Smarandache $\mathcal{N}$–subalgebras(resp. filters) of $\mathcal{C}I$–algebras

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Abstract. In this paper, we introduce the notions of $\mathcal{N}$-subalgebras and $\mathcal{N}$-filters based on Smarandache $\mathcal{C}I$-algebra and give a number of their properties. The relationship between $\mathcal{N}(Q, f)$-subalgebras(filters) and $\mathcal{N}$-subalgebras(filters) are also investigated.

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1 Introduction

Some recent researchers led to generalizations of the notion of fuzzy set that introduced by Zadeh in 1965 [15]. The generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the point $\{1\}$ into the interval $[0, 1]$. In order to provide a mathematical tool to deal with negative information, Jun et. al. introduced $\mathcal{N}$-structures, based on negative-valued functions [6]. In 1966, Y. Imai and K. Iseki [3] introduced two classes of abstract algebras: $\mathcal{B}C\mathcal{K}$-algebras and $\mathcal{B}C\mathcal{I}$-algebras. It is known that the class of $\mathcal{B}C\mathcal{K}$-algebras is a proper subclass of the class of $\mathcal{B}C\mathcal{I}$-algebras. H. S. Kim and Y. H. Kim defined a $\mathcal{B}E$-algebra [5]. Biao Long Meng, defined notion of $\mathcal{C}I$-algebra as a generalization of a $\mathcal{B}E$-algebra [9]. It is known that any $\mathcal{B}E$-algebra is a $\mathcal{C}I$-algebra. Hence, every $\mathcal{B}E$-algebra is
a weaker structure than CI-algebra, thus we can consider in any CI-algebra a weaker structure as BE-algebra. Jun et. al. discussed the notion of N-structures in BCH/BCK/BCI-algebras and investigated their properties in [6, 7]. They introduced the notions of N-ideals of subtraction algebras and N-closed ideals in BCK/BCI-algebras. We introduce the notions of N-subalgebras and N-filters in CI-algebras and give a number of their properties and The relationship between N-subalgebras and N-filters was discussed in [14]. Also, we discuss on Smarandache CI-algebra and investigated some of their useful properties in [2]. Beside, we introduced the notion of anti fuzzy set and stated the relationship with the N-function of CI-algebra X. We showed that every anti fuzzy filter is an anti fuzzy subalgebra in [1]. K. J. Lee and Y. B. Jun introduced the notion of N-subalgebras and N-ideals based on a sub-BCK-algebra of a BCI-algebras and their relations/properties are investigated in [8].

In the present paper, we continue study of CI-algebras and apply the N-structures to the filter theory in CI-algebras and Smarandache CI-algebras, also investigate the relationship between N-subalgebra and N-filters based on Smarandache CI-algebras. We show that any N(Q, f)-closed filter is an N(Q, ϱ)-subalgebra. We give some conditions for N-subalgebras(filters) to be N(Q, g)-subalgebras(resp. filters).

2 Preliminaries

In this section we review the basic definitions and some elementary aspects that are necessary for this paper.

Definition 2.1. [9] An algebra \((X; *, 1)\) of type \((2, 0)\) is called a CI-algebra if it satisfying the following axioms:

\[(CI1) \quad x \ast x = 1,\]
\[(CI2) \quad 1 \ast x = x,\]
\[(CI3) \quad x \ast (y \ast z) = y \ast (x \ast z), \text{ for all } x, y, z \in X.\]

A CI–algebra X satisfying the condition \(x \ast 1 = 1\) is called a BE-algebra. In any CI-algebra X one can define a binary relation “\(\leq\)" by \(x \leq y\) if and only if \(x \ast y = 1\).

A CI-algebra X has the following properties:

\[(i) \quad y \ast ((y \ast x) \ast x) = 1,\]
\[(ii) \quad (x \ast 1) \ast (y \ast 1) = (x \ast y) \ast 1,\]

\[(iii) \quad \text{if } 1 \leq x, \text{ then } x = 1, \text{ for all } x, y \in X.\]

A non-empty subset \(S\) of a \(CI\)-algebra \(X\) is called a subalgebra of \(X\) if \(x \ast y \in S\) whenever \(x, y \in S\). A mapping \(f : X \to Y\) of \(CI\)-algebra is called a homomorphism if \(f(x \ast y) = f(x) \ast f(y)\), for all \(x, y \in X\). A non-empty subset \(F\) of \(CI\)-algebra \(X\) is called a filter of \(X\) if (1) \(1 \in F\), (2) \(x \in F\) and \(x \ast y \in F\) implies \(y \in F\). A filter \(F\) of \(CI\)-algebra \(X\) is said to closed if \(x \in F\) implies \(x \ast 1 \in F\). A nonempty subset \(S\) of a \(CI\)-algebra \(X\) is called a subalgebra of \(X\) if \(x \ast y \in S\), for all \(x, y \in S\). For our convenience, the empty set \(\emptyset\) is regarded as a subalgebra of \(X\). Denote by \(Q(X, [-1, 0])\) the collection of functions from a set \(X\) to \([-1, 0]\). We say that an element of \(Q(X, [-1, 0])\) is a negative-valued function from \(X\) to \([-1, 0]\) (briefly, \(N\)-function on \(X\)). By an \(N\)-structure we mean an ordered pair \((X, f)\) of \(X\) and an \(N\)-function \(f\) on \(X\).

In what follows, let \(X\) denote a \(CI\)-algebra and \(f\) an \(N\)-function on \(X\) unless otherwise specified.

**Definition 2.2.** [14] By a subalgebra of \(X\) based on \(N\)-function \(f\) (briefly, \(N\)-subalgebra of \(X\)), we mean an \(N\)-structure \((X, f)\) in which \(f\) satisfies the following assertion:

\[f(x \ast y) \leq \max\{f(x), f(y)\}, \text{ for all } x, y \in X.\]

**Definition 2.3.** [14] By a filter of \(X\) based on \(N\)-function \(f\) (briefly, \(N\)-filter of \(X\)), we mean an \(N\)-structure \((X, f)\) in which \(f\) satisfies the following conditions:

\[(i) \quad f(1) \leq f(y),\]

\[(ii) \quad f(y) \leq \max\{f(x \ast y), f(x)\}, \text{ for all } x, y \in X.\]

**Definition 2.4.** [2] A Smarandache \(CI\)-algebra \(X\) is defined to be a \(CI\)-algebra \(X\) in which there exists a proper subset \(Q\) of \(X\) such that satisfies the following conditions:

\[(S1) \quad 1 \in Q \text{ and } |Q| \geq 2,\]

\[(S2) \quad Q \text{ is a } BE\text{-algebra under the operation of } X.\]
Example 2.1. [2] Let $X := \{1, a, b, c, d\}$ be a set with the following table.

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Then $X$ is a CI-algebra and $Q = \{1, a, b, c\}$ is a BE-algebra.

Definition 2.5. [2] A nonempty subset $F$ of CI-algebra $X$ is called a Smarandache filter of $X$ related to $Q$ (or briefly, $Q$-Smarandache filter of $X$) if it satisfies:

- (SF1) $1 \in F$,
- (SF2) $(\forall y \in Q)(\forall x \in F)(x \ast y \in F \Rightarrow y \in F)$.

Definition 2.6. [11] A fuzzy set $\mu : X \rightarrow [0, 1]$ is called an anti fuzzy subalgebra of $X$ if it satisfies:

$\mu(x \ast y) \leq \max\{\mu(x), \mu(y)\}$, for all $x, y \in X$.

Definition 2.7. [1] A fuzzy set $\mu : X \rightarrow [0, 1]$ is called an anti fuzzy filter of $X$ if it satisfies:

- (AFF1) $\mu(1) \leq \mu(x)$,
- (AFF2) $\mu(y) \leq \max\{\mu(x \ast y), \mu(x)\}$, for all $x, y \in X$.

3 Smarandache $N$-subalgebras

Definition 3.1. Let $X$ be a $Q$-Smarandache CI-algebra and $\varrho \in [-1, 0]$. An $N$-structure $(X, f)$ is called an $N$-subalgebra of $X$ based on $Q$ and $\varrho$ (briefly, $N(Q, \varrho)$-subalgebra of $X$) if it is an $N$-subalgebra of $X$ such that satisfies the following condition:

- (type 1) $(\forall x \in Q) \ (\forall y \in X \setminus Q) \ (f(x) \leq \varrho \leq f(y))$,
- (type 2) $(\forall x \in Q) \ (\exists y \in X \setminus Q) \ (f(x) \leq \varrho \leq f(y))$,
- (type 3) $(\exists x \in Q) \ (\forall y \in X \setminus Q) \ (f(x) \leq \varrho \leq f(y))$,
\[ (\text{type 4}) \quad \exists x \in Q \quad (\exists y \in X \setminus Q) \quad (f(x) \leq g \leq f(y)). \]

**Note.** If \( g := 0 \), then \( f(y) = 0 \), for all \( y \in X \setminus Q \). So, \((Q, f)\) is an \( N\)-subalgebra. If \( g := -1 \), then \( f(x) = -1 \), for all \( x \in Q \). And so \((X, f) = N(Q, g)\).

**Example 3.1.** a) In Example 2.1, an \( N\)-structure \((X, f)\) in which \( f \) is defined by \( f(1) = f(a) = -0.7, \ f(b) = -0.4, \ f(c) = -0.6 \) and \( f(d) = -0.3 \) is an \( N(Q, g)\)-subalgebra of all types on \( X \), for \( g \in [-0.4, -0.3] \) and \( Q = \{1, a, b, c\} \).

b) In Example 2.1, an \( N\)-structure \((X, g)\) in which \( g \) is defined by \( g(1) = g(a) = -0.7, \ g(b) = -0.2, \ g(c) = -0.6 \) and \( g(d) = -0.3 \) is not an \( N(Q, g)\)-subalgebra of \( X \) because \( g(d) = -0.3 \neq g(b) = -0.2 \).

c) In Example 2.1, an \( N\)-structure \((X, f)\) in which \( f \) is defined by \( f(1) = f(a) = -0.7, \ f(b) = -0.4, \ f(c) = -0.5 \) and \( f(d) = -0.3 \) is an \( N(Q, g)\)-subalgebra of type 2, type 3 and type 4 on \( X \), for \( g \in [-0.4, -0.3] \) and \( Q = \{1, a, b\} \), but it is not of type 1, because \( f(c) \not\in g \).

d) In Example 2.1, an \( N\)-structure \((X, f)\) in which \( f \) is defined by \( f(1) = f(a) = -0.7, \ f(b) = -0.2, \ f(c) = -0.3 \) and \( f(d) = -0.1 \) is an \( N(Q, g)\)-subalgebra of type 3 and type 4 on \( X \), for \( g \in [-0.7, -0.3] \) and \( Q = \{1, a, b\} \), but it is not of type 1 and type 2 on \( X \), because \( f(b) \not\in g \).

e) In Example 2.1, an \( N\)-structure \((X, f)\) in which \( f \) is defined by \( f(1) = f(a) = -0.7, \ f(b) = -0.2, \ f(c) = -0.5 \) and \( f(d) = -0.3 \) is an \( N(Q, g)\)-subalgebra of type 4 on \( X \), for \( g \in [-0.7, -0.3] \) and \( Q = \{1, a, b\} \), but it is not of type 1, type 2, type 3 on \( X \).

Now, in the following diagram we summarize the results of this definition. The mark \( A \rightarrow B \), means that \( A \) implies \( B \).

In this paper, we focus on \( N(Q, g)\)-subalgebra of type 1 and from now on \( X \) is a \( Q\)-Smarandache \( CI\)-algebra. The following example shows that there exists an \( N\)-structure \((X, f)\) in \( X \) such that it satisfies the condition (type 1), but it is not an \( N\)-subalgebra of \( X \).

**Example 3.2.** In Example 2.1, an \( N\)-structure \((X, f)\) in which \( f \) is defined by \( f(1) = -0.7, \ f(a) = -0.2, \ f(b) = -0.4, \ f(c) = -0.6 \) and \( f(d) = -0.3 \).
Then \((X, f)\) satisfies the condition (2.1) for \(\varrho \in [-0.2, -0.1]\), but it is not an \(\mathcal{N}\)-subalgebra. Because

\[
f(b * c) = f(a) = -0.2 \not> -0.4 = \max\{f(b), f(c)\}.
\]

**Proposition 3.1.** If an \(\mathcal{N}\)-structure \((X, f)\) satisfies the following condition:

\[
(\forall x \in Q)(\forall y \in X \setminus Q)(f(x) \leq f(y)),
\]

then \((X, f)\) is an \((Q, \varrho)\)-subalgebra of \(X\), for every \(\varrho \in [\bigvee_{x \in Q} f(x), \bigwedge_{y \in X \setminus Q} f(y)]\).

**Theorem 3.2.** Let \(\varrho \in [-1, 0]\). If \((X, f)\) is an \(\mathcal{N}(Q, \varrho)\)-subalgebra of \(X\), then

(i) \(Q \subseteq C(f; \varrho)\),

(ii) \((\forall \beta \in [-1, 0]) (\beta < \varrho \Rightarrow C(f; \beta) \text{ is a subalgebra of } Q)\).

**Proof.** Let \((X, f)\) be a \(\mathcal{N}(Q, \varrho)\)-subalgebra of \(X\). Obviously, \(Q \subseteq C(f; \varrho)\). If \(\beta \in [-1, 0]\) be such that \(\beta < \varrho\), then \(C(f; \beta) \subseteq Q\). Let \(x, y \in C(f; \beta)\). Then \(f(x) \leq \beta\) and \(f(x) \leq \beta\). Thus \(f(x \ast y) \leq \max\{f(x), f(y)\} \leq \beta\), and so \(x \ast y \in C(f; \beta)\). Thus \(C(f; \beta)\) is a subalgebra of \(Q\). \(\square\)

In the following theorem we give some conditions for an \(\mathcal{N}\)-subalgebra to be an \(\mathcal{N}(Q, \varrho)\)-subalgebra.

**Theorem 3.3.** Let \(\varrho \in [-1, 0]\). If \((X, f)\) is an \(\mathcal{N}\)-subalgebra of \(X\) satisfies the conditions (i) and (ii) in Theorem 3.2, then \((X, f)\) is an \(\mathcal{N}(Q, \varrho)\)-subalgebra of \(X\).

**Proof.** Let \(x \in Q\) and \(y \in X \setminus Q\). Then by Theorem 3.2(i), \(x \in C(f; \varrho)\), and so \(f(x) \leq \varrho\). Let \(f(y) = \beta\). If \(\beta < \varrho\), then by Theorem 3.2(ii), \(y \in C(f; \beta) \subseteq Q\), which is a contradiction. Hence \(f(x) \leq \varrho \leq \beta = f(y)\). Thus \((X, f)\) is an \(\mathcal{N}(Q, \varrho)\)-subalgebra of \(X\). \(\square\)

4 Smarandache \(\mathcal{N}\)-filters

**Definition 4.1.** Let \(X\) be a \(Q\)-Smarandache CI-algebra and \(\varrho \in [-1, 0]\). An \(\mathcal{N}\)-structure \((X, f)\) is called an \(\mathcal{N}\)-filter of \(X\) based on \(Q\) and \(\varrho\) (briefly, \(\mathcal{N}(Q, \varrho)\)-filter of \(X\)) if it satisfies the following conditions:

(i) \((\forall x \in Q) (\forall y \in X \setminus Q) (f(1) \leq f(x) \leq \varrho \leq f(y))\).
Example 4.1. In Example 2.1, an $\mathcal{N}$-structure $(X, f)$ in which $f$ is defined by $f(1) = -0.6$, $f(a) = -0.4$, $f(b) = -0.5$, $f(c) = -0.4$ and $f(d) = -0.3$ is an $\mathcal{N}(Q, g)$-filter of $X$ for $g \in [-0.4, -0.3]$.

Theorem 4.1. Let $\{\mathcal{N}(Q_i, g_i) : i \in \Delta\}$ be a family of $\mathcal{N}(Q_i, g_i)$-subalgebras (filters) of $X$ where $\Delta \neq \emptyset$ and $g_i \in [-1, 0]$, for all $i \in \Delta$.
Then $\mathcal{N}(\cap_{Q_i}, \min\{g_i\})_{i \in \Delta}$ is a subalgebra (filter) of $X$, too.

Example 4.2. Let $\varrho \in [-1, 0]$. If $(X, f)$ is an $\mathcal{N}(Q, g)$-filter of $X$, then

(i) $Q \subseteq C(f; g)$, 
(ii) $(\forall \beta \in [-1, 0]) (\beta < \varrho \Rightarrow C(f; \beta) \text{ is a filter of } Q)$.

Proof. Let $(X, f)$ be an $\mathcal{N}(Q, g)$-filter of $X$. Obviously, $Q \subseteq C(f; g)$. Let $\beta \in [-1, 0]$ be such that $\beta < \varrho$. If $x \in C(f; \beta)$, then $f(x) \leq \beta < \varrho$, and so $x \in Q$. Hence $C(f; \beta) \subseteq Q$. By Definition 4.1(i), $f(1) \leq f(x) \leq f(1)$, for all $x \in X$. Hence $f(1) \leq f(x) \leq \beta$ for all $x \in C(f; \beta)$, and so $1 \in C(f; \beta)$. Let $x, y \in Q$ be such that $x \ast y \in C(f; \beta)$ and $x \in C(f, \beta)$. Then $f(x \ast y) \leq \beta$ and $f(x) \leq \beta$. If $x, y \in C(f; \beta)$, then $f(x) \leq \beta$. Now by Definition 4.1(ii), $f(y) \leq \max\{f(x \ast y), f(x)\} \leq \beta$. Thus $y \in C(f; \beta)$. Therefore, $C(f; \beta)$ is a filter of $Q$.

For a $Q$-Smarandache $CI$-algebra $X$ and $\varrho \in [-1, 0]$, the following example shows that an $\mathcal{N}$-filter $(X, f)$ of $X$ may not be an $\mathcal{N}(Q, g)$-filter of $X$.

Example 4.2. Let $X := \{1, a, b, c\}$ be a set with the following table.

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Then $X$ is a $CI$-algebra and $Q := \{1, a\}$ is a $BE$-algebra [13]. Define an $\mathcal{N}$-structure $(X, f)$ in which $f$ is defined by $f(1) = -0.7$, $f(a) = -0.2$, $f(b) = -0.4$, $f(c) = -0.2$. Then $(X, f)$ is an $\mathcal{N}$-filter of $X$. But it is not an $\mathcal{N}(Q, \varrho)$ of $X$ for $\varrho \in [-0.7, -0.3]$. Because $f(a) = -0.2 > \varrho$.

In the following theorem we give conditions for an $\mathcal{N}$-filter to be an $\mathcal{N}(Q, \varrho)$-filter.
Theorem 4.3. Let \( \varrho \in [-1,0] \) and \((X,f)\) be an \(N\)-filter of \(X\) satisfies the conditions (i) and (ii) of Theorem 4.2. Then \((X,f)\) is an \(N(Q,\varrho)\)-filter of \(X\).

Proof. Let \(x \in Q\) and \(y \in X \setminus Q\). Then by Theorem 4.2(i), \(x \in C(f; \varrho)\), and so \(f(x) \leq \varrho\). Let \(f(y) = \beta\). If \(\beta < \varrho\), then by Theorem 4.2(ii), \(y \in C(f; \beta) \subseteq Q\), which is a contradiction. Hence \(\varrho \leq \beta = f(y)\). Since \(f\) is an \(N\)-filter of \(X\), the condition (ii) of Definition 4.1 is obvious. Therefore, \((X,f)\) is an \(N(Q,\varrho)\)-filter of \(X\).

The following example shows that an \(N(Q,\varrho)\)-subalgebra may not be an \(N(Q,\varrho)\)-filter.

Example 4.3. Let \(X := \{1, a, b, c, d\}\) be a set with the following table.

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Then \(X\) is a CI-algebra and \(Q = \{1, a, b, c\}\) is a BE-algebra. Define an \(N\)-structure \((X,f)\) in which \(f\) is defined by \(f(1) = -0.7\), \(f(a) = -0.3\), \(f(b) = -0.4\), \(f(c) = -0.2\) and \(f(d) = -0.1\). Then \((X,f)\) is an \(N\)-subalgebra, for \(\varrho \in [-0.2,0]\), but it is not an \(N\)-filter because

\[
f(c) = -0.2 \not\leq -0.3 = \max\{f(b*c), f(b)\}.
\]

Definition 4.2. An \(N\)-function on \(X\) is called closed \(N\)-filter if \(f\) satisfies:

\[
f(x*1) \leq f(x) \leq \max\{f(y*x, f(y))\}, \text{ for all } x,y \in X.
\]

Example 4.4. Let \(X := \{1, a, b\}\) be a set with the following table:

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Then \(X\) is a CI-algebra [10]. Define an \(N\)-function \(f : X \rightarrow [0,1]\) by \(f(1) = -0.7\), \(f(a) = -0.3\) and \(f(b) = -0.4\). Then \((X,f)\) is an \(N\)-filter of \(X\). But it is not an \(N\)-closed filter because

\[
f(b*1) = f(a) = -0.3 \not\leq f(b) = -0.4.
\]
Example 4.5. In Example 4.4, if define $N$-function $f : X \to [0, 1]$ by $f(1) = -0.7$, $f(a) = -0.4$ and $f(b) = -0.4$. Then $(X, f)$ is an $N$-closed filter of $X$.

Proposition 4.4. Let $(X, f)$ be an $N$-closed filter. Then $f(1) \leq f(x)$, for all $x \in X$.

Proof. Let $x \in X$. Now, by Definition 4.2, we have

$$f(1) \leq \max\{f(x * 1), f(x)\} \leq \max\{f(x), f(x)\} = f(x).$$

\[\square\]

Theorem 4.5. Let $(X, f)$ be an closed $N$-filter and $g \in [-1, 0]$. Then every $N(Q, \rho)$-filter is $N(Q, \rho)$-subalgebra of $X$.

Proof. Let $(X, f)$ be $N(Q, \rho)$-filter and $x, y \in X$. Then by ($C13$) and Definition 4.2, we have

$$f(x * y) \leq \max\{f(y * (x * y)), f(y)\}$$

$$= \max\{f(x * (y * y)), f(y)\}$$

$$= \max\{f(x * 1), f(y)\}$$

$$\leq \max\{f(x), f(y)\}.$$

Therefore, $(X, f)$ is an $N$-subalgebra of $X$. \[\square\]

Theorem 4.6. Let $(X, f)$ and $(X, g)$ be $N(Q_1, \rho_1)$ and $N(Q_2, \rho_2)$-subalgebra (filter) of $X$ respectively. Then $(X \times X, f \times g)$ is an $N(Q_1 \times Q_2, \max\{\rho_1, \rho_2\})$-subalgebra(filter) of $X \times X$.

Proof. Let $(x, y) \in (Q_1 \times Q_2)$ and $(z, t) \in (X \times X) \setminus (Q_1 \times Q_2)$. Then we have

$$(f \times g)(1, 1) = \max\{f(1), g(1)\} \leq \max\{f(x), g(y)\}$$

$$\leq \max\{\rho_1, \rho_2\}$$

$$\leq \max\{f(z), f(t)\} = (f \times g)(z, t).$$

Now, let $(x_1, x_2), (y_1, y_2) \in (Q_1 \times Q_2)$. Then

$$(f \times g)((x_1, x_2) * (y_1, y_2)) = (f \times g)((x_1 * y_1), (x_2 * y_2))$$

$$= \max\{f(x_1 * y_1), g(x_2 * y_2)\}$$

$$\leq \max\{\max\{f(x_1), f(y_1)\}, \max\{g(x_2), g(y_2)\}\}$$

$$= \max\{\max\{f(x_1), g(x_2)\}, \max\{f(y_1), g(y_2)\}\}$$

$$= \max\{(f \times g)(x_1, x_2), (f \times g)(y_1, y_2)\}.$$}

Hence $(X \times X, f \times g)$ is an $N(Q_1 \times Q_2, \max\{\rho_1, \rho_2\})$-subalgebra(resp. filter) of $X \times X$. \[\square\]
Proposition 4.7. Let $Q_1$ and $Q_2$ be two $BE$-algebras which are properly contained in $X$, $Q_1 \subseteq Q_2$ and $\varrho \in [-1, 0]$. Then every $N(Q_2, \varrho)$-subalgebra(filter) of $X$ is an $N(Q_1, \varrho)$-subalgebra(filter) of $X$.

Note. By the following example we show that the converse of above theorem is not correct in general.

Example 4.6. Let $X := \{1, a, b, c\}$ be a set with the following table.

\[
\begin{array}{c|cccc}
\ast & 1 & a & b & c \\
\hline
1 & 1 & a & b & c \\
a & 1 & 1 & b & c \\
b & 1 & a & 1 & c \\
c & c & c & c & 1 \\
\end{array}
\]

Then $Q_1 = \{1, a\}$, $Q_2 = \{1, a, b\}$ are $BE$-algebras which are properly contained in $X$ and $f(1) = -0.7$, $f(a) = -0.4$, $f(b) = -0.2$ and $f(c) = -0.1$. Then $(X, f)$ is an $N(Q_1, \varrho)$-subalgebra, for all $\varrho \in [-0.4, 0]$, but it is not an $N(Q_2, \varrho)$-subalgebra, because, if $\varrho := -0.3$, then $f(b) = -0.2 \not\leq -0.3$.

5 Conclusion

A Smarandache structure on a set $A$ means a week structure $W$ on $A$ such that there exist a proper subset $B$ of $A$ which is embedded with a strong structure $S$. It is that any $BE$-algebra is a $CI$-algebra. Hence, every $BE$-algebra is a weaker structure than $CI$-algebra, thus we can consider in any $CI$-algebra a weaker structure as $BE$-algebra.

In this paper, we have introduced the concept of $N$-subalgebra (filter) based on Smarandache $CI$-algebras and some related properties are investigated. We show that any $N(Q, f)$-closed filter is an $N(Q, f)$-subalgebra. We give some conditions for an $N$-subalgebras (filters) to be $N(Q, \varrho)$-subalgebras (filters).

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