Neutrosophic Sets and Systems, Vol. 59, 2023
University of New Mexico

On The Foundations of Symbolic 5-Plithogenic Number Theory

[1.] **Mohamed Soueycatt,** [2.] **Barbara Charchekhandra,** [3.] **Rashel Abu Hakmeh**

[1.] Department of Bioengineering, Al-Andalus Private University for Medical Sciences, Syria; **m.soueycatt55@au.edu.sy**

[2.] Jadavpur University, Department Of Mathematics, Kolkata, India; Charchekhandrabar32@yahoo.com

[3.] Faculty of Science, Mutah University, Jordan

Abstract: This paper is dedicated to study the properties of symbolic 5-plithogenic integers and number theory, where we present many number theoretical concepts such as symbolic 5-plithogenic Diophantine equations, symbolic 5-plithogenic congruencies, and symbolic 5-plithogenic Euler's function. Also, we present many examples to explain the validity and the scientific contribution of our work.

Keywords: symbolic 5-plithogenic integer, symbolic 5-plithogenic Euler's function, symbolic 5-plithogenic Pythagoras triple

Introduction

Symbolic n-plithogenic sets were defined for the first time by Smarandache in [4, 24-25], with many interesting algebraic properties.

In [1-3], the symbolic 2-plithogenic rings were defined as an extension of classical rings. Many results were obtained with respect to their ideals and homomorphisms. The symbolic 2-plithogenic rings and fields have many applications in generalizing other algebraic structures such as symbolic 2-plithogenic vector spaces, symbolic 2-plithogenic modules, and symbolic 2-plithogenic equations [5-7].

Laterally, many authors defined and studied symbolic 3-plithogenic algebraic structures, such as symbolic 3-plithogenic spaces and modules, see [8, 21-23].

zenodo.10031248/10.5281

In the literature, the extended integer systems were used in number theory, for example neutrosophic numbers have helped with neutrosophic number theory, refined neutrosophic numbers generated refined number theory and split-complex numbers generated split-complex number theory [9-20].

This has motivated many authors to study symbolic 2-plithogenic and symbolic 3-plithogenic number theoretical concepts such as congruencies, and Diophantine equations [26-36]. The generalized versions of number theoretical concepts are very applicable in other mathematical studies, especially in cryptography.

In this paper, we study the symbolic 5-plithogenic number theoretical concepts for the first time, and we illustrated many examples to clarify the novel approach.

Main discussion

Definition:

The ring of symbolic 5-plithogenic integers is defined as follows:

$5-SP_Z = \{x_0 + \sum_{i=1}^{5} x_i P_i ; x_i \in Z\}$, where $P_i \times P_j = p_{max(i,j)}, {P_i}^2 = P_i$.

Definition.

Let $X = x_0 + \sum_{i=1}^{5} x_i P_i, Y = y_0 + \sum_{i=1}^{5} y_i P_i, Z = z_0 + \sum_{i=1}^{5} z_i P_i \in 5 - SP_Z$, we say that:

1). $X \setminus Y$ if there exists $Z \in 5 - SP_Z$ such that $X.Z = Y$.

2). $X \equiv Y (mod\, Z)$ if $Z \setminus X - Y$.

3). $Z = gcd(X, Y)$ if $Z \setminus X, Z \setminus Y$ and if $T \setminus X, T \setminus Y$, then $T \setminus Z$.

4). X, Y are relatively prime if $gcd(X, Y) = 1$.

Theorem1.

Let $X = x_0 + \sum_{i=1}^{5} x_i P_i, Y = y_0 + \sum_{i=1}^{5} y_i P_i, Z = z_0 + \sum_{i=1}^{5} z_i P_i \in 5 - SP_Z$, then:

1). $Z = gcd(X, Y)$ if and only if:

$$\begin{cases} z_0 = gcd(x_0, y_0) \\ \sum_{i=0}^{j} z_i = gcd\left(\sum_{i=o}^{j} x_i, \sum_{i=0}^{j} y_i\right); 1 \le j \le 5 \end{cases}$$

2). $X \equiv Y (mod\, Z)$ if and only if $\sum_{i=0}^{j} x_i \equiv \sum_{i=0}^{j} y_i \left(mod \sum_{i=0}^{j} z_i\right),\ 0 \le j \le 5$.

3). If $X \setminus Y$ then $\sum_{i=o}^{j} x_i \setminus \sum_{i=0}^{j} y_i ; 0 \le j \le 5$.

Theorem2.

Let $X = x_0 + \sum_{i=1}^{5} x_i P_i, Y = y_0 + \sum_{i=1}^{5} y_i P_i, Z = z_0 + \sum_{i=1}^{5} z_i P_i, A = a_0 + \sum_{i=1}^{5} a_i P_i, B = b_0 + \sum_{i=1}^{5} b_i P_i, C = c_0 + \sum_{i=1}^{5} c_i P_i \in 5 - SP_Z$, then:

1). If $Z \setminus X, Z \setminus Y$, then $Z \setminus AX + BY$.

2). If $Z = gcd(X, Y)$, then there exists $A, B \in 5 - SP_Z$ such that $AX + BY = Z$.

3). If $X \equiv Y(mod\, Z)$, then:

$$\begin{cases} X + C = Y + C\ (mod\, Z) & (I) \\ X - C = Y - C\ (mod\, Z) & (II) \\ X.C = Y.C\ (mod\, Z) & (III) \end{cases}$$

4). X is invertible modulo Z if and only if $\sum_{i=o}^{j} x_i$ is invertible modulo$\sum_{i=o}^{j} z_i\, ; 0 \le j \le 5$, and:

$X^{-1}(mod\, Z) = {x_0}^{-1}(mod\, z_0) + P_1[(x_0 + x_1)^{-1}(mod\, z_0 + z_1) - {x_0}^{-1}(mod\, z_0)] + P_2[(x_0 + x_1 + x_2)^{-1}(mod\, z_0 + z_1 + z_2) - (x_0 + x_1)^{-1}(mod\, z_0 + z_1)] + P_3[(x_0 + x_1 + x_2 + x_3)^{-1}(mod\, z_0 + z_1 + z_2 + z_3) - (x_0 + x_1 + x_2)^{-1}(mod\, z_0 + z_1 + z_2)] + P_4[(x_0 + x_1 + x_2 + x_3 + x_4)^{-1}(mod\, z_0 + z_1 + z_2 + z_3 + z_4) - (x_0 + x_1 + x_2 + x_3)^{-1}(mod\, z_0 + z_1 + z_2 + z_3)] + P_5[(x_0 + x_1 + x_2 + x_3 + x_4 + x_5)^{-1}(mod\, z_0 + z_1 + z_2 + z_3 + z_4 + z_5) - (x_0 + x_1 + x_2 + x_3 + x_4)^{-1}(mod\, z_0 + z_1 + z_2 + z_3 + z_4)]$.

Theorem3.

Let $AX + BY = C$ be symbolic 5-plithogenic Diophantine equation in two variables, $A, B, C, X, Y \in 5 - SP_Z$, hence it is solvable if and only if:

$\sum_{i=0}^{j} a_i \sum_{i=0}^{j} x_i + \sum_{i=0}^{j} b_i \sum_{i=0}^{j} y_i = \sum_{i=0}^{j} c_i\, ; 0 \le j \le 5$ are solvable, i.e. $gcd\left(\sum_{i=0}^{j} a_i, \sum_{i=0}^{j} b_i\right) \setminus \sum_{i=0}^{j} c_i\, ; 0 \le j \le 5$.

Theorem4.

Let $X = x_0 + \sum_{i=1}^{5} x_i p_i \in 5 - SP_Z$, then:

$$X^n = {x_0}^n + P_1\left[\left(\sum_{i=0}^{1} x_i\right)^n - {x_0}^n\right] + P_2\left[\left(\sum_{i=0}^{2} x_i\right)^n - \left(\sum_{i=0}^{1} x_i\right)^n\right] + P_3\left[\left(\sum_{i=0}^{3} x_i\right)^n - \left(\sum_{i=0}^{2} x_i\right)^n\right] + P_4\left[\left(\sum_{i=0}^{4} x_i\right)^n - \left(\sum_{i=0}^{3} x_i\right)^n\right] + P_5\left[\left(\sum_{i=0}^{5} x_i\right)^n - \left(\sum_{i=0}^{4} x_i\right)^n\right]$$

Theorem5.

(X, Y, Z) is a symbolic 5-plithogenic Pythagoras triple i.e. it is a solution of the non linear Diophantine equation $X^2 + Y^2 = Z^2$, if and only if $\left(\sum_{i=0}^{j} x_i, \sum_{i=0}^{j} y_i, \sum_{i=0}^{j} z_i\right); 0 \le j \le 5$ is a Pythagoras triple in Z.

Theorem6.

(X, Y, Z, T) is a symbolic 5-plithogenic Pythagoras quadruple i.e. it is a solution of the non linear Diophantine equation $X^2 + Y^2 + Z^2 = T^2$, if and only if $\left(\sum_{i=0}^{j} x_i, \sum_{i=0}^{j} y_i, \sum_{i=0}^{j} z_i, \sum_{i=0}^{j} t_i\right); 0 \le j \le 5$ is a Pythagoras quadruple in Z.

Proof of theorem1.

1). We put

$$Z = z_0 + \sum_{i=1}^{5} z_i P_i, z_0 = gcd(x_0, y_0), \sum_{i=1}^{1} z_i = gcd\left(\sum_{i=1}^{1} x_i, \sum_{i=1}^{1} y_i\right), \sum_{i=1}^{2} z_i = gcd\left(\sum_{i=1}^{2} x_i, \sum_{i=1}^{2} y_i\right)$$

$$\sum_{i=1}^{3} z_i = gcd\left(\sum_{i=1}^{3} x_i, \sum_{i=1}^{3} y_i\right), \sum_{i=1}^{4} z_i = gcd\left(\sum_{i=1}^{4} x_i, \sum_{i=1}^{4} y_i\right), \sum_{i=1}^{5} z_i = gcd\left(\sum_{i=1}^{5} x_i, \sum_{i=1}^{5} y_i\right)$$

Assume that $T = t_0 + \sum_{i=1}^{5} t_i P_i$ with $T \setminus X, T \setminus Y$, hence:

$$\begin{cases} \sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} x_i, \sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} y_i; 0 \le j \le 5 \\ \sum_{i=0}^{j} t_i \setminus \sum_{i=0}^{j} x_i, \sum_{i=0}^{j} t_i \setminus \sum_{i=0}^{j} y_i; 0 \le j \le 5 \end{cases}$$

So that $\sum_{i=0}^{j} t_i \setminus \sum_{i=0}^{j} z_i; 0 \le j \le 5$, hence $T \setminus Z$ and $Z = gcd(X, Y)$.

2). $X \equiv Y(mod\ Z)$ if and only if $Z \setminus X - Y$, which is equivalent to

$\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j}(x_i - y_i)\ ; 0 \leq j \leq 5$, hence $\sum_{i=0}^{j} x_i \equiv \sum_{i=0}^{j} y_i \left(mod\ \sum_{i=0}^{j} z_i\right); 0 \leq j \leq 5$.

3). Assume that $X \setminus Y$, hence:

$$\begin{cases} x_0 z_0 = y_0 \quad (1) \\ x_0 z_1 + x_1 z_0 + x_1 z_1 = y_1 \quad (2) \\ x_0 z_2 + x_1 z_2 + x_2 z_2 + x_2 z_0 + x_2 z_1 = y_2 \quad (3) \\ x_0 z_3 + x_1 z_3 + x_2 z_3 + x_3 z_3 + x_3 z_0 + x_3 z_1 + x_3 z_2 = y_3 \quad (4) \\ x_0 z_4 + x_1 z_4 + x_2 z_4 + x_3 z_4 + x_4 z_4 + x_4 z_0 + x_4 z_1 + x_4 z_2 + x_4 z_3 = y_4 \quad (5) \\ x_0 z_5 + x_1 z_5 + x_2 z_5 + x_3 z_5 + x_4 z_5 + x_5 z_5 + x_5 z_0 + x_5 z_1 + x_5 z_2 + x_5 z_3 + x_5 z_4 = y_5 \quad (6) \end{cases}$$

By adding $(1)+(2), (1)+(2)+(3), (1)+(2)+(3)+(4), (1)+(2)+(3)+(4)+(5), (1)+(2)+(3)+(4)+(5)+(6)$, we get:

$$\begin{cases} x_0 z_0 = y_0 \\ \sum_{i=1}^{1} x_i \sum_{i=1}^{1} z_i = \sum_{i=1}^{1} y_i \\ \sum_{i=1}^{2} x_i \sum_{i=1}^{2} z_i = \sum_{i=1}^{2} y_i \\ \sum_{i=1}^{3} x_i \sum_{i=1}^{3} z_i = \sum_{i=1}^{3} y_i \\ \sum_{i=1}^{4} x_i \sum_{i=1}^{4} z_i = \sum_{i=1}^{4} y_i \\ \sum_{i=1}^{5} x_i \sum_{i=1}^{5} z_i = \sum_{i=1}^{5} y_i \end{cases}$$

Which means that $\sum_{i=0}^{j} x_i \setminus \sum_{i=0}^{j} y_i\ ; 0 \leq j \leq 5$

Example on theorem1.

Take $X = 3 + 2P_1 + 2P_2 + P_3 - P_4 + 4P_5, Y = 6 + P_1 + P_2 - P_3 - P_4 + 2P_5$

$$\begin{cases} gcd(x_0, y_0) = gcd(3,6) = 3 \\ gcd(x_0 + x_1, y_0 + y_1) = gcd(5,7) = 1 \\ gcd(x_0 + x_1 + x_2, y_0 + y_1 + y_2) = gcd(7,7) = 7 \\ gcd(x_0 + x_1 + x_2 + x_3, y_0 + y_1 + y_2 + y_3) = gcd(8,7) = 1 \\ gcd(x_0 + x_1 + x_2 + x_3 + x_4, y_0 + y_1 + y_2 + y_3 + y_4) = gcd(7,6) = 1 \\ gcd(x_0 + x_1 + x_2 + x_3 + x_4 + x_5, y_0 + y_1 + y_2 + y_3 + y_4 + y_5) = gcd(11,8) = 1 \end{cases}$$

Thus

$z_0 = 3, z_1 = 1 - 3 = -2, z_2 = 7 - 1 = 6, z_3 = 1 - 7 = -6, z_4 = 1 - 1 = 0, z_5 = 1 - 1 = 0$, hence:

$Z = gcd(X,Y) = 3 - 2P_1 + 6P_2 - 6P_3$

For $L = 1 + P_1 - P_2 + 2P_5$, we can see:

$L \setminus X - Y$, that is because:

$\begin{cases} 1 \setminus -3 \\ 2 \setminus -2 \\ 1 \setminus -1 \\ 1 \setminus 1 \\ 3 \setminus 3 \end{cases}$, thus $X \equiv Y(mod\ L)$.

Proof of theorem 2.

1). Assume that $Z \setminus X, Z \setminus Y$, then we get:

$\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} x_i$, and $\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} y_i\, ; 0 \le j \le 5$.

So that $\sum_{i=0}^{j} z_i \setminus \left(\sum_{i=0}^{j} a_i \sum_{i=0}^{j} x_i + \sum_{i=0}^{j} b_i \sum_{i=0}^{j} y_i\right)$ for $0 \le j \le 5$ and $Z \setminus AX + BY$.

2). Assume that $Z = gcd(X,Y)$, then $\sum_{i=0}^{j} z_i = gcd\left(\sum_{i=0}^{j} x_i\, , \sum_{i=0}^{j} y_i\right)$ for all $0 \le j \le 5$.

According to Bezout's theorem, we can write:

There exists $a_j, b_j \in Z$ such that $\sum_{i=0}^{j} z_i = a_j \sum_{i=0}^{j} x_i + b_j \sum_{i=0}^{j} y_i$

by putting

$A = a_0 + (a_1 - a_0)P_1 + (a_2 - a_1)P_2 + (a_3 - a_2)P_3 + (a_4 - a_3)P_4 + (a_5 - a_4)P_5$,

$B = b_0 + (b_1 - b_0)P_1 + (b_2 - b_1)P_2 + (b_3 - b_2)P_3 + (b_4 - b_3)P_4 + (b_5 - b_4)P_5$, we get:

$Z = AX + BY$.

3). Assume that $X \equiv Y(mod\ Z)$, then:

$\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j}(x_i - y_i)$ for all $0 \le j \le 5$, hence:

$$\begin{cases} \sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j}(x_i - c_i + c_i - y_i) \\ \sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j}(x_i + c_i - c_i + y_i) \end{cases}$$

Hence $X \pm C = Y \pm C(mod\ Z)$, also:

$\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j}(x_i - y_i) \sum_{i=0}^{j} c_i$ i.e. $\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} x_i \sum_{i=0}^{j} c_i - \sum_{i=0}^{j} y_i \sum_{i=0}^{j} c_i$

Hence $X.C \equiv Y.C(mod\ Z)$.

4). X is invertible modulo Z If and only if there exists $Y = y_0 + \sum_{i=1}^{j} y_i\, p_i \in 5 - SP_Z$ such that $X.Y \equiv 1(mod\ Z)$.

This equivalent to:

$\sum_{i=0}^{j} x_i \,.\sum_{i=0}^{j} y_i \equiv 1(mod\ Z)$ for $0 \le j \le 5$, hence:

$\sum_{i=0}^{j} x_i$ is invertible modulo $\sum_{i=0}^{j} z_i$ and:

$$X^{-1} = {x_0}^{-1}(mod\ z_0) + P_1\left[\left(\sum_{i=0}^{1} x_i\right)^{-1}\left(mod \sum_{i=0}^{1} z_i\right) - {x_0}^{-1}(mod\ z_0)\right]$$
$$+ P_2\left[\left(\sum_{i=0}^{2} x_i\right)^{-1}\left(mod \sum_{i=0}^{2} z_i\right) - \left(\sum_{i=0}^{1} x_i\right)^{-1}\left(mod \sum_{i=0}^{1} z_i\right)\right]$$
$$+ P_3\left[\left(\sum_{i=0}^{3} x_i\right)^{-1}\left(mod \sum_{i=0}^{3} z_i\right) - \left(\sum_{i=0}^{2} x_i\right)^{-1}\left(mod \sum_{i=0}^{2} z_i\right)\right]$$
$$+ P_4\left[\left(\sum_{i=0}^{4} x_i\right)^{-1}\left(mod \sum_{i=0}^{4} z_i\right) - \left(\sum_{i=0}^{3} x_i\right)^{-1}\left(mod \sum_{i=0}^{3} z_i\right)\right]$$
$$+ P_5\left[\left(\sum_{i=0}^{5} x_i\right)^{-1}\left(mod \sum_{i=0}^{5} z_i\right) - \left(\sum_{i=0}^{4} x_i\right)^{-1}\left(mod \sum_{i=0}^{4} z_i\right)\right]$$

Example on theorem 2.

Take:

$$X = 4 + 2P_1 - P_2 + 5P_3 - P_4 + P_5, Y = 2 + P_1 - P_2 + P_3 - P_4 + 4P_5, Z$$
$$= 2 - P_1 + P_2 - P_3 + P_4 + P_5, A = 1 + P_1, B = 2 - P_1 + 3P_2$$

we have $Z \setminus X$, that is because $2 \setminus 4, 1 \setminus 6, 2 \setminus 4, 1 \setminus 9, 2 \setminus 8, 3 \setminus 9$.

$Z \setminus Y$, that I because $2 \setminus 2, 1 \setminus 3, 2 \setminus 2, 1 \setminus 3, 2 \setminus 2, 3 \setminus 6$.

On the other hand,

$$AX + BY = (1 + P_1)(4 + 2P_1 - P_2 + 5P_3 - P_4 + P_5)$$
$$+ (2 - P_1 + 3P_2)(2 + P_1 - P_2 + P_3 - P_4 + 4P_5)$$
$$= 4 + 4P_1 + 2P_1 + 2P_1 - 2P_2 - 2P_2 + 5P_3 + 5P_3 - P_4 - P_4 + P_5 + P_5 + 4$$
$$+ 2P_1 - 2P_2 + 2P_3 - 2P_4 + 8P_5 - 2P_1 - P_1 + P_2 - P_3 + P_4 - 4P_5 + 6P_2$$
$$+ 3P_2 - 3P_4 + 12P_5 = 8 + 7P_1 + P_2 + 14P_3 - 6P_4 + 18P_5$$

$Z \setminus AX + BY$, that is because $2 \setminus 8, 1 \setminus 15, 2 \setminus 16, 1 \setminus 30, 2 \setminus 24, 3 \setminus 42$.

For $T = 3 + 2P_1 - 2P_2 - P_3 - P_4$, we can see:

$gcd(X,T) = gcd(4,3) + P_1[gcd(5,6) - gcd(4,3)] + P_2[gcd(3,4) - gcd(5,6)] + P_3[gcd(9,2) - gcd(3,4)] + P_4[gcd(8,1) - gcd(9,2)] + P_5[gcd(8,1) - gcd(9,1)]$,

hence X is invertible modulo T.

$$4^{-1}(mod\ 3) = 1, 6^{-1}(mod\ 5) = 1, 9^{-1}(mod\ 2) = 5, 8^{-1}(mod\ 1) = 1, 9^{-1}(mod\ 1) = 1, 4^{-1}(mod\ 3) = 1$$

$X^{-1}(mod\ T) = 1 + P_1[1 - 1] + P_2[1 - 1] + P_3[5 - 1] + P_4[1 - 5] + P_5[1 - 1] = 1 + 4P_3 - 4P_4$.

Proof of theorem3.

It is easy to check that $AX + BY = C$ is equivalent to:

$$\sum_{i=0}^{j} a_i \sum_{i=0}^{j} x_i + \sum_{i=0}^{j} b_i \sum_{i=0}^{j} y_i = \sum_{i=0}^{j} c_i\,; 0 \le j \le 5$$

The previous six Diophantine equations are solvable if and only if:

$$gcd\left(\sum_{i=0}^{j} a_i\,, \sum_{i=0}^{j} b_i\right) \backslash \sum_{i=0}^{j} c_i\,; 0 \le j \le 5$$

Example on theorem3.

Consider the following 5-plithogenic linear Diophantine equation in two variables:

$$(1 + P_2 - 3P_3 + 5P_4 + P_5)X + (1 - P_1 + P_2)Y = P_1 + P_2 - 3P_3 + 6P_4 + 2P_5$$

The equivalent system is:

$$\begin{cases} x_0 + y_0 = 0 \quad (1) \\ \sum_{i=0}^{1} x_i = 1 \quad (2) \\ \sum_{i=0}^{2} x_i + \sum_{i=0}^{2} y_i = 2 \quad (3) \\ -\sum_{i=0}^{3} x_i + \sum_{i=0}^{3} y_i = -1 \quad (4) \\ 4\sum_{i=0}^{4} x_i + \sum_{i=0}^{4} y_i = 5 \quad (5) \\ 5\sum_{i=0}^{5} x_i + \sum_{i=0}^{5} y_i = 7 \quad (6) \end{cases}$$

Equation (1) has a solution $x_0 = y_0 = 0$.

Equation (2) has a solution $x_0 + x_1 = 1$, hence $x_1 = 1, y_1 = 0$.

Equation (3) has a solution $x_0 + x_1 + x_2 = 1, y_0 + y_1 + y_2 = 1$, hence $x_2 = 0, y_2 = 0$.

Equation (4) has a solution $x_0 + x_1 + x_2 + x_3 = 1, y_0 + y_1 + y_2 + y_3 = 0$, hence $x_3 = 0, y_3 = 0$.

Equation (5) has a solution $x_0 + x_1 + x_2 + x_3 + x_4 = 1, y_0 + y_1 + y_2 + y_3 + y_4 = 1$, hence $x_4 = 0, y_4 = 1$.

Equation (6) has a solution $x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 1, y_0 + y_1 + y_2 + y_3 + y_4 + y_5 = 1$, hence $x_5 = 0, y_5 = 1$.

This means that $X = P_1, Y = P_4 + P_5$ is a solution.

proof on theorem4.

For $n = 1$, it holds directly.

We assume that it is true for k, we prove it for $k + 1$. $X^{k+1} = XX^k = \left(x_0 + \sum_{i=0}^{5} x_i\, p_i\right)\left[{x_0}^k + P_1((\sum_{i=0}^{1} x_i)^k - {x_0}^k) + P_2((\sum_{i=0}^{2} x_i)^k - (\sum_{i=0}^{1} x_i)^k) + P_3((\sum_{i=0}^{3} x_i)^k - (\sum_{i=0}^{2} x_i)^k) + P_4((\sum_{i=0}^{4} x_i)^k - (\sum_{i=0}^{3} x_i)^k) + P_5\left(\left(\sum_{i=0}^{5} x_i\right)^k - (\sum_{i=0}^{4} x_i)^k\right)\right] = {x_0}^{k+1} +$
$P_1[{x_0}^k(\sum_{i=0}^{1} x_i)^k - {x_0}^{k+1} + x_1{x_0}^k + x_1(\sum_{i=0}^{1} x_i)^k - x_1{x_0}^k] + P_2[x_0(\sum_{i=0}^{2} x_i)^k -$
$x_0(\sum_{i=0}^{1} x_i)^k + x_1(\sum_{i=0}^{2} x_i)^k - x_1(\sum_{i=0}^{1} x_i)^k + x_2{x_0}^k + x_1(\sum_{i=0}^{1} x_i)^k - x_2{x_0}^k +$
$x_2(\sum_{i=0}^{2} x_i)^k - x_2(\sum_{i=0}^{1} x_i)^k] + P_3[x_0(\sum_{i=0}^{3} x_i)^k - x_0(\sum_{i=0}^{2} x_i)^k + x_1(\sum_{i=0}^{3} x_i)^k -$
$x_1(\sum_{i=0}^{2} x_i)^k + x_2(\sum_{i=0}^{3} x_i)^k - x_2(\sum_{i=0}^{2} x_i)^k + x_2{x_0}^k + x_3(\sum_{i=0}^{1} x_i)^k - x_3{x_0}^k +$
$x_3(\sum_{i=0}^{2} x_i)^k - x_2(\sum_{i=0}^{1} x_i)^k + x_3(\sum_{i=0}^{3} x_i)^k - x_2(\sum_{i=0}^{2} x_i)^k] + \cdots = {x_0}^{k+1} +$
$+P_1[(\sum_{i=0}^{1} x_i)^{k+1} - {x_0}^{k+1}] + P_2[(\sum_{i=0}^{2} x_i)^{k+1} - (\sum_{i=0}^{1} x_i)^{k+1}] + \cdots$

And the proof holds.

Proof of theorem5.

$X^2 + Y^2 = Z^2$ implies that:

$$\begin{cases} {x_0}^2 + {y_0}^2 = {z_0}^2 \\ \left(\sum_{i=0}^{1} x_i\right)^2 + \left(\sum_{i=0}^{1} y_i\right)^2 = \left(\sum_{i=0}^{1} z_i\right)^2 \\ \left(\sum_{i=0}^{2} x_i\right)^2 + \left(\sum_{i=0}^{2} y_i\right)^2 = \left(\sum_{i=0}^{2} z_i\right)^2 \\ \left(\sum_{i=0}^{3} x_i\right)^2 + \left(\sum_{i=0}^{3} y_i\right)^2 = \left(\sum_{i=0}^{3} z_i\right)^2 \\ \left(\sum_{i=0}^{4} x_i\right)^2 + \left(\sum_{i=0}^{4} y_i\right)^2 = \left(\sum_{i=0}^{4} z_i\right)^2 \\ \left(\sum_{i=0}^{5} x_i\right)^2 + \left(\sum_{i=0}^{5} y_i\right)^2 = \left(\sum_{i=0}^{5} z_i\right)^2 \end{cases}$$

Which implies the proof.

Theorem 6 can be proved by the same argument.

Example on theorem5.

Consider $X = 3 + P_5, Y = 4 - P_5, Z = 5$, we have:

$X^2 + Y^2 = Z^2$, hence (X, Y, Z) is a Pythagoras triple.

We can see clearly that:

$$\begin{cases} x_0 = 3, y_0 = 4, z_0 = 5 \\ \sum_{i=0}^{1} x_i = 3, \sum_{i=0}^{1} y_i = 4 \\ \sum_{i=0}^{2} x_i = 3, \sum_{i=0}^{2} y_i = 4 \\ \sum_{i=0}^{3} x_i = 3, \sum_{i=0}^{3} y_i = 4 \\ \sum_{i=0}^{4} x_i = 3, \sum_{i=0}^{4} y_i = 4 \\ and \sum_{i=0}^{5} x_i = 3, \sum_{i=0}^{5} y_i = 4, \sum_{i=0}^{k} z_i = 4\,; 1 \le k \le 5 \end{cases}$$

Definition.

Let $X = x_0 + \sum_{i=0}^{5} x_i P_i \in 5 - SP_Z$, hence we say that $X > 0$ if and only if $x_0 > 0, \sum_{i=0}^{k} x_i > 0\,; 1 \le k \le 5$

For example: $X = 3 + P_1 - P_2 + 2P_3 - P_4 - P_5 > 0$, that is because:

$3 > 0{,}4 > 0{,}3 > 0{,}5 > 0{,}4 > 0{,}3 > 0$.

If $Y = y_0 + \sum_{i=0}^{5} y_i P_i \in 5 - SP_Z$, we ay that $X \geq Y$ if and only if $x_0 \geq y_0, \sum_{i=0}^{k} x_i \geq \sum_{i=0}^{k} y_i\,; 1 \leq k \leq 5$.

For $X = 2 + P_1 + 2P_2 + 5P_3 + P_4 + 6P_5, Y = 1 + P_1 + P_2 + P_3 + 3P_4 + P_5, X \geq Y$, that is because:

$2 \geq 1{,}3 \geq 2{,}5 \geq 3{,}10 \geq 4{,}11 \geq 7{,}17 \geq 8$

Definition.

Let $X = x_0 + \sum_{i=0}^{5} x_i P_i\,, y = y_0 + \sum_{i=0}^{5} y_i\, P_i \geq 0$, hence:

$$X^Y = {x_0}^{y_0} + P_1\left[\left(\sum_{i=0}^{1} x_i\right)^{\sum_{i=0}^{1} y_i} - {x_0}^{y_0}\right] + P_2\left[\left(\sum_{i=0}^{2} x_i\right)^{\sum_{i=0}^{2} y_i} - \left(\sum_{i=0}^{1} x_i\right)^{\sum_{i=0}^{1} y_i}\right]$$
$$+ P_3\left[\left(\sum_{i=0}^{3} x_i\right)^{\sum_{i=0}^{3} y_i} - \left(\sum_{i=0}^{2} x_i\right)^{\sum_{i=0}^{2} y_i}\right]$$
$$+ P_4\left[\left(\sum_{i=0}^{4} x_i\right)^{\sum_{i=0}^{4} y_i} - \left(\sum_{i=0}^{3} x_i\right)^{\sum_{i=0}^{3} y_i}\right]$$
$$+ P_5\left[\left(\sum_{i=0}^{5} x_i\right)^{\sum_{i=0}^{5} y_i} - \left(\sum_{i=0}^{4} x_i\right)^{\sum_{i=0}^{4} y_i}\right]$$

Example.

Let $X = 2 + 3P_1 - P_2 - P_3 - P_4 + P_5, Y = 1 + P_1 - P_2 + P_3 - P_4 + P_5$, we have:

$$\begin{cases} x_0 = 2, y_0 = 1, {x_0}^{y_0} = 2 \\ \sum_{i=0}^{1} x_i = 5, \sum_{i=0}^{1} y_i = 2, 5^2 = 25 \\ \sum_{i=0}^{2} x_i = 4, \sum_{i=0}^{2} y_i = 1, 4^1 = 4 \\ \sum_{i=0}^{3} x_i = 3, \sum_{i=0}^{3} y_i = 2, 3^2 = 9 \\ \sum_{i=0}^{4} x_i = 2, \sum_{i=0}^{4} y_i = 1, 2^1 = 2 \\ \sum_{i=0}^{5} x_i = 3, \sum_{i=0}^{5} y_i = 2, 3^2 = 9 \end{cases}$$

Hence $X^Y = 2 + (25-2)P_1 + (4-25)P_2 + (9-4)P_3 + (2-9)P_4 + (9-2)P_5 = 2 + 23P_1 - 21P_2 + 5P_3 - 7P_4 + 7P_5$

Definition.

Let $X = x_0 + \sum_{i=0}^{5} x_i P_i > 0$, then:

$$\varphi(X) = \varphi(x_0) + P_1\left[\varphi\left(\sum_{i=0}^{1} x_i\right) - \varphi(x_0)\right] + P_2\left[\varphi\left(\sum_{i=0}^{2} x_i\right) - \varphi\left(\sum_{i=0}^{1} x_i\right)\right] + P_3\left[\varphi\left(\sum_{i=0}^{3} x_i\right) - \varphi\left(\sum_{i=0}^{2} x_i\right)\right] + P_4\left[\varphi\left(\sum_{i=0}^{4} x_i\right) - \varphi\left(\sum_{i=0}^{3} x_i\right)\right] + P_5\left[\varphi\left(\sum_{i=0}^{5} x_i\right) - \varphi\left(\sum_{i=0}^{4} x_i\right)\right]$$

Where φ is Euler's function on Z.

Example.

Let $X = 3 + 2P_1 + P_2 + P_3 - P_4 + P_5$, then:

$$\varphi(x_0) = \varphi(3) = 2, \varphi\left(\sum_{i=0}^{1} x_i\right) = \varphi(5) = 4, \left(\sum_{i=0}^{2} x_i\right) = (6) = 2, \varphi\left(\sum_{i=0}^{3} x_i\right) = \varphi(7) = 6, \varphi\left(\sum_{i=0}^{4} x_i\right) = \varphi(8) = 2, \varphi\left(\sum_{i=0}^{5} x_i\right) = \varphi(7) = 6$$

$$\varphi(X) = 2 + (4-2)P_1 + (2-4)P_2 + (6-2)P_3 + (2-6)P_4 + (6-2)P_5 = 2 + 2P_1 - 2P_2 + 4P_3 - 4P_4 + 4P_5$$

Theorem.

Let $X = x_0 + \sum_{i=0}^{5} x_i P_{i_i}, Y = y_0 + \sum_{i=0}^{5} y_i P_i \in 5 - SP_Z, gcd(X,Y) = 1$ and $X, Y > 0$, hence:

$X^{\varphi(Y)} \equiv 1(mod\ Y)$

Proof.

$gcd(x_0, y_0) = 1$, hence $x_0^{\varphi(y_0)} \equiv 1(mod\ y_0)$.

$gcd(\sum_{i=0}^{1} x_i, \sum_{i=0}^{1} y_i) = 1$, hence $(\sum_{i=0}^{1} x_i)^{\varphi(\sum_{i=0}^{1} y_i)} \equiv 1(mod\ \sum_{i=0}^{1} y_i)$

By a similar argument, we get:

$$\left(\sum_{i=0}^{2} x_i\right)^{\varphi(\sum_{i=0}^{2} y_i)} \equiv 1\left(mod \sum_{i=0}^{2} y_i\right), \left(\sum_{i=0}^{3} x_i\right)^{\varphi(\sum_{i=0}^{3} y_i)} \equiv 1\left(mod \sum_{i=0}^{3} y_i\right)$$

$$\left(\sum_{i=0}^{4} x_i\right)^{\varphi(\sum_{i=0}^{4} y_i)} \equiv 1\left(mod \sum_{i=0}^{4} y_i\right), \left(\sum_{i=0}^{5} x_i\right)^{\varphi(\sum_{i=0}^{5} y_i)} \equiv 1\left(mod \sum_{i=0}^{5} y_i\right)$$

This implies

$X^{\varphi(Y)} \equiv 1 + (1-1)P_1 + (1-1)P_2 + (1-1)P_3 + (1-1)P_4 + (1-1)P_5 \equiv 1(mod\ Y)$.

Example.

Consider $X = 5 + 2P_1 + 4P_2 + 2P_3 - 2P_4 + 2P_5, Y = 7 + 4P_1 - 4P_2 + P_3 + P_4 + P_5$.

$$gcd(X,Y) = gcd(5,7) + +P_1[gcd(7,11) - gcd(5,7)] + P_2[gcd(11,7) - gcd(7,11)] + P_3[gcd(13,9) - gcd(11,7)] + P_4[gcd(11,9) - gcd(13,9)] + P_5[gcd(13,10) - gcd(11,9)] \equiv 1$$

Also, we have:

$$x_0 = 5, y_0 = 7, \varphi(y_0) = 6, x_0^{\varphi(y_0)} = 5^6 \equiv 1(mod\ 7)$$

$$\sum_{i=0}^{1} x_i = 7, \sum_{i=0}^{1} y_i = 11, \varphi\left(\sum_{i=0}^{1} y_i\right) = 10, 7^{10} \equiv 1(mod\ 11)$$

$$\sum_{i=0}^{2} x_i = 11, \sum_{i=0}^{2} y_i = 7, \varphi\left(\sum_{i=0}^{2} y_i\right) = 6, 11^6 \equiv 1(mod\ 7)$$

$$\sum_{i=0}^{3} x_i = 13, \sum_{i=0}^{3} y_i = 81, \varphi\left(\sum_{i=0}^{3} y_i\right) = 4, 13^4 \equiv 1(mod\ 8)$$

$$\sum_{i=0}^{4} x_i = 11, \sum_{i=0}^{4} y_i = 91, \varphi\left(\sum_{i=0}^{4} y_i\right) = 6, 11^6 \equiv 1(mod\ 9)$$

$$\sum_{i=0}^{5} x_i = 13, \sum_{i=0}^{5} y_i = 10, \varphi\left(\sum_{i=0}^{5} y_i\right) = 4, 13^4 \equiv 1(mod\ 10)$$

Hence $X^{\varphi(Y)} \equiv 1(mod\ Y)$

Remark.

We call the previous result by symbolic 5-plithogenic Euler's theorem.

Conclusion

In this work, we have studied the properties of symbolic 5-plithogenic integers for the first time, where concepts such as symbolic 5-plithogenic divisors, congruencies, and linear Diophantine equations were handled by many theorems and examples.

Also, we have presented the conditions of symbolic 5-plithogenic Pythagoras triples and quadruples in the corresponding symbolic 5-plithogenic ring of integers.

References

1. Nader Mahmoud Taffach , Ahmed Hatip., "A Review on Symbolic 2-Plithogenic Algebraic Structures " Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.
2. Nader Mahmoud Taffach , Ahmed Hatip.," A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations ", Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.
3. Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.
4. Smarandache, F., " Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)", Neutrosophic Sets and Systems, vol. 53, 2023.
5. Taffach, N., " An Introduction to Symbolic 2-Plithogenic Vector Spaces Generated from The Fusion of Symbolic Plithogenic Sets and Vector Spaces", Neutrosophic Sets and Systems, Vol 54, 2023.

6. Taffach, N., and Ben Othman, K., " An Introduction to Symbolic 2-Plithogenic Modules Over Symbolic 2-Plithogenic Rings", Neutrosophic Sets and Systems, Vol 54, 2023.
7. Merkepci, H., and Rawashdeh, A., " On The Symbolic 2-Plithogenic Number Theory and Integers ", Neutrosophic Sets and Systems, Vol 54, 2023.
8. Albasheer, O., Hajjari., A., and Dalla., R., " On The Symbolic 3-Plithogenic Rings and TheirAlgebraic Properties", Neutrosophic Sets and Systems, Vol 54, 2023.
9. Abobala, M., On Refined Neutrosophic Matrices and Their Applications In Refined Neutrosophic Algebraic Equations, Journal Of Mathematics, Hindawi, 2021
10. Olgun, N., Hatip, A., Bal, M., and Abobala, M., " A Novel Approach To Necessary and Sufficient Conditions For The Diagonalization of Refined Neutrosophic Matrices", International Journal of neutrosophic Science, Vol. 16, pp. 72-79, 2021.
11. Merkepci, M., Abobala, M., and Allouf, A., " The Applications of Fusion Neutrosophic Number Theory in Public Key Cryptography and the Improvement of RSA Algorithm ", Fusion: Practice and Applications, 2023.
12. Abobala, M., and Allouf, A., " On A Novel Security Scheme for The Encryption and Decryption Of 2×2 Fuzzy Matrices with Rational Entries Based on The Algebra of Neutrosophic Integers and El-Gamal Crypto-System", Neutrosophic Sets and Systems, vol.54, 2023.
13. Khaldi, A., " A Study On Split-Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.
14. Ahmad, K., " On Some Split-Complex Diophantine Equations", Neoma Journal Of Mathematics and Computer Science, 2023.

15. Ali, R., " On The Weak Fuzzy Complex Inner Products On Weak Fuzzy Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.

16. Merkepci, M., and Abobala, M., " Security Model for Encrypting Uncertain Rational Data Units Based on Refined Neutrosophic Integers Fusion and El Gamal Algorithm ", Fusion: Practice and Applications, 2023.
17. Ben Othman, K., Albasheer, O., Nadweh, R., Von Shtawzen, O., and Ali, R., "On The Symbolic 6-Plithogenic and 7-Plithogenic Rings", Galoitica Journal of Mathematical Structures and Applications, Vol.8, 2023.
18. Bisher Ziena, M., and Abobala, M., " On The Refined Neutrosophic Real Analysis Based on Refined Neutrosophic Algebraic AH-Isometry", Neutrosophic Sets and Systems, vol. 54, 2023.
19. M. B. Zeina and M. Abobala, "A Novel Approach of Neutrosophic Continuous Probability Distributions using AH-Isometry with Applications in Medicine," in *Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics*, Elsevier, 2023.
20. M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, "Introduction to Symbolic 2-Plithogenic Probability Theory," *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 1, 2023.
21. Ali, R., and Hasan, Z., "An Introduction To The Symbolic 3-Plithogenic Vector Spaces", Galoitica Journal Of Mathematical Structures and Applications, vol. 6, 2023.
22. Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.
23. Ben Othman, K., "On Some Algorithms For Solving Symbolic 3-Plithogenic Equations", Neoma Journal Of Mathematics and Computer Science, 2023.

24. Florentin Smarandache, Plithogenic Algebraic Structures. Chapter in "Nidus idearum Scilogs, V: joining the dots" (third version), Pons Publishing Brussels, pp. 123-125, 2019.
25. Florentin Smarandache: Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets – Revisited, *Neutrosophic Sets and Systems*, vol. 21, 2018, pp. 153-166.
26. Sarkis, M., " On The Solutions Of Fermat's Diophantine Equation In 3-refined Neutrosophic Ring of Integers", Neoma Journal of Mathematics and Computer Science, 2023.
27. Abobala, M., "On Some Algebraic Properties of n-Refined Neutrosophic Elements and n-Refined Neutrosophic Linear Equations", Mathematical Problems in Engineering, Hindawi, 2021
28. Abobala, M., "A Study Of Nil Ideals and Kothe's Conjecture In Neutrosophic Rings", International Journal of Mathematics and Mathematical Sciences, hindawi, 2021
29. Abobala, M., "On The Characterization of Maximal and Minimal Ideals In Several Neutrosophic Rings", Neutrosophic sets and systems, Vol. 45, 2021.
30. Abobala, M., Bal, M., Aswad, M., "A Short Note On Some Novel Applications of Semi Module Homomorphisms", International journal of neutrosophic science, 2022.
31. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Concept Of Symbolic 7-Plithogenic Real Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.
32. Ben Othman, K., Von Shtawzen, O., Khaldi, A., and Ali, R., "On The Symbolic 8-Plithogenic Matrices", Pure Mathematics For Theoretical Computer Science, Vol.1, 2023.

33. Abualkishik, A., Almajed, R., Thompson, W., "Improving The Performance of Fog-assisted Internet of Things Networks Using Bipolar Trapezoidal Neutrosophic Sets", Journal of Wireless and Ad Hoc Communication, 2023.
34. Ibrahim, M., and Abobala, M., "An Introduction To Refined Neutrosophic Number Theory", Neutrosophic Sets and Systems, Vol. 45, 2021.
35. Sankari, H., and Abobala, M., "Neutrosophic Linear Diophantine Equations With two Variables", Neutrosophic Sets and Systems, Vol. 38, pp. 22-30, 2020.
36. Merkepci, M., and Abobala, M., " On Some Novel Results About Split-Complex Numbers, The Diagonalization Problem And Applications To Public Key Asymmetric Cryptography", Journal of Mathematics, Hindawi, 2023.

Received 1/5/2023, Accepted 10/10/2023