Neutrosophic Ideal Theory

Neutrosophic Local Function and Generated Neutrosophic Topology

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ABSTRACT

Abstract In this paper we introduce the notion of ideals on neutrosophic set which is considered as a generalization of fuzzy and fuzzy intuitionistic ideals studies in [9,11], the important neutrosophic ideals has been given in [4]. The concept of neutrosophic local function is also introduced for a neutrosophic topological space. These concepts are discussed with a view to find new neutrosophic topology from the original one in [8]. The basic structure, especially a basis for such generated neutrosophic topologies and several relations between different neutrosophic ideals and neutrosophic topologies are also studied here. Possible application to GIS topology rules are touched upon.

KEYWORDS: Neutrosophic Set, Intuitionistic Fuzzy Ideal, Fuzzy Ideal, Neutrosophic Ideal, Neutrosophic Topology.

1-INTRODUCTION

The neutrosophic set concept was introduced by Smarandache [12, 13]. In 2012 neutrosophic sets have been investigated by Hanafy and Salama at el [4, 5, 6, 7, 8, 9, 10]. The fuzzy set was introduced by Zadeh [14] in 1965, where each element had a degree of membership. In 1983 the intuitionistic fuzzy set was introduced by K. Atanassov [1, 2, 3] as a generalization of fuzzy set, where besides the degree of membership and the degree of non-membership of each element. Salama at el [9] defined intuitionistic fuzzy ideal for a set and generalized the concept of fuzzy ideal concepts, first initiated by Sarker [10]. Neutrosophy has laid the foundation for a whole family of new mathematical theories generalizing both their classical and fuzzy counterparts. In this paper we will introduce the definitions of normal neutrosophic set, convex set, the concept of α-cut and neutrosophic ideals, which can be discussed as generalization of fuzzy and fuzzy intuitionistic studies.

2-TERMINOLOGIES

We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [12, 13], and Salama at el. [4, 5, 6, 7, 8, 9, 10].

3- NEUTROSOPHIC IDEALS [4].

Definition 3.1
Let X be non-empty set and L a non-empty family of NSs. We will call L is a neutrosophic ideal (NL for short) on X if

- \( A \in L \) and \( B \subseteq A \implies B \in L \) [heredity].
• \( A \in L \) and \( B \in L \Rightarrow A \lor B \in L \) [Finite additivity].

A neutrosophic ideal \( L \) is called a \( \sigma \)-neutrosophic ideal if \( A_j \mid j \in \mathbb{N} \leq L \), implies \( \lor_j A_j \in L \) (countable additivity).

The smallest and largest neutrosophic ideals on a non-empty set \( X \) are \( 0_N \) and \( NS \) on \( X \). Also, \( N.I \), \( N.I_c \) are denoting the neutrosophic ideals (NL for short) of neutrosophic subsets having finite and countable support of \( X \) respectively. Moreover, if \( A \) is a nonempty \( NS \) in \( X \), then \( B \in NS : B \subseteq A \) is an NL on \( X \). This is called the principal NL of all \( NS \)s denoted by \( NL(A) \).

Remark 3.1

• If \( 1_N \notin L \), then \( L \) is called neutrosophic proper ideal.

• If \( 1_N \in L \), then \( L \) is called neutrosophic improper ideal.

\( O_N \in L \).

Example 3.1

Any Initiationistic fuzzy ideal \( \ell \) on \( X \) in the sense of Salama is obviously and NL in the form
\[ L = A : A = \{ x, \mu_A, \sigma_A, v_A \} \in \ell. \]

Example 3.2

Let \( X = a, b, c \ A = \{ x, 0.2, 0.5, 0.6 \}, B = \{ x, 0.5, 0.7, 0.8 \}, \) and \( D = \{ x, 0.5, 0.6, 0.8 \}, \) then the family
\[ L = O_N, A, B, D \] of \( NS \)s is an NL on \( X \).

Example 3.3

Let \( X = a, b, c, d, e \) and \( A = \{ x, \mu_A, \sigma_A, v_A \} \) given by:

<table>
<thead>
<tr>
<th>( X )</th>
<th>( \mu_A )</th>
<th>( \sigma_A )</th>
<th>( v_A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>0.6</td>
<td>0.4</td>
<td>0.3</td>
</tr>
<tr>
<td>( b )</td>
<td>0.5</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>( c )</td>
<td>0.4</td>
<td>0.6</td>
<td>0.4</td>
</tr>
<tr>
<td>( d )</td>
<td>0.3</td>
<td>0.8</td>
<td>0.5</td>
</tr>
<tr>
<td>( e )</td>
<td>0.3</td>
<td>0.7</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Then the family \( L = O_N, A \) is an NL on \( X \).

Definition 3.3

Let \( L_1 \) and \( L_2 \) be two NL on \( X \). Then \( L_2 \) is said to be finer than \( L_1 \) or \( L_1 \) is coarser than \( L_2 \) if \( L_1 \subseteq L_2 \). If also \( L_1 \neq L_2 \) then \( L_2 \) is said to be strictly finer than \( L_1 \) or \( L_1 \) is strictly coarser than \( L_2 \).

Two NL said to be comparable, if one is finer than the other. The set of all NL on \( X \) is ordered by the relation \( L_1 \) is coarser than \( L_2 \) this relation is induced the inclusion in \( NS \)s.
The next Proposition is considered as one of the useful result in this sequel, whose proof is clear.

**Proposition 3.1**

Let \( L_j : j \in J \) be any non-empty family of neutrosophic ideals on a set \( X \). Then \( \bigcap_{j \in J} L_j \) and \( \bigcup_{j \in J} L_j \) are neutrosophic ideal on \( X \).

In fact \( L \) is the smallest upper bound of the set of the \( L_j \) in the ordered set of all neutrosophic ideals on \( X \).

**Remark 3.2**

The neutrosophic ideal by the single neutrosophic set \( O_N \) is the smallest element of the ordered set of all neutrosophic ideals on \( X \).

**Proposition 3.3**

A neutrosophic set \( A \) in neutrosophic ideal \( L \) on \( X \) is a base of \( L \) iff every member of \( L \) contained in \( A \).

**Proof**

(Necessity) Suppose \( A \) is a base of \( L \). Then clearly every member of \( L \) contained in \( A \).

(Sufficiency) Suppose the necessary condition holds. Then the set of neutrosophic subset in \( X \) contained in \( A \) coincides with \( L \) by the Definition 4.3.

**Proposition 3.4**

For a neutrosophic ideal \( L_1 \) with base \( A \), is finer than a fuzzy ideal \( L_2 \) with base \( B \) iff every member of \( B \) contained in \( A \).

**Proof**

Immediate consequence of Definitions

**Corollary 3.1**

Two neutrosophic ideals bases \( A, B \), on \( X \) are equivalent iff every member of \( A \), contained in \( B \) and vice versa.

**Theorem 3.1**

Let \( \eta = \{ \mu_j, \sigma_j, \gamma_j \} : j \in J \) be a non-empty collection of neutrosophic subsets of \( X \). Then there exists a neutrosophic ideal \( L(\eta) = \{ A \in NSs : A \subseteq \bigvee A_j \} \) on \( X \) for some finite collection \( \{ A_j : j = 1, 2, \ldots, n \subseteq \eta \} \).

**Proof**

Clear.

**Remark 3.3**

ii) The neutrosophic ideal \( L(\eta) \) defined above is said to be generated by \( \eta \) and \( \eta \) is called sub base of \( L(\eta) \).

**Corollary 3.2**

Let \( L_1 \) be an neutrosophic ideal on \( X \) and \( A \in NSs \), then there is a neutrosophic ideal \( L_2 \) which is finer than \( L_1 \)
and such that \( A \in L_2 \) iff
\[
A \lor B \in L_2 \text{ for each } B \in L_1.
\]

**Corollary 3.3**

Let \( A = \langle x, \mu_A \cdot \sigma_A, V_A \rangle \in L_1 \) and \( B = \langle x, \mu_B \cdot \sigma_B, V_B \rangle \in L_2 \), where \( L_1 \) and \( L_2 \) are neutrosophic ideals on the set \( X \).

then the neutrosophic set \( A^*B = \langle x, \mu_{A \cdot B} \cdot \sigma_{A \cdot B}, V_{A \cdot B} \rangle \in L_1 \lor L_2 \) on \( X \) where \( \mu_{A \cdot B} \subseteq \mu_A \cdot \mu_B \cdot \sigma_{A \cdot B} \subseteq x \in X \cdot \sigma_{A \cdot B} \cdot \sigma_{A \cdot B} \), may be \( \sigma_A(x) \land \sigma_B(x) \lor \sigma_A(x) \lor \sigma_B(x) \) and \( V_{A \cdot B} \subseteq V_A \cdot V_B \subseteq x \in X \).

4. Neutrosophic local Functions

**Definition 4.1.** Let \((X, \mathcal{T})\) be a neutrosophic topological spaces (NTS for short ) and \( L \) be neutrosophic ideal (NL, for short) on \( X \). Let \( A \) be any NS of \( X \). Then the neutrosophic local function \( NA^* \) of \( A \) is the union of all neutrosophic points (NP, for short) \( C \in \beta, \gamma \) such that if \( U \subseteq N \in C \beta, \gamma \) and \( NA^*(L, \mathcal{T}) = \sigma(C, \beta, \gamma \in X) : A \cup U \notin L \) for every \( \cup \) of \( C \beta, \gamma \), \( NA^*(L, \mathcal{T}) \) is called a neutrosophic local function of \( A \) with respect to \( \mathcal{T} \) and \( L \) which it will be denoted by \( NA^*(L, \mathcal{T}) \), or simply \( NA^* \).

**Example 4.1.** One may easily verify that.
If \( L = \{0, 1\} \), then \( NA^*(L, \mathcal{T}) = Ncl(A, \mathcal{T}) \), for any neutrosophic set \( A \subseteq NS \) on \( X \).

If \( L = \) all \( NS \)s on \( X \) then \( NA^*(L, \mathcal{T}) = 0, \) for any \( A \subseteq NS \) on \( X \).

**Theorem 4.1.** Let \( \mathcal{T} \) be a NTS and \( L_1, L_2 \) be two neutrosophic ideals on \( X \). Then for any neutrosophic sets \( A, B \) of \( X \) then the following statements are verified

i) \( A \subseteq B \) \( \Rightarrow \) \( NA^*(L_1, \mathcal{T}) \subseteq NB^*(L_1, \mathcal{T}) \),

ii) \( L_1 \subseteq L_2 \) \( \Rightarrow \) \( NA^*(L_2, \mathcal{T}) \subseteq NA^*(L_1, \mathcal{T}) \).

iii) \( NA^* = Ncl(A, \mathcal{T}) \subseteq Ncl(A) \).

iv) \( NA^* \subseteq NA \).

v) \( N \cup B \subseteq NB^* \).

vi) \( N(A \cup B) = NA^*(L_2, \mathcal{T}) \subseteq NA^*(L_1, \mathcal{T}) \).

vii) \( L \subseteq L \) \( \Rightarrow \) \( N \cup \subseteq NB^* \).

viii) \( NA^*(L_1, \mathcal{T}) \) is neutrosophic closed set .

**Proof.**

i) Since \( A \subseteq B \), let \( p = C, \beta, \gamma \subseteq NA^* \) \( \subseteq \) then \( A \cup U \notin L \) for every \( U \subseteq N \). By hypothesis we get \( B \cup U \notin L \), then \( p = C, \beta, \gamma \subseteq NB^* \).

ii) Clearly, \( L_1 \subseteq L_2 \) implies \( NA^*(L_1, \mathcal{T}) \subseteq NA^*(L_2, \mathcal{T}) \) as there may be other \( IFS \)s which belong to \( L_2 \) so that for \( \mathcal{GFP} \) \( p = C, \beta, \gamma \subseteq NA^* \) but \( C, \beta, \gamma \subseteq NB^* \).

iii) Since \( O_N \subseteq L \) for any \( NL \) on \( X \), therefore by (ii) and Example 3.1, \( NA^* \subseteq NA^* \) \( \subseteq \) \( Ncl(A, \mathcal{T}) \) for any \( NA \) on \( X \). Suppose \( p_1 = C, \beta, \gamma \subseteq Ncl(A, \mathcal{T}) \). So for every \( U \subseteq N \) \( \subseteq \), \( NA^* \cup U \notin O_N \), there exists \( p_2 = C, \beta, \gamma \subseteq A \cup U \) such that for every \( nbd \) of \( p_2 \) \( \subseteq N \) \( \subseteq \). Since \( A \cup U \notin L \), therefore \( p_1 = C, \beta, \gamma \subseteq A \cup U \subseteq L \).


and so \( Ncl(A^*) \subseteq NA^* \). While, the other inclusion follows directly. Hence \( NA^* = Ncl(NA^*) \). But the inequality \( NA^* \subseteq Ncl(NA^*) \).

iv) The inclusion \( NA^* \cap NB^* \subseteq N \oplus B \) follows directly by (i). To show the other implication, let \( p = C(\alpha, \beta, \gamma) \in N \oplus B \) for every \( U \in N(p) \), \( \exists B \neq U \subseteq L \), i.e., \( \exists U \sqcap B \neq U \sqcap B \). Then, we have two cases \( A \sqcap U \subseteq L \) and \( B \sqcap U \subseteq L \) or the converse, this means that \( \exists U_1, U_2 \in N \oplus C(\alpha, \beta, \gamma) \) such that \( A \sqcap U_1 \subseteq L \) and \( B \sqcap U_2 \subseteq L \) and \( A \sqcap U_1 \neq U_2 \subseteq L \) and \( B \sqcap U_1 \neq U_2 \subseteq L \). Then the inclusion \( NA^* \subseteq Ncl(NA^*) \).

vi) By (iii), we have \( NA^* = Ncl(NA^*) \subseteq Ncl(NA^*) = NA^* \).

Let \( \tilde{\mathcal{K}}, \tilde{\tau} \) be a GIFS and \( L \) be GIFL on \( X \). Let us define the neutrosophic closure operator \( cl^*(A) = A \cup A^* \) for any GIFS \( A \) of \( X \). Clearly, let \( Ncl^*(A) \) is a neutrosophic operator. Let \( N \tilde{\tau}^*(L) \) be a neutrosophic set. Let \( N \tilde{\tau}^*(L) \) be a neutrosophic set. Let \( N \tilde{\tau}^*(L) \) be a neutrosophic set. Let \( N \tilde{\tau}^*(L) \) be a neutrosophic set.

\[ \tilde{\tau}^*(A) = A \cup A^* \]

for every \( N \tilde{\tau}^*(L) \) be a neutrosophic set. Let \( N \tilde{\tau}^*(L) \) be a neutrosophic set. Let \( N \tilde{\tau}^*(L) \) be a neutrosophic set. Let \( N \tilde{\tau}^*(L) \) be a neutrosophic set. Let \( N \tilde{\tau}^*(L) \) be a neutrosophic set.

Theorem 4.2. Let \( \tilde{\tau}_1, \tilde{\tau}_2 \) be two neutrosophic topologies on \( X \). Then for any neutrosophic ideal \( L \) on \( X \), \( \tilde{\tau}_1 \leq \tilde{\tau}_2 \) implies \( NA^*(L, \tau_2) \subseteq NA^*(L, \tau_1) \), for every \( A \in L \) then \( N \tilde{\tau}^*_1 \subseteq N \tilde{\tau}^*_2 \).

Proof. Clear.

A basis \( \tilde{\mathcal{B}} \), \( \tilde{\tau}^* \) for \( N \tilde{\tau}^*(L) \) can be described as follows:

\[ \tilde{\tau}^*(A) = A \cup B \in L \]

Then we have the following theorem.

Theorem 4.3. \( \tilde{\mathcal{B}} \), \( \tilde{\tau} \) is a neutrosophic ideal \( L \) on \( X \). In particular, we have for two neutrosophic ideals \( \tilde{\tau}_1, \tilde{\tau}_2 \) on \( X \), \( \tilde{\tau}_1 \subseteq \tilde{\tau}_2 \).

Proof. Straight forward.

The relationship between \( \tilde{\tau} \) and \( N \tilde{\tau}^*(L) \) established throughout the following result which have an immediately proof.

Theorem 4.4. Let \( \tilde{\tau}_1, \tilde{\tau}_2 \) be two neutrosophic topologies on \( X \). Then for any neutrosophic ideal \( L \) on \( X \), \( \tilde{\tau}_1 \subseteq \tilde{\tau}_2 \) implies \( N \tilde{\tau}^*_1 \subseteq N \tilde{\tau}^*_2 \).

Theorem 4.5 : Let \( \tilde{\mathcal{F}}, \tilde{\mathcal{T}} \) be a neutrosophic on \( X \) and \( L_1, L_2 \) be two neutrosophic ideals on \( X \). Then for any neutrosophic set \( A \) in \( X \), we have

i) \( NA^*(A) \cup L_2, \tilde{\tau}_2 = NA^*(A) \cup L_1, \tilde{\tau}_1 \)

ii) \( N \tilde{\tau}^*(L_2) \) is a neutrosophic ideal \( L \) on \( X \). In particular, we have for two neutrosophic ideals \( L_1, L_2 \) on \( X \), \( N \tilde{\tau}^*_1 \subseteq N \tilde{\tau}^*_2 \).

Proof. Let \( p = C(\alpha, \beta, \gamma) \in \tilde{\mathcal{F}} \cup L_2, \tilde{\tau}_2 \) this means that there exists \( U_p \in N \tilde{\mathcal{F}} \) such that \( A \cup U_p \in \tilde{\mathcal{F}} \cup L_2 \). There exists \( \ell_2 \in L_1 \) and \( \ell_2 \in L_2 \) such that \( A \cup U_p \in \ell_2 \cap L_2 \). because of the heredity of \( L_1 \) and, assuming \( \ell_1 = \ell_2 \).

Thus we have \( A \cup U_p \subseteq A \cup L_2 \) and \( A \cup U_p \subseteq A \cup L_2 \) therefore \( U_p - \ell_1 \subseteq A \subseteq L_2 \), and \( U_p - \ell_1 \subseteq A \subseteq L_2 \).

Hence \( p = C(\alpha, \beta, \gamma) \in NA^*(A) \cup L_2, \tilde{\tau}_2 \) or \( p = C(\alpha, \beta, \gamma) \in NA^*(A) \cup L_2, \tilde{\tau}_2 \) because \( p \) must belong to either \( \ell_1 \) or \( \ell_2 \) but not to both. This gives \( NA^*(A) \cup L_2, \tilde{\tau}_2 \supseteq NA^*(A) \cup L_2, \tilde{\tau}_1 \).

To show the second inclusion, let us assume \( p = C(\alpha, \beta, \gamma) \in NA^*(A) \cup L_2, \tilde{\tau}_2 \) this implies that \( U_p \subseteq N \tilde{\mathcal{F}} \).

and \( \ell_2 \in L_2 \) such that \( A \cup U_p \subseteq A \subseteq L_1 \). By the heredity of \( L_2 \), if we assume that \( \ell_2 \leq A \) and define \( \ell_1 = U_p - \ell_2 \). Then we have \( A \cup U_p \subseteq \ell_2 \subseteq \ell_1 \cup L_2 \). Thus,
\[ N^* A_1 \cup L_2, \tau \leq N^* A_1, \tau^*(L_2) \wedge N^* A_2, \tau^*(L_2) \] and similarly, we can get \[ A_1^* \cup L_2, \tau \leq A_1^* , \tau^*(L_2) \]. This gives the other inclusion, which complete the proof.

**Corollary 4.1.** Let \( \mathfrak{A}, \tau \) be a NTS with neutrosophic ideal \( L \) on \( X \). Then

i) \( N^*(L, \tau) = N^*(L, \tau^*) \) and \( N^*(L) = N(N^*(L))^*(L) \).

ii) \( N^*(L_1 \cup L_2) = \mathfrak{A}^*(L_1) \cap \mathfrak{A}^*(L_2) \)

**Proof.** Follows by applying the previous statement.

**REFERENCES**


