Neutrosophic Soft Normed Linear Spaces

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Abstract: In this paper, the neutrosophic norm has been defined on a soft linear space which is hereafter called neutrosophic soft normed linear space (NSNLS). Several characteristics of sequences defined in this space have been investigated here. Moreover, the notion of convexity and the metric in NSNLS have been introduced and some of their properties are established.

Keywords: Neutrosophic Soft Norm; Neutrosophic Soft Normed Linear Space (NSNLS); Convergent Sequence; Cauchy Sequence; Convexity; Metric in NSNLS; Neutrosophic Soft Metric Space (NSMS).

1 Introduction

The concept of Neutrosophic Set (NS) was first introduced by Smarandache [4, 5] which is a generalisation of classical sets, fuzzy set, intuitionistic fuzzy set etc. Zadeh's [11] classical concept of fuzzy set is a strong mathematical tool to deal with the complexity generally arising from uncertainty in the form of ambiguity in real life scenario. For different specialized purposes, there are suggestions for nonclassical and higher order fuzzy sets since from the initiation of fuzzy set theory. Among several higher order fuzzy sets, intuitionistic fuzzy sets introduced by Atanassov [10] have been found to be very useful and applicable. But each of these theories has it's different difficulties as pointed out by Molodtsov [3]. The basic reason for these difficulties is inadequacy of parametrization tool of the theories.

Molodtsov [3] presented soft set theory as a completely generic mathematical tool which is free from the parametrization inadequacy syndrome of different theory dealing with uncertainty. Molodtsov successfully applied several directions for the applications of soft set theory, such as smoothness of functions, game theory, operation research, Riemann integration, Perron integration and probability etc. Now, soft set theory and it's applications are progressing rapidly in different fields. The concept of soft point was provided by so many authors but more authentic definition was given in [19]. There is a progressive development of norm linear spaces and inner product spaces over fuzzy set, intuitionistic fuzzy set and soft set by different researchers for instance Dinda and Samanta [1], Felbin [2], Yazar et al. [12], Issac and Maya K. [13], Saadati and Vaezpour [16], Cheng and Mordeson [17], Vijayablaji et al. [18], Das et al. [19-22], Samanta and Jebril [25], Bag and Samanta [26-29], Beaula and Priyanga [30] and many others.

In 2013, Maji [14] has introduced a combined concept Neutrosophic soft set (NS_s). Accordingly, several mathematicians have produced their research works in different mathematical structures for instance Deli [6, 7], Deli and Broumi [8, 9], Maji [15], Broumi et al. [23, 24], Bera and Mahapatra [31-35]. Later, this concept has been modified by Deli and Broumi [9].

In the present study, our aim is to define the neutrosophic norm on a soft linear space and investigate it's several characteristics. Section 2 gives some preliminary necessary definitions which will be used in rest of this paper. The notion of neutrosophic norm over soft linear space and the sequence of soft points in an NSNLS have been introduced in Section 3. Then, there is a study on Cauchy sequence in an NSNLS in Section 4. The concept of convexity and the metric in NSNLS have been developed in Section 5 and Section 6, respectively. Finally, the conclusion of the present work is briefly stated in Section 7.

2 Preliminaries

We recall some basic definitions related to fuzzy set, soft set, neutrosophic soft set for the sake of completeness.

2.1 Definition [33]

A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t - norm if $*$ satisfies the following conditions :
(i) $*$ is commutative and associative.
(ii) $*$ is continuous.
(iii) $a * 1 = 1 * a = a$, $\forall a \in [0,1]$.
(iv) $a * b \le c * d$ if $a \le c, b \le d$ with $a, b, c, d \in [0,1]$.

A few examples of continuous t-norm are $a * b = ab, a * b = \min\{a, b\}, a * b = \max\{a + b - 1, 0\}$.

2.2 Definition [33]

A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is continuous t-conorm (s-norm) if $\diamond$ satisfies the following conditions :
(i) $\diamond$ is commutative and associative.
(ii) $\diamond$ is continuous.
(iii) $a \diamond 0 = 0 \diamond a = a$, $\forall a \in [0,1]$.
(iv) $a \diamond b \le c \diamond d$ if $a \le c, b \le d$ with $a, b, c, d \in [0,1]$.

A few examples of continuous s-norm are $a \diamond b = a + b - ab, a \diamond b = \max\{a, b\}, a \diamond b = \min\{a + b, 1\}$.

2.3 Definition [4]

Let X be a space of points (objects), with a generic element in X denoted by x. A NS B on X is characterized by a truth-membership function T_B, an indeterminacy-membership function I_B and a falsity-membership function F_B where $T_B(x)$, $I_B(x)$ and $F_B(x)$ are real standard or non-standard subsets of $]^-0, 1^+[$ i.e., $T_B, I_B, F_B : X \to]^-0, 1^+[$. Thus the NS B over X is defined as : $B = \{< x, (T_B(x), I_B(x), F_B(x)) > | x \in X\}$.

There is no restriction on the sum of $T_B(x), I_B(x), F_B(x)$ and so, $^-0 \le \sup T_B(x) + \sup I_B(x) + \sup F_B(x) \le 3^+$. Here $1^+ = 1 + \epsilon$, where 1 is it's standard part and ϵ it's non-standard part. Similarly $^-0 = 0 - \epsilon$, where 0 is it's standard part and ϵ it's non-standard part.

From philosophical point of view, a NS takes the value from real standard or nonstandard subsets of $]^-0, 1^+[$. But to practice in real scientific and engineering areas, it is difficult to use NS with value from real standard or nonstandard subset of $]^-0, 1^+[$. Hence we consider the NS which takes the value from the subset of [0,1].

2.4 Definition [3]

Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denote the power set of U. Then for $A \subseteq E$, a pair (F, A) is called a soft set over U, where $F : A \to P(U)$ is a mapping.

2.5 Definition [14]

Let U be an initial universe set and E be a set of parameters. Let $NS(U)$ denote the power set of all NSs of U. Then for $A \subseteq E$, a pair (F, A) is called an NS_s over U, where $F : A \to NS(U)$ is a mapping.

This concept has been redefined by Deli and Broumi [9] as given below.

2.6 Definition [9]

Let U be an initial universe set and E be a set of parameters that describes the elements of U. Let $NS(U)$ denote the power set of all NSs over U. Then, a NS_s N over U is a set defined by a set valued function f_N representing a mapping $f_N : E \to NS(U)$ where f_N is called approximate function of N. In other words, the NS_s N is a parameterized family of some elements of the set $NS(U)$ and therefore it can be written as a set of ordered pairs i.e., $N = \{(e, f_N(e)) : e \in E\}$ where $f_N(e) = \{< x, (T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x)) > | x \in U\}$. Here $T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x) \in [0, 1]$ are respectively called the truth-membership, indeterminacy-membership, falsity-membership function of $f_N(e)$. Since supremum of each T, I, F is 1 so the inequality $0 \leq T_{f_N(e)}(x) + I_{f_N(e)}(x) + F_{f_N(e)}(x) \leq 3$ is obvious.

2.6.1 *Example*

Let $U = \{h_1, h_2, h_3\}$ be a set of houses and $E = \{e_1(\text{beautiful}), e_2(\text{wooden}), e_3(\text{costly})\}$ be a set of parameters with respect to which the nature of houses are described. Let,
$f_N(e_1) = \{< h_1, (0.5, 0.6, 0.3) >, < h_2, (0.4, 0.7, 0.6) >, < h_3, (0.6, 0.2, 0.3) >\}$;
$f_N(e_2) = \{< h_1, (0.6, 0.3, 0.5) >, < h_2, (0.7, 0.4, 0.3) >, < h_3, (0.8, 0.1, 0.2) >\}$;
$f_N(e_3) = \{< h_1, (0.7, 0.4, 0.3) >, < h_2, (0.6, 0.7, 0.2) >, < h_3, (0.7, 0.2, 0.5) >\}$;
Then $N = \{[e_1, f_N(e_1)], [e_2, f_N(e_2)], [e_3, f_N(e_3)]\}$ is a NS_s over (U, E). The tabular representation of the NS_s N is as :

Table 1 : Tabular form of NS_s N

	$f_N(e_1)$	$f_N(e_2)$	$f_N(e_3)$
h_1	(0.5,0.6,0.3)	(0.6,0.3,0.5)	(0.7,0.4,0.3)
h_2	(0.4,0.7,0.6)	(0.7,0.4,0.3)	(0.6,0.7,0.2)
h_3	(0.6,0.2,0.3)	(0.8,0.1,0.2)	(0.7,0.2,0.5)

2.6.2 Definition [9]

Let N_1 and N_2 be two NS_ss over the common universe (U, E). Then their union is denoted by $N_1 \cup N_2 = N_3$ and is defined as :

$$N_3 = \{(e, \{< x, T_{f_{N_3}(e)}(x), I_{f_{N_3}(e)}(x), F_{f_{N_3}(e)}(x) > | x \in U\}) | e \in E\}$$

where $T_{f_{N_3}(e)}(x) = T_{f_{N_1}(e)}(x) \diamond T_{f_{N_2}(e)}(x), I_{f_{N_3}(e)}(x) = I_{f_{N_1}(e)}(x) * I_{f_{N_2}(e)}(x)$, $F_{f_{N_3}(e)}(x) = F_{f_{N_1}(e)}(x) * F_{f_{N_2}(e)}(x)$.

Their intersection is denoted by $N_1 \cap N_2 = N_4$ and is defined as :

$$N_4 = \{(e, \{< x, T_{f_{N_4}(e)}(x), I_{f_{N_4}(e)}(x), F_{f_{N_4}(e)}(x) > | x \in U\}) | e \in E\}$$

where $T_{f_{N_4}(e)}(x) = T_{f_{N_1}(e)}(x) * T_{f_{N_2}(e)}(x), I_{f_{N_4}(e)}(x) = I_{f_{N_1}(e)}(x) \diamond I_{f_{N_2}(e)}(x)$, $F_{f_{N_4}(e)}(x) = F_{f_{N_1}(e)}(x) \diamond F_{f_{N_2}(e)}(x)$.

2.7 Definition [25]

An intuitionistic fuzzy norm on a linear space $V(F)$ is an object of the form $A = \{< (x, t), \mu(x, t), \nu(x, t) > | (x, t) \in V \times R^+\}$, where μ, ν are fuzzy functions on $V \times R^+$, μ denotes the degree of membership and ν denotes the degree of non-membership $(x, t) \in V \times R^+$ satisfying the following conditions :
(i) $\mu(x, t) + \nu(x, t) \leq 1, \forall (x, t) \in V \times R^+$.
(ii) $\mu(x, t) > 0$.
(iii) $\mu(x, t) = 1$ iff $x = \theta$.

(iv) $\mu(cx,t) = \mu(x, \frac{t}{|c|}), \forall c(\neq 0) \in F$.
(v) $\mu(x,s) * \mu(y,t) \leq \mu(x+y, s+t)$.
(vi) $\mu(x,\cdot)$ is non-decreasing function of R^+ and $\lim_{t\to\infty} \mu(x,t) = 1$.
(vii) $\nu(x,t) < 1$.
(viii) $\nu(x,t) = 0$ iff $x = \theta$.
(ix) $\nu(cx,t) = \nu(x, \frac{t}{|c|}), \forall c(\neq 0) \in F$.
(x) $\nu(x,s) \diamond \nu(y,t) \geq \nu(x+y, s+t)$.
(xi) $\nu(x,\cdot)$ is non-increasing function of R^+ and $\lim_{t\to\infty} \nu(x,t) = 0$.

2.8 Definition [19]

A soft set (F,E) over X is said to be a soft point if there is exactly one $e \in E$, such that $F(e) = \{x\}$ and $F(e') = \phi, \forall e' \in E - \{e\}$. It is denoted by x_e. Two soft points $x_e, y_{e'}$ are said to be equal if $e = e'$ and $x = y$.

2.9 Definition [22]

Let V be a vector space over a field K and let A be a parameter set. Let G be a soft set over V so that $G(\lambda)$ is a vector subspace of V, $\forall \lambda \in A$. Then G is called a soft vector space or soft linear space of V over K.

2.10 Definition [12]

Let $SV(\widetilde{X})$ be a soft vector space. Then a mapping $||\cdot|| : SV(\widetilde{X}) \to R^+(E)$ is said to be a soft norm on $SV(\widetilde{X})$, if $||\cdot||$ satisfies the following conditions :
(1) $||x_e|| \geq \tilde{0},\ \forall x_e \in SV(\widetilde{X})$ and $||x_e|| = \tilde{0} \Leftrightarrow x_e = \theta_0$.
(2) $||\tilde{r}x_e|| = |\tilde{r}|\,||x_e||,\ \forall x_e \in SV(\widetilde{X})$ and for every soft scalar $\tilde{r}$.
(3) $||x_e + y_{e'}|| \leq ||x_e|| + ||y_{e'}||,\ \forall x_e, y_{e'} \in SV(\widetilde{X})$.
(4) $||x_e.y_{e'}|| = ||x_e||\,||y_{e'}||,\ \forall x_e, y_{e'} \in SV(\widetilde{X})$.

The soft vector space $SV(\tilde{X})$ with a soft norm $||\cdot||$ on $\widetilde{X}$ is said to be a soft normed linear space and is denoted by $(\widetilde{X}, ||\cdot||)$.

2.11 Definition [32]

Let A be a NS over the universal set X. The (α,β,γ) -cut of A is a crisp subset $A_{(\alpha,\beta,\gamma)}$ of the neutrosophic set A and is defined as $A_{(\alpha,\beta,\gamma)} = \{x \in X : T_A(x) \geq \alpha, I_A(x) \leq \beta, F_A(x) \leq \gamma\}$ where $\alpha,\beta,\gamma \in [0,1]$ with $0 \leq \alpha+\beta+\gamma \leq 3$. This $A_{(\alpha,\beta,\gamma)}$ is called (α,β,γ)-level set or (α,β,γ)-cut set of the neutrosophic set A.

3 Neutrosophic soft norm

In this section, we have defined NSNLS with suitable examples, the convergence of sequence in NSNLS and have studied some related basic properties.

Unless otherwise stated, $V(K)$ is a vector space over the field K and E is treated as the parametric set through out this paper, $e \in E$ an arbitrary parameter.

3.1 Definition

Let $\widetilde{V}$ be a soft linear space over the field K and $R(E)$, $\Delta_{\widetilde{V}}$ denote respectively, the set of all soft real numbers and the set of all soft points on $\widetilde{V}$. Then, a neutrosophic subset N over $\Delta_{\widetilde{V}} \times R(E)$ is called a neutrosophic soft norm on $\widetilde{V}$ if for $x_e, y_{e'} \in \Delta_{\widetilde{V}}$ and $\tilde{c} \in K$ ($\tilde{c}$ being soft scalar), the following conditions hold.
(i) $0 \leq T_N(x_e, \tilde{t}),\ I_N(x_e, \tilde{t}),\ F_N(x_e, \tilde{t}) \leq 1,\ \forall \tilde{t} \in R(E)$.
(ii) $0 \leq T_N(x_e, \tilde{t}) + I_N(x_e, \tilde{t}) + F_N(x_e, \tilde{t}) \leq 3,\ \forall \tilde{t} \in R(E)$.
(iii) $T_N(x_e, \tilde{t}) = 0$ with $\tilde{t} \leq \tilde{0}$.
(iv) $T_N(x_e, \tilde{t}) = 1$ with $\tilde{t} > \tilde{0}$ iff $x_e = \tilde{\theta}$, the null soft vector.
(v) $T_N(\tilde{c}x_e, \tilde{t}) = T_N(x_e, \frac{\tilde{t}}{|\tilde{c}|}),\ \forall \tilde{c}(\neq \tilde{0}),\ \tilde{t} > \tilde{0}$.
(vi) $T_N(x_e, \tilde{s}) * T_N(y_{e'}, \tilde{t}) \leq T_N(x_e \oplus y_{e'}, \tilde{s} \oplus \tilde{t})\ \forall \tilde{s}, \tilde{t} \in R(E)$.
(vii) $T_N(x_e, \cdot)$ is a continuous non-decreasing function for $\tilde{t} > \tilde{0}$ and $\lim_{\tilde{t} \to \infty} T_N(x_e, \tilde{t}) = 1$.
(viii) $I_N(x_e, \tilde{t}) = 1$ with $\tilde{t} \leq \tilde{0}$.
(ix) $I_N(x_e, \tilde{t}) = 0$ with $\tilde{t} > \tilde{0}$ iff $x_e = \tilde{\theta}$, the null soft vector.
(x) $I_N(\tilde{c}x_e, \tilde{t}) = I_N(x_e, \frac{\tilde{t}}{|\tilde{c}|}),\ \forall \tilde{c}(\neq \tilde{0}),\ \tilde{t} > \tilde{0}$.
(xi) $I_N(x_e, \tilde{s}) \diamond I_N(y_{e'}, \tilde{t}) \geq I_N(x_e \oplus y_{e'}, \tilde{s} \oplus \tilde{t}),\ \forall \tilde{s}, \tilde{t} \in R(E)$.
(xii) $I_N(x_e, \cdot)$ is a continuous non-increasing function for $\tilde{t} > \tilde{0}$ and $\lim_{\tilde{t} \to \infty} I_N(x_e, \tilde{t}) = 0$.
(xiii) $F_N(x_e, \tilde{t}) = 1$ with $\tilde{t} \leq \tilde{0}$.
(xiv) $F_N(x_e, \tilde{t}) = 0$ with $\tilde{t} > \tilde{0}$ iff $x_e = \tilde{\theta}$, the null soft vector.
(xv) $F_N(\tilde{c}x_e, \tilde{t}) = F_N(x_e, \frac{\tilde{t}}{|\tilde{c}|}),\ \forall \tilde{c}(\neq \tilde{0}),\ \tilde{t} > \tilde{0}$.
(xvi) $F_N(x_e, \tilde{s}) \diamond F_N(y_{e'}, \tilde{t}) \geq F_N(x_e \oplus y_{e'}, \tilde{s} \oplus \tilde{t}),\ \forall \tilde{s}, \tilde{t} \in R(E)$.
(xvii) $F_N(x_e, \cdot)$ is a continuous non-increasing function for $\tilde{t} > \tilde{0}$ and $\lim_{\tilde{t} \to \infty} F_N(x_e, \tilde{t}) = 0$.

Then $(\widetilde{V}(K), N, *, \diamond)$ is a NSNLS.

3.1.1 *Example*

Let $(\widetilde{V}, ||\cdot||)$ be a soft normed linear space. Take $a * b = ab$ and $a \diamond b = a + b - ab$. Define,

$$T_N(x_e, \tilde{t}) = \begin{cases} \frac{\tilde{t}}{\tilde{t} \oplus ||x_e||} & \text{if } \tilde{t} > ||x_e|| \\ 0 & \text{otherwise .} \end{cases} \quad I_N(x_e, \tilde{t}) = \begin{cases} \frac{||x_e||}{\tilde{t} \oplus ||x_e||} & \text{if } \tilde{t} > ||x_e|| \\ 1 & \text{otherwise .} \end{cases} \quad F_N(x_e, \tilde{t}) = \begin{cases} \frac{||x_e||}{\tilde{t}} & \text{if } \tilde{t} > ||x_e|| \\ 1 & \text{otherwise .} \end{cases}$$

Then $(\widetilde{V}(K), N, *, \diamond)$ is an NSNLS.

Proof. All the conditions are well satisfied. We shall only verify the conditions (vi), (xi), (xvi) for $\tilde{s}, \tilde{t} > \tilde{0}$ because these are obvious for $\tilde{s}, \tilde{t} \leq \tilde{0}$. Now,

$$\begin{aligned} & T_N(x_e \oplus y_{e'}, \tilde{s} \oplus \tilde{t}) - T_N(x_e, \tilde{s}) * T_N(y_{e'}, \tilde{t}) \\ = \ & \frac{\tilde{s} \oplus \tilde{t}}{\tilde{s} \oplus \tilde{t} \oplus ||x_e \oplus y_{e'}||} - \frac{\tilde{s}\tilde{t}}{(\tilde{s} \oplus ||x_e||)(\tilde{t} \oplus ||y_{e'}||)} \\ \geq \ & \frac{\tilde{s} \oplus \tilde{t}}{\tilde{s} \oplus \tilde{t} \oplus ||x_e|| \oplus ||y_{e'}||} - \frac{\tilde{s}\tilde{t}}{(\tilde{s} \oplus ||x_e||)(\tilde{t} \oplus ||y_{e'}||)} \\ = \ & \{(\tilde{s} \oplus \tilde{t})(\tilde{s} \oplus ||x_e||)(\tilde{t} \oplus ||y_{e'}||) - \tilde{s}\tilde{t}(\tilde{s} \oplus \tilde{t} \oplus ||x_e|| \oplus ||y_{e'}||)\}/B \\ & \text{where } B = (\tilde{s} \oplus \tilde{t} \oplus ||x_e|| \oplus ||y_{e'}||)(\tilde{s} \oplus ||x_e||)(\tilde{t} \oplus ||y_{e'}||) \\ = \ & \{\tilde{t}^2||x_e|| \oplus \tilde{s}^2||y_e'|| \oplus (\tilde{s} \oplus \tilde{t})||x_e y_e'||\}/B \geq 0. \end{aligned}$$

Hence, $T_N(x_e,\tilde{s}) * T_N(y_{e'},\tilde{t}) \leq T_N(x_e \oplus y_{e'}, \tilde{s} \oplus \tilde{t}),\ \forall \tilde{s},\tilde{t} \in R(E)$. Next,

$$\begin{aligned}
& I_N(x_e,\tilde{s}) \diamond I_N(y_{e'},\tilde{t}) - I_N(x_e \oplus y_{e'}, \tilde{s} \oplus \tilde{t}) \\
= & \frac{||x_e||}{\tilde{s} \oplus ||x_e||} \oplus \frac{||y_{e'}||}{\tilde{t} \oplus ||y_{e'}||} - \frac{||x_e y_{e'}||}{(\tilde{s} \oplus ||x_e||)(\tilde{t} \oplus ||y_{e'}||)} - \frac{||x_e \oplus y_{e'}||}{||x_e \oplus y_{e'}|| \oplus \tilde{s} \oplus \tilde{t}} \\
= & \frac{||x_e y_{e'}|| \oplus \tilde{t}||x_e|| \oplus \tilde{s}||y_{e'}||}{(\tilde{s} \oplus ||x_e||)(\tilde{t} \oplus ||y_{e'}||)} - \frac{||x_e \oplus y_{e'}||}{||x_e \oplus y_{e'}|| \oplus \tilde{s} \oplus \tilde{t}} \\
= & \{(||x_e \oplus y_{e'}|| \oplus \tilde{s} \oplus \tilde{t})(\tilde{t}||x_e|| \oplus \tilde{s}||y_{e'}|| \oplus ||x_e y_{e'}||) \\
& -||x_e \oplus y_{e'}||(\tilde{s} \oplus ||x_e||)(\tilde{t} \oplus ||y_{e'}||)\}/D \\
& \text{where } D = (\tilde{s} \oplus \tilde{t} \oplus ||x_e \oplus y_{e'}||)(\tilde{s} \oplus ||x_e||)(\tilde{t} \oplus ||y_{e'}||) \\
= & \{(\tilde{s} \oplus \tilde{t})(\tilde{t}||x_e|| \oplus \tilde{s}||y_{e'}|| \oplus ||x_e y_{e'}||) - \tilde{s}\tilde{t}||x_e \oplus y_{e'}||\}/D \\
\geq & \{(\tilde{s} \oplus \tilde{t})(\tilde{t}||x_e|| \oplus \tilde{s}||y_{e'}|| \oplus ||x_e y_{e'}||) - \tilde{s}\tilde{t}(||x_e|| \oplus ||y_{e'}||)\}/D \\
= & \{\tilde{t}^2||x_e|| \oplus \tilde{s}^2||y_{e'}|| \oplus (\tilde{s} \oplus \tilde{t})||x_e y_{e'}||\}/D \geq 0.
\end{aligned}$$

Hence, $I_N(x_e,\tilde{s}) \diamond I_N(y_{e'},\tilde{t}) \geq I_N(x_e \oplus y_{e'}, \tilde{s} \oplus \tilde{t}),\ \forall \tilde{s},\tilde{t} \in R(E)$. Finally,

$$\begin{aligned}
& F_N(x_e,\tilde{s}) \diamond F_N(y_{e'},\tilde{t}) - F_N(x_e \oplus y_{e'}, \tilde{s} \oplus \tilde{t}) \\
= & \frac{||x_e||}{\tilde{s}} \oplus \frac{||y_{e'}||}{\tilde{t}} - \frac{||x_e y_{e'}||}{\tilde{s}\tilde{t}} - \frac{||x_e \oplus y_{e'}||}{\tilde{s} \oplus \tilde{t}} \\
= & \frac{\tilde{t}||x_e|| \oplus \tilde{s}||y_{e'}|| - ||x_e y_{e'}||}{st} - \frac{||x_e \oplus y_{e'}||}{\tilde{s} \oplus \tilde{t}} \\
\geq & \{\tilde{s}^2||y_{e'}|| \oplus \tilde{t}^2||x_e|| - (\tilde{s} \oplus \tilde{t})||x_e y_{e'}||\}/\tilde{s}\tilde{t}(\tilde{s} \oplus \tilde{t}) \\
= & \{\tilde{s}||y_{e'}||(\tilde{s} - ||x_e||) \oplus \tilde{t}||x_e||(\tilde{t} - ||y_{e'}||)\}/\tilde{s}\tilde{t}(\tilde{s} \oplus \tilde{t}) \geq 0 \text{ (as } \tilde{s} > ||x_e||,\ \tilde{t} > ||y_{e'}||).
\end{aligned}$$

Thus, $F_N(x_e,\tilde{s}) \diamond F_N(y_{e'},\tilde{t}) \geq F_N(x_e \oplus y_{e'}, \tilde{s} \oplus \tilde{t}),\ \forall \tilde{s},\tilde{t} \in R(E)$. This completes the proof.

3.2 Definition

Let $\{x^n_{e_n}\}$ be a sequence of soft points in a NSNLS $(\widetilde{V}(K), N, *, \diamond)$. Then the sequence converges to a soft point $x_e \in \widetilde{V}$ iff for given $r \in (0,1), \tilde{t} > \tilde{0}$ there exists $n_0 \in \mathbf{N}$ (the set of natural numbers) such that
$T_N(x^n_{e_n} - x_e, \tilde{t}) > 1 - r, I_N(x^n_{e_n} - x_e, \tilde{t}) < r, F_N(x^n_{e_n} - x_e, \tilde{t}) < r,\ \forall n \geq n_0$. Or,
$\lim_{n\to\infty} T_N(x^n_{e_n} - x_e, \tilde{t}) = 1,\ \lim_{n\to\infty} I_N(x^n_{e_n} - x_e, \tilde{t}) = 0,\ \lim_{n\to\infty} F_N(x^n_{e_n} - x_e, \tilde{t}) = 0$ as $\tilde{t} \to \infty$.
Then the sequence $\{x^n_{e_n}\}$ is called a convergent sequence in the NSNLS $(\widetilde{V}(K), N, *, \diamond)$.

3.3 Theorem

If the sequence $\{x^n_{e_n}\}$ in a NSNLS $(\widetilde{V}(K), N, *, \diamond)$ is convergent, then the point of convergence is unique.

Proof. Let $\lim_{n\to\infty} x^n_{e_n} = x_{e_j}$ and $\lim_{n\to\infty} x^n_{e_n} = y_{e_k}$ for $x_{e_j} \neq y_{e_k}$. Then for $\tilde{s}, \tilde{t} > \tilde{0}$,
$\lim_{n\to\infty} T_N(x^n_{e_n} - x_{e_j}, \tilde{s}) = 1, \lim_{n\to\infty} I_N(x^n_{e_n} - x_{e_j}, \tilde{s}) = 0, \lim_{n\to\infty} F_N(x^n_{e_n} - x_{e_j}, \tilde{s}) = 0$ as $\tilde{s} \to \infty$ and
$\lim_{n\to\infty} T_N(x^n_{e_n} - y_{e_k}, \tilde{t}) = 1, \lim_{n\to\infty} I_N(x^n_{e_n} - y_{e_k}, \tilde{t}) = 0, \lim_{n\to\infty} F_N(x^n_{e_n} - y_{e_k}, \tilde{t}) = 0$ as $\tilde{t} \to \infty$.

Now, $\quad T_N(x_{e_j} - y_{e_k}, \tilde{s} \oplus \tilde{t}) = T_N(x_{e_j} - x^n_{e_n} \oplus x^n_{e_n} - y_{e_k}, \tilde{s} \oplus \tilde{t}) \geq T_N(x^n_{e_n} - x_{e_j}, \tilde{s}) * T_N(x^n_{e_n} - y_{e_k}, \tilde{t})$

Taking limit as $n \to \infty$ and for $\tilde{s}, \tilde{t} \to \infty$,

$T_N(x_{e_j} - y_{e_k}, \tilde{s} \oplus \tilde{t}) \geq 1 * 1 = 1$ i.e., $T_N(x_{e_j} - y_{e_k}, \tilde{s} \oplus \tilde{t}) = 1$ (1)

Further, $I_N(x_{e_j} - y_{e_k}, \tilde{s} \oplus \tilde{t}) = I_N(x_{e_j} - x^n_{e_n} \oplus x^n_{e_n} - y_{e_k}, \tilde{s} \oplus \tilde{t}) \leq I_N(x^n_{e_n} - x_{e_j}, \tilde{s}) \diamond I_N(x^n_{e_n} - y_{e_k}, \tilde{t})$

Taking limit as $n \to \infty$ and for $\tilde{s}, \tilde{t} \to \infty$,
$I_N(x_{e_j} - y_{e_k}, \tilde{s} \oplus \tilde{t}) \leq 0 \diamond 0 = 0$ i.e., $I_N(x_{e_j} - y_{e_k}, \tilde{s} \oplus \tilde{t}) = 0$ (2)
Similarly, $F_N(x_{e_j} - y_{e_k}, \tilde{s} \oplus \tilde{t}) = 0$ (3)
Hence, $x_{e_j} = y_{e_k}$ and this completes the proof.

3.4 Theorem

In an NSNLS $(\widetilde{V}(K), N, *, \diamond)$, if $\lim_{n\to\infty} x^n_{e_n} = x_{e_j}$ and $\lim_{n\to\infty} y^n_{e_n} = y_{e_k}$ then $\lim_{n\to\infty}(x^n_{e_n} \oplus y^n_{e_n}) = x_{e_j} \oplus y_{e_k}$.

Proof. Here, for $\tilde{s}, \tilde{t} > \tilde{0}$,
$\lim_{n\to\infty} T_N(x^n_{e_n} - x_{e_j}, \tilde{s}) = 1, \lim_{n\to\infty} I_N(x^n_{e_n} - x_{e_j}, \tilde{s}) = 0, \lim_{n\to\infty} F_N(x^n_{e_n} - x_{e_j}, \tilde{s}) = 0$ as $\tilde{s} \to \infty$ and $\lim_{n\to\infty} T_N(y^n_{e_n} - y_{e_k}, \tilde{t}) = 1, \lim_{n\to\infty} I_N(y^n_{e_n} - y_{e_k}, \tilde{t}) = 0, \lim_{n\to\infty} F_N(y^n_{e_n} - y_{e_k}, \tilde{t}) = 0$ as $\tilde{t} \to \infty$. Now,

$$\lim_{n\to\infty} T_N[(x^n_{e_n} \oplus y^n_{e_n}) - (x_{e_j} \oplus y_{e_k}), \tilde{s} \oplus \tilde{t}] = \lim_{n\to\infty} T_N[(x^n_{e_n} - x_{e_j}) \oplus (y^n_{e_n} - y_{e_k}), \tilde{s} \oplus \tilde{t}]$$
$$\geq \lim_{n\to\infty} T_N(x^n_{e_n} - x_{e_j}, \tilde{s}) * \lim_{n\to\infty} T_N(y^n_{e_n} - y_{e_k}, \tilde{t}) \quad \text{[by (vi) in Definition 3.1]}$$
$$= 1 * 1 = 1 \text{ as } \tilde{s}, \tilde{t} \to \infty.$$

Hence, $\lim_{n\to\infty} T_N[(x^n_{e_n} \oplus y^n_{e_n}) - (x_{e_j} \oplus y_{e_k}), \tilde{s} \oplus \tilde{t}] = 1$ as $\tilde{s}, \tilde{t} \to \infty$. Again,

$$\lim_{n\to\infty} I_N[(x^n_{e_n} \oplus y^n_{e_n}) - (x_{e_j} \oplus y_{e_k}), \tilde{s} \oplus \tilde{t}] = \lim_{n\to\infty} I_N[(x^n_{e_n} - x_{e_j}) \oplus (y^n_{e_n} - y_{e_k}), \tilde{s} \oplus \tilde{t}]$$
$$\leq \lim_{n\to\infty} I_N(x^n_{e_n} - x_{e_j}, \tilde{s}) \diamond \lim_{n\to\infty} I_N(y^n_{e_n} - y_{e_k}, \tilde{t}) \quad \text{[by (xi) in Definition 3.1]}$$
$$= 0 \diamond 0 = 0 \text{ as } \tilde{s}, \tilde{t} \to \infty.$$

So, $\lim_{n\to\infty} I_N[(x^n_{e_n} \oplus y^n_{e_n}) - (x_{e_j} \oplus y_{e_k}), \tilde{s} \oplus \tilde{t}] = 0$ as $\tilde{s}, \tilde{t} \to \infty$.
Similarly, $\lim_{n\to\infty} F_N[(x^n_{e_n} \oplus y^n_{e_n}) - (x_{e_j} \oplus y_{e_k}), \tilde{s} \oplus \tilde{t}] = 0$ as $\tilde{s}, \tilde{t} \to \infty$ and this ends the theorem.

3.5 Theorem

If $\lim_{n\to\infty} x^n_{e_n} = x_e$ and $\tilde{0} \neq \tilde{c} \in K$, then $\lim_{n\to\infty} \tilde{c}x^n_{e_n} = \tilde{c}x_e$ in an NSNLS $(\widetilde{V}(K), N, *, \diamond)$.

Proof. Here,
$\lim_{n\to\infty} T_N(\tilde{c}x^n_{e_n} - \tilde{c}x_e, \tilde{t}) = \lim_{n\to\infty} T_N(x^n_{e_n} - x_e, \frac{\tilde{t}}{|\tilde{c}|}) = 1$, as $\frac{\tilde{t}}{|\tilde{c}|} \to \infty$.
$\lim_{n\to\infty} I_N(\tilde{c}x^n_{e_n} - \tilde{c}x_e, \tilde{t}) = \lim_{n\to\infty} I_N(x^n_{e_n} - x_e, \frac{\tilde{t}}{|\tilde{c}|}) = 0$, as $\frac{\tilde{t}}{|\tilde{c}|} \to \infty$.
$\lim_{n\to\infty} F_N(\tilde{c}x^n_{e_n} - \tilde{c}x_e, \tilde{t}) = \lim_{n\to\infty} F_N(x^n_{e_n} - x_e, \frac{\tilde{t}}{|\tilde{c}|}) = 0$, as $\frac{\tilde{t}}{|\tilde{c}|} \to \infty$.
Thus the theorem is proved.

4 Cauchy Sequence in NSNLS (Fundamental Sequence in NSNLS)

Here, we have defined the Cauchy sequence in NSNLS, Complete NSNLS and have investigated their several structural characteristics.

4.1 Definition

A sequence $\{x^n_{e_n}\}$ of soft points in an NSNLS $(\widetilde{V}(K), N, *, \diamond)$ is said to be bounded for $r \in (0,1)$ and $\tilde{t} > \tilde{0}$ if the followings hold :

$T_N(x^n_{e_n}, \tilde{t}) > 1 - r, I_N(x^n_{e_n}, \tilde{t}) < r, F_N(x^n_{e_n}, \tilde{t}) < r$, $\forall n \in$ **N** (the set of natural numbers).

4.2 Definition

1. A sequence $\{x^n_{e_n}\}$ of soft points in an NSNLS $(\widetilde{V}(K), N, *, \diamond)$ is said to be a Cauchy sequence if given $r \in (0,1), \tilde{t} > \tilde{0}$ there exists $n_0 \in$ **N** (the set of natural numbers) such that

$T_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) > 1 - r, I_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) < r, F_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) < r,\ \ \forall m, n \geq n_0$. Or,

$\lim_{n,m\to\infty} T_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 1$, $\lim_{n,m\to\infty} I_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 0$, $\lim_{n,m\to\infty} F_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 0$ as $\tilde{t} \to \infty$.

2. Let $\{x^n_{e_n}\}$ be a Cauchy sequence of soft points in a soft normed linear space $(\widetilde{V}, ||\cdot||)$. Then $\lim_{n,m\to\infty} ||x^n_{e_n} - x^m_{e_m}|| = 0$ hold.

4.2.1 *Example*

For $\tilde{t} > \tilde{0}$, let $T_N(x_e, \tilde{t}) = \frac{\tilde{t}}{\tilde{t} \oplus ||x_e||}$, $I_N(x_e, \tilde{t}) = \frac{||x_e||}{\tilde{t} \oplus ||x_e||}$, $F_N(x_e, \tilde{t}) = \frac{||x_e||}{\tilde{t}}$.

Then $(\widetilde{V}(K), N, *, \diamond)$ is an NSNLS. Now,

$$\lim_{n,m\to\infty} \frac{\tilde{t}}{\tilde{t} \oplus ||x^n_{e_n} - x^m_{e_m}||} = 1,\ \lim_{n,m\to\infty} \frac{||x^n_{e_n} - x^m_{e_m}||}{\tilde{t} \oplus ||x^n_{e_n} - x^m_{e_m}||} = 0, \lim_{n,m\to\infty} \frac{||x^n_{e_n} - x^m_{e_m}||}{\tilde{t}} = 0.$$
$$\Rightarrow \lim_{n,m\to\infty} T_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 1,\ \lim_{n,m\to\infty} I_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 0, \lim_{n,m\to\infty} F_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 0 \text{ as } \tilde{t} \to \infty.$$

This shows that $\{x^n_{e_n}\}$ is a Cauchy sequence in the NSNLS $(\widetilde{V}(K), N, *, \diamond)$.

4.3 Theorem

Every convergent sequence of soft points in a NSNLS $(\widetilde{V}(K), N, *, \diamond)$ is a Cauchy sequence.

Proof. Let $\{x^n_{e_n}\}$ be a convergent sequence of soft points in a NSNLS $(\widetilde{V}(K), N, *, \diamond)$ so that $\lim_{n\to\infty} x^n_{e_n} = x_e$. Then for $\tilde{t} > \tilde{0}$,

$$\lim_{n,m\to\infty} T_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = \lim_{n,m\to\infty} T_N(x^n_{e_n} - x^m_{e_m} \oplus x_e - x_e, \tilde{t}) = \lim_{n,m\to\infty} T_N[(x^n_{e_n} - x_e) \oplus (x_e - x^m_{e_m}), \tilde{t}]$$
$$\geq \lim_{n\to\infty} T_N(x^n_{e_n} - x_e, \frac{\tilde{t}}{2}) * \lim_{m\to\infty} T_N(x_e - x^m_{e_m}, \frac{\tilde{t}}{2}) \quad \text{[by (vi) in Definition 3.1]}$$
$$= \lim_{n\to\infty} T_N(x^n_{e_n} - x_e, \frac{\tilde{t}}{2}) * \lim_{m\to\infty} T_N(x^m_{e_m} - x_e, \frac{\tilde{t}}{2}) \quad \text{[by (v) in Definition 3.1]}$$
$$= 1 * 1 = 1 \text{ as } \tilde{t} \to \infty.$$

So, $\lim_{n,m\to\infty} T_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 1$. Again,

$$\lim_{n,m\to\infty} I_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = \lim_{n,m\to\infty} I_N(x^n_{e_n} - x^m_{e_m} \oplus x_e - x_e, \tilde{t}) = \lim_{n,m\to\infty} I_N[(x^n_{e_n} - x_e) \oplus (x_e - x^m_{e_m}), \tilde{t}]$$
$$\geq \lim_{n\to\infty} I_N(x^n_{e_n} - x_e, \frac{\tilde{t}}{2}) \diamond \lim_{m\to\infty} I_N(x_e - x^m_{e_m}, \frac{\tilde{t}}{2}) \quad \text{[by (xi) in Definition 3.1]}$$

$= \lim_{n\to\infty} I_N(x^n_{e_n} - x_e, \frac{\tilde{t}}{2}) \diamond \lim_{m\to\infty} I_N(x^m_{e_m} - x_e, \frac{\tilde{t}}{2})$ [by (x) in Definition 3.1]
$= 0 \diamond 0 = 0$ as $\tilde{t} \to \infty$.

So, $\lim_{n,m\to\infty} I_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 0$ and similarly, $\lim_{n,m\to\infty} F_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 0$.
Hence, $\{x^n_{e_n}\}$ is a Cauchy sequence.

4.3.1 *Example*

The following example will clarify that the inverse of the Theorem 4.3 may not be true.
Let $R_1 = \{\frac{1}{n} | n \in \mathbf{N}\}$ (**N** being the set of natural numbers) be a subset of real numbers and $||x|| = |x|$. With respect to the neutrosophic norm defined in [4.2.1], obviously $(\widetilde{R}_1(R), N, *, \diamond)$ is an NSNLS. Now,

$$\lim_{n,m\to\infty} \frac{\tilde{t}}{\tilde{t} \oplus ||x^n_{e_n} - x^m_{e_m}||} = \lim_{n,m\to\infty} \frac{\tilde{t}}{\tilde{t} \oplus |\frac{1}{n} - \frac{1}{m}|} = 1, \ \lim_{n,m\to\infty} \frac{||x^n_{e_n} - x^m_{e_m}||}{\tilde{t} \oplus ||x^n_{e_n} - x^m_{e_m}||} = \lim_{n,m\to\infty} \frac{|\frac{1}{n} - \frac{1}{m}|}{\tilde{t} \oplus |\frac{1}{n} - \frac{1}{m}|} = 0,$$
$$\text{and } \lim_{n,m\to\infty} \frac{||x^n_{e_n} - x^m_{e_m}||}{\tilde{t}} = \lim_{n,m\to\infty} \frac{|\frac{1}{n} - \frac{1}{m}|}{\tilde{t}} = 0.$$

Thus $\{x^n_{e_n}\}$ is a Cauchy sequence of soft point in the NSNLS $(\widetilde{R}_1(R), N, *, \diamond)$. But,

$\lim_{n\to\infty} I_N(x^n_{e_n} - x^k_{e_k}, \tilde{t}) = \lim_{n\to\infty} \frac{|\frac{1}{n} - \frac{1}{k}|}{\tilde{t} \oplus |\frac{1}{n} - \frac{1}{k}|} \neq 0.$
This shows that the Cauchy sequence $\{x^n_{e_n}\}$ is not convergent in that NSNLS.

4.4 Theorem

Every Cauchy sequence is bounded in an NSNLS $(\widetilde{V}(K), N, *, \diamond)$ if $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for any two real numbers $a, b \in [0, 1]$.

Proof. Let $\{x^n_{e_n}\}$ be a Cauchy sequence. Then given a fixed $r_0 \in (0, 1)$ and $\tilde{t}' > \tilde{0}$, there exists a natural number n_0 such that $T_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}') > 1 - r_0, \ \forall m, n \geq n_0$.
Since $\lim_{\tilde{t}\to\infty} T_N(x_e, \tilde{t}) = 1$, for each $x^i_{e_i}$ there exists $\tilde{t}_i > \tilde{0}$ such that $T_N(x^i_{e_i}, \tilde{t}) > 1 - r_0, \ \forall \tilde{t} \geq \tilde{t}_i, \ i = 1, 2, 3, \cdots$; Let $\tilde{t}_0 = \tilde{t}' + \max\{\tilde{t}_1, \tilde{t}_2, \cdots, \tilde{t}_{n_0}\}$. Then,

$T_N(x^n_{e_n}, \tilde{t}_0) > T_N(x^n_{e_n}, \tilde{t}' \oplus \tilde{t}_{n_0})$ [by (vii) in Definition (3.1) (i.e., the monotonicity property)]
$= T_N(x^n_{e_n} - x^{n_0}_{e_{n_0}} \oplus x^{n_0}_{e_{n_0}}, \tilde{t}' \oplus \tilde{t}_{n_0}) \geq T_N(x^n_{e_n} - x^{n_0}_{e_{n_0}}, \tilde{t}') * T_N(x^{n_0}_{e_{n_0}}, \tilde{t}_{n_0})$ [by (vi) in Definition (3.1)]
$> (1 - r_0) * (1 - r_0) = 1 - r_0, \ \forall n \geq n_0.$

Thus $T_N(x^n_{e_n}, \tilde{t}_0) > 1 - r_0, \ \forall n \geq n_0$ and further $T_N(x^n_{e_n}, \tilde{t}_0) \geq T_N(x^n_{e_n}, \tilde{t}_n) > 1 - r_0, \ \forall n = 1, 2, \cdots, n_0$.
Hence as a whole, $T_N(x^n_{e_n}, \tilde{t}_0) > 1 - r_0$. (4)

Next, for $r_0 \in (0, 1)$ and $\tilde{t}' > \tilde{0}$, there exists a natural number n_1 such that $I_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}') < r_0, \ \forall m, n \geq n_1$. Since $\lim_{\tilde{t}\to\infty} I_N(x_e, \tilde{t}) = 0$, for each $x^i_{e_i}$ there exists $\tilde{t}'_i > 0$ such that $I_N(x^i_{e_i}, \tilde{t}) < r_0, \ \forall \tilde{t} \geq \tilde{t}'_i, \ i = 1, 2, 3, \cdots$; Let $\tilde{t}'_0 = \tilde{t}' + \max\{\tilde{t}'_1, \tilde{t}'_2, \cdots \tilde{t}'_{n_1}\}$. Then,

$I_N(x^n_{e_n}, \tilde{t}'_0) < I_N(x^n_{e_n}, \tilde{t}' \oplus \tilde{t}'_{n_1})$ [by (xii) in Definition (3.1) (i.e., the monotonicity property)]
$= I_N(x^n_{e_n} - x^{n_1}_{e_{n_1}} \oplus x^{n_1}_{e_{n_1}}, \tilde{t}' \oplus \tilde{t}'_{n_1}) \leq I_N(x^n_{e_n} - x^{n_1}_{e_{n_1}}, \tilde{t}') \diamond I_N(x^{n_1}_{e_{n_1}}, \tilde{t}'_{n_1})$ [by (xi) in Definition (3.1)]
$< r_0 \diamond r_0 = r_0, \ \forall n \geq n_1.$

Thus $I_N(x^n_{e_n}, \tilde{t}'_0) < r_0,\ \forall n \geq n_1$ and further $I_N(x^n_{e_n}, \tilde{t}'_0) \leq I_N(x^n_{e_n}, \tilde{t}'_n) < r_0 \forall n = 1, 2, \cdots, n_1$.
Hence as a whole, $I_N(x^n_{e_n}, \tilde{t}'_0) < r_0$. (5)

Finally, for $r_0 \in (0, 1)$ and $\tilde{t}' > \tilde{0}$, there exists a natural number n_2 such that $F_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}') < r_0,\ \forall m, n \geq n_2$. Since $\lim_{\tilde{t}\to\infty} F_N(x_e, \tilde{t}) = 0$, for each $x^i_{e_i}$ there exists $\tilde{t}''_i > 0$ such that $F_N(x^i_{e_i}, \tilde{t}) < r_0,\ \forall \tilde{t} \geq \tilde{t}''_i,\ i = 1, 2, 3, \cdots$; Let $\tilde{t}''_0 = \tilde{t}' + \max\{\tilde{t}''_1, \tilde{t}''_2, \cdots \tilde{t}''_{n_2}\}$. Then,

$$F_N(x^n_{e_n}, \tilde{t}''_0) < F_N(x^n_{e_n}, \tilde{t}' \oplus \tilde{t}'_{n_2}) \text{ [by (xvii) in Definition (3.1) (i.e., the monotonicity property)]}$$
$$= F_N(x^n_{e_n} - x^{n_2}_{e_{n_2}} \oplus x^{n_2}_{e_{n_2}}, \tilde{t}' \oplus \tilde{t}''_{n_2}) \leq F_N(x^n_{e_n} - x^{n_2}_{e_{n_2}}, \tilde{t}') \diamond F_N(x^{n_2}_{e_{n_2}}, \tilde{t}''_{n_2}) \text{ [by (xvi) in Definition (3.1)]}$$
$$< r_0 \diamond r_0 = r_0\ \ \forall n \geq n_2$$

Thus $F_N(x^n_{e_{n_2}}, \tilde{t}''_0) < r_0,\ \forall n \geq n_2$ and further $F_N(x^n_{e_n}, \tilde{t}''_0) \leq F_N(x^n_{e_n}, \tilde{t}''_n) < r_0, \forall n = 1, 2, \cdots, n_2$.
Hence as a whole, $F_N(x^n_{e_n}, \tilde{t}''_0) < r_0$. (6)

This completes the theorem.

4.5 Theorem

In an NSNLS $(\widetilde{V}(K), N, *, \diamond)$, if $\{x^n_{e_n}\}$, $\{y^n_{e_n}\}$ are Cauchy sequences of soft vectors and $\{\tilde{\lambda}_n\}$ is a Cauchy sequence of soft scalars in an NSNLS $(\widetilde{V}(K), N, *, \diamond)$, then $\{x^n_{e_n} \oplus y^n_{e_n}\}$ and $\{\tilde{\lambda}_n y^n_{e_n}\}$ are also Cauchy sequences in NSNLS $(\widetilde{V}(K), N, *, \diamond)$.

Proof. For $\tilde{t} > \tilde{0}$, we have,
$\lim_{n,m\to\infty} T_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 1, \lim_{n,m\to\infty} I_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 0, \lim_{n,m\to\infty} F_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 0$ as $\tilde{t} \to \infty$ and
$\lim_{n,m\to\infty} T_N(y^n_{e_n} - y^m_{e_m}, \tilde{t}) = 1, \lim_{n,m\to\infty} I_N(y^n_{e_n} - y^m_{e_m}, \tilde{t}) = 0, \lim_{n,m\to\infty} F_N(y^n_{e_n} - y^m_{e_m}, \tilde{t}) = 0$ as $\tilde{t} \to \infty$.

$$\lim_{n,m\to\infty} T_N[(x^n_{e_n} \oplus y^n_{e_n}) - (x^m_{e_m} \oplus y^m_{e_m}), \tilde{t}] = \lim_{n,m\to\infty} T_N[(x^n_{e_n} - x^m_{e_m}) \oplus (y^n_{e_n} - y^m_{e_m}), \tilde{t}]$$
$$\geq \lim_{n,m\to\infty} T_N(x^n_{e_n} - x^m_{e_m}, \frac{\tilde{t}}{2}) * \lim_{n,m\to\infty} T_N(y^n_{e_n} - y^m_{e_m}, \frac{\tilde{t}}{2}) = 1 * 1 = 1 \text{ as } \tilde{t} \to \infty.$$

Hence, $\lim_{n,m\to\infty} T_N[(x^n_{e_n} \oplus y^n_{e_n}) - (x^m_{e_m} \oplus y^m_{e_m}), \tilde{t}] = 1$ as $\tilde{t} \to \infty$.

$$\lim_{n,m\to\infty} I_N[(x^n_{e_n} \oplus y^n_{e_n}) - (x^m_{e_m} \oplus y^m_{e_m}), \tilde{t}] = \lim_{n,m\to\infty} I_N[(x^n_{e_n} - x^m_{e_m}) \oplus (y^n_{e_n} - y^m_{e_m}), \tilde{t}]$$
$$\leq \lim_{n,m\to\infty} I_N(x^n_{e_n} - x^m_{e_m}, \frac{\tilde{t}}{2}) \diamond \lim_{n,m\to\infty} I_N(y^n_{e_n} - y^m_{e_m}, \frac{\tilde{t}}{2}) = 0 \diamond 0 = 0 \text{ as } \tilde{t} \to \infty$$

So, $\lim_{n,m\to\infty} I_N[(x^n_{e_n} \oplus y^n_{e_n}) - (x^m_{e_m} \oplus y^m_{e_m}), \tilde{t}] = 0$ as $\tilde{t} \to \infty$.
Similarly, $\lim_{n,m\to\infty} F_N[(x^n_{e_n} \oplus y^n_{e_n}) - (x^m_{e_m} \oplus y^m_{e_m}), \tilde{t}] = 0$ as $\tilde{t} \to \infty$.

This ends the first part. For the next part,

$$\lim_{m,n\to\infty} T_N[(\tilde{\lambda}_m y^m_{e_m} - \tilde{\lambda}_n y^n_{e_n}), \tilde{t}\,] = \lim_{m,n\to\infty} T_N[(\tilde{\lambda}_m y^m_{e_m} - \tilde{\lambda}_n y^n_{e_n}) \oplus (\tilde{\lambda}_m y^n_{e_n} - \tilde{\lambda}_m y^n_{e_n}), \tilde{t}]$$
$$= \lim_{m,n\to\infty} T_N[\tilde{\lambda}_m(y^m_{e_m} - y^n_{e_n}) \oplus y^n_{e_n}(\tilde{\lambda}_m - \tilde{\lambda}_n), \tilde{t}\,] \geq \lim_{m,n\to\infty} T_N[(y^m_{e_m} - y^n_{e_n}), \frac{\tilde{t}}{2|\tilde{\lambda}_m|}] * T_N(y^n_{e_n}, \frac{\tilde{t}}{2|\tilde{\lambda}_m - \tilde{\lambda}_n|})$$

Since $|\tilde{\lambda}_m - \tilde{\lambda}_n| \to \tilde{0}$ as $m, n \to \infty$, so $|\tilde{\lambda}_m - \tilde{\lambda}_n| \neq \tilde{0}$. Again $\{y^n_{e_n}\}$ being Cauchy sequence is bounded.

Hence, $\lim_{m,n\to\infty} T_N[(\tilde{\lambda}_m y^m_{e_m} - \tilde{\lambda}_n y^n_{e_n}), \tilde{t}] = 1$ as $\tilde{t} \to \infty$. Further,

$$\lim_{m,n\to\infty} I_N[(\tilde{\lambda}_m y^m_{e_m} - \tilde{\lambda}_n y^n_{e_n}), \tilde{t}] = \lim_{m,n\to\infty} I_N[(\tilde{\lambda}_m y^m_{e_m} - \tilde{\lambda}_n y^n_{e_n}) \oplus (\tilde{\lambda}_m y^n_{e_n} - \tilde{\lambda}_m y^n_{e_n}), \tilde{t}]$$
$$= \lim_{m,n\to\infty} I_N[\tilde{\lambda}_m(y^m_{e_m} - y^n_{e_n}) \oplus y^n_{e_n}(\tilde{\lambda}_m - \tilde{\lambda}_n), \tilde{t}] \leq \lim_{m,n\to\infty} I_N[(y^m_{e_m} - y^n_{e_n}), \frac{\tilde{t}}{2|\tilde{\lambda}_m|}] \diamond I_N(y^n_{e_n}, \frac{\tilde{t}}{2|\tilde{\lambda}_m - \tilde{\lambda}_n|})$$

By similar argument, $\lim_{m,n\to\infty} I_N[(\tilde{\lambda}_m y^m_{e_m} - \tilde{\lambda}_n y^n_{e_n}), \tilde{t}] = 0$ as $\tilde{t} \to \infty$ and finally, $\lim_{m,n\to\infty} F_N[(\tilde{\lambda}_m y^m_{e_m} - \tilde{\lambda}_n y^n_{e_n}), \tilde{t}] = 0$ as $\tilde{t} \to \infty$. Hence, the 2nd part is completed.

4.6 Definition

Let $(\widetilde{V}(K), N, *, \diamond)$ be a NSNLS and $\Delta_{\widetilde{V}}$ be the collection of all soft points on $\widetilde{V}$. Then $(\widetilde{V}(K), N, *, \diamond)$ is said to be a complete NSNLS if every Cauchy sequence of soft points in $\Delta_{\widetilde{V}}$ converges to a soft point of $\Delta_{\widetilde{V}}$.

4.7 Theorem

In an NSNLS $(\widetilde{V}(K), N, *, \diamond)$, if every Cauchy sequence has a convergent subsequence then $(\widetilde{V}(K), N, *, \diamond)$ is a complete NSNLS.

Proof. Let $\{x^{n_k}_{e_{n_k}}\}$ be a convergent subsequence of a Cauchy sequence $\{x^n_{e_n}\}$ in an NSNLS $(\widetilde{V}(K), N, *, \diamond)$ such that $\{x^{n_k}_{e_{n_k}}\} \to x_e \in \widetilde{V}$. Since $\{x^n_{e_n}\}$ be a Cauchy sequence in $(\widetilde{V}(K), N, *, \diamond)$, given $\tilde{t} > \tilde{0}$
$\lim_{n,k\to\infty} T_N(x^n_{e_n} - x^{n_k}_{e_{n_k}}, \frac{\tilde{t}}{2}) = 1, \lim_{n,k\to\infty} I_N(x^n_{e_n} - x^{n_k}_{e_{n_k}}, \frac{\tilde{t}}{2}) = 0, \lim_{n,k\to\infty} F_N(x^n_{e_n} - x^{n_k}_{e_{n_k}}, \frac{\tilde{t}}{2}) = 0$ as $\tilde{t} \to \infty$
Again since $\{x^{n_k}_{e_{n_k}}\}$ converges to x_e, then
$\lim_{k\to\infty} T_N(x^{n_k}_{e_{n_k}} - x_e, \frac{\tilde{t}}{2}) = 1, \lim_{k\to\infty} I_N(x^{n_k}_{e_{n_k}} - x_e, \frac{\tilde{t}}{2}) = 0, \lim_{k\to\infty} F_N(x^{n_k}_{e_{n_k}} - x_e, \frac{\tilde{t}}{2}) = 0$ as $\tilde{t} \to \infty$.
Now, $T_N(x^n_{e_n} - x_e, \tilde{t}) = T_N(x^n_{e_n} - x^{n_k}_{e_{n_k}} \oplus x^{n_k}_{e_{n_k}} - x_e, \tilde{t}) \geq T_N(x^n_{e_n} - x^{n_k}_{e_{n_k}}, \frac{\tilde{t}}{2}) * T_N(x^{n_k}_{e_{n_k}} - x_e, \frac{\tilde{t}}{2})$.
It implies $\lim_{n\to\infty} T_N(x^n_{e_n} - x_e, \tilde{t}) = 1$.
Further, $I_N(x^n_{e_n} - x_e, \tilde{t}) = I_N(x^n_{e_n} - x^{n_k}_{e_{n_k}} \oplus x^{n_k}_{e_{n_k}} - x_e, \tilde{t}) \leq I_N(x^n_{e_n} - x^{n_k}_{e_{n_k}}, \frac{\tilde{t}}{2}) \diamond I_N(x^{n_k}_{e_{n_k}} - x_e, \frac{\tilde{t}}{2})$.
It implies $\lim_{n\to\infty} I_N(x^n_{e_n} - x_e, \tilde{t}) = 0$. Similarly, $\lim_{n\to\infty} F_N(x^n_{e_n} - x_e, \tilde{t}) = 0$.
This shows that $\{x^n_{e_j}\}$ converges to $x_{e_j} \in \widetilde{V}$ and thus the theorem is proved.

5 Convexity of NSNLS

Here, the notion of convex NSNLS has been introduced along with the development of some basic theorems.

5.1 Definition

Let $(\widetilde{V}(K), N, *, \diamond)$ be a neutrosophic soft normed linear space and $x_{e_i}, y_{e_j} \in \Delta_{\widetilde{V}}$. Then the set of all soft points of the form $z_{e_k} = \tilde{c}x_{e_i} \oplus (\tilde{1} - \tilde{c})y_{e_j}$ such that $\tilde{c}(e) \in (0, 1), \forall e \in E$ is called the line segment joining the soft points x_{e_i}, y_{e_j}. A soft set $(\widetilde{\phi} \neq \widetilde{W}(K) \subset \widetilde{V}(K))$ is said to form a convex NSNLS with respect to the same neutrosophic soft norm as defined on $\widetilde{V}$ if all the line segments joining any two soft points of $\widetilde{W}$ are contained in $\widetilde{W}$ and satisfy all the neutrosophic soft norm axioms.

5.2 Definition

A soft subset $\widetilde{W}$ of $\widetilde{V}$ in an NSNLS $(\widetilde{V}(K), N, *, \diamond)$ is said to be bounded if for given $r \in (0,1)$ and $\tilde{t} > \tilde{0}$, the following inequalities hold.

$$T_N(x_e, \tilde{t}) > 1 - r, I_N(x_e, \tilde{t}) < r, F_N(x_e, \tilde{t}) < r, \ \forall x_e \in \widetilde{W}\,. \tag{7}$$

5.3 Definition

Let $(\widetilde{V}(K), N, *, \diamond)$ be a NSNLS and $\tilde{t} \in R^+(E)$ (the set of all non-negative soft real numbers). Then an open ball and a closed ball with centre at x_e and radius $r \in (0,1)$ are as follows :

$$OB(x_e, r, \tilde{t}) = \{y_{e'} \in \Delta_{\widetilde{V}} | T_N(x_e - y_{e'}, \tilde{t}) > 1 - r, I_N(x_e - y_{e'}, \tilde{t}) < r, F_N(x_e - y_{e'}, \tilde{t}) < r\}. \tag{8}$$

$$CB[x_e, r, \tilde{t}] = \{y_{e'} \in \Delta_{\widetilde{V}} | T_N(x_e - y_{e'}, \tilde{t}) \geq 1 - r, I_N(x_e - y_{e'}, \tilde{t}) \leq r, F_N(x_e - y_{e'}, \tilde{t}) \leq r\}. \tag{9}$$

5.4 Theorem

Every open ball (closed ball) in an NSNLS is convex and bounded if $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for any two real numbers $a, b \in [0, 1]$.

Proof. Let $OB(x_e, r, \tilde{t})$ be an open ball with centre x_e and radius r in an NSNLS $(\widetilde{V}(K), N, *, \diamond)$. Suppose $y_{e_j}, z_{e_k} \in OB(x_e, r, \tilde{t})$. Then,

$T_N(x_e - y_{e_j}, \tilde{t}) > 1 - r,\ I_N(x_e - y_{e_j}, \tilde{t}) < r,\ F_N(x_e - y_{e_j}, \tilde{t}) < r$ and

$T_N(x_e - z_{e_k}, \tilde{t}) > 1 - r,\ I_N(x_e - z_{e_k}, \tilde{t}) < r,\ F_N(x_e - z_{e_k}, \tilde{t}) < r$.

Now, for $\tilde{c} \in (\tilde{0}, \tilde{1})$ ($\tilde{c}$ being a soft scalar),

$$T_N[x_e - (\tilde{c}y_{e_j} \oplus (\tilde{1} - \tilde{c})z_{e_k}), \tilde{t}\,] = T_N[(\tilde{c} \oplus \tilde{1} - \tilde{c})x_e - (\tilde{c}y_{e_j} \oplus (\tilde{1} - \tilde{c})z_{e_k}), \tilde{t}\,]$$

$$= T_N[\tilde{c}(x_e - y_{e_j}) \oplus (\tilde{1} - \tilde{c})(x_e - z_{e_k}), \tilde{t}\,] \geq T_N[\tilde{c}(x_e - y_{e_j}), \frac{\tilde{t}}{2}] * T_N[(\tilde{1} - \tilde{c})(x_e - z_{e_k}), \frac{\tilde{t}}{2}]$$

$$= T_N[(x_e - y_{e_j}), \frac{\tilde{t}}{2|\tilde{c}|}] * T_N[(x_e - z_{e_k}), \frac{\tilde{t}}{2|(\tilde{1} - \tilde{c})|}] > (1 - r) * (1 - r) = 1 - r.$$

$$I_N[x_e - (\tilde{c}y_{e_j} \oplus (\tilde{1} - \tilde{c})z_{e_k}), \tilde{t}\,] = I_N[(\tilde{c} \oplus \tilde{1} - \tilde{c})x_e - (\tilde{c}y_{e_j} \oplus (\tilde{1} - \tilde{c})z_{e_k}), \tilde{t}\,]$$

$$= I_N[\tilde{c}(x_e - y_{e_j}) \oplus (\tilde{1} - \tilde{c})(x_e - z_{e_k}), \tilde{t}\,] \leq I_N[\tilde{c}(x_e - y_{e_j}), \frac{\tilde{t}}{2}] \diamond I_N[(\tilde{1} - \tilde{c})(x_e - z_{e_k}), \frac{\tilde{t}}{2}]$$

$$= I_N[(x_e - y_{e_j}), \frac{\tilde{t}}{2|\tilde{c}|}] \diamond I_N[(x_e - z_{e_k}), \frac{\tilde{t}}{2|(\tilde{1} - \tilde{c})|}] < r \diamond r = r.$$

Similarly, $F_N[x_e - (\tilde{c}y_{e_j} \oplus (\tilde{1} - \tilde{c})z_{e_k}), \tilde{t}] < r$.

This shows that $[\tilde{c}y_{e_j} \oplus (\tilde{1} - \tilde{c})z_{e_k}] \in B(x_e, r, \tilde{t})$ with respect to the neutrosophic soft norm N. Hence, the 1st part is completed.

For the 2nd part, let $y_{e_j} \in B(x_e, r, \tilde{t})$ an arbitrary soft point. Then $T_N(x_e - y_{e_j}, \tilde{t}) > 1 - r, I_N(x_e - y_{e_j}, \tilde{t}) < r$, $F_N(x_e - y_{e_j}, \tilde{t}) < r$. Now,

$$T_N(y_{e_j}, \tilde{t}) = T_N(y_{e_j} - x_e \oplus x_e, \tilde{t}) \geq T_N(y_{e_j} - x_e, \frac{\tilde{t}}{2}) * T_N(x_e, \frac{\tilde{t}}{2}) \text{ [by (vi) in Definition 3.1]}$$

$$= T_N(x_e - y_{e_j}, \frac{\tilde{t}}{2}) * T_N(x_e, \frac{\tilde{t}}{2}) \text{ [by (v) in Definition 3.1]}$$

$$> (1 - r) * T_N(x_e, \frac{\tilde{t}}{2}).$$

Since $\lim_{\tilde{t}\to\infty} T_N(x_e,\tilde{t}) = 1$, $\exists\,\tilde{t}_0 > \tilde{0}$ so that $T_N(x_e,\tilde{t}) \geq 1-r,\ \forall\tilde{t} \geq \tilde{t}_0$. Thus, $T_N(y_{e_j},\tilde{t}) \geq (1-r)*(1-r) = (1-r),\ \forall\tilde{t} \geq \tilde{t}_0$. Next,

$$I_N(y_{e_j},\tilde{t}) = I_N(y_{e_j} - x_e \oplus x_e,\tilde{t}) \leq I_N(y_{e_j} - x_e,\frac{\tilde{t}}{2}) \diamond I_N(x_e,\frac{\tilde{t}}{2}) \text{ [by (xi) in Definition 3.1]}$$
$$= I_N(x_e - y_{e_j},\frac{\tilde{t}}{2}) \diamond I_N(x_e,\frac{\tilde{t}}{2}) \text{ [by (x) in Definition 3.1]}$$
$$< r \diamond I_N(x_e,\frac{\tilde{t}}{2}).$$

Since $\lim_{\tilde{t}\to\infty} I_N(x_e,\tilde{t}) = 0$, $\exists\,\tilde{t}_1 > \tilde{0}$ so that $I_N(x_e,\tilde{t}) \leq r,\ \forall\tilde{t} \geq \tilde{t}_1$. Thus, $I_N(y_{e_j},\tilde{t}) \leq r \diamond r = r, \forall\tilde{t} \geq \tilde{t}_1$. Similarly, $F_N(y_{e_j},\tilde{t}) \leq r, \forall\tilde{t} \geq \tilde{t}_2$.

Hence, $T_N(y_{e_j},\tilde{t}) \geq 1-r,\ I_N(y_{e_j},\tilde{t}) \leq r,\ F_N(y_{e_j},\tilde{t}) \leq r,\ \forall y_{e_j} \in OB(x_e,r,\tilde{t})$ and $\forall\tilde{t} \geq \max\{\tilde{t}_0,\tilde{t}_1,\tilde{t}_2\}$ and this ends the 2nd part.

5.5 Theorem

The intersection of an arbitrary number of convex soft sets is also convex in an NSNLS.

Proof. Let $\{\widetilde{W}_i | i \in \Gamma\}$ be a collection of convex soft sets in the NSNLS $(\widetilde{V}(K), N, *, \diamond)$ such that each $\widetilde{W}_i \subset \widetilde{V}$. Then $\cap_i \widetilde{W}_i = \widetilde{W}$ (say) is obviously convex. Let $x_{e_i} = [\tilde{c}y_{e_j} \oplus (\tilde{1}-\tilde{c})z_{e_k}] \in \widetilde{W}$ for $y_{e_j}, z_{e_k} \in \widetilde{W}$ and $\tilde{c} \in (\tilde{0},\tilde{1})$. Since $\widetilde{W} \subset \widetilde{V}$, so $(\widetilde{W}(K), N, *, \diamond)$ is a convex NSNLS and this proves the theorem.

6 Metric in NSNLS

The metric of NSNLS is defined in this section. Some related theorems are developed also.

6.1 Definition

The set of all mappings $T_N : \Delta_{\widetilde{V}} \times \Delta_{\widetilde{V}} \times R(E) \to [0,1]$, $I_N : \Delta_{\widetilde{V}} \times \Delta_{\widetilde{V}} \times R(E) \to [0,1]$ and $F_N : \Delta_{\widetilde{V}} \times \Delta_{\widetilde{V}} \times R(E) \to [0,1]$ together is said to form a neutrosophic soft metric on the soft linear space $\widetilde{V}$ if $\{T_N, I_N, F_N\}$ satisfies the following axioms :

(i) $0 \leq T_N(x_{e_i},y_{e_j},\tilde{t}),\ I_N(x_{e_i},y_{e_j},\tilde{t}),\ F_N(x_{e_i},y_{e_j},\tilde{t}) \leq 1,\ \forall x_{e_i},y_{e_j} \in \Delta_{\widetilde{V}}$ and $\forall\tilde{t} \in R(E)$.
(ii) $T_N(x_{e_i},y_{e_j},\tilde{t}) + I_N(x_{e_i},y_{e_j},\tilde{t}) + F_N(x_{e_i},y_{e_j},\tilde{t}) \leq 3,\ \forall x_{e_i},y_{e_j} \in \Delta_{\widetilde{V}}$ and $\tilde{t} \in R(E)$.
(iii) $T_N(x_{e_i},y_{e_j},\tilde{t}) = 0$ with $\tilde{t} \leq \tilde{0}$.
(iv) $T_N(x_{e_i},y_{e_j},\tilde{t}) = 1$ with $\tilde{t} > \tilde{0}$ iff $x_{e_i} = y_{e_j}$
(v) $T_N(x_{e_i},y_{e_j},\tilde{t}) = T_N(y_{e_j},x_{e_i},\tilde{t})$ with $\tilde{t} > \tilde{0}$.
(vi) $T_N(x_{e_i},y_{e_j},\tilde{s}) * T_N(y_{e_j},z_{e_k},\tilde{t}) \leq T_N(x_{e_i},z_{e_k},\tilde{s}\oplus\tilde{t}),\ \forall\,\tilde{s},\tilde{t} > \tilde{0};\ x_{e_i},y_{e_j},z_{e_k} \in \Delta_{\widetilde{V}}$.
(vii) $T_N(x_{e_i},y_{e_j},\cdot) : [0,\infty) \to [0,1]$ is continuous, $\forall x_{e_i},y_{e_j} \in \Delta_{\widetilde{V}}$.
(viii) $\lim_{t\to\infty} T_N(x_{e_i},y_{e_j},\tilde{t}) = 1,\ \forall x_{e_i},y_{e_j} \in \Delta_{\widetilde{V}}, \tilde{t} > \tilde{0}$
(ix) $I_N(x_{e_i},y_{e_j},\tilde{t}) = 1$ with $\tilde{t} \leq \tilde{0}$.
(x) $I_N(x_{e_i},y_{e_j},\tilde{t}) = 0$ with $\tilde{t} > \tilde{0}$ iff $x_{e_i} = y_{e_j}$
(xi) $I_N(x_{e_i},y_{e_j},\tilde{t}) = I_N(y_{e_j},x_{e_i},\tilde{t})$ with $\tilde{t} > \tilde{0}$.
(xii) $I_N(x_{e_i},y_{e_j},\tilde{s}) \diamond I_N(y_{e_j},z_{e_k},\tilde{t}) \geq I_N(x_{e_i},z_{e_k},\tilde{s}\oplus\tilde{t}),\ \forall\,\tilde{s},\tilde{t} > \tilde{0};\ x_{e_i},y_{e_j},z_{e_k} \in \Delta_{\widetilde{V}}$.
(xiii) $I_N(x_{e_i},y_{e_j},\cdot) : [0,\infty) \to [0,1]$ is continuous, $\forall x_{e_i},y_{e_j} \in \Delta_{\widetilde{V}}$.

(xiv) $\lim_{t\to\infty} I_N(x_{e_i}, y_{e_j}, \tilde{t}) = 0, \ \forall x_{e_i}, y_{e_j} \in \Delta_{\widetilde{V}}, \tilde{t} > \tilde{0}$
(xv) $F_N(x_{e_i}, y_{e_j}, \tilde{t}) = 1$ with $\tilde{t} \leq \tilde{0}$.
(xvi) $F_N(x_{e_i}, y_{e_j}, \tilde{t}) = 0$ with $\tilde{t} > \tilde{0}$ iff $x_{e_i} = y_{e_j}$
(xvii) $F_N(x_{e_i}, y_{e_j}, \tilde{t}) = F_N(y_{e_j}, x_{e_i}, \tilde{t})$ with $\tilde{t} > \tilde{0}$.
(xviii) $F_N(x_{e_i}, y_{e_j}, \tilde{s}) \diamond F_N(y_{e_j}, z_{e_k}, \tilde{t}) \geq F_N(x_{e_i}, z_{e_k}, \tilde{s} \oplus \tilde{t}), \ \forall \tilde{s}, \tilde{t} > \tilde{0}; \ x_{e_i}, y_{e_j}, z_{e_k} \in \Delta_{\widetilde{V}}$.
(xix) $F_N(x_{e_i}, y_{e_j}, \cdot) : [0, \infty) \to [0, 1]$ is continuous, $\forall x_{e_i}, y_{e_j} \in \Delta_{\widetilde{V}}$.
(xx) $\lim_{t\to\infty} F_N(x_{e_i}, y_{e_j}, \tilde{t}) = 0, \ \forall x_{e_i}, y_{e_j} \in \Delta_{\widetilde{V}}, \tilde{t} > \tilde{0}$

Then $(\widetilde{V}(K), \{T_N, I_N, F_N\}, *, \diamond)$ is a neutrosophic soft metric space (NSMS).

6.1.1 *Example*

Let $(\widetilde{X}, d)$ be a soft metric space. Define $a * b = ab$, $a \diamond b = a + b - ab$ and $\forall x_{e_i}, y_{e_j} \in \widetilde{X}$, $\tilde{t} > \tilde{0}$,
$T_N(x_{e_i}, y_{e_j}, \tilde{t}) = \frac{\tilde{t}}{\tilde{t} \oplus d(x_{e_i}, y_{e_j})}$, $I_N(x_{e_i}, y_{e_j}, \tilde{t}) = \frac{d(x_{e_i}, y_{e_j})}{\tilde{t} \oplus d(x_{e_i}, y_{e_j})}$, $F_N(x_{e_i}, y_{e_j}, \tilde{t}) = \frac{d(x_{e_i}, y_{e_j})}{\tilde{t}}$;

Then $(\widetilde{X}, \{T_N, I_N, F_N\}, *, \diamond)$ is an NSMS.

Proof. We shall only verify the axioms (vi), (xii), (xviii). Others are straight forward.

$$
\begin{aligned}
& T_N(x_{e_i}, z_{e_k}, \tilde{s} \oplus \tilde{t}) - T_N(x_{e_i}, y_{e_j}, \tilde{s}) * T_N(y_{e_j}, z_{e_k}, \tilde{t}) \\
= \ & \frac{\tilde{s} \oplus \tilde{t}}{\tilde{s} \oplus \tilde{t} \oplus d(x_{e_i}, z_{e_k})} - \frac{\tilde{s}\tilde{t}}{(\tilde{s} \oplus d(x_{e_i}, y_{e_j}))(\tilde{t} \oplus d(y_{e_j}, z_{e_k}))} \\
= \ & \{(\tilde{s} \oplus \tilde{t})(\tilde{s} \oplus d(x_{e_i}, y_{e_j}))(\tilde{t} \oplus d(y_{e_j}, z_{e_k})) - \tilde{s}\tilde{t}(\tilde{s} \oplus \tilde{t} \oplus d(x_{e_i}, z_{e_k})\}/G \\
& \text{where } G = (\tilde{s} \oplus \tilde{t} \oplus d(x_{e_i}, z_{e_k}))(\tilde{s} \oplus d(x_{e_i}, y_{e_j}))(\tilde{t} \oplus d(y_{e_j}, z_{e_k})) \\
= \ & \{\tilde{s}\tilde{t}[d(x_{e_i}, y_{e_j}) \oplus d(y_{e_j}, z_{e_k})] \oplus \tilde{t}^2 d(x_{e_i}, y_{e_j}) \oplus \tilde{s}^2 d(y_{e_j}, z_{e_k}) \\
& \oplus (\tilde{s} \oplus \tilde{t}) d(x_{e_i}, y_{e_j}) d(y_{e_j}, z_{e_k}) - \tilde{s}\tilde{t} d(x_{e_i}, z_{e_k})\}/G \\
\geq \ & \{\tilde{t}^2 d(x_{e_i}, y_{e_j}) \oplus \tilde{s}^2 d(y_{e_j}, z_{e_k}) \oplus (\tilde{s} \oplus \tilde{t}) d(x_{e_i}, y_{e_j}) d(y_{e_j}, z_{e_k})\}/G \geq 0
\end{aligned}
$$

Hence, $T_N(x_{e_i}, y_{e_j}, \tilde{s}) * T_N(y_{e_j}, z_{e_k}, \tilde{t}) \leq T_N(x_{e_i}, z_{e_k}, \tilde{s} \oplus \tilde{t})$.

$$
\begin{aligned}
& I_N(x_{e_i}, y_{e_j}, \tilde{s}) \diamond I_N(y_{e_j}, z_{e_k}, \tilde{t}) - I_N(x_{e_i}, z_{e_k}, \tilde{s} \oplus \tilde{t}) \\
= \ & \frac{d(x_{e_i}, y_{e_j})}{\tilde{s} \oplus d(x_{e_i}, y_{e_j})} \oplus \frac{d(y_{e_j}, z_{e_k})}{\tilde{t} \oplus d(y_{e_j}, z_{e_k})} - \frac{d(x_{e_i}, y_{e_j}) d(y_{e_j}, z_{e_k})}{(\tilde{s} \oplus d(x_{e_i}, y_{e_j}))(\tilde{t} \oplus d(y_{e_j}, z_{e_k}))} - \frac{d(x_{e_i}, z_{e_k})}{(\tilde{s} \oplus \tilde{t}) \oplus d(x_{e_i}, z_{e_k})} \\
= \ & \frac{d(x_{e_i}, y_{e_j}) d(y_{e_j}, z_{e_k}) \oplus \tilde{t} d(x_{e_i}, y_{e_j}) \oplus \tilde{s} d(y_{e_j}, z_{e_k})}{(\tilde{s} \oplus d(x_{e_i}, y_{e_j}))(\tilde{t} \oplus d(y_{e_j}, z_{e_k}))} - \frac{d(x_{e_i}, z_{e_k})}{\tilde{s} \oplus \tilde{t} \oplus d(x_{e_i}, z_{e_k})} \\
= \ & \{(\tilde{s} \oplus \tilde{t})(\tilde{t} d(x_{e_i}, y_{e_j}) \oplus \tilde{s} d(y_{e_j}, z_{e_k}) \oplus d(x_{e_i}, y_{e_j}) d(y_{e_j}, z_{e_k})) - \tilde{s}\tilde{t} d(x_{e_i}, z_{e_k})\}/H \\
& \text{where } H = (\tilde{s} \oplus \tilde{t} \oplus d(x_{e_i}, z_{e_k}))(\tilde{s} \oplus d(x_{e_i}, y_{e_j}))(\tilde{t} \oplus d(y_{e_j}, z_{e_k})) \\
\geq \ & \{(\tilde{s} \oplus \tilde{t})(\tilde{t} d(x_{e_i}, y_{e_j}) \oplus \tilde{s} d(y_{e_j}, z_{e_k}) \oplus d(x_{e_i}, y_{e_j}) d(y_{e_j}, z_{e_k})) - \tilde{s}\tilde{t}[d(x_{e_i}, y_{e_j}) \oplus d(y_{e_j}, z_{e_k})]\}/H \\
= \ & \{\tilde{t}^2 d(x_{e_i}, y_{e_j}) \oplus \tilde{s}^2 d(y_{e_j}, z_{e_k}) \oplus (\tilde{s} \oplus \tilde{t}) d(x_{e_i}, y_{e_j}) d(y_{e_j}, z_{e_k})\}/H \geq 0
\end{aligned}
$$

Hence, $I_N(x_{e_i}, y_{e_j}, \tilde{s}) \diamond I_N(y_{e_j}, z_{e_k}, \tilde{t}) \geq I_N(x_{e_i}, z_{e_k}, \tilde{s} \oplus \tilde{t})$. Finally,

$$
\begin{aligned}
& F_N(x_{e_i}, y_{e_j}, \tilde{s}) \diamond F_N(y_{e_j}, z_{e_k}, \tilde{t}) - F_N(x_{e_i}, z_{e_k}, \tilde{s} \oplus \tilde{t}) \\
= \ & \frac{d(x_{e_i}, y_{e_j})}{\tilde{s}} \oplus \frac{d(y_{e_j}, z_{e_k})}{\tilde{t}} - \frac{d(x_{e_i}, y_{e_j}) d(y_{e_j}, z_{e_k})}{\tilde{s}\tilde{t}} - \frac{d(x_{e_i}, z_{e_k})}{\tilde{s} \oplus \tilde{t}}
\end{aligned}
$$

$$\begin{aligned}
&= \frac{\tilde{t}d(x_{e_i}, y_{e_j}) \oplus \tilde{s}d(y_{e_j}, z_{e_k}) - d(x_{e_i}, y_{e_j})d(y_{e_j}, z_{e_k})}{st} - \frac{d(x_{e_i}, z_{e_k})}{\tilde{s} \oplus \tilde{t}} \\
&\geq \{\tilde{s}^2 d(y_{e_j}, z_{e_k}) \oplus \tilde{t}^2 d(x_{e_i}, y_{e_j}) - (\tilde{s} \oplus \tilde{t})d(x_{e_i}, y_{e_j})d(y_{e_j}, z_{e_k})\}/\tilde{s}\tilde{t}(\tilde{s} \oplus \tilde{t}) \\
&= \{\tilde{s}d(y_{e_j}, z_{e_k})(\tilde{s} - d(x_{e_i}, y_{e_j})) \oplus \tilde{t}d(x_{e_i}, y_{e_j})(\tilde{t} - d(y_{e_j}, z_{e_k}))\}/\tilde{s}\tilde{t}(\tilde{s} \oplus \tilde{t}) \geq 0 \\
&\quad [\text{as } F_N \in [0,1] \text{ so } \tilde{s} \geq d(x_{e_i}, y_{e_j}),\ \tilde{t} \geq d(y_{e_j}, z_{e_k})]
\end{aligned}$$

Thus, $F_N(x_{e_i}, y_{e_j}, \tilde{s}) \diamond F_N(y_{e_j}, z_{e_k}, \tilde{t}) \geq F_N(x_{e_i}, z_{e_k}, \tilde{s} \oplus \tilde{t})$. This completes the proof.

6.2 Theorem

Every NSNLS is a NSMS.

Proof. Define a neutrosophic soft metric $\{T_N, I_N, F_N\}$ over an NSNLS $(\widetilde{V}(K), N, *, \diamond)$ as follows.
$T_N(x_{e_i}, y_{e_j}, \tilde{t}) = T_N(x_{e_i} - y_{e_j}, \tilde{t})$, $I_N(x_{e_i}, y_{e_j}, \tilde{t}) = I_N(x_{e_i} - y_{e_j}, \tilde{t})$, $F_N(x_{e_i}, y_{e_j}, \tilde{t}) = F_N(x_{e_i} - y_{e_j}, \tilde{t})$ for each of $x_{e_i}, y_{e_j} \in \Delta_{\widetilde{V}}$. We shall verify here the metric axioms (v), (vi) only. Rest axioms are satisfied in well manner.

(v) $T_N(x_{e_i}, y_{e_j}, \tilde{t}) = T_N(x_{e_i} - y_{e_j}, \tilde{t}) = T_N(y_{e_j} - x_{e_i}, \dfrac{\tilde{t}}{|-1|}) = T_N(y_{e_j} - x_{e_i}, \tilde{t}) = T_N(y_{e_j}, x_{e_i}, \tilde{t})$

(vi) $T_N(x_{e_i}, z_{e_k}, \tilde{s} \oplus \tilde{t}) = T_N(x_{e_i} - z_{e_k}, \tilde{s} \oplus \tilde{t}) = T_N(x_{e_i} - y_{e_j} \oplus y_{e_j} - z_{e_k}, \tilde{s} \oplus \tilde{t})$
$\geq T_N(x_{e_i} - y_{e_j}, \tilde{s}) * T_N(y_{e_j} - z_{e_k}, \tilde{t}) = T_N(x_{e_i}, y_{e_j}, \tilde{s}) * T_N(y_{e_j}, z_{e_k}, \tilde{t})$

The four metric axioms (xi), (xii), (xvii), (xviii) can be similarly verified.

6.3 Definition

A sequence $\{x^n_{e_n}\}$ of soft points in a NSMS $(\widetilde{X}, \{T_N, I_N, F_N\}, *, \diamond)$ is said to be a convergent sequence and converges to x_e if

$\lim_{n\to\infty} T_N(x^n_{e_n}, x_e, \tilde{t}) = 1$, $\lim_{n\to\infty} I_N(x^n_{e_n}, x_e, \tilde{t}) = 0$, $\lim_{n\to\infty} F_N(x^n_{e_n}, x_e, \tilde{t}) = 0$ as $\tilde{t} \to \infty$.

6.4 Theorem

The limit of a convergent sequence $\{x^n_{e_n}\}$ in a NSMS $(\widetilde{X}, \{T_N, I_N, F_N\}, *, \diamond)$ is unique.

Proof. If possible $\lim_{n\to\infty} x^n_{e_n} = x_{e_j}$ and $\lim_{n\to\infty} x^n_{e_n} = y_{e_k}$ for $x_{e_j} \neq y_{e_k}$. Then for $\tilde{s}, \tilde{t} > \tilde{0}$,
$\lim_{n\to\infty} T_N(x^n_{e_n}, x_{e_j}, \tilde{s}) = 1$, $\lim_{n\to\infty} I_N(x^n_{e_n}, x_{e_j}, \tilde{s}) = 0$, $\lim_{n\to\infty} F_N(x^n_{e_n}, x_{e_j}, \tilde{s}) = 0$ as $\tilde{s} \to \infty$ and $\lim_{n\to\infty} T_N(x^n_{e_n}, y_{e_k}, \tilde{t}) = 1$, $\lim_{n\to\infty} I_N(x^n_{e_n}, y_{e_k}, \tilde{t}) = 0$, $\lim_{n\to\infty} F_N(x^n_{e_n}, y_{e_k}, \tilde{t}) = 0$ as $\tilde{t} \to \infty$. Now,

$$T_N(x_{e_j}, y_{e_k}, \tilde{s} \oplus \tilde{t}) \geq T_N(x_{e_j}, x^n_{e_n}, \tilde{s}) * T_N(x^n_{e_n}, y_{e_k}, \tilde{t}) = T_N(x^n_{e_n}, x_{e_j}, \tilde{s}) * T_N(x^n_{e_n}, y_{e_k}, \tilde{t})$$

Taking limit as $n \to \infty$ and for $\tilde{s}, \tilde{t} \to \infty$, $T_N(x_{e_j}, y_{e_k}, \tilde{s} \oplus \tilde{t}) \geq 1 * 1 = 1$.
It implies $T_N(x_{e_j}, y_{e_k}, \tilde{s} \oplus \tilde{t}) = 1$. (10)

Next, $I_N(x_{e_j}, y_{e_k}, \tilde{s} \oplus \tilde{t}) \leq I_N(x_{e_j}, x^n_{e_n}, \tilde{s}) \diamond I_N(x^n_{e_n}, y_{e_k}, \tilde{t}) = I_N(x^n_{e_n}, x_{e_j}, \tilde{s}) \diamond I_N(x^n_{e_n}, y_{e_k}, \tilde{t})$

Taking limit as $n \to \infty$ and for $\tilde{s}, \tilde{t} \to \infty$, $I_N(x_{e_j}, y_{e_k}, \tilde{s} \oplus \tilde{t}) \leq 0 \diamond 0 = 0$.
This shows $I_N(x_{e_j} - y_{e_k}, \tilde{s} \oplus \tilde{t}) = 0$. (11)
Similarly, $F_N(x_{e_j}, y_{e_k}, \tilde{s} \oplus \tilde{t}) = 0$ (12)

Hence, $x_{e_j} = y_{e_k}$ and this completes the proof.

6.5 Definition

A sequence $\{x^n_{e_n}\}$ of soft points in a NSMS $(\widetilde{X}, \{T_N, I_N, F_N\}, *, \diamond)$ is said to be a Cauchy sequence if $\lim_{n,m\to\infty} T_N(x^n_{e_n}, x^m_{e_m}, \tilde{t}) = 1$, $\lim_{n,m\to\infty} I_N(x^n_{e_n}, x^m_{e_m}, \tilde{t}) = 0$, $\lim_{n,m\to\infty} F_N(x^n_{e_n}, x^m_{e_m}, \tilde{t}) = 0$ as $\tilde{t} \to \infty$ and $\forall x^n_{e_n}, x^m_{e_m} \in \widetilde{X}$.

6.6 Theorem

Every convergent sequence is a Cauchy sequence in a NSMS $(\widetilde{X}, \{T_N, I_N, F_N\}, *, \diamond)$.

Proof. Let $\{x^n_{e_n}\}$ be a convergent sequence in a NSMS $(\widetilde{X}, \{T_N, I_N, F_N\}, *, \diamond)$ with $\lim_{n\to\infty} x^n_{e_n} = x_e$. Then for $\tilde{t} > \tilde{0}$,

$$\lim_{n\to\infty} T_N(x^n_{e_n}, x^m_{e_m}, \tilde{t}) \geq \lim_{n\to\infty} T_N(x^n_{e_n}, x_e, \frac{\tilde{t}}{2}) * \lim_{n\to\infty} T_N(x_e, x^m_{e_m}, \frac{\tilde{t}}{2})$$
$$= \lim_{n\to\infty} T_N(x^n_{e_n}, x_e, \frac{\tilde{t}}{2}) * \lim_{n\to\infty} T_N(x^m_{e_m}, x_e, \frac{\tilde{t}}{2}) = 1 * 1 = 1$$

So, $\lim_{n\to\infty} T_N(x^n_{e_n}, x^m_{e_m}, \tilde{t}) = 1$. Next,

$$\lim_{n\to\infty} I_N(x^n_{e_n}, x^m_{e_m}, \tilde{t}) \leq \lim_{n\to\infty} I_N(x^n_{e_n}, x_e, \frac{\tilde{t}}{2}) \diamond \lim_{n\to\infty} I_N(x_e, x^m_{e_m}, \frac{\tilde{t}}{2})$$
$$= \lim_{n\to\infty} I_N(x^n_{e_n}, x_e, \frac{\tilde{t}}{2}) \diamond \lim_{n\to\infty} I_N(x^m_{e_m}, x_e, \frac{\tilde{t}}{2}) = 0 \diamond 0 = 0$$

So, $\lim_{n\to\infty} I_N(x^n_{e_n}, x^m_{e_m}, \tilde{t}) = 0$ and similarly, $\lim_{n\to\infty} F_N(x^n_{e_n}, x^m_{e_m}, \tilde{t}) = 0$.
Hence, $\{x^n_{e_n}\}$ is a Cauchy sequence.

6.7 Definition

A NSMS $(\widetilde{X}, \{T_N, I_N, F_N\}, *, \diamond)$ is said to be complete if every Cauchy sequence of soft points in $\widetilde{X}$ converges to a soft point of $\widetilde{X}$.

6.8 Theorem

In a NSMS $(\widetilde{X}, \{T_N, I_N, F_N\}, *, \diamond)$, if every Cauchy sequence has a convergent subsequence then the NSMS is complete.

Proof. Let $\{x^{n_k}_{e_{n_k}}\}$ be a subsequence of a Cauchy sequence $\{x^n_{e_n}\}$ in a NSMS $(\widetilde{X}, \{T_N, I_N, F_N\}, *, \diamond)$ such that $\{x^{n_k}_{e_{n_k}}\} \to x_e \in \widetilde{X}$. Since $\{x^n_{e_n}\}$ be a Cauchy sequence in $(\widetilde{X}, \{T_N, I_N, F_N\}, *, \diamond)$, given $\tilde{t} > \tilde{0}$,

$\lim_{n,k\to\infty} T_N(x^n_{e_n}, x^{n_k}_{e_{n_k}}, \frac{\tilde{t}}{2}) = 1$, $\lim_{n,k\to\infty} I_N(x^n_{e_n}, x^{n_k}_{e_{n_k}}, \frac{\tilde{t}}{2}) = 0$, $\lim_{n,k\to\infty} F_N(x^n_{e_n}, x^{n_k}_{e_{n_k}}, \frac{\tilde{t}}{2}) = 0$ as $\tilde{t} \to \infty$.
Since $\{x^{n_k}_{e_{n_k}}\}$ converges x_e, then as $\tilde{t} \to \infty$,

$\lim_{k\to\infty} T_N(x^{n_k}_{e_{n_k}}, x_e, \frac{\tilde{t}}{2}) = 1$, $\lim_{k\to\infty} I_N(x^{n_k}_{e_{n_k}}, x_e, \frac{\tilde{t}}{2}) = 0$, $\lim_{k\to\infty} F_N(x^{n_k}_{e_{n_k}}, x_e, \frac{\tilde{t}}{2}) = 0$.
Now, $T_N(x^n_{e_n}, x_e, \tilde{t}) \geq T_N(x^n_{e_n}, x^{n_k}_{e_{n_k}}, \frac{\tilde{t}}{2}) * T_N(x^{n_k}_{e_{n_k}}, x_e, \frac{\tilde{t}}{2}) \Rightarrow \lim_{n\to\infty} T_N(x^n_{e_n}, x_e, \tilde{t}) = 1$.

Next, $I_N(x^n_{e_n}, x_e, \tilde{t}) \leq I_N(x^n_{e_n}, x^{n_k}_{e_{n_k}}, \frac{\tilde{t}}{2}) \diamond I_N(x^{n_k}_{e_{n_k}}, x_e, \frac{\tilde{t}}{2}) \Rightarrow \lim_{n\to\infty} I_N(x^n_{e_n}, x_e, \tilde{t}) = 0$.
Similarly, $\lim_{n\to\infty} F_N(x^n_{e_n}, x_e, \tilde{t}) = 0$.

This shows that $\{x^n_{e_n}\}$ converges to $x_e \in \widetilde{X}$ and thus the theorem is proved.

6.9 *Remark*

The using of $\tilde{0}, \tilde{1}$ instead of 0, 1 in some equalities is meaning that the left side of equality represents a set of soft points, not a set of neutrosophic components.

6.10 Theorem

In a NSMS $(\widetilde{V}(K), \{T_N, I_N, F_N\}, *, \diamond)$, define

$$||x_{e_i} - y_{e_j}||^1_\alpha = \inf\{\tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{t}) \geq \alpha, \alpha \in (0,1)\} \quad (13)$$
$$||x_{e_i} - y_{e_j}||^2_\beta = \sup\{\tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \beta, \beta \in (0,1)\} \quad (14)$$
$$||x_{e_i} - y_{e_j}||^3_\gamma = \sup\{\tilde{t}|F_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \gamma, \gamma \in (0,1)\} \quad (15)$$

Then $\{||\cdot||^1_\alpha, ||\cdot||^2_\beta, ||\cdot||^3_\gamma\}$ are ascending family of norms on $\widetilde{V}$ if $a * b = \min\{a,b\}$ and $a \diamond b = \max\{a,b\}$ for any two real numbers $a, b \in [0,1]$.

Proof. For $||\cdot||^1_\alpha$, we have

(i) $T_N(x_{e_i}, y_{e_j}, \tilde{t}) = 0,\ \forall \tilde{t} \leq \tilde{0}$ [by (iii) in Definition 6.1]
$\Rightarrow \{\tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{t}) \geq \alpha, \alpha \in (0,1)\} = \tilde{0}$
$\Rightarrow \inf\{\tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{t}) \geq \alpha, \alpha \in (0,1)\} = \tilde{0}$
$\Rightarrow ||x_{e_i} - y_{e_j}||^1_\alpha = \tilde{0}$

(ii) $T_N(x_{e_i}, y_{e_j}, \tilde{t}) = 1,\ \forall \tilde{t} \geq \tilde{0}$ iff $x_{e_i} = y_{e_j}$
$\Rightarrow \{\tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{t}) \geq \alpha, \alpha \in (0,1)\} = \tilde{1}$
$\Rightarrow \inf\{\tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{t}) \geq \alpha, \alpha \in (0,1)\} = \tilde{1}$
$\Rightarrow ||x_{e_i} - y_{e_j}||^1_\alpha = \tilde{1}$

(iii) $||x_{e_i} - y_{e_j}||^1_\alpha = \inf\{\tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{t}) \geq \alpha, \alpha \in (0,1)\}$
$= \inf\{\tilde{t}|T_N(y_{e_j}, x_{e_i}, \tilde{t}) \geq \alpha, \alpha \in (0,1)\} = ||y_{e_j} - x_{e_i}||^1_\alpha$

(iv) $||x_{e_i} - y_{e_j}||^1_\alpha + ||y_{e_j} - z_{e_k}||^1_\alpha,\ \alpha \in (0,1)$
$= \inf\{\tilde{s}|T_N(x_{e_i}, y_{e_j}, \tilde{s}) \geq \alpha\} + \inf\{\tilde{t}|T_N(y_{e_j}, z_{e_k}, \tilde{t}) \geq \alpha\}$
$= \inf\{\tilde{s} \oplus \tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{s}) \geq \alpha, T_N(y_{e_j}, z_{e_k}, \tilde{t}) \geq \alpha\}$
$= \inf\{\tilde{s} \oplus \tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{s}) * T_N(y_{e_j}, z_{e_k}, \tilde{t}) \geq \alpha * \alpha\}$
$\leq \{\tilde{s} \oplus \tilde{t}|T_N(x_{e_i}, z_{e_k}, \tilde{s} \oplus \tilde{t}) \geq \alpha\}$ [by (vi) in Definition 6.1]
$= ||x_{e_i} - z_{e_k}||^1_\alpha$

Thus, $||\cdot||^1_\alpha$ is an $\alpha-$ norm induced by the fuzzy soft metric T_N on $\widetilde{V}$.

Finally, for $0 < \alpha_1 < \alpha_2$,

$$
\begin{aligned}
& \{\tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{t}) \geq \alpha_2\} \subset \{\tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{t}) \geq \alpha_1\} \\
\Rightarrow\ & \inf\{\tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{t}) \geq \alpha_1\} \leq \inf\{\tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{t}) \geq \alpha_2\} \\
\Rightarrow\ & ||x_{e_i} - y_{e_j}||^1_{\alpha_1} \leq ||x_{e_i} - y_{e_j}||^1_{\alpha_2}
\end{aligned}
$$

Hence, $|| \cdot ||^1_\alpha$ is an ascending norm on $\widetilde{V}$. Next, for $|| \cdot ||^2_\beta$, we have

$$
\begin{aligned}
\text{(v)}\quad & I_N(x_{e_i}, y_{e_j}, \tilde{t}) = 1,\ \forall \tilde{t} \leq \tilde{0} \text{ [by (ix) in Definition 6.1]} \\
\Rightarrow\ & \{\tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \beta, \beta \in (0,1)\} = \tilde{1} \\
\Rightarrow\ & \sup\{\tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \beta, \beta \in (0,1)\} = \tilde{1} \\
\Rightarrow\ & ||x_{e_i} - y_{e_j}||^1_\beta = \tilde{1} \\
\text{(vi)}\quad & I_N(x_{e_i}, y_{e_j}, \tilde{t}) = 0,\ \forall \tilde{t} \geq \tilde{0} \iff x_{e_i} = y_{e_j} \\
\Rightarrow\ & \{\tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \beta, \beta \in (0,1)\} = \tilde{0} \\
\Rightarrow\ & \sup\{\tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \beta, \beta \in (0,1)\} = \tilde{0} \\
\Rightarrow\ & ||x_{e_i} - y_{e_j}||^2_\beta = \tilde{0} \\
\text{(vii)}\quad & ||x_{e_i} - y_{e_j}||^2_\beta = \sup\{\tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \beta, \beta \in (0,1)\} \\
=\ & \sup\{\tilde{t}|I_N(y_{e_j}, x_{e_i}, \tilde{t}) \leq \beta, \beta \in (0,1)\} = ||y_{e_j} - x_{e_i}||^2_\beta \\
\text{(viii)}\quad & ||x_{e_i} - y_{e_j}||^2_\beta + ||y_{e_j} - z_{e_k}||^2_\beta,\ \beta \in (0,1) \\
=\ & \sup\{\tilde{s}|I_N(x_{e_i}, y_{e_j}, \tilde{s}) \leq \beta\} + \sup\{\tilde{t}|I_N(y_{e_j}, z_{e_k}, \tilde{t}) \leq \beta\} \\
=\ & \sup\{\tilde{s} \oplus \tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{s}) \leq \beta, I_N(y_{e_j}, z_{e_k}, \tilde{t}) \leq \beta\} \\
=\ & \sup\{\tilde{s} \oplus \tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{s}) \diamond I_N(y_{e_j}, z_{e_k}, \tilde{t}) \leq \beta \diamond \beta\} \\
\geq\ & \{\tilde{s} \oplus \tilde{t}|I_N(x_{e_i}, z_{e_k}, \tilde{s} \oplus \tilde{t}) \leq \beta\} \text{ [by (xi) in Definition 6.1]} \\
=\ & ||x_{e_i} - z_{e_k}||^2_\beta
\end{aligned}
$$

Thus, $|| \cdot ||^2_\beta$ is a $\beta-$ norm induced by the fuzzy soft metric I_N on $\widetilde{V}$. Finally, for $0 < \beta_1 < \beta_2$,

$$
\begin{aligned}
& \{\tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \beta_2\} \supset \{\tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \beta_1\} \\
\Rightarrow\ & \sup\{\tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \beta_1\} \leq \sup\{\tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \beta_2\} \\
\Rightarrow\ & ||x_{e_i} - y_{e_j}||^2_{\beta_1} \leq ||x_{e_i} - y_{e_j}||^2_{\beta_2}
\end{aligned}
$$

Hence, $|| \cdot ||^2_\beta$ is an ascending norm on $\widetilde{V}$.
In a similar manner, $|| \cdot ||^3_\gamma$ is also an ascending norm on $\widetilde{V}$ and this ends the theorem.

7 Conclusion

The motivation of the present paper is to define a neutrosophic norm on a soft linear space. The convergence of sequence, characteristics of Cauchy sequence, the concept of convexity and the metric in NSNLS have been introduced here. These are illustrated by suitable examples. Their several related properties and structural characteristics have been investigated. We expect, this paper will promote the future study on neutrosophic soft normed linear spaces and many other general frameworks.

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