right) = 0.106$$

$$S_3 = ln\left(1+\frac{1}{4}\left(|ln(1)| + |ln(0.94)| + |ln(0.96)| + |ln(0.94)|\right)\right) = 0.040$$

$$S_4 = ln\left(1+\frac{1}{4}\left(|ln(0.75)| + |ln(1)| + |ln(1)| + |ln(0.98)|\right)\right) = 0.074$$

Step 5. Now, calculate the performance of the alternatives by removing each criterion. The result in Table 5 using Eq. (9). But first let's present an example $S_{11}^{`}$.

$$S_{11}^{`} = ln\left(1+\frac{1}{4}\left(|\ln(0.93)| + |\ln(0.99)| + |\ln(1)|\right)\right) = 0.021$$

Table 5. Values of $S_{11}^{`}$

Alternatives	$C1$	$C2$	$C3$	$C4$
Alt1	0.021	0.072	0.086	0.088
Alt2	0.074	0.072	0.061	0.08
Alt3	0.040	0.025	0.030	0.025
Alt4	0.004	0.074	0.074	0.069

Step 6. Calculating the absolute value of the deviations using formula of Eq. (10).

$$E_1 = |0.021 - 0.088| + |0.072 - 0.106| + |0.086 - 0.040| + |0.088 - 0.074| = 0.161$$

$$E_2 = |0.074 - 0.088| + |0.072 - 0.106| + |0.061 - 0.040| + |0.08 - 0.074| = 0.075$$

$$E_3 = |0.040 - 0.088| + |0.025 - 0.106| + |0.030 - 0.040| + |0.025 - 0.074| = 0.188$$

$$E_4 = |0.040 - 0.088| + |0.074 - 0.106| + |0.074 - 0.040| + |0.069 - 0.074| = 0.155$$

Step 7. Finally, compute weight for each criterion using Eq. (11), as presented in Figure 4.

$$w_1 = 0.278$$

$$w_2 = 0.129$$

$$w_3 = 0.325$$

$$w_4 = 0.268$$

Figure 4. Weights of criteria.

Step 8. After calculating weights for every criterion we now using VIKOR method to rank alternatives but first we get Table 4.

Table 4. Normalized decision matrix.

Alternatives	$C1$	$C2$	$C3$	$C4$
Alt1	0.75	0. 93	0.99	1
Alt2	0.87	0.86	0.96	0.89
Alt3	1	0.94	0.96	0.94
Alt4	0.75	1	1	0.98

Step 9. Determine the PIS (best f_j^*) and NIS (worst f_j^-) by using Eq. (13) as presented in Table 5.

Table 5. PIS and NIS.

W_j	**0.278**	**0.129**	**0.325**	**0.268**
f_j^*	1	1	1	1
f_j^-	0.75	0.86	0.96	0.89

Step 10. Compute (S_i) and (R_i) of each alternative using Eq. (15) and Eq. (16) and the result will be founded in Table 6.

Step11. Calculate the value of VIKOR index, using Eq. (17) the result in Table 6. Notice that, v = 0.5**.**

Table 6. Final ranking of the alternatives.

Alternatives	$C1$	$C2$	$C3$	$C4$	S_i	R_i	Q_i	Rank
Alt1	0.278	0.06	0.081	0	0.419	0.278	0.103	3
Alt2	0.145	0.129	0.325	0.268	0.867	0.325	1	1
Alt3	0	0.055	0.325	0.146	0.526	0.325	0.684	2
Alt4	0.278	0	0	0.049	0.327	0.278	0	4
V= 0.5				S^*, R^*	0.327	0.278		
				S^-, R^-	0.867	0.325		

Step 13. After evaluating and ranking we found that the order for best alternatives of selecting the best smart city is A2, A3, and A4 as presented in Figure 5.

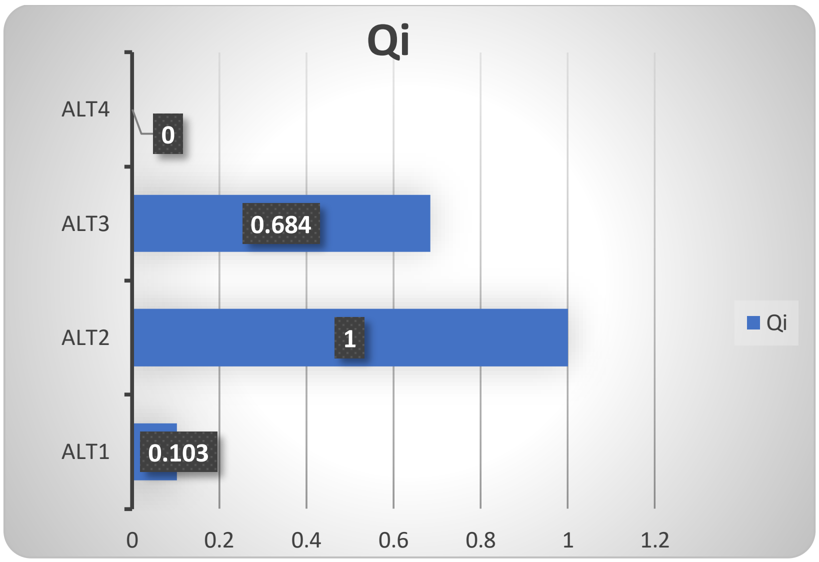

Figure 5. Ranking the alternatives.

4.4 Determining ranking of alternatives using MCDM methods

To deduce final best alternative, we use four methods (TOPSIS, CoCoSo, COPRAS)

4.4.1 TOPSIS Method

Ranking alternatives based on TOPSIS method shown in Table 7.

Table 7. Final ranking using TOPSIS method.

Alternatives	d^+	d^-	S_i	Rank
Alt1	0.587	0.0929	0.136	4
Alt2	0.02	0.02	0.5	2
Alt3	0.016	0.012	0.43	3
Alt4	0.026	0.038	0.59	1

4.4.2 CoCoSo Method

Ranking alternatives based on CoCoSo method shown in Table 8.

Table 8. Final ranking using CoCoSo method.

Alternatives	K_{ia}	K_{ib}	K_{ic}	K_i	Rank
Alt1	0.232	2.95	0.631	2.83	3
Alt2	0.368	5.195	1	4.059	1
Alt3	0.250	3.27	0.679	3.013	2
Alt4	0.149	2	0.404	2.218	4

4.4.3 COPRAS Method

Ranking alternatives based COPRAS method shown in Table 9.

Table 9. Final ranking using COPRAS method.

Alternatives	S^+	S^-	Q_i	Rank
Alt1	0.108	0.152	0.2515	2
Alt2	0.101	0.141	0.2557	1
Alt3	0.088	0.146	0.2374	4
Alt4	0.104	0.152	0.2465	3

4.5 Comparative Analysis

A comparative analysis can ensure experts to validate the outcomes by some changes in the essential model and clarify the robustness of the proposed methodology. Therefore, comparative analysis use comparing ranking results obtained by a MCDM methods used in this paper using Spearman's rank correlation coefficient.

The comparative analysis is compare of the ranking from MCDM techniques including COPRAS, TOPSIS, CoCoSo, and VIKOR. The final ranking of VIKOR, TOPSIS, CoCoSo and COPRAS methods is shown in Table 10 and Figure 6 represent the graphical chart of the ranking order for each method.

Table 10. Comparison of other MCDM methods.

Alternatives	VIKOR	TIPOSIS	CoCoSo	COPRAS
Alt1	3	4	3	2
Alt2	1	2	1	1
Alt3	2	3	2	4
Alt4	4	1	4	3

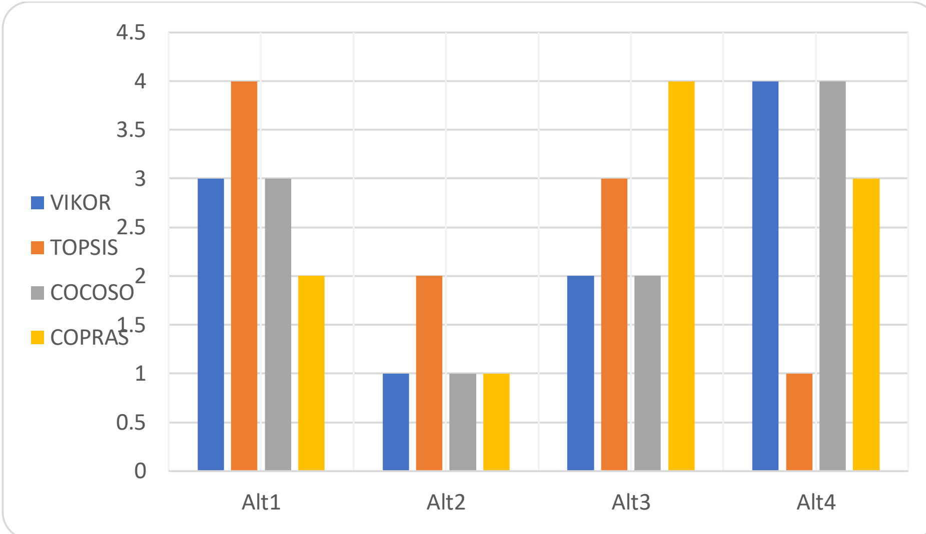

Figure 6. Final ranking of other MCDM methods.

Spearman's rank correlation coefficient, often denoted as r_s, is a measure of the strength and direction of a monotonic relationship between two variables. In other words, it assesses how well the variables are related, with direction and strength taken into account. Spearman's rank correlation coefficient ranges from -1 to 1:

- -1 indicates a perfect negative relationship (as one increases, the other decreases).
- 1 indicates a perfect positive relationship (as one increases, the other also increases).
- 0 indicates no relationship.

The Spearman's rank correlation coefficient can be calculated using, $r_s = 1 - \frac{6\sum {d_i}^2}{m.(m^2-1)}$, where d_i difference in ranking of the alternative by the two methods and m is the number of alternatives.

5. Conclusions

Smart buildings, or "smart structures," are becoming increasingly popular due to their potential to enhance energy efficiency, improve indoor air quality, and optimize building operation costs. While these benefits are widely recognized, it is crucial to address challenges associated with the integration of technology into building systems. The first challenge is ensuring reliable connectivity. To maximize the potential of smart buildings, it is necessary to establish a robust network infrastructure that can handle data transfer at a high rate. This requires a careful selection of connectivity hardware, software, and security measures. Additionally, connectivity speed and reliability should be taken into account, as delays in data transmission can significantly impact the effectiveness of smart building systems. Another challenge is ensuring compatibility and integration of different smart building systems and components. The ability to easily connect, integrate, and synchronize different technologies is crucial for creating a fully functioning smart building ecosystem. This can be achieved by implementing a consistent set of standards and protocols across all components, ensuring smooth communication and seamless data sharing. Another challenge is dealing with data privacy and security. Smart buildings contain sensitive data, such as occupant information, energy usage patterns, and environmental conditions. It is crucial to protect this data from unauthorized access and ensure its confidentiality and integrity. This can be achieved by implementing strong data encryption, secure access controls, and regular security audits. Moreover, there is the challenge of balancing energy efficiency, occupant comfort, and the integration of cutting-edge technology. Smart buildings must not only minimize energy consumption but also be able to accommodate and utilize new technologies without compromising their energy efficiency or occupant comfort. Despite these challenges, smart buildings hold immense potential for creating more sustainable, energy-efficient, and technologically advanced cities. By addressing the issues associated with smart building integration and focusing on innovative solutions, countries like Egypt can successfully navigate the complex path to building a smart future. After applying analysis on MCDM methods we found that VIKOR and CoCoSo methods have the same result in this study. In conclusion, after comparing the performance of the CoCoSo and VIKOR methods in the MCDM process, the two methods demonstrated comparable performance. The use of the CoCoSo method ensures consistency in decision-making, while the VIKOR method offers a more comprehensive understanding of the alternatives. Overall, these two methods can provide reliable guidance in selecting the best smart building technologies for the New Capital of EGYPT.

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Foundation of SuperHyperStructure

&

Neutrosophic SuperHyperStructure

(review paper)

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Abstract

In this paper we extend the SuperHyperAlgebra, SuperHyperGraph, SuperHyperTopology, SuperHyperSoft Set, endowed with SuperHyperOperations, SuperHyperAxioms, and SuperHyperFunctions, to the most general form of structure, from our real world, called SuperHyperStructure in any field of knowledge. A practical application of the SuperHyperStructure is presented at the end.

The prefix "Hyper" [Marty, 1934] stand for the codomain of the functions and operations to be *P(H)*, or the powerset of the set *H*. While the prefix "Super" [Smarandache, 2016] stands for using the $P^n(H)$, $n \geq 2$, or the *n-th* PowerSet of the Set *H* {because the *set* (or *system*) *H* (that may be a set of items, a company, institution, country, region, etc.) is organized in *sub-systems*, which in their turn are organized in *sub-sub-systems*, and so on} in the domain and/or codomain of the functions and operations.

Keywords: *n-th* PowerSet of a Set, SuperHyperAlgebra, SuperHyperGraph, SuperHyperTopology, SuperHyperSoft Set, SuperHyperOperations, SuperHyperAxioms, SuperHyperFunctions, HyperStructure, SuperHyperStructure, Neutrosophic SuperHyperStructure

1. From Classical Structure and HyperStructure to SuperHyperStructure

We present below the evolution of structures in all fields of knowledge:
Classical Structure, HyperStructure, SuperHyperStructure (none having indeterminacy);

and

Neutrosophic Classical Structure, Neutrosophic HyperStructure, Neutrosophic SuperHyperStructure (all having some indeterminacy, as in our everyday life).

2. Classical Structure

A **Classical Structure** is built on a non-empty set H, whose **Classical Operations** ($\#_0$) are defined as:

$\#_0 : H^m \to H$, for integer $m \geq 1$,

and with **Classical Axioms** acting on it.

3. HyperStructure

A **HyperStructure** (Marty [1], 1934) is built on a non-empty set H, whose **HyperOperations** ($\#_{H0}$) are defined as:

$\#_{H0} : H^m \to P \cdot (H)$, where $P \cdot (H)$ is the set of all non-emtpy subsets of H,

and with **HyperAxioms** acting on it.

4. Neutrosophic HyperStructure

As an extension of the HyperStructure, the **Neutrosophic HyperStructure** (Smarandache [2], 2016) is built on a non-empty set H, whose **Neutrosophic HyperOperations** ($\#_{NS0}$) are defined as:

$\#_{NS0} : H^m \to P(H)$, where $P(H)$ is the set of all non-emtpy and empty subsets of H,

and the axioms acting on it are called **Neutrosophic HyperAxioms**.

5. Definition of the n^th^-PowerSet $\boldsymbol{P_\star^n(H)}$ *without* Indeterminacy (no empty-set)

The n^th^-PowerSet $\boldsymbol{P_\star^n(H)}$ [2] of the set H, that the SuperHyperStructure is built on, describes a world that does not contain indeterminacy, where similarly the *set* (or *system*) H (that may be a set of items, a company, institution, country, region, etc.) is organized in sub-systems, which in turn are organized in sub-sub-systems, and so on.

The n^th^-PowerSet $\boldsymbol{P_*^n(H)}$ is also defined recursively:

$$\boldsymbol{P_\star^0(H)} \stackrel{\text{def}}{=} H$$

$$\boldsymbol{P_\star^1(H)} = P_\star(\boldsymbol{H})$$

$$\boldsymbol{P_\star^2(H)} = P_\star(P_\star(\boldsymbol{H}))$$

$$\boldsymbol{P_\star^3(H)} = P_\star(\boldsymbol{P_\star^2(H)}) = P_\star\big(P_\star(P_\star(\boldsymbol{H}))\big)$$

… … … ………………………………………....

$$\boldsymbol{P_\star^n(H)} = P_\star(\boldsymbol{P_\star^{n-1}(H)}) = \underbrace{P_\star\big(P_\star(\ldots P_\star(\boldsymbol{H}) \ldots)\big)}_{n},$$

where P is repeated n times into the last formula,

and the empty-set $\emptyset$ (that represents indeterminacy, uncertainty) is not allowed in none of the sequence terms:

$$\boldsymbol{H}, P_\star(\boldsymbol{H}), \boldsymbol{P_\star^2(H)}, \boldsymbol{P_\star^3(H)}, \ldots, \boldsymbol{P_\star^n(H)}.$$

6. Definition of the n^{th}-PowerSet $P^n(H)$ *with* Indeterminacy (represented by the empty-set)

The n^{th}-PowerSet $\boldsymbol{P^n(H)}$ of the set H, that the Neutrosophic SuperHyperStructure is built on, best describes our real world where always the indeterminacy occurs, and a *set* (or *system*) H (that may be a set of items, a company, institution, country, region, etc.) is organized in sub-systems, which in turn are organized in sub-sub-systems, and so on.

The n^{th}-PowerSet $\boldsymbol{P^n(H)}$ is defined recursively:

$$P^0(\boldsymbol{H}) \stackrel{\text{def}}{=} H$$

$$P^1(\boldsymbol{H}) = P(\boldsymbol{H})$$

$$P^2(\boldsymbol{H}) = P(P(\boldsymbol{H}))$$

$$P^3(\boldsymbol{H}) = P\big(P^2(\boldsymbol{H})\big) = P\big(P(P(\boldsymbol{H}))\big)$$

… … … ……………………………………..

$$P^n(\boldsymbol{H}) = P(P^{n-1}(\boldsymbol{H})) = \underbrace{P\big(P(\dots P(\boldsymbol{H})\dots)\big)}_{n},$$

where P is repeated *n* times into the last formula,

and the empty-set $\emptyset$ (that represents indeterminacy, uncertainty) is allowed in all sequence terms:

$$H, P(H), P^2(H), P^3(H), \dots, P^n(H).$$

The n^{th}-PowerSet $P_\star^n(H)$ and $P^n(H)$ of a non-empty set H were introduced by Smarandache [2] in 2016.

7. SuperHyperStructure

The SuperHyperStructure was founded by Smarandache in 2016 [2], who introduced the SuperHyperAlgebra in 2016 and developed it in 2022 [8], SuperHyperGraph in 2019, 2020, 2022 [3, 4, 5], SuperHyperFunction and SuperHyperTopology in 2022 [6], and respectively the SuperHyperOperations, and SuperHyperAxioms [2016-2022].

A SuperHyperStructure is built on the n-th powerset $P_\star^n(H)$ of a non-empty set H, for integer $n \geq 1$, whose **SuperHyperOperators** ($\#_{SH0}$) are defined as follows:

$$\#_{SH0}: (P_\star^r(H))^m \to P_\star^n(H),$$

where $P_\star^r(H)$ is the powerset of H, for integer $r \geq 1$, while similarly $P_\star^n(H)$ is the n-th powerset of H, both without any empty-sets, and the **SuperHyperAxioms** act on it.

8. Neutrosophic SuperHyperStructure

Similarly, a **Neutrosophic SuperHyperStructure** (2016) is built on the n^{th}-powerset $P^n(H)$ of a non-empty set H, for $n \geq 1$, whose Neutrosophic SuperHyperOperators ($\#_{NSH0}$) are defined as follows:

$\#_{NSH0}: \left(P^r(H)\right)^m \to P^n(H)$,

where $P^r(H)$ is the r-powerset of H, for integer $r \geq 1$, while $P^n(H)$ is the n[th]-powerset of H, both containing empty-sets.

9. The Triplet of HyperStructure

As an analogy of the neutrosophic triplet [9 – 19] presented between (2016, 2019 – 2023):

<Algebra, NeutroAlgebra, AntiAlgebra>,

we propose now the following triplet:

<HyperStructure, Neutro-HyperStructure, Anti-HyperStructure>,

that extends Marti's HyperStructure,

where:

— the HyperStructure has all axioms totally (100%) true;

— the Neutro-HyperStructure has at least one axiom which is partially true (*T*), partially indeterminate (*I*), and partially false (*F*); $(T, I, F) \in \{(1,0,0), (0,0,1)\}$, and no axiom is totally (100%) false;

— the Anti-HyperStructure has at least one axiom that is 100% false, or $(T, I, F) = (0, 0, 1)$, no matter how the other axioms are.

10. The Triplet of SuperHyperStructure

One has, as a further extension of the above, the following triplet:

<SuperHyperStructure, Neutro-SuperHyperStructure, Anti-SuperHyperStructure>,

where:

— the SuperHyperStructure has all axioms totally (100%) true;

— the Neutro-SuperHyperStructure has at least one axiom that is partially true (*T*), partially indeterminate (*I*), and partially false (*F*); or $(T, I, F) \in \{(1,0,0), (0,0,1)\}$, and no axiom is totally (100%) false;

— the Anti-SuperHyperStructure has at least one axiom that is totally (100%) false, or $(T, I, F) = (0, 0, 1)$, no matter how are the other axioms.

11. SuperHyperFunction of Many Variable [7]

$$f : (P_*^r S)^m \to P_*^n(S), \text{ for integers } m \geq 2 \text{ and } r, n \geq 0.$$

It is part of the SuperHyperStructure.

12. Example of SuperHyperFunction of Two Variables

Let's take *m = 2, r = 1,* and *n = 2.*

$$f : (P_*(S))^2 \to P_*^2(S)$$

x / y	{1}	{2}	{1, 2}
{1}	{{1}, {2}}	{1}	{{1}, {1 2}}
{2}	{{2}, {1, 2}}	{{1}, {1, 2}}	{2}
{1, 2}	{1, 2}		{{1}. {2}, {1, 2}}

Table 1 of Values of the above SuperHyperFunction of Two Variable f(x, y)

For example, *f({1}, {1, 2}) = {{1}, {1 2}}*, etc.

13. SuperHyperAlgebra

We recall our 2016 concepts of SuperHyperOperation, SuperHyperAxiom, SuperHyperAlgebra, and their corresponding Neutrosophic SuperHyperOperation Neutrosophic SuperHyperAxiom and Neutrosophic SuperHyperAlgebra [2] and developed later in (2019-2024), especially in [8] in 2022.

Let $P_*^n(H)$ be the n^th^-powerset of the set H such that none of $P(H), P^2(H), \ldots, P^n(H)$ contain the empty set ϕ.

Also, let $P^n(H)$ be the n^th^-powerset of the set H such that at least one of the $P^2(H), \ldots, P^n(H)$ contain the empty set ϕ.

The SuperHyperOperations are operations whose codomain is either $P_*^n(H)$ and in this case one has **classical-type SuperHyperOperations**, or $P^n(H)$ and in this case one has **Neutrosophic SuperHyperOperations**, for integer $n \geq 2$.

14. Classical-type Binary SuperHyperOperation

A classical-type Binary SuperHyperOperation $\circ_{(2,n)}^*$ is defined as follows:

$$\circ_{(2,n)}^* : H^2 \to P_*^n(H)$$

where $P_*^n(H)$ is the n^{th}-powerset of the set H, with no empty-set.

15. Examples of classical-type Binary SuperHyperOperation

1) Let $H = \{a, b\}$ be a finite discrete set; then its power set, without the empty-set ϕ, is:

$$P(H) = \{a, b, \{a, b\}\}, \text{ and:}$$

$$P^2(H) = P(P(H)) = P(\{a, b, \{a, b\}\}) = \{a, b, \{a, b\}, \{a, \{a, b\}\}, \{b, \{a, b\}\}, \{a, b, \{a, b\}\}\}.$$

$$\circ_{(2,2)}^* : H^2 \to P_*^2(H)$$

Table 2. Example 1 of classical-type Binary SuperHyperOperation.

$\circ^{*}_{(2,2)}$	a	b
a	$\{a,\{a,b\}\}$	$\{b,\{a,b\}\}$
b	a	$\{a,b,\{a,b\}\}$

16. Classical-type m-ary SuperHyperOperation {or more accurate denomination (m, n)-SuperHyperOperation}

Let U be a universe of discourse and a non-empty set H, $H \subset U$. Then:

$$\circ^{*}_{(m,n)} : H^m \to P_*^n(H)$$

where the integers $m, n \geq 1$,

$$H^m = \underbrace{H \times H \times \ldots \times H}_{m\ times},$$

and $P_*^n(H)$ is the n^{th}-powerset of the set H that includes the empty-set.

This SuperHyperOperation is a m-ary operation defined from the set H to the n^{th}-powerset of the set H.

17. Neutrosophic m-ary SuperHyperOperation {or more accurate denomination Neutrosophic (m, n)-SuperHyperOperation}

Let U be a universe of discourse and a non-empty set H, $H \subset U$. Then:

$$\circ_{(m,n)} : H^m \to P^n(H)$$

where the integers $m, n \geq 1$; $P^n(H)$ - the n-th powerset of the set H that includes the empty-set.

18. SuperHyperAxiom

A **classical-type SuperHyperAxiom** or more accurately a ***(m, n)*-SuperHyperAxiom** is an axiom based on classical-type SuperHyperOperations.

Similarly, a **Neutrosophic SuperHyperAxiom** {or Neutrosphic (m, n)-SuperHyperAxiom} is an axiom based on Neutrosophic SuperHyperOperations.

There are:

- **Strong SuperHyperAxioms**, when the left-hand side is equal to the right-hand side as in non-hyper axioms,
- and **Week SuperHyperAxioms**, when the intersection between the left-hand side and the right-hand side is non-empty.

19. SuperHyperAlgebra and Neutrosophic SuperHyperStructure

A **SuperHyperAlgebra** or more accurately ***(m-n)*-SuperHyperAlgebra** is an algebra dealing with SuperHyperOperations and SuperHyperAxioms.

Again, a **Neutrosophic SuperHyperAlgebra** {or Neutrosphic (m, n)-SuperHyperAlgebra} is an algebra dealing with Neutrosophic SuperHyperOperations and Neutrosophic SuperHyperOperations.

In general, we have **SuperHyperStructures** {or *(m-n)*-SuperHyperStructures}, and corresponding **Neutrosophic SuperHyperStructures**.

For example, there are SuperHyperGrupoid, SuperHyperSemigroup, SuperHyperGroup, SuperHyperRing, SuperHyperVectorSpace, etc.

20. SuperHyperGraph (or n-SuperHyperGraph)

Introduced by F. Smarandache [3, 4, 5] in 2019 and developed in 2020 - 2022.

Let $V = \{v_1, v_2, \dots, v_m\}$, for $1 \leq m \leq \infty$, be a set of finite or infinite number of vertices, that contains Single Vertices (the classical ones), Indeterminate Vertices (unclear, vague, partially known), and Null Vertices (totally unknown, empty).

Let $P(V)$ pe the power of set V, that includes the empty set □ too.

Then $P^n(V)$ be the n-power set of the set V, defined in a recurrent way, i.e.:

P(V), $P^2(V) = P(P(V))$, $P^3(V) = P(P^2(V)) = P(P(P(V)))$, …, $P^n(V) = P(P^{n-1}(V))$,

for $1 \leq n \leq \infty$, where by definition $P^0(V) \overset{def}{=} V$ and $P^1(V) \overset{def}{=} P(V)$.

Then, the SuperHyperGraph (SHG) [or n-SuperHyperGraph (n-SHG)] is an ordered pair:

$$\text{n-SHG} = (G_n, E_n),$$

where $G_n \subseteq P^n(V)$, and $E_n \subseteq P^n(V)$, for $1 \leq n \leq \infty$.

G_n is the set of vertices, and E_n is the set of edges.

The set of vertices G_n contains all possible types of vertices as in our real world:

- Singles Vertices (the classical ones);
- Indeterminate Vertices (unclear, vague, partially unknown);
- Null Vertices (totally unknown, empty);

and:

- SuperVertex (or SubsetVertex), i.e. two or more (single, indeterminate, or null) vertices put together as a group (organization).
- n-SuperVertex that is a collection of many vertices such that at least one is an $(n-1)$- SuperVertex and all the others into the collection are r-SuperVertices, if any, whose order $r \leq n-1$.

The set of edges E_n contains the following types of edges:

- Singles Edges (the classical ones);
- Indeterminate Edges (unclear, vague, partially unknown);
- Null Edges (totally unknown, empty);

and:

- HyperEdge (connecting three or more single vertices);

- SuperEdge (connecting two vertices, at least one of them being a SuperVertex);
- n-SuperEdge (connecting two vertices, at least one being a n-SuperVertex, and the other of order r-SuperVertex, with $r \leq n$);
- SuperHyperEdge (connecting three or more vertices, at least one being a SuperVertex);
- n-SuperHyperEdge (connecting three or more vertices, at least one being a n - SuperVertex, and the other r-SuperVertices with $r \leq n$;
- MultiEdges (two or more edges connecting the same two vertices);
- Loop (and edge that connects an element with

itself). and:

- Directed Graph (classical one);
- Undirected Graph (classical one);
- Neutrosophic Directed Graph (partially directed, partially undirected, partially indeterminate direction).

21. SuperHyperTopology [6, 7]

Let consider τ_{SHT} be a family of subsets of $P_*^n(H)$.

Then τ_{SHT} is called a SuperHyperTopology on $P_*^n(H)$, if it satisfies the following axioms:

(SHT-1) ϕ and $P_*^n(H)$ belong to τ_{SHT}.

(SHT-2) The intersection of any finite number of elements in τ_{SHT} is in τ_{SHT}.

(SHT-3) The union of any finite or infinite number of elements in τ_{SHT} is in τ_{SHT}.

Then $(P_*^n(H), \tau_{SHT})$ is called a SuperHyperTopological Space on $P_*^n(H)$.

22. Neutrosophic SuperHyperTopology [6, 7]

Let consider τ_{NSHT} be a family of subsets of $P^n(H)$.

Then τ_{NSHT} is called a Neutrosophic SuperHyperTopology on $P^n(H)$, if it satisfies the following axioms:

(NSHT-1) ϕ and $P^n(H)$ belong to τ_{NSHT}.

(NSHT-2) The intersection of any finite number of elements in τ_{NSHT} is in τ_{NSHT}.

(NSHT-3) The union of any finite or infinite number of elements in τ_{NSHT} is in τ_{NSHT}.

Then $(P^n(H), \tau_{NSHT})$ is called a Neutrosophic SuperHyperTopological Space on $P^n(H)$.

23. SuperHyperSoft Set

The SuperHyperSoft Set [22, 23] is an extension of the HyperSoft Set [21] and Soft Set [20].

Let $\mathcal{U}$ be a universe of discourse, $\mathcal{P}(\mathcal{U})$ the powerset of $\mathcal{U}$.

Let $a_1, a_2, \ldots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $A_1, A_2, \ldots, A_n$,

with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \ldots, n\}$.
Let $\mathcal{P}(A_1)$, $\mathcal{P}(A_2)$, …, $\mathcal{P}(A_n)$ be the powersets of the sets A_1, A_2, …, A_n respectively.

Then the pair $(F, \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \ldots \times \mathcal{P}(A_n)$, where × meaning Cartesian product, or:

$F: \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \ldots \times \mathcal{P}(A_n) \to \mathcal{P}(\mathcal{U})$

is called a SuperHyperSoft Set.

24. Example of SuperHyperSoft Set

If we define the function:

$F: \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \mathcal{P}(A_3) \times \mathcal{P}(A_4) \to \mathcal{P}(\mathcal{U})$.

we get a *SuperHyperSoft Set*.

Let's assume, from the previous example, that:

$F(\{\text{medium}, \text{tall}\}, \{\text{white}, \text{red}, \text{black}\}, \{\text{female}\}, \{\text{American}, \text{Italian}\}) = \{x_1, x_2\}$, which means that:

$F(\{\text{medium or tall}\} \text{ and } \{\text{white or red or black}\} \text{ and } \{\text{female}\} \text{ and } \{\text{American or Italian}\})$ = $\{x_1, x_2\}$.

Therefore, the SuperHyperSoft Set offers a larger variety of selections, so x_1 and x_2 may be:

either medium, or tall (but not small),

either white, or red, or black (but not yellow),

mandatory female (not male),

and either American, or Italian (but not French, Spanish, Chinese).

In this example there are:

Card{medium, tall}· Card{white, red, black}· Card{female}· Card{American, Italian} = 2·3·1·2 =12 possibilities, where Card{ } means cardinal of the set { }.

This is closer to our everyday life, since for example, when selecting something, we have not been too strict, but accepting some variations (for example: medium or tall, white or red or black, etc.).

25. Fuzzy-Extension-SuperHyperSoft Set

$F: \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \ldots \times \mathcal{P}(A_n) \to \mathcal{P}\left(\mathcal{U}\left(x(d^0)\right)\right)$

where $x(d^0)$ is the fuzzy or any fuzzy-extension degree of appurtenance of the element x to the set $\mathcal{U}$.

Fuzzy-Extensions mean all types of fuzzy sets [3], such as:

Fuzzy Set, Intuitionistic Fuzzy Set, Inconsistent Intuitionistic Fuzzy Set (Picture Fuzzy Set, Ternary Fuzzy Set), Pythagorean Fuzzy Set (Atanassov's Intuitionistic Fuzzy Set of second type), Fermatean Fuzzy Set, q-Rung Orthopair Fuzzy Set, Spherical Fuzzy Set, n-HyperSpherical Fuzzy Set, Neutrosophic Set, Spherical Neutrosophic Set, Refined Fuzzy/Intuitionistic_Fuzzy/Neutrosophic/other_fuzzy_extension Sets, Plithogenic Set, etc.

26. Example of Fuzzy_Extension SuperHyperSoft Set

In the previous example, taking the degree of a generic element $x(d^0)$ as neutrosophic, one gets the Neutrosophic SuperHyperSoft Set.
Assume, that: $F(\{\text{medium}, \text{tall}\}, \{\text{white}, \text{red}, \text{black}\}, \{\text{female}\}, \{\text{American}, \text{Italian}\}) =$

$= \{x_1(0.7, 0.4, 0.1), x_2(0.9, 0.2, 0.3)\}.$

Which means that: x_1 with respect to the attribute values
$(\{\text{medium or tall}\} \text{ and } \{\text{white or red or black}\} \text{ and } \{\text{female}\}, \text{and } \{\text{American or Italian}\})$
has the degree of appurtenance to the set 0.7, the indeterminate degree of appurtenance 0.4, and the degree of non-appurtenance 0.1.
While x_2 has the degree of appurtenance to the set 0.9, the indeterminate degree of appurtenance 0.2, and the degree of non-appurtenance 0.3.

27. Examples of HyperAlgebra and Neutrosophic HyperAlgebra

27.1. Commutative SemiHyperGroup

The SemiHyperGroup is a particular case of HyperAlgebra.
Let $\mathbb{Z}$ be the set of integers, $\mathbb{Z} = \{-\infty, \dots, -2, -1, 0, 1, 2, \dots, +\infty\}$.
Let's define the HyperLaw * as follows:
$*: Z \times Z \to P(Z), x \star y = \{x, y\} \in P(\mathbb{Z}),$
so the law is *well-defined*.
The law is *associative*, since:

$(x \star y) \star z = x \star (y \star z)$

$\{x, y\} \star z = x \star \{y, z\}$

$(x \star z) \cup (y \star z) = (x \star y) \cup (x \star z)$

$\{x, z\} \cup \{y, z\} = \{x, y\} \cup \{x, z\}$

$\{x, y, z\} = \{x, y, z\}$

The law is *commutative*, since
$x \star y = \{x, y\} = \{y, x\} = y \star x$

27.2. Commutative Neutrosophic SemiHyperGroup

The Neutrosophic SemiHyperGroup is a particular case of Neutrosophic HyperAlgebra.

Let the HyperLaw * be defined as:
$\star: (Z \cup \{\emptyset\}) \times (Z \cup \{\emptyset\}) \to P(Z \cup \{\emptyset\})$
where the empty set $\emptyset$ leaves room for indeterminacy, unknown etc.

$$x \star y = \begin{cases} \{x, y\}, \text{for both } x, y \neq \emptyset \\ \emptyset, \ \text{for } x, \text{or } y, \text{or both} = \emptyset \end{cases}$$

The law is well-defined, associative and commutative (proven as above for the SemiHyperGroup).

28. Examples of SuperHyperAlgebra and Neutrosophic SuperHyperAlgebra

28.1. Commutative SuperHyperGrupoid

Let again $\mathbb{Z}$ be the set of integers, $\mathbb{Z} = \{-\infty, \dots, -2, -1, 0, 1, 2, \dots, +\infty\}$.

For 2-nd powerset $P^2(\mathbb{Z}) = P(P(\mathbb{Z}))$ one has two rows of parentheses, one inside the other, of the form: $\{\dots\{\dots\}\dots\}$.

Let's define the binary SuperHyperLaw

$\star: \mathbb{Z}^2 \to P^2(\mathbb{Z})$

$x \star y = \{x, y, \{x, y\}\} \in P^2(\mathbb{Z})$

Clearly the law is *well-defined.*

The law $\star$ is also *commutative,* but *non-associative,* as proven below.

Commutativity:

$x \star y = \{x, y, \{x, y\}\} = \{y, x, \{y, x\}\} = y \star x.$

Non-Associativity:

$$(x \star y) \star z = \{x, y, \{x, y\} \star z\} = \{x \star z, y \star z, \{x, y\} \star z\} = \{x, z, \{x, z\}, y, z, \{y, z\}, x \star z, y \star z\}$$
$$= \{x, z, \{x, z\}, y, z, \{y, z\}, z, z, \{x, z\}, y, z, \{y, z\}\} = \{x, y, z, \{x, z\}, \{y, z\}\}$$
$$x \star (y \star z) = x \star \{y, z, \{y, z\}\} = \{x \star y, x \star z, x \star \{y, z\}\} = \{x, y, \{x, y\}, x, z, \{x, z\}, x \star y, x \star z\}$$
$$= \{x, y, \{x, y\}, x, z, \{x, z\}, x, y, \{x, y\}, x, z, \{x, z\}\} = \{x, y, z, \{x, y\}, \{x, z\}\}$$

Therefore $(x \star y) \star z \neq x \star (y \star z)$.

28.2. Commutative Neutrosophic SuperHyperGrupoid

Similarly, we define:

Let the Neutrosophic SuperHyperLaw * be defined as:

$*: (Z \cup \{\phi\}) \times (Z \cup \{\phi\}) \to P^2(Z \cup \{\phi\})$

where the empty set $\emptyset$ aslo leaves room for indeterminacy, unknown etc.

$$x \star y = \begin{cases} \{x, y, \{x, y\}\}, \text{for both } x, y \neq \emptyset \\ \emptyset, \text{ for } x, \text{or } y, \text{or both} = \emptyset \end{cases}$$

The law is well-defined, non-associative, and commutative (proven as above for the SuperHyperGroupoid).

29. Practical Application of the SuperHyperStructure

Let H be the set (system) that represent all inhabitants of the US country.

The set H is organized into 50 sub-sets, $H_1, H_2, \dots, H_{50}$

that represent the American <u>states</u>: where $H_1, H_2, ..., H_{50} \in P(H)$.

Each state $H_i, 1 \le i \le 50,$ is organized into <u>counties</u>, $H_{ij}, 1 \le j \le i_M$, where i_M is the maximum number of H_i state's counties, with all $H_{i,1}, H_{i,2}, ..., H_{i,i_M} \in P(H_i) \subset P(P(H)) = P^2(H)$.

(One uses commas in between indexes in order to separate them when the values of some indexes have two or more digits, for example $H_{3,12}$ means the 12th county of the 3rd state; which is different from $H_{31,2}$ that means the 2nd county of the 31st state.)

Further on, each $H_{i,j}$ county, for all *i* and *j* indexes,

is organized into <u>sub-counties</u>, $H_{i,j,k}, 1 \le k \le j_M$, where j_M is the maximum number of sub-counties of the county $H_{i,j}$. Therefore:

all $H_{i,j,1}, H_{i,j,2}, ..., H_{i,j,k_m} \in P(H_{i,j}) \subset P(P(H_i)) \subset P(P(P(H))) = P^3(H)$.

This shows the practical application of the n-th powerset of a set, for *n = 3* in this case (three levels of a SuperHyperStructure): country, states (one index *i*), counties (two indexes *i,j*), and sub-counties (three indexes *i,j,k*).

Surely, if needed, one can go *deeper in* (each <u>sub-county</u> is formed by <u>towns</u>, each town by <u>districts</u>, and so on); or *deeper out* (each <u>country</u> is part of a <u>continent</u>, each continent is part of a <u>planet</u>, each planet is part of a <u>solar system</u>, and so on).

The following SuperHyperStructure (denoted below by *A*), with three levels of structures, has been formed as:

$$A = \{H_i, H_{i,j}, H_{i,j,k}, 1 \le i \le 50, 1 \le j \le i_M, 1 \le k \le j_M\} \subset P^3(H).$$

(i) In the real world, this is a Neutrosophic SuperHyperStructure, because it has a lot of indeterminacy: for example the *population* of the country *H* is dynamic, in a continuous change: new people are born while others die as we speak, there are millions of illegal emigrants that are not counted as citizens, others have dual or triple citizenship so they only partially belong to *H*'s population; other citizen live outside the country.

(ii) Many laws are as well neutrosophic, because they apply to some states H_i (as examples: the law of abortion, or the law of bearing arms, or the law of consuming marijuana, etc.), but not to others. We call them NeutroLaws in the NeutroAlgebraic Structures (we mean: laws that are partially true and partially false in the same space, and sometime also partially indeterminate).

(iii) Let's endow this SuperHyperStructure with some SuperHyperLaw (called SuperHyper because it is built on a 3-rd PowerSet of the Set *H*):

$\# : A \times A \to A$

$x \# y = \{x \wedge (x \subseteq y) \to y\}$

{If x and $x \subseteq y$, then y}

Let $x, y \in A$. If x and $x \subseteq y$, then y.

Consider a cyber company that provides internet connection to people from the sub-county $H_{2,3,5}$, which is included in the county $H_{2,3}$, then the company will provide internet connection to the county $H_{2,3}$ as well.
Let $H_{2,3,5}$ and $H_{2,3} \in A$. If $H_{2,3,5}$ gets internet connection and because $H_{2,3,5} \subset H_{2,3}$, then $H_{2,3}$ also gets internet connection.

30. Conclusion

We have extended the SuperHyperAlgebra and its correspondents (SuperHyperGraph, SuperHyperTopology, etc.) to the SuperHyperStructure in general, for any field of knowledge and on any type of space. The SuperHyperStructures were inspired from, and they perfectly fit, our real world. See the last practical application.

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