Symbolic Neutrosophic and Plithogenic Marshall-Olkin Type I Class of Distributions

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Abstract: In this paper we present the symbolic neutrosophic and plithogenic Marshall-Olkin type I class of distributions. We derive the formal form of the cumulative distribution function and probability density function of neutrosophic and plithogenic Marshall-Olkin Type I class of distributions. As a special case of the mentioned class of distributions we study the generalized uniform distribution in both neutrosophic and plithogenic forms, we derive its PDF and CDF then present an algorithm of random numbers generation according to it, then we estimate its parameters using maximum likelihood estimation and support the results with a simulation study to show the efficiency of the calculated parameters and study its asymptotic properties including unbiasedness and consistency.

Keywords: Marshall-Olkin Type I Class of Distributions; Neutrosophic; Plithogenic; AH Isometry; Maximum Likelihood Estimation; Random Numbers Generation.

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1. Introduction

Neutrosophic Probability Theory and Plithogenic Probability Theory are both intriguing extensions of traditional probability theory that deal with uncertainty and ambiguity in a more nuanced and comprehensive manner. These theories were developed to address situations where classical probability theory falls short in capturing the complexity of real-world uncertainties.

Neutrosophic Probability Theory is an extension of classical probability theory that introduces the concept of "Neutrosophy." Neutrosophy deals with indeterminacy, ambiguity, and imprecision that arise in various fields such as philosophy, mathematics, and decision-making[1]–[31].

Plithogenic Probability Theory is another extension of classical probability theory that aims to address the limitations of traditional probability theory in handling complex uncertainties. It introduces the concept of "Plithogeny," which deals with the multitude of conditions that contribute to the occurrence or non-occurrence of an event. Unlike classical probability theory, where events are often treated as independent and isolated, plithogenic probability theory recognizes that events are influenced by a multitude of interconnected factors. It also focuses on understanding how various conditions interact and contribute to the overall probability of an event. This theory is particularly useful in scenarios involving interdependent events, network analysis, and systems with intricate dependencies.[32]–[47]

zenodo.10031226/10.5281

In this paper we will deal with symbolic neutrosophic sets and symbolic plithogenic sets where the elements of these sets take the form $N = a + bI; I^2 = I$ for neutrosophic sets and $S = a + bP_1 + cP_2\,; P_1^2 = P_1, P_2^2 = P_2, P_1 \cdot P_2 = P_2 \cdot P_1 = P_2$ for plithogenic sets and we will generalize the well know Marshall Olkin class of distributions [48]–[55] to both neutrosophic and plithogenic class.

2. Preliminaries

Definition 2.1

Let $R(I) = \{a + bI\,; a, b \in R\}$ be the neutrosophic field of reals where $I^2 = I$. One-dimensional AH-isometry between $R(I)$ and R^2 and its inverse are given by:

$$T: R(I) \to R^2\,;\ T(a + bI) = (a, a + b) \quad (1)$$

$$T^{-1}: R^2 \to R(I)\,;\ T^{-1}(a, b) = a + (b - a)I \quad (2)$$

Note:

T is an algebraic isomorphism and it preserves distances.

Definition 2.2

A neutrosophic random variable X_N is defined as follows:

$$X_N = X_1 + X_2 I\,; I^2 = I$$

Where X_1, X_2 are classical random variables.

Definition 2.3

Let $f: R(I) \to R(I); f = f(x_N), x_N \in R(I)$ then f is called a neutrosophic real function with one neutrosophic variable.

Definition 2.4

Let $a_N = a_1 + a_2 I, b_N = b_1 + b_2 I \in R(I)$ be neutrosophic numbers. We say that $a_N \geq_N b_N$ if:

$$a_1 \geq b_1\,, a_1 + a_2 \geq b_1 + b_2$$

Definition 2.5

Let $R(P_1, P_2) = \{a_0 + a_1 P_1 + a_2 P_2\,; a_0, a_1, a_2 \in R,\ P_1^2 = P_1,\ P_2^2 = P_2,\ P_1 \cdot P_2 = P_2 \cdot P_1 = P_2\}$ be the Plithogenic field of reals. One-dimensional isometry between $R(P_1, P_2)$ and R^3 and its inverse are defined as follows:

$$T: R(P_1, P_2) \to R^3\,;\ T(a_0 + a_1 P_1 + a_2 P_2) = (a_0\,, a_0 + a_1\,, a_0 + a_1 + a_2) \quad (3)$$

$$T^{-1}: R^3 \to R(P_1, P_2)\,;\ T^{-1}(a_0, a_1, a_2) = a_0 + (a_1 - a_0)P_1 + (a_2 - a_1)P_2 \quad (4)$$

Definition 2.6

A Plithogenic random variable X_P is defined as follows:

$X_P = X_0 + X_1 P_1 + X_2 P_2\,; P_1^2 = P_1, P_2^2 = P_2, P_1 \cdot P_2 = P_2 \cdot P_1 = P_2$ where X_0, X_1, X_2 are classical random variables.

Definition 2.7

Let $f: R(P_1, P_2) \to R(P_1, P_2); f = f(x_P)\,, x_P \in R(P_1, P_2)$ then f is called a Plithogenic real function with one plithogenic variable.

Definition 2.8

Let $a_P = a_0 + a_1 P_1 + a_2 P_2, b_P = b_0 + b_1 P_1 + b_2 P_2 \in R(P_1, P_2)$ be two plithogenic numbers. We say that $a_P \geq_P b_P$ if:

$$a_0 \geq b_0\,, a_0 + a_1 \geq b_0 + b_1, a_0 + a_1 + a_2 \geq b_0 + b_1 + b_2$$

3. Neutrosophic Marshall-Olkin Type I Class of Distributions:

In this section we are going to derive the neutrosophic form of Marshall-Olkin Type I class of distributions depending on its cumulative probability distribution function and probability distribution function and some generalized distributions according to it.

Definition 3.1

Neutrosophic Marshall Olkin Type I cumulative distribution function is classical Marshall-Olkin Type I cumulative distribution function but defined on $R(I)$, taking values in $R(I)$ and with parameters from $R(I)$, that is its CDF is:

$$G(x_N;\rho_N)=\frac{F(x_N)}{F(x_N)(1-\rho_N)+\rho_N};\ x_N\in R(I)\ \&\ 0<_N\rho_N<_N 1 \tag{5}$$

Theorem 3.1

The neutrosophic formal form of (5) is:

$$G(x_N;\rho_N)=\frac{F(x_1)}{F(x_1)(1-\rho_1)+\rho_1}+I\left[\frac{F(x_1+x_2)}{F(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)}-\frac{F(x_1)}{F(x_1)(1-\rho_1)+\rho_1}\right] \tag{6}$$

Proof

$$T[G(x_N;\rho_N)]=\frac{T[F(x_N)]}{T[F(x_N)]T[1-\rho_N]+T[\rho_N]}$$

$$=\frac{\big(F(x_1),F(x_1+x_2)\big)}{\big(F(x_1),F(x_1+x_2)\big)\big(1-\rho_1,1-(\rho_1+\rho_2)\big)+(\rho_1,\rho_1+\rho_2)}$$

$$=\left(\frac{\big(F(x_1),F(x_1+x_2)\big)}{\Big(F(x_1)(1-\rho_1)+\rho_1,\big(F(x_1+x_2)\big)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)\Big)}\right)$$

$$=\left(\frac{F(x_1)}{F(x_1)(1-\rho_1)+\rho_1},\frac{F(x_1+x_2)}{F(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)}\right)$$

So:

$$G(x_N;\rho_N)=T^{-1}\left(\left(\frac{F(x_1)}{F(x_1)(1-\rho_1)+\rho_1},\frac{F(x_1+x_2)}{F(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)}\right)\right)$$

$$=\frac{F(x_1)}{F(x_1)(1-\rho_1)+\rho_1}+I\left[\frac{F(x_1+x_2)}{F(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)}-\frac{F(x_1)}{F(x_1)(1-\rho_1)+\rho_1}\right]$$

Note:

Neutrosophic probability distribution function of Marshall-Olkin Type I class of distributions can be derived by direct derivation of equation (5).

$$g(x_N;\rho_N)=\frac{\rho_N f(x_N)}{[(1-\rho_N)F(x_N)+\rho_N]^2};\ x_N\in R(I)\ \&\ 0<_N\rho_N<_N 1 \tag{7}$$

Theorem 3.2

The neutrosophic formal form of (7) is:

$$g(x_N;\rho_N)=\frac{\rho_1 f(x_1)}{[(1-\rho_1)F(x_1)+\rho_1]^2}+I\left[\frac{(\rho_1+\rho_2)f(x_1+x_2)}{\left[(1-(\rho_1+\rho_2))F(x_1+x_2)+\rho_1+\rho_2\right]^2}-\frac{\rho_1 f(x_1)}{[(1-\rho_1)F(x_1)+\rho_1]^2}\right] \quad (8)$$

Proof

$$T[g(x_N;\rho_N)]=\frac{T[\rho_N]T[f(x_N)]}{\left[T[(1-\rho_N)]T[F(x_N)]+T[\rho_N]\right]^2}=\frac{(\rho_1,\rho_1+\rho_2)\big(f(x_1),f(x_1+x_2)\big)}{\left[(1-\rho_1,1-(\rho_1+\rho_2))\big(F(x_1),F(x_1+x_2)\big)+(\rho_1,\rho_1+\rho_2)\right]^2}$$

$$=\frac{\big(\rho_1 f(x_1),(\rho_1+\rho_2)f(x_1+x_2)\big)}{\left[\big((1-\rho_1)F(x_1)+\rho_1,(1-(\rho_1+\rho_2))F(x_1+x_2)+\rho_1+\rho_2\big)\right]^2}$$

$$=\left(\frac{\rho_1 f(x_1)}{[(1-\rho_1)F(x_1)+\rho_1]^2},\frac{(\rho_1+\rho_2)f(x_1+x_2)}{\left[(1-(\rho_1+\rho_2))F(x_1+x_2)+\rho_1+\rho_2\right]^2}\right)$$

So:

$$g(x_N;\rho_N)=T^{-1}\left(\frac{\rho_1 f(x_1)}{[(1-\rho_1)F(x_1)+\rho_1]^2},\frac{(\rho_1+\rho_2)f(x_1+x_2)}{\left[(1-(\rho_1+\rho_2))F(x_1+x_2)+\rho_1+\rho_2\right]^2}\right)$$

$$=\frac{\rho_1 f(x_1)}{[(1-\rho_1)F(x_1)+\rho_1]^2}+I\left[\frac{(\rho_1+\rho_2)f(x_1+x_2)}{\left[(1-(\rho_1+\rho_2))F(x_1+x_2)+\rho_1+\rho_2\right]^2}-\frac{\rho_1 f(x_1)}{[(1-\rho_1)F(x_1)+\rho_1]^2}\right]$$

Theorem 3.3

Equation (8) represents probability density function in classical sense.

Proof

$$\int_{-\infty}^{+\infty} g(x_N;\rho_N)dx_N=\int_{-\infty}^{+\infty}\frac{\rho_1 f(x_1)}{[(1-\rho_1)F(x_1)+\rho_1]^2}dx_1$$

$$+I\left[\int_{-\infty}^{+\infty}\frac{(\rho_1+\rho_2)f(x_1+x_2)}{\left[(1-(\rho_1+\rho_2))F(x_1+x_2)+\rho_1+\rho_2\right]^2}d(x_1+x_2)-\int_{-\infty}^{+\infty}\frac{\rho_1 f(x_1)}{[(1-\rho_1)F(x_1)+\rho_1]^2}dx_1\right]$$

$$=\int_{-\infty}^{+\infty}d\left(\frac{F(x_1)}{F(x_1)(1-\rho_1)+\rho_1}\right)$$

$$+I\left[\int_{-\infty}^{+\infty}d\left(\frac{F(x_1+x_2)}{F(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)}\right)-\int_{-\infty}^{+\infty}d\left(\frac{F(x_1)}{F(x_1)(1-\rho_1)+\rho_1}\right)\right]=1$$

Also, it is easy to see that $T[g(x_N;\rho_N)]$ presents two continuous and positive functions. Depending on [3], [25] we conclude that the given neutrosophic function is a neutrosophic probability density function in classical sense.

4. Neutrosophic Marshall-Olkin Type I Uniform distribution:

Definition 4.1

The neutrosophic cumulative distribution function of the neutrosophic Marshall-Olkin Type I uniform distribution is defined as follows:

$$G(x_N;\rho_N,a_N,b_N)=\frac{x_N-a_N}{x_N(1-\rho_N)+\rho_N b_N-a_N}\ ;a_N<_N x_N<_N b_N\ ,0<_N\rho_N<_N 1 \quad (9)$$

Theorem 4.1

The neutrosophic formal form of (9) is:

$$G(x_N;\rho_N,a_N,b_N)=\frac{x_1-a_1}{x_1(1-\rho_1)+\rho_1b_1-a_1}+I\left[\frac{(x_1+x_2)-(a_1+a_2)}{(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)}-\frac{x_1-a_1}{x_1(1-\rho_1)+\rho_1b_1-a_1}\right] \quad (10)$$

Proof

$$T[G(x_N;\rho_N,a_N,b_N)]=\frac{T[x_N]-T[a_N]}{T[x_N]T[1-\rho_N]+T[\rho_N]\cdot T[b_N]-T[a_N]}$$

$$=\frac{(x_1,x_1+x_2)-(a_1,a_1+a_2)}{(x_1,x_1+x_2)\big(1-\rho_1,1-(\rho_1+\rho_2)\big)+(\rho_1,\rho_1+\rho_2)\cdot(b_1,b_1+b_2)-(a_1,a_1+a_2)}$$

$$=\left(\frac{\big(x_1-a_1,(x_1+x_2)-(a_1+a_2)\big)}{\Big(x_1(1-\rho_1)+\rho_1b_1-a_1,(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)\Big)}\right)$$

$$=\left(\frac{x_1-a_1}{x_1(1-\rho_1)+\rho_1b_1-a_1},\frac{(x_1+x_2)-(a_1+a_2)}{(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)}\right)$$

$$G(x_N;\rho_N,a_N,b_N)=T^{-1}\left(\frac{x_1-a_1}{x_1(1-\rho_1)+\rho_1b_1-a_1},\frac{(x_1+x_2)-(a_1+a_2)}{(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)}\right)$$

$$=\frac{x_1-a_1}{x_1(1-\rho_1)+\rho_1b_1-a_1}$$

$$+I\left[\frac{(x_1+x_2)-(a_1+a_2)}{(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)}-\frac{x_1-a_1}{x_1(1-\rho_1)+\rho_1b_1-a_1}\right]$$

Definition 4.2

The neutrosophic probability distribution function of neutrosophic Marshall-Olkin Type I uniform distribution is defined as follows:

$$g(x_N;\rho_N,a_N,b_N)=\frac{(b_N-a_N)\rho_N}{[x_N(1-\rho_N)+\rho_Nb_N-a_N]^2}\ ;\ a_N<_N x_N<_N b_N\,,0<_N\rho_N<_N 1 \quad (11)$$

Theorem 4.2

The neutrosophic formal form of (11) is:

$$g(x_N;\rho_N,a_N,b_N)=\frac{(b_1-a_1)\rho_1}{[x_1(1-\rho_1)+\rho_1b_1-a_1]^2}+I\left[\frac{\big((b_1+b_2)-(a_1+a_2)\big)(\rho_1+\rho_2)}{\big[(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)\big]^2}-\frac{(b_1-a_1)\rho_1}{[x_1(1-\rho_1)+\rho_1b_1-a_1]^2}\right] \quad (12)$$

Proof

$$T[g(x_N;\rho_N,a_N,b_N)]=\frac{(T[b_N]-T[a_N])T[\rho_N]}{\big[T[x_N]T[(1-\rho_N)]+T[\rho_N]\cdot T[b_N]-T[a_n]\big]^2}$$

$$=\frac{\big((b_1,b_1+b_2)-(a_1,a_1+a_2)\big)(\rho_1,\rho_1+\rho_2)}{\big[(x_1,x_1+x_2)\big(1-\rho_1,1-(\rho_1+\rho_2)\big)+(\rho_1,\rho_1+\rho_2)\cdot(b_1,b_1+b_2)-(a_1,a_1+a_2)\big]^2}$$

$$=\frac{\Big((b_1-a_1)\rho_1,\big((b_1+b_2)-(a_1+a_2)\big)(\rho_1+\rho_2)\Big)}{\Big[\Big(x_1(1-\rho_1)+\rho_1b_1-a_1,(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)\Big)\Big]^2}$$

$$=\left(\frac{(b_1-a_1)\rho_1}{[x_1(1-\rho_1)+\rho_1b_1-a_1]^2},\frac{\big((b_1+b_2)-(a_1+a_2)\big)(\rho_1+\rho_2)}{\big[(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)\big]^2}\right)$$

So:

$$g(x_N;\rho_N,a_N,b_N)=T^{-1}\left(\frac{(b_1-a_1)\rho_1}{[x_1(1-\rho_1)+\rho_1 b_1-a_1]^2},\frac{\big((b_1+b_2)-(a_1+a_2)\big)(\rho_1+\rho_2)}{\big[(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)\big]^2}\right)$$

$$=\frac{(b_1-a_1)\rho_1}{[x_1(1-\rho_1)+\rho_1 b_1-a_1]^2}$$

$$+I\left[\frac{\big((b_1+b_2)-(a_1+a_2)\big)(\rho_1+\rho_2)}{\big[(x_1+x_2)\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)\big]^2}-\frac{(b_1-a_1)\rho_1}{[x_1(1-\rho_1)+\rho_1 b_1-a_1]^2}\right]$$

4.1 Parameters' estimation using neutrosophic maximum likelihood estimation method:

Let $\mathbb{X}_N$ a neutrosophic random sample drawn from neutrosophic Marshall-Olkin Type I uniform distribution with PDF defined in (11) then the neutrosophic likelihood function will be:

$$L_N=L(\mathbb{X}_N;\Theta)=\prod_{i=1}^{n}f(X_{iN};a_N,b_N,\rho_N)=\prod_{i=1}^{n}\frac{(b_N-a_N)\rho_N}{[x_{iN}(1-\rho_N)+\rho_N b_N-a_N]^2}$$

$$=\frac{(b_N-a_N)^n\rho_N^n}{\prod_{i=1}^{n}[x_{iN}(1-\rho_N)+\rho_N b_N-a_N]^2} \qquad (13)$$

By taking log of (13), we get the loglikelihood function as follows:

$$\mathcal{L}_N=\ln L(\mathbb{X}_N;\Theta)=n\ln(b_N-a_N)+n\ln\rho_N-2\sum_{i=1}^{n}\ln[x_{iN}(1-\rho_N)+\rho_N b_N-a_N] \qquad (14)$$

Taking partial derivatives of previous equation according to a_N,b_N,ρ_N yields to:

$$\frac{\partial}{\partial a_N}\mathcal{L}_N=\frac{-n}{b_N-a_N}+2\sum_{i=1}^{n}\frac{1}{[x_{iN}(1-\rho_N)+\rho_N b_N-a_N]} \qquad (15)$$

$$\frac{\partial}{\partial b_N}\mathcal{L}_N=\frac{n}{b_N-a_N}-2\sum_{i=1}^{n}\frac{\rho_N}{[x_{iN}(1-\rho_N)+\rho_N b_N-a_N]} \qquad (16)$$

$$\frac{\partial}{\partial a_N}\mathcal{L}_N=\frac{n}{\rho_N}+2\sum_{i=1}^{n}\frac{x_{iN}-b_N}{[x_{iN}(1-\rho_N)+\rho_N b_N-a_N]} \qquad (17)$$

Using the AH-Isometry we get:

$$\begin{cases}\dfrac{\partial}{\partial a_1}\mathcal{L}_1=\dfrac{-n}{b_1-a_1}+2\displaystyle\sum_{i=1}^{n}\frac{1}{[x_{i1}(1-\rho_1)+\rho_1 b_1-a_1]}\\ \dfrac{\partial}{\partial(a_1+a_2)}(\mathcal{L}_1+\mathcal{L}_2)=\dfrac{-n}{(b_1+b_2)-(a_1+a_2)}+2\displaystyle\sum_{i=1}^{n}\frac{1}{\big[(x_{i1}+x_{i2})\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)\big]}\end{cases} \qquad (18)$$

$$\begin{cases}\dfrac{\partial}{\partial b_1}\mathcal{L}_1=\dfrac{n}{b_1-a_1}-2\displaystyle\sum_{i=1}^{n}\frac{\rho_1}{[x_{i1}(1-\rho_1)+\rho_1 b_1-a_1]}\\ \dfrac{\partial}{\partial(b_1+b_2)}(\mathcal{L}_1+\mathcal{L}_2)=\dfrac{n}{(b_1+b_2)-(a_1+a_2)}-2\displaystyle\sum_{i=1}^{n}\frac{(\rho_1+\rho_2)}{\big[(x_{i1}+x_{i2})\big(1-(\rho_1+\rho_2)\big)+(\rho_1+\rho_2)(b_1+b_2)-(a_1+a_2)\big]}\end{cases} \qquad (19)$$

$$\begin{cases} \frac{\partial}{\partial \rho_1}\mathcal{L}_1 = \frac{n}{\rho_1} + 2\sum_{i=1}^{n}\frac{x_{i1} - b_1}{[x_{i1}(1-\rho_1) + \rho_1 b_1 - a_1]} \\ \frac{\partial}{\partial(\rho_1+\rho_2)}(\mathcal{L}_1 + \mathcal{L}_2) = \frac{n}{(\rho_1+\rho_2)} + 2\sum_{i=1}^{n}\frac{(x_{i1}+x_{i2}) - (b_1+b_2)}{\left[(x_{i1}+x_{i2})\left(1-(\rho_1+\rho_2)\right) + (\rho_1+\rho_2)(b_1+b_2) - (a_1+a_2)\right]} \end{cases} \quad (20)$$

Solving equations (18-20) numerically yields to the desired estimators.

4.2 Simulation and random numbers generation:

Solving equation (9) with respect to x_N yields to:

$$x_N = \frac{y_N \rho_N b_N - a_N y_N + a_N}{1 - y_N(1-\rho_N)} \quad (21)$$

Where $y_N = F_N(x_N)$ is neutrosophic uniformly distributed on [0,1]
By taking AH-isometry to (21) we get:

$$x_1 = \frac{y_1 \rho_1 b_1 - a_1 y_1 + a_1}{1 - y_1(1-\rho_1)} \quad (22)$$

$$x_1 + x_2 = \frac{(y_1+y_2)(\rho_1+\rho_2)(b_1+b_2) - (a_1+a_2)(y_1+y_2) + (a_1+a_2)}{1 - (y_1+y_2)\left(1-(\rho_1+\rho_2)\right)} \quad (23)$$

Using equations (22-23), we can generate classical random numbers following classical Marshall-Olkin type I uniform distribution with chosen parameters, then using T^{-1} we will get neutrosophic Marshall-Olkin type I uniform numbers.
Monte Carlo simulation is done using Maple software with total replication of $N = 1000$ times and with sample sizes of $15, 50, 100, 150$ and fixed parameters $a_N = 1 + 2I, b_N = 2 + 5I, \rho_N = 0.5 + 0.1I$. We can check goodness of our estimations based on bias of the estimators and mean square error of it using the following equations:

$$Bias = \frac{\sum_{i=1}^{n}\left|\hat{\theta}_{iN} - \theta_N\right|}{n} \quad (24)$$

$$MSE = \frac{\sum_{i=1}^{n}\left(\hat{\theta}_{iN} - \theta_N\right)^2}{n} \quad (25)$$

Table 1. Simulation results of neutrosophic Marshall-Olkin type I uniform distribution.

	$\boldsymbol{a_N = 1 + 2I}$		
$\boldsymbol{n}$	$\boldsymbol{\hat{a}_N}$	$\boldsymbol{Bias\ \hat{a}_N}$	$\boldsymbol{MSE\ \hat{a}_N}$
15	$1.03255 + 2.12169I$	$0.03255 + 0.12169I$	$0.00214 + 0.04528I$
50	$1.00994 + 2.03760I$	$0.00994 + 0.03760I$	$0.00020 + 0.00431I$
100	$1.00500 + 2.01894I$	$0.00500 + 0.01894I$	$0.00005 + 0.00446I$
150	$1.00339 + 2.01285I$	$0.00339 + 0.01285I$	$0.00002 + 0.00049I$
	$\boldsymbol{b_N = 2 + 5I}$		
$\boldsymbol{n}$	$\boldsymbol{\hat{b}_N}$	$\boldsymbol{Bias\ \hat{b}_N}$	$\boldsymbol{MSE\ \hat{b}_N}$
15	$1.88284 + 4.71224I$	$0.11716 + 0.28776I$	$0.02464 + 0.27718I$
50	$1.96282 + 4.91173I$	$0.03718 + 0.08827I$	$0.00267 + 0.02810I$
100	$1.98031 + 4.95365I$	$0.01969 + 0.04635I$	$0.00072 + 0.00740I$

150	$1.98730 + 4.97020I$	$0.01270 + 0.02980I$	$0.00032 + 0.00323I$
	$\boldsymbol{\rho_N = 0.5 + 0.1I}$		
$\boldsymbol{n}$	$\boldsymbol{\hat{\rho}_N}$	$\boldsymbol{Bias\ \hat{\rho}_N}$	$\boldsymbol{MSE\ \hat{\rho}_N}$
15	$0.61207 + 0.10236I$	$0.23605 + 0.03591I$	$0.11861 + 0.03663I$
50	$0.53516 + 0.10201I$	$0.10746 + 0.01942I$	$0.01990 + 0.00782I$
100	$0.51818 + 0.10096I$	$0.07376 + 0.01414I$	$0.00945 + 0.00391I$
150	$0.51377 + 0.10103I$	$0.06017 + 0.01164I$	$0.00591 + 0.00249I$

Table (1) shows that as sample size n increases, bias of the estimators and mean square error of it decrease which means that our estimators are asymptotically unbiased and consistent.

5. Plithogenic Marshall-Olkin Type I Class of Distributions

In this section we are going to construct plithogenic form of Marshall-Olkin Type I class of distributions, cumulative probability distribution function, probability distribution function and uniform generalized distribution according to it.

Definition 5.1

The plithogenic form of cumulative distribution function of the first type of Marshall-Olkin Type I class of distributions is defined as follows:

$$G(x_P;\rho_P) = \frac{F(x_P)}{F(x_P)(1-\rho_P)+\rho_P}\ ; x_P \in R(P_1,P_2)\ , 0 <_P \rho_P <_P 1 \tag{26}$$

Where $P_1^2 = P_1,\ P_2^2 = P_2,\ P_1 \cdot P_2 = P_2 \cdot P_1 = P_2$.

Theorem 5.1

The plithogenic formal form of (26) is:

$$\begin{aligned} G(x_P;\rho_P) &= \frac{F(x_0)}{F(x_0)(1-\rho_0)+\rho_0} + P_1\left[\frac{F(x_0+x_1)}{F(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)} - \frac{F(x_0)}{F(x_0)(1-\rho_0)+\rho_0}\right] \\ &+ P_2\left[\frac{F(x_0+x_1+x_2)}{F(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)}\right. \\ &\left. - \frac{F(x_0+x_1)}{F(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)}\right] \end{aligned} \tag{27}$$

Proof

$$T[G(x_P;\rho_P)] = \frac{T[F(x_P)]}{T[F(x_P)]T[1-\rho_P]+T[\rho_P]}$$

$$= \frac{\big(F(x_0),F(x_0+x_1),F(x_0+x_1+x_2)\big)}{\big(F(x_0),F(x_0+x_1),F(x_0+x_1+x_2)\big)\Big(1-\rho_0,1-(\rho_0+\rho_1),\big(1-(\rho_0+\rho_1+\rho_2)\big)\Big)+\big(\rho_0,(\rho_0+\rho_1),(\rho_0+\rho_1+\rho_2)\big)}$$

$$= \left(\frac{\big(F(x_0),F(x_0+x_1),F(x_0+x_1+x_2)\big)}{\Big(F(x_0)(1-\rho_0)+\rho_0,F(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1),F(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)\Big)}\right)$$

$$= \left(\frac{F(x_0)}{F(x_0)(1-\rho_0)+\rho_0},\frac{F(x_0+x_1)}{F(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)},\frac{F(x_0+x_1+x_2)}{F(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)}\right)$$

So:

$$G(x_P;\rho_P)$$
$$= T^{-1}\left(\frac{F(x_0)}{F(x_0)(1-\rho_0)+\rho_0},\frac{F(x_0+x_1)}{F(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)},\frac{F(x_0+x_1+x_2)}{F(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)}\right)$$
$$= \frac{F(x_0)}{F(x_0)(1-\rho_0)+\rho_0} + P_1\left[\frac{F(x_0+x_1)}{F(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)} - \frac{F(x_0)}{F(x_0)(1-\rho_0)+\rho_0}\right]$$
$$+ P_2\left[\frac{F(x_0+x_1+x_2)}{F(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)} - \frac{F(x_0+x_1)}{F(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)}\right]$$

Note:

Plithogenic probability distribution function of Marshall-Olkin Type I class of distributions can be derived by direct derivation of equation (26).

$$g(x_P;\rho_P) = \frac{\rho_P f(x_P)}{[(1-\rho_P)F(x_P)+\rho_P]^2}\ ; x_P \in R(P_1,P_2), 0 <_P \rho_P <_P 1 \quad (28)$$

Where $P_1^2 = P_1,\ P_2^2 = P_2,\ P_1 \cdot P_2 = P_2 \cdot P_1 = P_2$.

Theorem 5.2

The Plithogenic formal form of (28) is:

$$g(x_P;\rho_P) = \frac{\rho_0 f(x_0)}{[(1-\rho_0)F(x_0)+\rho_0]^2} + P_1\left[\frac{(\rho_0+\rho_1)f(x_0+x_1)}{\big[\big(1-(\rho_0+\rho_1)\big)F(x_0+x_1)+(\rho_0+\rho_1)\big]^2} - \frac{\rho_0 f(x_0)}{[(1-\rho_0)F(x_0)+\rho_0]^2}\right]$$
$$+ P_2\left[\frac{(\rho_0+\rho_1+\rho_2)f(x_0+x_1+x_2)}{\big[\big(1-(\rho_0+\rho_1+\rho_2)\big)F(x_0+x_1+x_2)+(\rho_0+\rho_1+\rho_2)\big]^2} - \frac{(\rho_0+\rho_1)f(x_0+x_1)}{\big[\big(1-(\rho_0+\rho_1)\big)F(x_0+x_1)+(\rho_0+\rho_1)\big]^2}\right] \quad (29)$$

Proof

$$T[g(x_P;\rho_P)] = \frac{T[\rho_P]T[f(x_P)]}{\big[T[(1-\rho_P)]T[F(x_P)]+T[\rho_P]\big]^2}$$
$$= \frac{(\rho_0,\rho_0+\rho_1,\rho_0+\rho_1+\rho_2)\big(f(x_0),f(x_0+x_1),f(x_0+x_1+x_2)\big)}{\big[\big(1-\rho_0,1-(\rho_0+\rho_1),1-(\rho_0+\rho_1+\rho_2)\big)\big(F(x_0),F(x_0+x_1),F(x_0+x_1+x_2)\big)+(\rho_0,\rho_0+\rho_1,\rho_0+\rho_1+\rho_2)\big]^2}$$
$$= \frac{\big(\rho_0 f(x_0),(\rho_0+\rho_1)f(x_0+x_1),(\rho_0+\rho_1+\rho_2)f(x_0+x_1+x_2)\big)}{\Big[\Big((1-\rho_0)F(x_0)+\rho_0,\big(1-(\rho_0+\rho_1)\big)F(x_0+x_1)+(\rho_0+\rho_1),\big(1-(\rho_0+\rho_1+\rho_2)\big)F(x_0+x_1+x_2)+(\rho_0+\rho_1+\rho_2)\Big)\Big]^2}$$
$$= \left(\frac{\rho_0 f(x_0)}{[(1-\rho_0)F(x_0)+\rho_0]^2},\frac{(\rho_0+\rho_1)f(x_0+x_1)}{\big[\big(1-(\rho_0+\rho_1)\big)F(x_0+x_1)+(\rho_0+\rho_1)\big]^2},\frac{(\rho_0+\rho_1+\rho_2)f(x_0+x_1+x_2)}{\big[\big(1-(\rho_0+\rho_1+\rho_2)\big)F(x_0+x_1+x_2)+(\rho_0+\rho_1+\rho_2)\big]^2}\right)$$

So:

$$g(x_N;\rho_N)$$
$$= T^{-1}\left(\frac{\rho_0 f(x_0)}{[(1-\rho_0)F(x_0)+\rho_0]^2},\frac{(\rho_0+\rho_1)f(x_0+x_1)}{\big[\big(1-(\rho_0+\rho_1)\big)F(x_0+x_1)+(\rho_0+\rho_1)\big]^2},\frac{(\rho_0+\rho_1+\rho_2)f(x_0+x_1+x_2)}{\big[\big(1-(\rho_0+\rho_1+\rho_2)\big)F(x_0+x_1+x_2)+(\rho_0+\rho_1+\rho_2)\big]^2}\right)$$
$$= \frac{\rho_0 f(x_0)}{[(1-\rho_0)F(x_0)+\rho_0]^2} + P_1\left[\frac{(\rho_0+\rho_1)f(x_0+x_1)}{\big[\big(1-(\rho_0+\rho_1)\big)F(x_0+x_1)+(\rho_0+\rho_1)\big]^2} - \frac{\rho_0 f(x_0)}{[(1-\rho_0)F(x_0)+\rho_0]^2}\right]$$
$$+ P_2\left[\frac{(\rho_0+\rho_1+\rho_2)f(x_0+x_1+x_2)}{\big[\big(1-(\rho_0+\rho_1+\rho_2)\big)F(x_0+x_1+x_2)+(\rho_0+\rho_1+\rho_2)\big]^2} - \frac{(\rho_0+\rho_1)f(x_0+x_1)}{\big[\big(1-(\rho_0+\rho_1)\big)F(x_0+x_1)+(\rho_0+\rho_1)\big]^2}\right]$$

Theorem 5.3

Equation (29) represents probability density function in classical sense.

Proof

$$T\left[\int_{-\infty}^{+\infty} g(x_P;\rho_P)dx_P\right]$$
$$= \int_{-\infty}^{+\infty} \frac{\rho_0 f(x_0)}{\left[(1-\rho_0)F(x_0)+\rho_0\right]^2}dx_0$$
$$+ P_1\left[\int_{-\infty}^{+\infty} \frac{(\rho_0+\rho_1)f(x_0+x_1)}{\left[\left(1-(\rho_0+\rho_1)\right)F(x_0+x_1)+(\rho_0+\rho_1)\right]^2}d(x_0+x_1)\right.$$
$$\left.-\int_{-\infty}^{+\infty} \frac{\rho_0 f(x_0)}{\left[(1-\rho_0)F(x_0)+\rho_0\right]^2}dx_0\right]$$
$$+ P_2\left[\int_{-\infty}^{+\infty} \frac{(\rho_0+\rho_1+\rho_2)f(x_0+x_1+x_2)}{\left[(1-(\rho_0+\rho_1+\rho_2))F(x_0+x_1+x_2)+(\rho_0+\rho_1+\rho_2)\right]^2}d(x_0+x_1+x_2)\right.$$
$$\left.-\int_{-\infty}^{+\infty} \frac{(\rho_0+\rho_1)f(x_0+x_1)}{\left[\left(1-(\rho_0+\rho_1)\right)F(x_0+x_1)+(\rho_0+\rho_1)\right]^2}d(x_0+x_1)\right]$$
$$= \int_{-\infty}^{+\infty} d\left(\frac{F(x_0)}{F(x_0)(1-\rho_0)+\rho_0}\right)$$
$$+ P_1\left[\int_{-\infty}^{+\infty} d\left(\frac{F(x_0)}{F(x_0)(1-\rho_0)+\rho_0}\right)-\int_{-\infty}^{+\infty} d\left(\frac{F(x_0+x_1)}{F(x_0+x_1)(1-(\rho_0+\rho_1))+(\rho_0+\rho_1)}\right)\right]$$
$$+ P_2\left[\int_{-\infty}^{+\infty} d\left(\frac{F(x_0+x_1+x_2)}{F(x_0+x_1+x_2)(1-(\rho_0+\rho_1+\rho_2))+(\rho_0+\rho_1+\rho_2)}\right)\right.$$
$$\left.-\int_{-\infty}^{+\infty} d\left(\frac{F(x_0+x_1)}{F(x_0+x_1)(1-(\rho_0+\rho_1))+(\rho_0+\rho_1)}\right)\right] = 1$$

Also, it is easy to see that $T[g(x_P;\rho_P)]$ presents three continuous and positive functions. Depending on this we can see that the given plithogenic function is a plithogenic probability density function in classical sense.

6. Plithogenic Marshall-Olkin Type I Uniform distribution:

Definition 6.1

Plithogenic cumulative distribution function of the Marshall-Olkin Type I uniform distribution is defined as follows:

$$G(x_P;\rho_P,a_P,b_P) = \frac{x_P-a_P}{x_P(1-\rho_P)+\rho_P b_P - a_P};\, a_P <_P x_P <_P b_P\,, 0 <_P \rho_P <_P 1 \quad (30)$$

Where $P_1^2 = P_1, P_2^2 = P_2, P_1 \cdot P_2 = P_2 \cdot P_1 = P_2$.

Theorem 6.1

The plithogenic formal form of (30) is:

$$G(x_P;\rho_P,a_P,b_P)=\frac{x_0-a_0}{x_0(1-\rho_0)+\rho_0 b_0-a_0}$$
$$+P_1\left[\frac{(x_0+x_1)-(a_0+a_1)}{(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)}-\frac{x_0-a_0}{x_0(1-\rho_0)+\rho_0 b_0-a_0}\right]$$
$$+P_2\left[\frac{(x_0+x_1+x_2)-(a_0+a_1+a_2)}{(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)}\right.$$
$$\left.-\frac{(x_0+x_1)-(a_0+a_1)}{(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)}\right] \quad (31)$$

Proof

$$T[G(x_P;\rho_P,a_P,b_P)]=\frac{T[x_P]-T[a_P]}{T[x_P]T[1-\rho_P]+T[\rho_P]\cdot T[b_P]-T[a_P]}$$

$$=\frac{(x_0,x_0+x_1,x_0+x_1+x_2)-(a_0,a_0+a_1,a_0+a_1+a_2)}{(x_0,x_0+x_1,x_0+x_1+x_2)\big(1-\rho_0,1-(\rho_0+\rho_1),1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0,\rho_0+\rho_1,\rho_0+\rho_1+\rho_2)\cdot(b_0,b_0+b_1,b_0+b_1+b_2)-(a_0,a_0+a_1,a_0+a_1+a_2)}$$

$$=\left(\frac{\big(x_0-a_0,(x_0+x_1)-(a_0+a_1),(x_0+x_1+x_2)-(a_0+a_1+a_2)\big)}{\big(x_0(1-\rho_0)+\rho_0 b_0-a_0,(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1),(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)\big)}\right)$$

$$=\left(\frac{x_0-a_0}{x_0(1-\rho_0)+\rho_0 b_0-a_0},\frac{(x_0+x_1)-(a_0+a_1)}{(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)},\right.$$
$$\left.\frac{(x_0+x_1+x_2)-(a_0+a_1+a_2)}{(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)}\right)$$

$$G(x_P;\rho_P,a_P,b_P)=T^{-1}\left(\frac{x_0-a_0}{x_0(1-\rho_0)+\rho_0 b_0-a_0},\frac{(x_0+x_1)-(a_0+a_1)}{(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)},\right.$$
$$\left.\frac{(x_0+x_1+x_2)-(a_0+a_1+a_2)}{(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)}\right)$$

$$=\frac{x_0-a_0}{x_0(1-\rho_0)+\rho_0 b_0-a_0}+P_1\left[\frac{(x_0+x_1)-(a_0+a_1)}{(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)}-\frac{x_0-a_0}{x_0(1-\rho_0)+\rho_0 b_0-a_0}\right]$$
$$+P_2\left[\frac{(x_0+x_1+x_2)-(a_0+a_1+a_2)}{(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)}\right.$$
$$\left.-\frac{(x_0+x_1)-(a_0+a_1)}{(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)}\right]$$

Definition 6.2

Plithogenic probability distribution function of Marshall-Olkin Type I uniform distribution is defined as follows:

$$g(x_P;\rho_P,a_P,b_P)=\frac{(b_P-a_P)\rho_P}{\big[x_P(1-\rho_P)+\rho_P b_p-a_P\big]^2};a_P<_P x_P<_P b_P\,,0<_P\rho_P<_P 1 \quad (32)$$

Where $P_1^2=P_1,P_2^2=P_2,P_1\cdot P_2=P_2\cdot P_1=P_2$.

Theorem 6.2

Plithogenic formal form of (32) is:

$$g(x_P;\rho_P,a_P,b_P)=\frac{(b_0-a_0)\rho_0}{[x_0(1-\rho_0)+\rho_0b_0-a_0]^2}+P_1\left[\frac{\big((b_0+b_1)-(a_0+a_1)\big)(\rho_0+\rho_1)}{\big[(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)\big]^2}-\frac{(b_0-a_0)\rho_0}{[x_0(1-\rho_0)+\rho_0b_0-a_0]^2}\right]+P_2\left[\frac{\big((b_0+b_1+b_2)-(a_0+a_1+a_2)\big)(\rho_0+\rho_1+\rho_2)}{\big[(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)\big]^2}-\frac{\big((b_0+b_1)-(a_0+a_1)\big)(\rho_0+\rho_1)}{\big[(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)\big]^2}\right] \quad (33)$$

Proof

$$T[g(x_P;\rho_P,a_P,b_P)]=\frac{(T[b_P]-T[a_P])T[\rho_P]}{\big[T[x_P]T[(1-\rho_P)]+T[\rho_P]\cdot T[b_P]-T[a_P]\big]^2}$$

$$=\frac{\big((b_0,b_0+b_1,b_0+b_1+b_2)-(a_0,a_0+a_1,a_0+a_1+a_2)\big)(\rho_0,\rho_0+\rho_1,\rho_0+\rho_1+\rho_2)}{\big[(x_0,x_0+x_1,x_0+x_1+x_2)\big(1-\rho_0,1-(\rho_0+\rho_1),1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0,\rho_0+\rho_1,\rho_0+\rho_1+\rho_2)\cdot(b_0,b_0+b_1,b_0+b_1+b_2)-(a_0,a_0+a_1,a_0+a_1+a_2)\big]^2}$$

$$=\frac{\Big((b_0-a_0)\rho_0,\big((b_0+b_1)-(a_0+a_1)\big)(\rho_0+\rho_1),\big((b_0+b_1+b_2)-(a_0+a_1+a_2)\big)(\rho_0+\rho_1+\rho_2)\Big)}{\Big[\Big(x_0(1-\rho_0)+\rho_0b_0-a_0,(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1),(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)\Big)\Big]^2}$$

$$=\left(\frac{(b_0-a_0)\rho_0}{[x_0(1-\rho_0)+\rho_0b_0-a_0]^2},\frac{\big((b_0+b_1)-(a_0+a_1)\big)(\rho_0+\rho_1)}{\big[(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)\big]^2},\frac{\big((b_0+b_1+b_2)-(a_0+a_1+a_2)\big)(\rho_0+\rho_1+\rho_2)}{\big[(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)\big]^2}\right)$$

So:

$$g(x_P;\rho_P,a_P,b_P)=T^{-1}\left(\frac{(b_0-a_0)\rho_0}{[x_0(1-\rho_0)+\rho_0b_0-a_0]^2},\frac{\big((b_0+b_1)-(a_0+a_1)\big)(\rho_0+\rho_1)}{\big[(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)\big]^2},\frac{\big((b_0+b_1+b_2)-(a_0+a_1+a_2)\big)(\rho_0+\rho_1+\rho_2)}{\big[(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)\big]^2}\right)$$

$$=\frac{(b_0-a_0)\rho_0}{[x_0(1-\rho_0)+\rho_0b_0-a_0]^2}+P_1\left[\frac{\big((b_0+b_1)-(a_0+a_1)\big)(\rho_0+\rho_1)}{\big[(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)\big]^2}-\frac{(b_0-a_0)\rho_0}{[x_0(1-\rho_0)+\rho_0b_0-a_0]^2}\right]+P_2\left[\frac{\big((b_0+b_1+b_2)-(a_0+a_1+a_2)\big)(\rho_0+\rho_1+\rho_2)}{\big[(x_0+x_1+x_2)\big(1-(\rho_0+\rho_1+\rho_2)\big)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)\big]^2}-\frac{\big((b_0+b_1)-(a_0+a_1)\big)(\rho_0+\rho_1)}{\big[(x_0+x_1)\big(1-(\rho_0+\rho_1)\big)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)\big]^2}\right]$$

6.1 Parameters' estimation of plithogenic Marshall-Olkin Type I Uniform distribution:

Let $\mathbb{X}_P$ a plithogenic random sample drawn from plithogenic Marshall-Olkin Type I uniform distribution with PDF defined in (32) then the plithogenic likelihood function will be:

$$L_P = L(\mathbb{X}_P;\Theta) = \prod_{i=1}^{n} f(X_{iP}; a_P, b_P, \rho_P) = \prod_{i=1}^{n} \frac{(b_P - a_P)\rho_P}{[x_{iP}(1-\rho_P) + \rho_P b_P - a_P]^2}$$

$$= \frac{(b_P - a_P)^n \rho_P^n}{\prod_{i=1}^{n}[x_{iP}(1-\rho_P) + \rho_P b_P - a_P]^2} \quad (34)$$

By taking log of (34), we get the loglikelihood function as follows:

$$\mathcal{L}_P = \ln L(\mathbb{X}_P;\Theta) = n\ln(b_P - a_P) + n\ln\rho_P - 2\sum_{i=1}^{n}\ln[x_{iP}(1-\rho_P) + \rho_P b_P - a_P] \quad (35)$$

Taking partial derivatives of previous equation according to a_P, b_P, ρ_P yields to:

$$\frac{\partial}{\partial a_P}\mathcal{L}_P = \frac{-n}{b_P - a_P} + 2\sum_{i=1}^{n}\frac{1}{[x_{iP}(1-\rho_P) + \rho_P b_P - a_P]} \quad (36)$$

$$\frac{\partial}{\partial b_P}\mathcal{L}_P = \frac{n}{b_P - a_P} - 2\sum_{i=1}^{n}\frac{\rho_P}{[x_{iP}(1-\rho_P) + \rho_P b_P - a_P]} \quad (37)$$

$$\frac{\partial}{\partial a_P}\mathcal{L}_P = \frac{n}{\rho_P} + 2\sum_{i=1}^{n}\frac{x_{iP} - b_P}{[x_{iP}(1-\rho_P) + \rho_P b_P - a_P]} \quad (38)$$

Using the AH-Isometry equations (36-38) become:

$$\frac{\partial}{\partial a_0}\mathcal{L}_0 = \frac{-n}{b_0 - a_0} + 2\sum_{i=1}^{n}\frac{1}{[x_{i0}(1-\rho_0) + \rho_0 b_0 - a_0]}$$

$$\frac{\partial}{\partial(a_0+a_1)}(\mathcal{L}_0+\mathcal{L}_1) = \frac{-n}{(b_0+b_1)-(a_0+a_1)} + 2\sum_{i=1}^{n}\frac{1}{\left[(x_{i0}+x_{i1})\big(1-(\rho_0+\rho_1)\big) + (\rho_0+\rho_1)(b_0+b_1) - (a_0+a_1)\right]}$$

$$\frac{\partial}{\partial(a_0+a_1+a_2)}(\mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2)$$

$$= \frac{-n}{(b_0+b_1+b_2)-(a_0+a_1+a_2)}$$

$$+ 2\sum_{i=1}^{n}\frac{1}{\left[(x_{i0}+x_{i1}+x_{i2})\big(1-(\rho_0+\rho_1+\rho_2)\big) + (\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2) - (a_0+a_1+a_2)\right]}$$

$$\frac{\partial}{\partial b_0}\mathcal{L}_0 = \frac{n}{b_0 - a_0} - 2\sum_{i=1}^{n}\frac{\rho_0}{[x_{i0}(1-\rho_0) + \rho_0 b_0 - a_0]}$$

$$\frac{\partial}{\partial(b_0+b_1)}(\mathcal{L}_0+\mathcal{L}_1) = \frac{n}{(b_0+b_1)-(a_0+a_1)} - 2\sum_{i=1}^{n}\frac{(\rho_0+\rho_1)}{\left[(x_{i0}+x_{i1})\big(1-(\rho_0+\rho_1)\big) + (\rho_0+\rho_1)(b_0+b_1) - (a_0+a_1)\right]}$$

$$\frac{\partial}{\partial(b_0+b_1+b_2)}(\mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2)$$
$$=\frac{n}{(b_0+b_1+b_2)-(a_0+a_1+a_2)}$$
$$-2\sum_{i=1}^{n}\frac{(\rho_0+\rho_1+\rho_2)}{\left[(x_{i0}+x_{i1}+x_{i2})\left(1-(\rho_0+\rho_1+\rho_2)\right)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)\right]}$$

$$\frac{\partial}{\partial\rho_0}\mathcal{L}_0=\frac{n}{\rho_0}+2\sum_{i=1}^{n}\frac{x_{i0}-b_0}{[x_{i0}(1-\rho_0)+\rho_0 b_0-a_0]}$$

$$\frac{\partial}{\partial(\rho_0+\rho_1)}(\mathcal{L}_0+\mathcal{L}_1)=\frac{n}{(\rho_0+\rho_1)}+2\sum_{i=1}^{n}\frac{(x_{i0}+x_{i1})-(b_0+b_1)}{\left[(x_{i0}+x_{i1})\left(1-(\rho_0+\rho_1)\right)+(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)\right]}$$

$$\frac{\partial}{\partial(\rho_0+\rho_1+\rho_2)}(\mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2)$$
$$=\frac{n}{(\rho_0+\rho_1+\rho_2)}$$
$$+2\sum_{i=1}^{n}\frac{(x_{i0}+x_{i1}+x_{i2})-(\rho_0+\rho_1+\rho_2)}{\left[(x_{i0}+x_{i1}+x_{i2})\left(1-(\rho_0+\rho_1+\rho_2)\right)+(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)\right]}$$

Solving previous equations numerically give us the desired estimations.

6.2 Simulation and random numbers generating:

Random numbers generating can be done using the following equation:

$$x_P=\frac{y_P\rho_P b_P-a_P y_P+a_P}{1-y_P(1-\rho_P)} \tag{39}$$

By taking AH-isometry to (39) we get:

$$x_0=\frac{y_0\rho_0 b_0-a_0 y_0+a_0}{1-y_0(1-\rho_0)} \tag{40}$$

$$x_0+x_1=\frac{(y_0+y_1)(\rho_0+\rho_1)(b_0+b_1)-(a_0+a_1)(y_0+y_1)+(a_0+a_1)}{1-(y_0+y_1)\left(1-(\rho_0+\rho_1)\right)} \tag{41}$$

$$x_0+x_1+x_2$$
$$=\frac{(y_0+y_1+y_2)(\rho_0+\rho_1+\rho_2)(b_0+b_1+b_2)-(a_0+a_1+a_2)(y_0+y_1+y_2)+(a_0+a_1+a_2)}{1-(y_0+y_1+y_2)\left(1-(\rho_0+\rho_1+\rho_2)\right)} \tag{42}$$

By using equations (40-42), we can generate random numbers following classical Marshall-Olkin type I uniform distribution, then using T^{-1} we will get plithogenic Marshall-Olkin type I uniform distribution generated numbers.

performance of maximum likelihood estimators based on Monte Carlo simulation using Maple software with total replication of $N=1000$ times and with sample size of $15,50,100,150$ with fixed parameters $a_P=1.5+0.3P_1+0.5P_2, b_P=2+0.5P_1+1.2P_2, \rho_P=1+0.7P_1-0.4P_2$ is checked based on bias of the estimators and mean square error of it using the following equations:

$$Bias=\frac{\sum_{i=1}^{n}\left|\hat{\theta}_{iP}-\theta_P\right|}{n} \tag{43}$$

$$MSE = \frac{\sum_{i=1}^{n}\left(\hat{\theta}_{iP} - \theta_P\right)^2}{n} \quad (44)$$

Table 2. Simulation results of plithogenic Marshall-Olkin type I uniform distribution.

	$a_P = 1.5 + 0.3P_1 + 0.5P_2$		
n	$\hat{a}_P$	$Bias\ \hat{a}_P$	$MSE\ \hat{a}_P$
15	$1.53060 + 0.33686P_1 + 0.54019P_2$	$0.03060 + 0.03686P_1 + 0.04025P_2$	$0.00179 + 0.00643P_1 + 0.01337P_2$
50	$1.50975 + 0.31286P_1 + 0.51248P_2$	$0.00975 + 0.01286P_1 + 0.01248P_2$	$0.00019 + 0.00079P_1 + 0.00142P_2$
100	$1.50495 + 0.30667P_1 + 0.50629P_2$	$0.00495 + 0.00667P_1 + 0.00629P_2$	$0.00005 + 0.00021P_1 + 0.00036P_2$
150	$1.50337 + 0.30457P_1 + 0.50427P_2$	$0.00337 + 0.00457P_1 + 0.00427P_2$	$0.00002 + 0.00010P_1 + 0.00017P_2$
	$b_P = 2 + 0.5P_1 + 1.2P_2$		
n	$\hat{b}_P$	$Bias\ \hat{b}_P$	$MSE\ \hat{b}_P$
15	$1.96711 + 0.50424P_1 + 1.15560P_2$	$0.03289 - 0.00424P_1 + 0.04441P_2$	$0.00212 - 0.00043P_1 + 0.00904P_2$
50	$1.99035 + 0.50158P_1 + 1.18710P_2$	$0.00965 - 0.00157P_1 + 0.01289P_2$	$0.00019 - 0.00006P_1 + 0.00076P_2$
100	$1.99499 + 0.50085P_1 + 1.19331P_2$	$0.00501 - 0.00085P_1 + 0.00668P_2$	$0.00005 - 0.00002P_1 + 0.00019P_2$
150	$1.99679 + 0.50055P_1 + 1.19572P_2$	$0.00321 - 0.00055P_1 + 0.00428P_2$	$0.00002 - 0.00001P_1 + 0.00009P_2$
	$\rho_P = 1 + 0.7P_1 - 0.4P_2$		
n	$\hat{\rho}_P$	$Bias\ \hat{\rho}_P$	$MSE\ \hat{\rho}_P$
15	$1.11156 + 0.65361P_1 - 0.36750P_2$	$0.41604 + 0.25464P_1 - 0.14580P_2$	$0.34855 + 0.50312P_1 - 0.31230P_2$
50	$1.04020 + 0.69063P_1 - 0.39240P_2$	$0.20518 + 0.13733P_1 - 0.07870P_2$	$0.07171 + 0.12394P_1 - 0.07800P_2$
100	$1.02096 + 0.69566P_1 - 0.39636P_2$	$0.14470 + 0.09968P_1 - 0.05702P_2$	$0.03574 + 0.06449P_1 - 0.04076P_2$
150	$1.01753 + 0.69967P_1 - 0.39901P_2$	$0.11841 + 0.08157P_1 - 0.04664P_2$	$0.02270 + 0.04158P_1 - 0.02633P_2$

Table (2) shows that as sample size n increases, bias of the estimators and mean square error of it decrease which means that our estimators are asymptotically unbiased and consistent.

7. Conclusions

We have studied and derived neutrosophic Marshall Olkin (I) class of distributions and plithogenic Marshall Olkin (I) class of distribution and found its cumulative distribution functions and probability distribution functions. Also, we studied a special case of these new classes that is uniformly generalized distribution and estimated its parameters using maximum likelihood estimation method and made a simulation study to show the power and efficiency of our estimators and the simulation results show that our estimators are unbiased and consistent. In future researches we are looking forward to study more special distributions generalized by Marshall Olkin class and study its applications in reliability theory and queueing theory.

Funding: "This research received no external funding".

Acknowledgments: We are very grateful to editor in chief and reviewers for their hard work and special thanks to the guest editor Mohammad Abobala.

Conflicts of Interest: "The authors declare no conflict of interest."

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Received: 12/7/2023, Accepted: 11/10/2023