Absolutely Harmonious Labeling of Graphs

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Abstract: Absolutely harmonious labeling $f$ is an injection from the vertex set of a graph $G$ with $q$ edges to the set $\{0, 1, 2, ..., q-1\}$, if each edge $uv$ is assigned $f(u) + f(v)$ then the resulting edge labels can be arranged as $a_0, a_1, a_2, ..., a_{q-1}$ where $a_i = q - i$ or $q + i$, $0 \leq i \leq q - 1$. However, when $G$ is a tree one of the vertex labels may be assigned to exactly two vertices. A graph which admits absolutely harmonious labeling is called absolutely harmonious graph. In this paper, we obtain necessary conditions for a graph to be absolutely harmonious and study absolutely harmonious behavior of certain classes of graphs.

Key Words: Graph labeling, Smarandachely $k$-labeling, harmonious labeling, absolutely harmonious labeling.

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§1. Introduction

A vertex labeling of a graph $G$ is an assignment $f$ of labels to the vertices of $G$ that induces a label for each edge $xy$ depending on the vertex labels. For an integer $k \geq 1$, a Smarandachely $k$-labeling of a graph $G$ is a bijective mapping $f : V \rightarrow \{1, 2, \cdots, k|V(G)| + |E(G)|\}$ with an additional condition that $|f(u) - f(v)| \geq k$ for $uv \in E$. Particularly, if $k = 1$, i.e., such a Smarandachely 1-labeling is the usually labeling of graph. Among them, labelings such as those of graceful labeling, harmonious labeling and mean labeling are some of the interesting vertex labelings found in the dynamic survey of graph labeling by Gallian [2]. Harmonious labeling is one of the fundamental labelings introduced by Graham and Sloane [3] in 1980 in connection with their study on error correcting code. Harmonious labeling $f$ is an injection from the vertex set of a graph $G$ with $q$ edges to the set $\{0, 1, 2, ..., q-1\}$, if each edge $uv$ is assigned $f(u) + f(v) \pmod{q}$ then the resulting edge labels are distinct. However, when $G$ is a tree one of the vertex labels may be assigned to exactly two vertices. Subsequently a few variations of harmonious labeling, namely, strongly c-harmonious labeling [1], sequential labeling [5], elegant labeling [1] and felicitous labeling [4] were introduced. The later three labelings were introduced to avoid such exceptions for the trees given in harmonious labeling. A strongly

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1-harmonious graph is also known as strongly harmonious graph.

It is interesting to note that if a graph $G$ with $q$ edges is harmonious then the resulting edge labels can be arranged as $a_0, a_1, a_2, \ldots, a_{q-1}$ where $a_i = i$ or $q+i$, $0 \leq i \leq q-1$. That is for each $i$, $0 \leq i \leq q-1$ there is a distinct edge with label either $i$ or $q+i$. An another interesting and natural variation of edge label could be $q-i$ or $q+i$. This prompts to define a new variation of harmonious labeling called absolutely harmonious labeling.

**Definition 1.1** An absolutely harmonious labeling $f$ is an injection from the vertex set of a graph $G$ with $q$ edges to the set $\{0, 1, 2, \ldots, q-1\}$, if each edge $uv$ is assigned $f(u) + f(v)$ then the resulting edge labels can be arranged as $a_0, a_1, a_2, \ldots, a_{q-1}$ where $a_i = q-i$ or $q+i$, $0 \leq i \leq q-1$. However, when $G$ is a tree one of the vertex labels may be assigned to exactly two vertices. A graph which admits absolutely harmonious labeling is called absolutely harmonious graph.

The result of Graham and Sloane [3] states that $C_n, n \equiv 1 (mod 4)$ is harmonious, but we show that $C_n, n \equiv 1 (mod 4)$ is not an absolutely harmonious graph. On the other hand, we show that $C_4$ is an absolutely harmonious graph, but it is not harmonious. We observe that a strongly harmonious graph is an absolutely harmonious graph.

To initiate the investigation on absolutely harmonious graphs, we obtain necessary conditions for a graph to be an absolutely harmonious graph and prove the following results:

1. Path $P_n, n \geq 3$, a class of banana trees, and $P_n \odot K_m^c$ are absolutely harmonious graphs.
2. Ladders, $C_n \odot K_m$, Triangular snakes, Quadrilateral snakes, and $mK_4$ are absolutely harmonious graphs.
3. Complete graph $K_n$ is absolutely harmonious if and only if $n = 3$ or $4$.
4. Cycle $C_n, n \equiv 1$ or $2 (mod 4), C_m \times C_n$ where $m$ and $n$ are odd, $mK_3, m \geq 2$ are not absolutely harmonious graphs.

§2. Necessary Conditions

**Theorem 2.1** If $G$ is an absolutely harmonious graph, then there exists a partition $(V_1, V_2)$ of the vertex set $V(G)$, such that the number of edges connecting the vertices of $V_1$ to the vertices of $V_2$ is exactly $\left\lceil \frac{q}{2} \right\rceil$.

**Proof** If $G$ is an absolutely harmonious graph, then the vertices can be partitioned into two sets $V_1$ and $V_2$ having respectively even and odd vertex labels. Observe that among the $q$ edges $q\left\lfloor \frac{q}{2} \right\rfloor$ edges are labeled with odd numbers, according as $q$ is even or $q$ is odd. For an edge to have odd label, one end vertex must be odd labeled and the other end vertex must be even labeled. Thus, the number of edges connecting the vertices of $V_1$ to the vertices of $V_2$ is exactly $\left\lceil \frac{q}{2} \right\rceil$. \hfill $\Box$

**Remark 2.2** A simple and straight forward application of Theorem 2.1 identifies the non absolutely harmonious graphs. For example, complete graph $K_n$ has $\frac{n(n-1)}{2}$ edges. If we assign
m vertices to the part \( V_1 \), there will be \( m(n - m) \) edges connecting the vertices of \( V_1 \) to the vertices of \( V_2 \). If \( K_n \) has an absolutely harmonious labeling, then there is a choice of \( m \) for which \( m(n - m) = \left\lfloor \frac{n^2 - n}{4} \right\rfloor \). Such a choice of \( m \) does not exist for \( n = 5, 7, 8, 10, \ldots \).

A graph is called even graph if degree of each vertex is even.

**Theorem 2.3** If an even graph \( G \) is absolutely harmonious then \( q \equiv 0 \) or \( 3 \) (mod 4).

**Proof** Let \( G \) be an even graph with \( q \equiv 1 \) or \( 2 \) (mod 4) and \( d(v) \) denotes the degree of the vertex \( v \) in \( G \). Suppose \( f \) be an absolutely harmonious labeling of \( G \). Then the resulting edge labels can be arranged as \( a_0, a_1, a_2, \ldots, a_q \) where \( a_i = q - i \) or \( q + i \), \( 0 \leq i \leq q - 1 \). In other words, for each \( i \), the edge label \( a_i \) is \((q - i) + 2ib_i, 0 \leq i \leq q - 1 \) where \( b_i \in \{0, 1\} \). Evidently

\[
\sum_{v \in V(G)} d(v)f(v) - 2\sum_{k=0}^{q-1} kb_k = \left(\frac{q + 1}{2}\right).
\]

As \( d(v) \) is even for each \( v \) and \( q \equiv 1 \) or \( 2 \) (mod 4),

\[
\sum_{v \in V(G)} d(v)f(v) - 2\sum_{k=0}^{q-1} kb_k \equiv 0 \text{ (mod 2)}
\]

but \( \left(\frac{q + 1}{2}\right) \equiv 1 \text{ (mod 2)} \). This contradiction proves the theorem.

**Corollary 2.4** A cycle \( C_n \) is not an absolutely harmonious graph if \( n \equiv 1 \) or \( 2 \) (mod 4).

**Corollary 2.5** A grid \( C_m \times C_n \) is not an absolutely harmonious graph if \( m \) and \( n \) are odd.

**Theorem 2.6** If \( f \) is an absolutely harmonious labeling of the cycle \( C_n \), then edges of \( C_n \) can be partitioned into two sub sets \( E_1, E_2 \) such that

\[
\sum_{uv \in E_1} |f(u) + f(v) - n| = \frac{n(n + 1)}{4} \quad \text{and} \quad \sum_{uv \in E_2} |f(u) + f(v) - n| = \frac{n(n - 3)}{4}.
\]

**Proof** Let \( v_1v_2v_3\ldots v_nv_1 \) be the cycle \( C_n \), where \( e_i = v_{i-1}v_i, 2 \leq i \leq n \) and \( e_1 = v_nv_1 \). Define \( E_1 = \{uv \in E \mid f(u) + f(v) - n \text{ is non negative}\} \) and \( E_2 = \{uv \in E \mid f(u) + f(v) - n \text{ is negative}\} \). Since \( f \) is an absolutely harmonious labeling of the cycle \( C_n \),

\[
\sum_{uv \in E} |f(u) + f(v) - n| = \frac{n(n - 1)}{2}.
\]

In other words,

\[
\sum_{uv \in E_1} |f(u) + f(v) - n| + \sum_{uv \in E_2} |f(u) + f(v) - n| = \frac{n(n - 1)}{2}.
\]

Since \( \sum_{uv \in E} (f(u) + f(v) - n) = -n \), we have

\[
\sum_{uv \in E_1} |f(u) + f(v) - n| - \sum_{uv \in E_2} |f(u) + f(v) - n| = -n.
\]
Solving equations (1) and (2), we get the desired result. \(\square\)

**Remark 2.7** If \(n \equiv 1 \text{ or } 2 \pmod{4}\) then both \(\frac{n(n+1)}{4}\) and \(\frac{n(n-3)}{4}\) cannot be integers. Thus the cycle \(C_n\) is not an absolutely harmonious graph if \(n \equiv 1 \text{ or } 2 \pmod{4}\).

**Remark 2.8** Observe that the conditions stated in Theorem 2.1, Theorem 2.3, and Theorem 2.6 are necessary but not sufficient. Note that \(C_8\) satisfies all the conditions stated in Theorems 2.1, 2.3, and 2.6 but it is not an absolutely harmonious graph. For, checking each of the \(\frac{8!}{2}\) possibilities reveals the desired result about \(C_8\).

§3. Absolutely Harmonious Graphs

**Theorem 3.1** The path \(P_{n+1}\), where \(n \geq 2\) is an absolutely harmonious graph.

**Proof** Let \(P_{n+1}: v_1v_2\ldots v_{n+1}\) be a path, \(r = \lceil \frac{n}{2} \rceil\), \(s = \begin{cases} \left\lfloor \frac{s}{2} \right\rfloor + 1 & \text{if } n \equiv 0 \pmod{4} \\ \left\lfloor \frac{s}{2} \right\rfloor & \text{otherwise} \end{cases}\)

\(t = \begin{cases} s - 1 & \text{if } n \equiv 0 \text{ or } 1 \pmod{4} \\ s & \text{if } n \equiv 2 \text{ or } 3 \pmod{4} \end{cases}\), \(T_1 = n\), \(T_2 = \begin{cases} 2t + 2 & \text{if } n \equiv 0 \text{ or } 1 \pmod{4} \\ 2t + 1 & \text{if } n \equiv 2 \text{ or } 3 \pmod{4} \end{cases}\)

Then \(r + s + t = n + 1\). Define \(f: V(P_{n+1}) \rightarrow \{0, 1, 2, 3, \ldots, n - 1\}\) by:

\(f(v_i) = T_1 - i\) if \(1 \leq i \leq r\), \(f(v_{r+i}) = T_2 - 2i\) if \(1 \leq i \leq s\) and \(f(v_{r+s+i}) = T_3 + 2i\) if \(1 \leq i \leq t\).

Evidently \(f\) is an absolutely harmonious labeling of \(P_{n+1}\). For example, an absolutely harmonious labeling of \(P_{12}\) is shown in Fig.3.1.

\[\text{Fig.3.1} \]

The tree obtained by joining a new vertex \(v\) to one pendant vertex of each of the \(k\) disjoint stars \(K_{1,n_1}, K_{1,n_2}, K_{1,n_3}, \ldots, K_{1,n_k}\) is called a banana tree. The class of all such trees is denoted by \(BT(n_1, n_2, n_3, \ldots, n_k)\).

**Theorem 3.2** The banana tree \(BT(n, n, \ldots, n)\) is absolutely harmonious.
Proof Let \( V(BT(n, n, n, \cdots, n)) = \{v\} \cup \{v_j, v_{jr} : 1 \leq j \leq k \text{ and } 1 \leq r \leq n\} \) where \( d(v_j) = n \) and \( E(BT(n, n, n, \ldots, n)) = \{vv_jn : 1 \leq j \leq k\} \cup \{v_jv_{jr} : 1 \leq j \leq k, 1 \leq r \leq n\} \). Clearly \( BT(n, n, \cdots, n) \) has order \((n + 1)k + 1\) and size \((n + 1)k\). Define

\[
 f : V(BT(n, n, \cdots, n)) \rightarrow \{1, 2, 3, \ldots, (n + 1)k - 1\}
\]
as follows:

\[
f(v) = 1, \ f(v_j) = (n + 1)(j - 1) : 1 \leq j \leq k, \ f(v_{jr}) = (n + 1)(j - 1) + r : 1 \leq r \leq n.
\]

It can be easily verified that \( f \) is an absolutely harmonious labeling of \( BT(n, n, n, \ldots, n) \). For example an absolutely harmonious labeling of \( BT(4, 4, 4) \) is shown in Fig. 3.2. \( \square \)

The corona \( G_1 \circ G_2 \) of two graphs \( G_1(p_1, q_1) \) and \( G_2(p_2, q_2) \) is defined as the graph obtained by taking one copy of \( G_1 \) and \( p_1 \) copies of \( G_2 \) and then joining the \( i^{th} \) vertex of \( G_1 \) to all the vertices in the \( i^{th} \) copy of \( G_2 \).

**Theorem 3.3** The corona \( P_n \circ K_m^C \) is absolutely harmonious.

Proof Let \( V(P_n \circ K_m^C) = \{u_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\} \) and \( E(P_n \circ K_m^C) = \{u_iu_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_{ij}u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\} \). We observe that \( P_n \circ K_m^C \) has order \((m + 1)n\) and size \((m + 1)n - 1\). Define \( f : V(P_n \circ K_m^C) \rightarrow \{0, 1, 2, \ldots, mn + n - 2\} \) as follows:

\[
 f(u_i) = \begin{cases} 
 0 & \text{if } i = 1, \\
 (m + 1)(i - 1) & \text{if } i = \left\lceil \frac{n}{2} \rightceil - 1, \\
 (m + 1)(i - 1) - 1 & \text{otherwise,}
\end{cases} 
\]

and for \( 1 \leq j \leq m - 1 \),

\[
 f(u_{ij}) = \begin{cases} 
 (m + 1)(i - 1) + j & \text{if } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1, \\
 (m + 1)(i - 1) + j - 1 & \text{if } \left\lceil \frac{n}{2} \right\rceil \leq i \leq n.
\end{cases} 
\]

It can be easily verified that \( f \) is an absolutely harmonious labeling of \( P_n \circ K_m^C \). For example an absolutely harmonious labeling of \( P_5 \circ K_3^C \) is shown in Fig. 3.3. \( \square \)
Theorem 3.4 The corona $C_n \odot K^C_m$ is absolutely harmonious.

Proof Let $V(C_n \odot K^C_m) = \{u_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(C_n \odot K^C_m) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nu_1\} \cup \{u_iu_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$. We observe that $C_n \odot K^C_m$ has order $(m+1)n$ and size $(m+1)n$. Define $f : V(C_n \odot K^C_m) \rightarrow \{0, 1, 2, ..., mn+1\}$ as follows:

- $f(u_i) = \begin{cases} 0 & \text{if } i = 1, \\ (m+1)(i-1) - 1 & \text{if } 2 \leq i \leq \frac{n-1}{2}, \\ (m+1)(i-1) & \text{otherwise}, \end{cases}$

- $f(u_{im}) = \begin{cases} (m+1)i & \text{if } 1 \leq i \leq \frac{n-3}{2}, \\ (m+1)i - 1 & \text{otherwise}. \end{cases}$

and for $1 \leq j \leq m-1$

- $f(u_{ij}) = \begin{cases} (m+1)(i-1) + j & \text{if } 1 \leq i \leq \left\lceil \frac{n}{4} \right\rceil - 1, \\ (m+1)(i-1) + j - 1 & \text{if } \left\lceil \frac{n}{4} \right\rceil \leq i \leq n. \end{cases}$

It can be easily verified that $f$ is an absolutely harmonious labeling of $C_n \odot K^C_m$. For example an absolutely harmonious labeling of $C_5 \odot K^C_3$ is shown in Figure 3.4.

Fig.3.3

Theorem 3.5 The ladder $P_n \times P_2$, where $n \geq 2$ is an absolutely harmonious graph.

Proof Let $V(P_n \times P_2) = \{u_1, u_2, u_3, ..., u_n\} \cup \{v_1, v_2, v_3, ..., v_n\}$ and $E(P_n \times P_2) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 1 \leq i \leq n\}$. We note that $P_n \times P_2$ has order
$2n$ and size $3n - 2$.

**Case 1.** $n \equiv 0 \pmod{4}$.

Define $f : V(P_n \times P_2) \rightarrow \{0, 1, 2, ..., 3n - 3\}$ by

\[
f(u_i) = \begin{cases} 
3i - 2 & \text{if } i \text{ is odd}, \\
3i - 2 & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n-4}{2}, \\
3i - 1 & \text{if } i \text{ is even and } i = \frac{n}{2}, \\
3i - 3 & \text{if } i \text{ is even and } \frac{n+4}{2} \leq i \leq n,
\end{cases}
\]

\[
f(v_1) = 0, \quad f(v_{\frac{n+2}{2}}) = \frac{3n - 6}{2}, \quad f(v_{i+1}) = f(u_i) + 1 \text{ if } 1 \leq i \leq n - 1 \text{ and } i \neq \frac{n}{2}.
\]

**Case 2.** $n \equiv 1 \pmod{4}$.

Define $f : V(P_n \times P_2) \rightarrow \{0, 1, 2, ..., 3n - 3\}$ by

\[
f(u_i) = \begin{cases} 
3i - 2 & \text{if } i \text{ is odd and } 1 \leq i \leq \frac{n-3}{2}, \\
3i - 1 & \text{if } i = \frac{n+1}{2}, \\
3i - 3 & \text{if } i \text{ is odd and } \frac{n+5}{2} \leq i \leq n,
\end{cases}
\]

\[
f(v_1) = 0, \quad f(v_{\frac{n+3}{2}}) = \frac{3n - 3}{2}, \quad f(v_{i+1}) = f(u_i) + 1 \text{ if } 1 \leq i \leq n - 1 \text{ and } i \neq \frac{n+1}{2}.
\]

**Case 3.** $n \equiv 2 \pmod{4}$.

Define $f : V(P_n \times P_2) \rightarrow \{0, 1, 2, ..., 3n - 3\}$ by

\[
f(u_i) = \begin{cases} 
3i - 2 & \text{if } i \text{ is odd}, \\
3i - 2 & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n-2}{2}, \\
3i - 3 & \text{if } i \text{ is even and } \frac{n+2}{2} \leq i \leq n,
\end{cases}
\]

\[
f(v_1) = 0, \quad f(v_{i+1}) = f(u_i) + 1 \text{ if } 1 \leq i \leq n - 1.
\]

**Case 4.** $n \equiv 3 \pmod{4}$.

Define $f : V(P_n \times P_2) \rightarrow \{0, 1, 2, ..., 3n - 3\}$ by

\[
f(u_i) = \begin{cases} 
3i - 2 & \text{if } i \text{ is odd and } 1 \leq i \leq \frac{n-1}{2}, \\
3i - 3 & \text{if } i \text{ is odd and } \frac{n+3}{2} \leq i \leq n,
\end{cases}
\]

\[
f(v_1) = 0, \quad f(v_{i+1}) = f(u_i) + 1 \text{ if } 1 \leq i \leq n - 1.
\]
In all four cases, it can be easily verified that \( f \) is an absolutely harmonious labeling of \( P_n \times P_2 \). For example, an absolutely harmonious labeling of \( P_9 \times P_2 \) is shown in Fig. 3.5.

\[
\begin{array}{cccccccccc}
1 & 4 & 7 & 10 & 14 & 16 & 18 & 22 & 24 \\
\hline
20 & 14 & 8 & 1 & 5 & 9 & 15 & 21 & 0 \\
19 & 13 & 7 & 11 & 12 & 17 & 19 & 23 & 18 \\
1 & 3 & 6 & 10 & 11 & 14 & 15 & 20 & 2 \\
23 & 18 & 12 & 6 & 2 & 4 & 11 & 17 & 23 \\
\end{array}
\]

**Fig. 3.5**

A \( K_n \)-snake has been defined as a connected graph in which all blocks are isomorphic to \( K_n \) and the block-cut point graph is a path. A \( K_3 \)-snake is called triangular snake.

**Theorem 3.6** A triangular snake with \( n \) blocks is absolutely harmonious if and only if \( n \equiv 0 \text{ or } 1 \pmod{4} \).

**Proof** The necessity follows from Theorem 2.3. Let \( G_n \) be a triangular snake with \( n \) blocks on \( p \) vertices and \( q \) edges. Then \( p = 2n - 1 \) and \( q = 3n \). Let \( V(G_n) = \{u_i : 1 \leq i \leq n + 1 \} \cup \{v_i : 1 \leq i \leq n \} \) and \( E(G_n) = \{u_iu_{i+1}, u_iv_i, u_{i+1}v_i : 1 \leq i \leq n \} \).

**Case 1.** \( n \equiv 0 \pmod{4} \).

Let \( m = \frac{n}{4} \). Define \( f : V(G_n) \rightarrow \{0, 1, 2, \ldots, 3n - 1\} \) as follows:

\[
f(u_i) = \begin{cases} 
0 & \text{if } i = 1, \\
2i - 2 & \text{if } 2 \leq i \leq 3m \text{ and } i \equiv 0 \text{ or } 2 \pmod{3}, \\
2i - 1 & \text{if } 2 \leq i \leq 3m \text{ and } i \equiv 1 \pmod{3}, \\
6i - 3n - 7 & \text{otherwise},
\end{cases}
\]

\[
f(v_i) = \begin{cases} 
1 & \text{if } i = 1, \\
2i - 1 & \text{if } 2 \leq i \leq 3m - 1 \text{ and } i \equiv \text{ or } 2 \pmod{3}, \\
2i - 2 & \text{if } 2 \leq i \leq 3m - 1 \text{ and } i \equiv 1 \pmod{3}, \\
6m + 1 & \text{if } i = 3m, \\
6i - 3n - 3 & \text{otherwise}.
\end{cases}
\]

**Case 2.** \( n \equiv 1 \pmod{4} \).

Let \( m = \frac{n - 1}{4} \). Define \( f : V(G_n) \rightarrow \{0, 1, 2, \ldots, 3n - 1\} \) as follows:
\[ f(u_i) = \begin{cases} 
0 & \text{if } i = 1, \\
2i - 2 & \text{if } 2 \leq i \leq 3m + 2 \text{ and } i \equiv 0 \text{ or } 2 \pmod{3}, \\
2i - 1 & \text{if } 2 \leq i \leq 3m + 2 \text{ and } i \equiv 1 \pmod{3}, \\
6i - 3n - 7 & \text{otherwise},
\end{cases} \]

\[ f(v_i) = \begin{cases} 
1 & \text{if } i = 1, \\
2i - 1 & \text{if } 2 \leq i \leq 3m + 1 \text{ and } i \equiv 0 \text{ or } 2 \pmod{3}, \\
2i - 2 & \text{if } 2 \leq i \leq 3m + 1 \text{ and } i \equiv 1 \pmod{3}, \\
6i - 3n - 3 & \text{otherwise}.
\end{cases} \]

In both cases, it can be easily verified that \( f \) is an absolutely harmonious labeling of the triangular snake \( G_n \). For example, an absolutely harmonious labeling of a triangular snake with five blocks is shown in Fig.3.6.

Theorem 3.7  \( K_4 \)-snakes are absolutely harmonious.

Proof Let \( G_n \) be a \( K_4 \)-snake with \( n \) blocks on \( p \) vertices and \( q \) edges. Then \( p = 3n + 1 \) and \( q = 6n \). Let \( V(G_n) = \{u_i, v_i, w_i : 1 \leq i \leq n\} \cup \{v_{n+1}\} \) and \( E(G_n) = \{u_iv_i, u_iw_i, v_iw_i : 1 \leq i \leq n\} \cup \{u_iv_{i+1}, v_iv_{i+1}, w_iv_{i+1} : 1 \leq i \leq n\} \) Define \( f : V(G_n) \longrightarrow \{0, 1, 2, ..., 6n - 1\} \) as follows:

\[ f(u_i) = 3i - 3, f(v_i) = 3i - 2, f(w_i) = 3i - 1 \]

where \( 1 \leq i \leq n \), and \( f(v_{n+1}) = 3n + 1 \). It can be easily verified that \( f \) is an absolutely harmonious labeling of \( G_n \) and hence \( K_4 \)-snakes are absolutely harmonious. For example, an absolutely harmonious labeling of a \( K_4 \)-snake with five blocks is shown in Fig.3.7.
A quadrilateral snake is obtained from a path $u_1u_2\ldots u_{n+1}$ by joining $u_i, u_{i+1}$ to new vertices $v_i, w_i$ respectively and joining $v_i$ and $w_i$.

**Theorem 3.8** All quadrilateral snakes are absolutely harmonious.

**Proof** Let $G_n$ be a quadrilateral snake with $V(G_n) = \{u_i : 1 \leq i \leq n+1\} \cup \{v_i, w_i : 1 \leq i \leq n\}$ and $E(G_n) = \{u_iu_{i+1}, u_iv_i, u_{i+1}w_i, v_iw_i : 1 \leq i \leq n\}$. Then $p = 3n + 1$ and $q = 4n$. Let

$$m = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{n-1}{2} & \text{if } n \equiv 1 \pmod{2} \end{cases}.$$

Define $f : V(G_n) \rightarrow \{0, 1, 2, \ldots, 4n-1\}$ as follows:

$$f(u_i) = \begin{cases} 0 & \text{if } i = 1, \\ 4i - 6 & \text{if } 2 \leq i \leq m + 1, \\ 4i - 7 & \text{if } m + 2 \leq i \leq n + 1 \end{cases}, \quad f(v_i) = \begin{cases} 4i - 3 & \text{if } 1 \leq i \leq m, \\ 4i - 2 & \text{if } m + 1 \leq i \leq n, \\ 4i - 1 & \text{if } m + 1 \leq i \leq n. \end{cases}$$

$$f(w_i) = \begin{cases} 4i & \text{if } 1 \leq i \leq m, \\ 4i - 1 & \text{if } m + 1 \leq i \leq n. \end{cases}$$

It can be easily verified that $f$ is an absolutely harmonious labeling of the quadrilateral snake $G_n$ and hence quadrilateral snakes are absolutely harmonious. For example, an absolutely harmonious labeling of a quadrilateral snake with six blocks is shown in Fig. 3.8.
Theorem 3.9 The disjoint union of $m$ copies of the complete graph on four vertices, $mK_4$ is absolutely harmonious.

Proof Let $u_i^j$ where $1 \leq i \leq 4$ and $1 \leq j \leq m$ denotes the $i^{th}$ vertex of the $j^{th}$ copy of $mK_4$. We note that that $mK_4$ has order $4m$ and size $6m$. Define $f : V(mK_4) \to \{0, 1, 2, \ldots, 6m - 1\}$ as follows: $f(u_1^1) = 0, f(u_2^1) = 1, f(u_3^1) = 2, f(u_4^1) = 4, f(u_1^2) = q - 3, f(u_2^2) = q - 4, f(u_3^2) = q - 5, f(u_4^2) = q - 7, f(u_1^{i+2}) = f(u_1^i) + 6$ if $j$ is odd and $f(u_1^{i+2}) = f(u_1^i) - 6$ if $j$ is even, where $1 \leq i \leq 4$ and $1 \leq j \leq m - 2$. Clearly $f$ is an absolutely harmonious labeling. For example, an absolutely harmonious labeling of $5K_4$ is shown in Figure 11. Box

Observation 3.10 If $f$ is an absolutely harmonious labeling of a graph $G$, which is not a tree, then

1. Each $x$ in the set $\{0, 1, 2\}$ has inverse image.
2. Inverse images of 0 and 1 are adjacent in $G$.
3. Inverse images of 0 and 2 are adjacent in $G$.

Theorem 3.11 The disjoint union of $m$ copies of the complete graph on three vertices, $mK_3$ is absolutely harmonious if and only if $m = 1$.

Proof Let $u_i^j$, where $1 \leq i \leq 3$ and $1 \leq j \leq m$ denote the $i^{th}$ vertex of the $j^{th}$ copy of $mK_3$. Assignments of the values 0, 1, 2 to the vertices of $K_3$ gives the desired absolutely harmonious labeling of $K_3$. For $m \geq 2$, $mK_3$ has $3m$ vertices and $3m$ edges. If $mK_3$ is an absolutely harmonious graph, we can assign the numbers $\{0, 1, 2, 3m - 1\}$ to the vertices of $mK_3$ so that its edges receive each of the numbers $a_0, a_1, \ldots, a_{q - 1}$, where $a_i = q - i$ or $q + i, 0 \leq i \leq q - 1$. By Observation 3.10, we can assume, without loss of generality that $f(u_1^1) = 0, f(u_2^1) = 1, f(u_3^1) = 2$. Thus we get the edge labels $a_{q-1}, a_{q-2}$ and $a_{q-3}$. In order to have an edge labeled $a_{q-4}$, we must have two adjacent vertices labeled $q - 1$ and $q - 3$. We can assume without loss of generality that $f(u_1^2) = q - 1$ and $f(u_2^2) = q - 3$. In order to have an edge labeled $a_{q-5}$, we must have $f(u_3^2) = q - 4$. There is now no way to obtain an edge labeled $a_{q-6}$. This contradiction proves the theorem.

Theorem 3.12 A complete graph $K_n$ is absolutely harmonious graph if and only if $n = 3$ or 4.

Proof From the definition of absolutely harmonious labeling, it can be easily verified that $K_1$ and $K_2$ are not absolutely harmonious graphs. Assignments of the values 0, 1, 2 and 0, 1, 2, 4 respectively to the vertices of $K_3$ and $K_4$ give the desired absolutely harmonious labeling of them. For $n > 4$, the graph $K_n$ has $q \geq 10$ edges. If $K_n$ is an absolutely harmonious graph, we can assign a subset of the numbers $\{0, 1, 2, q - 1\}$ to the vertices of $K_n$ so that the edges receive each of the numbers $a_0, a_1, \ldots, a_{q-1}$ where $a_i = q - i$ or $q + i, 0 \leq i \leq q - 1$. By Observation 3.10, 0, 1, and 2 must be vertex labels. With vertices labeled 0, 1, and 2, we have edges labeled $a_{q-1}, a_{q-2}$ and $a_{q-3}$. To have an edge labeled $a_{q-4}$, we must adjoin the vertex label 4. Had we adjoined the vertex label 3 to induce $a_{q-4}$, we would have two edges labeled $a_{q-3}$, namely, between 0 and 3, and between 1 and 2. Had we adjoined the vertex labels $q - 1$
and \( q - 3 \) to induce \( a_{q-4} \), we would have three edges labeled \( a_1 \), namely, between \( q - 1 \) and 0, between \( q - 1 \) and 2, and between \( q - 3 \) and 2. With vertices labeled 0, 1, 2, and 4, we have edges labeled \( a_{q-1} \), \( a_{q-2} \), \( a_{q-3} \), \( a_{q-4} \), \( a_{q-5} \), and \( a_{q-6} \). Note that for \( K_4 \) with \( q = 6 \), this gives the absolutely harmonious labeling. To have an edge labeled \( a_{q-7} \), we must adjoin the vertex label 7; all the other choices are ruled out. With vertices labeled 0, 1, 2, 4, and 7, we have edges labeled \( a_{q-1} \), \( a_{q-2} \), \( a_{q-3} \), \( a_{q-4} \), \( a_{q-5} \), \( a_{q-6} \), \( a_{q-7} \), \( a_{q-8} \), \( a_{q-9} \), and \( a_{q-11} \). There is now no way to obtain an edge labeled \( a_{q-10} \), because each of the ways to induce \( a_{q-10} \) using two numbers contains at least one number that cannot be assigned as a vertex label. We may easily verify that the following boxed numbers are not possible choices as vertex labels:

\[
\begin{array}{cccc}
0 & 10 \\
1 & 9 \\
2 & 8 \\
3 & 7 \\
4 & 6 \\
q - 1 & q - 9 \\
q - 2 & q - 8 \\
q - 3 & q - 7 \\
q - 4 & q - 6 \\
\end{array}
\]

This contradiction proves the theorem.

□

References


