R. Muller

SMARANDACHE FUNCTION

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Vol. 1

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Editorial

Florentin Smarandache, a mathematician from Eastern Europe, escaped from his country because the communist authorities had prohibited the publication of his research papers and his participation in international congresses. After two years of waiting in a political refugee camp in Turkey, he emigrated to the United States.

As research workers, receiving our co-worker, we decided to publish a selection of his papers.

R. Muller, Editor

Readers are encouraged to submit to the Editor manuscripts concerning this function and/or its properties, relations, applications, etc.

A profound knowledge of this function would contribute to the study of prime numbers, in accordance with the following property: **If p is a number greater than 4, then p is prime if and only if η (p) = p.**

The manuscripts may be in the format of remarks, conjectures, (un)solved and/or open problems, notes, research papers, etc.

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A FUNCTION IN THE NUMBER THEORY

Summary

In this paper I shall construct a function η having the following properties:

(1) $\forall\, n \in Z \quad n \neq 0 \quad (\eta(n))! = M\, n.$

(2) $\eta(n)$ is the smallest natural number with the property (1).

We consider: $N = \{0, 1, 2, 3, \ldots\}$ and $N^* = \{1, 2, 3, \ldots\}$.

Lemma 1. $\forall\, k, p \in N^*, p \neq 1$, k is uniquely written under the shape: $k = t_1 a_{n_1}^{(p)} + \ldots + t_\ell a_{n_\ell}^{(p)}$ where $a_{n_i}^{(p)} = \dfrac{p^{n_i}-1}{p-1}$, $i = \overline{1,\ell}$, $n_1 > n_2 > \ldots > n_\ell > 0$ and $1 \leq t_j \leq p - 1$, $j = \overline{1,\ell-1}$, $1 \leq t_\ell \leq p$, $n_i, t_i \in N$, $i = \overline{1,\ell}$, $\ell \in N^*$.

Proof. The string $(a_n^{(p)})_{n\in N^*}$ consists of strictly increasing infinite natural numbers and $a_{n+1}^{(p)} - 1 = p \cdot a_n^{(p)}$, $\forall\, n \in N^*$, p is fixed,

$$a_1^{(p)} = 1,\ a_2^{(p)} = 1 + p,\ a_3^{(p)} = 1 + p + p^2,\ \ldots$$

$\Rightarrow N^* = \bigcup_{n\in N^*} ([a_n^{(p)}, a_{n+1}^{(p)}) \cap N^*)$ where $[a_n^{(p)}, a_{n+1}^{(p)}) \cap$

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$$\cap \; [a_{n+1}^{(p)}, a_{n+2}^{(p)}) = \emptyset$$

because $a_n^{(p)} < a_{n+1}^{(p)} < a_{n+2}^{(p)}$.

Let $k \in N^*$, $N^* = \bigcup_{n \in N^*} ([a_n^{(p)}, a_{n+1}^{(p)}) \cap N^*) \rightarrow \exists! n_1 \in N^*: \; k \in$

$\in \left[a_{n_1}^{(p)}, a_{n_1+1}^{(p)} \right) \rightarrow$ k is uniquely written under

the shape $k = \left[\frac{k}{a_{n_1}^{(p)}} \right] a_{n_1}^{(p)} + r_1$ (integer division theorem).

We note $\left[\frac{k}{a_{n_1}^{(p)}} \right] = t_1 \rightarrow k = t_1 a_{n_1}^{(p)} + r_1, \; r_1 < a_{n_1}^{(p)}$.

If $r_1 = 0$, as $a_{n_1}^{(p)} \le k \le a_{n_1+1}^{(p)} - 1 \rightarrow 1 \le t_1 \le p$ and Lemma 1 is proved.

If $r_1 \neq 0 \rightarrow \exists \, ! \; n_2 \in N^*: \; r_1 \in \left[a_{n_2}^{(p)}, a_{n_2+1}^{(p)} \right)$;

$a_{n_1}^{(p)} > r_1 \rightarrow n_1 > n_2$, $r_1 \neq 0$ and $a_{n_1}^{(p)} \le k \le a_{n_1+1}^{(p)} - 1 \rightarrow 1 \le t_1 \le$ $\le p - 1$ because we have $t_1 \le (a_{n_1+1}^{(p)} - 1 - r_1) : a_n^{(p)} < p$.

The procedure continues similarly. After a finite number of steps ℓ, we achieve $r_\ell = 0$, as k = finite, $k \in N^*$

and $k > r_1 > r_2 > \dots > r_\ell = 0$ and between 0 and k there is only a finite number of distinct natural numbers.

Thus:

k is uniquely written: $k = t_1 a_{n_1}^{(p)} + r_1$, $1 \le t_1 \le p - 1$,

r is uniquely written: $r_1 = t_2 a_{n_2}^{(p)} + r_2$, $n_2 < n_1$,

$$1 \le t_2 \le p - 1,$$

⋮

$r_{\ell-1}$ is uniquely written: $r_{\ell-1} = t_\ell a_{n_\ell}^{(p)} + r_\ell$ and $r_\ell = 0$,

$$n_\ell < n_{\ell-1},\ 1 \le t_\ell \le p,$$

$\Rightarrow$k is uniquely written under the shape $k = t_1 a_{n_1}^{(p)} + \dots +$

$$+ \dots + t_\ell a_{n_\ell}^{(p)}$$

with $n_1 > n_2 > \dots > n_\ell > 0$; $n_\ell > 0$ because $n_\ell \in N^*$, $1 \le t_j \le$ $\le p - 1$, $j = \overline{1, \ell-1}$, $1 \le t_\ell \le p$, $\ell \ge 1$.

Let $k \in N^*$, $k = t_1 a_{n_1}^{(p)} + \dots + t_\ell a_{n_\ell}^{(p)}$ with $a_{n_i}^{(p)} = \dfrac{p^{n_i} - 1}{p-1}$,

$i = \overline{1,\ell}$, $\ell \geq 1$, n_i, $t_i \in N^*$, $i = \overline{1,\ell}$, $n_1 > n_2 > \ldots > n_\ell >$

$1 \leq t_j \leq p - 1$, $j = \overline{1,\ell-1}$, $1 \leq t_\ell \leq p$.

I construct the function η_p, p = prime > 0, $\eta_p: N^* \to N$ thus:

$$\forall n \in N^* \quad \eta_p(a_n^{(p)}) = p^n ,$$

$$\eta_p(t_1 a_{n_1}^{(p)} + \ldots + t_\ell a_{n_\ell}^{(p)}) = t_1 \eta_p(a_{n_1}^{(p)}) + \ldots$$

$$+ t_\ell \eta_p (a_{n_\ell}^{(p)}) .$$

NOTE 1. The function η_p is well defined for each natural number.

Proof

LEMMA 2. $\forall k \in N^* \Rightarrow k$ is uniquely written as $k = t_1 a_{n_1}^{(p)} + \ldots + t_\ell a_{n_\ell}^{(p)}$ with the conditions from Lemma 1 $\Rightarrow \exists!\ t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell} = \eta_p (t_1 a_{n_1}^{(p)} + \ldots + t_\ell a_{n_\ell}^{(p)})$ and $t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell} \in N^*$.

LEMMA 3. $\forall\, k \in N^*$, $\forall\, p \in N$, p = prime $\Rightarrow k = t_1 a_{n_1}^{(p)} + \ldots + t_\ell a_{n_\ell}^{(p)}$ with the conditions from Lemma 2 $\Rightarrow \eta_p(k) = t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell}$.

It is known that $\left[\frac{a_1 + \ldots + a_n}{b}\right] \geq \left[\frac{a_1}{b}\right] + \ldots + \left[\frac{a_n}{b}\right]$ $\forall\, a_i,\, b \in N^*$ where through $[\alpha]$ we have written the integer side of the number α. I shall prove that p's powers sum from the natural numbers which make up the result factors $(t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell})!$ is $\geq k$;

$$\left[\frac{t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell}}{p}\right] \geq \left[\frac{t_1 p^{n_1}}{p}\right] + \ldots + \left[\frac{t_\ell p^{n_\ell}}{p}\right] = t_1 p^{n_1 - 1} + \ldots + t_\ell p^{n_\ell - 1}$$

$\vdots$

$$\left[\frac{t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell}}{p^n}\right] \geq \left[\frac{t_1 p^{n_1}}{p^{n_\ell}}\right] + \ldots + \left[\frac{t_\ell p^{n_\ell}}{p^{n_\ell}}\right] = t_1 p^{n_1 - n_\ell} + \ldots + t_\ell p^0$$

$$\left[\frac{t_1 p^{n_1} + \dots + t_\ell p^{n_\ell}}{p^{n_1}}\right] \geq \left[\frac{t_1 p^{n_1}}{p^{n_1}}\right] + \dots + \left[\frac{t_\ell p^{n_\ell}}{p^{n_1}}\right] = t_1 p^0 + \dots + \left[\frac{t_\ell p^{n_\ell}}{p^{n_1}}\right] .$$

Adding → p's powers sum is $\geq t_1(p^{n_1-1} + \dots + p^0) + \dots +$

$+ t_\ell (p^{n_\ell-1} + \dots + p^0) = t_1 a_{n_1}^{(p)} + \dots + t_\ell a_{n_\ell}^{(p)} = k$.

THEOREM 1. the function η_p, p = prime, defined previously, has the following properties:

(1) $\forall\, k \in N^*$, $(\eta_p(k))! = \mathcal{M} p^k$.

(2) $\eta_p(k)$ is the smallest number with the property (1).

Proof

(1) results from Lemma 3.

(2) $\forall\, k \in N^*$, $p \geq 2 \rightarrow k = t_1 a_{n_1}^{(p)} + \dots + t_\ell a_{n_\ell}^{(p)}$ (by Lemma 2) is uniquely written, where:

$$n_i,\ t_i \in N^*,\ n_1 > n_2 > \ldots > n_\ell > 0,\ a_{n_i}^{(p)} = \frac{p^{n_i}-1}{p-1} \in N^*,$$

$$i = \overline{1,\ell}\ ,\ 1 \le t_j \le p-1,\ j = \overline{1,\ell-1}\ ,\ 1 < t_\ell < p.$$

$$\Rightarrow \eta_p(k) = t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell}.\quad \text{I note:}\quad z = t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell}.$$

Let us prove that z is the smallest natural number with the property (1). I suppose by the method of reductio ad absurdum that $\exists\ \gamma \in N,\ \gamma < z$:

$\gamma! = Mp^k$;

$\gamma < z \Rightarrow \gamma \le z-1 \Rightarrow (z-1)! = Mp^k$.

$z - 1 = t_1 p^{n_1} + \ldots + t_\ell\, p_\ell^{n_\ell} - 1;\ n_1 > n_2 > \ldots > n_\ell \ge 1$ and

$n_j \in N,\ j = \overline{1,\ell}$;

$$\left[\frac{z-1}{p}\right] = t_1 p^{n_1-1} + \ldots + t_{\ell-1} p^{n_{\ell-1}-1} + t_\ell p^{n_\ell-1} - 1 \text{ as } \left[\frac{-1}{p}\right] = -1$$

because $p \ge 2$,

$$\left[\frac{z-1}{p^{n_\ell}} \right] = t_1 p^{n_1 - n_\ell} + \ldots + t_{\ell-1} p^{n_{\ell-1} - n_\ell} + t_\ell p^0 - 1 \text{ as } \left[\frac{-1}{p^{n_\ell}} \right] = -1$$

as $p \geq 2$, $n_\ell \geq 1$,

$$\left[\frac{z-1}{p^{n_\ell+1}} \right] = t_1 p^{n_1 - n_\ell - 1} + \ldots + t_{\ell-1} p^{n_{\ell-1} - n_\ell - 1} + \left[\frac{t_\ell p^{n_\ell} - 1}{p^{n_\ell+1}} \right] =$$

$$= t_1 p^{n_1 - n_\ell - 1} + \ldots + t_{\ell-1} p^{n_{\ell-1} - n_\ell - 1} \quad \text{because}$$

$$0 < t_\ell p^{n_\ell} - 1 \leq p \cdot p^{n_\ell} - 1 < p^{n_\ell+1} \quad \text{as } t_\ell < p \text{ ;}$$

$\vdots$

$$\left[\frac{z-1}{p^{n_{\ell-1}}} \right] = t_1 p^{n_1 n_{\ell-1}} + \ldots + t_{\ell-1} p^0 + \left[\frac{t_\ell p^{n_\ell} - 1}{p^{n_{\ell-1}}} \right] = t_1 p^{n_1 - n_{\ell-1}} +$$

$$+ \ldots + t_{\ell-1} p^0 \text{ as } n_{\ell-1} > n_\ell \text{ ,}$$

$\vdots$

$$\left[\frac{z-1}{p^{n_1}} \right] = t_1 p^0 + \left[\frac{t_2 p^{n_2} + \ldots + t_\ell p^{n_\ell} - 1}{p^{n_1}} \right] = t_1 p^0 \text{ .}$$

Because $0 < t_2 p^{n_2} + \ldots + t_\ell p^{n_\ell} - 1 \leq (p-1)\, p^{n_2} + \ldots +$

$$+ (p-1)p^{n_{\ell-1}} + p\cdot p^{n_\ell} - 1 \le (p-1)\cdot \sum_{i=n_{\ell-1}}^{n_2} p^i + p^{n_\ell+1} - 1 \le$$

$$\le (p-1)\frac{p^{n_2+1}}{p-1} = p^{n_2+1} - 1 < p^{n_1} - 1 < p^{n_1} \Rightarrow$$

$$\Rightarrow \left[\frac{t_2 p^{n_2} + \ldots + t_\ell p^{n_\ell} - 1}{p^{n_1}}\right] = 0$$

$$\left[\frac{z-1}{p^{n_1+1}}\right] = \left[\frac{t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell} - 1}{p^{n_1+1}}\right] = 0 \text{ because:}$$

$0 < t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell} - 1 < p^{n_1+1} - 1 < p^{n_1+1}$ according to a reasoning similar to the previous one.

Adding $\Rightarrow$ p's powers sum in the natural numbers which make up the product factors $(z-1)!$ is:

$$t_1 (p^{n_1-1} + \ldots + p^0) + \ldots + t_{\ell-1}' (p^{n_{\ell-1}-1} + \ldots + p^0) +$$

$$+ t_\ell (p^{n_\ell-1} + \ldots + p^0) - 1\cdot n_\ell = k - n_\ell < k - 1 < k \text{ because}$$

$n_\ell > 1 - (z-1)! \neq \dot{M}p^k$, this contradicts the supposition made.

- $\eta_p(k)$ is the smallest natural number with the property $(\eta_p(k))! = \dot{M}p^k$.

I construct a new function $\eta: Z\setminus\{0\} \to N$ defined as follows:

$$\begin{cases} \eta(\pm 1) = 0, \\ \forall n = \epsilon p_1^{\alpha_1} \ldots p_s^{\alpha_s} \text{ with } \epsilon = \pm 1,\ p_i = \text{prime}, \\ p_i \neq p_j \text{ for } i \neq j,\ \alpha_i \geq 1,\quad i = \overline{1,s}\ ,\ \eta(n) = \\ = \max\limits_{i=\overline{1,s}} \{\eta_{p_i}(\alpha_i)\}. \end{cases}$$

NOTE $\underline{2}$. η is well defined and defined overall.

<u>Proof</u>

(a) $\forall n \in Z$, $n \neq 0$, $n \neq \pm 1$, n is uniquely written, independent of the order of the factors, under the shape of $n = \epsilon p_1^{\alpha_1} \ldots p_s^{\alpha_s}$ with $\epsilon = \pm 1$ where p_i = prime, $p_i \neq p_j$, $\alpha_i \geq 1$ (decompose into prime factors in Z = factorial ring)).

- $\exists!\ \eta(n) = \max\limits_{\overline{1,s}} \{\eta_{p_i}(\alpha_i)\}$ as s = finite and $\eta_{p_i}(\alpha_i) \in N^*$

and $\exists \max_{\overline{i=1,s}} \{\eta_{p_i}(\alpha_i)\}$

(b) $n = \pm 1 \Rightarrow \exists ! \ \eta(n) = 0.$

THEOREM 2. The function η previously defined has the following properties:

(1) $(\eta(n))! = \mathcal{M}n, \ \forall n \in z \setminus \{0\}$;

(2) $\eta(n)$ is the smallest natural number with this property.

Proof

(a) $\eta(n) = \max_{\overline{i=1,s}} \{\eta_{p_i}(\alpha_i)\}, \ n = \epsilon \cdot p_1^{\alpha_1} \ldots p_s^{\alpha_s}$,

$(n \neq \pm 1)$,

$(\eta_{p_1}(\alpha_1))! = \mathcal{M}p_1^{\alpha_1}$,

$\vdots$

$(\eta_{p_s}(\alpha_s))! = \mathcal{M}p_s^{\alpha_s}$.

Supposing $\max_{\overline{i=1,s}} \{\eta_{p_i}(\alpha_1)\} = \eta_{p_{i_0}}(\alpha_{i_0}) \Rightarrow (\eta_{p_{i_0}}(\alpha_{i_0}))! =$

$= \mathcal{M}p_{i_0}^{\alpha_{i_0}}), \eta_{p_{i_0}}(\alpha_{i_0}) \in N^*$ and because $(p_i, p_j) = 1, \ i \neq j$,

$$\Rightarrow (\eta_{p_{i_0}}(\alpha_{i_0}))! = \mathcal{M}p_j^{\alpha_j}, \; j = \overline{1,s}.$$

$$\Rightarrow (\eta_{p_{i_0}}(\alpha_{i_0}))! = \mathcal{M}p_1^{\alpha_1} \dots p_s^{\alpha_s}.$$

(b) $n = \pm 1 \Rightarrow \eta(n) = 0; \; 0! = 1, \; 1 = \mathcal{M}\epsilon \cdot 1 = \mathcal{M}n.$

(2) (a) $n \neq \pm 1 \Rightarrow n = \epsilon\, p_1^{\alpha_1} \dots p_s^{\alpha_s} \Rightarrow \eta(n) = \max_{i=\overline{1,s}} \eta_{p_i}.$

Let $\max_{i=\overline{1,s}} \{\eta_{p_i}(\alpha_i)\} = \eta_{p_{i_0}}(\alpha_{i_0}), \; 1 \leq i \leq s;$

$\eta_{p_{i_0}}(\alpha_{i_0})$ is the smallest natural number with the property:

$$(\eta_{p_{i_0}}(\alpha_{i_0}))! = \mathcal{M}\, p_{i_0}^{\alpha_{i_0}} \Rightarrow \forall \gamma \in N, \; \gamma < \eta_{p_{i_0}}(\alpha_{i_0}) \Rightarrow$$

$$\gamma! \neq \mathcal{M}p_{i_0}^{\alpha_{i_0}} \Rightarrow \gamma! \neq \mathcal{M}\,\epsilon \cdot p_1^{\alpha_1} \dots p_{i_0}^{\alpha_{i_0}} \dots p_s^{\alpha_s} = \mathcal{M}\, n$$

$\Rightarrow \eta_{p_{i_0}}(\alpha_{i_0})$ is the smallest natural number with the property.

(b) $n = \pm 1 \Rightarrow \eta(n) = 0$ and it is the smallest natural number $\Rightarrow$ 0 is the smallest natural number with the property $0! = \mathcal{M}(\pm 1)$.

NOTE 3. The functions η_p are increasing, not injective, on $N^* \to \{p^k \mid k = 1, 2, \ldots\}$ they are surjective.

The function η is increasing, it is not injective, it is surjective on $Z \setminus \{0\} \to N \setminus \{1\}$.

CONSEQUENCE. Let $n \in N^*$, $n > 4$. Then $n = \text{prime} \Leftrightarrow \eta(n) = n$.

Proof

"$\Rightarrow$"

$n = \text{prime}$ and $n \geq 5 \Rightarrow \eta(n) = \eta_n(1) = n$.

"$\Leftarrow$"

Let $\eta(n) = n$ and suppose by absurd that $n \neq \text{prime} \Rightarrow$

(a) or $n = p_1^{\alpha_1} \ldots p_s^{\alpha_s}$ with $s \geq 2$, $\alpha_i \in N^*$, $i = \overline{1,s}$,

$$\eta(n) = \max_{i=\overline{1,s}} \{\eta_{p_i}(\alpha_i)\} = \eta_{p_{i_0}}(\alpha_{i_0}) < \alpha_{i_0} p_{i_0} < n$$

contradicts the assumption; or

(b) $n = p_1^{\alpha_1}$ with $\alpha_1 \geq 2 \Rightarrow \eta(n) = \eta_{p_1}(\alpha_1) \leq p_1 \cdot \alpha_1 < p_1^{\alpha_1} = n$

because $\alpha_1 \geq 2$ and $n > 4$ and it contradicts the hypothesis.

Application

1. Find the smallest natural number with the property:

$$n! = \mathcal{M}(\pm 2^{31} \cdot 3^{27} \cdot 7^{13}).$$

Solution

$$\eta(\pm\ 2^{31} \cdot 3^{27} \cdot 7^{13}) = \max\ \{\eta_2(31),\ \eta_3(27),\ \eta_7(13)\}.$$

Let us calculate $\eta_2(31)$; we make the string $(a_n^{(2)})_{n \in N^*} =$
$= 1, 3, 7, 15, 31, 63, \ldots$
$31 = 1 \cdot 31 \Rightarrow \eta_2(31) = \eta_2(1 \cdot 31) = 1 \cdot 2^5 = 32.$
Let's calculate $\eta_3(27)$ making the string $(a_n^{(3)})_{n \in N^*} =$
$= 1, 4, 13, 40, \ldots; 27 = 2 \cdot 13 + 1 \Rightarrow \eta_3(27) = \eta_3(2 \cdot 13 + 1 \cdot 1) =$
$= 2 \cdot \eta_3(13) + 1 \cdot \eta_3(1) = 2 \cdot 3^3 + 1 \cdot 3^1 = 54 + 3 = 57.$
Let's calculate $\eta_7(13)$; making the string $(a_n^{(7)})_{n \in N^*} =$
$= 1, 8, 57, \ldots; 13 = 1 \cdot 8 + 5 \cdot 1 \Rightarrow \eta_7(13) = 1 \cdot \eta_7(8) + 5 \cdot \eta_7(1)$
$= 1 \cdot 7^2 + 5 \cdot 7^1 = 49 + 35 = 84 \Rightarrow \eta(\pm\ 2^{31} \cdot 3^{27} \cdot 7^{13}) = \max\ \{32, 57, 84\} = 84 \Rightarrow 84! = M(\pm\ 2^{31} \cdot 3^{27} \cdot 7^{13})$ and 84 is the smallest number with this property.

2. Which are the numbers with the factorial ending in 1000 zeros?

Solution

$n = 10^{1000}$, $(\eta(n))! = M10^{1000}$ and it is the smallest number with this property.

$$\eta(10^{1000}) = \eta(2^{1000} \cdot 5^{1000}) = \max\ \{\eta_2\ (1000),\ \eta_5\ (1000)\} =$$

$$= \eta_5(1000) = \eta_5(1 \cdot 781 + 1 \cdot 156 + 2 \cdot 31 + 1) = 1 \cdot 5^5 + 1 \cdot 5^4 +$$

$+ 2\cdot5^3 + 1\cdot5^7 = 4005$, 4005 is the smallest number with this property. 4006, 4007, 4008, 4009 verify the property but 4010 does not because 4010! = 4009! 4010 has 1001 zeros.

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AN INFINITY OF UNSOLVED PROBLEMS CONCERNING A FUNCTION IN THE NUMBER THEORY

§1. Abstract

W. Sierpiński has asserted to an international conference that if mankind lasted for ever and numbered the unsolved problems, then in the long run all these unsolved problems would be solved.

The purpose of our paper is that making an infinite number of unsolved problems to prove his supposition is not true. Moreover, the author considers the unsolved problems proposed in this paper can never be all solved!

Every period of time has its unsolved problems which were not previously recommended until recent progress. Number of new unsolved problems are exponentially increasing in comparison with ancient unsolved ones which are solved at present. Research into one unsolved problem may produce many new interesting problems. The reader is invited to exhibit his works about them.

§2. Introduction

We have constructed (*) a function η which associates to each non-null integer n the smallest positive integer m such that m! is a multiple of n. Thus, if n has the standard form:

$n = \epsilon p_1^{a_1} \ldots p_r^{a_r}$, with all p_i distinct primes, all $a_i \in N^*$, and $\epsilon = \pm 1$, then $\eta(n) = \max_{1 \le i \le r} \{\eta_{p_i}(a_i)\}$, and $\eta(\pm 1) = 0$.

Now, we define the η_p functions: let p be a prime and $a \in N^*$; then $\eta_p(a)$ is the smallest positive integer b such that b! is a multiple of p^a. Constructing the sequence:

$$\alpha_k^{(p)} = \frac{p^k - 1}{p-1}, \quad k = 1, 2, \ldots$$

we have $\eta_p(\alpha_k^{(p)}) = p^k$, for all prime p, and all k = 1, 2, ... Because any $a \in N^*$ is uniquely written in the form:

$$a = t_1\alpha_{n_1}^{(p)} + \ldots + t_e\alpha_{n_e}^{(p)}, \text{ where } n_1 > n_2 > \ldots > n_e > 0,$$

and $1 \le t_j \le p - 1$ for $j = 0, 1, \ldots, e - 1$, and $1 \le t_e \le p$, with all n_i, t_i from N, the author proved that

$$\eta_p(a) = \sum_{i=1}^{e} t_i \, \eta_p(\alpha_{n_i}^{(p)}) = \sum_{i=1}^{e} t_i p^{n_i}.$$

§3. Some Properties of the Function η

Clearly, the function η is even: $\eta(-n) = \eta(n)$, $n \in Z^*$. If $n \in N^*$ we have:

$$(1) \quad \frac{-1}{(n-1)!} \leq \frac{\eta(n)}{n} \leq 1 \, ,$$

and: $\frac{\eta(n)}{n}$ is maximum if and only if n is prime or n = 4;

$\frac{\eta(n)}{n}$ is minimum if and only if n = k! .

Clearly η is not a periodical function. For p prime, the functions η_p are increasing, not injective but on $N^* - \{p^k \mid k = 1, 2, \ldots\}$ they are surjective. From (1) we find that $\eta = o(n^{1+\epsilon})$, $\epsilon > 0$, and $\eta = O(n)$.

The function η is <u>generally increasing</u> on N*, that is: $(\forall)\ n \in N^*$, $(\exists)\ m_0 \in N^*$, $m_0 = m_0(n)$, such that for all $m \geq m_0$ we have $\eta(m) \geq \eta(n)$ (and generally decreasing on Z_-^*); it is not injective, but it is surjective on $Z\setminus\{0\} \to N\setminus\{1\}$.

The number n is called a <u>barrier</u> for a number-theoretic function f(m) if, for all m < n, $m + f(m) \leq n$ (P. Erdös and J. L. Selfridge). Does $\epsilon\, \eta(m)$ have infinitely many barriers, with $0 < \epsilon \leq 1$? [No, because there is a $m_0 \in N$ such that for all $n - 1 \geq m_0$ we have $\eta(n-1) \geq$ $\geq \frac{2}{\epsilon}$ (η is generally increasing), whence $n - 1 + \epsilon\, \eta(n-1) \geq$ $\geq n + 1$.]

$\sum_{n\geq 2} 1/\eta(n)$ is divergent, because $1/\eta(n) \geq 1/n$.

$$\eta\Big(\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{k\text{ times}}\Big) = 2 + \underbrace{2^{\cdot^{\cdot^{\cdot^{2^{n}}}}}}_{k-1\text{ times}}$$

Proof: Let

$a_m^{(2)} = 2^m - 1$, where $m = \underbrace{2^{\cdot^{\cdot^{\cdot^{2^{n}}}}}}_{k-2\text{ times}}$;

then $\eta\left(2^{2^m}\right) = \eta_2\,(2^m) = \eta_2\,(1 + a_m^{(2)}) = \eta_2\,(1) + \eta_2\,(a_m^{(2)}) =$
$= 2 + 2^m$.

§4. Glossary of Symbols and Notions

A-sequence: an integer sequence $1 \le a_1 < a_2 < \dots$ so that no a_i is the sum of distinct members of the sequence other than a_i (R. K. Guy);

Average Order: if $f(n)$ is an arithmetical function and $g(n)$ is any simple function of n such that $f(1) + \dots + f(n) \sim g(1) + \dots + g(n)$ we say that $f(n)$ is of the average order of $g(n)$;

$d(x)$: number of positive divisors of x;

d_x: difference between two consecutive primes: $p_{x+1} - p_x$; .

Dirichlet Series: a series of the form $F(s) = \sum_{n=1}^{\infty} \frac{\alpha_n}{n^s}$, s may be real or complex;

Generating Function: any function $F(s) = \sum_{n=1}^{\infty} \alpha_n u_n(s)$ is considered as a generating function of α_n; the most usual form of $u_n(s)$ is: $u_n(s) = e^{-\lambda_n \cdot s}$, where λ_n is a sequence of positive numbers which increases steadily to infinity;

Log x: Napierian logarithm of x, to base e;

Normal Order: f(n) has the normal order F(n) if f(n) is approximately F(n) for almost all values of n, i.e. (2), $(\forall)\ \epsilon > 0$, $(1 - \epsilon) \cdot F(n) < f(n) < (1 + \epsilon) \cdot F(n)$ for almost all values of n; "almost all" n means that the numbers less than n which do not possess the property (2) is o (x);

Lipschitz-Condition: a function f verifies the Lipschitz-condition of order $\alpha \in (0, 1]$ if $(\exists)\ k > 0: |f(x)-f(y)| \leq k\ |x-y|^{\alpha}$; if $\alpha = 1$, f is called a k Lipschitz-function; if $k < 1$, f is called a contractant function;

Multiplicative Function: a function $f: N^* \to C$ for which $f(1) = 1$, and $f(m \cdot n) = f(m) \cdot f(n)$ when $(m, n) = 1$;

p(x): largest prime factor of x;

Uniformly Distributed:	a set of points in (a, b) is uniformly distributed if every sub-interval of (a, b) contains its proper quota of points;
Incongruent Roots:	two integers x, y which satisfy the congruence $f(x) \equiv f(y) \equiv 0 \pmod{m}$ and so that $x \not\equiv y \pmod{m}$;
s-additive sequence:	a sequence of the form: $a_1 = \ldots = a_s = 1$ and $a_{n+s+1} = a_{n+1} + \ldots + a_{n+s}$, $n \in N^*$ (R. Queneau);
s(n):	sum of aliquot parts (divisors of n other than n) of n; $\sigma(n) - n$;
$s^k(n)$:	k^{th} iterate of s(n);
$s^*(n)$:	sum of unitary aliquot parts of n;
$r_k(n)$:	least number of numbers not exceeding n, which must contain a k-term arithmetic progression;
$\pi(x)$:	number of primes not exceeding x;
$\pi(x; a, b)$:	number of primes not exceeding x and congruent to a modulo b;
$\sigma(n)$:	sum of divisors of n; $\sigma_1(n)$;
$\sigma_k(n)$:	sum of k-th powers of divisors of n;
$\sigma^k(n)$:	k-th iterate of $\sigma(n)$;
$\sigma^*(n)$:	sum of unitary divisors of n;

$\varphi(n)$: Euler's totient function; number of numbers not exceeding n and prime to n;

$\varphi^k(n)$: k-th iterate of $\varphi(n)$;

$\bar{\varphi}(n)$: $= n \prod (1 + p^{-1})$, where the product is taken over the distinct prime divisors of n;

$\Omega(n)$: number of prime factors of n, counting repetitions;

$\omega(n)$: number of distinct prime factors of n;

$\lfloor a \rfloor$: floor of a; greatest integer not greater than a;

(m, n): g.c.d. (greatest common divisor) of m and n;

$[m, n]$: l.c.d. (least common multiple) of m and n;

$|f|$: modulus or absolute value of f;

$f(x) \sim g(x)$: $f(x)/g(x) \to 1$ as $x \to \infty$; f is asymptotic t g;

$f(x) = o(g(x))$: $f(x)/g(x) \to 0$ as $x \to \infty$;

$f(x) = O(g(x))$, $f(x) \ll g(x)$; there is a constant c such that $|f(x)| < cg(x)$, for any x;

$\Gamma(x)$: Euler's function of first case (gamma function); $\Gamma : R^*_+ \to R$, $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$. We have $\Gamma(x+1) = x\,\Gamma(x)$. If $x \in N^*$, $\Gamma(x) = (x - 1)!$

$\beta(x)$: Euler's function of second degree (beta function); $\beta : R^*_+ \times R^*_+ \to R$,

$$\beta(u, v) = \Gamma(u)\,\Gamma(v)/\Gamma(u + v) = \int_0^1 t^{u-1} \cdot (1 - t)^{v-1}\,dt;$$

$\mu(x)$: Möbius' function; $\mu : N \to N$ $\mu(1) = 1$; $\mu(n) = (-1)^k$ if n is the product of $k > 1$ distinct primes; $\mu(n) = 0$ in all other cases;

$\theta(x)$: Tchebycheff θ-function; $\theta : R_+ \to R$,

$\theta(x) = \Sigma \log p$

where the summation is taken over all primes p not exceeding x;

$\Psi(x)$: Tchebycheff's Ψ-function; $\Psi(x) =$

$$= \sum_{n \le x} \Lambda(n), \text{ with}$$

$$\Lambda(n) = \begin{cases} \log p, \text{ if } n \text{ is an integer power of the prime } p; \\ 0, \text{ in all other cases.} \end{cases}$$

This glossary can be continued with OTHER (ARITHMETICAL) FUNCTIONS.

§5. General Unsolved Problems Concerning the Function η

(1) Is there a closed expression for $\eta(n)$?

(2) Is there a good asymptotic expression for $\eta(n)$? (If yes, find it.)

(3) For a fixed non-null integer m, does $\eta(n)$ divide n-m? (Particularly when m = 1.) Of course, for m = 0 it is trivial: we find n = k!, or n is a squarefree, etc.

(4) Is η an algebraic function? (If no, is there the max Card $\{n \in Z^* \mid (\exists)\ p \in R[x, y]$, p non-null polynomial, with $p(n, \eta(n)) = 0$ for all these $n\}$?) More generally we introduce the notion: g is a f-function if $f(x, g(x)) = 0$ for all x, and $f \in R[x, y]$, f non-null. Is η a f-function? (If no, is there the max Card $\{n \in Z^* \mid (\exists)\ f \in R[x, y]$, f non-null, $f(n, \eta(n)) = 0$ for all these $n\}$?)

(5) Let A be a set of consecutive integers from N*. Find max Card A for which η is monotonous. For example, Card A $\geq$ 5, because for A = {1, 2, 3, 4, 5} η is 0, 2, 3, 4, 5, respectively.

(6) A number is called an *η-algebraic number of degree n* $\in$ N* if it is a root of the polynomial

(p) $p_\eta(x) = \eta(n)\ x^n + \eta(n-1)\ x^{n-1} + \ldots + \eta(1)\ x^1 = 0.$

An *η-algebraic field* $\mathbb{M}$ is the aggregate of all numbers

$$R_\eta(\upsilon) = \frac{A(\upsilon)}{B(\upsilon)},$$

where υ is a given η-algebraic number, and $A(\upsilon)$, $B(\upsilon)$ are polynomials in υ of the form (p) with $B(\upsilon) \neq 0$. Study $\mathbb{M}$.

(7) Are the points $p_n = \eta(n)/n$ uniformly distributed in the interval (0, 1)?

(8) Is 0.0234537465114..., where the sequence of digits is $\eta(n)$, $n \geq 1$, an irrational number?

*

Is it possible to represent all integer n under the form:

(9) $n = \pm \eta(a_1)^{a_2} \pm \eta(a_2)^{a_3} \pm \ldots \pm \eta(a_k)^{a_1}$, where the integers k, $a_1, \ldots, a_k$, and the signs are conveniently chosen?

(10) But as $n = \pm a_1^{\eta(a_1)} \pm \ldots \pm a_k^{\eta(a_k)}$?

(11) But as $n = \pm a_1^{\eta(a_2)} \pm a_2^{\eta(a_3)} \pm \ldots \pm a_k^{\eta(a_1)}$?

*

Find the smallest k for which: $(\forall)\ n \in N^*$ at least one of the numbers $\eta(n), \eta(n + 1), \ldots, \eta(n + k - 1)$ is:

(12) A perfect square.

(13) A divisor of k^n.

(14) A multiple of a fixed nonzero integer p.

(15) A factorial of a positive integer.

*

(16) Find a general form of the continued fraction expansion of $\eta(n)/n$, for all $n \geq 2$.

(17) Are there integers m, n, p, q, with $m \neq n$ or $p \neq q$, for which: $\eta(m) + \eta(m + 1) + \ldots + \eta(m + p) = \eta(n) + \eta(n + 1) + \ldots + \eta(n + q)$?

(18) Are there integers m, n, p, k with $m \neq n$ and $p > 0$, such that:

$$\frac{\eta(m)^2 + \eta(m + 1)^2 + \ldots + \eta(m + p)^2}{\eta(n)^2 + \eta(n + 1)^2 + \ldots + \eta(n + p)^2} = k\ ?$$

(19) How many primes have the form:

$$\overline{\eta(n)\ \eta(n+1)\ \ldots\ \eta(n+k)}\ ,$$

for a fixed integer k? For example:

$$\overline{\eta(2)\ \eta(3)} = 23,\ \overline{\eta(5)\ \eta(6)} = 53 \text{ are primes.}$$

(20) Prove that $\eta(x^n) + \eta(y^n) = \eta(z^n)$ has an infinity of integer solutions, for any $n \geq 1$. Look, for example, at the solution (5, 7, 2048) when n = 3. (On Fermat's last theorem.) More generally: the diophantine equation $\sum_{i=1}^{k} \eta(x_i^s) = \sum_{j=1}^{m} \eta(y_j^t)$ has an infinite number of solutions.

(21) Are there m, n, k non-null positive integers, $m * 1 * n$, for which $\eta(m \cdot n) = m^k \cdot \eta(n)$? Clearly, η is not homogenous to degree k.

(22) Is it possible to find two distinct numbers k, n for which $\log_{\eta(k^n)} \eta(n^k)$ be an integer? (The base is $\eta(k^n)$.)

(23) Let the congruence be: $h_\eta(x) = c_n\, x^{\eta(n)} + \ldots + c_1 \cdot x^{\eta(1)} \equiv 0 \pmod{m}$. How many incongruent roots has h_η, for some given constant integers n, $c_1, \ldots, c_n$?

(24) We know that $e^x = \sum_{n=0}^{\infty} x^n/n!$. Calculate $\sum_{n=1}^{\infty} x^{\eta(n)} / n!$, $\sum_{n=1}^{\infty} x^n / \eta(n)!$ and eventually some of their properties.

(25) Find the average order of $\eta(n)$.

(26) Find some $u_n(s)$ for which F(s) be a generating function of $\eta(n)$, and F(s) have at all a simple form.

Particularly, calculate Dirichlet series $F(s) = \sum_{n=1}^{\infty} \eta(n)/n^s$, with $s \in R$ (or $s \in C$).

(27) Does $\eta(n)$ have a normal order?

(28) We know that Euler's constant is

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n \right).$$

Is $\lim_{n\to\infty} \left[1 + \sum_{k=2}^{n} 1/\eta(k) - \log \eta(n) \right]$ a constant? If yes, find it.

(29) Is there an m for which $\eta^{-1}(m) = \{a_1, a_2, \dots, a_{pq}\}$ such that the numbers $a_1, a_2, \dots, a_{pq}$ can constitute a matrix of p rows and q columns with the sum of elements on each row and each column is constant? Particularly when the matrix is square.

*

(30) Let $\{x_n^{(s)}\}_{n\geq 1}$ be a s-additive sequence. Is it possible to have $\eta(x_n^{(s)}) = x_m^{(s)}$, $n \neq m$? But $x_{\eta(n)}^{(s)} = \eta(x_n^{(s)})$?

(31) Does η verify a Lipschitz Condition?

(32) Is η a k-Lipschitz Condition?

(33) Is η a contractant function?

(34) Is it possible to construct an A-sequence $a_1, \ldots, a_n$ such that $\eta(a_1), \ldots, \eta(a_n)$ be an A-sequence, too? Yes, for example 2, 3, 7, 31, ... Find such an infinite sequence.

*

Find the greatest n such that: if $a_1, \ldots, a_n$ constitute a p-sequence then $\eta(a_1), \ldots, \eta(a_n)$ constitute a p-sequence, too; where a p-sequence means:

(35) Arithmetical progression.

(36) Geometrical progression.

(37) A complete system of modulo n residues.

Remark: let p be a prime, and $p, p^2, \ldots, p^p$ a geometrical progression, then $\eta(p^i) = ip$, $i \in \{1, 2, \ldots, p\}$, constitute an arithmetical progression of length p. In this case $n \to \infty$.

(38) Let's use the sequence $a_n = \eta(n)$, $n \geq 1$. Is there a recurring relation of the form $a_n = f(a_{n-1}, a_{n-2}, \ldots)$ for any n?

(39) Are there blocks of consecutive composite numbers $m + 1, \ldots, m + n$ such that $\eta(m + 1), \ldots, \eta(m + n)$ be composite numbers, too? Find the greatest n.

(40) Find the number of partitions of n as sum of $\eta(m)$, $2 < m \leq n$.

MORE UNSOLVED GENERAL PROBLEMS CONCERNING THE FUNCTION η

§6. Unsolved Problems Concerning the Function η and Using the Number Sequences

41-2065) Are there non-null and non-prime integers a_1, a_2, ..., a_n in the relation P, so that $\eta(a_1)$, $\eta(a_2)$, ..., $\eta(a_n)$ be in the relation R? Find the greatest n with this property. (Of course, all a_i are distinct.) Where each P, R can represent one of the following number sequences:

(1) Abundant numbers; $a \in N$ is abundant if $\sigma(a) > 2a$.

(2) Almost perfect numbers; $a \in N$, $\sigma(a) = 2a - 1$.

(3) Amicable numbers; in this case we take $n = 2$; a, b are called amicable if $a \neq b$ and $\sigma(a) = \sigma(b) = a + b$.

(4) Augmented amicable numbers; in this case $n = 2$; a, b are called augmented amicable if $\sigma(a) = \sigma(b) = a + b - 1$ (Walter E. Beck and Rudolph M. Najar).

(5) Bell numbers: $b_n = \sum_{k=1}^{n} S(n, k)$, where $S(n, k)$ are stirling numbers of second case.

(6) Bernoulli numbers (Jacques 1st): B_n, the coefficients of the development in integer sequence of

$$\frac{t}{e^t-1} = 1 - \frac{t}{2} + \frac{B_1}{2!}t^2 - \frac{B_2}{4!}t^4 + \ldots + (-1)^{n-1}\frac{B_n}{(2n)!}t^{2n} + \ldots,$$

for $0 < |t| < 2\pi$; (here we always take $\lfloor 1/B_n \rfloor$).

(7) Catalan numbers: $c_1 = 1$, $c_n = \frac{1}{n}\binom{2n-2}{n-1}$ for $n \geq 2$.

(8) Carmichael numbers; an odd composite number a, which is a pseudoprime to base b for every b relatively prime to a, is called a Carmichael number.

(9) Congruent numbers; let n = 3, and the numbers a, b, c; we must have $a \equiv b \pmod{c}$.

(10) Cullen numbers: $C_n = n \cdot 2^n + 1$, $n \geq 0$.

(11) C_1-sequence of integers; the author introduced a sequence $a_1, a_2, \ldots$ so that:

$(\forall)\ i \in N^*, (\exists)\ j, k \in N^*, j \neq i \neq k \neq j, : a_i \equiv a_j \pmod{a_k}$

(12) C_2-sequence of integers; the author defined other sequence $a_1, a_2, \ldots$ so that:

$(\forall)\ i \in N^*, (\exists)\ j, k \in N^*, i \neq j \neq k \neq i, : a_j \equiv a_k \pmod{a_i}$.

(13) Deficient numbers; $a \in N^*$, $\sigma(a) < 2a$.

(14) Euler numbers: the coefficients E_n in the expansion of $\sec x = \sum_{n\geq 0} E_n x^n/n!$; here we will take $|E_n|$.

(15) Fermat numbers: $F_n = 2^{2^n} + 1$, $n \geq 0$.

(16) Fibonacci numbers: $f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$, $n \geq 3$.

(17) Genocchi numbers: $G_n = 2(2^{2n} - 1) B_n$, where B_n are Bernoulli numbers; always $G_n \in Z$.

(18) Harmonic mean; in this case every member of the sequence is the harmonic mean of the preceding members.

(19) Harmonic numbers; a number n is called harmonic if the harmonic mean of all divisors of n is an integer (C. Pomerance).

(20) Heteromeous numbers: $h_n = n(n + 1)$, $n \in N^*$.

(21) K-hyperperfect numbers; a is k-hyperperfect if $a = 1 + \Sigma d_i$, where the numeration is taken over all proper divisors, $1 < d_i < a$, or $k\,\sigma(a) = (k + 1)\,a + k - 1$ (Daniel Minoli and Robert Bear).

(22) Kurepa numbers: $!n = 0! + 1! + 2! + \ldots + (n - 1)!$

(23) Lucas numbers: $L_1 = 1$, $L_2 = 3$, $L_n = L_{n-1} + L_{n-2}$, $n \geq 3$.

(24) Lucky numbers: from the natural numbers strike out all even numbers, leaving the odd numbers; apart from 1, the first remaining number is 3; strike out every third member in the new sequence; the next member remaining is 7; strike out every seventh member in this sequence; next 9 remains; etc. (V. Gardiner, R. Lazarus, N. Metropolis, S. Ulam).

(25) Mersenne numbers: $M_p = 2^p - 1$.

(26) m-perfect numbers; a is m-perfect if $\sigma^m(a) = 2a$ (D. Bode).

(27) Multiply perfect (or k-fold perfect) numbers; a is k-fold perfect if $\sigma(a) = k\,a$.

(28) Perfect numbers; a is perfect if $\sigma(a) = 2a$.

(29) Polygonal numbers (represented on the perimeter of a polygon): $p_n^k = k(n-1)$.

(30) Polygonal numbers (represented on the closed surface of a polygon): $p_n^k = \frac{(k-2)\,n^2 - (k-4)\,n}{2}$.

(31) Primitive abundant numbers; a is primitive abundant if it is abundant, but none of its proper divisors are.

(32) Primitive pseudoperfect numbers; a is primitive pseudoperfect if it is pseudoperfect, but none of its proper divisors are.

(33) Pseudoperfect numbers; a is pseudoperfect if it is equal to the sum of some of its proper divisors (W. Sierpiński).

(34) Pseudoprime numbers to base b; a is pseudoprime to base b if a is an odd composite number for which $b^{a-1} \equiv 1 \pmod{a}$ (C. Pomerance, J. L. Selfridge, S. Wagstaff).

(35) Pyramidal numbers: $\pi_n = \frac{1}{6} n(n+1)(n+2)$, $n \in N^*$.

(36) Pythagorian numbers; let n = 3 and a, b, c be integers; then it must have the relation: $a^2 = b^2 + c^2$.

(37) Quadratic residues of a fixed prime p: the nonzero numbers r for which the congruence $r \equiv x^2 \pmod{p}$ has solutions.

(38) Quasi perfect numbers; a is quasi perfect if $\sigma(a) = 2a + 1$.

(39) Reduced amicable numbers; we take n = 2; two integers a, b for which $\sigma(a) = \sigma(b) = a + b + 1$ are called reduced amicable numbers (Walter E. Beck and Rudolph M. Najar).

(40) Stirling numbers of first case: $s(0, 0) = 1$, and $s(n, k)$ is the coefficient of x^k from the development $x(x - 1) \ldots (x - n + 1)$.

(41) Stirling numbers of second case: $S(0, 0) = 1$, and $S(n, k)$ is the coefficient of the polynom $x^{(k)} = x(x - 1) \ldots (x - k + 1)$, $1 \le k \le n$, from the development (which is uniquely written):

$$x^n = \sum_{k=1}^{n} S(n, k)\, x^{(k)} .$$

(42) Superperfect numbers; a is superperfect if $\sigma^2(a) = 2a$ (D. Suryanarayana).

(43) Untouchable numbers; a is untouchable if $s(x) = 1$ has no solution (Jack Alanen).

(44) U-numbers: starting from arbitrary u_1 and u_2 continues with those numbers which can be expressed in just

one way as the sum of two distinct earlier members of the sequence (S. M. Ulam).

(45) Weird numbers; a is called weird if it is abundant but not pseudoperfect (S. J. Benkoski).

MORE NUMBER SEQUENCES

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The unsolved problem No. 41 is obtained by taking P = = (1) and R = (1).

The unsolved problem No. 42 is obtained by taking P = = (1), R = (2).

The unsolved problem No. 2065 is obtained by taking P = (45) and R = (45).

OTHER UNSOLVED PROBLEMS CONCERNING THE FUNCTION η AND USING NUMBER SEQUENCES

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§7. Unsolved Diophantine Equations Concerning the Function η

2066) Let $0 < k \leq 1$ be a rational number. Does the diophantine equation $\eta(n)/n = k$ always have solutions? Find all k so that this equation has an infinite number of solutions. (For example, if $k = 1/r$, $r \in N^*$, then $n = rp_{a+h}$, $h = 1, 2, \ldots$, all p_{a+h} are primes, and a is a chosen index such that $p_{a+1} > r$.)

2067) Let $\{a_n\}_{n\geq 0}$ be a sequence, $a_0 = 1$, $a_1 = 2$, and $a_{n+1} = a_{\eta(n)} + \eta(a_n)$. Are there infinitely many pairs (m, n), $m \neq n$, for which $a_m = a_n$? (For example: $a_9 = a_{13} = 16$.)

2068) Conjecture: the equation $\eta(x) = \eta(x + 1)$ has no solution.

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Let m, n be fixed integers. Solve the diophantine equations:

2069) $\eta(m\,x + n) = x$.

2070) $\eta(m\,x + n) = m + n\,x$.

2071) $\eta(m\,x + n) = x!$

2072) $\eta(x^m) = x^n$.

2073) $\eta(x)^m = \eta(x^n)$.

2074) $\eta(m\,x + n) = \eta(x)^y$.

2075) $\eta(x) + y = x + \eta(y)$, x and y are not primes.

2076) $\eta(x) + \eta(y) = \eta(x + y)$, x and y are not twin primes. (Generally, η is not additive.)

2077) $\eta(x + y) = \eta(x)\cdot\eta(y)$. (Generally, η is not an exponential function.)

2078) $\eta(xy) = \eta(x)\eta(y)$. (Generally, η is not a multiplicative function.)

2079) $\eta(m\,x + n) = x^y$.

2080) $\eta(x)\,y = x\,\eta(y)$, x and y are not primes.

2081) $\eta(x)/y = x/\eta(y)$, x and y are not primes. (Particularly when $y = 2^k$, $k \in N$, i.e., $\eta(x)/2^k$ is a dyadic rational number.)

2082) $\eta(x)^y = x^{\eta(y)}$, x and y are not primes.

2083) $\eta(x)^{\eta(y)} = \eta(x^y)$.

2084) $\eta(x^y) - \eta(z^w) = 1$, with $y \neq 1 \neq w$. (On Catalan's problem.)

2085) $\eta(x^y) = m$, $y \geq 2$.

2086) $\eta(x^x) = y^y$. (A trivial solution: $x = y = 2$.)

2087) $\eta(x^y) = y^x$. (A trivial solution: $x = y = 2$.)

2088) $\eta(x) = y!$ (An example: $x = 9$, $y = 3$.)

2089) $\eta(m\,x) = m\,\eta(x)$, $m \geq 2$.

2090) $m^{\eta(x)} + \eta(x)^n = m^n$.

2091) $\eta(x^2)/m \pm \eta(y^2)/n = 1$.

2092) $\eta(x_1^{y_1} + \ldots + x_r^{y_r}) = \eta(x_1)^{y_1} + \ldots + \eta(x_r)^{y_r}$.

2093) $\eta(x_1! + \ldots + x_r!) = \eta(x_1)! + \ldots + \eta(x_r)!$.

2094) $(x, y) = (\eta(x), \eta(y))$, x and y are not primes.

2095) $[x, y] = [\eta(x), \eta(y)]$, x and y are not primes.

OTHER UNSOLVED DIOPHANTINE EQUATIONS CONCERNING

THE FUNCTION η ONLY

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§8. Unsolved Diophantine Equations Concerning the Function η in Correlation with Other Functions

Let m, n be fixed integers. Solve the diophantine equations:

2096-2102) $\eta(x) = d(m\,x + n)$

$\eta(x)^m = d(x^n)$

$\eta(x) + y = x + d(y)$

$\eta(x) \cdot y = x \cdot d(y)$

$\eta(x)/y = d(y)/x$

$\eta(x)^{y} = x^{d(y)}$

$\eta(x)^{y} = d(y)^{x}$

2103-2221) Same equations as before, but we substitute the function d(x) with d_x, p(x), s(x), $s^k(x)$, $s*(x)$, $r_k(x)$, $\pi(x)$, $\pi(x; m, n)$, $\sigma_k(x)$, $\sigma^k(x)$, $\sigma*(x)$, $\varphi(x)$, $\varphi^k(x)$, $\bar{\varphi}(x)$, $\Omega(x)$, $\omega(x)$ respectively.

2222) $\eta(s(x, y)) = s(\eta^{(x)}, \eta(y))$.

2223) $\eta(S(x, y)) = S(\eta(x), \eta(y))$.

2224) $\eta(\lfloor x \rfloor) = \lfloor \Gamma(x) \rfloor$.

2225) $\eta(\lfloor x - y \rfloor) = \lfloor \beta(x, y) \rfloor$.

2226) $\beta(\eta(\lfloor x \rfloor), y) = \beta(x, \eta(\lfloor y \rfloor))$.

2227) $\eta(\lfloor \beta(x, y) \rfloor) = \lfloor \beta(\eta(\lfloor x \rfloor), \eta(\lfloor y \rfloor)) \rfloor$.

2228) $\mu(\eta(x)) = \mu(\varphi(x))$.

2229) $\eta(x) = \lfloor \theta(x) \rfloor$.

2230) $\eta(x) = \lfloor \psi(x) \rfloor$.

2231) $\eta(m\,x + n) = A_x^n = x(x - 1) \ldots (x - n + 1)$.

2232) $\eta(m\,x + n) = A_x^m$.

2233) $\eta(m\,x + n) = \binom{x}{n} = \dfrac{x!}{n!(x-n)!}$.

2234) $\eta(m\,x + n) = \binom{x}{m}$.

2235) $\eta(m\,x + n) = p_x$ = the x-th prime.

2236) $\eta(m\,x + n) = \lfloor 1/B_x \rfloor$.

2237) $\eta(m\,x + n) = G_x$.

2238) $\eta(m\ x + n) = k_x = \left(\quad n \quad \right)$.

2239) $\eta(m\ x + n) = k_x^m$.

2240) $\eta(m\ x + n) = s(m, x)$.

2241) $\eta(m\ x + n) = s(x, n)$.

2242) $\eta(m\ x + n) = S(m, x)$.

2243) $\eta(m\ x + n) = S(x, n)$.

2244) $\eta(m\ x + n) = \pi_x$.

2245) $\eta(m\ x + n) = b_x$.

2246) $\eta(m\ x + n) = |E_x|$.

2247) $\eta(m\ x + n) = !\ x$.

2248) $\eta(x) \equiv \eta(y) \pmod{m}$.

2249) $\eta(xy) \equiv x \pmod{y}$.

2250) $\eta(x)\ (x + m) + \eta(y)\ (y + m) = \eta(z)\ (z + m)$.

2251) $\eta(m\ x + n) = f_x$.

2252) $\eta(m\ x + n) = F_x$.

2253) $\eta(m\ x + n) = M_x$.

2254) $\eta\ (m\ x + n) = c_x$.

2255) $\eta\ (m\ x + n) = C_x$.

2256) $\eta\ (m\ x + n) = h_x$.

2257) $\eta\ (m\ x + n) = L_x$.

More unsolved diophantine equations concerning the function η in correlation with other functions.

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§9. Unsolved Diophantine Equations Concerning the Function η in Composition with Other Functions

2258) $\eta(d(x)) = d(\eta(x))$, x is not prime.

2259-2275) Same equations as this, but we substitute the function d(x) with d_x, p(x), ..., ω(x) respectively.

More unsolved diophantine equations concerning the function η in composition with other functions. (For example: $\eta(\pi(4(x))) = \varphi(\eta(\pi(x)))$, etc.)

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§10. Unsolved Diophantine Inequations Concerning the Function η

Let m, n be fixed integers. Solve the following diophantine inequalities:

2276) $\eta(x) \geq \eta(y)$.

2277) is $0 < \{x/\eta(x)\} < \{\eta(x)/x\}$ infinitely often? where {a} is the fractional part of a.

2278) $\eta(m\,x + n) < d(x)$.

2279-2300) Same (or similar) inequations as this, but we substitute the function d(x) with d_x, p(x), ..., ω(x), Γ(x), β(x, x), μ(x), θ(x), Ψ(x), respectively.

More unsolved diophantine inequations concerning the function η in correlation (or composition, etc.) with other functions. (For example: $\theta(\eta(\lfloor x \rfloor)) < \eta(\lfloor \theta(x) \rfloor)$, etc.)

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§11. Arithmetic Functions Constructed by Means of the Function η

UNSOLVED PROBLEMS CONCERNING THESE NEW FUNCTIONS

I. The function $S_\eta : N^* \to N$, $S_\eta(x) = \sum_{0<n\leq x} \eta(n)$.

2301) Is $\sum_{x\geq 2} S_\eta(x)^{-1}$ a convergent series?

2302) Find the smallest k for which $(\underbrace{S_\eta \circ \ldots \circ S_\eta}_{k \text{ times}})(m) \geq n$, for m, n fixed integers.

2303-4602) Study S_η. The same (or similar questions for S_η as for η.

II. The function $C_\eta : N^* \to Q$, $C_\eta(x) = \frac{1}{x}(\eta(1) + \eta(2) + \ldots + \eta(x))$ (sum of Cesaro concerning the function η).

4603) Is $\sum_{x\geq 1} C_\eta(x)^{-1}$ a convergent series?

4604) Find the smallest k for which $(\underbrace{C_\eta \circ \ldots \circ C_\eta}_{k \text{ times}})(m) \geq n$, for m, n fixed integers.

4605)-6904) Study C_η. The same (or similar) questions for C_η as for η.

III. The function $E_\eta : N^* \to N$, $E_\eta (x) = \sum_{k=1}^{k_0} \eta^{(k)} (x)$, where

$\eta^{(1)} = \eta$ and $\eta^{(k)} = \eta o \ldots o \eta$ of k times, and k_0 is the smallest integer k for which $\eta^{(k+1)} (x) = \eta^{(k)} (x)$.

6905) Is $\sum_{x \geq 2} E_\eta (x)^{-1}$ a convergent series?

6906) Find the smallest x for which $E_\eta (x) > m$, where m is a fixed integer.

6907-9206) Study E_η. The same (or similar) questions for S_η as for η.

IV. The function $F_\eta : N \backslash \{0, 1\} \to N$, $F_\eta(x) = \sum_{\substack{0<p\leq x \\ p \text{ prime}}} \eta_p (x)$.

9207) Is $\sum_{x \geq 2} F_\eta (x)^{-1}$ a convergent series?

9208-11507) Study the function F_η. The same (or similar questions for F_η as for η.

V. The function $\alpha_\eta : N^* \to N$, $\alpha_\eta (x) = \sum_{n=1}^{x} \beta(n)$, where

$$\beta(n) = \begin{cases} 0, & \text{if } \eta(n) \text{ is even;} \\ 1, & \text{if } \eta(n) \text{ is old.} \end{cases}$$

11508) Let $n \in N^*$. Find the smallest k for which

$$\underbrace{(\alpha_\eta o \ldots o\alpha_\eta)}_{k \text{ times}} (n) = 0.$$

11509-13808) Study α_η. The same (or similar) questions for α_η as for η.

VI. The function $m_\eta : N^* \to N$, $m_\eta(j) = a_j$, $1 \le j \le n$, fixed integers, and $m_\eta(n+1) = \min_i \{\eta(a_i + a_{n-i})\}$, etc.

13809) Is $\sum_{x\ge1} m_\eta(x)^{-1}$ a convergent series?

13810-16109) Study m_η. The same (or similar) questions for m_η as for η.

VII. The function $M_\eta : N^* \to N$. A given finite positive integer sequence $a_1, \ldots, a_n$ is successively extended by:

$M_\eta(n+1) = \max_i \{\eta(a_i + a_{n-i})\}$, etc.

$M_\eta(j) = a_j$, $1 \le j \le n$.

16110) Is $\sum_{x\ge1} M_\eta(x)^{-1}$ a convergent series?

16111-18410) Study M_η. The same (or similar) questions for M_η as for η.

VIII. The function $\eta^{-1}_{min} : N\setminus\{1\} \to N$, $\eta^{-1}_{min}(x) = \min\{\eta^{-1}(x)\}$, where $\eta^{-1}(x) = \{a \in N \mid \eta(a) = x\}$. For example $\eta^{-1}(6) = \{2^4, 2^4 \cdot 3, 2^4 \cdot 3^2, 3^2, 3^2 \cdot 2, 3^2 \cdot 2^2, 3^2 \cdot 2^3\}$, whence $\eta^{-1}_{min}(6) = 9$.

18411) Find the smallest k for which $(\underbrace{\eta^{-1}_{min} \circ \ldots \circ \eta^{-1}_{min}}_{k \text{ times}})$

18412-20711) Study η^{-1}_{min}. The same (or similar) questions for η^{-1}_{min} as for η.

IX. The function $\eta^{-1}_{card} : N \to N$, $\eta^{-1}_{card}(x) = \text{Card}\,(\eta^{-1}(x))$, where Card A means the number of elements of the set A.

20712) Find the smallest k for which

$$\Big(\underbrace{\eta^{-1}_{card} \circ \ldots \circ \eta^{-1}_{card}}_{k \text{ times}}\Big)(m) \geq n, \text{ for m, n fixed integers.}$$

20713-23012) Study η^{-1}_{card}. The same (or similar) questions for η^{-1}_{card} as for η.

X. The function $d_\eta : N^* \to N$, $d_\eta(x) = |\eta(x+1) - \eta(x)|$.

Let $d_\eta^{(k+1)}(x) = |d_\eta^{(k)}(x+1) - d_\eta^{(k)}(x)|$, for all $k \in N^*$, where $d_\eta^{(1)}(x) = d_\eta(x)$.

23013) Conjecture: $d_\eta^{(k)}(1) = 1$ or 0, for all $k \geq 2$. (This reminds us of Gillreath's conjecture on primes.) For example:

```
η(1) = 0
          2
η(2) = 2    1
          1   1
η(3) = 3    0   1
          1   0   0
η(4) = 4    0   1   1
          1   1   1   0
η(5) = 5    1   0   1   0
          2   1   0   0   1
η(6) = 3    2   0   1   1   0
          4   1   1   1   1   1
η(7) = 7    1   1   0   2   1   0
          3   0   1   3   2   1   1
η(8) = 4    1   0   3   0   0   1   0
          2   0   4   3   2   0   1   1
η(9) = 6    1   4   0   2   0   0   1   1
          1   4   4   1   2   0   2   0   1
η(10)= 5    5   0   1   0   0   2   1   0   1
          6   4   3   1   2   2   1   0   0   0
η(11)=11    1   3   0   2   2   1   1   0   1
          7   1   3   3   0   1   0   0   1
η(12)= 4    2   0   3   2   1   1   1   1
          9   1   0   1   1   0   1   1
η(13)=13    3   0   2   1   1   0   0
          6   1   2   0   0   0   1
η(14)= 7    4   2   2   1   1   1
          2   3   4   1   1   1
η(15)= 5    1   6   1   0   0
          1   9   5   1   1
η(16)= 6   10   1   2   1
         11  10   7   2
η(17)=17    0   8   0
         11   2   7
η(18)= 6    2   1
         13   1
η(19)=19    1
         14
η(20)= 5
  .
  .
  .
```

23014-25313) Study $d_\eta^{(k)}$. The same (or similar) questions for $d_\eta^{(k)}$ as for η.

XI. The function $\omega_\eta : N^* \to N$, $\omega_\eta(x)$ is the number of m, with $0 < m \leq x$, so that $\eta(m)$ divide x. Hence, $\omega_\eta(x) \geq \omega(x)$, and we have equality if $x = 1$ or x is a prime.

25314) Find the smallest k for which $(\underbrace{\omega_\eta \circ \ldots \circ \omega_\eta}_{k \text{ times}})(x) = 0$, for a fixed integer x.

25315-27614) Study ω_η. The same (or similar) questions for ω_η as for η.

XII. The function $M_\eta : N^* \to N$, $M_\eta(x)$ is the number of m, with $0 < m \le x^x$, so that $\eta(m)$ is a multiple of x. For example $M_\eta(3) = \text{Card}\{1, 3, 6, 9, 12, 27\} = 6$. If p is a prime, $M_\eta(p) = \text{Card}\{1, a_2, \ldots, a_r\}$, then all a_i, $2 \le i \le r$, are multiples of p.

27615) Let m, n be integer numbers. Find the smallest k for which $(\underbrace{M_\eta \circ \ldots \circ M_\eta}_{k \text{ times}})(m) \ge n$.

27616-29915) Study M_η. The same (or similar questions for M_η as for η.

XIII. The function $\sigma_\eta : N^* \to N$, $\sigma_\eta(x) = \sum_{\substack{d|x \\ d>0}} \eta(d)$.

For example $\sigma_\eta(18) = \eta(1) + \eta(2) + \eta(3) + \eta(6) + \eta(9) + \eta(18) = 20$, $\sigma_\eta(9) = 9$.

29916) Are there an infinity of nonprimes n so that $\sigma_\eta(n) = n$?

29917-32216) Study σ_η. The same (or similar) questions for σ_η as for η..

XIV. The function $\pi_\eta : N \to N$, $\pi_\eta(x)$ is the number of numbers n so that $\eta(n) \le x$. If $p_1 < p_2 < \ldots < p_k \le n < p_{k+1}$ is the primes sequence, and for all i = 1, 2, ..., k we have $p_i^{a_i}$ divides n! but $p_i^{a_i+1}$ does not divide n!, then:

$$\pi_\eta (n) = (a_1 + 1) \ldots (a_k + 1).$$

32217-34516) Study π_η. The same (or similar) question for π_η as for η.

XV. The function $\varphi_\eta : N^* \to N$, $\varphi_\eta(x)$ is the number of m, with $0 < m \le x$, having the property $(\eta(m), x) = 1$.

34517) Is always true that $\varphi_\eta(x) < \varphi(x)$?

34518) Find x for which $\varphi_\eta(x) \ge \varphi(x)$.

34519) Find the smallest k so that $(\underbrace{\varphi_\eta \circ \ldots \circ \varphi_\eta}_{k \text{ times}})(x) = 1$, for a fixed integer x.

34520-36819) Study φ_η. The same (or similar) questions for φ_η as for η.

More unsolved problems concerning these 15 functions.

More new (arithmetic) functions constructed by means of the function η, and new unsolved problems concerning them.

36820 → ∞. We can continue these recurring sequences of unsolved problems in number theory to infinity. Thus, we construct an infinity of more new functions: Using the functions S_η, C_η, ..., φ_η construct the functions f_{11}, f_{12}, ..., f_{1n_1} (by varied combinations between S_η, C_η, ..., φ_η; for example: $S_\eta^{(i+1)}(x) = \sum_{0<n\leq x} S_\eta^{(i)}$ for all $x \in N^*$, $S_\eta^{(i)} : N^* \to N$ for all $i = 0, 1, 2, \ldots$, where $S_\eta^{(0)} = S_\eta$. Or:

$$SC_\eta(x) = \frac{1}{x} \sum_{n=1}^{x} S_\eta(n),$$ $SC_\eta : N^* \to Q$, SC_η being a combination between S_η and C_η; etc.); analogously by means of the functions f_{11}, f_{12}, ... f_{1n_1} we construct the functions f_{21}, f_{22}, ..., f_{2n_2} etc. The method to obtain new functions continues to infinity. For each function we have at least 2300 unsolved problems, and we have an infinity of thus functions. The method can be represented in the following way:

$$\eta \xrightarrow{\text{produces}} S_\eta, C_\eta, \ldots, \varphi_\eta \to f_{11}, f_{12}, \ldots, f_{1n_1}$$

$$f_{11}, f_{12}, \ldots, f_{1n_1} \longrightarrow f_{21}, f_{22}, \ldots, f_{2n_2}$$

$$f_{21}, f_{22}, \ldots, f_{2n_2} \longrightarrow f_{31}, f_{32}, \ldots, f_{3n_3}$$

$$f_{i1}, f_{i2}, \ldots, f_{in_i} \longrightarrow f_{i+1,1}, f_{i+1,2}, \ldots, f_{i+1,n_{i+1}}$$

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Other recurring methods to make new unsolved problems.

§12. Conclusion

With this paper the author wants to prove that we can construct infinitely many unsolved problems, especially in number theory: you "rock and roll" the numbers until you create interesting scenarios! Some problems in this paper could effect the subsequent development of mathematics.

The world is in a general crisis. Do the unsolved problems really constitute a mathematical crisis, or contrary to that, do their absence lead to an intellectual stagnation? Mankind will always have problems to solve, they even must again solve previously solved problems(!) For example, this paper shows that people will be more and more overwhelmed by (open) unsolved problems. [It is easier to ask than to answer.]

Here, there are proposed (un)solved problems which are enough for ever!! Suppose you solve an infinite number of problems, there will always be an infinity of problems remaining. Do not assume those proposals are trivial and non-important, rather, they are very substantial.

§13. **References** (books and papers which have inspired the author)

[1] *Arnoux Gabriel, Arithmétique graphique. Introduction à l'étude des fonctions arithmétique, Gauthiers-Villars, Paris, 1906.*

[2] *Blanchard A., Initiation à la théorie analitique des nombres primiers, Dunod. Paris, 1969.*

[3] *Borevitch Z.I. and Shafarevitch I.R., Number Theory, Academic Press. New York. 1966.*

[4] *BouvierAlain et George Michel (sous la direction de Francois Le Lionnais), Dictionaire des Mathématiques, Presses Universitaires de France, Paris, 1979.*

[5] *Carmichael R. D., Theory of Numbers, Mathematical Monographs. No. 13. New York. Wiley, 1914.*

[6] *Chandrasekharan K., Introduction to Analytic Number Theory, Springer-Verlag. 1968.*

[7] *Davenport H., Higher Arithmetic, London, Hutchison, 1952.*

[8] *Dickson L. E., Introdution to the Theory of Numbers, Chicago Univ. Press, 1929.*

[9] *Estermann T., Introduction to Modern Prime Number Theory, Cambridge Tracts in Mathematics. No. 41. 1952.*

[10] *Erdös P., Problems and Results in Combinatorial Number Theory. Bordeaux. 1974.*

[11] *Fourrey E., Récréactions Arithmétiques, Troisième Édition, Vuibert et Nony, Paris. 1904.*

[12] *"Gamma" Journal. Unsolved Problems Corner, Braşov. 1985.*

Goodstein, R. L., Recursive Number Theory. A Development of Recursive Arithmetic in a Logic-Free Equation Calculus, North-Holland Publishing Company, 1964.

Grosswald, Emil and Hagis, Peter, Arithmetic Progressions Consisting Only of Primes, Math. Comput. 33, 1343-1352, 1979.

Guy, Richard K., Unsolved Problems in Number Theory, Springer-Verlag, New York, Heidelberg, Berlin, 1981.

Halberstam, H. and Roth, K. F., Sequences, Oxford U.P., 1966.

Hardy, G. H. and Wright, E. M., An Introduction to the Theory of Numbers, Clarendon Press, Oxford, Fifth Edition, 1984.

Hasse, H., Number Theory, Akademie-Verlag, Berlin, 1977.

Landau, Edmund, Elementary Number Theory, with Exercises by Paul T. Bateman and Eugene E. Kohlbecker, Chelsea, New York, 1958.

Mordell, L. J., Diophantine Equations, Academic Press, London, 1969.

Nagell, T., Introduction to Number Theory, New York, Wiley, 1951.

Niven, I., Irrational Numbers, Carus Math. Monographs, No. 11, Math. Assoc. of America, 1956.

Ogilvy, C. S., Unsolved Problems for the Amateur, Tomorrow's Math., Oxford Univ. Press, New York, 1962.

Ore, O., Number Theory and Its History, McGraw-Hill, New York, 1978.

Report of Institute in the Theory of Numbers, Univ. of Colorado, Boulder, 1959.

Shanks, Daniel, Solved and Unsolved Problems in Number Theory, Spartan, Washington, D. C., 1962.

Sierpiński, W., On Some Unsolved Problems of Arithmetics, Scripta Mathematica, Vol. 25, 1960.

Smarandache, Florentin, A Function in the Number Theory *, in Analele Univ. Timisoara, Vol. XVIII, Fasc. 1, pp. 79-88, 1980; M. R. 83c: 10008.

Smarandache, Florentin, Problèmes Avec et Sans ... Problèmes!, Somipress, Fès, Morocco, 1983; M.R. 84k: 00003.

Ulam, S., A Collection of Mathematical Problems, Interscience, New York, 1960.

Vinogradov, I. M., An Introduction to the Theory of Numbers, Translated by Helen Popova, Pergamon Press, London and New York, 1955.

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Corduneanu, Department of Mathematics, University of Texas at Arlington, in "Libertas Mathematica," tomus IX, p. 175, The Grid, Arlington, Texas. Another abstract had been published in the proceedings of the short communications, International Congress of Mathematicians, Berkeley, California, 1986.]

SOLVING PROBLEMS BY USING A FUNCTION IN THE NUMBER THEORY

Let $n \geq 1$, $h \geq 1$, and $a \geq 2$ be integers. For which values of a and n is $(n + h)!$ a multiple of a^n ?
(A generalization of the problem $n^0 = 1270$, Mathematics Magazine, Vol. 60, No. 3, June 1987, p. 179, proposed by Roger B. Eggleton, The University of Newcastle, Australia.)

Solution

(For $h = 1$ the problem $n^0 = 1270$ is obtained.)

§1. Introduction

We have constructed a function η (see [1]) having the following properties:

(a) For each non-null integer n, $\eta(n)!$ is a multiple of n;

(b) $\eta(n)$ is the smallest natural number with the property (a).

It is easy to prove:

Lemma 1. $(\forall)$ $k, p \in N^*$, $p \neq 1$, k is uniquely written in the form:

$$k = t_1 a_{n_1}^{(p)} + \dots + t_\ell a_{n_\ell}^{(p)} ,$$

where $a_{n_i}^{(p)} = (p^{n_i} - 1) / (p - 1)$, $i = 1, 2, \dots, \ell$, $n_1 > n_2 > \dots > n_\ell > 0$ and $1 \le t_j \le p - 1$, $j = 1, 2, \dots, \ell - 1$, $1 \le t_\ell \le p$, n_i, $t_i \in N$, $i = 1, 2, \dots, \ell$, $\ell \in N^*$.

We have constructed the function η_p, p prime > 0, $\eta_p : N^* \to N^*$, thus:

$$(\forall)\ n \in N^*,\ \eta_p\left(a_n^{(p)}\right) = p^n, \text{ and}$$

$$\eta_p\left(t_1 a_{n_1}^{(p)} + \dots + t_\ell a_{n_\ell}^{(p)}\right) =$$

$$= t_1 \eta_p\left(a_{n_1}^{(p)}\right) + \dots + t_\ell \eta_p\left(a_{n_\ell}^{(p)}\right) .$$

Of course:

Lemma 2.

(a) $(\forall)\ k \in N^*,\ \eta_p(k)! = \mathcal{M}p^k$.

(b) $\eta_p(k)$ is the smallest number with the property (a). Now, we construct another function:

$\eta : Z\setminus\{0\} \to N$ defined as follows:

$$\begin{cases} \eta(\pm 1) = 0, \\ (\forall)\ n = \epsilon\, p_1^{\alpha_1} \ldots p_s^{\alpha_s} \text{ with } \epsilon = \pm 1,\ p_i \text{ prime and} \\ p_i \neq p_j \text{ for } i \neq j, \text{ all } \alpha_i \in N^*,\ \eta(n) = \\ = \max\limits_{1 \le i \le s} \{\eta_{p_i}(\alpha_i)\}. \end{cases}$$

It is not difficult to prove η has the demanded properties of §1.

§2. Now, let $a = p_1^{\alpha_1} \ldots p_s^{\alpha_s}$, with all $\alpha_i \in N^*$ and all p_i distinct primes. By the previous theory we have:

$$\eta(a) = \max_{1 \le i \le s} \{n_{p_i}(\alpha_i)\} = \eta_p(\alpha) \text{ (by notation)}.$$

Hence $\quad \eta(a) = \eta(p^\alpha),\ \eta(p^\alpha)! = Mp^\alpha.$

We know:

$$(t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell})! = Mp^{t_1 \frac{p^{n_1}-1}{p-1} + \ldots + t_\ell \frac{p^{n_\ell}-1}{p-1}}.$$

We put:

$$t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell} = n + h$$

and $t_1 \frac{p^{n_1}-1}{p-1} + \ldots + t_\ell \frac{p^{n_\ell}-1}{p-1} = \alpha\, n.$

Whence

$$\frac{1}{\alpha}\left[\frac{p^{n_1}-1}{p-1} + \ldots + t_\ell \frac{p^{n_\ell}-1}{p-1}\right] \geq t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell} - h$$

or

$$(1)\quad \alpha\,(p-1)\,h \geq (\alpha\, p - \alpha - 1)\,[t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell}] +$$

$$+ (t_1 + \ldots + t_\ell).$$

On this condition we take $n_0 = t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell} - h$

(see Lemma 1), hence $n = \begin{cases} n_0, & n_0 > 0; \\ 1, & n_0 \leq 0 \end{cases}$.

Consider giving $a \neq 2$, we have a finite number of n. There are an infinite number of n if and only if $\alpha p - \alpha - 1 = 0$, i.e., $\alpha = 1$ and $p = 2$, i.e., $a = 2$.

§3. Particular Case

If $h = 1$ and $a \neq 2$, because

$$t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell} \geq p^{n_\ell} > 1$$

and $t_1 + \ldots + t_\ell \geq 1$, it follows from (1) that:

(1') $(\alpha p - \alpha) > (\alpha p - \alpha - 1) \cdot 1 + 1 = \alpha p - \alpha$,

which is impossible. If $h = 1$ and $a = 2$ then $\alpha = 1$, $p = 2$, or

(1") $1 \geq t_1 + \ldots + t_\ell$,

hence $\ell = 1$, $t_1 = 1$ whence $n = t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell} - h =$

$= 2^{n_1} - 1$, $n_1 \in N^*$ (the solution to problem 1270).

Example 1. Let $h = 16$ and $a = 3^4 \cdot 5^2$. Find all n such that

$$(n + 16)\,! = M\ 2025^n.$$

Solution

$\eta\ (2025) = \max\ \{\eta_3\ (4),\ \eta_5\ (2)\} = \max\ \{9,\ 10\} = 10 =$
$= \eta_5\ (2) = \eta\ (5^2)$. Whence $\alpha = 2$, $p = 5$. From (1) we have:

$$128 \geq 7[t_1 5^{n_1} + \ldots + t_\ell\, 5^{n_\ell}] + t_1 + \ldots + t_\ell.$$

Because $5^4 > 128$ and $7\ [t_1\ 5^{n_1} + \ldots + t_\ell\ 5^{n_\ell}] < 128$ we find $\ell = 1$,

$$128 \geq 7\, t_1\, 5^{n_1} + t_1\ ,$$

whence $n_1 \leq 1$, i.e., $n_1 = 1$, and $t_1 = 1, 2, 3$. Then $n_0 = t_1 5 - 16 < 0$, hence we take $n = 1$.

Example 2

$(n + 7)! = M\ 3^n$ when $n = 1, 2, 3, 4, 5$.

$(n + 7)! = M\ 5^n$ when $n = 1$.

$(n + 7)! = M\ 7^n$ when $n = 1$.

But $(n + 7)! \neq M\ p^n$, for p prime > 7, $(\forall)\ n \in N^*$.

$(n + 7)! = M\ 2^n$ when

$$n_0 = t_1\, 2^{n_1} + \ldots + t_\ell\, 2^{n_\ell} - 7,$$

$$t_1, \ldots, t_{\ell-1} = 1,$$

$$1 \leq t_\ell \leq 2,\ t_1 + \ldots + t_\ell \leq 7$$

and
$$n = \begin{cases} n_0, & n_0 > 0; \\ 1, & n_0 \leq 0. \end{cases}$$

etc.

Exercise for Readers

If $n \in N^*$, $a \in N^* \setminus \{1\}$, find all values of a and n such that:

$(n + 7)!$ be a multiple of a^n.

<u>Some Unsolved Problems</u> (see [2])

Solve the diophantine equations:

(1) $\eta(x) \cdot \eta(y) = \eta(x + y)$.

(2) $\eta(x) = y!$ (A solution: $x = 9$, $y = 3$).

(3) Conjecture: the equation $\eta(x) = \eta(x + 1)$ has no solution.

<u>References</u>

[1] Florentin Smarandache, "A Function in the Number Theory," Analele Univ. Timisoara, Fasc. 1, Vol. XVIII, pp. 79-88, 1980, MR: 83c: 10008.

[2] Idem, Un Infinity of Unsolved Problems Concerning a Function in Number Theory, International Congress of Mathematicians, Univ. of Berkeley, CA, August 3-11, 1986.

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[A comment about this generalization was published in "Mathematics Magazine", Vol. 61, No. 3, June 1988, p. 202: "Smarandache considered the general problem of finding positive integers n, a, and k, so that $(n + k)!$ should be a multiple of a^n. Also, for positive integers p and k, with p prime, he found a formula for determining the smallest integer f(k) with the property that $(f(k))!$ is a multiple of p^k."]

SOME LINEAR EQUATIONS INVOLVING A FUNCTION IN THE NUMBER THEORY

We have constructed a function η which associates to eac non-null integer m the smallest positive n such that n! is a multiple of m.

(a) Solve the equation $\eta(x) = n$, where $n \in N$.

*(b) Solve the equation $\eta(mx) = x$, where $m \in Z$. Discussion.

(c) Let $\eta^{(i)}$ note $\eta \circ \eta \circ \ldots \circ \eta$ of i times. Prove tha there is a k for which

$$\eta^{(k)}(m) = \eta^{(k+1)}(m) = n_m, \text{ for all } m \in Z^*\setminus\{1\}.$$

**Find n_m and the smallest k with this property.

Solution

(a) The cases n = 0, 1 are trivial.

We note the increasing sequence of primes less or equal than n by $P_1, P_2, \ldots, P_k$, and

$$\beta_t = \sum_{h\geq 1} [n/p_t^h], \quad t = 1, 2, \ldots, k;$$

where [y] is the greatest integer less or equal than y.

Let $n = p_{i_1}^{\alpha_{i_1}} \dots p_{i_s}^{\alpha_{i_s}}$, where all p_{i_j} are distinct primes and all α_{i_j} are from N.

Of course we have $n \leq x \leq n!$

Thus $x = p_1^{\sigma_1} \dots p_k^{\sigma_k}$ where $0 \leq \sigma_t \leq \beta_t$ for all $t = 1, 2, \dots, k$ and there exists at least a $j \in \{1, 2, \dots, s\}$ for which

$$\sigma_{i_j} \in \{\beta_{i_j}\ \beta_{i_j}^{-1}, \dots, \beta_{i_j} - \alpha_{i_j} + 1\} .$$

Clearly n! is a multiple of x, and is the smallest one.

(b) See [1] too. We consider $m \in N^*$.

Lemma 1. $\eta(m) \leq m$, and $\eta(m) = m$ if and only if $m = 4$ or m is a prime.

Of course m! is a multiple of m.

If $m \neq 4$ and m is not a prime, the Lemma is equivalent to there are m_1, m_2 such that $m = m_1 \cdot m_2$ with $1 < m_1 \leq m_2$ and ($2 m_2 < m$ or $2 m_1 < m$). Whence $\eta(m) \leq 2 m_2 < m$, respectively $\eta(m) \leq \max(m_2, 2m) < m$.

Lemma 2. Let p be a prime ≥ 5. Then $\eta(p x) = x$ if and only if x is a prime $> p$, or $x = 2p$.

Proof: $\eta(p) = p$. Hence $x > p$.

Analogously: x is not a prime and $x \neq 2p \Rightarrow x = x_1 x_2$, $1 < x_1 \leq x_2$ and ($2 x_2 < x$, $x_2 \neq p$, and $2 x_1 < x$) $\Rightarrow \eta(p x) \leq$

$\leq \max\{p, 2x_2\} < x$ respectively $\eta(px) \leq \max\{p, 2x_1, x_2\}$ $< x$.

Observations

$\eta(2x) = x \Leftrightarrow x = 4$ or x is an odd prime.

$\eta(3x) = x \Leftrightarrow x = 4, 6, 9$ or x is a prime > 3.

Lemma 3. If $(m, x) = 1$ then x is a prime $> \eta(m)$.

Of course, $\eta(mx) = \max\{\eta(m), \eta(x)\} = \eta(x) = x$. And $x \neq \eta(m)$, because if $x = \eta(m)$ then $m \cdot \eta(m)$ divides $\eta(m)!$ that is m divides $(\eta(m) - 1)!$ whence $\eta(m) \leq \eta(m) - 1$.

Lemma 4. If x is not a prime then $\eta(m) < x \leq 2\eta(m)$ and $x = 2\eta(m)$ if and only if $\eta(m)$ is a prime.

Proof: If $x > 2\eta(m)$ there are x_1, x_2 with $1 < x_1 \leq x_2$, $x = x_1 x_2$. For $x_1 < \eta(m)$ we have $(x - 1)!$ is a multiple of $m\,x$. Same proof for other cases.

Let $x = 2\eta(m)$; if $\eta(m)$ is not a prime, then $x = 2ab$, $1 < a \leq b$, but the product $(\eta(m) + 1)(\eta(m) + 2) \ldots (2\eta(m) - 1)$ is divided by x.

If $\eta(m)$ is a prime, $\eta(m)$ divides m, whence $m \cdot 2\eta(m)$ is divided by $\eta(m)^2$, it results in $\eta(m \cdot 2\eta(m)) \geq 2 \cdot \eta(m)$, but $(\eta(m) + 1)(\eta(m) + 2) \ldots (2\eta(m))$ is a multiple of $2\eta(m)$, that is $\eta(m \cdot 2\eta(m)) = 2\eta(m)$.

Conclusion

All x, prime number > η (m), are solutions.

If η (m) is prime, then x = 2 η (m) is a solution.

*If x is not a prime, η (m) < x < 2 η (m), and x does not divide (x - 1)!/m then x is a solution (semi-open question). If m = 3 it adds x = 9 too. (No other solution exists yet.)

(c)

Lemma 5. η (a b) $\leq$ η (a) + η (b).

Of course, η (a) = a' and η (b) = b' involves (a' + + b')! = b'! (b' + 1) ... (b' + a'). Let a' $\leq$ b'. Then η(ab) $\leq$ a' + b', because the product of a' consecutive positive integers is a multiple of a'!

Clearly, if m is a prime then k = 1 and n_m = m.

If m is not a prime then η (m) < m, whence there is a k for which $\eta^{(k)}$ (m) = $\eta^{(k+1)}$ (m).

If m $\neq$ 1 then 2 $\leq$ n_m $\leq$ m.

Lemma 6. n_m = 4 or n_m is a prime.

If n_m = n_1 n_2, 1 < n_1 $\leq$ n_2, then η (n_m) < n_m. Absurd. n_m $\neq$ 4.

(**) This question remains open.

Reference

[1] F. Smarandache, A Function in the Number Theory, An. Univ. Timisoara, seria st. mat., Vol. XVIII, fasc. 1, pp. 79-88, 1980; Mathematical Reviews: 83c: 10008.

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Brasov, 1987.]

CONTENTS

A collection of papers concerning Smarandache type functions, numbers, sequences, integer algorithms, paradoxes, experimental geometries, algebraic structures, neutrosophic probability, set, and logic, etc.