Abstract. The purpose of this paper is to define the so-called "neutrosophic crisp points" and "neutrosophic crisp ideals", and obtain their fundamental properties. Possible application to GIS topology rules are touched upon.

Keywords: Neutrosophic Crisp Point, Neutrosophic Crisp Ideal.

1 Introduction

Neutrosophy has laid the foundation for a whole family of new mathematical theories, generalizing both their crisp and fuzzy counterparts. The idea of "neutrosophic set" was first given by Smarandache [12, 13]. In 2012 neutrosophic operations have been investigated by Salama at el. [4 -10]. The fuzzy set was introduced by Zadeh [13]. The intuitionistic fuzzy set was introduced by Atanassov [1, 2, 3]. Salama at el. [9] defined intuitionistic fuzzy ideal for a set and generalized the concept of fuzzy ideal concepts, first initiated by Sarker [11]. Here we shall present the crisp version of these concepts.

2 Terminologies

We recollect some relevant basic preliminaries, and in particular the work of Smarandache in [12, 13], and Salama at el. [4 -10].

3 Neutrosophic Crisp Points

One can easily define a natural type of neutrosophic crisp set in X, called "neutrosophic crisp point" in X, corresponding to an element $p \in X$:

3.1 Definition

Let X be a nonempty set and $p \in X$. Then the neutrosophic crisp point $p_N$ defined by $p_N = \{\{p\}, \emptyset, \{p\}^C\}$ is called a neutrosophic crisp point (NCP for short) in X, where NCP is a triple (\{only one element in X\}, the empty set,\{the complement of the same element in X\}).

Neutrosophic crisp points in X can sometimes be inconvenient when expressing a neutrosophic crisp set in X in terms of neutrosophic crisp points. This situation will occur if $A = \{A_1, A_2, A_3\}$, and $p \in A_1$, where $A_1, A_2, A_3$ are three subsets such that $A_1 \cap A_2 = \emptyset$, $A_1 \cap A_3 = \emptyset$, $A_2 \cap A_3 = \emptyset$. Therefore we define the vanishing neutrosophic crisp points as follows:

3.2 Definition

Let $X = \{a, b, c, d\}$ and $p = b \in X$. Then $p_N = \{\emptyset, \emptyset, \{a, c, d\}\}$.

Now we shall present some types of inclusions of a neutrosophic crisp point to a neutrosophic crisp set:

3.3 Definition

Let $p_N = \{\{p\}, \emptyset, \{p\}^C\}$ be a NCP in X and $A = \{A_1, A_2, A_3\}$ a neutrosophic crisp set in X.

(a) $p_N$ is said to be contained in $A$ (denoted by $p_N \in A$ for short) iff $p \in A_1$.

(b) Let $p_N$ be a VNCP in X, and $A = \{A_1, A_2, A_3\}$ a neutrosophic crisp set in X. Then $p_N$ is said to be contained in $A$ (denoted by $p_N \in A$ for short) iff $p \not\in A_3$.

3.1 Proposition

Let $\{D_j : j \in J\}$ be a family of NCSs in X. Then

(a) $p_N \in \bigcap_{\rho j} D_j$ iff $p_N \in D_j$ for each $j \in J$.

(b) $p_N \in \bigcap_{\rho j} D_j$ iff $p_N \in D_j$ for each $j \in J$.
(b2) \( p_{Nk} \in \bigcap_{j \in J} D_j \) iff \( \exists j \in J \) such that \( p_{Nk} \in D_j \).

**Proof**

Straightforward.

### 3.2 Proposition

Let \( A = \{A_1, A_2, A_3\} \) and \( B = \{B_1, B_2, B_3\} \) be two neutrosophic crisp sets in \( X \). Then

a) \( A \subseteq B \) iff for each \( p_N \) we have \( p_N \in A \Rightarrow p_N \in B \) and for each \( p_{NN} \) we have \( p_N \in A \Rightarrow p_{NN} \in B \).

b) \( A = B \) iff for each \( p_N \) we have \( p_N \in A \Rightarrow p_N \in B \) and for each \( p_{NN} \) we have \( p_{NN} \in A \Leftrightarrow p_{NN} \in B \).

**Proof**

Obvious.

### 3.4 Proposition

Let \( A = \{A_1, A_2, A_3\} \) be a neutrosophic crisp set in \( X \). Then

\[
A = (\bigcup\{p_N : p_N \in A\}) \cup (\bigcup\{p_{NN} : p_{NN} \in A\}).
\]

**Proof**

It is sufficient to show the following equalities:

\[
A_1 = (\bigcup\{p_N \} : p_N \in A) \cup (\bigcup\{p_{NN} : p_{NN} \in A\}),
\]

and \( A_3 = (\bigcap\{\{q\} : q \in A\}) \cap (\bigcap\{\{q\} : p_{NN} \in A\}) \),

which are fairly obvious.

### 3.4 Definition

Let \( f : X \rightarrow Y \) be a function.

(a) Let \( p_N \) be a neutrosophic crisp point in \( X \). Then the image of \( p_N \) under \( f \), denoted by \( f(p_N) \), is defined by \( f(p_N) = \{\{q\}, \phi, \{q\}^c\} \), where \( q = f(p) \).

(b) Let \( p_{NN} \) be a VNCP in \( X \). Then the image of \( p_{NN} \) under \( f \), denoted by \( f(p_{NN}) \), is defined by \( f(p_{NN}) = \{\phi, \{q\}, \{q\}^c\} \), where \( q = f(p) \).

It is easy to see that \( f(p_N) \) is indeed a NCP in \( Y \), namely \( f(p_N) = q_N \), where \( q = f(p) \), and it is exactly the same meaning of the image of a NCP under the function \( f \).

\( f(p_{NN}) \) is also a VNCP in \( Y \), namely \( f(p_{NN}) = q_{NN} \), where \( q = f(p_N) \).

### 3.4 Proposition

Any NCS \( A \) in \( X \) can be written in the form \( A = A \cup \cup \cup A \), where \( A = \cup\{p_N : p_N \in A\} \), \( A = \phi_N \) and \( A = \cup\{p_{NN} : p_{NN} \in A\} \). It is easy to show that, if \( A = \{A_1, A_2, A_3\} \), then \( A = \{x, A_1, \phi, A_1^c\} \) and \( A_N = \{x, \phi, A_2, A_3\} \).

### 3.5 Proposition

Let \( f : X \rightarrow Y \) be a function and \( A = \{A_1, A_2, A_3\} \) be a neutrosophic crisp set in \( X \). Then we have \( f(A) = f(A) \cup f(A) \cup f(A) \).

**Proof**

This is obvious from \( A = A \cup A \cup A \).

## 4 Neutrosophic Crisp Ideal Subsets

### 4.1 Definition

Let \( X \) be non-empty set, and \( L \) a non–empty family of NCSDs. We call \( L \) a neutrosophic crisp ideal (NCL for short) on \( X \) if

i. \( B \subseteq A \Rightarrow B \subseteq A \) [heredity],

ii. \( B \subseteq A \Rightarrow B \cup B \subseteq L \) [Finite additivity].

A neutrosophic crisp ideal \( L \) is called a \( \sigma \)-neutrosophic crisp ideal if \( \{M_j\} \subseteq L \), implies \( \bigcup_{j \in J} M_j \subseteq L \) (countable additivity).

The smallest and largest neutrosophic crisp ideals on a non-empty set \( X \) are \( \{\phi\} \) and the NSSDs on \( X \). Also, \( NCL_{f} \), \( NCL_{e} \) are denoting the neutrosophic crisp ideals (NCL for short) of neutrosophic subsets having finite and countable support of \( X \) respectively. Moreover, if \( A \) is a non-empty NS in \( X \), then \( \{B \in NSSD : B \subseteq A\} \) is an NCL on \( X \). This is called the principal NCL of all NSSDs, denoted by \( NCL(A) \).
4.1 Remark
   i. If \( X_N \notin L \), then L is called neutrosophic proper ideal.
   
   ii. If \( X_N \in L \), then L is called neutrosophic improper ideal.
   
   iii. \( \phi_N \in L \).

4.1 Example
Let \( \{ a, b, c \} = \{ \emptyset, \{ a \}, \{ b, c \}, \{ c \} \} \),
\( B = \{ \{ a \}, \{ b \}, \{ c \} \} \),
\( C = \{ \{ a \}, \{ b \}, \{ c \} \} \),
\( D = \{ \{ a \}, \{ c \}, \{ d \} \} \),
\( E = \{ \{ a \}, \{ b \}, \{ c \} \} \),
\( F = \{ \{ a \}, \{ b \}, \{ c \} \} \),
\( G = \{ \{ a \}, \{ b \}, \{ c \} \} \).
Then the family \( L = \{ \phi_N, A, B, C, D, E, F, G \} \) of NCSs is an NCL on X.

4.2 Definition
Let \( L_1 \) and \( L_2 \) be two NCLs on X. Then \( L_2 \) is said to be finer than \( L_1 \), or \( L_1 \) is coarser than \( L_2 \), if \( L_1 \subseteq L_2 \). If also \( L_1 \neq L_2 \), then \( L_2 \) is said to be strictly finer than \( L_1 \), or \( L_1 \) is strictly coarser than \( L_2 \).

Two NCLs are said to be comparable, if one is finer than the other. The set of all NCLs on X is ordered by the relation: \( L_1 \) is coarser than \( L_2 \); this relation is induced by the inclusion in NCSs.

The next Proposition is considered as one of the useful results in this sequel, whose proof is clear. \( L_j = \{ A_{j_1}, A_{j_2}, A_{j_3} \} \).

4.1 Proposition
Let \( \{ L_j : j \in J \} \) be any non-empty family of neutrosophic crisp ideals on a set X. Then \( \bigcap_{j \in J} L_j \) and \( \bigcup_{j \in J} L_j \) are neutrosophic crisp ideals on X, where
\[ \bigcap_{j \in J} L_j = \{ \emptyset \} \cup \bigcup_{j \in J} A_{j_1} \cap \bigcup_{j \in J} A_{j_2} \cap \bigcup_{j \in J} A_{j_3} \] or
\[ \bigcap_{j \in J} L_j = \{ \emptyset \} \cup \bigcup_{j \in J} A_{j_1} \cap \bigcup_{j \in J} A_{j_2} \cup \bigcup_{j \in J} A_{j_3} \] and
\[ \bigcup_{j \in J} L_j = \{ \emptyset \} \cup \bigcup_{j \in J} A_{j_1} \cup \bigcup_{j \in J} A_{j_2} \cup \bigcup_{j \in J} A_{j_3} \] or
\[ \bigcup_{j \in J} L_j = \{ \emptyset \} \cup \bigcup_{j \in J} A_{j_1} \cup \bigcup_{j \in J} A_{j_2} \cup \bigcup_{j \in J} A_{j_3} \].

In fact, L is the smallest upper bound of the sets of the \( L_j \) in the ordered set of all neutrosophic crisp ideals on X.

4.2 Remark
The neutrosophic crisp ideal defined by the single neutrosophic set \( \phi_N \) is the smallest element of the ordered set of all neutrosophic crisp ideals on X.

4.2 Proposition
A neutrosophic crisp set \( A = \{ A_1, A_2, A_3 \} \) in the neutrosophic crisp ideal L on X is a base of L iff every member of L is contained in A.

Proof
(Necessity) Suppose A is a base of L. Then clearly every member of L is contained in A.
(Sufficiency) Suppose the necessary condition holds. Then the set of neutrosophic crisp subsets in X contained in A coincides with L by the Definition 4.3.

4.3 Proposition
A neutrosophic crisp ideal \( L_1 \) with base \( \{ A_1, A_2, A_3 \} \), is finer than a fuzzy ideal \( L_2 \) with base \( \{ B_1, B_2, B_3 \} \), iff every member of \( B \) is contained in A.

Proof
Immediate consequence of the definitions.

4.1 Corollary
Two neutrosophic crisp ideals bases A, B, on X, are equivalent iff every member of A is contained in B and vice versa.

4.1 Theorem
Let \( \eta = \{ \emptyset, \{ A_1 \}, \{ A_2 \}, \{ A_3 \} \} \) be a non-empty collection of neutrosophic crisp subsets of X. Then there exists a neutrosophic crisp ideal \( L(\eta) = \{ A \in NCS : A \subseteq \bigcup_{j \in J} A_j \} \) on X for some finite collection \( \{ A_j : j = 1, 2, ..., n \} \).

Proof
It’s clear.

4.3 Remark
The neutrosophic crisp ideal \( L(\eta) \) defined above is said to be generated by \( \eta \) and \( \eta \) is called sub-base of \( L(\eta) \).
4.2 Corollary

Let \( L_1 \) be a neutrosophic crisp ideal on \( X \) and \( A \in \text{NCSs} \), then there is a neutrosophic crisp ideal \( L_2 \) which is finer than \( L_1 \) and such that \( A \in L_2 \) iff \( A \cup B \in L_2 \) for each \( B \in L_1 \).

Proof

It’s clear.

4.2 Theorem

If \( L = \left\{ \phi_N, \left\langle A_1, A_2, A_3 \right\rangle \right\} \) is a neutrosophic crisp ideals on \( X \), then:

i) \( \left\{ L = \left\{ \phi_N, \left\langle A_1, A_2, A_3 \right\rangle \right\} \right\} \) is a neutrosophic crisp ideals on \( X \).

ii) \( \left\{ L = \left\{ \phi_N, \left\langle A_3, A_2, A_1 \right\rangle \right\} \right\} \) is a neutrosophic crisp ideals on \( X \).

Proof

Obvious.

4.3 Theorem

Let \( A = \left\langle A_1, A_2, A_3 \right\rangle \in L_1 \), and 
\( B = \left\langle B_1, B_2, B_3 \right\rangle \in L_2 \), where \( L_1 \) and \( L_2 \) are neutrosophic crisp ideals on \( X \), then \( A*B \) is a neutrosophic crisp set:

\[ A*B = \left\langle A_1*B_1, A_2*B_2, A_3*B_3 \right\rangle \]

where

\[ A_1*B_1 = \cup \left\langle A_1 \cap B_1, A_2 \cap B_2, A_3 \cap B_3 \right\rangle \]

\[ A_2*B_2 = \cap \left\langle A_1 \cap B_1, A_2 \cap B_2, A_3 \cap B_3 \right\rangle \] and

\[ A_3*B_3 = \cap \left\langle A_1 \cap B_1, A_2 \cap B_2, A_3 \cap B_3 \right\rangle \].

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References