Neutrosophic soft relations and some properties

Irfan Deli, Said Broumi

Received 25 February 2014; Revised 19 July 2014; Accepted 24 July 2014

Abstract. In this work, we first define a relation on neutrosophic soft sets which allows to compose two neutrosophic soft sets. It is devised to derive useful information through the composition of two neutrosophic soft sets. Then, we examine symmetric, transitive and reflexive neutrosophic soft relations and many related concepts such as equivalent neutrosophic soft set relation, partition of neutrosophic soft sets, equivalence classes, quotient neutrosophic soft sets, neutrosophic soft composition are given and their propositions are discussed. Finally a decision making method on neutrosophic soft sets is presented.

2010 AMS Classification: 03E72, 08A72

Keywords: Neutrosophic soft sets, neutrosophic soft relation, neutrosophic soft composition, partition of neutrosophic soft sets, equivalence classes, quotient neutrosophic soft sets.

1. Introduction

In 1965, Zadeh[29] proposed the theory of fuzzy set theory which is applied in many real applications to handle uncertainty. After Zadeh, Smarandache proposed the theory of neutrosophic set[25] that is the generalization of many theory such as; fuzzy set[29], intuitionistic fuzzy set[1], and neutrosophic set[1]. The concept of neutrosophic set handle indeterminate data whereas fuzzy set theory and intuitionistic fuzzy set theory failed when the relation are indeterminate.

In 1999, Molodotsov[20] introduced the theory of soft set which is free from the parameterization inadequacy syndrome of fuzzy set theory, rough set theory[24], probability theory for dealing with uncertainty. Presently work on the soft set theory is progressing rapidly such as; on the operations (e.g. [8]) and on the applications (e.g. [15]). In recent years, soft set theory have been expanded by embedding the ideas of fuzzy sets (e.g. [9, 10, 13, 18]), intuitionistic fuzzy sets (e.g. [4, 17, 14, 28]), interval-valued intuitionistic fuzzy set rough (e.g. [22]), neutrosophic sets (e.g. [6,
11, 16, 19]), interval neutrosophic sets (e.g. [5, 7]). Also, many authors studied on relations in soft set [2, 3, 23, 27], in fuzzy soft set [26] and in intuitionistic fuzzy soft set [12, 21].

This paper is an attempt to extend the concept of intuitionistic fuzzy soft relation proposed by Dinda and Samanta [12] to neutrosophic soft relation. The organization of this paper is as follow: In section 2, we give the basic definitions and results of neutrosophic set theory [25] soft set theory [20] and neutrosophic soft set theory [16] that are useful for subsequent discussions. In section 3, neutrosophic soft relations and their propositions are proposed. In section 4, a decision making method on neutrosophic soft sets is presented.

2. Preliminary

In this section, we give the basic definitions and results of neutrosophic set theory [25], soft set theory [20] and neutrosophic soft set theory [16] that are useful for subsequent discussions.

**Definition 2.1.** [25] Let $U$ be a space of points (objects), with a generic element in $U$ denoted by $u$. A neutrosophic set (N-set) $A$ in $U$ is characterized by a truth-membership function $T_A$, a indeterminacy-membership function $I_A$ and a falsity-membership function $F_A$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $[0, 1]^+$. It can be written as

$$A = \{ x, (T_A(x), I_A(x), F_A(x)) : x \in U, T_A(u), I_A(x), F_A(x) \subseteq [0, 1] \}.$$ 

There is no restriction on the sum of $T_A(u)$, $I_A(u)$ and $F_A(u)$, so $-0 \leq \sup T_A(u) + \sup I_A(u) + \sup F_A(u) \leq 3^+$. Here, $1^+ = 1 + \varepsilon$, where 1 is its standard part and $\varepsilon$ its non-standard part. Similarly, $-0 = 1 - \varepsilon$, where 0 is its standard part and $\varepsilon$ its non-standard part.

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $[-0, 1^+]$. So instead of $[-0, 1^+]$ we need to take the interval $[0, 1]$ for technical applications, because $[-0, 1^+]$ will be difficult to apply in the real applications such as in scientific and engineering problems.

**Definition 2.2.** [20] Let $U$ be a universe, $E$ be a set of parameters that are describe the elements of $U$ and $A \subseteq E$. Then, a soft set $F_A$ over $U$ is a set defined by a set valued function $f_A$ representing a mapping

$$f_A : E \rightarrow P(U) \text{ such that } f_A(x) = \emptyset \text{ if } x \in E - A$$

where $f_A$ is called approximate function of the soft set $F_A$. In other words, the soft set is a parametrized family of subsets of the set $U$ and therefore it can be written a set of ordered pairs

$$F_A = \{(x, f_A(x)) : x \in E, f_A(x) = \emptyset \text{ if } x \in E - A \}.$$ 

The subscript $A$ in the $f_A$ indicates that $f_A$ is the approximate function of $F_A$. The value $f_A(x)$ is a set called $x$-element of the soft set for every $x \in E$. 

2
Definition 2.3. [16] Let $U$ be a universe, $N(U)$ be the set of all neutrosophic sets on $U$, $E$ be a set of parameters that are describe the elements of $U$ and $A \subseteq E$. Then, a neutrosophic soft set $N$ over $U$ is a set defined by a set valued function $f_N$ representing a mapping $f_N : E \to N(U)$ such that $f_N(x) = \emptyset$ if $x \in E - A$ where $f_N$ is called approximate function of the neutrosophic soft set $N$. In other words, the neutrosophic soft set is a parametrized family of some elements of the set $P(U)$ and it can be written as; $N = \{ (x, f_N(x)) : x \in E, f_N(x) = \emptyset \text{ if } x \in E - A \}$

Definition 2.4. [16] Let $N_1$ and $N_2$ be two neutrosophic soft sets over neutrosophic soft universes $(U, A)$ and $(U, B)$, respectively.

1. $N_1$ is said to be neutrosophic soft subset of $N_2$ if $A \subseteq B$ and $T_{f_{N_1}(x)}(u) \leq T_{f_{N_2}(x)}(u)$, $I_{f_{N_1}(x)}(u) \leq I_{f_{N_2}(x)}(u)$, $F_{f_{N_1}(x)}(u) \geq F_{f_{N_2}(x)}(u)$, $\forall x \in A, u \in U$.

2. $N_1$ and $N_2$ are said to be equal if $N_1$ neutrosophic soft subset of $N_2$ and $N_2$ neutrosophic soft subset of $N_1$.

Definition 2.5. [16] Let $E = \{ e_1, e_2, \ldots \}$ be a set of parameters. The NOT set of $E$ is denoted by $\neg E$ is defined by $\neg E = \{ \neg e_1, \neg e_2, \ldots \}$ where $\neg e_i = \text{not } e_i, \forall i$.

Definition 2.6. [16] Let $N_1$ and $N_2$ be two neutrosophic soft sets over soft universes $(U, A)$ and $(U, B)$, respectively,

1. The complement of a neutrosophic soft set $N_1$ denoted by $N_1^c$ and is defined by a set valued function $f_{N_1}^c : E \to N(U)$ representing a mapping $f_{N_1}^c = \{ (x, < F_{f_{N_1}(x)}(u), I_{f_{N_1}(x)}(u), T_{f_{N_1}(x)}(u) >) : x \in E, u \in U \}$.

2. Then the union of $N_1$ and $N_2$ is denoted by $N_1 \cup N_2$ and is defined by $N_3(C = A \cup B)$, where the truth-membership, indeterminacy-membership and falsity-membership of $N_3$ are as follows: $\forall u \in U$

$$T_{f_{N_3}(x)}(u) = \begin{cases} T_{f_{N_1}(x)}(u), & \text{if } x \in A - B \\ T_{f_{N_2}(x)}(u), & \text{if } x \in B - A \\ \max\{T_{f_{N_1}(x)}(u), T_{f_{N_2}(x)}(u)\}, & \text{if } x \in A \cap B \end{cases}$$

$$I_{f_{N_3}}(u) = \begin{cases} I_{f_{N_1}(x)}(u), & \text{if } x \in A - B \\ I_{f_{N_2}(x)}(u), & \text{if } x \in B - A \\ \frac{(I_{f_{N_1}(x)}(u), I_{f_{N_2}(x)}(u))}{2}, & \text{if } x \in A \cap B \end{cases}$$

$$F_{f_{N_3}(x)}(u) = \begin{cases} F_{f_{N_1}(x)}(u), & \text{if } x \in A - B \\ F_{f_{N_2}(x)}(u), & \text{if } x \in B - A \\ \min\{I_{f_{N_1}(x)}(u), I_{f_{N_2}(x)}(u)\}, & \text{if } x \in A \cap B \end{cases}$$

3. Then the intersection of $N_1$ and $N_2$ is denoted by $N_1 \cap N_2$ and is defined by $N_3(C = A \cap B)$, where the truth-membership, indeterminacy-membership and falsity-membership of $N_3$ are as follows: $\forall u \in U$

$$T_{f_{N_3}(x)}(u) = \min\{T_{f_{N_1}(x)}(u), T_{f_{N_2}(x)}(u)\}, I_{f_{N_3}}(u) = \frac{(I_{f_{N_1}(x)}(u), I_{f_{N_2}(x)}(u))}{2}$$

$$F_{f_{N_3}}(u) = \min\{I_{f_{N_1}(x)}(u), I_{f_{N_2}(x)}(u)\}$$
and $F_{f_{N_3(x)}}(u) = \max\{F_{f_{N_1(x)}}(u), F_{f_{N_2(x)}}(u)\}, \forall x \in C$.

3. Relations on the Neutrosophic Soft Sets

In this section, after given the cartesian products of two neutrosophic soft sets, we define a relations on neutrosophic soft sets and study their desired properties. The relation extend the concept of intuitionistic fuzzy soft relation proposed by Dinda and Samanta[12] to neutrosophic soft relation. Some of it is quoted from [2, 12, 21, 23, 26, 27].

**Definition 3.1.** Let $N_1$ and $N_2$ be two neutrosophic soft sets over neutrosophic soft universes $(U, A)$ and $(U, B)$, respectively. Then the cartesian product of $N_1$ and $N_2$ is denoted by $N_1 \times N_2 = N_3$ is defined by

$$N_3 = \{((x, y), f_{N_3(x,y)}) : (x, y) \in A \times B\}$$

where the truth-membership, indeterminacy-membership and falsity-membership of $N_3$ are as follows: $\forall u \in U, \forall (x, y) \in A \times B,$

$$T_{f_{N_3(x,y)}}(u) = \min\{T_{f_{N_1(x)}}(u), T_{f_{N_2(y)}}(u)\},$$

$$I_{f_{N_3(x,y)}}(u) = \frac{(I_{f_{N_1(x)}}(u), I_{f_{N_2(y)}}(u))}{2}$$

and

$$F_{f_{N_3(x,y)}}(u) = \max\{F_{f_{N_1(x)}}(u), F_{f_{N_2(y)}}(u)\}$$

**Example 3.2.** Let $U = \{u_1, u_2, u_3, u_4\}$, $E = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $A = \{x_1, x_2, x_3\}$ and $B = \{x_3, x_6\}$ be two subsets of $E$. $N_1$ and $N_2$ be two neutrosophic soft sets over neutrosophic soft universes $(U, A)$ and $(U, B)$, respectively, as

$$N_1 = \left\{(x_1, \{< u_1, (0.7, 0.6, 0.7) >, < u_2, (0.4, 0.2, 0.8) >, < u_3, (0.9, 0.1, 0.5) >, \right\}$$

and

$$N_2 = \left\{(x_3, \{< u_1, (0.8, 0.9, 0.6) >, < u_2, (0.7, 0.8, 0.8) >, < u_3, (0.5, 0.6, 0.4) >, \right\}$$

Then, the cartesian product of $N_1$ and $N_2$ is obtained as follows
\[
\Gamma_X \times \Gamma_Y = \left\{ \left( (x_1, x_3), \{ < u_1, (0.7, 0.75, 0.7) >, < u_2, (0.4, 0.5, 0.8) >, < u_3, (0.5, 0.35, 0.5) >, < u_4, (0.3, 0.5, 0.7) > \} \right), \left( (x_1, x_6), \{ < u_1, (0.7, 0.5, 0.7) >, < u_2, (0.4, 0.2, 0.8) >, < u_3, (0.6, 0.25, 0.6) >, < u_4, (0.4, 0.7, 0.7) > \} \right), \left( (x_2, x_3), \{ < u_1, (0.5, 0.8, 0.8) >, < u_2, (0.5, 0.85, 0.8) >, < u_3, (0.5, 0.6, 0.8) >, < u_4, (0.5, 0.55, 0.6) > \} \right) \right\}
\]

**Definition 3.3.** Let \( N_1, N_2, \ldots, N_n \) be \( n \) neutrosophic soft sets over neutrosophic soft universes \((U, A_1), (U, A_2), \ldots, (U, A_n)\), respectively. Then the cartesian product of \( N_1, N_2, \ldots, N_n \) is denoted by \( N_1 \times N_2 \times \ldots \times N_n = N_{x^n} \) is defined by

\[
N_{x^n} = \{(x_1, x_2, \ldots, x_n) \mid f_{N_{x^n}}(x_1, x_2, \ldots, x_n) : (x_1, x_2, \ldots, x_n) \in A_1 \times A_2 \times \ldots \times A_n \}
\]

where the truth-membership, indeterminacy-membership and falsity-membership of \( N_{x^n} \) are as follows: \( \forall u \in U, \forall (x_1, x_2, \ldots, x_n) \in A_1 \times A_2 \times \ldots \times A_n \),

\[
T_{f_{N_{x^n}}(x_1, x_2, \ldots, x_n)}(u) = \min\{T_{f_{N_1}(x_1)}(u), T_{f_{N_2}(x_2)}(u), \ldots, T_{f_{N_n}(x_n)}(u)\},
\]

\[
I_{f_{N_{x^n}}(x_1, x_2, \ldots, x_n)}(u) = \frac{(I_{f_{N_1}(x_1)}(u), I_{f_{N_2}(x_2)}(u), \ldots, I_{f_{N_n}(x_n)}(u))}{n}
\]

and

\[
F_{f_{N_{x^n}}(x_1, x_2, \ldots, x_n)}(u) = \max\{F_{f_{N_1}(x_1)}(u), F_{f_{N_2}(x_2)}(u), \ldots, F_{f_{N_n}(x_n)}(u)\}
\]

**Definition 3.4.** Let \( N_1 \) and \( N_2 \) be two neutrosophic soft sets over soft universes \((U, A)\) and \((U, B)\), respectively. Then an neutrosophic soft relation from \( N_1 \) to \( N_2 \) is an neutrosophic soft subset of \( N_1 \times N_2 \). In other words, an neutrosophic soft relation from \( N_1 \) to \( N_2 \) is of the form \((R, C)\), \( C \subseteq A \times B \) where and \( R(x, y) \subseteq N_1 \times N_2 \) \( \forall (x, y) \in C \).

**Example 3.5.** Let us consider the Example 3.2. Then, we define a neutrosophic soft relation \( R \), from \( N_1 \) to \( N_2 \), as follows

\[
R = \left\{ \left( (x_1, x_3), \{ < u_1, (0.7, 0.75, 0.7) >, < u_2, (0.4, 0.5, 0.8) >, < u_3, (0.5, 0.35, 0.5) >, < u_4, (0.3, 0.5, 0.7) > \} \right), \left( (x_2, x_3), \{ < u_1, (0.5, 0.8, 0.8) >, < u_2, (0.5, 0.85, 0.8) >, < u_3, (0.5, 0.6, 0.8) >, < u_4, (0.5, 0.55, 0.6) > \} \right), \left( (x_2, x_6), \{ < u_1, (0.5, 0.55, 0.8) >, < u_2, (0.5, 0.55, 0.8) >, < u_3, (0.5, 0.75, 0.5) >, < u_4, (0.5, 0.75, 0.5) > \} \right), \left( (x_3, x_3), \{ < u_1, (0.8, 0.75, 0.9) >, < u_2, (0.5, 0.85, 0.9) >, < u_3, (0.5, 0.55, 0.4) >, < u_4, (0.3, 0.6, 0.4) > \} \right) \right\}
\]

**Definition 3.6.** Let \( R \) be an neutrosophic soft relation from \( N_1 \) to \( N_2 \) then \( R^{-1} \) is defined as

\[
R^{-1}(x, y) = R(y, x), \forall (x, y) \in A \times B
\]
Example 3.7. Let us consider the Example 3.5. Then, we define an neutrosophic soft relation $R^{-1}$, from $N_2$ to $N_1$, as follows

$$R^{-1} = \{(x_3, x_1), \{< u_1, (0.7, 0.75, 0.7) >, < u_2, (0.4, 0.5, 0.8) >, < u_3, (0.5, 0.35, 0.5) >, < u_4, (0.3, 0.5, 0.7) >\}\}.$$

Theorem 3.8. If $R$ be a neutrosophic soft relation from $N_1$ to $N_2$ then $R^{-1}$ is a neutrosophic soft relation from $N_2$ to $N_1$.

Proof: $R^{-1}(x, y) = R(y, x) = f_{N_2}(y) \cap f_{N_1}(x) = f_{N_1}(x) \cap f_{N_2}(y), \forall (x, y) \in A \times B$.
Hence $R^{-1}$ is a neutrosophic soft relation from $N_2$ to $N_1$.

Proposition 3.9. Let $R_1$ and $R_2$ be two neutrosophic soft relations. Then

1. $(R_1^{-1})^{-1} = R_1$
2. $R_1 \subseteq R_2 \Rightarrow R_1^{-1} \subseteq R_2^{-1}$

Proof:

1. $(R_1^{-1})^{-1}(x, y) = R_1^{-1}(y, x) = R_1(x, y)$
2. $R_1(x, y) \subseteq R_2(x, y) \Rightarrow R_1^{-1}(y, x) \subseteq R_2^{-1}(y, x) \Rightarrow R_1^{-1} \subseteq R_2^{-1}$

Definition 3.10. Let $N_1$ and $N_2$ be two neutrosophic soft sets over soft universes $(U, A)$ and $(U, B)$, respectively. $R$ be an neutrosophic soft relation from $N_1$ to $N_2$. Then domain $D(R)$ and range $R(R)$ of $R$ respectively is defined as the neutrosophic soft sets

$$D(R) = \{x, f_{N_1}(x)) \in N_1 : R(x, y) \in R\}$$

$$R(R) = \{(y, f_{N_2}(y)) \in N_2 : R(x, y) \in R\}.$$

Example 3.11. Let us consider the Example 3.5.

$$D(R_F) = \{(x_1, \{< u_1, (0.7, 0.6, 0.7) >, < u_2, (0.4, 0.2, 0.8) >, < u_3, (0.9, 0.1, 0.5) >, < u_4, (0.4, 0.7, 0.7) >\}, (x_2, < u_1, (0.5, 0.7, 0.8) >, < u_2, (0.5, 0.9, 0.3) >, < u_3, (0.5, 0.6, 0.8) >, < u_4, (0.5, 0.8, 0.5) >\}, (x_3, \{< u_1, (0.8, 0.6, 0.9) >, < u_2, (0.5, 0.9, 0.9) >, < u_3, (0.7, 0.5, 0.4) >, < u_4, (0.3, 0.5, 0.6) >\}\})$$

$$R(R_F) = \{(x_3, \{< u_1, (0.8, 0.9, 0.6) >, < u_2, (0.7, 0.8, 0.8) >, < u_3, (0.5, 0.6, 0.4) >, < u_4, (0.3, 0.3, 0.6) >\}, (x_6, \{< u_1, (0.8, 0.4, 0.6) >, < u_2, (0.6, 0.2, 0.8) >, < u_3, (0.6, 0.4, 0.6) >, < u_4, (0.5, 0.7, 0.4) >\}\})$$

Proposition 3.12. Let $R_1$ and $R_2$ be two neutrosophic soft relations. Then

1. $R_1 \subseteq R_2 \Rightarrow R(R_1) \subseteq R(R_2)$
2. $R_1 \subseteq R_2 \Rightarrow D(R_1) \subseteq D(R_2)$
The composition of two neutrosophic soft relations \( R_1 \) and \( R_2 \) is defined by \((R_1 \circ R_2)(x, z) = R_1(x, y) \wedge R_2(y, z)\) where \( R_1 \) is a neutrosophic soft relation form \( N_1 \) to \( N_2 \) and \( R_2 \) is a neutrosophic soft relation from \( N_2 \) to \( N_3 \).

**Proposition 3.14.** If \( R_1 \) and \( R_2 \) are two neutrosophic soft relation form \( N_1 \) to \( N_2 \), then \((R_1 \circ R_2)^{-1} = R_2^{-1} \circ R_1^{-1}\)

**Proof:**

\[
((R_1 \circ R_2)(x, z))^{-1} = (R_1 \circ R_2)(z, x) = R_1(z, y) \wedge R_2(y, x) = R_2(y, x) \wedge R_1(z, y) = R_2^{-1}(x, y) \wedge R_1^{-1}(y, z) = R_2^{-1} \circ R_1^{-1}
\]

Then, the proof is valid.

**Definition 3.15.** Let \( R \) be an neutrosophic soft relation from \( N_1 \) to \( N_1 \),

1. its neutrosophic soft symmetric relation if \( R(x, y) = R(y, x) \ \forall x, y \in A \)
2. its neutrosophic soft transitive relation if \( R \circ R \subseteq R \)
3. its neutrosophic soft reflexive relation if \( R(x, x) \subseteq R(x, x) \ \forall x, y \in A \)
4. its neutrosophic soft equivalence relation if it is symmetric, transitive and reflexive.

**Example 3.16.** Let \( U = \{u_1, u_2, u_3\} \), \( E = \{x_1, x_2\} \). Assume that a neutrosophic soft set on \( U \) as;

\[
N_1 = \{(x_1, \{< u_1, (0.2, 0.8, 0.7) >, < u_2, (0.5, 0.7, 0.8) >, < u_3, (0.4, 0.3, 0.7) >\}), (x_2, \{< u_1, (0.4, 0.7, 0.9) >, < u_2, (0.5, 0.3, 0.8) >, < u_3, (0.5, 0.6, 0.7) >\})\}
\]

Then, we get a neutrosophic soft relation \( R \) on \( N_1 \) as follows

\[
R = \{(x_1, x_1), \{< u_1, (0.2, 0.8, 0.7) >, < u_2, (0.5, 0.7, 0.8) >, < u_3, (0.4, 0.3, 0.7) >\}), (x_1, x_2), \{< u_1, (0.2, 0.8, 0.9) >, < u_2, (0.5, 0.7, 0.8) >, < u_3, (0.4, 0.6, 0.7) >\}), (x_2, x_1), \{< u_1, (0.2, 0.8, 0.9) >, < u_2, (0.5, 0.7, 0.8) >, < u_3, (0.4, 0.6, 0.7) >\}), (x_2, x_2), \{< u_1, (0.2, 0.8, 0.9) >, < u_2, (0.5, 0.7, 0.8) >, < u_3, (0.4, 0.6, 0.7) >\})\}
\]

\( R \) on \( N_1 \) is a neutrosophic soft equivalence relation because it is symmetric, transitive and reflexive.

**Proposition 3.17.** Let \( R \) be an neutrosophic soft relation from \( N_1 \) to \( N_1 \).

1. If \( R \) is symmetric if and only if \( R^{-1} = R \)
2. \( R \) is symmetric if and only if \( R^{-1} = R \)
3. If \( R_1 \) and \( R_2 \) are symmetric relations on \( N_1 \), then \( R_1 \circ R_2 \) is symmetric on \( N_1 \) if and only if \( R_1 \circ R_2 = R_2 \circ R_1 \)

**Proof:**
(1) Assume that $R$ is symmetric. Then, we have
\[
R^{-1}(x, y) = R(y, x) = R(x, y) = R^{-1}(y, x)
\]

So, $R^{-1}$ is symmetric.

Conversely, assume that $R^{-1}$ is symmetric. Then, we have
\[
R(x, y) = R(y, x) = R^{-1}(x, y) = R(y, x)
\]

So, $R$ is symmetric.

The proof of (2) and (3) can be made similarly.

**Corollary 3.18.** If $R$ is symmetric, then $R^n_F$ is symmetric for all positive integer $n$, where $R^n_F = R \circ R \circ \ldots \circ R$.

**Proposition 3.19.** Let $R$ be an neutrosophic soft relation from $N_1$ to $N_1$.

1. If $R$ is transitive, then $R^{-1}$ is also transitive.
2. If $R$ is transitive then $R \circ R$ is so.
3. If $R$ is reflexive then $R^{-1}$ is so.
4. If $R$ is symmetric and transitive, then $R$ is reflexive.

**Proof:**

1. \[
R^{-1}(x, y) = R(y, x) \supseteq R \circ R(y, x)
\]
\[
= R(y, z) \land R(z, x)
\]
\[
= R(z, x) \land R(y, z)
\]
\[
= R^{-1}(x, z) \land R^{-1}(y, z)
\]
\[
= R^{-1} \circ R^{-1}(x, y)
\]

So, $R^{-1} \circ R^{-1} \subseteq R^{-1}$. The proof is completed.

The proof of (2), (3) and (4) can be made similarly.

**Definition 3.20.** Let $R$ be an neutrosophic soft relation from $N_1$ to $N_1$, then equivalence class of $(x, f_{N_1}(x))$ denoted by $[(x, f_{N_1}(x))]_R$ is defined as
\[
[(x, f_{N_1}(x))]_R = \{(y, f_{N_1}(y)) : R(x, y) \in R\}
\]

**Example 3.21.** Let us consider the Example 3.5. Then,
\[
[(x, f_{N_1}(x))]_R = \{(x_3, \langle u_1, (0.8, 0.9, 0.6) \rangle, < u_2, (0.7, 0.8, 0.8) \rangle, < u_3, (0.5, 0.6, 0.4) \rangle, < u_4, (0.3, 0.3, 0.6) \rangle)\}
\]

**Proposition 3.22.** Let $R$ be an equivalence relation on neutrosophic soft relation from $N_1$ to $N_1$.

For any $(x, f_{N_1}(x)), (y, f_{N_1}(y)) \in N_1$, $R(x, y) \in R$ iff $[(x, f_{N_1}(x))]_R = [(y, f_{N_1}(y))]_R$.

**Proof:** Suppose $[(x, f_{N_1}(x))]_R = [(y, f_{N_1}(y))]_R$. Since $R$ is reflexive $R(y, y) \in R$.

Hence $(y, f_{N_1}(y)) \in [(y, f_{N_1}(y))]_R = [(x, f_{N_1}(x))]_R$ which gives $R(x, y) \in R$.

Conversely suppose $R(x, y) \in R$. Let $(x_1, f_{N_1}(x_1)) \in [(x, f_{N_1}(x))]_R$. Then $R(x_1, y) \in R$. Using the transitive property of $R$ this gives $(x_1, f_{N_1}(x_1)) \in [(y, f_{N_1}(y))]_R$. 


A collection of nonempty neutrosophic soft subsets $P = \{N_i : i \in I\}$ of a neutrosophic soft set $N$ is called a partition of $N$.

**Definition 3.23.** A collection of nonempty neutrosophic soft subsets $P = \{N_i : i \in I\}$ of a neutrosophic soft set $N$ is called a partition of $N$.

1. $N_i \neq \emptyset$
2. $N = \bigcup_i N_i$
3. $N_i \cap N_j = \emptyset \text{ if } i \neq j$

Here elements of the partition are called a block of $N$.

Moreover corresponding to a partition $N_i$ of a neutrosophic soft set $N$, we can define a neutrosophic soft set relation on $N$ by $R(x, y)$ if $(x, f_{N_i}(x))$ and $(y, f_{N_i}(y))$ belong to the same block. In follows, we will prove that the relation defined in this manner is an equivalence relation.

**Proposition 3.24.** Let $P = \{N_i : i \in I\}$ be a partition of neutrosophic soft set $N$ the neutrosophic soft set relation defined on $N$ as $R(x, y)$ if $(x, f_{N_i}(x))$ and $(y, f_{N_i}(y))$ are the elements of the same block is an equivalence relation.

**Proof:** Reflexive: Let $(x, f_{N}(x))$ be any element of $N$ it is clear that $(x, f_{N}(x))$ is in the same block itself. Hence $R(x, x) \in R$.

Symmetric: If $R(x, y) \in R$, then $(x, f_{N}(x))$ and $(y, f_{N}(y))$ are in the same block. Therefore $R(y, x) \in R$.

Transitive: If $R(x, y) \in R$ and $R(y, z) \in R$ then $(x, f_{N}(x))$, $(y, f_{N}(y))$ and $(z, f_{N}(z))$ must lie in the same block. Therefore $R(x, z) \in R$.

**Remark 3.25.** The equivalence neutrosophic soft relation defined in the above theorem is called an equivalence neutrosophic soft set relation determined by the partition $P$.

**Proposition 3.26.** Corresponding to every equivalence relation defined on a neutrosophic soft set $N$ there exists a partition on $N$ and this partition precisely consists of the equivalence classes of $R$.

**Proof:** Let be $[(x, f_{N}(x))]$ equivalence class of $R$ on $N$. Let $A_x$ denote all those elements in $A$ corresponding to $[(x, f_{N}(x))]$, i.e. $A_x = \{y \in A : R(x, y) \in R \}$. Thus we can denote $[(x, f_{N}(x))]$ as $N_x$ on $A_x$. So we have to show that the collection $\{(x, f_{N}(x))\}$ of such distinct sets forms a partition $P$ of $N$. In order to prove this we should prove

1. $N = \bigcup_x N_x$
2. If $A_x$, $A_y$ are not identical then $A_x \cap A_y \neq \emptyset$.

Since $R$ is reflexive $R(x, x) \in R \forall x \in A$ so that part (1) can prove easily.

Now for the second part, Let $x \in A_x \cap A_y$. Then $(x, f_{N}(x)) \in N_x$ and $(x, f_{N}(x)) \in N_y$. Using the transitive property of $R$ we have $R(x, y) \in R$. Now using the Proposition 3.22 we have $[(x, f_{N_y}(x))] = [(y, f_{N_y}(y))]$. This gives $A_x = A_y$.

**Remark 3.27.** The partition constructed in the above theorem therefore consists of all equivalence classes of $R$ and is called the quotient neutrosophic soft sets of $N$ and is denoted by $N/R$. 


4. Decision Making Method

In this section, we construct a soft neutrosophication operator and a decision making method on relations. Some of it is quoted from in [11, 16, 19].

Now we can construct a decision making method on neutrosophic soft relation by the following algorithm;

1. Input the neutrosophic soft $N_1$ and $N_2$
2. Obtain the neutrosophic soft matrix $R$ (relational Table) corresponding to cartesian product of $N_1$ and $N_2$ respectively.
3. Compute the comparison Table using the following formula;

$$T + J - F.$$ 

4. Select the highest numerical grades from comparison table for each row
5. Find the score table which having the following form:

<table>
<thead>
<tr>
<th>Objects</th>
<th>$h_1$</th>
<th>...</th>
<th>...</th>
<th>$h_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Highest numerical grade</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Where $x_n$ denotes the parameters of $N_1$ and $y_n$ denotes the parameters of $N_2$.

6. Compute the score of each objects by taking the sum of these numerical grades.
7. Find $m$, for which $s_m = maxs_j$, Then $s_m$ is the highest score , if $m$ has more than one values, you can choose any one value $s_j$.

Now we use this algorithm to find the best choice in decision making system.

Example 4.1. Let $U = \{u_1, u_2, u_3, u_4\}$ be the set of four shirts. Suppose that two friends want to buy a shirt for a mutual friend among these four shirts according to their choice parameters $E_1 = \{x_1, x_2, x_3\} = \{\text{Expensive, moderate, inexpensive}\}$ and $E_2 = \{y_1, y_2, y_3\} = \{\text{Green, black, Red}\}$ respectively, then we select a shirt on the basis of the sets of friend’s parameters by using the neutrosophic soft relation decision making method.

(1) We input the neutrosophic soft $N_1$ and $N_2$ as:

$$N_1 = \begin{cases}
\left\{ x_1 \begin{pmatrix}
  u_1 &=& (0.7, 0.6, 0.7) \\
  u_2 &=& (0.4, 0.2, 0.8) \\
  u_3 &=& (0.9, 0.1, 0.5) \\
  u_4 &=& (0.4, 0.7, 0.7)
\end{pmatrix}
\right.
\end{cases}$$

$$N_2 = \begin{cases}
\left\{ y_1 \begin{pmatrix}
  u_1 &=& (0.8, 0.9, 0.6) \\
  u_2 &=& (0.7, 0.8, 0.9) \\
  u_3 &=& (0.5, 0.9, 0.3) \\
  u_4 &=& (0.3, 0.6, 0.4)
\end{pmatrix}
\right.
\end{cases}$$
(2) In Table I, we obtain the neutrosophic soft matrix $R$ (relational Table I) corresponding to Cartesian product of $N_1$ and $N_2$, respectively.

<table>
<thead>
<tr>
<th>R</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x_1, y_1)$</td>
<td>$(0.7, 0.75, 0.7)$</td>
<td>$(0.4, 0.5, 0.8)$</td>
<td>$(0.5, 0.35, 0.7)$</td>
<td>$(0.3, 0.5, 0.7)$</td>
</tr>
<tr>
<td>$(x_1, y_2)$</td>
<td>$(0.7, 0.5, 0.7)$</td>
<td>$(0.4, 0.2, 0.8)$</td>
<td>$(0.6, 0.25, 0.6)$</td>
<td>$(0.4, 0.7, 0.7)$</td>
</tr>
<tr>
<td>$(x_1, y_3)$</td>
<td>$(0.3, 0.5, 0.8)$</td>
<td>$(0.4, 0.45, 0.8)$</td>
<td>$(0.8, 0.2, 0.6)$</td>
<td>$(0.3, 0.7, 0.7)$</td>
</tr>
<tr>
<td>$(x_2, y_1)$</td>
<td>$(0.5, 0.8, 0.8)$</td>
<td>$(0.5, 0.85, 0.8)$</td>
<td>$(0.5, 0.6, 0.8)$</td>
<td>$(0.5, 0.55, 0.6)$</td>
</tr>
<tr>
<td>$(x_2, y_2)$</td>
<td>$(0.5, 0.55, 0.8)$</td>
<td>$(0.5, 0.55, 0.8)$</td>
<td>$(0.5, 0.5, 0.8)$</td>
<td>$(0.5, 0.75, 0.5)$</td>
</tr>
<tr>
<td>$(x_2, y_3)$</td>
<td>$(0.3, 0.55, 0.8)$</td>
<td>$(0.4, 0.8, 0.8)$</td>
<td>$(0.5, 0.45, 0.8)$</td>
<td>$(0.3, 0.75, 0.5)$</td>
</tr>
<tr>
<td>$(x_3, y_1)$</td>
<td>$(0.8, 0.75, 0.9)$</td>
<td>$(0.5, 0.85, 0.9)$</td>
<td>$(0.5, 0.55, 0.4)$</td>
<td>$(0.3, 0.6, 0.4)$</td>
</tr>
<tr>
<td>$(x_3, y_2)$</td>
<td>$(0.8, 0.5, 0.9)$</td>
<td>$(0.5, 0.55, 0.9)$</td>
<td>$(0.6, 0.45, 0.6)$</td>
<td>$(0.3, 0.6, 0.6)$</td>
</tr>
<tr>
<td>$(x_3, y_3)$</td>
<td>$(0.3, 0.5, 0.9)$</td>
<td>$(0.5, 0.8, 0.9)$</td>
<td>$(0.7, 0.4, 0.6)$</td>
<td>$(0.3, 0.7, 0.6)$</td>
</tr>
</tbody>
</table>

Table I: Neutrosophic soft matrix $R$ (relational Table)

(3) By using the Table I, we compute the comparison Table II as;

<table>
<thead>
<tr>
<th>R</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x_1, y_1)$</td>
<td>0.65</td>
<td>0.56</td>
<td>0.15</td>
<td>0.1</td>
</tr>
<tr>
<td>$(x_1, y_2)$</td>
<td>0.5</td>
<td>-0.2</td>
<td>0.25</td>
<td>0.4</td>
</tr>
<tr>
<td>$(x_1, y_3)$</td>
<td>0</td>
<td>0.05</td>
<td>0.4</td>
<td>0.3</td>
</tr>
<tr>
<td>$(x_2, y_1)$</td>
<td>0.5</td>
<td>0.55</td>
<td>0.3</td>
<td>0.45</td>
</tr>
<tr>
<td>$(x_2, y_2)$</td>
<td>0.25</td>
<td>0.25</td>
<td>0.2</td>
<td>0.75</td>
</tr>
<tr>
<td>$(x_2, y_3)$</td>
<td>0.05</td>
<td>0.4</td>
<td>0.15</td>
<td>0.55</td>
</tr>
<tr>
<td>$(x_3, y_1)$</td>
<td>0.65</td>
<td>0.45</td>
<td>0.65</td>
<td>0.5</td>
</tr>
<tr>
<td>$(x_3, y_2)$</td>
<td>0.4</td>
<td>0.15</td>
<td>0.45</td>
<td>0.3</td>
</tr>
<tr>
<td>$(x_3, y_3)$</td>
<td>-0.1</td>
<td>0.4</td>
<td>0.5</td>
<td>0.4</td>
</tr>
</tbody>
</table>

Table II: Comparison table

(4) we select the highest numerical grades from Table II for each row in Table III as;

<table>
<thead>
<tr>
<th>R</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x_1, y_1)$</td>
<td><strong>0.65</strong></td>
<td>0.56</td>
<td>0.15</td>
<td>0.1</td>
</tr>
<tr>
<td>$(x_1, y_2)$</td>
<td><strong>0.5</strong></td>
<td>-0.2</td>
<td>0.25</td>
<td>0.4</td>
</tr>
<tr>
<td>$(x_1, y_3)$</td>
<td>0</td>
<td><strong>0.55</strong></td>
<td>0.4</td>
<td>0.3</td>
</tr>
<tr>
<td>$(x_2, y_1)$</td>
<td>0.5</td>
<td>0.55</td>
<td>0.3</td>
<td>0.45</td>
</tr>
<tr>
<td>$(x_2, y_2)$</td>
<td>0.25</td>
<td>0.25</td>
<td>0.2</td>
<td><strong>0.75</strong></td>
</tr>
<tr>
<td>$(x_2, y_3)$</td>
<td>0.05</td>
<td>0.4</td>
<td>0.15</td>
<td><strong>0.55</strong></td>
</tr>
<tr>
<td>$(x_3, y_1)$</td>
<td><strong>0.65</strong></td>
<td>0.45</td>
<td><strong>0.65</strong></td>
<td>0.5</td>
</tr>
<tr>
<td>$(x_3, y_2)$</td>
<td>0.4</td>
<td>0.15</td>
<td><strong>0.45</strong></td>
<td>0.3</td>
</tr>
<tr>
<td>$(x_3, y_3)$</td>
<td>-0.1</td>
<td>0.4</td>
<td><strong>0.5</strong></td>
<td>0.4</td>
</tr>
</tbody>
</table>

Table III

11
(5) we find the score table which having the following form:

<table>
<thead>
<tr>
<th></th>
<th>(x_1 , y_1)</th>
<th>(x_1 , y_2)</th>
<th>(x_2 , y_1)</th>
<th>(x_2 , y_2)</th>
<th>(x_2 , y_3)</th>
<th>(x_3 , y_1)</th>
<th>(x_3 , y_2)</th>
<th>(x_3 , y_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>u_1</td>
<td>u_1</td>
<td>u_3</td>
<td>u_1</td>
<td>u_4</td>
<td>u_4</td>
<td>u_1,u_3</td>
<td>u_3</td>
<td>u_3</td>
</tr>
<tr>
<td>0.65</td>
<td>0.5</td>
<td>0.4</td>
<td>0.55</td>
<td>0.75</td>
<td>0.55</td>
<td>0.65</td>
<td>0.45</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table IV: Score table

(6) we compute the score of each objects by taking the sum of these numerical grades as;

\[
\begin{align*}
  u_1 & : 0.65 + 0.5 + 0.65 = 1.8 \\
  u_2 & : 0.55 \\
  u_3 & : 0.4 + 0.65 + 0.45 + 0.5 = 2 \\
  u_4 & : 0.75 + 0.55 = 1.3
\end{align*}
\]

(7) \( s_j = 1.3 \), so the two friends will select the shirt with the highest score, hence, they will choose shirt \( u_4 \).

5. Conclusion

In this paper, we first give basic definition and operations of neutrosophic sets, soft sets and neutrosophic soft sets. We then presented neutrosophic soft relation on the neutrosophic soft set theory. Also, we give some properties for neutrosophic soft relation. Finally a decision making method on neutrosophic soft sets is presented. It can be applied to problems of many fields that contain uncertainty such as computer science, decision making and so on.

References


Irfan Deli (irfandeli@kilis.edu.tr) –

Muallim Rifat Faculty of Education, Kilis 7 Aralık University, 79000 Kilis, Turkey.
BROUMI SAID (broumisaid78@gmail.com) –
Administrator of Faculty of Arts and Humanities, Hay El Baraka Ben M’sik Casablanca
B.P. 7951, Hassan II University Mohammedia-Casablanca, Morocco