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# SMARANDACHE NOTIONS

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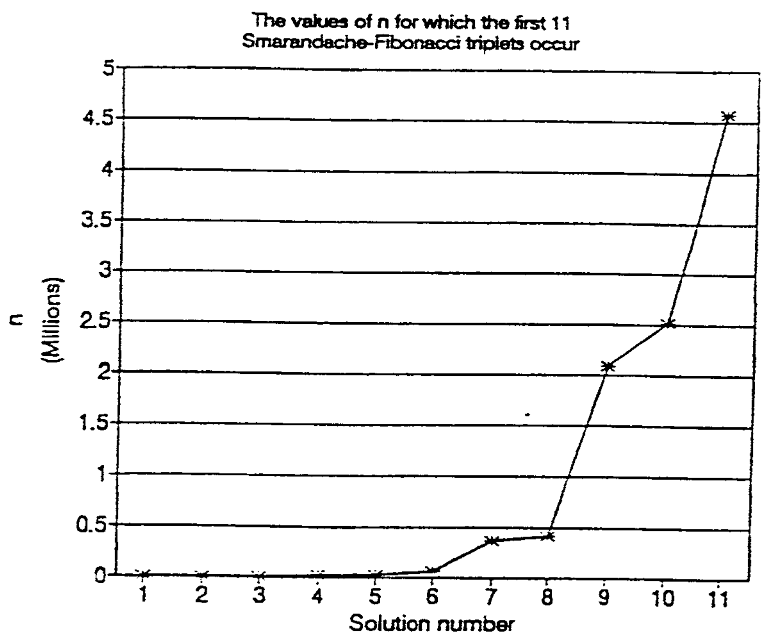


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**A collection of papers concerning Smarandache type functions, numbers, sequences, integer algorithms, paradoxes, experimental geometries, algebraic structures, neutrosophic probability, set, and logic, etc. is published this year.**


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# ON CERTAIN INEQUALITIES INVOLVING THE SMARANDACHE FUNCTION

by

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**1.** The Smarandache function satisfies certain elementary inequalities which have importance in the deduction of properties of this (or related) functions. We quote here the following relations which have appeared in the Smarandache Function Journal:

Let p be a prime number. Then

$$S(p^x) \le S(p^y) \quad \text{for } x \le y \tag{1}$$

$$\frac{S(p^a)}{p^a} \ge \frac{S(p^{a+1})}{p^{a+1}} \quad \text{for } a \ge 0 \tag{2}$$

where x, y, a are nonnegative integers;

$$S(p^a) \le S(q^a) \quad \text{for } p \le q \text{ primes;} \tag{3}$$

$$(p-1)a + 1 \le S(p^a) \le pa\,; \tag{4}$$

If $p > \frac{a}{2}$ and $p \le a-1$ $(a \ge 2)$, then

$$S(p^a) \le p(a-1) \tag{5}$$

For inequalities (3), (4), (5), see [2], and for (1), (2), see [1].

We have also the result ([4]):

$$\text{For composite } n \ne 4, \quad \frac{S(n)}{n} \le \frac{2}{3} \tag{6}$$

Clearly, $1 \le S(n)$ for $n \ge 1$ and $1 < S(n)$ for $n \ge 2$ (7)

and $S(n) \le n$ (8)

which follow easily from tfe definition $S(n) = \min\{k \in N^* : n \text{ divides } k!\}$

**2.** Inequality (2), written in the form $S(p^{a+1}) \leq pS(p^a)$, gives by successive application $S(p^{a+2}) \leq pS(p^{a+1}) \leq p^2 S(p^a)$, ..., that is

$$S(p^{a+c}) \leq p^c \cdot S(p^a) \qquad (9)$$

where a and c are natural numbers (For c = 0 there is equality, and for a = 0 this follows by (8)).

Relation (9) suggest the following result:

### Theorem 1.

For all positive integers m and n holds true the inequality

$$S(mn) \leq m \cdot S(n) \qquad (10)$$

**Proof.**

For a general proof, suppose that m and n have a canonical factorization

$$m = p_1^{a_1} \dots p_r^{a_r} \cdot q_1^{b_1} \dots q_s^{b_s}, \quad n = p_1^{c_1} \dots p_r^{c_r} \cdot t_1^{d_1} \dots t_k^{d_k} \quad ,$$

where $p_i\,(i = \overline{1,r})$, $q_j\,(j = \overline{1,s})$, $t_p\,(p = \overline{1,k})$ are distinct primes and $a_i \geq 0$, $c_j \geq 0$, $b_j \geq 1$, $d_p \geq 1$ are integers.

By a well known result of Smarandache (see [3]) we can write

$$S(m \cdot n) = \max\{S(p_1^{a_1+c_1}), \dots, S(p_r^{a_r+c_r}), S(q_1^{b_1}), \dots, S(q_s^{b_s}), S(t_1^{d_1}), \dots, S(t_k^{d_k})\}$$

$$\leq \max\{p_1^{a_1} S(p_1^{c_1}), \dots, p_r^{a_r} S(p_r^{c_r}), S(q_1^{b_1}), \dots, S(q_s^{b_s}), \dots, S(t_k^{d_k})\}$$

by (9). Now, a simple remark and inequality (8) easily give

$$S(m \cdot n) \leq p_1^{a_1} \dots p_r^{a_r} q_1^{b_1} \dots q_s^{b_s} \cdot \max\{S(p_1^{c_1}), \dots, S(p_r^{c_r}), S(t_1^{d_1}), \dots, S(t_k^{d_k})\} = mS(n)$$

proving relation (10).

**Remark.**

For (m,n)=1, inequality (10) appears as

$$\max\{S(m), S(n)\} \leq mS(n)$$

This can be proved more generally, for all m and n

### Theorem 2.

For all m, n we have:

$$\max\{S(m), S(n)\} \leq mS(n) \qquad (11)$$

**Proof.**

The proof is very simple. Indeed, if $S(m) \geq S(n)$, then $S(m) \leq mS(n)$ holds, since $S(n) \geq 1$ and $S(m) \leq m$, see (7), (8). For $S(m) \leq S(n)$ we have $S(n) \leq mS(n)$ which is t me by $m \geq 1$. In all cases, relation (11) follows.

This proof has an independent interest. As we shall see, Theorem 2 will follow also from Theorem 1 and the following result:

### Theorem 3.

For all m, n we have

$$S(mn) \geq \max \{S(m), S(n)\} \qquad (12)$$

**Proof.**

Inequality (1) implies that $S(p^a) \leq S(p^{a+c})$, $S(p^c) \leq S(p^{a+c})$, so by using the representations of m and n, as in the proof of Theorem 1, by Smarandache's theorem and the above inequalities we get relation (12).

We note that, equality holds in (12) only when all $a_i = 0$ or all $c_i = 0$ $(i = \overline{1,r})$, i.e. when m and n are coprime.

**3.** As an application of (10), we get:

### Corollary 1.

a) $\frac{S(a)}{a} \leq \frac{S(b)}{b}$, if $b \mid a$ (13)

b) If a has a composite divisor $b \neq 4$, then

$$\frac{S(a)}{a} \leq \frac{S(b)}{b} \leq \frac{2}{3} \qquad (14)$$

**Proof.**

Let $a = b \cdot k$. Then $\frac{S(bk)}{bk} \leq \frac{S(b)}{b}$ is equivalent with $S(kb) \leq kS(b)$, which is relation (10) for m=k, n=b.

Relation (14) is a consequence of (13) and (6). We note that (14) offers an improvement of inequality (6).

We now prove:

### Corollary 2.

Let m, n, r, s be positive integers. Then:

$$S(mn) + S(rs) \geq \max \{ S(m) + S(r), S(n) + S(s) \} \qquad (15)$$

**Proof.**

We apply the known relation:

$$\max \{ a + c, b + d \} \leq \max \{ a, b \} + \max \{ c, d \} \qquad (16)$$

By Theorem 3 we can write $S(mn) \geq \max \{S(m), S(n)\}$ and $S(rs) \geq \max \{S(r), S(s)\}$, so by consideration of (16) with

$$a \equiv S(m),\ b \equiv S(r),\ c \equiv S(n),\ d \equiv S(s)$$

we get the desired result.

**Remark.**

Since (16) can be generalized to n numbers $(n \geq 2)$, and also Theorem 1-3 do hard for the general case (which follow by induction; however these results are based essentially on (10) - (15), we can obtain extensions of these theorems to n numbers.

## Corollary 3.

*Let* a, b composite numbers, $a \neq 4$, $b \neq 4$. Then

$$\frac{S(ab)}{ab} \leq \frac{S(a)+S(b)}{a+b} \leq \frac{2}{3};$$

and

$$S^2(ab) \leq ab[S^2(a) + S^2(b)]$$

where $S^2(a) = (S(a))^2$, etc.

**Proof.**

By (10) we have $S(a) \geq \frac{S(ab)}{b}$, $S(b) \geq \frac{S(ab)}{a}$, so by addition

$S(a) + S(b) \geq S(ab)\left(\frac{1}{a} + \frac{1}{b}\right)$, giving the first part of (16).

For the second, we have by (6):

$S(a) \leq \frac{2}{3}a$, $S(b) \leq \frac{2}{3}b$, so $S(a)+S(b) \leq \frac{2}{3}(a+b)$, yielding the second part of (16).

For the proof of (17), remark that by $2(n^2 + r^2) \geq (n + r)^2$, the first past of (16), as well as the inequality $2ab \leq (a + b)^2$ we can write successively:

$$S^2(ab) \leq \frac{a^2b^2}{(a+b)^2} \cdot [S(a)+S(b)]^2 \leq \frac{2a^2b^2}{(a+b)^2} \cdot [S^2(a)+S^2(b)] \leq ab[S^2(a)+S^2(b)]$$

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# *ON SOME CONVERGENT SERIES*

*by Emil Burton*

*Notations :*

$N^*$ set of integers 1 , 2 , 3 , . . .

$d(n)$ the number of divisors of $n$ .

$S(n)$ the Smarandache function $S : N^* \to N^*$ .

$S(n)$ is the smallest integer $m$ with the property that $m!$ is divisible by $n$

$R$ set of real numbers.

In this article we consider the series $\sum_{k=1}^{\infty} f(S(k))$.

$f : N^* \to R$ is a function which satisfies any conditions.

*Proposition 1.* Let $f : N \to R$ be a function which satisfies condition :

$$f(t) \le \frac{c}{t^{\alpha}(d(t!)-d((t-1)!))}$$

for every $t \in N^*$ , $\alpha > 1$ constant, $c > 0$ constant.

Then the series $\sum_{t=1}^{\infty} f(S(k))$ is convergent.

*Proof:* Let us denote by $m_t$ the number of elements of the set

$M_t = \{ k \in N^* ; S(k) = t \} = \{ k \in N^* ; k \mid t! \text{ and } k \nmid (t-1)! \}$.

It follows that $m_t = d(t!) - d((t-1)!)$ .

$$\sum_{k=1}^{\infty} f(S(k)) = \sum_{t=1}^{\infty} m_t f(t)$$

We have $m_t \cdot f(t) \le m_t \cdot \frac{c}{t^{\alpha} m_t} = \frac{c}{t^{\alpha}}$ .

It is well - known that $\sum_{t=1}^{\infty} \frac{1}{t^{\alpha}}$ is convergent if $\alpha > 1$.

Therefore $\sum_{k=1}^{\infty} f(S(k)) < \infty$ .

It is known that $d(n) < 2\sqrt{n}$ if $n \in N^*$ (2)

and it is obvious that $m_t < d(t!)$ (3)

We can show that

$$\sum_{k=1}^{\infty} ( S(k)^p \sqrt{S(k)!} )^{-1} < \infty \text{ , } p > 1 \qquad (4)$$

$$\sum_{k=1}^{\infty} ( S(k)! )^{-1} < \infty \qquad (5)$$

$$\sum_{k=1}^{\infty} ( S(1)\,!\, S(2)\,! \ldots S(k)\,! )^{-1/k} < \infty \qquad (6)$$

$$\sum_{k=2}^{\infty} ( S(k) \sqrt{S(k)!}\, (\log S(k))^p )^{-1} < \infty \,,\ p > 1 \qquad (7)$$

Write $f(S(k)) = (S(k)^p \cdot \sqrt{S(k)!}\,)^{-1}$, $f(t) = (t^p \cdot \sqrt{t!}\,)^{-1} = 2(t^p \cdot \sqrt{t!}\,)^{-1} <$
$< 2(t^p \cdot d(t!))^{-1} < 2(t^p \cdot (d(t!) - d((t-1)!)))^{-1}$.
Now use the proposition 1 to get (4).
The convergence of ( 5 ) follows from inequality $t\sqrt{t!} < t\,!$ if $p \in R$, $p > 1$, $t > t_* = [e^{2p+1}]$, $t \in N^*$. Here $[e^{2p+1}]$ means the greatest integer $\le e^{2p+1}$.
The details are left to the reader. To show (6) we can use the Carleman's Inequality: Let $(x_n)_{n \in N^*}$ be a sequence of positive real numbers and $x_n \neq 0$ for some $n$. Then

$$\sum_{k=1}^{\infty} ( x_1 x_2 \cdots x_k )^{1/k} < e \sum_{k=1}^{\infty} x_k \qquad (8)$$

Write $x_k = ( S(k)\,! )^{-1}$ and use ( 8 ) and ( 5 ) to get ( 6 ).
It is well-known that

$$\sum_{n=2}^{\infty} (n(\log n)^p)^{-1} < \infty \ \text{ if and only if } p > 1,\ p \in R. \qquad (9)$$

Write $f(t) = (t\sqrt{t!}\,(\log t)^p)^{-1}$, $t \ge 2$, $t \in N^*$. We have

$$\sum_{k=2}^{\infty} (S(k)\sqrt{S(k)!}\,(\log S(k))^p)^{-1} = \sum_{k=2}^{\infty} m_t f(t).$$

$m_t f(t) < d(t\,!)\, f(t) < 2\sqrt{t!}\,(t\sqrt{t!}\,(\log t)^p)^{-1} = 2\,(t\,(\log t)^p)^{-1}$.
Now use ( 9 ) to get ( 7 ).
*Remark 1.* Apply ( 5 ) and Cauchy's Condensation Test to see that

$\sum_{k=0}^{\infty} 2^k (S(2^k)\,!)^{-1} < \infty$. This implies that $\lim_{k \to \infty} 2^k (S(2^k)\,!)^{-1} = 0$.

A problem: Test the convergence behaviour of the series

$$\sum_{n=1}^{\infty} (S(k)^p \sqrt{(S(k)-1)\,!}\,)^{-1}. \qquad (10)$$

*Remark 2.* This problem is more powerful than ( 4 ).
Let $p_n$ denote the n-th prime number ($p_1=2$, $p_2=3$, $p_3=5$, $p_4=7$, ...).

It is known that $\sum_{n=1}^{\infty} 1/p_n = \infty$. (11)

We next make use of (11) to obtain the following result:

$$\sum_{n=1}^{\infty} S(n)/n^2 = \infty. \qquad (12)$$

We have $\sum_{n=1}^{\infty} S(n)/n^2 \geq \sum_{k=1}^{\infty} S(p_k)/p_k^2 = \sum_{k=1}^{\infty} p_k/p_k^2 = \sum_{k=1}^{\infty} 1/p_k$ (13)

Now apply (13) and (11) to get (12).

We can also show that

$$\sum_{n=1}^{\infty} S(n)/n^{1+p} < \infty \text{ if } p > 1, p \in R. \quad (14)$$

Indeed, $\sum_{n=1}^{\infty} S(n)/n^{1+p} \leq \sum_{n=1}^{\infty} n/n^{1+p} = \sum_{n=1}^{\infty} 1/n^p < \infty$.

If $0 \leq p \leq 2$, we have $S(n)/n^p \geq S(n)/n^2$.

Therefore $\sum_{n=1}^{\infty} S(n)/n^p = \infty$ if $0 \leq p \leq 2$.

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# ON SOME DIOPHANTINE EQUATIONS

by

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Let $S(n)$ be defined as the smallest integer such that $(S(n))!$ is divisible by $n$ (Smarandache Function). We shall assume that $S : N^* \to N^*$, $S(1) = 1$. Our purpose is to study a variety of Diophantine equations involving the Smarandache function. We shall determine all solutions of the equations (1), (3) and (8).

(1) $x^{S(x)} = S(x)^x$

(2) $x^{S(y)} = S(y)^x$

(3) $x^{S(x)} + S(x) = S(x)^x + x$

(4) $x^{S(y)} + S(y) = S(y)^x + x$

(5) $S(x)^x + x^2 = x^{S(x)} + S(x)^2$

(6) $S(y)^x + x^2 = x^{S(y)} + S(y)^2$

(7) $S(x)^x + x^3 = x^{S(x)} + S(x)^3$

(8) $S(y)^x + x^3 = x^{S(y)} + S(y)^3$.

For example, let us solve equation (1):

We observe that if $x = S(x)$, then (1) holds.

But $x = S(x)$ if and only if $x \in \{1, 2, 3, 4, 5, 7, ..\} = \{x \in N^*; x\text{-prime}\} \cup \{1, 4\}$.

If $x \geq 6$ is not a prime integer, then $S(x) < x$. We can write $x = S(x) + t$, $t \in N^*$, which implies that $S(x)^{S(x)+t} = (S(x) + t)^{S(x)}$. Thus we have $S(x)^t = (1 + \frac{t}{S(t)})^{S(x)}$.

Applying the well-known result $(1 + \frac{k}{n})^n < 3^k$, for $n, k \in N^*$, we have $S(x)^t < 3^t$ which implies that $S(x) < 3$ and consequently $x < 3$. This contradicts our choice of $x$.

Thus, the solutions of (1) are $A_1 = \{x \in N^* ; x = \text{prime}\} \cup \{1, 4\}$.

Let us denote by $A_k$ the set of all solutions of the equation (k).

To find $A_8$ for example, we see that $(S(n), n) \in A_8$ for $n \in N^*$.

Now suppose that $x \neq S(y)$. We can show that $(x, y)$ does not belongs to $A_8$ as follows: $1 < S(y) < x \Rightarrow S(y) \geq 2$ and $x \geq 3$. On the other hand,

$S(y)^x - x^{S(y)} > S(y)^x - x^x = (S(y) - x)(S(y)^{x-1} + xS(y)^{x-2} + \ldots + x^{x-1}) \geq$

$(S(y) - x)(S(y)^2 + xS(y) + x^2) = S(y)^3 - x^3$.

Thus, $A_8 = \{(x, y) ; y = n, x = S(n), n \in N^*\}$.

To find $A_3$, we se that $x = 1$ implies $S(x) = 1$ and (3) holds.

If $S(x) = x$, (3) also holds.

If $x \geq 6$ is not a prime number, then $x > S(x)$.

Write $x = S(x) + t$, $t \in N^* = \{1, 2, 3, \ldots\}$.

Combining this with (3) yields

$S(x)^{S(x)+t} + S(x) + t = (S(x) + t)^{S(x)} + S(x) \Leftrightarrow S(x)^t + t/S(x)^{S(x)} = (1 + t/S(x))^{S(x)} < 3^t$

which implies $S(x) < 3$. This contradicts our choice of $x$.

Thus $A_3 = \{ x \in N^* ; x = \text{prime} \} \cup \{ 1 , 4 \}$.
Now, we suppose that the reader is able to find $A_2 , A_4 , \ldots , A_7$.
We next determine all positive integers $x$ such that $x = \sum_{k^2 < x} k^2$

Write $1^2 + 2^2 + \ldots + s^2 = x$ (1)

$s^2 < x$ (2)

$(s + 1)^2 \geq x$ (3)

(1) implies $x = s(s+1)(2s+1)/6$. Combining this with (2) and (3) we have $6s^2 < s(s+1)(2s+1)$ and $6(s+1)^2 \geq s(s+1)(2s+1)$. This implies that $s \in \{2 , 3\}$.
$s = 2 \Rightarrow x = 5$ and $s = 3 \Rightarrow x = 14$.
Thus $x \in \{ 5 , 14 \}$.
In a similar way we can solve the equation $x = \sum_{k^3 < x} k^3$

We find $x \in \{ 9 , 36 , 100 \}$.
We next show that the set $M_p = \{ n \in N^* ; n = \sum_{k^p < n} k^p , p \geq 2 \}$ has at least $[ p/\ln 2] - 2$ elements.
Let $m \in N^*$ such that $m - 1 < p/\ln 2$ (4)
and $p/\ln 2 < m$ (5)
Write (4) and (5) as:
$2 < e^{p/m-1}$ (6)
$e^{p/m} < 2$ (7)
Write $x_k = ( 1 + 1/k )^k$, $y_k = ( 1 + 1/k )^{k+1}$.
It is known that $x_s < e < y_t$ for every $s , t \in N^*$.
Combining this with (6) and (7) we have
$x_s^{p/m} < e^{p/m} < 2 < e^{p/m-1} < y_t^{p/m-1}$ for every $s , t \in N^*$.
We have $2 < y_t^{p/m-1} = ((t+1)/t)^{(t+1)p/m-1} \leq ((t+1)/t)^p$ if $(t+1)/(m-1) \leq 1$.
So, if $t \leq m - 2$ we have $2 < ((t+1)/t)^p \Leftrightarrow 2 t^p < (t+1)^p \Leftrightarrow (t+1)^p - t^p > t^p$ (8).
Let $A_p(s)$ denote the sum $1^p + 2^p + \ldots\ldots + s^p$.
Proposition 1. $(t+1)^p > A_p(t)$ for every $t \leq m-2 , t \in N^*$.
Proof. Suppose that $A_p(t) \geq (t+1)^p \Leftrightarrow A_p(t-1) > (t+1)^p - t^p > t^p \Leftrightarrow$
$A_p(t-2) > t^p - (t-1)^p > (t-1)^p \Leftrightarrow \ldots \Leftrightarrow A_p(1) > 2^p$ which is not true.
It is obvious that $A_p(t) > t^p$ if $t \in N^*$, $2 \leq t \leq m-2$ which implies $A_p(t) \in M_p$ for every $t \in N^*$ and $2 \leq t \leq m-2$.
Therefore card $M_p > m-3 = (m-1) - 2 = [ p/\ln 2 ] - 2$.

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# ABOUT THE SMARANDACHE COMPLEMENTARY PRIME FUNCTION

by

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Let $c : N \to N$ be the function defined by the condition that $n + c(n) = p_i$, where $p_i$ is the smallest prime number, $p_i \geq n$.

**Example**

$c(0) = 2$, $c(1) = 1$, $c(2) = 0$, $c(3) = 0$, $c(4) = 1$, $c(5) = 0$, $c(6) = 1$, $c(7) = 0$ and so on.

1 ) If $p_k$ and $p_{k-1}$ are two consecutive primes and $p_k < n \leq p_{k-1}$, then :

$c(n) \in \{ p_{k-1} - p_k - 1, p_{k-1} - p_k - 2, \ldots, 1, 0 \}$, because :

$c(p_k + 1) = p_{k-1} - p_k - 1$ and so on, $c(p_{k-1}) = 0$.

2 ) $c(p) = c(p-1) - 1 = 0$ for every p prime, because $c(p) = 0$ and $c(p-1) = 1$.

We also can observe that $c(n) \neq c(n+1)$ for every $n \in N$.

**1. Property**

The equation $c(n) = n$, $n > 1$ has no solutions.

**Proof**

If n is a prime it results $c(n) = 0 < n$.

It is wellknown that between n and 2n, $n > 1$ there exists at least a prime number. Let $p_k$ be the smallest prime of them. Then if n is a composite number we have :

$c(n) = p_k - n < 2n - n = n$, therefore $c(n) < n$.

It results that for every $n \neq p$, where p is a prime, we have $\frac{1}{n} \leq \frac{c(n)}{n} < 1$, therefore $\sum\limits_{\substack{n \neq p \\ p \text{ prime}}} \frac{c(n)}{n}$ diverges. Because for the primes $c(p)/p = 0$ we can say that $\sum\limits_{n \geq 1} \frac{c(n)}{n}$ diverges.

**2. Property**

If n is a composite number, then $c(n) = c(n-1) - 1$.

**Proof**

Obviously.

It results that for n and $(n-1)$ composite numbers we have $\frac{c(n)}{c(n-1)} > 1$. Now, if $p_k < n < p_{k-1}$ where $p_k$ and $p_{k-1}$ are consecutive primes, then we have :

$c(n)\,c(n+1)\ldots c(p_{k-1}-1) = (p_{k-1}-n)!$

and if $n \leq p_k < p_{k-1}$ then $c(n)\,c(n+1)\ldots c(p_{k-1}-1) = 0$.

Of course, every $\prod\limits_{n=k}^{r} c(n) = 0$ if there exists a prime number p, $k \leq p \leq r$.

If $n = p_k$ is any prime number, then $c(n) = 0$ and because $c(n+1) = p_{k-1} - n - 1$ it results that $|c(n) - c(n+1)| = 1$ if and only if n and $(n+2)$ are primes ( friend prime numbers )

**3. Property**

For every k - th prime number $p_k$ we have :

$c(p_k + 1) < (\log p_k)^2 - 1$.

**Proof**

Because $c(p_k + 1) = p_{k-1} - p_k - 1$ we have $p_{k-1} - p_k = c(p_k + 1) + 1$.

But, on the other hand we have $p_{k-1} - p_k < (\log p_k)^2$, then the assertion follows.

**4. Property**

$c(c(n)) < c(n)$ and $c^m(n) < c(n) < n$, for every $n > 1$ and $m \geq 2$ .

**Proof**

If we denote $c(n) = r$ then we have :

$c(c(n)) = c(r) < r = c(n)$.

Then we suppose that the assertion is true for m : $c^m(n) < c(n) < n$, and we prove it for ( m + 1 ), too :

$c^{m+1}(n) = c(c^m(n)) < c^m(n) < c(n) < n.$

**5. Property**

For every prime p we have $(c(p-1))^n \le c((p-1)^n)$.

**Proof**

$c(p-1) = 1 \Rightarrow (c(p-1))^n = 1$ while $(p-1)^n$ is a composite number, therefore $c((p-1)^n) \ge 1$.

**6. Property**

The following kind of Fibonacci equation :

$$c(n) + c(n+1) = c(n+2) \qquad (1)$$

has solutions.

**Proof**

If n and ( n + 1 ) are both composite numbers, then $c(n) > c(n+1) \ge 1$. If ( n + 2 ) is a prime, then $c(n+2) = 0$ and we have no solutions in this case. If ( n + 2 ) is also a composite number, then :

$c(n) > c(n+1) > c(n+2) \ge 1$, therefore $c(n) + c(n+1) > c(n+2)$

and we have no solutions also in this case.

Therefore n and ( n + 1 ) are not both composite numbers in the equality ( 1 ).

If n is a prime, then ( n + 1 ) is a composite number and we must have :

$0 + c(n+1) = c(n+2)$, wich is not possible ( see ( 2 ) ).

We have only the case when ( n + 1 ) is a prime; in this case we must have :

$1 + 0 = c(n+2)$ but this implies that ( n + 3 ) is a prime number, so the only solutions are when ( n + 1 ) and ( n + 2 ) are friend prime numbers.

**7. Property**

The following equation:

$$\frac{c(n) + c(n+2)}{2} = c(n+1) \qquad (2)$$

has an infinite number of solutions.

**Proof**

Let $p_k$ and $p_{k+1}$ be two consecutive prime numbers, but not friend prime numbers.

Then, for every integer i between $p_k + 1$ and $p_{k+1} - 1$ we have:

$$\frac{c(i-1)+c(i+1)}{2} = \frac{(p_{k+1}-i+1)+(p_{k+1}-i-1)}{2} = p_{k+1}-i = c(i).$$

So, for the equation ( 2 ) all positive integer n between $p_k + 1$ and $p_{k+1} - 1$ is a solution.

If n is prime, the equation becomes $\frac{c(n+2)}{2} = c(n+1)$.

But ( n + 1 ) is a composite number, therefore $c(n+1) \neq 0 \Rightarrow c(n+2)$ must be composite number. Because in this case $c(n+1) = c(n+2) + 1$ and the equation has the form $\frac{c(n+2)}{2} = c(n+2) + 1$, so we have no solutions.

If ( n + 1 ) is prime, then we must have $\frac{c(n)+c(n+2)}{2} = 0$, where n and ( n + 2 ) are composite numbers. So we have no solutions in this case, because $c(n) \geq 1$ and $c(n+2) \geq 1$.

If ( n + 2 ) is a prime, the equation has the form $\frac{c(n)}{2} = c(n+1)$, where ( n + 1 ) is a composite number, therefore $c(n+1) \neq 0$. From ( 2 ) it rezults that $c(n) \neq 0$, so n is also a composite number. This case is the same with the first considered case.

Therefore the only solutions are for $\overline{p_k, p_{k+1}-2}$, where $p_k$, $p_{k+1}$ are consecutive primes, but not friend consecutive primes.

**8. Property**

The greatest common divisor of n and c ( x ) is 1 :

$(x, c(x)) = 1$, for every composite number x.

**Proof**

Taking into account of the definition of the function c, we have $x + c(x) = p$, where p is a prime number.

If there exists $d \neq 1$ so that $d / x$ and $d / c(x)$, then it implies that $d / p$. But p is a prime number, therefore $d = p$.

This is not possibile because $c(x) < p$.

If p is a prime number, then $(p, c(p)) = (p, 0) = p$.

**9. Property**

The equation $[x, y] = [c(x), c(y)]$, where $[x, y]$ is the least common multiple of $x$ and $y$ has no solutions for $x, y > 1$.

**Proof**

Let us suppose that $x = dk_1$ and $y = dk_2$, where $d = (x, y)$. Then we must have :

$[x, y] = dk_1k_2 = [c(x), c(y)]$.

But $(x, c(x)) = (dk_1, c(x)) = 1$, therefore $dk_1$ is given in the least common multiple $[c(x), c(y)]$ by $c(y)$.

But $(y, c(y)) = (dk_1, c(y)) = 1 \Rightarrow d = 1 \Rightarrow (x, y) = 1 \Rightarrow$

$\Rightarrow [x, y] = xy > c(x)c(y) \geq [c(x), c(y)]$, therefore the above equation has no solutions, for $x, y > 1$.

For $x = 1 = y$ we have $[x, y] = [c(x), c(y)] = 1$.

**10. Property**

The equation :

$$(x, y) = (c(x), c(y)) \qquad (3)$$

has an infinite number of solutions.

**Proof**

If $x = 1$ and $y = p - 1$ then $(x, y) = 1$ and $(c(x), c(y)) = (1, 1) = 1$, for an arbitrary prime p.

Easily we observe that every pair $(n, n + 1)$ of numbers is a solutions for the equation $(3)$, if n is not a prime.

**11. Property**

The equation :

$$c(x) + x = c(y) + y \qquad (4)$$

has an infinite number of solutions.

**Proof**

From the definition of the function c it results that for every x and y satisfying

$p_k < x \le y \le p_{k-1}$ we have $c(x) + x = c(y) + y = p_{k-1}$. Therefore we have $(p_{k-1} - p_k)^2$ couples $(x, y)$ as different solutions. Then, until the n-th prime $p_n$, we have $\sum_{k=1}^{n-1} (p_{k-1} - p_k)^2$ different solutions.

**Remark**

It seems that the equation $c(x) + y = c(y) + x$ has no solutions $x \neq y$, but it is not true.

Indeed, let $p_k$ and $p_{k-1}$ be consecutive primes such that $p_{k-1} - p_k = 6$ ( is possibile : for example $29 - 23 = 6$, $37 - 31 = 6$, $53 - 47 = 6$ and so on ) and $p_k - 2$ is not a prime.

Then $c(p_k - 2) = 2$, $c(p_k - 1) = 1$, $c(p_k) = 0$, $c(p_k + 1) = 5$, $c(p_k + 2) = 4$, $c(p_k + 3) = 3$ and we have :

1. $c(p_k + 1) - c(p_k - 2) = 5 - 2 = 3 = (p_k + 1) - (p_k - 2)$
2. $c(p_k + 2) - c(p_k - 1) = 3 = (p_k + 2) - (p_k - 1)$
3. $c(p_k + 3) - c(p_k) = 3 = (p_k + 3) - p_k$, thus

$c(x) - c(y) = x - y$ ( $\Leftrightarrow c(x) + y = c(y) + x$ ) has the above solutions if $p_k - p_{k-1} > 3$

If $p_k - p_{k-1} = 2$ we have only the two last solutions.

In the general case, when $p_{k-1} - p_k = 2h$, $h \in N^*$, let $x = p_k - u$ and $y = p_k + v$, $u, v \in N$ be the solutions of the above equation.

Then $c(x) = c(p_k - u) = u$ and $c(y) = c(p_k + v) = 2h - v$.

The equation becomes:

$u + (p_k + v) = (2h - v) + (p_k - u)$, thus $u + v = h$.

Therefore, the solutions are $x = p_k - u$ and $y = p_k + h - u$, for every $u = \overline{0, h}$ if $p_k - p_{k-1} > h$ and $x = p_k - u$, $y = p_k + h - u$, for every $u = \overline{0, l}$ if $p_k - p_{k-1} = l + 1 \le h$.

**Remark**

$c(p_k + 1)$ is an odd number, because if $p_k$ and $p_{k-1}$ are consecutive primes, $p_k > 2$, then $p_k$ and $p_{k-1}$ are, of course, odd numbers; then $p_{k-1} - p_k - 1 = c(p_k + 1)$ are **always** odd.

**12. Property**

The sumatory function of c, $F_c(n) \overset{def}{=} \sum_{\substack{d \in N \\ d/n}} c(d)$ has the properties :

a) $F_c(2p) = 1 + c(2p)$

b) $F_c(pq) = 1 + c(pq)$, where p and q are prime numbers.

**Proof**

a) $F_c(2p) = c(1) + c(2) + c(p) + c(2p) = 1 + c(2p)$.

b) $F_c(pq) = c(1) + c(p) + c(q) + c(pq) = 1 + c(pq)$.

**Remark**

The function c is not multiplicative : $0 = c(2) \cdot c(p) < c(2p)$.

**13. Property**

$$c^k(p) = \begin{cases} 0 & \text{for } k \text{ odd number} \\ 2 & \text{for } k \text{ even number}, \quad k \geq 1 \end{cases}$$

**Proof**

We have :

$c^1(p) = 0$;

$c^2(p) = c(c(p)) = c(0) = 2$;

$c^3(p) = c(2) = 0$;

$c^4(p) = c(0) = 2$.

Using the complete mathematical induction, the property holds.

**Consequences**

1 ) We have $\dfrac{c^k(p) + c^{k+1}(p)}{2} = 1$ for every $k \geq 1$ and p prime number.

2 ) $\sum_{k=1}^{r} c^k(p) = \left[\dfrac{r}{2}\right] \cdot 2$, where [ x ] is the integer part of x, and

$\sum_{\substack{k=2 \\ k \text{ even}}}^{r} \dfrac{1}{c^k(p)} = \left[\dfrac{r}{2}\right] \cdot \dfrac{1}{2}$, thus $\sum_{k\geq 1} c^k(p)$ and $\sum_{\substack{k\geq 2 \\ k \text{ even}}} \dfrac{1}{c^k(p)}$ are divergent series.

**Remark**

$c^k(p-1) = c^{k-1}(c(p-1)) = c^{k-1}(1) = 1$, for every prime $p > 3$ and $k \in N^*$,

therefore $c^{k_1}(p_1 - 1) = c^{k_2}(p_2 - 1)$ for every primes $p_1, p_2 > 3$ and $k_1, k_2 \in N^*$.

**14.Property**

The equation :

$$c(x) + c(y) + c(z) = c(x)\,c(y)\,c(z) \qquad (5)$$

has an infinite number of solutions.

**Proof**

The only non-negative solutions for the diofantine equation $a + b + c = abc$ are $a = 1$, $b = 2$ and $c = 3$ and all circular permutations of $\{ 1, 2, 3 \}$.

Then :

$c ( x ) = 1 \Rightarrow x = p_k - 1$, $p_k$ prime number, $p_k > 3$

$c ( y ) = 2 \Rightarrow y = p_k - 2$ , where $p_{r-1}$ and $p_r$ are consecutive prime numbers such that $p_r - p_{r-1} \geq 3$

$c ( z ) = 3 \Rightarrow z = p_t - 3$, where $p_{t-1}$ and $p_t$ are consecutive prime numbers such that $p_t - p_{t-1} \geq 4$

and all circular permutations of the above values of x, y and z.

Of course, the equation $c ( x ) = c ( y )$ has an infinite number of solutions.

**Remark**

We can consider $c^{\leftarrow} ( y )$, for every $y \in N^*$, defined as $c^{\leftarrow} ( y ) = \{ x \in N \mid c ( x ) = y \}$. For example $c^{\leftarrow} ( 0 )$ is the set of all primes, and $c^{\leftarrow} ( 1 )$ is the set $\{ 1, p_{k-1} \}_{\substack{p_k \text{ prime} \\ p_k > 3}}$ and so on.

A study of these sets may be interesting.

**Remark**

If we have the equation :

$$c^k ( x ) = c ( y ), k \geq 2 \qquad (6)$$

then, using property 13, we have two cases.

If x is prime and k is odd, then $c^k ( x ) = 0$ and ( 5 ) implies that y is prime.

In the case when x is prime and k is even it results $c^k ( x ) = 2 = c ( y )$, which implies that y is a prime, such that $y - 2$ is not prime.

If $x = p$, $y = q$, p and q primes, $p,q > 3$, then $( p - 1, q - 1 )$ are also solutions, because $c^k ( p - 1 ) = 1 = c ( q - 1 )$, so the above equation has an infinite number of couples as solutions.

Also a study of $( c^k ( x ) )^{\leftarrow}$ seems to be interesting.

**Remark**

The equation :

$$c(n)+c(n+1)+c(n+2)=c(n-1) \qquad (7)$$

has solutions when $c(n-1)=3$, $c(n)=2$, $c(n+1)=1$, $c(n+2)=0$, so the solutions are $n = p - 2$ for every p prime number such that between $p - 4$ and p there is not another prime.

The equation :

$$c(n-2)+c(n-1)+c(n+1)+c(n+2)=4c(n) \qquad (8)$$

has as solutions $n = p - 3$, where p is a prime such that between $p - 6$ and p there is not another prime, because $4c(n) = 12$ and $c(n-2)+c(n-1)+c(n+1)+c(n+2)=12$.

For example $n = 29 - 3 = 26$ is a solution of the equation ( 7 ).

The equation :

$$c(n)+c(n-1)+c(n-2)+c(n-3)+c(n-4)=2c(n-5) \qquad (9)$$

( see property 7 ) has as solution $n = p - 5$, where p is a prime, such that between $p - 6$ and p there is not another prime. Indeed we have $0 + 1 + 2 + 3 + 4 = 2\cdot 5$.

Thus, using the properties of the function c we can decide if an equation, which has a similar form with the above equations, has or has not solutions.

But a difficult problem is : " For any even number a, can we find consecutive primes such that $p_{k-1} - p_k = a$? "

The answer is useful to find the solutions of the above kind of equations, but is also important to give the answer in order to solve another open problem :

" Can we get, as large as we want, but finite decreasing sequence k, k - 1, ..., 2, 1, 0 ( **odd k** ), included in the sequence of the values of c?"

If someone gives an answer to this problem, then it is easy to give the answer ( it will be the same ) at the similar following problem :

" Can we get, as large as we want, but finite decreasing sequence k, k - 1, ..., 2, 1, 0 ( **even k**), included in the sequence of the values of c?"

We suppose the answer is negative.

In the same order of ideea, it is interesting to find $\max_n \frac{c(n)}{n}$.

It is wellknown ( see [ 4 ], page 147 ) that $p_{n+1} - p_n < ( \ln p_n )^2$, where $p_n$ and $p_{n+1}$ are two consecutive primes.

Moreover, $\frac{c(n)}{n}$, $p_k < n \leq p_{k+1}$ reaches its maximum value for $n = p_k + 1$, where $p_k$ is a prime.

So, in this case :

$$\frac{c(n)}{n} = \frac{p_{k+1} - p_k - 1}{p_k + 1} < \frac{(\ln p_k)^2 - 1}{p_k + 1} \overset{k\to\infty}{\rightarrow} 0$$

Using this result, we can find the maximum value of $\frac{c(n)}{n}$

For $p > 100$ we have $\frac{(\ln p)^2 - 1}{p+1} < \frac{(\ln 100)^2 - 1}{101} < \frac{1}{4}$.

Using the computer, by a straight forward computation, it is easy to prove that $\max_{2\leq n\leq 100} \frac{c(n)}{n} = \frac{3}{8}$, wich is reached for $n = 8$.

Because $\frac{c(n)}{n} < \frac{1}{4}$ for every $n>100$ it results that $\max_{n\geq 2} \frac{c(n)}{n} = \frac{3}{8}$ reached for $n = 8$.

**Remark**

There exists an infinite number of finite sequences $\{ c ( k_1 ), c ( k_1 + 1), ..., c ( k_2 ) \}$ such that $\sum_{k=k_1}^{k_2} c(k)$ is a three-cornered number for $k_1 , k_2 \in N^*$ ( the n-th three-cornered number is $T_n \overset{def}{=} \frac{n(n+1)}{2}$, $n \in N^*$ ).

For example, in the case $k_1 = p_k$ and $k_2 = p_{k+1}$ , two consecutive primes, we have the finite sequence $\{ c ( p_k ), c ( p_k + 1), ..., c ( p_{k+1} - 1), c ( p_{k+1} ) \}$ and

$$\sum_{k=p_k}^{p_{k+1}} c(k) = 0 + (p_{k+1} - p_k - 1) + .. + 2 + 1 + 0 = \frac{(p_{k+1} - p_k - 1)(p_{k+1} - p_k)}{2} = T_{p_{k+1}-p_k-1}$$

Of course, we can define the function $c' : N \setminus \{ 0, 1 \} \rightarrow N$, $c' ( n ) = n - k$, where k is the smallest natural number such that $n - k$ is a prime number, but we shall give some properties of this function in another paper.

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# THE FUNCTIONS $\theta_s(x)$ AND $\bar{\theta}_s(x)$

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In this paper we define the function $\theta_s: N\setminus\{0,1\} \to N$ and $\bar{\theta}_s: N\setminus\{0,1,2\} \to N$ as follows :

$$\theta_s(x) = \sum_{\substack{p/x \\ 0<p\le x \\ p-\text{prime}}} S(p^x);\quad \bar{\theta}_s(x) = \sum_{\substack{p\nmid x \\ 0<p\le x \\ p-\text{prime}}} S(p^x),$$

where $S(p^x)$is the Smarandache function defined in [3] (S(n) is the smollest integer m such that m! is divisible by n).

For the begining we give some properties of the $\theta$ function. Let us observe that, from the definition of $\theta_s$ it results:

$\theta_s(2) = S(2^2) = 4$,
$\theta_s(3) = S(3^3) = 6$,
$\theta_s(4) = S(2^4) = 6$,
$\theta_s(5) = S(5^5) = 25$,
$\theta_s(6) = S(2^6) + S(3^6) = 7 + 15 = 22$,
$\theta_s(7) = S(7^7) = 49$,
$\theta_s(8) = S(2^8) = 10$,
$\theta_s(9) = S(3^9) = 21$,
$\theta_s(10) = S(2^{10}) + S(5^{10}) = 12 + 45 = 57$,
$\theta_s(11) = S(11^{11}) = 121$,
$\theta_s(12) = S(2^{17}) + S(3^{12}) = 43$.

We note also that if p-prime than $\theta_s(p^p) = p^2$.

**Proposition 1.** *The series* $\sum_{x\ge2}(\theta_s(x))^{-1}$ *is convergent.*

*Proof.*
$$\sum_{x\ge2}(\theta_s(x))^{-1} = \frac{1}{S(2^2)} + \frac{1}{S(3^3)} + \frac{1}{S(5^5)} + \frac{1}{S(2^6)+S(3^6)} +$$

$$+\frac{1}{S(7^7)} + \frac{1}{S(2^8)} + \frac{1}{S(3^9)} + \frac{1}{S(2^{10})+S(5^{10})} + \frac{1}{S(11^{11})} + \cdots$$

$$\le \sum_{\substack{i=2 \\ p_i/x}}^{\infty}\left(\frac{1}{p_i^2} + \frac{1}{(p_{V(x)}-1)V(x)}\right) = \sum_{i=2}^{\infty}\frac{1}{p_i^2} + \sum_{p_{V(x)}/x}\frac{1}{(p_{V(x)}-1)V(x)},$$

where V(x) denote the number of the primes less or equal with x and divide by x.

Of course the series $\sum_{i=2}^{\infty}\frac{1}{p_i^2}$ and $\sum_{p_{V(x)}/x}\frac{1}{(p_{V(x)}-1)V(x)}$ are convergent, so the propositionis proved.

**Proposition 2.** *Let the sequence* $T(n)=1-\lg\theta_s(n)+\sum_{i=2}^{n}\frac{1}{\theta_s(i)}$. *Then* $\lim_{n\to\infty}T(n)=-\infty$.

The proof is imediate because the series $\sum_{n=2}^{\infty}\frac{1}{\theta_s(n)}$ is convergent according by the proposition 1.

**Proposition 3.** *The equation* $\theta_s(x)=\theta_s(x+1)$ (0) *has no solution if x is a prime.*

*Proof.* If x is a prime number the equation become

$x^2=\theta_s(x+1)$, where

$\theta_s(x+1)=S(p_{i_1}^{x+1})+S(p_{i_2}^{x+1})+\cdots+S(p_{i_{V(x+1)}}^{x+1})$.

Using the inequality

$$(p-1)\alpha<S(p^{\alpha})\le p\alpha \tag{1}$$

given in [4], we have

$$\theta_s(x+1)\le(x+1)(p_{i_1}+p_{i_2}+\cdots+p_{i_{V(x+1)}})$$

Let us presume that the equation (0) has solution. We have the following relation:

$$x^2\le(x+1)(p_{i_1}+p_{i_2}+\cdots+p_{i_{V(x+1)}}) \tag{2}$$

and we prove that

$$p_{i_1}+p_{i_2}+\cdots+p_{i_{V(x+1)}}\le x-1 \tag{3}$$

for $x\ge 9$.

Let $\eta=p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdot\ldots\cdot p_r^{\alpha_r}$, $p_i\ne p_j$, $i\ne j$, the decomposition of n into primes. We define the function $f(n)=1+\alpha_1p_1+\cdots+\alpha_rp_r$ and we show that $f(n)\le n-2$ for $n\ge 9$. If $1\le n<9$ the precedent inequality is verified by calculus). For $n\ge 9$, we prove the inequality by induction:

$f(9) = 7$, $f(10) = 8$, $f(12) = 8 < 10$, true.

Now let us suppose that $f(n) \le n - 2$, $\forall\, n \ge 12$, and we show that $f(n+1) \le n - 1$.
In this case we have three different situations:
I) $n + 1 = h = k_1 \cdot k_2$, where $k_1, k_2$ are composed members. Using the true relation, $f(h) = f(k_1 \cdot k_2) = f(k_1) + f(k_2) - 1$, we have

$$f(h) = f(n+1) = f(k_1) + f(k_2) - 1 \le k_1 - 2 + k_2 - 2 - 1 = k_1 \cdot k_2 - 8 - 5 \le$$
$$\le h - 2 = n + 1 - 2 = n - 1 \Rightarrow f(n+1) \le n - 1.$$

II) $n + 1 = h = k_1 \cdot k_2$, where, $k_1$ – prime, $k_2$ – compounded,

$$f(h) = f(k_1) + f(k_2) - 1 \le k_1 + 1 + k_2 - 2 - 1 \le k_1 \cdot k_2 - 2 = n - 1.$$

III) $n + 1 = h = k_1 \cdot k_2$, where $k_1, k_2$ – prime,

$$f(h) = 1 + k_1 + k_2 = k_1 k_2 + 2 - (k_1 - 1)(k_2 - 1) \le h + 2 - 4 = h - 2 = n - 1.$$

**Conclusion**: $f(n) \le n - 2$, $\forall\, n \ge 9$.
Then $f(n) \le n + 2 \Rightarrow 1 + \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_r p_r \le n - 2 \Rightarrow$

$$\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_r p_r \le n - 3.$$

We obtain

$$p_1 + p_2 + \cdots + p_r \le \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_r p_r \le n - 3 \le n - 2$$

Using (3) in (2) we have

$$x^2 \le (x+1)(x-1) \Rightarrow x^2 < x^2 - 1, \text{ imposible.}$$

**Proposition 4**. *The equation* $\theta_s(x) = \theta_s(x+1)$ *has no solution for* $(x+1)$ *- prime.*

*Proof*. We have $\theta_s(x+1) = (x+1)^2$.
We suppose that the equation has solution and with the inequalityes (1) is must that

$$(x+1)^2 \le x(p_{i_1}^x + p_{i_2}^x + \cdots + p_{i_{v(x)}}^x) \le x^2, \text{ imposible.}$$

We give some particular value for $\bar{\theta}_s(x) = \sum_{\substack{p^x \mid x \\ p\text{-prime}}} S(p^x)$;

$\bar{\theta}_s(3) = S(2^3) = 4$

$\bar{\theta}_s(4) = S(3^4) = 9$

$\bar{\theta}_s(5) = S(2^5) + S(3^5) = 20$

$\bar{\theta}_s(6) = S(5^6) = 30$

$\bar{\theta}_s(7) = S(2^7) + S(3^7) + S(5^7) =$
$=8+18+36=62$

$\bar{\theta}_s(8) = S(3^8) + S(5^8) + S(7^8) = 18 + 35 + 49 = 102$

$\bar{\theta}_s(9) = S(2^9) + S(5^9) + S(7^9) = 12 + 40 + 56 = 108$

$\bar{\theta}_s(10) = S(3^{10}) + S(7^{10}) = 24 + 63 = 87$

$\bar{\theta}_s(11) = S(2^{11}) + S(3^{11}) + S(5^{11}) + S(7^{11}) = 16 + 27 +$
$+50+73=163$

$\bar{\theta}_s(12) = S(2^{12}) + S(7^{12}) + S(11^{12}) = 50+77+$
$+121=248$

$\bar{\theta}_s(13) = S(2^{13}) + S(3^{13}) + S(5^{13}) + S(7^{13}) + S(11^{13}) =$
$=16+27+60+84+132=319.$

**Proposition 5.** *The series* $\sum_{x\geq 3} (\bar{\theta}_s(x))^{-1}$ *is convergent.*

*Proof.*

$$\sum_{x\geq 3} (\bar{\theta}_s(x))^{-1} = \frac{1}{S(2^3)} + \frac{1}{S(3^4)} + \frac{1}{S(2^5)+S(3^5)} + \frac{1}{S(2^8)+S(5^8)+S(7^8)} + \frac{1}{S(2^9)+S(5^9)+S(7^9)}$$

$$+\cdots \leq \frac{1}{S(2^3)} + \frac{1}{S(3^4)} + \frac{1}{S(5^6)} + \frac{1}{S(2^7)} + \cdots \leq \sum_{\substack{x\geq 3 \\ p_i \nmid x \\ p_i - \text{prime}}} \frac{1}{S(p_i^x)} \leq \sum_{\substack{x\geq 3 \\ p_i \nmid x \\ p_i - \text{prime}}} \frac{1}{p_i^2} \leq \sum_{x\geq 3} \frac{1}{x^2}.$$

Because the series $\sum_{x\geq 3} \frac{1}{x^2}$ is convergent, we have that our series is convergent.

**Proposition 6.** *If* $T(n) = 1 - \lg \bar{\theta}_s(n) + \sum_{i=3}^{n} \frac{1}{\bar{\theta}_s(i)}$ *then* $\lim_{n\to\infty} T(n) = -\infty.$

**Proposition 7.** *The equation* $\bar{\theta}_s(x) = \bar{\theta}_s(x+1)$ *has no solution if* x+1=p-*prime.*

*Proof.* If x+1 is prime be wouldn't divide with any of prime numbers then him

$$\bar{\theta}_s(x+1) = \sum_{\substack{p \nmid x+1 \\ 0<p\le x+1}} S(p^{x+1}) = S(p_{i_1}^{x+1}) + S(p_{i_2}^{x+1}) + \cdots + S(p_{i_{\pi(x)}}^{x+1}).$$

The number x is divisible with at least two prime numbers then him. In the case $\bar{\theta}_s(x) = \sum_{\substack{p \nmid x \\ 0<p\le x}} S(p^x)$ will have at least two terms $S(p_{i_k}^x)$ less then they are in $\bar{\theta}_s(x+1)$.

Moreover $S(p_i^x) \le S(p_i^{x+1})$ and it results that $\bar{\theta}_s(x) < \bar{\theta}_s(x+1)$.

**Proposition 8**. *The equation* $\bar{\theta}_s(x) = \bar{\theta}_s(x+1)$ *has no solution if* x=p-prime, $x \ge 9$.

*Proof.* using the function $F_s(x) = \sum_{\substack{0<p\le x \\ p-\text{prime}}} S(p^x)$ defined in [2] we have

$$F_s(x) = \theta_s(x) + \bar{\theta}_s(x)$$

$$F_s(x+1) = \theta_s(x+1) + \bar{\theta}_s(x+1).$$

If our equation have solution $\bar{\theta}_s(x) = \bar{\theta}_s(x+1)$ then

$$F_s(x) - F_s(x+1) = \theta_s(x) - \theta_s(x+1)$$

or

$$F_s(x) - F_s(x+1) = x^2 - \theta_s(x+1).$$

Is known [2] that $F_s(x) - F_s(x+1) < 0$. We have $x^2 - \theta_s(x+1) < 0 \Rightarrow x^2 < \theta_s(x+1)$.
Using (3) we have

$\theta_s(x+1) \le (x+1)(x-1) = x^2 - 1$, therefore $x^2 < x^2 - 1$, imposible.

For x<9 is verified by calculus that the equation $\bar{\theta}_s(x) = \bar{\theta}_s(x+1)$ has not solution.

*Proposed problem*

1. $\theta_s(x) = \theta_s(x+1)$, x, x+1 are composed numbers.
2. $\bar{\theta}_s(x) = \bar{\theta}_s(x+1)$, x, x+1, are composed numbers.

*Calculate*

3. $\lim_{n\to\infty} \frac{\theta_s(n)}{n^\alpha}$, $\alpha \in \mathbf{R}$.

4. $\lim_{n\to\infty} \frac{\bar{\theta}_s(n)}{n^\alpha}$, $\alpha \in \mathbf{R}$.

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# The function $\Pi_S(x)$

by

**Vasile Seleacu and Stefan Zanfir**

In this paper are studied some properties of the numerical function $\Pi_S : N^* \to N$, $\Pi_S(x) = \{ m \in (0, x] / S(m) = \text{prime number} \}$, where $S(m)$ is the Smarandache function, defined in [1].

Numerical example :

$\Pi_S(1) = 0$, $\Pi_S(2) = 1$, $\Pi_S(3) = 2$, $\Pi_S(4) = 2$, $\Pi_S(5) = 3$, $\Pi_S(6) = 4$, $\Pi_S(7) = 5$, $\Pi_S(8) = 5$, $\Pi_S(9) = 5$, $\Pi_S(10) = 6$, $\Pi_S(11) = 7$, $\Pi_S(12) = 7$, $\Pi_S(13) = 8$, $\Pi_S(14) = 9$, $\Pi_S(15) = 10$, $\Pi_S(16) = 10$, $\Pi_S(17) = 11$, $\Pi_S(18) = 11$, $\Pi_S(19) = 12$, $\Pi_S(20) = 13$.

**Proposion 1.**

According to the definition we have :

a) $\Pi_S(x) \leq \Pi_S(x+1)$,

b) $\Pi_S(x) = \Pi_S(x-1) + 1$, if x is a prime,

c) $\Pi_S(x) \leq \varphi(x)$, if x is a prime,

where $\varphi(x)$ is the Euler's totient function.

**Proposition 2.**

The equation $\Pi_S(x) = \left[\frac{x}{2}\right]$, in the hypothesis $x \neq 1$ and $\Pi_S(x+1) = \Pi_S(x)$ has no solution in the following situation :

a) x is a prime,

b) x is a composite number, odd

c) x + 1 is the square of a positiv integer and x is odd.

**Proof.**

Using the reduction ad absurdum method we suppose that the equation $\Pi_S(x) = [\frac{x}{2}]$ has solution. Then $\Pi_S(x+1) = [\frac{x+1}{2}]$. Using the hypothesis we have :

$[\frac{x}{2}] = [\frac{x+1}{2}]$, false.

Because $x + 1$ is a perfect square we deduce that x is a composite number and because x is an uneven we obtain ( b ).

**Proposition 3.**

$\forall$ $a \geq 2$ and $k \geq 2$ $S(a^k)$ is not a prime.

**Proof.**

If we suppose that $S(a^k) = p$ is a prime, then $p! = a^k p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n} p$ and $(a^k, p) = 1$. We deduce that $a^k / (p-1)! \Rightarrow$

$S(a^k) \leq p - 1 < p$, false.

**Proposition 4.**

$\forall$ $x \in N^*$, we have :

$$\left[\frac{x}{2}\right] \leq \Pi_S(x) \leq x - [\sqrt{x}]$$

**Proof.**

We used the mathematical induction. In the particular case $x \in \{1, 2, 3, 4\}$ our inequality is verified by direct calculus.

We suppose that the inequality is verified for $x \in N^*$ and we proved it for $x + 1$.

We have the following cases :

1) $x + 1$ the prime number, with the subcases :

a) x is not a square of some integer. Then $\Pi_S(x+1) = \Pi_S(x) + 1$.

We suppose that $\Pi_S(x) \leq x - [\sqrt{x}]$

Let prove that $\Pi_S(x+1) \leq x + 1 - [\sqrt{x+1}]$.

It results that $\Pi_S(x+1) \leq x + 1 - [\sqrt{x+1}] \Leftrightarrow \Pi_S(x) \leq x - [\sqrt{x+1}]$.

It's enough to prove that $x - [\sqrt{x}] \leq x - [\sqrt{x+1}]$. This relation is true because from our hypothesis it results that $[\sqrt{x}] = [\sqrt{x+1}]$.

For the left side of the inequality we have $\Pi_S(x) \geq \left[\frac{x}{2}\right]$, true, and let prove that $\Pi_S(x+1) > \left[\frac{x+1}{2}\right]$.

Because $\Pi_S(x+1) = \Pi_S(x) + 1$ we have to prove that $\Pi_S(x) + 1 \geq \left[\frac{x+1}{2}\right]$

Therefore $\Pi_S(x) \geq \left[\frac{x+1}{2}\right] - 1$, that is a true relation.

b) x perfect square.

We suppose that $\Pi_S(x) \leq x - [\sqrt{x}]$ is true. Then :

$\Pi_S(x) \le x + 1 - [\sqrt{x+1}] \Rightarrow \Pi_S(x) + 1 \le x + 1 - [\sqrt{x+1}] \Leftrightarrow \Pi_S(x) \le x - [\sqrt{x+1}]$.
That is a true relation because $[\sqrt{x}] = [\sqrt{x+1}]$. For the left inequality the demonstration is analogous with ( a )

2) x prime

a) $x - 1$ is not a perfect square.

We suppose that $\Pi_S(x) \le x - [\sqrt{x}]$ is true.

Let prove that $\Pi_S(x+1) \le x + 1 - [\sqrt{x+1}]$.

In this case we have the following two situations :

( i ) If $\Pi_S(x+1) = \Pi_S(x) + 1$, then we must prove that :

$\Pi_S(x) + 1 \le x + 1 - [\sqrt{x+1}]$.

Supposing that $\Pi_S(x) \ge \left[\frac{x}{2}\right]$ is true, let show that $\Pi_S(x+1) \ge \left[\frac{x+1}{2}\right]$ or $\Pi_S(x) + 1 \ge \left[\frac{x+1}{2}\right]$, therefore $\Pi_S(x) \ge \left[\frac{x+1}{2}\right] - 1$ and that results from the hypothesis.

(ii ) If $\Pi_S(x+1) = \Pi_S(x)$. We have to prove that $\Pi_S(x) \le x+1 - [\sqrt{x+1}]$

Of course this inequality is true. For the left side of the inequality we have to prove that $\Pi_S(x) \ge \left[\frac{x+1}{2}\right]$. If we admit $\left[\frac{x}{2}\right] \le \Pi_S(x) < \left[\frac{x+1}{2}\right]$ we obtain that $\Pi_S(x) = \left[\frac{x}{2}\right], x \ne 1$.

According to the *Proposition 2*, this inequality can't be true.

Therefore we have $\Pi_S(x) \ge \left[\frac{x+1}{2}\right]$.

Let observe that $x + 1$ is not a perfect square, if $x > 3$ is a prime number. For $x = 3$ the inequality is verified by calculus.

3) x is an even composed number. Then :

a) If $x + 1$ is a prime.

We know that $\Pi_S(x+1) = \Pi_S(x) + 1$. Then supposing $\Pi_S(x) \le x - [\sqrt{x}]$.
We have to prove that $\Pi_S(x+1) \le x + 1 - [\sqrt{x+1}]$ or $\Pi_S(x) = x - \left[\sqrt{x+1}\right]$.

This is true, because $[\sqrt{x}] = \left[\sqrt{x+1}\right]$.

For the left inequality we have to show $\Pi_S(x+1) \ge \left[\frac{x+1}{2}\right]$,
or $\Pi_S(x) + 1 \ge \left[\frac{x+1}{2}\right]$. But $\Pi_S(x) \ge \left[\frac{x+1}{2}\right] - 1$, is true.

b) If $x + 1$ is an odd composite number, then

( i ) If $\Pi_S(x+1) = \Pi_S(x) + 1$, the demonstration is the same as at ( a ).

( ii ) If $\Pi_S(x+1) = \Pi_S(x)$, we have to prove that $\Pi_S(x) \le x + 1 - \left[\sqrt{x+1}\right]$

Obvious.

The left inequality is obvious.

c) $x + 1$ perfect square.

Using *Proposition 3* we have only the case $\Pi_S(x+1) = \Pi_S(x)$. Then if we consider to be true the relation $\Pi_S(x) \le x - [\sqrt{x}]$.

Let prove that $\Pi_S(x+1) < x+1-\left[\sqrt{x+1}\right]$.

But $\Pi_S(x) \le x+1-\left[\sqrt{x+1}\right]$ is true.

For the left inequality we suppose that $\Pi_S(x) \ge \left[\frac{x}{2}\right]$ is true. We have to prove that $\Pi_S(x+1) \ge \left[\frac{x+1}{2}\right]$.

Because $\Pi_S(x+1) = \Pi_S(x)$ it results $\Pi_S(x) \ge \left[\frac{x+1}{2}\right]$.

So, we must have $\left[\frac{x}{2}\right] \ge \left[\frac{x+1}{2}\right]$. This is true, because $x+1$ is an odd number.

4) $x$ is an odd composed number.

a) If $x+1$ is even composed number the proof is the same as in ( 2a ).

For the right inequality we have :

( i ) If $\Pi_S(x+1) = \Pi_S(x) + 1$ and we suppose that $\Pi_S(x) \le x - [\sqrt{x}]$, let to prove that $\Pi_S(x+1) \le x+1-\left[\sqrt{x+1}\right]$.

This relation lead us to $\Pi_S(x) \le x - \left[\sqrt{x+1}\right]$. This is true because $[\sqrt{x}] = \left[\sqrt{x+1}\right]$.

( ii ) If $\Pi_S(x+1) = \Pi_S(x)$ the proof is obvious.

b) If $x+1$ is a perfect square.

In this case according to the *Proposition 3* we have only the situation $\Pi_S(x+1) = \Pi_S(x)$. The right sided inequality is obvious and the left side inequality has the same proof as for ( 2a ).

5) If $x$ is a perfect square.

a) If $x$ is a prime and the only situation is that $\Pi_S(x+1) = \Pi_S(x) + 1$. The demonstration is obvious.

b) If $x+1$ is a composite number.

For the right inequality we have :

( i ) If $\Pi_S(x+1) = \Pi_S(x+1)$, the proof is analogous as in the preceding case.

( ii ) If $\Pi_S(x+1) = \Pi_S(x)$ the proof is obvious.

For the left inequality :

If $x+1$ is an odd composite number the relation is obvious.

If $x+1$ is an even composite number then :

if $\Pi_S(x+1) = \Pi_S(x) + 1$, the proof is analogous with ( a ).

if $\Pi_S(x+1) = \Pi_S(x)$ then $x$ can be just an odd perfect square.

We suppose that $\Pi_S(x) \ge \left[\frac{x}{2}\right]$ is true.

To show that $\Pi_S(x) \ge \left[\frac{x+1}{2}\right]$, if we suppose, again, that $\Pi_S(x) < \left[\frac{x+1}{2}\right]$

it results

$$\left[\frac{x}{2}\right] \le \Pi_S(x) < \left[\frac{x+1}{2}\right], \text{ and we have } \Pi_S = \left[\frac{x}{2}\right].$$

**Proposition 5.**

$$\lim_{n\to\infty} [\ \Pi_S(2n) - \Pi_S(n)\ ] = \infty.$$

**Proof.**

According to the *Proposition 4* we have :

$$\Pi_S(n) \le n - \left[\sqrt{n+1}\right] < n < \Pi_S(2n) \Rightarrow$$

$$\Pi_S(2n) - \Pi_S(n) > \left[\sqrt{n+1}\right] \text{ and } \lim_{n\to\infty} \left[\sqrt{n+1}\right] = \infty.$$

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# ON THREE NUMERICAL FUNCTIONS

by

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In this paper we define the numerical functions $\varphi_S$, $\varphi_S^*$, $\omega_S$ and we prove some properties of these functions.

**1. Definition.** If S(n) is the Smarandache function, and (m, n) is the greatest common divisor of m and n, then the functions $\varphi_S$, $\varphi_S^*$ and $\omega_S$ are defined on the set **N*** of the positive integers, with values in the set N of all the non negative integers, such that:

$\varphi_S(x) = \text{Card}\{m \in \mathbf{N}^* / 0 < m \le x, (S(m), x) = 1\}$

$\varphi_S^*(x) = \text{Card}\{m \in \mathbf{N}^* / 0 < m \le x, (S(m), x) \ne 1\}$

$\omega_S(x) = \text{Card}\{m \in \mathbf{N}^* / 0 < m \le x, \text{ and } S(m) \text{ divides } x\}$.

From this definition it results that:

$$\varphi_S(x) + \varphi_S^*(x) = x \text{ and } \omega_S(x) \le \varphi_S^*(x) \quad (1)$$

for all $x \in \mathbf{N}^*$.

**2. Proposition.** For every prime number $p \in \mathbf{N}^*$ we have

$\varphi_S(p) = p - 1 = \varphi(p)$, $\varphi_S(p^2) = p^2 - p = \varphi(p^2)$

where $\varphi$ is Euler's totient function.

**Proof.** Of course, if p is a prime then for all integer a satisfying $0 < a \le p - 1$ we have $(S(a), p) = 1$, because $S(a) \le a$. So, if we note $M_1(x) = \{m \in \mathbf{N}^* / 0 < m \le p, (S(m), p) = 1\}$ then $a \in M_1(p)$.

At the same time, because $S(p) = p$, it results that $(S(p), p) = p \ne 1$ and so $p \notin M_1(p)$.

Then we have $\varphi_S(p) = p - 1 = \varphi(p)$.

The positive integers a, not greater than $p^2$ and not belonging to the set $M_1(p^2)$ are: $p, 2p, \ldots, (p-1)p, p^2$.

For $p = 2$ this assertion is evidently true, and if $p$ is an odd prime number then for all $h < p$ it results $S(h \cdot p) = p$.

Now, if $m < p^2$ and $m \neq hp$ then $(S(m), p^2) = 1$. Indeed, if for $m = q_1^{\alpha_1} \cdot q_2^{\alpha_2} \cdots q_r^{\alpha_r}$, $q_i \neq p$ we have $(S(m), p^2) \neq 1$, then it exists a divisor $q^\alpha$ of $m$ such that $S(m) = S(q^\alpha) = q(\alpha - i_\alpha)$, with $i_\alpha \in \left[0; \left[\frac{\alpha - 1}{q}\right]\right]$.

From $(q(\alpha - i_\alpha), p^2) \neq 1$ it results $(q(\alpha - i_\alpha), p) \neq 1$ and because $q \neq p$ it results $(\alpha - i_\alpha, p) \neq 1$, so $(\alpha - i_\alpha, p) = p$. But $p$ does not divide $\alpha - i_\alpha$ because $\alpha < p$.

Indeed, we have:

$$q^\alpha < p^2 \Leftrightarrow \alpha < 2\log_q p \leq 2 \cdot \frac{p}{2} = p$$

because we have:

$$\log_q p \leq \frac{p}{2} \text{ for } q \geq 2 \text{ and } p \geq 3.$$

So,

$$\varphi_S(p^2) = p^2 - \text{Card}\,\{1 \cdot p, 2 \cdot p, \dots, (p-1)p, p^2\} = p^2 - p = \varphi(p^2).$$

**3. Proposition.** For every $x \in \mathbf{N}^*$ we have:

$$\varphi_S(x) \leq x - \tau(x) + 1$$

where $\tau(x)$ is the number of the divisors of $x$.

**Proof.** From (1) it results that $\varphi_S(x) = x - \varphi_S{}^*(x)$, and of course, from the definition of $\varphi_S{}^*$ and $\tau$ it results $\varphi_S{}^*(x) \geq \tau(x) - 1$. Then $\varphi_S(x) \leq x - \tau(x) + 1$. Particularly, if $x$ is a prime then $\varphi_S(x) \leq x - 1$, because in this case $\tau(x) = 2$.

If $x$ is a composite number, it results that $\varphi_S(x) \leq x - 2$.

**4. Proposition.** If $p < q$ are two consecutive primes then :

$$\varphi_S(pq) = \varphi(pq).$$

**Proof.** Evidently, $\varphi(pq) = (p-1)(q-1)$ and

$$\varphi_S(pq) = \text{Card}\{m \in \mathbf{N}^* \,/\, 0 < m \leq pq,\ (S(m), pq) = 1\}.$$

Because $p$ and $q$ are consecutive primes and $p < q$ it results that the multiples of $p$ and $q$ which are not greater than $pq$ are exactly given by the set:

$$M = \{p, 2p, \dots, p^2, (p+1)p, \dots, (q-1)p, qp, q, 2q, \dots, (p-1)q\}.$$

These are in number of $p + q - 1$.

Evidently, $(S(m), pq) \neq 1$ for $m \in \{p, 2p, \ldots, (p-1)p, p^2, q, 2q, \ldots, (p-1)q\}$.

Let us calculate $S(m)$ for $m \in \{(p+1)p, (p+2)p, \ldots, (q-1)p\}$.

Evidently, $(p+i, p) = 1$ for $1 \leq i \leq q - p - 1$, and so $[p+i, p] = p(p+i)$.

It results that $S(p(p+i) = S([p, p+i]) = \max\{S(p), S(p+i)\} = S(p)$.
Indeed, to estimate $S(p+i)$ let $p + i = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_h^{\alpha_h} < q < 2p$.
Then $p_1^{\alpha_1} < p, p_2^{\alpha_2} < p, \ldots, p_h^{\alpha_h} < p$.
It results that:
$S(p+i) = S(p_j^{\alpha_j}) < S(p)$, for some $j = \overline{1,h}$.
It results that:

$(S(p(p+i), pq) = (p, pq) = p \neq 1$.

In the following we shall prove that if $0 < m \leq pq$ and m is not a multiple of p or q then $(S(m), pq) = 1$.

It is said that if $m < p^2$ is not a multiple of p then $(S(m), p) = 1$.

If $m \leq q^2$ is not a multiple of q then it results also $(S(m), q) = 1$.

Now, if $m < p^2$ (and of course $m < q^2$) is not a multiple either of p and q then from $(S(m), p) = 1$ and $(S(m), q) = 1$ it results $(S(m), pq) = 1$.

Finally, for $p^2 < m < pq < q^2$, with m not a multiple either of p and q, if the decomposition of m into primes is $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_S^{\alpha_S}$ then $S(m) = S(p_k^{\alpha_k}) < S(p) = p$ so $(S(m), p) = 1$.

Analogously, $(S(m), q) = 1$, and so $(S(m), pq) = 1$.

Consequently,

$$\varphi_S(pq) = pq - p - q + 1 = \varphi_S(pq).$$

**5. Proposition**

**(i)** If $p > 2$ is a prime number then $\omega_S(p) = 2$, $\omega_S(p^2) = p$.

**(ii)** If x is a composite number then $\omega_S(x) \geq 3$.

**Proof.** From the definition of the function $\omega_S$ it results that $\omega_S(p) = 2$.

If $1 \le m \le p^2$, from the condition that S(m) divides $p^2$ it results m = 1 or m = kp, with

$k \le p - 1$, so :

$m \in \{1, p, 2p, \ldots, (p-1)p\}$ and $\omega_S(p^2) = p$.

If x is a composite number, let p be one of its prime divisors.

Then, of course, $1, p, 2p \in \{m \,/\, 0 < m \le x\}$.

If p > 3 then :

S(1) = 1 divides x, S(p) = p divides x and S(2p) = S(p) = p divides x.

It rezults $\omega_S(x) \ge 3$.

If $x = 2^\alpha$, with $\alpha \ge 2$ then :

S(1) = 1 divides x, S(2) = 2 divides x and S(4) = 4 divides x ,

so we have also $\omega_S(x) \ge 3$.

**6. Proposition.** For every positive integer x we have :

$\omega_S(x) \le x - \varphi(x) + 1$. (2)

**Proof.** We have $\varphi(x) = x$ - Card A, when

$A = \{m \,/\, 0 < m \le x, (m, x) \ne 1\}$.

Evidently, the inequality (2) is valid for all the prime numbers.

If x is a composite number it results that at least a proper divisor of m is also a divisor of S(m) and of x. So $(m, x) \ne 1$ and consequently $m \in A$.

So, $\{m \,/\, 0 < m \le x, S(m) \text{ divides } x\} \subset A \cup \{1\}$ and it results that :

Card $\{m \,/\, 0 < m \le x, S(m) \text{ divides } x\} \le$ Card A - 1, or

$\omega_S(x) \le 1 +$ Card A,

and from this it results (2).

**7. Proposition.** The equation $\omega_S(x) = \omega_S(x+1)$ has not a solution between the prime numbers.

**Proof.** Indeed, if x is a prime then $\omega_S(x) = 2$ and because x + 1 is a composite number it results $\omega_S(x+1) \ge 3$.

Let us observe that the above equation has solutions between the primes. For instance, $\omega_S(35) = \omega_S(36) = 11$.

**8. Proposition.** The function $\varphi_S(x)$ has all the primes as local maximal points.

**Proof.** We have $\varphi_S(p) = p - 1$, $\varphi_S(p - 1) \leq p - 3 < \varphi_S(p)$ and $\varphi_S(p + 1) \leq \varphi_S(p)$, because $p + 1$ being a composite number has at least two divisors.

Let us mention now the following unsolved problems:

**(UP$_1$)** There exists $x \in \mathbf{N}^*$ such that $\varphi_S(x) < \varphi(x)$.

**(UP$_2$)** For all $x \in \mathbf{N}^*$ is valid the inequality

$$\omega_S(x) \geq \tau(x)$$

where $\tau(x)$ is the number of the divisors of x.

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# THE MONOTONY OF SMARANDACHE FUNCTIONS OF FIRST KIND

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Smarandache functions of first kind are defined in [1] thus:

$$S_n : N^* \to N^*, \quad S_1(k) = 1 \text{ and } S_n(k) = \max_{1\le j\le r}\{S_{p_j}(i_j k)\},$$

where $n = p_1^{i_1} \cdot p_2^{i_2} \cdots p_r^{i_r}$ and $S_{p_j}$ are functions defined in [4].

They $\Sigma_1$ - standardise $(N^*,+)$ in $(N^*,\le,+)$ in the sense that

$$\Sigma_1: \quad \max\{S_n(a), S_n(b)\} \le S_n(a+b) \le S_n(a) + S_n(b)$$

for every $a,b \in N^*$ and $\Sigma_2$ - standardise $(N^*,+)$ in $(N^*,\le,\cdot)$ by

$$\Sigma_2: \quad \max\{S_n(a), S_n(b)\} \le S_n(a+b) \le S_n(a)\cdot S_n(b), \text{ for every } a,b \in N^*$$

In [2] it is prooved that the functions $S_n$ are increasing and the sequence $\{S_{p^i}\}_{i\in N^*}$ is also increasing. It is also proved that if $p,q$ are prime numbers, then

$$p\cdot i < q \Rightarrow S_{p^i} < S_q \text{ and } i < q \Rightarrow S_i < S_q,$$

where $i \in N^*$.

It would be used in this paper the formula

$$S_p(k) = p(k - i_k), \text{ for same } i_k \text{ satisfying } 0 \le i_k \le \left[\frac{k-1}{p}\right], \text{ (see [3])} \tag{1}$$

**1. Proposition.** *Let $p$ be a prime number and $k_1, k_2 \in N^*$. If $k_1 < k_2$ then $i_{k_1} \le i_{k_2}$, where $i_{k_1}, i_{k_2}$ are defined by* (1).

*Proof.* It is known that $S_p : \mathrm{N}^* \to \mathrm{N}^*$ and $S_p(k) = pk$ for $k \le p$. If $S_p(k) = mp^\alpha$ with $m, \alpha \in \mathrm{N}^*, (m,p) = 1$, there exist $\alpha$ consecutive numbers:

$n, n+1, \ldots, n+\alpha-1$ so that
$k \in \{n, n+1, \ldots, n+\alpha-1\}$ and
$S_p(n) = S_p(n+1) = \cdots = S(n+\alpha-1)$,

this means that $S_p$ is stationed the $\alpha-1$ steps $(k \to k+1)$.

If $k_1 < k_2$ and $S_p(k_1) = S_p(k_2)$, because $S_p(k_1) = p(k_1 - ik_1)$, $S_p(k_2) = p(k_2 - ik_2)$ it results $i_{k_1} < i_{k_2}$.

If $k_1 < k_2$ and $S_p(k_1) < S_p(k_2)$, it is easy to see that we can write:

$$i_{k_1} = \beta_1 + \sum_{\substack{\alpha \\ mp^\alpha < S_p(k_1)}} (\alpha - 1), \quad \text{where} \quad \beta_1 = 0 \text{ for } S_p(k_1) \neq mp^\alpha, \quad \text{if } S_p(k_1) = mp^\alpha$$

then $\beta_1 \in \{0,1,2,\dots,\alpha-1\}$
and

$$i_{k_2} = \beta_2 + \sum_{\substack{\alpha \\ mp^\alpha < S_p(k_2)}} (\alpha - 1), \quad \text{where} \quad \beta_2 = 0 \text{ for } S_p(k_2) \neq mp^\alpha, \text{ if } S_p(k_2) = mp^\alpha \text{ then}$$

$\beta_2 \in \{0,1,2,\dots,\alpha-1\}$.

Now is obviously that $k_1 < k_2$ and $S_p(k_1) < S_p(k_2) \Rightarrow i_{k_1} \le i_{k_2}$. We note that, for $k_1 < k_2$, $i_{k_1} = i_{k_2}$ iff $S_p(k_1) < S_p(k_2)$ and $\{mp^\alpha \mid \alpha > 1 \text{ and } mp^\alpha \le S_p(k_1)\} = \{mp^\alpha \mid \alpha > 1 \text{ and } mp^\alpha < S_p(k_2)\}$

**2. Proposition.** *If $p$ is a prime number and $p \ge 5$, then $S_p > S_{p-1}$ and $S_p > S_{p+1}$.*

*Proof.* Because $p-1 < p$ it results that $S_{p-1} < S_p$. Of course $p+1$ is even and so:

(i) if $p+1 = 2^i$, then $i > 2$ and because $2i < 2^i - 1 = p$ we have $S_{p+1} < S_p$.

(ii) if $p+1 \neq 2^i$, let $p+1 = p_1^{i_1} \cdot p_2^{i_2} \cdots p_r^{i_r}$, then $S_{p+1}(k) = \max_{1\le j\le r}\{S_{p_j^{i_j}}(k)\} = S_{p_m^{i_m}}(k) =$
$= S_{p_m}(i_m \cdot k)$.

Because $p_m \cdot i_m \le p_m^{i_m} \le \dfrac{p+1}{2} < p$ it results that $S_{p_m^{i_m}}(k) < S_p(k)$ for $k \in N^*$, so that $S_{p+1} < S_p$.

**3. Proposition.** *Let $p,q$ be prime numbers and the sequences of functions*

$$\{S_{p^i}\}_{i\in N^*}, \quad \{S_{q^j}\}_{j\in N^*}$$

If $p < q$ and $i \le j$, then $S_{p^i} < S_{q^j}$.

*Proof.* Evidently, if $p < q$ and $i \le j$, then for every $k \in N^*$

$$S_{p^i}(k) \le S_{p^j}(k) < S_{q^j}(k)$$

so,

$$S_{p^i} < S_{q^j}$$

**4. Definition.** *Let $p,q$ be prime numbers. We consider a function $S_{q^j}$, a sequence of functions $\{S_{p^i}\}_{i\in N^*}$, and we note:*

$$i_{(j)} = \max_i \left\{ i \mid S_{p^i} < S_{q^j} \right\}$$

$$i^{(j)} = \min_i \left\{ i \middle| S_{q^j} < S_{p^i} \right\},$$

*then* $\{k \in N | i_{(j)} < k < i^{(j)}\} = \Delta_{p^j(q^j)} = \Delta_{i(j)}$ *defines the interference zone of the function* $S_{q^j}$ *with the sequence* $\left\{S_{p^i}\right\}_{i \in N^*}$.

5. ***Remarque.***

a) If $S_{q^j} < S_{p^i}$ for $i \in \mathbf{N}^*$, then now exists $i_{(j)}$ and $i^{(j)} = 1$, and we say that $S_{q^j}$ is separately of the sequence of functions $\left\{S_{p^i}\right\}_{i \in N^*}$.

b) If there exist $k \in \mathbf{N}^*$ so that $S_{p^k} < S_{q^j} < S_{p^{k+1}}$, then $\Delta_{p^j(q^j)} = \varnothing$ and say that the function $S_{q^j}$ does not interfere with the sequence of functions $\left\{S_{p^i}\right\}_{i \in N^*}$.

6. **Definition.** *The sequence* $\{x_n\}_{n \in N^*}$ *is generaly increasing if*

$$\forall n \in N^* \; \exists m_0 \in N^* \text{ so that } x_m \geq x_n \text{ for } m \geq m_0.$$

7. ***Remarque.*** If the sequence $\{x_n\}_{n \in N^*}$ with $x_n \geq 0$ is generaly increasing and boundled, then every subsequence is generaly increasing and boundled.

8. **Proposition.** *The sequence* $\{S_n(k)\}_{n \in N^*}$, *where* $k \in N^*$, *is in generaly increasing and boundled.*

*Proof.* Because $S_n(k) = S_{n^k}(1)$, it results that $\{S_n(k)\}_{n \in N^*}$ is a subsequence of $\{S_m(1)\}_{m \in N^*}$.

The sequence $\{S_m(1)\}_{m \in N^*}$ is generaly increasing and boundled because:

$$\forall m \in N^* \; \exists t_0 = m! \text{ so that } \forall t \geq t_0 \; S_t(1) \geq S_{t_0}(1) = m \geq S_m(1).$$

From the remarque 7 it results that the sequence $\{S_n(k)\}_{n \in N^*}$ is generaly increasing boundled.

9. **Proposition.** *The sequence of functions* $\{S_n\}_{n \in N^*}$ *is generaly increasing boundled.*

*Proof.* Obviously, the zone of interference of the function $S_m$ with $\{S_n\}_{n \in N^*}$ is the set

$$\Delta_{n(m)} = \{k \in N^* | n_{(m)} < k < n^{(m)}\} \text{ where}$$

$$n_{(m)} = \max\{n \in N^* | S_n < S_m\}$$

$$n^{(m)} = \min\{n \in N^* | S_m < S_n\}.$$

The interference zone $\Delta_{n(m)}$ is nonemty because $S_m \in \Delta_{n(m)}$ and finite for $S_1 \le S_m \le S_p$, where $p$ is one prime number greater than $m$.

Because $\{S_n(1)\}$ is generaly increasing it results:

$$\forall m \in N^* \; \exists t_0 \in N^* \text{ so that } S_t(1) \ge S_m(1) \text{ for } \forall t \ge t_0.$$

For $r_0 = t_0 + n^{(m)}$ we have

$$S_r \ge S_m \ge S_m(1) \text{ for } \forall r \ge r_0,$$

so that $\{S_n\}_{n\in N^*}$ is generaly increasing boundled.

**10. *Remarque.***

a) For $n = p_1^{i_1} \cdot p_2^{i_2} \cdots p_r^{i_r}$ are posible the following cases:

1) $\exists\, k \in \{1,2,\ldots,r\}$ so that

$$S_{p_j^{i_j}} \le S_{p_k^{i_k}} \text{ for } j \in \{1,2,\ldots,r\},$$

then $S_n = S_{p_k^{i_k}}$ and $p_k^{i_k}$ is named the dominant factor for $n$.

2) $\exists\, k_1, k_2, \ldots, k_m \in \{1,2,\ldots,r\}$ so that :

$\forall t \in \overline{1,m} \quad \exists q_t \in N^*$ so that $S_n(q_t) = S_{p_{k_t}^{i_{k_t}}}(q_t)$ and

$$\forall\, l \in N^* \quad S_n(l) = \max_{1\le t\le m}\left\{ S_{p_{k_t}^{i_{k_t}}}(l) \right\}.$$

We shall name $\{p_{k_t}^{i_{k_t}} \,|\, t \in \overline{1,m}\}$ the active factors, the others wold be name passive factors for $n$.

b) We consider

$$N_{p_1 p_2} = \{ n = p_1^{i_1} \cdot p_2^{i_2} \,|\, i_1, i_2 \in N^* \}, \text{ where } p_1 < p_2 \text{ are prime numbers.}$$

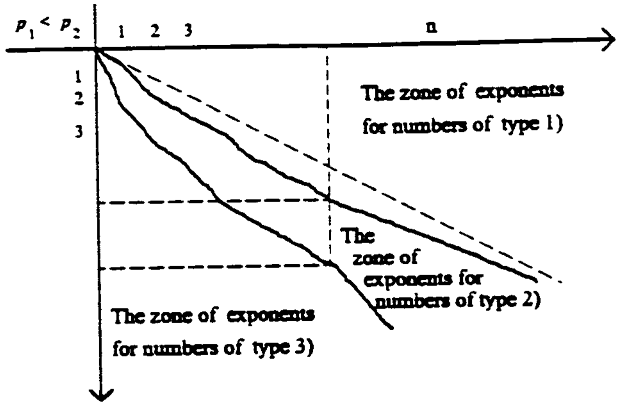
For $n \in N_{p_1 p_2}$ appear the following situations:

1) $i_1 \in (0, i_1^{(i_2)}]$, this means that $p_1^{i_1}$ is a pasive factor and $p_2^{i_2}$ is an active factor.

2) $i_1 \in (i_{1(i_2)}, i_1^{(i_2)})$ this means that $p_1^{i_1}$ and $p_2^{i_2}$ are active factors.

3) $i_1 \in [i_1^{(i_2)}, \infty)$ this means that $p_1^{i_1}$ is a active factor and $p_2^{i_2}$ is a pasive factor.

For $p_1 < p_2$ the repartion of exponents is represently in following scheme:




For numbers of type 2) $i_1 \in (i_{1(i_2)}, i_1^{(i_2)})$ and $i_2 \in (i_{2(i_1)}, i_2^{(i_1)})$

c) I consider that

$$N_{p_1 p_2 p_3} = \{n = p_1^{i_1} \cdot p_2^{i_2} \cdot p_3^{i_3} | i_1, i_2, i_3 \in N^*\},$$

where $p_1 < p_2 < p_3$ are prime numbers.

Exist the following situations:

1) $n \in N^{p_j}, j = 1,2,3$ this means that $p_j^{i_j}$ is active factor.

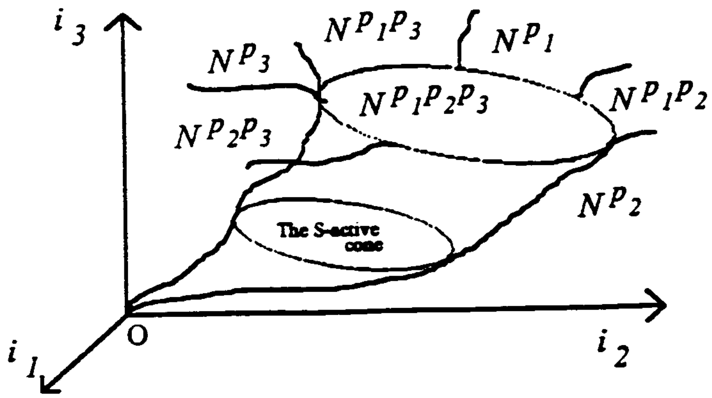
2) $n \in N^{p_j p_k}, j \neq k;\ j,k \in \{1,2,3\}$, this means that $p_j^{i_j}, p_k^{i_k}$ are active factors.

3) $n \in N^{p_1 p_2 p_3}$, this means that $p_1^{i_1}, p_2^{i_2}, p_3^{i_3}$ are active factors. $N^{p_1 p_2 p_3}$ is named the S-active cone for $N_{p_1 p_2 p_3}$.

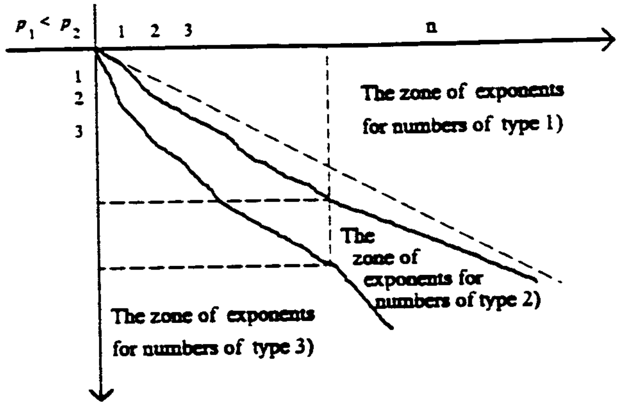
Obviously

$$N^{p_1 p_2 p_3} = \{n = p_1^{i_1} p_2^{i_2} p_3^{i_3} | i_1, i_2, i_3 \in N^* \text{ and } i_k \in (i_{k(i_j)}, i_k^{(i_j)}) \text{ where } j \neq k; j,k \in \{1,2,3\}\}.$$

The repartision of exponents is represented in the following scheme:



For $p_1 < p_2$ the repartion of exponents is representy in following scheme:




For numbers of type 2) $i_1 \in (i_{1(i_2)}, i_1^{(i_2)})$ and $i_2 \in (i_{2(i_1)}, i_2^{(i_1)})$

c) I consider that

$$N_{p_1 p_2 p_3} = \{n = p_1^{i_1} \cdot p_2^{i_2} \cdot p_3^{i_3} \mid i_1, i_2, i_3 \in N^*\},$$

where $p_1 < p_2 < p_3$ are prime numbers.

Exist the following situations:

1) $n \in N^{p_j}, j = 1,2,3$ this means that $p_j^{i_j}$ is active factor.

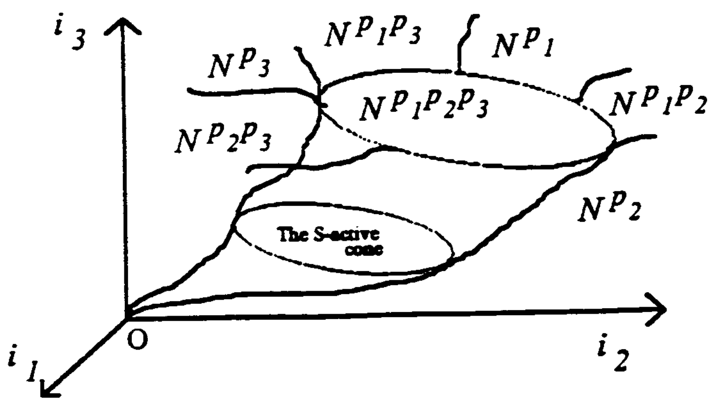
2) $n \in N^{p_j p_k}, j \neq k;\ j,k \in \{1,2,3\}$, this means that $p_j^{i_j}, p_k^{i_k}$ are active factors.

3) $n \in N^{p_1 p_2 p_3}$, this means that $p_1^{i_1}, p_2^{i_2}, p_3^{i_3}$ are active factors. $N^{p_1 p_2 p_3}$ is named the S-active cone for $N_{p_1 p_2 p_3}$.

Obviously

$$N^{p_1 p_2 p_3} = \{n = p_1^{i_1} p_2^{i_2} p_3^{i_3} \mid i_1, i_2, i_3 \in N^* \text{ and } i_k \in (i_{k(i_j)}, i_k^{(i_j)}) \text{ where } j \neq k; j,k \in \{1,2,3\}\}.$$

The repartision of exponents is represented in the following scheme:



d) Generaly, I consider $N_{p_1 p_2 \cdots p_r} = \{n = p_1^{i_1} \cdot p_2^{i_2} \cdot \cdots \cdot p_r^{i_r} \mid i_1, i_2, \ldots, i_r \in N^*\}$, where $p_1 < p_2 < \cdots < p_r$ are prime numbers.

On $N_{p_1 p_2 \cdots p_r}$ exist the following relation of equivalence:

$$n \rho m \Leftrightarrow n \text{ and } m \text{ have the same active factors.}$$

This have the following clases:

- $N^{p_{j_1}}$, where $j_1 \in \{1,2,\ldots,r\}$.

$n \in N^{p_{j_1}} \Leftrightarrow n$ hase only $p_{j_1}^{i_{j_1}}$ active factor

- $N^{p_{j_1} p_{j_2}}$, where $j_1 \neq j_2$ and $j_1, j_2 \in \{1,2,\ldots,r\}$.

$n \in N^{p_{j_1} p_{j_2}} \Leftrightarrow n$ has only $p_{j_1}^{i_{j_1}}$, $p_{j_2}^{i_{j_2}}$ active factors.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

$N^{p_1 p_2 \cdots p_r}$ wich is named S-active cone.

$N^{p_1 p_2 \cdots p_r} = \{n \in N_{p_1 p_2 \cdots p_r} \mid n \text{ has } p_1^{i_1}, p_2^{i_2}, \ldots, p_r^{i_r} \text{ active factors}\}$.

Obviously, if $n \in N^{p_1 p_2 \cdots p_r}$, then $i_k \in (i_{k(i_j)}, i_k^{(i_j)})$ with $k \neq j$ and $k, j \in \{1,2,\ldots,r\}$.

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# THE SMARANDACHE NEAR-TO-PRIMORIAL (S.N.T.P.) FUNCTION

by

M. R. Mudge

### *Definition A.*

The PRIMORIAL Function, $p^*$, of a prime number, p, is defined be the product of the prime numbers less than or equal to p. e.g. $7^* = 2 \cdot 3 \cdot 5 \cdot 7 = 210$ similarly $11^* = 2310$. A number, q, is said to be near to prime if and only if *either* q+1 *or* q-1 are primes it is said to be the mean-of-a-prime-pair if and only if *both* q+1 *and* q-1 are prime.

p such that $p^*$ is near to prime: 2, 7, 13, 37, 41, 53, 59, 67, 71, 79, 83, 89, ...

p such that $p^*$ is mean-of-a-prime-pair: 3, 5, 11, 31, ...

TABLE I

| p | 2 | 3 | 5 | 7 | 11 | 13 |
|---|---|---|---|---|---|---|
| $p^*-1$ | 1 | 5 | 29p | $209=11 \cdot 19$ | 2309p | 30029p |
| $p^*$ | 2 | 6 | 30 | 210 | 2310 | 30030 |
| $p^*+1$ | 3 | 7 | 31p | 211p | 2311p | $30031=59 \cdot 509$ |

### *Definition B.*

The SMARANDACHE Near-To-Primorial Function, SPr(n), is defined as the smallest prime p such that either $p^*$ or $p^* \pm 1$ is divisible by n.

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | ...59... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| SPr(n) | 2 | 2 | 2 | 5 | 3 | 3 | 3 | 5 | ? | 5 | 11 | 13 |

Questions relating to this function include:

1. Is SPr(n) defined for all positive integers n ?
2. What is the distribution of values of SPr(n) ?
3. Is this problem fundamentally altereted by replacing $p^* \pm 1$ by $p^* \pm 3, 5, \ldots$

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# A Note on the Smarandache Near-To-Primorial Function

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In a brief paper passed on to the author[1], Michael R. Mudge used the definition of the Primorial function:

**Definition:** For p any prime, the Primorial function of p, p* is the product of all prime numbers less than or equal to p.

Examples:

$3^* = 2 * 3 = 6$
$11^* = 2 * 3 * 5 * 7 * 11 = 2310$

To define the Smarandache Near-To-Primorial Function SPr(n)

**Definition:** For n a positive integer, the Smarandache Near-To-Primorial Function SPr(n) is the smallest prime p such that either p* or p* + 1 or p* - 1 is divisible by n.

A table of initial values is also given

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 . . . 59 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| SPr(n) | 2 | 2 | 2 | 5 | 3 | 3 | 3 | 5 | ? | 5 | 11 . . . 13 |

and the following questions posed:

1) Is SPr(n) defined for all positive integers n?
2) What is the distribution of values of SPr(n)?
3) Is this problem fundamentally altered by replacing $p^* \pm 1$ by $p^* \pm k$ for k = 3,5,...?

The purpose of this paper is to address these questions.

We start with a simple but important result that is presented in the form of a lemma.

**Lemma 1:** If the prime factorization of n contains more than one instance of a prime as a factor, then n cannot divide q* for q any prime.

**Proof:** Suppose that n contains at least one prime factor to a power greater than one, for reference purposes, call that prime p1. The list of prime factors of n contains a largest prime and we can call that prime p2. If we choose another arbitrary prime q, there are two cases to consider.

Case 1: $q < p2$. Then p2 cannot divide $q^*$, as $q^*$ contains no instances of p2 by definition.

Case 2: $q \geq p2$. In this case, each prime factor of n will divide $q^*$, but since p1 appears only once in $q^*$, $p1^2$ cannot divide $q^*$. Therefore, n cannot divide $q^*$ as well. □

We are now in a position to answer the first question.

**Theorem 1:** If n contains more than one instance of 2 as a factor, then SPr(n) does not exist.

**Proof:** Choose n to be a number having more than one instance of 2 as a factor. By lemma 1, there is no prime q such that n divides $q^*$. Furthermore, since 2 is a prime, $q^*$ is always even. Therefore, $q^* \pm 1$ is always odd and n cannot evenly divide it. □

The negative answer to the first question also points out two errors in the Mudge table. SPr(4) and SPr(8) do not exist, and an inspection of the given values verifies this. The Primorial of 5 is 2*3*5 = 30 and no element in the set { 29,30,31 } is evenly divisible by 4.

By definition, the range of SPr(n) is a set of prime numbers. The obvious question is then whether the range of SPr(n) is in fact the set of all prime numbers, and we state the answer as a theorem.

**Theorem 2:** The range of SPr(n) is the set of all prime numbers.

**Proof:** The first few values are by inspection.

SPr(1) = 2, SPr(5) = 3, SPr(10) = 5

Choose an arbitrary prime $p > 5$ and construct the number $p^* - 1$. Obviously, $p^* - 1$ divides $p^* - 1$. It is also clear that there is no prime $q < p$ such that $q^*$, $q^* + 1$ or $q^* - 1$ is divisible by $p^* - 1$. Therefore, $SPr(p^* - 1) = p$ and p is in the range of SPr(n). □

Which answers the second question posed by M. Mudge.

It is easy to establish an algorithmic process to determine if SPr(n) is defined for values of n containing more than one instance of a prime greater than 2.

The first step is to prove another lemma.

**Lemma 2:** If n contains a prime p that appears more than once as a factor of n, and q is any prime $q \geq p$, then n does not divide $q^* \pm 1$.

**Proof:** Let n, p and q have the stated properties. Clearly, p divides $q^*$ and since q is greater than 1, p cannot divide $q^* \pm 1$, forcing the conclusion that n cannot divide $q^* \pm 1$ as well. Combining this with lemma 1 gives the desired result. □

**Corollary:** If n contains some prime p more than once as a factor and SPr(n) exists, then the prime q such that n divides $q^* \pm 1$ must be less than p.

**Proof:** Clear. □

The next lemma deals with some of the instances where SPr(n) is defined.

**Lemma 3:** If $n = p_1 p_2 \ldots p_k$, where $k \geq 1$ and all $p_i$ are primes, then SPr(n) is defined.

**Proof:** Let q denote the largest prime factor of n. By definition, $q^*$ contains one instance of all primes less than or equal to q, so n must divide $q^*$. Given the existence of one such number, there must also be a minimal one. □

Combining all previous results, we can create a simple algorithm that can be used to determine if SPr(n) exists for any positive integer n.

Input: A positive integer n.
Output: Yes, if SPr(n) exists, No otherwise.

Step 1: Factor n into prime factors, $p_1 p_2 \ldots p_k$.
Step 2: If all primes appear to the first power, terminate with the message "Yes".
Step 3: If 2 appears to a power greater than 1, terminate with the message "No".
Step4: Set q = 2, the smallest prime.
Step 5: Compute $q^* + 1$ and $q^* - 1$.
Step 6: If n divides $q^* + 1$ or $q^* - 1$, terminate with the message "Yes".
Step 7: Increment q to the next largest prime.
Step 8: If $q \geq p$, terminate with the message "No".
Step 9: Goto step 5.

And this algorithm can be used to resolve the question mark in the Mudge table. Since 9 does not divide $2^* \pm 1$, SPr(9) is not defined. Furthermore, 3 to any power greater than 2 also cannot divide $2^* \pm 1$, so the conclusion is stronger in that SPr(n) is not defined for n any power of 3 greater than 3.

Note that modifications of this algorithm could be made so that it also returns the value of SPr(n) when defined.

These conclusions can be used to partially answer the third question. The conclusion of lemma 3 concerning all prime factors to the first power is unaffected. However, if $q \geq 3$ and q prime, then $q^* \pm 3$ is also divisible by 3, making solutions possible for higher powers of 3. Such results do indeed occur, as

$$3^* + 3 = 9$$

so that the modified $SPr(9) = 9$.

## Reference

1. **The Smarandache Near-To-Primorial Function**, personal correspondence by Michael R. Mudge.

# PRIMES BETWEEN CONSECUTIVE SMARANDACHE NUMBERS

by

G. Suggett

I assume that the range between S(n) and S(n+1) should be interpreted as including the endpoints ? If one is looking for cases in which there are no primes in the open interval between the two consecutive values, then the range of exceptions is much larger, including n = 1, 2, 3, 4, 5, 9, 14, 15, ... Using the closed interval gives a much smaller list of exceptions, starting, as you state, with n = 224. I have confirmed that the next value is n = 2057, but to go further on a systematic basis would be far too time-consuming. However, taking the hint about prime pairs, I have found the following:

Associated with the prime pair (101, 103): 265225, 265226
Associated with the prime pair (107, 109): 67697937, 67697938
Associated with the prime pair (149, 151): 843637, 843638
Associated with the prime pair (461, 463): 24652435, 24652436
Associated with the prime pair (521, 523): 35558770, 35558771
Associated with the prime pair (569, 571): 46297822, 46297823
Associated with the prime pair (821, 823): 138852445, 138852446
Associated with the prime pair (857, 859): 157906534, 157906535
Associated with the prime pair (881, 883): 171531580, 171531581
Associated with the prime pair (1061, 1063): 299441785, 299441786
Associated with the prime pair (1301, 1303): 551787925, 551787926
Associated with the prime pair (1697, 1699): 1223918824, 1223918825
Associated with the prime pair (1721, 1723): 1276553470, 1276553471
Associated with the prime pair (1787, 1789): 5108793239997, 5108793239998
Associated with the prime pair (1871, 1873): 6138710055036, 6138710055037
Associated with the prime pair (1877, 1879): 1655870629, 1655870630
Associated with the prime pair (1949, 1951): 1853717287, 1853717288
Associated with the prime pair (1997, 1999): 1994004499, 1994004500
Associated with the prime pair (2081, 2083): 2256222280, 2256222281
Associated with the prime pair (2111, 2113): 9945866761776, 9945866761777
Associated with the prime pair (2237, 2239): 2802334639, 2802334640
Associated with the prime pair (2381, 2383): 3378819955, 3378819956

Associated with the prime pair (2657, 2659): 4694666584, 4694666585
Associated with the prime pair (2729, 2731): 5086602202, 5086602203
Associated with the prime pair (2801, 2803): 5499766300, 5499766301
Associated with the prime pair (3251, 3253): 55912033969191, 55912033969192
Associated with the prime pair (3257, 3259): 8645559934, 8645559935
Associated with the prime pair (3461, 3463): 10373399185, 10373399186
Associated with the prime pair (3557, 3559): 11260501609, 11260501610
Associated with the prime pair (3581, 3583): 11489910655, 11489910656
Associated with the prime pair (3671, 3673): 90891127331586, 90891127331587
Associated with the prime pair (3917, 3919): 15036031219, 15036031220
Associated with the prime pair (3929, 3931): 15174611302, 15174611303
Associated with the prime pair (4001, 4003): 16024009000, 16024009001
Associated with the prime pair (4127, 4129): 145169740720152, 145169740720153
Associated with the prime pair (4217, 4219): 18761158894, 18761158895
Associated with the prime pair (4241, 4243): 19083231940, 19083231941
Associated with the prime pair (4421, 4423): 21617036545, 21617036546
Associated with the prime pair (4517, 4519): 23055716569, 23055716570
Associated with the prime pair (4547, 4549): 213896677247667, 213896677247668
Associated with the prime pair (4649, 4651): 25136152762, 25136152763
Associated with the prime pair (4721, 4723): 26321940220, 26321940221
Associated with the prime pair (5009, 5011): 31437871492, 31437871493
Associated with the prime pair (5021, 5023): 31664313895, 31664313896
Associated with the prime pair (5099, 5101): 338226861243825, 338226861243826
Associated with the prime pair (6089, 6091): 56466627682, 56466627683
Associated with the prime pair (6197, 6199): 59524353949, 59524353950
Associated with the prime pair (6569, 6571): 70898343322, 70898343323
Associated with the prime pair (6701, 6703): 75258100075, 75258100076
Associated with the prime pair (6869, 6871): 81060670597, 81060670598
Associated with the prime pair (7457, 7459): 103706773384, 103706773385
Associated with the prime pair (7589, 7591): 109311364057, 109311364058
Associated with the prime pair (7757, 7759): 116731835059, 116731835060

and so on. I am reaching the limits of my computational power, but with no obvious end in sight to the list. Do you have a copy of Radu's proof that the set is finite ? Does it give an upper bound on the values in the set ? I am intrigued.

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# Introducing the SMARANDACHE-KUREPA and SMARANDACHE-WAGSTAFF Functions

by

M. R. Mudge

*Definition A.*

The left-factorial function is defined by D.Kurepa thus:

$$!n = 0! + 1! + 2! + 3! + \dots + (n-1)!$$

whilst S.S.Wagstaff prefers:

$$B_n = !(n+1) - 1 = 1! + 2! + 3! + \dots + n!$$

The following properties should be observed:

(i) !n is only divisible by n when n = 2.
(ii) 3 is a factor of $B_n$ if n is greater than 1.
(iii) 9 is a factor of $B_n$ if n is greater than 4.
(iv) 99 is a factor of $B_n$ if n is greater than 9.

There are no other such cases of divisibility ob $B_n$ for n less than a thousand.

The tabulated values of these two functions together with their prime factors begin:

TABLE I.

| n | !n | $B_n$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 2 | 3 |
| 3 | 4=2·2 | 9=3·3 |
| 4 | 10=2·5 | 33=3·11 |
| 5 | 34=2·17 | 153=3·3·17 |
| 6 | 154=2·7·11 | 873=3·3·97 |
| 7 | 8742·19·23 | 5913=3·3·3·3·73 |
| 8 | 5914=2·2957 | 46233=3·3·11·467 |
| 9 | 46234=2·23117 | 409113=3·3·131·347 |
| 10 | 409114=2·204557 | |

"Intuitive Thought": There appear to be a disproportionate (unexpectedly high) number of large primes in this table?

***Definition B.***

For prime p not equal to 3 define the SMARANDACHE-KUREPA Function, SK(p), as the smallest integer such that !SK(p) is divisible by p.
For prime p not equal to 2 or 5 define the SMARANDACHE-WAGSTAFF Function, SW(p), as the smallest integer such that $B_{SW(p)}$ is divisible by p.

The tabulation of these two functions begins:

TABLE II.

| p | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 131 |
|---|---|---|---|---|---|---|---|---|---|---|
| SK(p) | 2 | * | 4 | 6 | 6 | ? | 5 | 7 | 7 | ? |
| SW(p) | * | 2 | * | ? | 4 | ? | 5 | ? | ? | 9 |

Where the entry * denotes that the value is not defined and the entry ? denotes not avaible from TABLE I above.

Some unanswered questions:

1. Are there other (*) - entries i.e. undefined values in the above table.
2. What is the distribution function of integers in both SK(p), SW(p) and their union ?
3. When, in general, is SK(p) = SW(p) ?

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# ABOUT THE SMARANDACHE COMPLEMENTARY CUBIC FUNCTION

by Marcela Popescu and Mariana Nicolescu

DEFINITION. *Let* $g: N^* \to N^*$ *be a numerical function defined by* $g(n) = k$, *where* k *is the smallest natural number such that* nk *is a perfect cube:* $nk = s^3, s \in N^*$.

*Examples*: 1) $g(7)=49$ because 49 is the smallest natural number such that $7 \cdot 49 = 7 \cdot 7^2 = 7^3$;

2) $g(12) = 18$ because 18 is the smallest natural number such that $12 \cdot 18 = (2^2 \cdot 3) \cdot (2 \cdot 3^2) = 2^3 \cdot 3^3 = (2 \cdot 3)^3$;

3) $g(27) = g(3^3) = 1$;

4) $g(54) = g(2 \cdot 3^3) = 2^2 = g(2)$.

PROPERTY 1. *For every* $n \in N^*$, $g(n^3) = 1$ *and for every prime* $p$ *we have* $g(p) = p^2$.

PROPERTY 2. *Let n be a composite natural number and* $n = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots \cdot p_{i_r}^{\alpha_{i_r}}$, $0 < p_{i_1} < p_{i_2} < \cdots < p_{i_r}$, $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_r} \in N^*$ *its prime factorization. Then* $g(n) = p_{i_1}^{d(\overline{\alpha_{i_1}})} \cdot p_{i_2}^{d(\overline{\alpha_{i_2}})} \cdot \ldots \cdot p_{i_r}^{d(\overline{\alpha_{i_r}})}$, *where* $\overline{\alpha_{i_j}}$ *is the remainder of the division of* $\alpha_{i_j}$ *by 3 and* $d: \{0, 1, 2\} \to \{0, 1, 2\}$ *is the numerical function defined by* $d(0) = 0, d(1) = 2$ *and* $d(2) = 1$.

If we take into account of the above definition of the function g, it is easy to prove the above properties.

OBSERVATION: $d(\overline{\alpha_{i_j}}) = \overline{3 - \overline{\alpha_{i_j}}}$, for every $\alpha_{i_j} \in N^*$, and in the sequel we use this writing for its simplicity.

REMARK 1. *Let* $m \in N^*$ *be a fixed natural number. If we consider now the numerical function* $\tilde{g}: N^* \to N^*$ *defined by* $\tilde{g}(n) = k$, *where k is the smallest natural number such that* $nk = s^m, s \in N^*$, *then we can observe that* $\tilde{g}$ *generalize the function g, and we also have:* $\tilde{g}(n^m) = 1$, $\forall n \in N^*$, $\tilde{g}(p) = p^{m-1}$, $\forall p$ *prime and* $\tilde{g}(n) = p_{i_1}^{\overline{m - \overline{\alpha_{i_1}}}} \cdot p_{i_2}^{\overline{m - \overline{\alpha_{i_2}}}} \cdot \ldots \cdot p_{i_r}^{\overline{m - \overline{\alpha_{i_r}}}}$, *where* $n = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots \cdot p_{i_r}^{\alpha_{i_r}}$ *is the prime factorization of n and* $\overline{\alpha_{i_j}}$ *is the remainder of the division of* $\alpha_{i_j}$ *by m, therefore the both above properties holds for* $\tilde{g}$, *too.*

REMARK 2. *Because* $1 \le g(n) \le n^2$, *for every* $n \in N^*$, *we have:* $\frac{1}{n} \le \frac{g(n)}{n} \le n$, *thus* $\sum_{n \ge 1} \frac{g(n)}{n}$ *is a divergent serie.*

*In a similar way, using that we have* $1 \le \tilde{g}(n) \le n^{m-1}$ *for every* $n \in N^*$, *it results that* $\sum_{n\ge1} \frac{\tilde{g}(n)}{n}$ *is also divergent.*

PROPERTY 3. *The function* $g: N^* \to N^*$ *is multiplicative:* $g(x \cdot y) = g(x) \cdot g(y)$ *for every* $x, y \in N^*$ *with* $(x,y) = 1$.

*Proof.* For $x = 1 = y$ we have $(x,y) = 1$ and $g(1 \cdot 1) = g(1) \cdot g(1)$. Let $x = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots \cdot p_{i_r}^{\alpha_{i_r}}$ and $y = q_{j_1}^{\beta_{j_1}} \cdot q_{j_2}^{\beta_{j_2}} \cdot \ldots \cdot q_{j_s}^{\beta_{j_s}}$ be the prime factorization of x and y, repectively, so that $x \cdot y = 1$.

Because $(x,y) = 1$ we have $p_{i_h} \ne q_{j_k}$, for every $h = \overline{1,r}$ and $k = \overline{1,s}$.

Then $g(x \cdot y) = p_{i_1}^{\overline{3-\overline{\alpha}_{i_1}}} \cdot p_{i_2}^{\overline{3-\overline{\alpha}_{i_2}}} \cdot \ldots \cdot p_{i_r}^{\overline{3-\overline{\alpha}_{i_r}}} \cdot q_{j_1}^{\overline{3-\overline{\beta}_{j_1}}} \cdot q_{j_2}^{\overline{3-\overline{\beta}_{j_2}}} \cdot \ldots \cdot q_{j_s}^{\overline{3-\overline{\beta}_{j_s}}} = g(x) \cdot g(y)$.

REMARK 3. *The property holds also for the function* $\tilde{g}: \tilde{g}(x \cdot y) = \tilde{g}(x) \cdot \tilde{g}(y)$, *where* $(x,y) = 1$.

PROPERTY 4. *If* $(x,y) = 1$, x *and* y *are not perfect cubes and* x,y>1, then *the equation* $g(x) = g(y)$ *has not natural solutions.*

*Proof.* Let $x = \prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}$ and $y = \prod_{k=1}^{s} q_{j_k}^{\beta_{j_k}}$ (where $p_{i_h} \ne q_{j_k}, \forall h = \overline{1,r}, k = \overline{1,s}$, because $(x,y) = 1$) be their prime factorizations. Then $g(x) = \prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{\alpha}_{i_h}}}$ and $g(y) = \prod_{k=1}^{s} q_{j_k}^{\overline{3-\overline{\beta}_{j_k}}}$ and there exist at least $\overline{\alpha}_{i_h} \ne 0$ and $\overline{\beta}_{j_k} \ne 0$ (because x and y are not perfect cubes), therefore $1 \ne p_{i_h}^{\overline{3-\overline{\alpha}_{i_h}}} \ne q_{j_k}^{\overline{3-\overline{\beta}_{j_k}}} \ne 1$, so $g(x) \ne g(y)$.

CONSEQUENCE 1. *The equation* $g(x) = g(x+1)$ *has not natural solutions because for* $x \ge 1$, *x and x+1 are not both perfect cubes and* $(x, x+1) = 1$.

REMARK 4. *The property and the consequence is also true for the function* $\tilde{g}$: if $(x,y) = 1$, $x > 1$, $y > 1$, *and it does not exist* $a, b \in N^*$ *so that* $x = a^m$, $y = b^m$ *(where m is fixed and has the above significance), then the equation* $\tilde{g}(x) = \tilde{g}(y)$ *has not natural solutions; the equation* $\tilde{g}(x) = \tilde{g}(x+1)$, $x \ge 1$ *has not natural solutions, too.*

*It is easy to see that the proofs are similary, but in this case we denote by* $\overline{\alpha}_{ij} = \alpha_{ij}$ *(mod m) and we replace* $\overline{3-\overline{\alpha}_{i_1}}$ *by* $\overline{m-\overline{\alpha}_{i_j}}$.

PROPERTY 5. *We have* $g(x \cdot y^3) = g(x)$, *for every* $x, y \in N^*$.

*Proof.* If $(x,y) = 1$, then $(x,y^3) = 1$ and using property 1 and property 3, we have: $g(x \cdot y^3) = g(x) \cdot g(y^3) = g(x)$.

If $(x,y)=1$ we can write: $x=\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}\cdot\prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}}$ and $y=\prod_{k=1}^{s} q_{j_k}^{\beta_{j_k}}\cdot\prod_{t=1}^{n} d_{l_t}^{\beta_{l_t}}$ where $p_{i_h}\neq d_{l_t}; q_{j_k}=d_{l_t}; p_{i_h}=q_{j_k}; \forall h=\overline{1,r}, k=\overline{1,s}$, $t=\overline{1,n}$. We have $g(x\cdot y^3)=g(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}\cdot\prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}}\cdot \prod_{k=1}^{s} q_{j_k}^{3\beta_{j_k}}\cdot\prod_{t=1}^{n} d_{l_t}^{3\beta_{l_t}})=g(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}\cdot\prod_{k=1}^{s} q_{j_k}^{3\beta_{j_k}}\cdot\prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}+3\beta_{l_t}})=g(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}\cdot\prod_{k=1}^{s} q_{j_k}^{3\beta_{j_k}})\cdot g(\prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}+3\beta_{l_t}})=$

$=\prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{\alpha_{i_h}}}}\cdot\prod_{k=1}^{s} q_{j_k}^{\overline{3-\overline{3\beta_{j_k}}}}\cdot\prod_{t=1}^{n} d_{l_t}^{\overline{3-\overline{\alpha_{l_t}+3\beta_{l_t}}}}=\prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{\alpha_{i_h}}}}\cdot\prod_{t=1}^{n} d_{l_t}^{\overline{3-\overline{\alpha_{l_t}}}}=g(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}})\cdot g(\prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}})=$

$=g(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}\cdot\prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}})=g(x).$

We used that $(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}},\prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}})=1$ and $(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}\cdot\prod_{k=1}^{s} q_{j_k}^{3\beta_{j_k}},\prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}+3\beta_{l_t}})=1$ and the above properties.

**REMARK 5.** *It is easy to see that we also have* $\tilde{g}(x\cdot y^m)=\tilde{g}(x)$, *for every* $x,y\in N^*$.

**OBSERVATION** . *If* $\frac{x}{y}=\frac{u^3}{v^3}$, *where* $\frac{u}{v}$ *is a simplified fraction, then g(x)=g(y). It is easy to prove this because* $x=kn^3$ *and* $y=kv^3$, *and using the above property we have:*

$$g(x)=g(k\cdot u^3)=g(k)=g(k\cdot v^3)=g(y)$$

**OBSERVATION.** If $\frac{x}{y}=\frac{u^m}{v^m}$ where $\frac{u}{v}$ is a simplified fraction, then, using remark 5, we have $\tilde{g}(x)=\tilde{g}(y)$, too.

**CONSEQUENCE 2.** *For every* $x\in N^*$ *and* $n\in N$,

$$g(x^n)=\begin{cases}1, \text{ if } n=3k;\\ g(x), \text{ if } n=3k+1;\\ g^2(x), \text{ if } n=3k+2,\ k\in N,\end{cases}$$

*where* $g^2(x)=g(g(x))$.

*Proof.* If n=3k, then $x^n$ is a perfect cube, therefore $g(x^n)=1$.
If n=3k+1, then $g(x^n)=g(x^{3k}\cdot x)=g(x^{3k})\cdot g(x)=g(x)$.
If n=3k+2, then $g(x^n)=g(x^{3k}\cdot x^2)=g(x^{3k})\cdot g(x^2)=g(x^2)$.

**PROPERTY 6.** $g(x^2)=g^2(x)$, for every $x\in \mathbf{N}^*$.

*Proof.* Let $x=\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}$ be the prime factorization of x. Then $g(x^2)=g(\prod_{h=1}^{r} p_{i_h}^{2\alpha_{i_h}})=\prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{2\alpha_{i_h}}}}$ and $g^2(x)=g(g(x))=g(\prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{\alpha_{i_h}}}})=\prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{3-\overline{\alpha_{i_h}}}}}$, but it is easy to observe that $\overline{3-\overline{2\alpha_{i_h}}}=\overline{3-\overline{3-\overline{\alpha_{i_h}}}}$, because for :

$\overline{\alpha_{i_h}} = 0 \quad \overline{3-2\overline{\alpha_{i_h}}} = \overline{3-0} = 0$ and $\overline{3-\overline{3-\overline{\alpha_{i_h}}}} = \overline{3-\overline{3-0}} = \overline{3-0} = 0$

$\overline{\alpha_{i_h}} = 1 \quad \overline{3-2\overline{\alpha_{i_h}}} = \overline{3-2} = 1$ and $\overline{3-\overline{3-\overline{\alpha_{i_h}}}} = \overline{3-\overline{3-1}} = \overline{3-2} = 1$

$\overline{\alpha_{i_h}} = 2 \quad \overline{3-2\overline{\alpha_{i_h}}} = \overline{3-\overline{4}} = \overline{3-1} = 2$ and $\overline{3-\overline{3-\overline{\alpha_{i_h}}}} = \overline{3-\overline{3-2}} = \overline{3-1} = 2$,

therefore $g(x^2) = g^2(x)$.

**REMARK 6.** *For the function* $\tilde{g}$ *is not true that* $\tilde{g}(x^2) = \tilde{g}^2(x)$, $\forall x \in N^*$. *For example, for* $m=5$ *and* $x = 3^2$, $\tilde{g}(x^2) = \tilde{g}(3^4) = 3$ *while* $\tilde{g}(\tilde{g}(3^2)) = \tilde{g}(3^3) = 3^2$.

More generally $\tilde{g}(x^k) = \tilde{g}^k(x)$, $\forall x \in N^*$ is not true. But for particular values of m,k and x the above equality is possible to be true. For example for $m = 6$, $x = 2^2$ and $k = 2$ : $\tilde{g}(x^2) = \tilde{g}(2^4) = 2^2$ and $\tilde{g}^2(x) = \tilde{g}(\tilde{g}(2^2)) = \tilde{g}(2^4) = 2^2$.

**REMARK 6'.** *a)* $\tilde{g}(x^{m-1}) = \tilde{g}^{m-1}(x)$ *for every* $x \in N^*$ *iff m is an odd number, because we have* $\overline{m-\overline{(m-1)\alpha_{i_h}}} = \underbrace{\overline{m-\overline{m-\ldots-\overline{m-\overline{\alpha_{i_h}}}}}}_{\text{m-1 times}}$, *for every* $\alpha_{i_h} \in N$.

Example: *For* $m = 5$, $\tilde{g}(x^4) = \tilde{g}^4(x)$, *for every* $x \in N^*$.

*b)* $\tilde{g}(x^{m-1}) = \tilde{g}^{m}(x)$, *for every* $x \in N^*$ *iff m is an even number, because we have* $\overline{m-\overline{(m-1)\alpha_{i_h}}} = \underbrace{\overline{m-\overline{m-\ldots-\overline{m-\overline{\alpha_{i_h}}}}}}_{\text{m times}}$, *for every* $\alpha_{i_h} \in N$.

Example: *For* $m = 4$, $\tilde{g}(x^3) = \tilde{g}^4(x)$, *for every* $x \in N^*$.

**PROPERTY 7.** *For every* $x \in N^*$ *we have* $g^3(x) = g(x)$.

*Proof.* Let $x = \prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}$ be the prime factorization of x. We saw that $g(x) = \prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{\alpha_{i_h}}}}$ and

$$g^3(x) = g(g^2(x)) = g(\prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{3-\overline{\alpha_{i_h}}}}}) = \prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{3-\overline{3-\overline{\alpha_{i_h}}}}}}.$$

But $\overline{3-\overline{\alpha_{i_h}}} = \overline{3-\overline{3-\overline{3-\overline{\alpha_{i_h}}}}}$, for every $\alpha_{i_h} \in N$, because for:

$\overline{\alpha_{i_h}} = 0 \quad \overline{3-\overline{\alpha_{i_h}}} = 0$ and $\overline{3-\overline{3-\overline{3-\overline{\alpha_{i_h}}}}} = \overline{3-\overline{3-\overline{3-0}}} = \overline{3-\overline{3-0}} = \overline{3-0} = 0$

$\overline{\alpha_{i_h}} = 1 \quad \overline{3-\overline{\alpha_{i_h}}} = 2$ and $\overline{3-\overline{3-\overline{3-\overline{\alpha_{i_h}}}}} = \overline{3-\overline{3-\overline{3-1}}} = \overline{3-\overline{3-2}} = \overline{3-1} = 2$

$\overline{\alpha_{i_h}} = 2 \quad \overline{3-\overline{\alpha_{i_h}}} = 1$ and $\overline{3-\overline{3-\overline{3-\overline{\alpha_{i_h}}}}} = \overline{3-\overline{3-\overline{3-2}}} = \overline{3-\overline{3-1}} = \overline{3-2} = 1$,

therefore $g^3(x) = g(x)$, for every $x \in N^*$.

REMARK 7. *For every* $x \in N^*$ *we have* $\tilde{g}^3(x) = \tilde{g}(x)$ because $\overline{m-\overline{\alpha_{i_h}}} = \overline{m-\overline{m-\overline{m-\overline{\alpha_{i_h}}}}}$, for every $\alpha_{i_h} \in \mathbf{N}$. For $\overline{\alpha_{i_h}} = a \in \{1,\dots,m-1\} = A$, we have $\overline{m-\overline{\alpha_{i_h}}} = m-a \in A$, therefore $\overline{m-\overline{m-\overline{\alpha_{i_h}}}} = \overline{m-(m-a)} = \bar{a} = a$, so that $\overline{m-\overline{m-\overline{m-\overline{\alpha_{i_h}}}}} = \overline{m-a} = \overline{m-\overline{\alpha_{i_h}}}$, which is also true for $\overline{\alpha_{i_h}} = 0$, therefore it is true for every $\alpha_{i_h} \in N^*$.

PROPERTY 8. *For every* $x, y \in N^*$ *we have* $g(x\cdot y) = g^2(g(x)\cdot g(y))$.

*Proof.* Let $x = \prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}} \cdot \prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}}$ and $y = \prod_{k=1}^{s} q_{j_k}^{\beta_{j_k}} \cdot \prod_{t=1}^{n} d_{l_t}^{\beta_{l_t}}$ be the prime factorization of x and y, respectively, where $p_{i_h} \neq d_{l_t}, q_{j_k} \neq d_{l_t}, p_{i_h} \neq q_{j_k}, \forall h = \overline{1,r}, k = \overline{1,s}, t = \overline{1,n}$. Of course $x\cdot y = \prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}} \cdot \prod_{k=1}^{s} q_{j_k}^{\beta_{j_k}} \cdot \prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}+\beta_{l_t}}$, so $g(x\cdot y) = \prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{\alpha_{i_h}}}} \cdot \prod_{k=1}^{s} q_{j_k}^{\overline{3-\overline{\beta_{j_k}}}} \cdot \prod_{t=1}^{n} d_{l_t}^{\overline{3-\overline{(\alpha_{l_t}+\beta_{l_t})}}}$. On the other hand, $g(x) = \prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{\alpha_{i_h}}}} \cdot \prod_{t=1}^{n} d_{l_t}^{\overline{3-\overline{\alpha_{l_t}}}}$ and $g(y) = \prod_{k=1}^{s} q_{j_k}^{\overline{3-\overline{\beta_{j_k}}}} \cdot \prod_{t=1}^{n} d_{l_t}^{\overline{3-\overline{\beta_{l_t}}}}$, so that

$$g^2(g(x)\cdot g(y)) = g^2\Big(\prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{\alpha_{i_h}}}} \cdot \prod_{k=1}^{s} q_{j_k}^{\overline{3-\overline{\beta_{j_k}}}} \cdot \prod_{t=1}^{n} d_{l_t}^{\overline{3-\overline{\alpha_{l_t}}}+\overline{3-\overline{\beta_{l_t}}}}\Big) = \prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{3-\overline{3-\overline{\alpha_{i_h}}}}}} \cdot \prod_{k=1}^{s} q_{j_k}^{\overline{3-\overline{3-\overline{3-\overline{\beta_{j_k}}}}}} \cdot$$

$$\cdot \prod_{t=1}^{n} d_{l_t}^{\overline{3-\overline{3-(\overline{3-\overline{\alpha_{l_t}}}+\overline{3-\overline{\beta_{l_t}}})}}} = \prod_{h=1}^{r} p_{i_h}^{\overline{3-\overline{\alpha_{i_h}}}} \cdot \prod_{k=1}^{s} q_{j_k}^{\overline{3-\overline{\beta_{j_k}}}} \cdot \prod_{t=1}^{n} d_{l_t}^{\overline{3-\overline{(\alpha_{l_t}+\beta_{l_t})}}} = g(x\cdot y),$$ because

$\overline{3-\overline{3-\overline{3-\overline{a}}}} = \overline{3-\overline{a}}$ and $\overline{3-\overline{3-(\overline{3-\overline{a}}+\overline{3-\overline{b}})}} = \overline{3-\overline{(a+b)}}, \forall a,b \in \mathbf{N}$.

REMARK 8. *In the case when* $(x,y) = 1$ *we obtain more simply the same result. Because* $(x,y) = 1 \Rightarrow (g(x),g(y)) = 1 \Rightarrow (g^2(x), g^2(y)) = 1$ *so we have:*

$$g^2(g(x)\cdot g(y)) = g(g(g(x)\cdot g(y))) = g(g(g(x))\cdot g(g(y))) = g(g^2(x)\cdot g^2(y)) =$$
$$= g(g^2(x))\cdot g(g^2(y)) = g^3(x)\cdot g^3(y) = g(x)\cdot g(y) = g(x\cdot y).$$

REMARK 9. *If* $(x,y) = 1$, *then* $g(xyz) = g^2(g(xy)\cdot g(z)) = g^2(g(x)g(y)g(z))$ *and this property can be extended for a finite number of factors, therefore if* $(x_1,x_2) = (x_2,x_3) = \dots = (x_{n-2},x_{n-1}) = 1$, *then* $g(\prod_{i=1}^{n} x_i) = g^2(\prod_{i=1}^{n} g(x_i))$.

PROPERTY 9. *The function g has not fixed points* $x \neq 1$.

*Proof.* We must prove that the equation $g(x) = x$ has not solutions $x>1$.

Let $x = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \dots \cdot p_{i_r}^{\alpha_{i_r}}, \alpha_{i_j} \geq 1, j = \overline{1,r}$ be the prime factorization of x. Then $g(x) = \prod_{j=1}^{r} p_{i_j}^{\overline{3-\overline{\alpha_{i_j}}}}$ implies that $\alpha_{i_j} = \overline{3-\overline{\alpha_{i_j}}}, \forall j \in \overline{1,r}$ which is not possible.

REMARK 10. *The function* $\tilde{g}$ *has fixed points only in the case* $m = 2k$, $k \in N^*$. *These points are* $x = p_{i_1}^{k} \cdot p_{i_2}^{k} \cdot \dots \cdot p_{i_r}^{k}$, *where* $p_{i_j}, j = \overline{1,r}$ *are prime numbers.*

**PROPERTY 10.** *If* $\left(\frac{x}{(x,y)},y\right)=1$ *and* $\left(\frac{y}{(x,y)},x\right)=1$ *then we have* $g((x,y))=(g(x),g(y))$, *where we denote by* $(x,y)$ *the greatest common divisor of* $x$ *and* $y$.

*Proof.* Because $\left(\frac{x}{(x,y)},y\right)=1$ and $\left(\frac{y}{(x,y)},x\right)=1$, we have $\left(\frac{x}{(x,y)},(x,y)\right)=1$ *and* $\left(\frac{y}{(x,y)},(x,y)\right)=1$, then x and y have the following prime factorization: $x=\prod_{h=1}^{r}p_{i_h}^{\alpha_{i_h}}\cdot\prod_{t=1}^{n}d_{l_t}^{\alpha_{l_t}}$ and $y=\prod_{k=1}^{s}q_{j_k}^{\beta_{j_k}}\cdot\prod_{t=1}^{n}d_{l_t}^{\alpha_{l_t}}$, $\quad p_{i_h}\neq d_{l_t}, q_{j_k}\neq d_{l_t}, \quad p_{i_h}\neq q_{j_k};\quad \forall h=\overline{1,r}, k=\overline{1,s}, t=\overline{1,n}$. Then $(x,y)=\prod_{t=1}^{n}d_{l_t}^{\alpha_{l_t}}$, therefore $g((x,y))=\prod_{t=1}^{n}d_{l_t}^{\overline{\overline{3-\alpha_{l_t}}}}$. On the other hand

$$(g(x),g(y))=\left(\prod_{h=1}^{r}p_{i_h}^{\overline{\overline{3-\alpha_{i_h}}}}\cdot\prod_{t=1}^{n}d_{l_t}^{\overline{\overline{3-\alpha_{l_t}}}},\prod_{k=1}^{s}q_{j_k}^{\overline{\overline{3-\beta_{j_k}}}}\cdot\prod_{t=1}^{n}d_{l_t}^{\overline{\overline{3-\alpha_{l_t}}}}\right)=\prod_{t=1}^{n}d_{l_t}^{\overline{\overline{3-\alpha_{l_t}}}}$$ and the assertion follows.

**REMARK 11.** *In the same conditions,* $\tilde{g}((x,y))=(\tilde{g}(x),\tilde{g}(y)), \forall x,y\in N^*$.

**PROPERTY 11.** *If* $\left(\frac{x}{(x,y)},y\right)=1$ *and* $\left(\frac{y}{(x,y)},x\right)=1$ *then we have:* $g([x,y])=[g(x),g(y)]$, *where* $(x,y)$ *has the above significance and* $[x,y]$ *is the least common multiple of* $x$ *and* $y$.

*Proof.* We have the prime factorization of x and y used in the proof of the above property, therefore:

$$g([x\cdot y])=g\left(\prod_{h=1}^{r}p_{i_h}^{\alpha_{i_h}}\cdot\prod_{k=1}^{s}q_{j_k}^{\beta_{j_k}}\cdot\prod_{t=1}^{n}d_{l_t}^{\alpha_{l_t}}\right)=\prod_{h=1}^{r}p_{i_h}^{\overline{\overline{3-\alpha_{i_h}}}}\cdot\prod_{k=1}^{s}q_{j_k}^{\overline{\overline{3-\beta_{j_k}}}}\cdot\prod_{t=1}^{n}d_{l_t}^{\overline{\overline{3-\alpha_{l_t}}}}$$ and

$$[g(x),g(y)]=\left[\prod_{h=1}^{r}p_{i_h}^{\overline{\overline{3-\alpha_{i_h}}}}\cdot\prod_{t=1}^{n}d_{l_t}^{\overline{\overline{3-\alpha_{l_t}}}},\prod_{k=1}^{s}q_{j_k}^{\overline{\overline{3-\beta_{j_k}}}}\cdot\prod_{t=1}^{n}d_{l_t}^{\overline{\overline{3-\alpha_{l_t}}}}\right]=$$

$$=\prod_{h=1}^{r}p_{i_h}^{\overline{\overline{3-\alpha_{i_h}}}}\cdot\prod_{k=1}^{s}q_{j_k}^{\overline{\overline{3-\beta_{j_k}}}}\cdot\prod_{t=1}^{n}d_{l_t}^{\overline{\overline{3-\alpha_{l_t}}}},$$

so we have $g([x,y])=[g(x),g(y)]$.

**REMARK 12.** *In the same conditions,* $\tilde{g}([x,y])=[\tilde{g}(x),\tilde{g}(y)], \forall x,y\in N^*$.

**CONSEQUENCE 4.** *If* $\left(\frac{x}{(x,y)},y\right)=1$ *and* $\left(\frac{y}{(x,y)},x\right)=1$, *then* $g(x)\cdot g(y)=$ $=g((x,y))\cdot g([x,y])$ *for every* $x,y\in N^*$.

*Proof.* Because $[x,y]=\dfrac{xy}{(x,y)}$ we have $[g(x),g(y)]=\dfrac{g(x)\cdot g(y)}{(g(x),g(y))}$ and using the last two properties we have:

$$g(x)\cdot g(y)=(g(x),g(y))\cdot[g(x),g(y)]=g((x,y))\cdot g([x,y]).$$

REMARK 13. *In the same conditions, we also have* $\tilde{g}(x)\cdot\tilde{g}(y)=\tilde{g}((x,y))\cdot\tilde{g}([x,y])$ *for every* $x,y\in N^*$.

PROPERTY 13. *The sumatory numerical function of the function g is*

$$F(n)=\prod_{j=1}^{k}\left(\frac{\alpha_{i_j}+\overline{3-\overline{\alpha_{i_j}}}}{3}(1+p_{i_j}+p_{i_j}^2)+h_{p_{i_j}}(\alpha_{i_j})\right),$$

*where* $n=p_{i_1}^{\alpha_{i_1}}\cdot p_{i_2}^{\alpha_{i_2}}\cdot\ldots\cdot p_{i_k}^{\alpha_{i_k}}$ *is the prime factorization of n, and* $h_p:N\to N$ *is the numerical function defined by* $h_p(\alpha)=\begin{cases}1 & \text{for } \alpha=3k\\ -p & \text{for } \alpha=3k+1\\ 0 & \text{for } \alpha=3k+2\end{cases}$, *where p is a given number.*

*Proof.* Because the sumatory function of g is defined as $F(n)=\sum_{d/n}g(d)$ and because $(p_{i_1}^{\alpha_{i_1}},\prod_{t=2}^{k}p_{i_t}^{\alpha_{i_t}})=1$ and g is a multiplicative function, we have:

$$F(n)=\left(\sum_{d_1/p_{i_1}^{\alpha_1}}g(d_1)\right)\cdot\left(\sum_{d_2/p_{i_2}^{\alpha_2}\ldots p_{i_k}^{\alpha_k}}g(d_2)\right)$$

and so on, making a finite number of steps we obtain: $F(n)=\prod_{j=1}^{k}F(p_{i_j}^{\alpha_{i_j}})$.

But it is easy to prove that:

$$F(p^{\alpha})=\begin{cases}\frac{\alpha}{3}(1+p+p^2)+1 & \text{for } \alpha=3k;\\ \frac{\alpha+2}{3}(1+p+p^2)-p & \text{for } \alpha=3k+1;\\ \frac{\alpha+1}{3}(1+p+p^2) & \text{for } \alpha=3k,\ k\in\mathbf{N}, \text{ for every prime } p\end{cases}$$

Using the function $h_p$, we can write $F(p^{\alpha})=\dfrac{\overline{3-\overline{\alpha}}}{3}(1+p+p^2)+h_p(\alpha)$, therefore we have the demanded expresion of $F(n)$.

REMARK 14. *The expresion of F(n), where F is the sumatory function of* $\tilde{g}$, *is similary, but it is necessary to replace*

$\dfrac{\alpha_{i_j} + \overline{\overline{3 - \alpha_{i_j}}}}{3}$ *by* $\dfrac{\alpha_{i_j} + \overline{\overline{m - \alpha_{i_j}}}}{m}$ *(where* $\overline{\alpha_{i_j}}$ *is now the remainder of the division of* $\alpha_{i_j}$ *by m and the sum* $1 + p_{i_j} + p_{i_j}^2$ *by* $\sum_{k=0}^{m-1} p_{i_j}^k$*) and to define an adapted function* $h_p$.

In the sequel we study some equations which involve the function g.

1. *Find the solutions of the equations* $x \cdot g(x) = a$, *where* $x, a \in N^*$.

If a is not a perfect cube, then the above equation has not solutions.

If a is a perfect cube, $a = b^3, b \in N^*$, where $b = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots \cdot p_{i_k}^{\alpha_{i_k}}$ is the prime factorization of b, then, taking into account of the definition of the function g, we have the solutions $x = b^3 / d_{i_1 i_2 \ldots i_k}$ where $d_{i_1 i_2 \ldots i_k}$ can be every product $p_{i_1}^{\beta_1} p_{i_2}^{\beta_2} \cdots p_{i_k}^{\beta_k}$ where $\beta_1, \beta_2, \ldots, \beta_k$ take an arbitrary value which belongs of the set $\{0, 1, 2\}$.

In the case when $\beta_1 = \beta_2 = \cdots = \beta_k = 0$ we find the special solution $x = b^3$, when $\beta_1 = \beta_2 = \cdots = \beta_k = 1$, the solution $p_{i_1}^{3\beta_1 - 1} p_{i_2}^{3\beta_2 - 1} \cdots p_{i_k}^{3\beta_k - 1}$ and when $\beta_1 = \beta_2 = \cdots = \beta_k = 2$, the solution $p_{i_1}^{3\beta_1 - 2} p_{i_2}^{3\beta_2 - 2} \cdots p_{i_k}^{3\beta_k - 2}$.

We find in this way $1 + 2C_k^1 + 2^2 C_k^1 + \cdots + 2^k C_k^k = 3^k$ different solutions, where k is the number of the prime divisors of b.

2. *Prove that the following equations have not natural solutions:*
$xg(x) + yg(y) + zg(z) = 4$ *or* $xg(x) + yg(y) + zg(z) = 5$. *Give a generalization.*

Because $xg(x) = a^3, yg(y) = b^3, zg(z) = c^3$ and the equations $a^3 + b^3 + c^3 = 4$ or $a^3 + b^3 + c^3 = 5$ have not natural solutions, then the assertion holds.

We can also say thet the equations $(xg(x))^n + (yg(y))^n + (zg(z))^n = 4$ or $(xg(x))^n + (yg(y))^n + (zg(z))^n = 5$ have not natural solutions, because the equations $a^{3n} + b^{3n} + c^{3n} = 4$ or $a^{3n} + b^{3n} + c^{3n} = 5$ have not.

3. *Find all solutions of the equation* $xg(x) - yg(y) = 999$.

Because $xg(x) = a^3$ and $yg(y) = b^3$ we must give the solutions of the equation $a^3 - b^3 = 999$, wh²ich are (a=10, b=1) and (a=12,b=9).

*In the first case*: a=10, b=1 we have $xa(x) = 10^3 = 2^3 \cdot 5^3$

$$\Rightarrow x_0 \in \{10^3, 2^2 \cdot 5^3, 2^3 \cdot 5^2, 2 \cdot 5^3, 2^3 \cdot 5, 2^2 \cdot 5^2, 2^2 \cdot 5, 2 \cdot 5^2, 2 \cdot 5\}$$

and yb(y)=1 $\Rightarrow y_0 = 1$ so we have 9 different solutions $(x_0, y_0)$.

*In the second case*: a=12, b=9 we have $xa(x) = 12^3 = 2^6 \cdot 3^3$

$$\Rightarrow x_0 \in \{2^6 \cdot 3^3, 2^5 \cdot 3^3, 2^6 \cdot 3^2, 2^4 \cdot 3^3, 2^6 \cdot 3, 2^5 \cdot 3^2, 2^4 \cdot 3^2, 2^5 \cdot 3, 2^4 \cdot 3\}$$

and yb(y)=$9^3 = 3^9 \Rightarrow y_0 \in \{3^9, 3^8, 3^7\}$ so we have another $9 \cdot 3 = 27$ different solutions $(x_0, y_0)$.

4. *It is easy to observe that the equation g(x)=1 has an infinite number of solutions: all perfect cube numbers.*

5. *Find the solutions of the of the equation* $g(x)+g(y)+g(z)=g(x)g(y)g(z)$.

*The same problem when the function is* $\tilde{g}$.

It is easy to prove that the solutions are, in the first case, the permutations of the sets $\{u^3, 4v^3, 9t^3\}$, where $u, v, t \in N^*$, and in the second case $\{u^m, 2^{m-1}v^m, 3^{m-1}t^m\}$, $u, v, t \in N^*$.

Using the same ideea of [1], it is easy to find the solutions of the following equations which involve the function g:

a) $g(x) = kg(y)$, $k \in N^*, k > 1$

b) $Ag(x) + Bg(y) + Cg(z) = 0$, $A, B, C \in Z^*$

c) $Ag(x) + Bg(y) = C$, $A, B, C \in Z^*$, and to find also the solutions of the above equations when we replace the function g by $\tilde{g}$.

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# Some Considerations Concerning the Sumatory Function Associated to Smarandache Function

by
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The Smarandache Function [4] ia a numerical function $S:N^* \to N^*$ defined by $S(n) = \min\{m | m! \, divisible \; by \; n\}$.
From the definition it results that if

$$n = p_1^{r_1} p_2^{r_2} \dots p_t^{r_t} \tag{1}$$

is the decomposition of $n$ into primes, then

$$S(n) = \max\{S(p_i^{r_i}) | i = 1,2,\dots,t\} \tag{2}$$

It is said that for every function $f$ it can be attashed the sumatory function

$$F(n) = \sum_{d/n} f(d) \tag{3}$$

If $f$ is the Smarandache function and $n = p^a$, then

$$F_s(p^a) = \sum_{j=0}^{a} S(p^j) = \sum_{j=0}^{a} S_p(j) \tag{4}$$

In [2] it is proved that

$$S(p^j) = (p-1)j + \sigma_{[p]}(j) \tag{5}$$

where

$$j = \sum_{i=1}^{t_j} k_i^j a_i(p) \tag{6}$$

and

$$\sigma_{[p]}(j) = \sum_{i=1}^{t_j} k_i^j \tag{7}$$

is the sum of the digits of the integer $j$, written in the generalised scale
$[p] \;\; : a_1(p), a_2(p), \dots, a_k(p), \dots$
with

$$a_n(p) = \frac{p^n - 1}{p-1} \quad , n = 1,2,\dots$$

For example

$$p = p \cdot a_1(p);$$
$$p^p = (p-1) \cdot a_p(p) + 1 \cdot a_1(p);$$
$$p^t = (p-1) \cdot a_t(p) + 1;$$

and

$$\sigma_{[p]}(p^p) = p;$$
$$S(p^p) = p^2; \quad S\left(p^{p^p}\right) = (p-1)p^p + p .$$

In [3] it is proved that

$$F_s(p^a) = (p-1)\frac{a(a+1)}{2} + \sum_{j=1}^{a} \sigma_{[p]}(j) \tag{8}$$

In the following we give an algorithm to calculate the sumatory function, associated to the Smarandache function:

1. Calculating the generalised scale $[p]$ $: a_1(p), a_2(p), \ldots, a_n(p), \ldots$
2. Calculating the expresion of $a$ in the scale $[p]$. Let $a_{[p]} = \overline{k_s k_{s-1} \ldots k_1}$.
3. For $i = 1,2,\ldots,s$

   3.1. If $k_i \neq 0$

   then

   3.1.1. $v_i = a - a_i(p) + 1$

   3.1.2. $z_i = \left(\overline{k_s k_{s-1} \ldots k_{i+1}}\right)_{u=a_i(p)}$

   3.1.3. $h_i = v_i - z_i$

   else

   3.1.4. $b = \overline{k_s k_{s-1} \ldots k_{i+1} - 1p00\ldots0}$

   3.1.5. $v_i = b - a_i(p) + 1$

   3.1.6. $z_i = \left(\overline{k_s k_{s-1} \ldots k_i}\right)_{u=a_i(p)}$

   3.1.7. $h_i = v_i - z_i$

   3.2. $A_i = \left[\dfrac{h_i}{a_{i+1}(p) - a_i(p)}\right]$

   3.3. $r_i = h_i - A_i\left(a_{i+1}(p) - a_i(p)\right)$

   3.4. $B_i = \left[\dfrac{r_i}{a_i(p)}\right]$

   3.5. $q_i = r_i - B_i * a_i(p)$

   3.6. $S_i = A_i a_i(p)\dfrac{p(p-1)}{2} + A_i p + a_i(p)\dfrac{B_i(B_i+1)}{2} + q_i(B_i+1)$

4. Calculating $F_s(p^a) = (p-1)\dfrac{a(a+1)}{2} + \sum_{i=1}^{a} S_i$ , $\left(\sum_{j=1}^{a} \sigma_{[p]}(j) = \sum_{i \geq 1} S_i(a)\right)$.

A Pascal program has been designed to the calculus of $F_s(p^a)$ :

```
uses dos,crt;
        type tablou=array[1..100] of real;
        var a,k,amare,bmare,niu,z,alfaa,betaa,ro,r,s:tablou;
alfa,p,ik,amax,beta,suma,fsuma,u:real;
i,dim,max,j:longint;
 hour,min,sec,sec100:word;
{*****************************************************}
 {Calc. scale p right}
 procedure bazapd(var b:tablou;var p:real;var a:real; var dim:longint);
 var i:longint;
 begin
    for i:=1 to 100 do
    bi]:=0;
    b[1]:=1;
    i:=0;
```

```
      repeat
        i:=i+1;
          b[i]:=b[i-1]*p+1;
      until b[i]>a;
   dim:=i;
end;

{*****************************************************}
{write  alfa in the scale p right}
procedure nrbazapd(var a:tablou; var p:real;
var alfa:real; var k:tablou; var max:longint);
var m,i:longint;
d,r,prod:real;
begin
for i:=1 to 100 do
    k[i]:=0;
d:=alfa;
max:=trunc(ln((p-1)*d+1)/ln(p));
repeat
  m:=trunc(ln((p-1)*d+1)/ln(p));
   k[m]:=trunc(d/a[m]);
   r:=d-a[m]*k[m];
   d:=r ;
until r<p;
if r<>0 then
  k[1]:=r;
end;
{*******************************************}
{calc. z for given i }

procedure calcz(var k:tablou;var a:tablou;
var i:longint;var u:real; var z:tablou; var p:real);
var j,i1,ind:longint;
prod:real;
begin
  z[i]:=0;
     ind:=1;
  for j:=i+1 to max do
    begin
     if k[j]<>0 then
      begin
         prod:=1;
         if ind>1 then
            begin
            for i1:=1 to ind-1 do
              prod:=prod*p; {****}
            prod:=prod*u+a[ind-1];
          end
         else
           prod:=u;
         z[i]:=z[i]+k[j]*prod;
```

```
    end;
    ind:=ind+1;
  end;
end;
{*************************************************}
begin
clrscr;
write('       give p=');
readln(p);
write('       give alfa=');
readln(alfa);
gettime(hour,min,sec,sec100);
writeln('       Timp Start:',hour,':',min,':',sec,':',sec100);
bazapd(a,p,alfa,dim);
nrbazapd(a,p,alfa,k,max);
for i:=1 to max do
  begin
   if k[i]<>0 then
    begin
     niu[i]:=alfa-a[i]+1;
     u:=a[i];
     calcz(k,a,i,u,z,p);
     alfaa[i]:=niu[i]-z[i];
    end
   else
    begin
     for j:=1 to max do
     betaa[j]:=k[j];
     betaa[i]:=p;
     betaa[i+1]:=betaa[i+1]-1;
     for j:=1 to i-1 do
      betaa[j]:=0;
       {Write beta in the scale 10}
       beta:=0;
        for j:=1 to max do
     beta:=beta+betaa[j]*a[j];
     niu[i]:=beta-a[i]+1;
     u:=a[i];
     calcz(betaa,a,i,u,z,p);
     alfaa[i]:=niu[i]-z[i];
     end;
     amare[i]:=int(alfaa[i]/(a[i+1]-a[i]));
     r[i]:=alfaa[i]-amare[i]*(a[i+1]-a[i]);
     bmare[i]:=int(r[i]/a[i]);
     ro[i]:=r[i]-bmare[i]*a[i];
     s[i]:=amare[i]*a[i]*(p*(p-1)/2)+amare[i]*p;
     s[i]:=s[i]+a[i]*(bmare[i]*(bmare[i]+1)/2);
     s[i]:=s[i]+ro[i]*(bmare[i]+1);
   end;
suma:=0;
```

```
for i:=1 to max do
 suma:=suma+s[i];
fsuma:=(p-1)*((alfa*(alfa+1))/2)+suma;
writeln('        fsuma=',fsuma);
gettime(hour,min,sec,sec100);
writeln('        Timp Stop:',hour,':',min,':',sec,':',sec100);
end.
```

We applied the algoritm for $p=3$ and $a=300$ we obtain

TIMES START: 10:34:1:56

TIMES STOP: 10:34:1:57

We applied the formulas [4] for $p=3$ and $a=300$ we obtain

TIMES START : 10:33:31:2

TIMES STOP: 10:33:31:95

A consequence of this work is that the proposed algoritm is faster then formula [4] .

From the Legendre formula it results that [1]

$$S_p(j)=p\left(j-i_p(j)\right) \text{ with } 0\le i_p(j)\le\left[\frac{j-1}{p}\right],$$

and

$$F_s\left(p^a\right)=\sum_{j=0}^{a}p\left(j-i_p(j)\right)=p\sum_{j=0}^{a}j-p\sum_{j=0}^{a}i_p(j)\ ,$$

consequently

$$F_s(p^a)=\frac{pa(a+1)}{2}-p\sum_{i=0}^{a}i_p(j) \qquad (9)$$

In [1] it is showed that

$$i_p(j)=\frac{j-\sigma_{[p]}(j)}{p}\ .$$

In particular,

$$i_p(p)=0\ ;\quad i_p(p^t)=p^{t-1}-1$$

and

$$\sum_{j=0}^{a}i_p(j)=\frac{1}{p}\left[\sum_{j=0}^{a}j-\sum_{i=1}^{t_j}k_i^j\right].$$

If $p\ge a$, then

$$j=j\cdot a_1(p),\quad \sigma_{[p]}(j)=j,\quad (j=1,2,\ldots,a),\quad S(p^a)=pa$$

and

$$\sum_{j=0}^{a}\sigma_{[p]}(j)=\frac{a(a+1)}{2};\quad \sum_{j=0}^{a}i_p(j)=0$$

$$F_s(p^a)=\frac{pa(a+1)}{2} \qquad (10)$$

For example

$$F_s(11^3)=s(1)+S(11)+S(11^2)+S(11^3)=66 \text{ or}$$

$$F_s(11^3)=\frac{11\cdot3\cdot4}{2}=66\ .$$

In particular,

$$F_s(p^p)=\frac{p^2(p+1)}{2} \tag{11}$$

If $p \le a$, then $a = pQ + R$ with $0 \le R \le p$, and

$$\sum_{j=0}^{a} i_p(j) = \sum_{j=0}^{a}\left\{\left[\frac{i}{p}\right]-\left[\frac{\sigma_p(j)}{p}\right]\right\} \Rightarrow$$

$$\sum_{j=0}^{a} i_p(j) = \frac{pQ(Q-1)}{2} + Q(R+1) - \sum_{j=p}^{a}\left[\frac{\sigma_p(j)}{p}\right],$$

consequently,

$$F_s(p^a)=\frac{pa(a+1)}{2}-\frac{p^2Q(Q-1)}{2}-pQ(R+1)+p\sum_{j=0}^{a}\left[\frac{\sigma_{[p]}(p)}{p}\right] \tag{12}$$

In particular, for $p = a$ then $Q = 1$, $R = 0$ and

$$F_s(p^p)=\frac{p^2(p+1}{2} \tag{13}$$

For example,

$F_s(3^3)=18$; $F_s(5^5)=75$;

$F_s(n^n)=\dfrac{2(2^p+1)^2(2^{p-2}+1)}{27}$, for $n=\dfrac{2^p+1}{3}$, with $3<p\le 31$ and $p$ prime.

If $n = p^a q^b$ with $p < q$ and $p^a < q$, then

$$F_s(p^a q^b)= \sum_{d/p^a q^b} S(d) = \sum_{i=0}^{a}\sum_{j=0}^{a} S(p^i q^j) = (a+1)\sum_{j=0}^{b} S(q^j) = (a+1)F_s(q^b)$$

Then :

I. If $q \ge b$,

$$F_s(p^a q^b)=\frac{qb(a+1)(b+1)}{2} \tag{14}$$

II. If $q \le b$,

$$F_s(p^a q^b)=\frac{(a+1)qb(b+1)}{2}-\frac{(a+1)q^2\overline{Q}(\overline{Q}-1)}{2}-q(a+1)\overline{Q}(\overline{R}+1)+ q(a+1)\sum_{j=q}^{b}\left[\frac{\sigma_{[q]}(j)}{q}\right] \tag{15}$$

where $b = q\overline{Q} + \overline{R}$, with $0 \le \overline{R} \le q$.

If $n = p^a q$, then

$$F_s(p^a q)=\sum_{i=0}^{a} S(p^i) + \sum_{i=0}^{a} S(qp^i) .$$

For $p > q$, then $p^i > q$ and $S(qp^i) = S(p^i)$ with $i \ge 1$ consequently,

$$F_s(p^a q) = 2F_s(p^a) + S(q) - 1 \tag{16}$$

For $p < q$, there exists $x < a$ with $p^{x-1} < q < p^x$ and

$$S(qp^i) = \{S(q),\quad i = 0,1,\ldots,x-1$$
$$S(p^i),\quad i = x,\ldots,a$$

consequently,

$$F_s(p^a q)=\sum_{i=0}^{x-1} S(p^i) + xS(q) + 2\sum_{i=x}^{a} S(p^i)$$

or

$$F_s(p^a q) = F_s(p^{x-1}) + xS(q) + (a+x)(a-x+1)(p-1) + 2\sum_{j=x}^{a} \sigma_{[p]}(j) \qquad (17)$$

For example, if $p \geq a$, then

$$F_S(p^a q) = F_s(p^{x-1}) + xS(q) + p(x+a)(a-x+1)$$

If $n = p^a q^k$ with $p > q$, then

$$F_s(p^a q^2) = \sum_{i=0}^{a} S(p^i) + \sum_{i=0}^{a} S(qp^i) + \sum_{i=0}^{a} S(q^2 p^i).$$

But $S(q^k p^i) = S(p^i)$ for $i \geq k$, because $\max(S(p^i), S(q^k)) = S(p^i)$ for $i \geq k$ consequently,

$$F_s(p^a q^2) = F_s(p^a q) + F_s(p^a) + S(q^2) + S(q^2 p) - p - 1 =$$
$$= 3F_s(p^a) + S(q) + S(q^2) + S(q^2 p) - p - 2$$

In short

$$F_s(p^a q) = 2F_s(p^a) + S(q) - 1$$
$$F_s(p^a q^2) = F_s(p^a q) + F_s(p^a) + S(q^2) + S(q^2 p) - p - 1$$
$$F_s(p^a q^3) = F_s(p^a q^2) + F_s(p^a) + S(q^3) + S(q^3 p) + S(q^3 p^2) - p - 2p - 1$$

...............

$$F_s(p^a q^k) = F_s(p^a q^{k-1}) + F_s(p^a) + S(q^k) + S(q^k p) + S(q^k p^2) +$$
$$+ ... + S(q^k p^{k-1}) - p - 2p - ... - (k-1)p - 1$$

Hence

$$F_s(p^a q^k) = (k+1)F_s(p^a) + \sum_{i=1}^{k} s(q^i) + \sum_{i=2}^{k} S(q^i p) + \sum_{i=3}^{k} S(q^i p^2) +$$
$$+ ... + \sum_{i=k-1}^{k} S(q^i p^{k-2}) + S(q^k p^{k-1}) - k - \frac{pk(k^2-1)}{6} \qquad (18)$$

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# SOME ELEMENTARY ALGEBRAIC CONSIDERATIONS INSPIRED BY SMARANDACHE'S FUNCTION (II)

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In this paper we continue the algebraic consideration begun in [2]. As it was sun, two of the proprieties of Smarandache's function are hold:

(1) $S$ is a surjective function;

(2) $S([m,n]) = \max\{S(m), S(n)\}$, where $[m,n]$ is the smallest common multiple of $m$ and $n$.

That is on $\aleph$ there are considered both of the divisibility order "$\preceq_d$" having the known properties and the total order with the usual order $\leq$ with all its properties. $\aleph$ has also the algebric usual operations "$+$" and "$\cdot$". For instance:

$$a \leq b \Longleftrightarrow (\exists)\, u \in \aleph \text{ so that } b = a + u.$$

Here we can stand out:

: the universal algebra $(\aleph^*, \Omega)$, the set of operations is $\Omega = \{\vee_d, \varphi_0\}$ where $\vee_d : (\aleph^*)^2 \to \aleph^*$ is given by $m \vee_d n = [m,n]$, and $\varphi_0 : (\aleph^*)^0 \to \aleph^*$ the null operation that fixes 1-unique particular element with the role of neutral element for "$\vee_d$"-that means $\varphi_0(\{\emptyset\}) = 1$ and $1 = e_{\vee_d}$;

: the universal algebra $(\aleph^*, \Omega')$, the set of operations is $\Omega' = \{\vee, \psi_0\}$ where $\vee : \aleph^2 \to \aleph$ is given by $x \vee y = \sup\{x,y\}$ and $\psi_0 : \aleph^0 \to \aleph$ a null operation with $\psi_0(\{\emptyset\}) = 0$ the unique particular element with the role of neutral element for $\vee$, so $0 = e_\vee$.

We observe that the universal algebras $(\aleph^*, \Omega)$ and $(\aleph^*, \Omega')$ are of the same type:

$$\begin{pmatrix} \vee_d & \varphi_0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} \vee & \psi_0 \\ 2 & 0 \end{pmatrix}$$

and with the similarity (bijective) $\vee_d \Longleftrightarrow \vee$ and $\varphi_0 \Longleftrightarrow \psi_0$, Smarandache's function $S : \aleph^* \to \aleph$ is a morphism surjective between them

$$\begin{aligned} S(x \vee_d y) &= S(x) \vee S(y), \forall x, y \in \aleph^* \text{ from (2) and} \\ S(\varphi_0(\{\emptyset\})) &= \psi_0(\{\emptyset\}) \Longleftrightarrow S(1) = 0. \end{aligned}$$

**Problem 3.** If $S : \aleph^* \to \aleph$ is Smarandache's function defined as we know by

$$S(n) = m \Longleftrightarrow m = \min\{k : n \text{ divides } k!\}$$

and $I$ is a some set, then there exists an unique $s : (\aleph^*)^I \to \aleph^I$ a surjective morphisme between the universal algebras $\left((\aleph^*)^I, \Omega\right)$ and $\left(\aleph^I, \Omega'\right)$ so that $p_i \circ s = S \circ \tilde{p}_i$, for $i \in I$, where $p_j : \aleph^I \to \aleph$ defined by $a = \{a_i\}_{i\in I} \in \aleph^I$, $p_j(a) = a_j$, for each $j \in I$, $p_j$ are the canonical projections, morphismes between $\left(\aleph^I, \Omega'\right)$ and $(\aleph, \Omega')$-universal algebras of the same kind and $\tilde{p}_j : (\aleph^*)^I \to \aleph^*$ analogously between $\left((\aleph^*)^I, \Omega\right)$ and $(\aleph^*, \Omega)$. We shall go over the following three steps in order to justify the assumption:

**Theorem 0.1.** *Let by $(\aleph, \Omega)$ is an universal algebra more complexe with*

$$\Omega = \{\vee_d, \wedge_d, \varphi_0, \overline{\varphi}_0\}$$

*of the kind $\tau : \Omega \to \aleph$ given by*

$$\tau = \begin{pmatrix} \vee_d & \wedge_d & \varphi_0 & \overline{\varphi}_0 \\ 2 & 2 & 0 & 0 \end{pmatrix}$$

*where $\vee_d$ and $\varphi_0$ are defined as above and $\wedge_d : \aleph^2 \to \aleph$, for each $x, y \in \aleph, x \wedge_d y = (x, y)$ where $(x, y)$ is the biggest common divisor of $x$ and $y$ and $\overline{\varphi}_0 : \aleph^0 \to \aleph$ is the null operation that fixes 0-an unique particular element having the role of the neutral element for "$\wedge_d$" i.e. $\overline{\varphi}_0(\{\emptyset\}) = 0$ so $0 = e_{\wedge_d}$ and $I$ a set. Then $\left(\aleph', \tilde{\Omega}\right)$ with $\tilde{\Omega} = \{\omega_1, \omega_2, \omega_0, \overline{\omega}_0\}$ becomes an universal algebra of the same kind as $(\aleph, \Omega)$ and the canonical projections become surjective morphismes between $\left(\aleph^I, \tilde{\Omega}\right)$ and $(\aleph, \Omega)$, an universal algebra that satisfies the following property of universality:*
*$(\mathcal{U})$ : for every $\left(A, \overline{\Omega}\right)$ with $\overline{\Omega} = \{\top, \perp, \sigma_0, \overline{\sigma}_0\}$ an universal algebra of the same kind*

$$\tau = \begin{pmatrix} \top & \perp & \sigma_0 & \overline{\sigma}_0 \\ 2 & 2 & 0 & 0 \end{pmatrix}$$

*and $u_i : A \to \aleph$, for each $i \in I$, morphismes between $\left(A, \overline{\Omega}\right)$ and $(\aleph, \Omega)$, exists an unique $u : A \to \aleph^I$ morphism between the universal algebras $\left(A, \overline{\Omega}\right)$ and $\left(\aleph^I, \tilde{\Omega}\right)$ so that $p_j \circ u = u_j$, for each $j \in I$, where $p_j : \aleph^I \to \aleph$ with each $a = \{a_i\}_{i\in I} \in \aleph^I, p_j(a) = a_j$, for each $j \in I$ are the canonical projections morphismes between $\left(\aleph^I, \tilde{\Omega}\right)$ and $(\aleph, \Omega)$.*

Proof. Indeed $\left(\aleph^I, \tilde{\Omega}\right)$ with $\tilde{\Omega} = \{\omega_1, \omega_2, \omega_0, \overline{\omega}_0\}$ becomes an universal algebra because we can well define:

$\omega_1$ : $(\aleph^I)^2 \rightarrow \aleph^I$ by each $a = \{a_i\}_{i\in I}, b = \{b_i\}_{i\in I} \in \aleph; \omega_1(a,b) = \{a_i \vee_d b_i\}_{i\in I} \in \aleph^I$ and

$\omega_2$ : $(\aleph^I)^2 \rightarrow \aleph^I$ by $\omega_2(a,b) = \{a_i \wedge_d b_i\}_{i\in I} \aleph^I$ and also

$\omega_0$ : $(\aleph^I)^0 \rightarrow \aleph^I$ with $\omega_0(\{\emptyset\}) = \{e_i = 1\}_{i\in I} \in \aleph^I$

an unique particular element (the family with all the components equal with 1) fixed by $\omega_0$ and having the role of neutral for the operation $\omega_1$ noted with $e_{\omega_1}$ and then $\overline{\omega}_0 : (\aleph^I)^0 \rightarrow \aleph^I$ with $\overline{\omega}_0(\{\emptyset\}) = \{\overline{e}_i = 0\}_{i\in I}$ an unique particular element fixed by $\overline{\omega}_0$ but hawing the role of neutral for the operation $\omega_2$ noted $\overline{e}_{\omega_2}$ (the verifies are imediate).

The canonical projections $p_j : \aleph^I \rightarrow \aleph$, defined as above, become morphismes between $\left(\aleph^I, \tilde{\Omega}\right)$ and $(\aleph, \Omega)$. Indeed the two universal algebras are of the same kind

$$\begin{pmatrix} \omega_1 & \omega_2 & \omega_0 & \overline{\omega}_0 \\ 2 & 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \vee_d & \wedge_d & \varphi_0 & \overline{\varphi}_0 \\ 2 & 2 & 0 & 0 \end{pmatrix}$$

and with the similairity (bijective) $\omega_1 \Longleftrightarrow \vee_d; \omega_2 \Longleftrightarrow \wedge_d; \omega_0 \Longleftrightarrow \varphi_0; \overline{\omega}_0 \Longleftrightarrow \overline{\varphi}_0$ we observe first that for each $a, b \in \aleph^I, p_j(\omega_1(a,b)) = p_j(a) \vee_d p_j(b)$, for each $j \in I$ because $a = \{a_i\}_{i\in I}, b = \{b_i\}_{i\in I}, p_j(\omega_1(a,b)) = p_j\left(\{a_i \vee_d b_i\}_{i\in I}\right) = a_j \vee_d b_j$ and $p_j(a) \vee_d p_j(b) = p_j(\{a_i\}_{i\in I}) \vee_d p_j\left(\{b_i\}_{i\in I}\right) = a_j \vee_d b_j$ and then $p_j(\omega_0(\{\emptyset\})) = \varphi_0(\{\emptyset\}) \Longleftrightarrow p_j\left(\{e_i = 1\}_{i\in I}\right) = 1 \Longleftrightarrow p_j(e_{\omega_1}) = e_{\vee_d}$; analogously we prove that $p_j$, for each $j \in I$ keeps the operations $\omega_2$ and $\overline{\omega}_0$, too. So, it was built the universal algebra $\left(\aleph^I, \tilde{\Omega}\right)$ with $\tilde{\Omega} = \{\omega_1, \omega_2, \omega_0, \overline{\omega}_0\}$ of the kind $\tau$ described above.

We prove the property of universality $(\mathcal{U})$.

We observe for this purpose that the $u_i$ morphismes for each $i \in I$, presumes the coditions: for each $x, y \in S, u_i(x \top y) = u_i(x) \vee_d u_i(y); u_i(x \bot y) = u_i(x) \wedge_d u_i(y); u_i(\sigma_0(\{\emptyset\})) = \varphi_0(\{\emptyset\}) \Longleftrightarrow u_i(e_\top) = e_{\vee_d} = 1$ and $u_i(\overline{\sigma}_0(\{\emptyset\})) = \overline{\varphi}_0(\{\emptyset\}) \Longleftrightarrow u_i(\overline{e}_\bot) = e_{\wedge_d} = 0$ which show also the similarity (bijective) between $\overline{\Omega}$ and $\Omega$. We also observe that $(S, \overline{\Omega})$ and $\left(\aleph^I, \tilde{\Omega}\right)$ are of the same kind and there is a similarity (bijective) between $\overline{\Omega}$ and $\tilde{\Omega}$ given by $\top \Longleftrightarrow \omega_1; \bot \Longleftrightarrow \omega_2; \sigma_0 \Longleftrightarrow \omega_0; \overline{\sigma}_0 \Longleftrightarrow \overline{\omega}_0$.

We define the corespondance $u : A \rightarrow \aleph^I$ by $u(x) = \{u_i(x)\}_{i\in I}$.

$u$ is the function:

- for each $x \in A, (\exists)\, u_i(x) \in \aleph$ for each $i \in I$ ($u_i$-functions) so $(\exists)\, \{u_i(x)\}_{i\in I}$ that can be imagines for $x$;

- $x_1 = x_2 \Longrightarrow u(x_1) = u(x_2)$ because $x_1 = x_2$ and $u_i$-functions lead to $u_i(x_1) = u_i(x_2)$ for each $i \in I \Longrightarrow \{u_i(x_1)\}_{i\in I} = \{u_i(x_2)\}_{i\in I} \Longrightarrow u(x_1) = u(x_2)$.

u is a morphisme: for each $x, y \in A$, $u(x\top y) = \{u_i(x\top y)\}_{i\in I} = \{u_i(x) \vee_d u_i(y)\}_{i\in I} = \omega_1\left(\{u_i(x)\}_{i\in I}, \{u_i(y)\}_{i\in I}\right) = \omega_1(u(x), u(y))$. Then $u(\sigma_0(\{\emptyset\})) = \omega_0(\{\emptyset\}) \Longleftrightarrow u(e_\top) = e_{\omega_1}$ because for each $\{a_i\}_{i\in I} \in \aleph^I$, $\omega_1(\{a_i\}_{i\in I}, \{u_i(e_\top)\}_{i\in I}) = \{a_i \vee_d u_i(e_\top)\}_{i\in I} = \{a_i \vee_d 1\}_{i\in I} = \{a_i\}_{i\in I}$.

Analogously we prove that $u$ keeps the operations: $\perp$ and $\overline{\sigma}_0$.

Besides the condition $p_j \circ u = u_j$, for each $j \in I$ is verified (by the definition: for each $x \in S$, $(p_j \circ u)(x) = p_j(u(x)) = p_j(\{u_i(x)\}_{i\in I}) = u_j(x)$).

For the singleness of $u$ we consider $u$ and $\overline{u}$, two morphismes so that $p_j \circ u = u_j$ (1) and $p_j \circ \overline{u} = u_j$ (2), for every $j \in I$. Then for every $x \in A$, if $u(x) = \{u_i(x)\}_{i\in I}$ and $\overline{u}(x) = \{z_i\}_{i\in I}$ we can see that $y_j = u_j(x) = (p_j \circ \overline{u})(x) = p_j(\{z_i\}_{i\in I}) = z_j$, for every $j \in I$ i.e. $u(x) = \overline{u}(x)$, for every $x \in A \Longleftrightarrow u = \overline{u}$.

Consequence . Particularly, taking $A = \aleph^I$ and $u_i = p_i$ we obtain: the morphisme $u : \aleph^I \to \aleph^i$ verifies the condition $p_j \circ u = p_j$, for every $j \in I$, if and only if, $u = 1_{\aleph^I}$.

The property of universality establishes the universal algebra $\left(\aleph^I, \tilde{\Omega}\right)$ until an isomorphisme as it results from:

**Theorem 0.2.** *If $(P, \Omega)$ is an universal algebra of the same kind as $(\aleph, \Omega)$ and $p'_i : P \to \aleph$, $i \in I$ a family of morphismes between $(P, \Omega)$ and $(\aleph, \Omega)$ so that for every universal algebra $\left(A, \overline{\Omega}\right)$ and every morphisme $u_i : A \to \aleph$, for every $i \in I$ between $\left(A, \overline{\Omega}\right)$ and $(\aleph, \Omega)$ it exists an unique morphisme $u : A \to P$ with $p'_j \circ u = u_i$, for every $i \in I$, then it exists an unique isomorphisme $f : P \to \aleph^I$ with $p_i \circ f = p'_i$, for every $i \in I$.*

Proof. From the property of universality of $\left(\aleph^I, \tilde{\Omega}\right)$ it results an unique $f : P \to \aleph^I$ so that for every $i \in I, p_i \circ f = p'_i$ with $f$ morphisme between $(P, \Omega)$ and $\left(\aleph^I, \tilde{\Omega}\right)$. Applying now the same property of universality to $(P, \Omega) \Longrightarrow$exists an unique $\overline{f} : \aleph^I \to P$ so that $p'_i \circ \overline{f} = p_i$, for every $i \in I$ with $\overline{f}$ morphisme between $\left(\aleph^I, \tilde{\Omega}\right)$ and $(P, \Omega)$. Then $p'_j \circ \overline{f} = p_j \Longleftrightarrow p_j \circ \left(f \circ \overline{f}\right) = p_j$, using the last consequence, we get $f \circ \overline{f} = 1_{\aleph^I}$. Analogously, we prove that $f \circ \overline{f} = 1_P$ from where $\overline{f} = f^{-1}$ and the morphisme $f$ becomes isomorphisme.

We could emphasize other properties (a family of finite support or the case $I-$filter) but we remain at these which are strictly necessary to prove the proposed assertion (Problem 3).

b) Firstly it was built $\left(\aleph^I, \tilde{\Omega}\right)$ being an universal algebra more complexe (with four operations). We try now a similar construction starting from $(\aleph, \Omega^*)$ with $\Omega^* =$

$(\vee, \wedge, \psi_0)$ with "$\vee$" and "$\psi_0$" defined as above and $\wedge : \aleph^2 \to \aleph$ with $x \wedge y = \inf\{x, y\}$ for every $x, y \in \aleph$. ■

**Theorem 0.3.** *Let by $(\aleph, \Omega^*)$ the above universal algebra and $I$ a set. Then:*
*(i) $\left(\aleph^I, \theta\right)$ with $\theta = \{\theta_1, \theta_2, \theta_0\}$ becomes an universal algebra of the same kind $\tau$ as $(\aleph, \Omega^*)$ so $\tau : \theta \to \aleph$ is*

$$\tau = \begin{pmatrix} \theta_1 & \theta_2 & \theta_0 \\ 2 & 2 & 0 \end{pmatrix};$$

*(ii) For every $j \in I$ the canonical projection $p_j : \aleph^I \to \aleph$ defined by every $a = \{a_i\}_{i \in I} \in \aleph^I, p_j(a) = a_j$ is a surjective morphisme between $\left(\aleph^I, \theta\right)$ and $(\aleph, \Omega^*)$ and $\ker p_j = \left\{a \in \aleph^I : a = \{a_i\}_{i \in I} \text{ and } a_j = 0\right\}$ where by definition we have $\ker p_j = \left\{a \in \aleph^I : p_j(a) = e_\vee\right\}$;*
*(iii) For every $j \in I$ the canonical injection $q_j : \aleph \to \aleph^I$ for every $x \in \aleph, q_j(x) = \{a_i\}_{i \in I}$ where $a_i = 0$ if $i \neq j$ and $a_j = x$ is an injective morphisme between $(\aleph, \Omega^*)$ and $\left(\aleph^I, \theta\right)$ and $q_j(\aleph) = \left\{\{a_i\}_{i \in I} : a_i = 0, \forall i \in I - \{j\}\right\}$;*
*(iv) If $j, k \in I$ then:*

$$p_j \circ q_k = \begin{cases} O\text{-the null morphisme} & \text{for } j \neq k, \\ 1_\aleph\text{-the identical morphisme} & \text{for } j = k. \end{cases}$$

Proof. (*i*) We well define the operations $\theta_1 : \left(\aleph^I\right)^2 \to \aleph^I$ by $\forall a = \{a_i\}_{i \in I} \in \aleph^I$ and $b = \{b_i\}_{i \in I} \in \aleph^I, \theta_1(a, b) = \{a_i \vee b_i\}_{i \in I}; \theta_2 : \left(\aleph^I\right)^2 \to \aleph^I$ by $\theta_2(a, b) = \{a_i \wedge b_i\}_{i \in I}$ and $\theta_0 : \left(\aleph^I\right)^0 \to \aleph^I$ by $\theta_0(\{\emptyset\}) = \{e_i = 0\}_{i \in I}$ an unique particular element fixed by $\theta_0$, but with the role of neutral element for $\theta_1$ and noted $e_{\theta_1}$ (the verifications are immediate).

(*ii*) The canonical projections are proved to be morphismes (see the step a)), they keep all the operations and

$$\ker p_j = \left\{a = \{a_i\}_{i \in I} \in \aleph^I : p_j(a) = e_\vee\right\} = \left\{a \in \aleph^I : a_j = 0\right\}.$$

(*iii*) For every $x, y \in \aleph, q_j(x \vee y) = \{c_i\}_{i \in I}$ where $c_i = 0$ for every $i \neq j$ and $c_j = x \vee y$ and

$$\theta_1\left(\left\{\begin{array}{ll} a_i = 0, & \forall i \neq j \\ a_j = x & \end{array}\right\}, \left\{\begin{array}{ll} b_i = 0, & \forall i \neq j \\ b_j = y & \end{array}\right\}\right) = \left\{\begin{array}{ll} c_i = 0 & , \forall i \neq j \\ c_j = x \vee y & \end{array}\right\}$$

i.e. $q_j(x \vee y) = \theta_1(q_j(x), q_j(y))$ with $j \in I$, therefore $q_j$ keeps the operation "$\vee$" for every $j \in I$. Then $q_j(\psi(\{\emptyset\}) = \theta_0(\{\emptyset\}) \iff q_j(e_\vee) = \{e_i = 0\}_{i \in I} \iff q_j(0) = \{e_i = 0\}_{i \in I} = e_{\theta_1}$ because $\forall a = \{a_i\}_{i \in I} \in \aleph^I, \theta_1(q_j(0), a) = \theta_1\left(\{e_i = 0\}_{i \in I}, \{a_i\}_{i \in I}\right) =$

$\{e_i \vee a_i\}_{i\in I} = \{a_i\}_{i\in I} = a$ enough for $q_j(0) = e_{\theta_1}$ because $\theta_1$ is obviously comutative -this observation refers to all the similar situations met before. Analogously we also prove that $\theta_2$ is kept by $q_j$ and this one for every $j \in I$.

(*iv*) For every $x \in \aleph$, $(p_j \circ q_k)(x) = p_j(q_k(x)) = p_j\left(\left\{\begin{array}{ll} a_i = 0 & ,\forall i \neq k \\ a_k = x & \end{array}\right\}\right) = 0 \Longrightarrow p_j \circ q_k = \mathcal{O}$ for $j \neq k$ and $(p_j \circ q_j)(x) = p_j(q_j(x)) = p_j\left(\left\{\begin{array}{ll} a_i = 0 & ,\forall i \neq j \\ a_j = x & \end{array}\right\}\right) = x \Longrightarrow p_j \circ q_k = 1_{\mathbf{N}}$ for $j = k$. ■

The universal algebra $\left(\aleph^I, \theta\right)$ satisfies the following property of universality:

**Theorem 0.4.** *For every* $\left(A, \bar{\theta}\right)$ *with* $\bar{\theta} = \{\top, \bot, \theta_0\}$ *an universal algebra of the some kind* $\tau : \bar{\theta} \to \aleph$

$$\tau = \left(\begin{array}{ccc} \top & \bot & \theta_0 \\ 2 & 2 & 0 \end{array}\right)$$

*as* $\left(\aleph^I, \theta\right)$ *and* $u_i : A \to \aleph$ *for every* $i \in I$ *morphismes between* $\left(A, \bar{\theta}\right)$ *and* $(\aleph, \Omega^*)$, *exists an unique* $u : A \to \aleph^I$ *morphisme between the universal algebras* $\left(A, \bar{\theta}\right)$ *and* $\left(\aleph^I, \theta\right)$ *so that* $p_j \circ u = u_j$, *for every* $j \in I$ *with* $p_j : \aleph^I \to \aleph, \forall \mathrm{a} = \{a_i\}_{i\in I} \in \aleph^I, p_j(a) = a_j$ *the canonical projections morphismes between* $\left(\aleph^I, \theta\right)$ *and* $(\aleph, \Omega^*)$.

Proof. The proof repeats the other one from the Theorem 1, step a). ■

The property of universality establishes the universal algebra $\left(\aleph^I, \theta\right)$ until an isomorphisme, which we can state by:

If $(P, \Omega^*)$ it is an universal algebra of the same kind as $(\aleph, \Omega^*)$ and $p'_i : P \to \aleph$ for every $i \in I$ a family of morphismes between $(P, \Omega^*)$ and $(\aleph, \Omega^*)$ so that for every universal algebra $\left(A, \bar{\theta}\right)$ and every morphismes $u_i : A \to \aleph, \forall i \in I$ between $\left(A, \bar{\theta}\right)$ and $(\aleph, \Omega^*)$ exists an unique morphisme $u : A \to P$ with $p'_i \circ u = u_i$, for every $i \in I$ then it exists an unique isomorphisme $f : P \to \aleph^I$ with $p_i \circ f = p'_i$, for every $i \in I$.

c) This third step contains the proof of the stated proposition (Problem 3).

As $(\aleph^*, \Omega)$ with $\Omega = (V_d, l_0\}$ is an universal algebra, in accordance with step a) it exists an universal algebra$\left((\aleph^*)^I, \Omega\right)$ with $\Omega = \{\omega_1, \omega_0\}$ defined by:

$$\begin{array}{rcl} \omega_1 & : & ((\aleph^*)^I)^2 \to (\aleph^*)^I \text{ by every } a = \{a_i\}_{i\in I} \text{ and } b = \{b_i\}_{i\in I} \in (\aleph^*)^I, \\ \omega_1(a, b) & = & \{a_i V_d b_i\}_{i\in I} \end{array}$$

and

$$\omega_0 : ((\aleph^*)^I)^0 \to (\aleph^*)^I \text{ by } \omega_0(\{\emptyset\}) = \{e_i = 1\}_{i\in I} = e_{\omega_1},$$

the canonical projections being certainly morphismes between $\left((\aleph^*)^I, \Omega\right)$ and $(\aleph^*, \Omega)$.

As $\left(\aleph, \Omega'\right)$ with $\Omega' = \{V, \Psi_0\}$ is an universal algebra, in accordance with step b) it exists an universal algebra $\left(\aleph^I, \Omega'\right)$ with $\Omega' = \{\theta_1, \theta_0\}$ defined by:

$$\theta_1 : (\aleph^I)^2 \to \aleph^I \text{ by every } a = \{a_i\}_{i\in I}, b = \{b_i\}_{i\in I} \in \aleph^I, \theta_1(a,b) = \{a_i V_d b_i\}_{i\in I}$$

and

$$\theta_0 : (\aleph^I)^0 \to \aleph^I \text{ by } \theta_0(\{\emptyset\}) = \{e_i = 0\}_{i\in I} = e_{\theta_1}$$

The universal algebras $\left((\aleph^*)^I, \Omega\right)$ and $\left(\aleph^I, \Omega'\right)$ are of the same kind

$$\begin{matrix} \omega_1 & \omega_2 \\ 2 & 0 \end{matrix} = \begin{matrix} \theta_1 & \theta_0 \\ 2 & 0 \end{matrix}$$

We use the property of universality for universal algebra $\left(\aleph', \Omega'\right)$: an universal algebra $(A, \Omega)$ can be $\left((\aleph^*)^I, \Omega\right)$ because they are the same kind; the morphismes $u_i : A \to \aleph$ from the assumption will be $\bar{s}_i : (\aleph^*)^I \to \aleph^*$ by every $a = \{a_i\}_{i\in I} \in (\aleph^*)^I, \bar{s}_j(a) = \bar{s}_j\left(\{a_i\}_{i\in I}\right) = s(a_j) \Longleftrightarrow \bar{s}_j = s \circ \bar{p}_j$ for every $j \in I$ where $s : \aleph^* \to \aleph$ is Smarandache's function and $\bar{p}_j : (\aleph^*)^I \to \aleph^*$ the canonical projections, morphismes between $\left((\aleph^*)^I, \Omega\right)$ and $(\aleph^*, \Omega)$. As $s$ is a morphisme berween $(\aleph^*, \Omega)$ and $\left(\aleph, \Omega'\right)$, $\bar{s}_j$ are morphismes (as a composition of morphismes) for every $j \in I$. The assumptions of the property of universality being provided $\Longrightarrow$ exists an unique $s : (\aleph^*)^I \to \aleph^I$ morphism between $\left((\aleph^*)^I, \Omega\right)$ and $\left(\aleph^I, \Omega\right)$ so that $p_j \circ s = \bar{s}_j \Longleftrightarrow p_j \circ s = S \circ \bar{p}_j$, for every $j \in I$. We finish the proof noticing that $s$ is also surjection: $p_j \circ S$ surjection (as a composition of surjections) $\Longrightarrow s$ surjection.

Remark: The proof of the step 3 can be done directly. As the universal algebras from the statement are built, we can define a correspondence $s : (\aleph^*)^I \to (\aleph^*)^I$ by every $a = \{a_i\}_{i\in I} \in (\aleph^*)^I, s(a) = \{S(a_i)\}_{i\in I}$, which is a function, then morphisme between the universal algebra of the same kind $\left((\aleph^*)^I, \Omega\right)$ and $\left(\aleph^I, \Omega'\right)$ and is also surjective, the required conditions being satisfied evidently.

The stated Problem finds a prolongation $s$ of the Smarandache function $S$ to more comlexe sets (for $I = \{1\} \Rightarrow s = S$). The properties of the function $s$ for the limitation to $\aleph^*$ could bring new properties for the Smarandache function.

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# A LINEAR COMBINATION WITH SMARANDACHE FUNCTION TO OBTAIN THE IDENTITY[1]

*by*

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In this paper we consider a numerical function $i_p : N^* \to N$ (p is an arbitrary prime number) associated with a particular Smarandache Function $S_p : N^* \to N$ such that $(1/p)S_p(a) + i_p(a) = a$.

**1. INTRODUCTION.** In [7] is defined a numerical function $S: N^* \to N$, $S(n)$ is the smallest integer such that $S(n)!$ is divisible by $n$. This function may be extended to all integers by defining $S(-n) = S(n)$.

If a and b are relatively prime then $S(a \cdot b) = \max\{S(a), S(b)\}$, and if $[a, b]$ is the last common multiple of a and b then $S([a \cdot b]) = \max\{S(a), S(b)\}$.

Suppose that $n = p_1^{a_1} p_2^{a_2} \ldots . p_r^{a_r}$ is the factorization of n into primes. In this case,

$$S(n) = \max\{S(p_i^{a_i} | i = 1, \ldots, r\} \qquad (1)$$

Let $a_n(p) = (p^n - 1)/(p - 1)$ and $[p]$ be the generalized numerical scale generated by $(a_n(p))_{n \in N}$ :

$$[p] : a_1(p), a_2(p), \ldots, a_n(p), \ldots$$

By $(p)$ we shall note the standard scale induced by the net $b_n(p) = p^n$ :

$$(p) : 1, p, p^2, p^3, \ldots, p^n, \ldots$$

In [2] it is proved that

$$S(p^a) = p\left(a_{[p]}\right)_{[p]} \qquad (2)$$

That is the value of $S(p^a)$ is obtained multiplying by p the number obtained writing the exponent a in the generalized scale [p] and "reading" it in the standard scale $(p)$.

Let us observe that the calculus in the generalized scale [p] is different from the calculus in the standard scale (p), because

$$a_{n+1}(p) = pa_n(p) + 1 \text{ and } b_{n+1}(p) = pb_n(p) \qquad (3)$$

We have also

$$a_m(p) \le a \Leftrightarrow (p^m - 1)/(p - 1) \le a \Leftrightarrow p^m \le (p - 1) \cdot a + 1 \Leftrightarrow m \le \log_p((p - 1) \cdot a + 1)$$

so if

$$a_{[p]} = v_t a_t(p) + v_{t-1} a_{t-1}(p) + \ldots + v_1 a_1(p) = \overline{v_t v_{t-1} \ldots v_{1[p]}}$$

is the expression of a in the scale [p] then t is the integer part of $\log_p((p - 1) \cdot a + 1)$

$$t = \left[\log_p((p - 1) \cdot a + 1)\right]$$

and the digit $v_t$ is obtained from $a = v_t a_t(p) + r_{t-1}$.

In [1] it is proved that

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$$S(p^a)=(p-1)\cdot a+\sigma_{[p]}(a) \tag{4}$$

where $\sigma_{[p]}(a)=v_1+v_2+...+v_\upsilon$.

A Legendre formula asert that

$$a!=\prod_{\substack{p_i\le a\\ p_i\ \text{prim}}} p_i^{E_{p_i}(a)}$$

where $E_p(a)=\sum_{j\ge1}\left[\frac{a}{p^j}\right]$.

We have also that ([5])

$$E_p(a)=\frac{\left(a-\sigma_{[p]}(a)\right)}{p-1} \tag{5}$$

and ([1]) $E_p(a)=\left(\left[\frac{a}{p}\right]_{(p)}\right)_{[p]}$.

In [1] is given also the following relation between the function $E_p$ and the Smarandache function

$$S(p^a)=\frac{(p-1)^2}{p}\left(E_p(a)+a\right)+\frac{p-1}{p}\sigma_{[p]}(a)+\sigma_{[p]}(a)$$

There exist a great number of problems concerning the Smarandache function. We present some of these problem.

P. Gronas find ([3]) the solution of the diophantine equation $F_S(n)=n$, where $F_S(n)=\sum_{d|n}S(d)$. The solution are n=9, n=16 or n=24, or n=2p, where p is a prime number.

T. Yau ([8]) find the triplets which verifies the Fibonacci relationship

$$S(n)=S(n+1)+S(n+2).$$

Checking the first 1200 numbers, he find just two triplets which verifies this relationship: (9,10,11) and (119,120,121). He can't find theoretical proof.

The following conjecture that: "the equation $S(x)=S(x+1)$, has no solution", was not completely solved until now.

**2. The Function $i_p(a)$.** In this section we shall note $S(p^a)=S_p(a)$. From the Legendre formula it results ([4]) that

$$S_p(a)=p\left(a-i_p(a)\right) \text{ with } 0\le i_p(a)\le\left[\frac{a-1}{p}\right]. \tag{6}$$

That is we have

$$\frac{1}{p}S_p(a)+i_p(a)=a \tag{7}$$

and so for each function $S_p$ there exists a function $i_p$ such that we have the linear combination (7) to obtain the identity.

In the following we keep out some formulae for the calculus of $i_p$. We shall obtain a duality relation between $i_p$ and $E_p$.

Let $a_{(p)}=\overline{u_k u_{k-1}.....u_1 u_0}=u_k p^k+u_{k-1}p^{k-1}+.....+u_1 p+u_0$.

Then

$$a=(p-1)\left(u_k\frac{p^k-1}{p-1}+u_{k-1}\frac{p^{k-1}-1}{p-1}+....+u_1\frac{p-1}{p-1}\right)+(u_k+u_{k-1}+...+u_1)+u_0=$$

$$(p-1)\left(\left[\frac{a}{p}\right]_{(p)}\right)_{[p]}+\sigma_{(p)}(a)=(p-1)E_{p(a)}+\sigma_{(p)}(a) \tag{8}$$

From (4) it results

$$a=\frac{S_p(a)-\sigma_{[p]}(a)}{p-1} \tag{9}$$

From (8) and (9) we deduce

$$(p-1)E_p(a)+\sigma_{(p)}(a)=\frac{S_p(a)-\sigma_{[p]}(a)}{p-1}.$$

So,

$$S_p(a)=(p-1)^2E_p(a)+(p-1)\sigma_{(p)}(a)+\sigma_{[p]}(a) \tag{10}$$

From (4) and (7) it results

$$i_p(a)=\frac{a-\sigma_{[p]}(a)}{p} \tag{11}$$

and it is easy to observe a complementary with the equality (5).
Combining (5) and (11) it results

$$i_p(a)=\frac{(p-1)E_p(a)+\sigma_{(p)}(a)-\sigma_{[p]}}{p} \tag{12}$$

From

$$a=\overline{\upsilon_t\upsilon_{t-1}...\upsilon_{1[p]}}=\upsilon_t(p^{t-1}+p^{t-2}+.........+p+1)+\upsilon_{t-1}(p^{t-2}+p^{t-3}+.........+p+1)+$$
$$+......+\upsilon_2(p+1)+\upsilon_1$$

it results that

$$a=(\upsilon_t p^{t-1}+\upsilon_{t-1}p^{t-2}+......+\upsilon_2 p+\upsilon_1)+\upsilon_t(p^{t-2}+p^{t-1}+....+1)+\upsilon_{t-1}(p^{t-3}+p^{t-4}+...+1)+.....+$$
$$\upsilon_3(p+1)+\upsilon_2=(a_{[p]})_{(p)}+\left[\frac{a}{p}\right]-\left[\frac{\sigma_{[p]}(a)}{p}\right]$$

because

$$\left[\frac{a}{p}\right]=\left[\upsilon_t(p^{t-2}+p^{t-3}+...+p+1)+\frac{\upsilon_t}{p}+\upsilon_{t-1}(p^{t-3}+p^{t-4}+....+p+1)+\frac{\upsilon_{t-1}}{p}+....+\right.$$
$$\left.+\upsilon_3(p+1)+\frac{\upsilon_3}{p}+\upsilon_2+\frac{\upsilon_2}{p}+\frac{\upsilon_1}{p}\right]=\upsilon_t(p^{t-2}+p^{t-3}+...+p+1)+$$
$$+\upsilon_{t-1}(p^{t-3}+p^{t-4}+...+p+1)+...+\upsilon_3(p+1)+\upsilon_2+\left[\frac{\sigma_{[p]}(a)}{p}\right]$$

we have $[n+x]=n+[x]$.
Then

$$a=(a_{[p]})_{(p)}+\left[\frac{a}{p}\right]-\left[\frac{\sigma_{[p]}(a)}{p}\right] \tag{13}$$

or

$$a=\frac{S_p(a)}{p}+\left[\frac{a}{p}\right]-\left[\frac{\sigma_{[p]}(a)}{p}\right]$$

It results that

$$S_p(a)=p\left(a-\left(\left[\frac{a}{p}\right]-\left[\frac{\sigma_{[p]}(a)}{p}\right]\right)\right) \tag{14}$$

From (11) and (14) we obtain

$$i_p(a)=\left[\frac{a}{p}\right]-\left[\frac{\sigma_{[p]}(a)}{p}\right] \tag{15}$$

It is know that there exists $m,n\in N$ such that the relation

$$\left[\frac{m-n}{p}\right]=\left[\frac{m}{p}\right]-\left[\frac{n}{p}\right] \tag{16}$$

is not verifies.

But if $\frac{m-n}{p}\in N$ then the relation (16) is satisfied.

From (11) and (15) it results

$$\left[\frac{a-\sigma_{[p]}(a)}{p}\right]=\left[\frac{a}{p}\right]-\left[\frac{\sigma_{[p]}(a)}{p}\right].$$

This equality results also by the fact that $i_p(a)\in N$.

From (2) and (11) or from (13) and (15) it results that

$$i_p(a)=a-\left(a_{[p]}\right)_{(p)} \tag{17}$$

From the condition on $i_p$ in (6) it results that $\Delta=\left[\frac{a-1}{p}\right]-i_p(a)\geq 0$.

To calculate the difference $\Delta=\left[\frac{a-1}{p}\right]-i_p(a)$ we observe that

$$\Delta=\left[\frac{a-1}{p}\right]-i_p(a)=\left[\frac{a-1}{p}\right]-\left[\frac{a}{p}\right]+\left[\frac{\sigma_{[p]}(a)}{p}\right] \tag{18}$$

For $a\in[kp+1,kp+p-1]$ we have $\left[\frac{a-1}{p}\right]=\left[\frac{a}{p}\right]$ so

$$\Delta=\left[\frac{a-1}{p}\right]-i_p(a)=\left[\frac{\sigma_{[p]}(a)}{p}\right] \tag{19}$$

If $a=kp$ then $\left[\frac{a-1}{p}\right]=\left[\frac{kp-1}{p}\right]=\left[k-\frac{1}{p}\right]=k-1$ and $\left[\frac{a}{p}\right]=k$.

So, (18) becomes

$$\Delta=\left[\frac{a-1}{p}\right]-i_p(a)=\left[\frac{\sigma_{[p]}(a)}{p}\right]-1 \tag{20}$$

Analogously, if $a=kp+p$, we have

$$\left[\frac{a-1}{p}\right]=\left[\frac{p(k+1)-1}{}p\right]=\left[k+1-\frac{1}{p}\right]=k \quad \text{and} \quad \left[\frac{a}{p}\right]=k+1$$

so, (18) has the form (20).

For any number a, for which $\Delta$ is given by (19) or by (20), we deduce that $\Delta$ is maximum when $\sigma_{[p]}(a)$ is maximum, so when

$$a_M=\overline{\underbrace{(p-1)(p-1)\ldots(p-1)p}_{t\ \text{terms}}}_{[p]} \tag{21}$$

That is

$$a_M = (p-1)a_t(p) + (p-1)a_{t-1}(p) + \ldots + (p-1)a_2(p) + p =$$
$$= (p-1)\left(\frac{p^t-1}{p-1} + \frac{p^{t-1}-1}{p-1} + \ldots + \frac{p^2-1}{p-1}\right) + p =$$
$$= (p^t + p^{t-1} + \ldots + p^2 + p) - (t-1) = pa_t(p) - (t-1)$$

It results that $a_M$ is not multiple of p if and only if $t-1$ is not a multiple of p.
In this case $\sigma_{[p]}(a) = (t-1)(p-1) + p = pt - t + 1$ and

$$\Delta = \left[\frac{\sigma_{[p]}(a)}{p}\right] = \left[t - \frac{t-1}{p}\right] = t - \left[\frac{t-1}{p}\right].$$

So $i_p(a_M) \geq \left[\frac{a_M - 1}{p}\right] - t$ or $i_p(a_M) \in \left[\left[\frac{a_M-1}{p}\right] - t, \left[\frac{a_M-1}{p}\right]\right]$. If $t-1 \in (kp, kp+p)$ then $\left[\frac{t-1}{p}\right] = k$ and $k(p-1)+1 < \Delta(a_M) < k(p-1)+p+1$ so $\lim_{a_M \to \infty} \Delta(a_M) = \infty$.

We also observe that

$$\left[\frac{a_M-1}{p}\right] = a_t(p) - \left[\frac{t-1}{p}\right] = \frac{p^{t+1}-1}{p-1} - \left[\frac{t-1}{p}\right] \in \left[\frac{p^{kp+1}-1}{p-1} - k, \frac{p^{kp+p+1}-1}{p-1} - k\right].$$

Then if $a_M \to \infty$ (as $p^x$), it results that $\Delta(a_M) \to \infty$ (as x).

From $\frac{i_p(a_M)}{\left[\frac{a_M-1}{p}\right]} = \frac{a_t(p) - t}{a_t(p) - \left[\frac{t-2}{p}\right]} \to 1$ it results $\lim_{a \to \infty} \frac{i_p(a)}{[a-1]p} = 1$.

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# EXAMPLES OF SMARANDACHE MAGIC SQUARES

by

M.R. Mudge

For $n \geq 2$, let A be a set of $n^2$ elements, and l a n-ary law defined on A.

As a generalization of the XVI-th - XVII-th centuries magic squares, we present the *Smarandache magic square of order n*, which is: 2 square array of rows of elements of A arranged so that the law l applied to each horizontal and vertical row and diagonal give the same result.

If A is an arithmetical progression and l the addition of n numbers, then many magic squares have been found. Look at Durer's 1514 engraving "Melancholia" 's one:

| | | | |
|---|---|---|---|
| 16 | 3 | 2 | 13 |
| 5 | 10 | 11 | 8 |
| 9 | 6 | 7 | 12 |
| 4 | 15 | 14 | 1 |

1. Can you find a such magic square of order at least 3 or 4, when A is a set of prime numbers and l the addition?
2. Same question when A is a set of square numbers, or cube numbers, or special numbers [for example: Fibonacci or Lucas numbers, triangular numbers, Smarandache quotients (i.e. q(m) is the smallest k such that mk is a factorial), etc.].

A similar definition for the *Smarandache magic cube of order n*, where the elements of A are arranged in the form of a cube of lenth n:

a. either each element inside of a unitary cube (that the initial cube is divided in)

b. either each element on a surface of a unitary cube

c. either each element on a vertex of a unitary cube.

3. Study similar questions for this case, which is much more complex.

An interesting law may be $l(a_1, a_2, \ldots, a_n) = a_1 + a_2 - a_3 + a_4 - a_5 + \ldots$

Now some examples of Smarandache Magic Squares: if A is a set of PRIME NUMBERS and l is the operation of addition, for orders at least 3 or 4.

Some examples, with the constant in brackets, elements drawn from the first hundred PRIME NUMBERS are :

| | | |
|---|---|---|
| 83 | 89 | 41 |
| 29 | 71 | 113 |
| 101 | 53 | 59 |

(213)

| | | |
|---|---|---|
| 101 | 491 | 251 |
| 431 | 281 | 131 |
| 311 | 71 | 461 |

(843)

| | | |
|---|---|---|
| 71 | 461 | 311 |
| 521 | 281 | 41 |
| 251 | 101 | 491 |

(843)

| | | |
|---|---|---|
| 113 | 149 | 257 |
| 317 | 173 | 29 |
| 89 | 197 | 233 |

(519)

| | | | | |
|---|---|---|---|---|
| 97 | 907 | 557 | 397 | 197 |
| 367 | 167 | 67 | 877 | 677 |
| 997 | 647 | 337 | 137 | 37 |
| 107 | 157 | 967 | 617 | 307 |
| 587 | 277 | 227 | 127 | 937 |

(2155)

Now recall the year A.D. 1987 and consider the following .. all elements are primes congruent to seven modulo ten ....

| | | |
|---|---|---|
| 967 | 1987 | 2017 |
| 2707 | 1657 | 607 |
| 1297 | 1327 | 2347 |

(4971)

| | | | |
|---|---|---|---|
| 1987 | 9907 | 11677 | 5237 |
| 4877 | 12037 | 9547 | 2347 |
| 10627 | 2707 | 4517 | 10957 |
| 11317 | 4157 | 3067 | 10267 |

(28808)

| | | | | |
|---|---|---|---|---|
| 7 | 2707 | 5237 | 937 | 947 |
| 4157 | 1297 | 227 | 1087 | 3067 |
| 1307 | 1447 | 1987 | 4517 | 577 |
| 2347 | 3797 | 1657 | 1667 | 367 |
| 2017 | 587 | 727 | 1627 | 4877 |

(9835)

What about the years 1993, 1997, & 1999 ?

In Personal Computer World, May 1991, page 288, I examine:
A multiplication magic square such as:

| | | |
|---|---|---|
| 18 | 1 | 12 |
| 4 | 6 | 9 |
| 3 | 36 | 2 |

with constant 216 obtained by multiplication of the elements in any row/column/principal diagonal.

A geometric magic square is obtained using elements which are a given base raised to the powers of the corresponding elements of a magic square .. it is clearly a multiplication magic square.

e.g. from

| | | |
|---|---|---|
| 8 | 1 | 6 |
| 3 | 5 | 7 |
| 4 | 9 | 2 |

C=15

and base 2 obtain

| | | |
|---|---|---|
| 256 | 2 | 64 |
| 8 | 32 | 128 |
| 16 | 512 | 4 |

where $M = 2^{15} = 32768$

Note that Henry Nelson of California has found an order three magic square consisting of *consecutive ten-digit prime numbers*. But "How did he do that" ???

A particular case:

TALISMAN MAGIC SQUARES are a relatively new concept, contain the integers from 1 to $n^2$ in such a way that the difference between any integer and its neighbours (either row-, column- or diagonal-wise) is greater than some given constant, D say.

e.g.

| | | | |
|---|---|---|---|
| 5 | 15 | 9 | 12 |
| 10 | 1 | 6 | 3 |
| 13 | 16 | 11 | 14 |
| 2 | 8 | 4 | 7 |

illustrates D=2.

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## Base Solution
## (The Smarandache Function)

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### Definition of the Smarandache function S(n)

S(n) = the smallest positive integer such that S(n)! is divisible by n.

### Problem A: Ashbacher's problem

*For what triplets n, n-1, n-2 does the Smarandache function satisfy the Fibonacci reccurrence: S(n)=S(n-1)+S(n-2). Solutions have been found for n=11, 121, 4902, 26245, 32112, 64010, 368140 and 415664. Is there a pattern that would lead to the proof that there is an infinite family of solutions?*

The next three triplets n, n-1, n-2 for which the Smarandache funtion S(n) satisfies the relation S(n)=S(n-1)+S(n-2) occur for n=2091206, n=2519648 and n=4573053. Apart from the triplet obtained from n=26245 the triplets have in common that one member is 2 times a prime and the other two members are primes.

This leads to a search for triplets restricted to integers which meet the following requirements:

$$n = xp^a \text{ with } a \le p+1 \text{ and } S(x) < ap \quad (1)$$

$$n-1 = yq^b \text{ with } b \le q+1 \text{ and } S(y) < bq \quad (2)$$

$$n-2 = zr^c \text{ with } c \le r+1 \text{ and } S(z) < cr \quad (3)$$

p,q and r are primes. With then have S(n)=ap, S(n-1)=bq and S(n-2)=cr. From this and by subtracting (2) from (1) and (3) from (2) we get

$$ap = bq + cr \tag{4}$$

$$xp^a - yq^b = 1 \tag{5}$$

$$yq^b - zr^c = 1 \tag{6}$$

Each solution to (4) generates infinitely many solutions to (5) which can be written in the form:

$$x = x_0 + q^b t, \qquad y = y_0 - p^a t \tag{5'}$$

where t is an integer and $(x_0, y_0)$ is the principal solution, which can be otained using Euclid's algorithm.

Solutions to (5') are substituted in (6') in order to obtain integer solutions for z.

$$z = (yq^b - 1)/r^c \tag{6'}$$

Implementation:
Solutions were generated for (a,b,c)=(2,1,1), (a,b,c)=(1,2,1) and (a,b,c)=(1,1,2) with the parameter t restricted to the interval $-9 \leq t \leq 10$. The output is presented on page 5. Since the correctness of these calculations are easily verfied from factorisations of S(n), S(n-1), and S(n-2) some of these are given in an annex. This study strongly indicates that the set of solutions is infinite.

**Problem B: Radu's problem**

*Show that, except for a finite set of numbers, there exists at least one prime number between S(n) and S(n+1).*

The immediate question is <u>what would be this finite set?</u> I order to examine this the following more stringent problem (which replaces "between" with the requirement that S(n) and S(n+1) must also be composite) will be considered.

*Find the set of consecutive integers n and n+1 for which two consecutive primes $p_k$ and $p_{k+1}$ exists so that $p_k < Min(S(n),S(n+1))$ and $p_{k+1} > Max(S(n),S(n+1))$.*

Consider

$$n+1 = xp_r^s$$

$$n = y p_{r+1}^s$$

where $p_r$ and $p_{r+1}$ are consecutive primes. Subtract

$$x p_r^s - y p_{r+1}^s = 1 \qquad (1)$$

The greatest common divisor $(p_r^s, p_{r+1}^s) = 1$ divides the right hand side of (1) which is the condition for this diophantine equation to have infinitely many integer solutions. We are interested in positive integer solutions (x,y) such that the following conditions are met.

I. $S(n+1) = sp_r$, i.e $S(x) < sp_r$

II. $S(n) = sp_{r+1}$, i.e $S(y) < sp_{r+1}$

in addition we require that the interval

III. $sp_r^s < q < sp_{r+1}^s$ is prime free, i.e. q is not a prime.

Euclid's algorithm has been used to obtain principal solutions $(x_0, y_0)$ to (1). The general set of solutions to (1) are then given by

$$x = x_0 + p_{r+1}^s t, \qquad y = y_0 - p_r^s t$$

with t an integer.

Implementation:
The above algorithms have been implemented for various values of the parameters $d = p_{r+1} - p_r$, s and t. A very large set of solutions was obtained. There is no indication that the set would be finite. A pair of primes may produce several solutions. Within the limits set by the design of the program the largest prime difference for which a solution was found is d=42 and the largest exponent which produced solutions is 4. Some numerically large examples illustrating the above facts are given on page 6.

**Problem C: Stuparu's problem**

*Consider numbers written in Smarandache Prime Base 1,2,3,5,7,11,.... given the example that 101 in Smarandache base means* $1 \cdot 3 + 0 \cdot 2 + 1 \cdot 1 = 4_{10}$.

As this leads to several ways to translate a base 10 number into a Base Smarandache number it seems that further precisions are needed. Example

$$111_{\text{Smarandache}} = 1\cdot 3+1\cdot 2+1\cdot 1 = 6_{10}$$

$$1001_{\text{Smarandache}} = 1\cdot 5+0\cdot 3+0\cdot 2+1\cdot 1 = 6_{10}$$

**Equipment and programs**

Computer programs for this study were written in UBASIC ver. 8.77. Extensive use was made of NXTPRM(x) and PRMDIV(n) which are very convenient although they also set an upper limit for the search routines designed in the main program. Programs were run on a dtk 486/33 computer. Further numerical outputs and program codes are available on request.

Smarandache - Ashbacher's problem.

| # | N | S(N) | S(N-1) | S(N-2) | t |
|---|---|---|---|---|---|
| 1 | 11 | 11 | 5 | 6 | 0 |
| 2 | 121 | 22 | 5 | 17 | 0 |
| 3 | 4902 | 43 | 29 | 14 | -4 |
| 4 | 32112 | 223 | 197 | 26 | -1 |
| 5 | 64010 | 173 | 46 | 127 | -1 |
| 6 | 368140 | 233 | 82 | 151 | -1 |
| 7 | 415664 | 313 | 167 | 146 | -8 |
| 8 | 2091206 | 269 | 202 | 67 | -1 |
| 9 | 2519648 | 1109 | 202 | 907 | 0 |
| 10 | 4573053 | 569 | 106 | 463 | -3 |
| 11 | 7783364 | 2591 | 202 | 2389 | 0 |
| 12 | 79269727 | 2861 | 2719 | 142 | 10 |
| 13 | 136193976 | 3433 | 554 | 2879 | -1 |
| 14 | 321022289 | 7589 | 178 | 7411 | 5 |
| 15 | 445810543 | 1714 | 761 | 953 | -1 |
| 16 | 559199345 | 1129 | 662 | 467 | -5 |
| 17 | 670994143 | 6491 | 838 | 5653 | -1 |
| 18 | 836250239 | 9859 | 482 | 9377 | 1 |
| 19 | 893950202 | 2213 | 2062 | 151 | 0 |
| 20 | 1041478032 | 2647 | 1286 | 1361 | -1 |
| 21 | 1148788154 | 2467 | 746 | 1721 | 3 |
| 22 | 1305978672 | 5653 | 1514 | 4139 | 0 |
| 23 | 1834527185 | 3671 | 634 | 3037 | -5 |
| 24 | 2390706171 | 6661 | 2642 | 4019 | 0 |
| 25 | 2502250627 | 2861 | 2578 | 283 | -1 |
| 26 | 3969415464 | 5801 | 1198 | 4603 | -2 |
| 27 | 3970638169 | 2066 | 643 | 1423 | -6 |
| 28 | 6493607750 | 3049 | 1262 | 1787 | 5 |
| 29 | 6964546435 | 2161 | 1814 | 347 | -4 |
| 30 | 11329931930 | 3023 | 2026 | 997 | -4 |
| 31 | 13429326313 | 4778 | 1597 | 3181 | 1 |
| 32 | 13849559620 | 6883 | 2474 | 4409 | 1 |
| 33 | 14988125477 | 3209 | 2986 | 223 | 2 |
| 34 | 17560225226 | 4241 | 3118 | 1123 | -2 |
| 35 | 25184038673 | 5582 | 1951 | 3631 | -2 |
| 36 | 69481145903 | 6301 | 3722 | 2579 | 3 |
| 37 | 155205225351 | 8317 | 4034 | 4283 | -5 |
| 38 | 196209376292 | 7246 | 3257 | 3989 | -5 |
| 39 | 344645609138 | 7226 | 2803 | 4423 | 9 |
| 40 | 401379101876 | 32122 | 653 | 31469 | 2 |
| 41 | 484400122414 | 16811 | 12658 | 4153 | -1 |
| 42 | 533671822944 | 21089 | 18118 | 2971 | 0 |
| 43 | 561967733244 | 21722 | 7159 | 14563 | -1 |
| 44 | 703403257356 | 13147 | 10874 | 2273 | -2 |
| 45 | 859525157632 | 14158 | 3557 | 10601 | -5 |
| 46 | 898606860813 | 19973 | 13402 | 6571 | 1 |
| 47 | 1185892343342 | 18251 | 12022 | 6229 | -2 |
| 48 | 1188795217601 | 29242 | 13049 | 16193 | 0 |
| 49 | 1294530625810 | 17614 | 5807 | 11807 | -3 |
| 50 | 1517767218627 | 11617 | 8318 | 3299 | -8 |
| 51 | 2677290337914 | 33494 | 3631 | 29863 | -3 |
| 52 | 3043063820555 | 14951 | 12202 | 2749 | 5 |
| 53 | 6344309623744 | 22978 | 7451 | 15527 | 6 |
| 54 | 16738688950356 | 30538 | 6977 | 23561 | 10 |
| 55 | 19448047080036 | 34186 | 17027 | 17159 | -4 |

Ashbacher's problem (ASHEDIT.UB), 951206, Henry Ibstedt

Smarandache ' Radu's problem.
q=p(j), p=p(j+1), d=p-q, N=x*q^s or y*p^s, k solutions to x*q^s - y*p^s = +/-1 will examined.

Parameters for this run: d = 2 , s = 2 , k = 15 .

| x | y | q | p | x*q^s | y*p^s |
|---|---|---|---|---|---|
| 13039 | 12198 | 59 | 61 | 45388759 | 45388758 |
| 1876 | 1755 | 59 | 61 | 6530356 | 6530355 |
| 7975 | 7544 | 71 | 73 | 40201975 | 40201976 |
| 26 | 25 | 101 | 103 | 265226 | 265225 |
| 5913 | 5698 | 107 | 109 | 67697937 | 67697938 |
| 113967 | 110968 | 149 | 151 | 2530181367 | 2530181368 |
| 38 | 37 | 149 | 151 | 843638 | 843637 |
| 49063 | 48438 | 311 | 313 | 4745422423 | 4745422422 |
| 636720 | 628609 | 311 | 313 | 61584195120 | 61584195121 |
| 60988 | 60291 | 347 | 349 | 7343504092 | 7343504091 |
| 182614 | 180527 | 347 | 349 | 21988369126 | 21988369127 |
| 1071729 | 1062490 | 461 | 463 | 227764918809 | 227764918810 |
| 116 | 115 | 461 | 463 | 24652436 | 24652435 |
| 214485 | 212636 | 461 | 463 | 45582566685 | 45582566684 |
| 1071961 | 1062720 | 461 | 463 | 227814223681 | 227814223680 |
| 131 | 130 | 521 | 523 | 35558771 | 35558770 |
| 1914834 | 1900217 | 521 | 523 | 519764455794 | 519764455793 |
| 143 | 142 | 569 | 571 | 46297823 | 46297822 |
| 3386439 | 3370000 | 821 | 823 | 2282598729999 | 2282598730000 |
| 206 | 205 | 821 | 823 | 138852446 | 138852445 |
| 2709522 | 2696369 | 821 | 823 | 1826328918402 | 1826328918401 |
| 215 | 214 | 857 | 859 | 157906535 | 157906534 |
| 1475977 | 1469112 | 857 | 859 | 1084029831673 | 1084029831672 |
| 3689620 | 3672459 | 857 | 859 | 2709837719380 | 2709837719379 |
| 221 | 220 | 881 | 883 | 171531581 | 171531580 |
| 2339288 | 2328703 | 881 | 883 | 1815664113368 | 1815664113367 |
| 5649579 | 5628340 | 1061 | 1063 | 6359849721459 | 6359849721460 |
| 266 | 265 | 1061 | 1063 | 299441786 | 299441785 |
| 5650111 | 5628870 | 1061 | 1063 | 6360448605031 | 6360448605030 |
| 597051 | 594868 | 1091 | 1093 | 710658461331 | 710658461332 |
| 664416 | 662113 | 1151 | 1153 | 880218981216 | 880218981217 |
| 1993825 | 1986914 | 1151 | 1153 | 2641421353825 | 2641421353826 |
| 7311461 | 7286118 | 1151 | 1153 | 9686230844261 | 9686230844262 |
| 9970279 | 9935720 | 1151 | 1153 | 13208635589479 | 13208635589480 |
| 8488719 | 8462680 | 1301 | 1303 | 14368014268119 | 14368014268120 |
| 5093101 | 5077478 | 1301 | 1303 | 8620587845701 | 8620587845702 |
| 326 | 325 | 1301 | 1303 | 551787926 | 551787925 |
| 5093753 | 5078128 | 1301 | 1303 | 8621691421553 | 8621691421552 |
| 8489371 | 8463330 | 1301 | 1303 | 14369117843971 | 14369117843970 |
| 2617231 | 2609312 | 1319 | 1321 | 4553356421791 | 4553356421792 |
| 1393198 | 1389861 | 1667 | 1669 | 3871542597022 | 3871542597021 |
| 9749046 | 9725695 | 1667 | 1669 | 27091516689894 | 27091516689895 |
| 14432580 | 14398621 | 1697 | 1699 | 41563073777220 | 41563073777221 |
| 2886176 | 2879385 | 1697 | 1699 | 8311635620384 | 8311635620385 |
| 425 | 424 | 1697 | 1699 | 1223918825 | 1223918824 |
| 431 | 430 | 1721 | 1723 | 1276553471 | 1276553470 |
| 2969160 | 2962271 | 1721 | 1723 | 8794179823560 | 8794179823559 |
| 1600708 | 1597131 | 1787 | 1789 | 5111651305252 | 5111651305251 |
| 1599813 | 1596238 | 1787 | 1789 | 5108793239997 | 5108793239998 |
| 24003460 | 23949821 | 1787 | 1789 | 76651905056740 | 76651905056741 |
| 19295178 | 19253993 | 1871 | 1873 | 67545491209098 | 67545491209097 |
| 1753596 | 1749853 | 1871 | 1873 | 6138710055036 | 6138710055037 |
| 470 | 469 | 1877 | 1879 | 1655870630 | 1655870629 |
| 14123034 | 14092985 | 1877 | 1879 | 49757270653386 | 49757270653385 |
| 7612314 | 7596715 | 1949 | 1951 | 28916143572714 | 28916143572715 |
| 3805913 | 3798114 | 1949 | 1951 | 14457144927713 | 14457144927714 |
| 488 | 487 | 1949 | 1951 | 1853717288 | 1853717287 |
| 19032493 | 18993492 | 1949 | 1951 | 72296846942293 | 72296846942292 |
| 11987503 | 11963528 | 1997 | 1999 | 47806269851527 | 47806269851528 |
| 500 | 499 | 1997 | 1999 | 1994004500 | 1994004499 |
| 521 | 520 | 2081 | 2083 | 2256222281 | 2256222280 |

Smarandache - Radu's problem.
q=p(j), p=p(j+1), d=p-q, N=x*q^s or y*p^s, k solutions to x*q^s - y*p^s = +/-1 will examined.

Parameters for this run: d = 2 , s = 2 , k = 15 .

| x | y | q | p | x*q^s | y*p^s |
|---|---|---|---|---|---|
| 4339410 | 4331081 | 2081 | 2083 | 18792079709010 | 18792079709009 |
| 11162451 | 11141330 | 2111 | 2113 | 49743464802771 | 49743464802770 |
| 2232913 | 2228688 | 2111 | 2113 | 9950577093073 | 9950577093072 |
| 2231856 | 2227633 | 2111 | 2113 | 9945866761776 | 9945866761777 |
| 15626163 | 15596596 | 2111 | 2113 | 69635198326323 | 69635198326324 |
| 33485239 | 33421880 | 2111 | 2113 | 149220973745719 | 149220973745720 |
| 35091287 | 35028624 | 2237 | 2239 | 175602730575503 | 175602730575504 |
| 10025682 | 10007779 | 2237 | 2239 | 50170207068258 | 50170207068259 |
| 560 | 559 | 2237 | 2239 | 2802334640 | 2802334639 |
| 2574748 | 2570211 | 2267 | 2269 | 13232374074172 | 13232374074171 |
| 38612140 | 38544101 | 2267 | 2269 | 198438946368460 | 198438946368461 |
| 22714160 | 22676049 | 2381 | 2383 | 128770230019760 | 128770230019761 |
| 596 | 595 | 2381 | 2383 | 3378819956 | 3378819955 |
| 17036663 | 17008078 | 2381 | 2383 | 96583585449743 | 96583585449742 |
| 16809771 | 16783850 | 2591 | 2593 | 112848716268651 | 112848716268650 |
| 21210178 | 21178283 | 2657 | 2659 | 149736411907522 | 149736411907523 |
| 665 | 664 | 2657 | 2659 | 4694666585 | 4694666584 |
| 14141227 | 14119962 | 2657 | 2659 | 99832099049323 | 99832099049322 |
| 10846754 | 10830625 | 2687 | 2689 | 78313227630626 | 78313227630625 |
| 40482708 | 40423043 | 2711 | 2713 | 297528512582868 | 297528512582867 |
| 11041232 | 11024959 | 2711 | 2713 | 81147766449872 | 81147766449871 |
| 683 | 682 | 2729 | 2731 | 5086602203 | 5086602202 |
| 52209210 | 52132769 | 2729 | 2731 | 388825011131610 | 388825011131609 |
| 39283344 | 39227305 | 2801 | 2803 | 308201442969744 | 308201442969745 |
| 701 | 700 | 2801 | 2803 | 5499766301 | 5499766300 |
| 23571128 | 23537503 | 2801 | 2803 | 184929665407928 | 184929665407927 |
| 31427937 | 31383104 | 2801 | 2803 | 246571053955137 | 246571053955136 |
| 4871101 | 4864860 | 3119 | 3121 | 47386854775261 | 47386854775260 |
| 68783872 | 68699319 | 3251 | 3253 | 726976811951872 | 726976811951871 |
| 5291818 | 5285313 | 3251 | 3253 | 55929229733818 | 55929229733817 |
| 5290191 | 5283688 | 3251 | 3253 | 55912033969191 | 55912033969192 |
| 79364254 | 79266695 | 3251 | 3253 | 838800879890254 | 838800879890255 |
| 815 | 814 | 3257 | 3259 | 8645559935 | 8645559934 |
| 53106220 | 53041059 | 3257 | 3259 | 563353383964780 | 563353383964779 |
| 5687721 | 5680978 | 3371 | 3373 | 64633219552161 | 64633219552162 |
| 866 | 865 | 3461 | 3463 | 10373399186 | 10373399185 |
| 890 | 889 | 3557 | 3559 | 11260501610 | 11260501609 |
| 64188549 | 64116910 | 3581 | 3583 | 823125773602989 | 823125773602990 |
| 38512771 | 38469788 | 3581 | 3583 | 493870868197531 | 493870868197532 |
| 896 | 895 | 3581 | 3583 | 11489910656 | 11489910655 |
| 6744546 | 6737203 | 3671 | 3673 | 90891127331586 | 90891127331587 |
| 46074703 | 46027688 | 3917 | 3919 | 706919053836967 | 706919053836968 |
| 980 | 979 | 3917 | 3919 | 15036031220 | 15036031219 |
| 983 | 982 | 3929 | 3931 | 15174611303 | 15174611302 |
| 15453744 | 15438023 | 3929 | 3931 | 238560079731504 | 238560079731503 |
| 30906505 | 30875064 | 3929 | 3931 | 477104984851705 | 477104984851704 |
| 48071026 | 48023003 | 4001 | 4003 | 769521032279026 | 769521032279027 |
| 1001 | 1000 | 4001 | 4003 | 16024009001 | 16024009000 |
| 48073028 | 48025003 | 4001 | 4003 | 769553080297028 | 769553080297027 |
| 25129997 | 25105444 | 4091 | 4093 | 420582691321157 | 420582691321156 |
| 25127950 | 25103399 | 4091 | 4093 | 420548432153950 | 420548432153951 |
| 125643844 | 125521085 | 4091 | 4093 | 2102810679104164 | 2102810679104165 |
| 8525353 | 8517096 | 4127 | 4129 | 145204912066537 | 145204912066536 |
| 8523288 | 8515033 | 4127 | 4129 | 145169740720152 | 145169740720153 |
| 76717852 | 76643549 | 4127 | 4129 | 1306668351866908 | 1306668351866909 |
| 1055 | 1054 | 4217 | 4219 | 18761158895 | 18761158894 |
| 89000860 | 88916499 | 4217 | 4219 | 1582710214456540 | 1582710214456539 |
| 54008086 | 53957183 | 4241 | 4243 | 971393809450966 | 971393809450967 |
| 1061 | 1060 | 4241 | 4243 | 19083231941 | 19083231940 |
| 97813539 | 97725100 | 4421 | 4423 | 1911789192817899 | 1911789192817900 |
| 19561823 | 19544136 | 4421 | 4423 | 382340544934343 | 382340544934344 |
| 1106 | 1105 | 4421 | 4423 | 21617036546 | 21617036545 |

Smarandache - Radu's problem.
q=p(j), p=p(j+1), d=p-q, N=x*q^s or y*p^s, k solutions to x*q^s - y*p^s = +/-1 will examined.

Parameters for this run: d = 2 , s = 2 , k = 15 .

| x | y | q | p | x*q^s | y*p^s |
|---|---|---|---|---|---|
| 19564035 | 19546346 | 4421 | 4423 | 382383779007435 | 382383779007434 |
| 97815751 | 97727310 | 4421 | 4423 | 1911832426890991 | 1911832426890990 |
| 1130 | 1129 | 4517 | 4519 | 23055716570 | 23055716569 |
| 113814843 | 113714786 | 4547 | 4549 | 2353145666327187 | 2353145666327186 |
| 10347838 | 10338741 | 4547 | 4549 | 213943713348142 | 213943713348141 |
| 10345563 | 10336468 | 4547 | 4549 | 213896677247667 | 213896677247668 |
| 113812568 | 113712513 | 4547 | 4549 | 2353098630226712 | 2353098630226713 |
| 43262439 | 43225240 | 4649 | 4651 | 935039789857239 | 935039789857240 |
| 1163 | 1162 | 4649 | 4651 | 25136152763 | 25136152762 |
| 86528367 | 86453966 | 4649 | 4651 | 1870154988172767 | 1870154988172766 |
| 1181 | 1180 | 4721 | 4723 | 26321940221 | 26321940220 |
| 12168478 | 12158613 | 4931 | 4933 | 295873634303758 | 295873634303757 |
| 36500500 | 36470909 | 4931 | 4933 | 887500933880500 | 887500933880501 |
| 158172945 | 158044714 | 4931 | 4933 | 3845937354341145 | 3845937354341146 |
| 86419606 | 86350053 | 4967 | 4969 | 2132065790970934 | 2132065790970933 |
| 37035199 | 37005392 | 4967 | 4969 | 913698690661711 | 913698690661712 |
| 125549352 | 125449153 | 5009 | 5011 | 3150043411177512 | 3150043411177513 |
| 100439231 | 100359072 | 5009 | 5011 | 2520028441367711 | 2520028441367712 |
| 1253 | 1252 | 5009 | 5011 | 31437871493 | 31437871492 |
| 75331616 | 75271495 | 5009 | 5011 | 1890076347300896 | 1890076347300895 |
| 176612447 | 176471832 | 5021 | 5023 | 4452477674959127 | 4452477674959128 |
| 1256 | 1255 | 5021 | 5023 | 31664313896 | 31664313895 |
| 117092180 | 117000379 | 5099 | 5101 | 3044373378656180 | 3044373378656179 |
| 13011376 | 13001175 | 5099 | 5101 | 338293186736176 | 338293186736175 |
| 13008825 | 12998626 | 5099 | 5101 | 338226861243825 | 338226861243826 |
| 15979618 | 15968313 | 5651 | 5653 | 510289941268018 | 510289941268017 |
| 111846018 | 111766891 | 5651 | 5653 | 3571668481454418 | 3571668481454419 |
| 155004692 | 154899067 | 5867 | 5869 | 5335523301564788 | 5335523301564787 |
| 51666274 | 51631067 | 5867 | 5869 | 1778440415416786 | 1778440415416787 |
| 258337240 | 258161201 | 5867 | 5869 | 8892404132398360 | 8892404132398361 |
| 51880712 | 51845431 | 5879 | 5881 | 1793134423680392 | 1793134423680391 |
| 17294551 | 17282790 | 5879 | 5881 | 597745357469191 | 597745357469190 |
| 17291610 | 17279851 | 5879 | 5881 | 597643708742010 | 597643708742011 |
| 51877771 | 51842492 | 5879 | 5881 | 1793032774953211 | 1793032774953212 |
| 190222415 | 190093056 | 5879 | 5881 | 6574589039798015 | 6574589039798016 |
| 224808576 | 224655697 | 5879 | 5881 | 7769978106009216 | 7769978106009217 |
| 1523 | 1522 | 6089 | 6091 | 56466627683 | 56466627682 |
| 185502928 | 185381127 | 6089 | 6091 | 6877691903796688 | 6877691903796687 |
| 1550 | 1549 | 6197 | 6199 | 59524353950 | 59524353949 |
| 76856752 | 76807167 | 6197 | 6199 | 2951515167416368 | 2951515167416367 |
| 115284353 | 115209976 | 6197 | 6199 | 4427242988947577 | 4427242988947576 |
| 1643 | 1642 | 6569 | 6571 | 70898343323 | 70898343322 |
| 86357725 | 86305164 | 6569 | 6571 | 3726487909703725 | 3726487909703724 |
| 66555047 | 66515086 | 6659 | 6661 | 2951202596042207 | 2951202596042206 |
| 1676 | 1675 | 6701 | 6703 | 75258100076 | 75258100075 |
| 161508670 | 161413581 | 6791 | 6793 | 7448405321794270 | 7448405321794269 |
| 116589810 | 116521529 | 6827 | 6829 | 5434009586603490 | 5434009586603489 |
| 69954569 | 69913600 | 6827 | 6829 | 3260437585177601 | 3260437585177600 |
| 1718 | 1717 | 6869 | 6871 | 81060670598 | 81060670597 |
| 120719765 | 120650286 | 6947 | 6949 | 5826033521189885 | 5826033521189886 |
| 127058385 | 126987104 | 7127 | 7129 | 6453819998221665 | 6453819998221664 |
| 347241454 | 347051445 | 7307 | 7309 | 18540002175090046 | 18540002175090045 |
| 133551875 | 133478796 | 7307 | 7309 | 7130634964416875 | 7130634964416876 |
| 80661167 | 80617174 | 7331 | 7333 | 4335018348995687 | 4335018348995686 |
| 26888278 | 26873613 | 7331 | 7333 | 1445071808877958 | 1445071808877957 |
| 26884611 | 26869948 | 7331 | 7333 | 1444874731239771 | 1444874731239772 |

Radu's problem, PCW Oct. 95, 951128, Henry Ibstedt

Smarandache - Radu's problem.

q=p(j), p=p(j+1), d=p-q, N=x*q^s or y*p^s. Principal solution to x*q^s - y*p^s = +/-1: x0,y0.
General solutions: x = x0 + t*p^s, y = y0 + t*q^s.

| N,N+1 | S(N),S(N+1) | d | s | t | q,p |
|---|---|---|---|---|---|
| 11822936664715339578483018 | 3225562 | 42 | 2 | -2 | 1612781 |
| 11822936664715339578483017 | 3225646 | | | | 1612823 |
| 111579064978581002637386836334 | 165999 | 4 | 3 | 0 | 55333 |
| 111579064978581002637386836335 | 166011 | | | | 55337 |
| 175498652132211624135022362227 | 165999 | 4 | 3 | -1 | 55333 |
| 175498652132211624135022362226 | 166011 | | | | 55337 |
| 27032997592120525363470705182284857039131 4 | 669764 | 2 | 4 | 0 | 167441 |
| 270329975921205253634707051822848570391313 | 669772 | | | | 167443 |

Radu's problem (RADUpres.UB), 951129, Henry Ibstedt

Factorisations:

11822936664715339578483018 = 2 * 3 * 89 * 193 * 431 * 1612781 ^ 2
11822936664715339578483017 = 509 * 3253 * 1612823 ^ 2

111579064978581002637386836334 = 2 * 7 * 37 ^ 2 * 56671 * 55333 ^ 3
111579064978581002637386836335 = 3 * 5 * 11 * 19 ^ 2 * 16433 * 55337 ^ 3

175498652132211624135022362227 = 3 * 11 ^ 2 * 307 * 12671 * 55333 ^ 3
175498652132211624135022362226 = 2 * 23 * 37 * 71 * 419 * 743 * 55337 ^ 3

270329975921205253634707051822848570391314 = 2 * 3 ^ 3 * 47 * 1289 * 2017 * 119983 * 167441 ^ 4
270329975921205253634707051822848570391313 = 37 * 23117 * 24517 * 38303 * 167443 ^ 4

Radufact, 951129, Henry Ibstedt

Adjacent primes:

Smarandache function values in the above examples: S1 and S2.
P1 and P2 are consecutive primes below and above S1 and S2 respectively. Prime gap = G.

| P1 | S1 | S2 | P2 | G |
|---|---|---|---|---|
| 3225539 | 3225562 | 3225646 | 3225647 | 108 |
| 165983 | 165999 | 166011 | 166013 | 30 |
| 669763 | 669764 | 669772 | 669787 | 24 |

Raduadj, 951130, Henry Ibstedt

Factorisations: Ashbacher - Fibonacci

N = 1185892343342 = 2 * 7 ^ 2 * 47 * 14107 * 18251 ^ 1
N-1 = 1185892343341 = 23 * 1427 * 6011 ^ 2
N-2 = 1185892343340 = 2 ^ 2 * 3 * 5 * 523 * 6067 * 6229 ^ 1
S(N) = 18251 = 18251 ^ 1
S(N-1) = 12022 = 2 * 6011 ^ 1
S(N-2) = 6229 = 6229 ^ 1

N = 1188795217601 = 67 * 83 * 14621 ^ 2
N-1 = 1188795217600 = 2 ^ 6 * 5 ^ 2 * 97 * 587 * 13049 ^ 1
N-2 = 1188795217599 = 3 ^ 2 * 11 * 17 * 181 * 241 * 16193 ^ 1
S(N) = 29242 = 2 * 14621 ^ 1
S(N-1) = 13049 = 13049 ^ 1
S(N-2) = 16193 = 16193 ^ 1

N = 1294530625810 = 2 * 5 * 1669 * 8807 ^ 2
N-1 = 1294530625809 = 3 ^ 2 * 101 * 103 * 2381 * 5807 ^ 1
N-2 = 1294530625808 = 2 ^ 4 * 7 * 19 * 67 * 769 * 11807 ^ 1
S(N) = 17614 = 2 * 8807 ^ 1
S(N-1) = 5807 = 5807 ^ 1
S(N-2) = 11807 = 11807 ^ 1

N = 1517767218627 = 3 * 11 * 107 * 163 * 227 * 11617 ^ 1
N-1 = 1517767218626 = 2 * 73 * 601 * 4159 ^ 2
N-2 = 1517767218625 = 5 ^ 3 * 7 * 17 * 157 * 197 * 3299 ^ 1
S(N) = 11617 = 11617 ^ 1
S(N-1) = 8318 = 2 * 4159 ^ 1
S(N-2) = 3299 = 3299 ^ 1

N = 2677290337914 = 2 * 3 * 37 * 43 * 16747 ^ 2
N-1 = 2677290337913 = 479 * 739 * 2083 * 3631 ^ 1
N-2 = 2677290337912 = 2 ^ 3 * 17 ^ 3 * 2281 * 29863 ^ 1
S(N) = 33494 = 2 * 16747 ^ 1
S(N-1) = 3631 = 3631 ^ 1
S(N-2) = 29863 = 29863 ^ 1

N = 3043063820555 = 5 * 11 * 571 * 6481 * 14951 ^ 1
N-1 = 3043063820554 = 2 * 41 * 997 * 6101 ^ 2
N-2 = 3043063820553 = 3 * 53 * 73 * 283 * 337 * 2749 ^ 1
S(N) = 14951 = 14951 ^ 1
S(N-1) = 12202 = 2 * 6101 ^ 1
S(N-2) = 2749 = 2749 ^ 1

N = 6344309623744 = 2 ^ 6 * 751 * 11489 ^ 2
N-1 = 6344309623743 = 3 ^ 3 * 7 ^ 2 * 13 * 31 * 1597 * 7451 ^ 1
N-2 = 6344309623742 = 2 * 107 * 211 * 9049 * 15527 ^ 1
S(N) = 22978 = 2 * 11489 ^ 1
S(N-1) = 7451 = 7451 ^ 1
S(N-2) = 15527 = 15527 ^ 1

N = 16738688950356 = 2 ^ 2 * 3 * 31 * 193 * 15269 ^ 2
N-1 = 16738688950355 = 5 * 197 * 1399 * 1741 * 6977 ^ 1
N-2 = 16738688950354 = 2 * 7 ^ 2 * 19 * 23 * 53 * 313 * 23561 ^ 1
S(N) = 30538 = 2 * 15269 ^ 1
S(N-1) = 6977 = 6977 ^ 1
S(N-2) = 23561 = 23561 ^ 1

N = 19448047080036 = 2 ^ 2 * 3 ^ 2 * 43 ^ 2 * 17093 ^ 2
N-1 = 19448047080035 = 5 * 7 * 19 * 37 * 61 * 761 * 17027 ^ 1
N-2 = 19448047080034 = 2 * 97 * 1609 * 3631 * 17159 ^ 1
S(N) = 34186 = 2 * 17093 ^ 1
S(N-1) = 17027 = 17027 ^ 1
S(N-2) = 17159 = 17159 ^ 1

# ON RADU'S PROBLEM

by H. Ibstedt

For a positive integer n, the Smarandache function S(n) is defined as the smallest positive integer such that S(n)! is divisible by n. Radu [1] noticed that for nearly all values of n up to 4800 there is always at least one prime number between S(n) and S(n+1) including possibly S(n) and S(n+1). The exceptions are n=224 for which S(n)=8 and S(n+1)=10 and n=2057 for which S(n)=22 and S(n+1)=21. Radu conjectured that, except for a finite set of numbers, there exists at least one prime number between S(n) and S(n+1). The conjecture does not hold if there are infinitely many solutions to the following problem.

*Find consecutive integers n and n+1 for which two consecutive primes $p_k$ and $p_{k+1}$ exist so that $p_k < Min(S(n),S(n+1))$ and $p_{k+1} > Max(S(n),S(n+1))$.*

Consider

$$n+1 = xp_r^s \qquad (1)$$

and

$$n = yp_{r+1}^s \qquad (2)$$

where $p_r$ and $p_{r+1}$ are consecutive prime numbers. Subtract (2) form (1).

$$xp_r^s - yp_{r+1}^s = 1 \qquad (3)$$

The greatest common divisor $(p_r^s,p_{r+1}^s) = 1$ divides the right hand side of (3) which is the condition for this diophantine equation to have infinitely many integer solutions. We are interested in positive integer solutions (x,y) such that the following conditions are met.

$$S(n+1) = sp_r, \text{ i.e } S(x) < sp_r \qquad (4)$$

$$S(n) = sp_{r+1}, \text{ i.e } S(y) < sp_{r+1} \qquad (5)$$

In addition we require that the interval

$$sp_r^s < q < sp_{r+1}^s \text{ is prime free, i.e. q is not a prime.} \qquad (6)$$

Euclid's algorithm is used to obtain principal solutions $(x_0,y_0)$ to (3). The general set of solutions to (3) are then given by

$$x = x_0 + p_{r+1}^s t, \qquad y = y_0 - p_r^s t \qquad (7)$$

with t an integer.

These algorithms were implemented for different values of the parameters $d=p_{r+1}-p_r$, s and t resulted in a very large number of solutions. Table 1 shows the 20 smallest (in respect of n) solutions found. There is no indication that the set would be finite. A pair of primes may produce several solutions.

Table 1. The 20 smallest solutions which ocurred for s=2 and d=2.

| # | n | S(n) | S(n+1) | P1 | P2 | t |
|---|---|---|---|---|---|---|
| 1 | 265225 | 206 | 202 | 199 | 211 | 0 |
| 2 | 843637 | 302 | 298 | 293 | 307 | 0 |
| 3 | 6530355 | 122 | 118 | 113 | 127 | -1 |
| 4 | 24652435 | 926 | 922 | 919 | 929 | 0 |
| 5 | 35558770 | 1046 | 1042 | 1039 | 1049 | 0 |
| 6 | 40201975 | 142 | 146 | 139 | 149 | 1 |
| 7 | 45388758 | 122 | 118 | 113 | 127 | -4 |
| 8 | 46297822 | 1142 | 1138 | 1129 | 1151 | 0 |
| 9 | 67697937 | 214 | 218 | 211 | 223 | 0 |
| 10 | 138852445 | 1646 | 1642 | 1637 | 1657 | 0 |
| 11 | 157906534 | 1718 | 1714 | 1709 | 1721 | 0 |
| 12 | 171531580 | 1766 | 1762 | 1759 | 1777 | 0 |
| 13 | 299441785 | 2126 | 2122 | 2113 | 2129 | 0 |
| 14 | 551787925 | 2606 | 2602 | 2593 | 2609 | 0 |
| 15 | 1223918824 | 3398 | 3394 | 3391 | 3407 | 0 |
| 16 | 1276553470 | 3446 | 3442 | 3433 | 3449 | 0 |
| 17 | 1655870629 | 3758 | 3754 | 3739 | 3761 | 0 |
| 18 | 1853717287 | 3902 | 3898 | 3889 | 3907 | 0 |
| 19 | 1994004499 | 3998 | 3994 | 3989 | 4001 | 0 |
| 20 | 2256222280 | 4166 | 4162 | 4159 | 4177 | 0 |

Within the limits set by the design of the program the largest prime difference for which a solution was found is d=42 and the largest exponent which produced solutions is s=4. Some numerically large examples illustrating the these facts are given in table 2.

Table 2.

| n/n+1 | S(n)/ S(n+1) | d | s | t | $p_r/p_{r+1}$ |
|---|---|---|---|---|---|
| 11822936664715339578483018 | 3225562 | 42 | 2 | -2 | 1612781 |
| 11822936664715339578483017 | 3225646 | | | | 1612823 |
| 11157906497858100263738683634 | 165999 | 4 | 3 | 0 | 55333 |
| 11157906497858100263738683635 | 166011 | | | | 55337 |
| 17549865213221162413502236227 | 16599 | 4 | 3 | -1 | 55333 |
| 17549865213221162413502236226 | 166011 | | | | 55337 |
| 270329975921205253634707051822848570391314 | 669764 | 2 | 4 | 0 | 167441 |
| 270329975921205253634707051822848570391313 | 669772 | | | | 167443 |

To see the relation between these large numbers and the corresponding values of the Smarandache function in table 2 the factorisations of these large numbers are given below:

$1182293666471533957848301 8 = 2 \cdot 3 \cdot 89 \cdot 193 \cdot 431 \cdot 1612781^2$
$11822936664715339578483017 = 509 \cdot 3253 \cdot 1612823^2$

$1115790649785810026373868363 4 = 2 \cdot 7 \cdot 37^2 \cdot 56671 \cdot 55333^3$
$11157906497858100263738683635 = 3 \cdot 5 \cdot 11 \cdot 19^2 \cdot 16433 \cdot 55337^3$

$17549865213221162413502236227 = 3 \cdot 11^2 \cdot 307 \cdot 12671 \cdot 55333^3$
$17549865213221162413502236226 = 2 \cdot 23 \cdot 37 \cdot 71 \cdot 419 \cdot 743 \cdot 55337^3$

$270329975921205253634707051822848570391314 = 2 \cdot 3^3 \cdot 47 \cdot 1289 \cdot 2017 \cdot 119983 \cdot 167441^4$
$270329975921205253634707051822848570391313 = 37 \cdot 23117 \cdot 24517 \cdot 38303 \cdot 167443^4$

It is also interesting to see which are the nearast smaller $P_k$ and nearast bigger $P_{k+1}$ primes to $S_1 = Min(S(n),S(n+1))$ and $S_2 = Max(S(n),S(n+1))$ respectively. This is shown in table 3 for the above examples.

Table 3.

| $P_k$ | $S_1$ | $S_2$ | $P_{k+1}$ | $G=P_{k+1}-P_k$ |
|---|---|---|---|---|
| 3225539 | 3225562 | 3225646 | 3225647 | 108 |
| 165983 | 165999 | 166011 | 166013 | 30 |
| 669763 | 669764 | 669772 | 669787 | 24 |

**Conclusion:** There are infintely many intervals {Min(S(n),S(n-1)),Max(S(n),S(n-1))} which are prime free.

**References:**

I. M. Radu, *Mathematical Spectrum, Sheffield Univeristy, UK*, Vol. 27, No.2, 1994/5, p. 43.

# SOME CONVERGENCE PROBLEMS INVOLVING THE SMARANDACHE FUNCTION

by

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In this paper we consider same series attashed to the Smarandache function (Dirichlet series and other (numerical) series). Asimptotic behaviour and convergence of these series is etablished.

1. INTRODUCTION. The Smarandache function $S : N^* \to N^*$ is defined [3] such that $S(n)$ is the smallest integer n with the property that n! is divisible by n. If

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots p_t^{\alpha_t} \qquad (1.1)$$

is the decomposition into primes of the positiv integer n, then

$$S(n) = \max_i S(p_i^{\alpha_i}) \qquad (1.2)$$

and more general if $n_1 \overset{d}{\vee} n_2$ is the smallest commun multiple of $n_1$ and $n_2$ then .

$$S(n_1 \overset{d}{\vee} n_2) = \max(S(n_1), S(n_2))$$

Let us observe that on the set N of non-negative integers, there are two latticeal structures generated respectively by $\vee = \max$, $\wedge = \min$ and $\overset{d}{\vee}$ = the last commun multiple, $\underset{d}{\wedge}$ = the greatest commun division. if we denote by $\leq$ and $\leq_d$ the induced orders in these lattices, It results

$$S(n_1 \overset{d}{\vee} n_2) = S(n_1) \vee S(n_2)$$

The calculus of $S(p^a)$ depends closely of two numerical scale, namely the standard scale

$$(p) : 1, p, p^2, \ldots, p^a, \ldots$$

and the generalised numerical scale [p]

$$[p] : a_1(p), a_2(p), \ldots, a_n(p), \ldots$$

where $a_k(p) = (p^k-1)/(p-1)$. The dependence is in the sens that

$$S(p^\alpha) = p(\alpha_{[p]})_{(p)} \quad (1.3)$$

so, $S(p^\alpha)$ is obtained multiplyng p by the number obtained writing $\alpha$ in the scale [p] and "reading" it in the scale (p).

Let us observe that if $b_n(p) = p^n$ then the calculus in the scale [p] is essentialy different from the standard scale (p), because :

$$b_{n+1}(p) = pb_n(p) \quad \text{but } a_{n+1}(p) = pa_n(p) + 1$$

(for more details see [2] ).

We have also [1] that

$$S(p^\alpha) = (p-1)\alpha + \sigma_{[p]}(\alpha) \quad (1.4)$$

where $\sigma_{[p]}(\alpha)$ is the sum of digits of the number $\alpha$ writen in the scale [p].

In [4] it is showed that if $\varphi$ is Euler's totient function and we note $S_p(\alpha) = S(p^\alpha)$ then

$$S_p(p^{\alpha-1}) = \varphi(p^\alpha) + p \quad (1.5)$$

It results that $\varphi(p_i^{\alpha_i}) = S(p_i^{p_i^{\alpha_i-1}}) - p$ so

$$\varphi(n) = \prod_{i=1}^{r} \left( S(p_i^{p_i^{\alpha_i-1}}) - p_i \right) ..$$

In the same paper [4] the function S is extended to the set Q of rational numbers.

2. GENERATING FUNCTIONS. It is known that we may attashe to each numerical function f:N*-->C the Dirichlet serie :

$$D_f(z) = \sum_{n=1}^{\infty} \frac{f(n)}{n^z} \quad (2.1)$$

which for some $z = x + iy$ may be convergent or not.
The simplest Dirichlet series is:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad (2.2)$$

called Riemann's function or zeta function where is convergent for $Re(z) > 1$.

It is said for instance that if f is Möbius function ( $\mu(1) = 1$, $\mu(p_1 p_2 \cdots p_r) = (-1)^r$ and $\mu(n) = 0$ if n is divisible by the squar of a prime number ) then $D_\mu(z) = 1/\zeta(z)$ for x>1, and if f is Euler's totient function ( $\varphi(n)$ = the number of positive integers not greater than and prime to the positive integer n ) then $D_\varphi(z) = \zeta(z-1)/\zeta(z)$ for x>2 ).

We have also $D_d(z) = \zeta^2(z)$, for $x > 1$, where d(n) is the number of divisors of n, including 1 and n, and $D_{\sigma_k}(n) = \zeta(z)\cdot\zeta(z-k)$ (for $x > 1$, $x > k+1$), where $\sigma_k(n)$ is the sum of the k-th powers of the divisors of n. We write $\sigma(n)$ for $\sigma_1(n)$.

In the sequel let we suppose that z is a real number, so z = x.

For the Smarandache function we have:

$$D_s(x) = \sum_{n=1}^{\infty} \frac{S(n)}{n^x}$$

If we note :

$$F_f^*(n) = \sum_{k \leq_d n} f(k)$$

it is said that Möbius function make a connection between f and $F_f^*$ by the inversion formula:

$$f(n) = \sum_{k \leq_d n} F_f^*(k)\mu(\tfrac{n}{k}) \qquad (2.3)$$

The functions $F_f^*$ are also called generating functions.

In [4] the Smarandache functions is regarded as a generating function and is constructed the function $s_o$ such that:

$$s_o(n) = \sum_{k \leq_d n} S(k)\mu(\tfrac{n}{k})$$

2.1. PROPOSITION. For all x > 2 we have :

(i) $\zeta(x) \leq D_S(x) \leq \zeta(x-1)$

(ii) $1 \leq D_{so}(x) \leq D_\varphi(x)$

(iii) $\zeta^2(x) \leq D_{FS}(x) \leq \zeta(x)\cdot\zeta(x-1)$

Proof. (i) The asertion results from the fact that $1 \leq S(n) \leq n$.

(ii) Using the multiplication of Dirichlet series we have:

$$\frac{1}{\zeta(x)} \cdot D_s(x) = \left(\sum_{k=1}^{\infty} \frac{\mu(k)}{k^x}\right)\left(\sum_{k=1}^{\infty} \frac{S(k)}{k^x}\right) = \mu(1)S(1) + \frac{\mu(1)S(2)+\mu(2)S(1)}{2^x} +$$

$$+ \frac{\mu(1)S(3)+\mu(3)S(1)}{3^x} + \frac{\mu(1)S(4)+\mu(2)S(2)+\mu(4)S(1)}{4^x} + \ldots = \sum_{k=1}^{\infty} \frac{s_o(k)}{k^x} = D_{so}(x)$$

and the asertion result using (i).

(iii) We have

$$\zeta(x)\cdot D_S(x) = \left(\sum_{k=1}^{\infty} \frac{1}{k^x}\right)\left(\sum_{k=1}^{\infty} \frac{S(k)}{k^x}\right) = S(1) + \frac{S(1)+S(2)}{2^x} + \frac{S(1)+S(3)}{3^x} + \ldots = D_{FS}(x)$$

so the inequalities holds using (i).

Let us observe that (iii) is equivalent to $D_d(x) \leq D_{FS} < D_\sigma(x)$. These inequalities can be deduced also observing that from $1 \leq S(n) \leq n$ it result:

$$\sum_{k \leq_d n} 1 \leq \sum_{k \leq_d n} S(k) \leq \sum_{k \leq_d n}$$

so,

$$d(n) \leq F_s(n) \leq \sigma(n) \qquad (2.4)$$

But from the fact that $F_S < n + 4$ (proved in [5]) we deduce

$$d(n) \leq F_s(n) \leq n + 4 \qquad (2.5)$$

Until now it is not known a closed formula for the calculus of the functions $D_s(x)$, $D_{s_0}(x)$ or $D_{rs}(x)$, but we can deduce asimptotic behaviour of these functions using the following well known results:

2.2. THEOREM. (i) $\zeta(z) = \frac{1}{z-1} + O(1)$

(ii) $\ln\zeta(z) = \ln\frac{1}{z-1} + O(z-1)$

(iii) $\zeta'(z) = -\frac{1}{(z-1)^2} + O(1)$

for all complex number.

Then from the proposition 2.1 we can get inequalities as the fallowings:

(i) $\frac{1}{x-1} + O(1) \leq D_s(x) \leq \frac{1}{x-2} + O(1)$

(ii) $1 \leq D_{s_0}(x) \leq \frac{x-1}{e^{A}(x-2)}$ for some positive constant A

(iii) $-\frac{1}{(x-1)^2} + O(1) \leq D'_s(x) \leq -\frac{1}{(x-2)^2} + O(1)$.

The Smarandache functions S may be extended to all the nonnegative integers defining S(-n) = S(n).

In [3] it is proved that the serie

$$\sum_{k=1}^{\infty} \frac{S(k)}{k!}$$

is convergent and has the sum q ∈ (e-1,2).

We can consider the function

$$f(z) = \sum_{k=1}^{\infty} \frac{S(k)}{(k+1)!} z^k$$

convergent for all z ∈ C because

$$\frac{a_{n+1}}{a_n} = \frac{S(n+1)}{(n+2)S(n)} \leq \frac{n+1}{(n+2)S(n)} \leq \frac{1}{S(n)}$$

and so $\frac{a_{n+1}}{a_n} \to 0$

2.3. PROPOSITION. The function f statisfies |f(z)| ≤ qz an the unit disc U(0,1) = {z | |z| < 1}.

Proof. A lema does to Schwartz asert that if the function f is olomorphe on the unit disc U(0,1) = { z | |z| < 1} and satisfies f(0) = 0, |f(z)| ≤ 1 for z ∈ U(0,1) then |f(z)| ≤ |z| on U(0,1) and |f '(0)| ≤ 1.

For |z| < 1 we fave |f(z)| < q so (1/q) f(z) satisfies the conditions of Schwartz lema.

3. SERIES INVOLVING THE SMARANDACHE FUNCTION. In this section we shall studie the convergence of some series concerning the function S.

Let b: N*-->N* be the function defined by: b(n) is the complement of n until the smallest factorial. From this definition it results that b(n) = (S(n)!)/n for all n ∈ N*.

3.1. PROPOSITION. The sequences $(b(n))_{n\geq 1}$ and also $(b(n)/n^k)_{n\geq 1}$ for $k\in R$, are divergent.

Proof. (i) The asertion results from the fact that $b(n!) = 1$ and if $(p_n)_{n\geq 1}$ is the sequence of prime members then

$$b(p_n) = \frac{S(p_n)!}{p_n} = \frac{p_n!}{p_n} = (p_n - 1)!$$

(ii) Let we note $x_n = b(n)/n^k$. Then

$$x_n = \frac{S(n)!}{n^{k+1}}$$

and for k > 0 it results

$$x_{n!} = \frac{S(n!)!}{(n!)^{k+1}} = \frac{n!}{(n!)^{k+1}} \to 0$$

$$x_{p_n} = \frac{p_n!}{(p_n)^{k+1}} = \frac{(p-1)!}{(p_n)^{k+1}} > \frac{p_1 \cdot p_2 \cdots p_{n-1}}{p_n^{k+1}} > p_n$$

because it in said [6] that $p_1\, p_2 \cdots p_{n-1} > p_n^{k+2}$ for n sufficiently large.

3.2. PROPOSITION. The sequence $T(n) = 1 + \sum_{i=2}^{n} \frac{1}{b(n)} - \ln b(n)$ is divergent.

Proof. If we suppose that $\lim_{n\to\infty} T(n) = l < \infty$, then because $\sum_{i=2}^{\infty} \frac{1}{b(n)} = \infty$ (see [3]) it results the contradiction $\lim_{n\to\infty} \ln b(n) = \infty$.

If we suppose $\lim_{n\to\infty} T(n) = -\infty$, from the equality $\ln b(n) = 1 + \sum_{i=2}^{\infty} \frac{1}{b(n)} - T(n)$ it results $\lim_{n\to\infty} \ln b(n) = \infty$.

We can't have $\lim_{n\to\infty} T(n) = +\infty$ because $T(n) < 0$. Indeed, from $i \leq S(i)!$ for $i \geq 2$ it results

$$i / S(i)! \leq 1 \text{ for all } i \geq 2$$

so

$$T(p_n) = 1 + \frac{2}{S(2)!} + \ldots + \frac{p_n}{S(p_n)!} - \ln((p_n - 1)!) < 1 + (p_n - 1) - \ln((p_n - 1)!) =$$

$$= p_n - \ln((p_n - 1)!).$$

But for k sufficiently large we have $e^k<(k-1)!$ that is there exists $m\in N$ so that $p_n < \ln((p_n - 1)!)$ for $n \geq m$. It results $p_n - \ln((p_n - 1)!) < 0$ for $n \geq m$, and so $T(n) < 0$.

Let now be the function

$$H_b(x) = \sum_{2\leq n\leq x} b(n).$$

3.3. PROPOSITION. The serie

$$\sum_{n\geq 2} H_b^{-1}(n) \tag{3.1}$$

is convergent.

Proof. the sequence $(b(2)+b(3)+ \ldots + b(n))_n$ is strictley increasing to $\infty$ and

$$\frac{S(2)!}{2}+\frac{S(3)!}{3}>\frac{S(2)!}{2}$$

$$\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}+\frac{S(5)!}{5}>\frac{S(5)!}{5}$$

$$\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}+\frac{S(5)!}{5}+\frac{S(6)!}{6}>\frac{S(5)!}{5}$$

$$\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}+\frac{S(5)!}{5}+\frac{S(6)!}{6}+\frac{S(7)}{7}>\frac{S(7)!}{7}$$

..........................................................................

so we have

$$\sum_{n\geq 2} H_b^{-1}(n)=\frac{1}{\frac{S(2)!}{2}}+\frac{1}{\frac{S(2)!}{2}+\frac{S(3)!}{3}}+\frac{1}{\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}}+\frac{1}{\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}+\frac{S(5)!}{5}}+\dots$$

$$\frac{1}{\frac{S(2)!}{2}+\frac{S(3)!}{3}+\dots+\frac{S(n)!}{n}}+\dots<$$

$$<\frac{2}{\frac{S(2)!}{2}}+\frac{1}{\frac{S(3)!}{3}}+\frac{2}{\frac{S(5)!}{5}}+\frac{4}{\frac{S(7)!}{7}}+\frac{2}{\frac{S(11)!}{11}}+\dots+\frac{p_{k+1}-p_k}{\frac{S(p_k)!}{p_k}}+\dots\,.$$

$$<1+\sum_{k\geq 2}\frac{p_k(p_{k+1}-p_k)}{S(p_k)!}=1+\sum_{k\geq 2}\frac{p_k(p_{k+1}-p_k)}{p_k!}=1+\frac{1}{2}+\frac{1}{12}+\sum_{k\geq 2}\frac{p_k(p_{k+1}-p_k)}{p_k!}.$$

But $(p_n-1)!>p_1p_2\dots p_n$ for $n\geq 4$ and then

$$\sum_{n\geq 2} H_b^{-1}(n)<\frac{19}{12}+\sum_{k\geq 4} a_k$$

where $a_k=\frac{p_k(p_{k+1}-p_k)}{p_k!}=\frac{p_{k+1}-p_k}{1\cdot 2\cdot 3\cdot\ldots\cdot(p_k-1)}<\frac{p_{k+1}-p_k}{p_1p_2\dots p_k}<\frac{p_{k+1}}{p_1p_2\dots p_k}$

Because $p_1p_2\dots p_k>p^3_{k+1}$ for k sufficiently large, it results

$$a_k<\frac{p_{k+1}}{p^3_{k+1}}=\frac{1}{p^2_{k+1}}\text{ for }k\geq k_0$$

and the convergence of the serie (3.1) follows from the convergence of the serie $\sum_{k\geq k_0}\frac{1}{p^2_{k+1}}$.

In the followings we give an elementary proof of the convergence of the series $\sum_{k=2}^{\infty}\frac{1}{S(k)\sqrt[\alpha]{S(k)!}}$, $\alpha\in R$, $\alpha>1$ provides information on the convergence behavior of the series $\sum_{k=2}^{\infty}\frac{1}{S(k)!}$.

3.4. PROPOSITION. The series $\sum_{k=2}^{\infty}\frac{1}{S(k)\sqrt[\alpha]{S(k)!}}$, converges if $\alpha\in R$ and $\alpha>1$.

Proof.

$$\sum_{k=2}^{\infty}\frac{1}{S(k)\sqrt[\alpha]{S(k)!}}=\frac{1}{2\sqrt[\alpha]{2!}}+\frac{1}{3\sqrt[\alpha]{3!}}+\frac{1}{4\sqrt[\alpha]{4!}}+\frac{1}{5\sqrt[\alpha]{5!}}+\frac{1}{3\sqrt[\alpha]{3!}}+\frac{1}{7\sqrt[\alpha]{7!}}+\frac{1}{4\sqrt[\alpha]{4!}}+\dots=\sum_{t=2}^{\infty}\frac{m_t}{t\sqrt[\alpha]{t!}}$$

where $m_t$ denotes the number of elements of the set

$M_t$ { $k\in N^*$, $S(k)=t$ } = { $k\in N^*$, $k\mid t$ and $k\nmid(t-1)!$ }.

It follows that $M_t$ { k∈N*, k | t } and there fore $m_t$<d(t!).
Hence $m_t < 2\sqrt{t!}$ and consequently we have

$$\sum_{t=2}^{\infty} \frac{m_t}{t^\alpha \sqrt{t!}} < \sum_{t=2}^{\infty} \frac{2\sqrt{t!}}{t^\alpha \sqrt{t!}} = 2\sum_{t=2}^{\infty} \frac{1}{t^\alpha}$$

So, $\sum_{t=2}^{\infty} \frac{m_t}{t^\alpha \sqrt{t!}}$ converges.

3.5. PROPOSITION. $t^\alpha \sqrt{t!} < t!$ if $\alpha \in R$, $\alpha > 1$ and $t > t_\bullet = [e^{2\alpha+1}]$, $t \in N^*$. (where [x] means the integer part of x).

Proof. $t^\alpha \sqrt{t!} < t! \Leftrightarrow t^{2\alpha} t! < (t!)^2 \Leftrightarrow t^{2\alpha} < t!$ (2)

On the other hand $t^{2\alpha} < (\frac{t}{e})^t \Leftrightarrow (e \cdot \frac{t}{e})^{2\alpha} < (\frac{t}{e})^t \Leftrightarrow e^{2\alpha} \cdot (\frac{t}{e})^{2\alpha} < (\frac{t}{e})^t \Rightarrow e^{2\alpha} < (\frac{t}{e})^{t-2\alpha}$ (3)

If t>$e^{2\alpha+1}$ => $(\frac{t}{e})^{t-2\alpha} > (\frac{e^{2\alpha+1}}{e})^{t-2\alpha} = (e^{2\alpha})^{t-2\alpha} > (e^{2\alpha})^{e^{2\alpha+1}-2\alpha}$

Applyng the well-known result that $e^x > 1 + x$ if x > 0 for x = 2α we have
$(e^{2\alpha})^{e^{2\alpha+1}-2\alpha} > (e^{2\alpha})^{2\alpha+1+1-2\alpha} = (e^{2\alpha})^2 = e^{4\alpha} > e^{2\alpha}$.
So, if t > $e^{2\alpha+1}$ we fave $e^{2\alpha} < (\frac{t}{e})^{t-2\alpha}$ (4)
It is well known that $(\frac{t}{e})^t < t!$ if t∈N*. (5)

Now, the proof of the proposition is obtained as follows:

If $t > t_\bullet = [e^{2\alpha+1}]$, $t \in N^*$ we have $e^{2\alpha} < (\frac{t}{e})^{t-2\alpha} \Leftrightarrow t^{2\alpha} < (\frac{t}{e})^t < t!$. Hence $t^{2\alpha} < t!$ if t >$t_\bullet$ and this proves the proposition.

CONSEQUENCE. The series $\sum_{k=2}^{\infty} \frac{1}{S(k)!}$ converges.

Proof. $\sum_{k=2}^{\infty} \frac{1}{S(k)!} = \sum_{t=2}^{\infty} \frac{m_t}{t!}$ where $m_t$ is defined as above.

If t > $t_\bullet$ we have $t^\alpha \sqrt{t!} < t! \Leftrightarrow \frac{1}{t^\alpha \sqrt{t!}} > \frac{1}{t!} \Leftrightarrow \frac{m_t}{t^\alpha \sqrt{t!}} > \frac{m_t}{t!}$.

Since $\sum_{t=2}^{\infty} \frac{m_t}{t^\alpha \sqrt{t!}}$ converges it results that $\sum_{t=2}^{\infty} \frac{m_t}{t!}$ also converges.

REMARQUE. From the definition of the Smarandache function it results that

card { k∈N*: S(k)=t } = card { k∈N*: k | t and k | (t-1)!} = d(t!)-d((t-1)!)

so we get

$$\sum_{t=2}^{n} car(dS^{-1}(t)) = \sum_{k=2}^{n} (d(t!) - d((t-1)!)) = d(n!) - 1$$

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# ON THE SMARANDACHE FUNCTION AND THE FIXED - POINT THEORY OF NUMBERS

by

Albert A. Mullin

This brief note points out several basic connections between the Smarandache function, fixed-point theory [1] and prime-number theory. First recall that fixed-point theory in function spaces provides elegent, if not short, proofs of the existence of solutions to many kinds of differential equations, integral equations, optimization problems and game-theoretic problems. Further, fixed-point theory in the ring of rational integers and fixed-lattice-point theory provide many results on the existence of solutions in diophantine theory. Here are four fundamental examples of fixed-point theory in number theory. (1) There is the well-known basic result that for $p>4$, p is prime iff $S(p) = p$. (2) Recall that the present author defined [2] the number-theoretic function $\Psi(n)$ as the *product* of the primes alone in the *mosaic* of n, where the mosaic of n is obtained from n by recursively applying the unique factorization theorem/fundamental theorem of arithmetic to itself! Now the asymptotic density of fixed points of $\Psi(n)$ is $7/\pi^2$, just as the asymptotic density of square-free numbers is $6/\pi^2$. Indeed, (3) the theory of *perfect* numbers is also connected to fixed-point theory, since if one puts $f(n) = \delta(n) - n$, where $\delta(n)$ is the sum of the divisors on n, then n is perfect iff $f(n) = n$. Finally, (4) the present author defined [2] the number-theoretic function $\Psi^*(n)$ as the *sum* of the primes alone in the *mosaic* of n. Here we have a striking similarity to the Smarandache function itself (see example (1) above), since $\Psi^*(n) = n$ iff $n = 4$ or $n = p$ for some prime p; i.e., if $> 4$, n is prime iff $\Psi^*(n) = n$. Thus, the distribution function for the fixed points of $S(n)$ or of $\Psi^*(n)$ is essentially the distribution function for the primes, $\Pi(n)$.

## *Problems*

(1) Put $S^2(n) = S(S(n))$ and define $S^m(n)$ recursively, where $S(n)$ is the Smarandache function. (*Note*: This approach aligns Smarandache function theory more closely with recursive function theory/computer theory.) For each n, determine the *least* m for which $S^m(n)$ is prime.

(2) Prove that $S(n) = S(n+3)$ for only finitely many n.

(3) Prove that $S(n) = S(n+2)$ for only finitely many n.

(4) Prove that $S(n) = S(n+1)$ for no n.

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# ON THE CALCULUS OF SMARANDACHE FUNCTION

*by*

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**Introduction.** The Smarandache function $S : \mathbf{N}^* \longrightarrow \mathbf{N}^*$ is defined [5] by the condition that $S(n)$ is the smallest integer $m$ such that $m!$ is divisible by $n$.So,we have $S(1) = 1, S(2^{12}) = 16$.

Considering on the set $\mathbf{N}^*$ two laticeal structures $\mathcal{N} = (\mathbf{N}^*, \wedge, \vee)$ and $\mathcal{N}_d = (\mathbf{N}^*, \underset{d}{\wedge}, \overset{d}{\vee})$,where $\wedge = \min, \vee = \max$, $\underset{d}{\wedge}=$ the greattest common divisor, $\overset{d}{\vee}=$ the smallest common multiple,it results that $S$ has the followings properties:

$$(s_1) \quad S(n_1 \overset{d}{\vee} n_2) = S(n_1) \vee S(n_2)$$
$$(s_2) \quad n_1 \leq_d n_2 \Longrightarrow S(n_1) \leq S(n_2)$$

where $\leq$ is the order in the lattice $\mathcal{N}$ and $\leq_d$ is the order in the lattice $\mathcal{N}_d$.It is said that

$$n_1 \leq_d n_2 \Longleftrightarrow n_1 \text{ divides } n_2$$

From these properties we deduce that in fact on must consider

$$S : \mathcal{N}_d \longrightarrow \mathcal{N}$$

**Methods for the calculus of S.** If

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_t^{\alpha_t} \tag{1}$$

is the decomposition of $n$ into primes,from $(s_1)$ it results

$$S(n) = \vee S(p_i^{\alpha_i})$$

so the calculus of $S(n)$ is reduced to the calculus of $S(p^\alpha)$.

If $e_p(n)$ is the exponent of the prime $p$ in the decomposition into primes of $n!$:

$$n! = \prod_{j=1}^{t} p_j^{e_{p_j}(n)}$$

by Legendre's formula it is said that

$$e_p(n) = \sum_{i \geq 1} [\frac{n}{p^i}]$$

Also we have

$$e_p(n) = \frac{n - \sigma_{(p)}(n)}{p - 1} \tag{2}$$

where $[x]$ is the integer part of $x$ and $\sigma_{(p)}(n)$ is the sum of digits of $n$ in the numerical scale

$$(p) : 1 \ , \ p \ , \ p^2 \ , ..., \ p^i ...$$

For the calculus of $S(p^\alpha)$ we need to consider in addition a generalised numerical scale $[p]$ given by:

$$[p] : \ a_1(p) \ , \ a_2(p) \ , \dots , a_i(p) \ , \ \dots$$

where $a_i(p) = (p^i - 1)/(p - 1)$.Then in [3] it is showed that

$$S(p^\alpha) = p(\alpha_{[p]})_{(p)} \tag{3}$$

that is the value of $S(p^\alpha)$ is obtained multiplying $p$ by the number obtained writing the exponent $\alpha$ in the generalised scale $[p]$ and "reading" it in the usual scale $(p)$.

Let us observe that the calculus in the generalised scale $[p]$ is essentially different from the calculus in the scale $(p)$.That is because if we note

$$b_n(p) = p^n$$

then for the usual scale $(p)$ it results the recurence relation

$$b_{n+1}(p) = p \cdot b_n(p)$$

and for the generalised scale $[p]$ we have

$$a_{n+1}(p) = p \cdot a_n(p) + 1$$

For this,to add some numbers in the scale $[p]$ we do as follows:

1) We start to add from the digits of "decimals", that is from the column corresponding to $a_2(p)$.

2) If adding some digits it is obtained $pa_2(p)$, then we utilise an *unit* from the classe of "units" (the column corresponding to $a_1(p)$) to obtain $p \cdot a_2(p) + 1 = a_3(p)$. Continuing to add, if agains it is obtained $p \cdot a_2(p)$, then a new unit must be used from the classe of units, etc.

*Example.* If

$$m_{[5]} = 442 = 4a_3(5) + 4a_2(5) + 2a_1(5)\ ,\ n_{[5]} = 412\ ,\ r_{[5]} = 44$$

then

$$\begin{array}{rcr} m+n+r & = & 442\ + \\ & & 412 \\ & & 44 \\ & & dcba \end{array}$$

To find the digits $a, b, c, d$ we start to add from the column corresponding to $a_2(5)$:

$$4a_2(5) + a_2(5) + 4a_2(5) = 5a_2(5) + 4a_2(5)$$

Now, if we take an unit from the first column we get:

$$5a_2(5) + 4a_2(5) + 1 = a_3(5) + 4a_2(5)$$

so $b = 4$.

Continuing the addition we have:

$$4a_3(5) + 4a_3(5) + a_3(5) = 5a_3(5) + 4a_3(5)$$

and using a new unit (from the first column) it results:

$$4a_3(5) + 4a_3(5) + a_3(5) + 1 = a_4(5) + 4a_3(5)$$

so $c = 4$ and $d = 1$.

Finaly, adding the remained units:

$$4a_1(5) + 2a_1(5) = 5a_1(5) + a_1(5) = 5a_1(5) + 1 = a_2(5)$$

it results that the digit $b = 4$ must be changed in $b = 5$ and $a = 0$.

So

$$m_{[5]} + n_{[5]} + r_{[5]} = 1450_{[5]} = a_4(5) + 4a_3(5) + 5a_2(5)$$

*Remarque.* As it is showed in [5],writing a positive integer $\alpha$ in the scale $[p]$ we may find the first non-zero digit on the right equals to $p$.Of course,that is no possible in the standard scale $(p)$.

Let us return now to the presentation of the formulae for the calculus of the Smarandache function. For this we expresse the exponent $\alpha$ in both the scales $(p)$ and $[p]$:

$$\alpha_{(p)} = c_u p^u + c_{u-1}p^{u-1} + \ldots + c_1 p + c_0 = \sum_{i=0}^{u} c_i p^i \tag{4}$$

and

$$\alpha_{[p]} = k_v a_v(p) + k_{v-1}a_{v-1}(p) + \ldots + k_1 a_1(p) = \sum_{j=1}^{v} k_j a_j(p) =$$
$$= \sum_{j=1}^{v} k_j \frac{p^j - 1}{p-1}$$

It results

$$(p-1)\alpha = \sum_{j=1}^{v} k_j p^j - \sum_{j=1}^{v} k_j \tag{5}$$

so,because $\sum_{j=1}^{v} k_j p^j = p(\alpha_{[p]})_{(p)}$, we get:

$$S(p^\alpha) = (p-1)\alpha + \sigma_{[p]}(\alpha) \tag{6}$$

From (4) we deduce

$$p\alpha = \sum_{i=0}^{u} c_i(p^{i+1} - 1) + \sum_{i=0}^{u} c_i$$

and

$$\frac{p}{p-1}\alpha = \sum_{i=0}^{u} c_i a_{i+1}(p) + \frac{1}{p-1}\sigma_{(p)}(\alpha)$$

Consequently

$$\alpha = \frac{p-1}{p}(\alpha_{(p)})_{[p]} + \frac{1}{p}\sigma_{(p)}(\alpha) \tag{7}$$

Replacing this expression of $\alpha$ in (6) we get:

$$S(p^{\alpha}) = \frac{(p-1)^2}{p}(\alpha_{(p)})_{[p]} + \frac{p-1}{p}\sigma_{(p)}(\alpha) + \sigma_{[p]}(\alpha) \qquad (8)$$

*Example.*To find $S(3^{89})$ we shall utilise the equality (3).For this we have:

$$\begin{array}{l} (3)\ :\ 1\ ,\ 3\ ,\ 9\ ,\ 27\ ,\ 81\ ,\ \ldots \\ [3]\ :\ 1\ ,\ 4\ ,\ 13\ ,\ 40\ ,\ 121\ ,\ \ldots \end{array}$$

and $89_{[3]} = 2021$,so $S(3^{89}) = 3(2021)_{(3)} = 183$.That is 183! is divisible by $3^{89}$ and it is the smallest factorial with this property.

We shall use now the equality (6) to calculate the same value $S(3^{89})$.For this we observe that $\sigma_{[3]}(89) = 5$ and, so $S(3^{89}) = 2 \cdot 89 + 5 = 183$.

Using (8) we get $89_{(3)} = 10022$ and :

$$S(3^{89}) = \frac{4}{3}(10022)_{[3]} + \frac{2}{3}\cdot 5 + 5 = 183$$

It is possible to expresse $S(p^{\alpha})$ by mins of the exponent $e_p(\alpha)$ in the following way: from (2) and (7) it results

$$e_p(\alpha) = (\alpha_{(p)})_{[p]} - \alpha \qquad (9)$$

and then from (8) and (9) it results

$$S(p^{\alpha}) = \frac{(p-1)^2}{p}(e_p(\alpha) + \alpha) + \frac{p-1}{p}\sigma_{(p)}(\alpha) + \sigma_{[p]}(\alpha) \qquad (10)$$

*Remarque.*From (3) and (8) we deduce a connection between the integer $\alpha$ writen in the scale $[p]$ and readed in the scale $(p)$ and the same integer writed in the scale $(p)$ and readed in the scale $[p]$.Namely:

$$p^2(\alpha_{[p]})_{(p)} - (p-1)^2(\alpha_{(p)})_{[p]} = p\sigma_{[p]}(\alpha) + (p-1)\sigma_{(p)}(\alpha) \qquad (11)$$

**The function $i_p(\alpha)$.** In the followings let we note $S(p^{\alpha}) = S_p(\alpha)$.Then from Legendre's formula it results:

$$(p-1)\alpha < S_p(\alpha) \leq p\alpha$$

that is $S(p^{\alpha}) = (p-1)\alpha + x = p\alpha - y$.

From (6) it results that $x = \sigma_{[p]}(\alpha)$ and to find $y$ let us write $S_p(\alpha)$ under the forme

$$S_p(\alpha) = p(\alpha - i_p(\alpha)) \tag{12}$$

As it is showed in [4] we have $0 \leq i_p(\alpha) \leq [\frac{\alpha-1}{p}]$.

Then it results that for each function $S_p$ there exists a function $i_p$ so that we have the linear combination

$$\frac{1}{p}S_p(\alpha) + i_p(\alpha) = \alpha \tag{13}$$

In [1] it is proved that

$$i_p(\alpha) = \frac{\alpha - \sigma_{[p]}(\alpha)}{p} \tag{14}$$

and so it is an evident analogy between the expression of $e_p(\alpha)$ given by the equality (2) and the expression of $i_p(\alpha)$ in (14).

In [1] it is also showed that

$$\alpha = (\alpha_{[p]})_{(p)} + [\frac{\alpha}{p}] - [\frac{\sigma_{[p]}(\alpha)}{p}] = (\alpha_{[p]})_{(p)} + \frac{\alpha - \sigma_{[p]}(\alpha)}{p}$$

and so

$$S(p^\alpha) = p(\alpha - [\frac{\alpha}{p}] + [\frac{\sigma_{[p]}(\alpha)}{p}]) \tag{15}$$

Finaly,let us observe that from the definition of Smarandache function it results that

$$(S_p \circ e_p)(\alpha) = p[\frac{\alpha}{p}] = \alpha - \alpha_p$$

where $\alpha_p$ is the remainder of $\alpha$ modulus $p$.Also we have

$$(e_p \circ S_p)(\alpha) \geq \alpha \text{ and } e_p(S_p(\alpha) - 1) < \alpha$$

so using (2) it results

$$\frac{S_p(\alpha) - \sigma_{(p)}(S_p(\alpha))}{p-1} \geq \alpha \text{ and } \frac{S_p(\alpha) - 1 - \sigma_{(p)}(S_p(\alpha) - 1)}{p-1} < \alpha$$

Using (6) we obtains that $S(p^\alpha)$ is the unique solution of the system

$$\sigma_{(p)}(x) \leq \sigma_{[p]}(\alpha) \leq \sigma_{(p)}(x-1)+1$$

The calculus of card($S^{-1}$(n)). Let $q_1, q_2, ..., q_k$ be all the prime itegers smallest then $n$ and non dividing $n$. Let also denote shortly $e_{q_j}(n) = f_j$. A solution $x_0$ of the equation

$$S(x) = n$$

has the property that $x_0$ divides $n!$ and non divides $(n-1)!$. Now, if $d(n)$ is the number of positive divisors of $n$, from the inclusion

$$\{m \ / \ m \text{ divides } (n-1)!\} \subset \{m \ / \ m \text{ divides } n!\}$$

and using the definition of Smarandache function it results that

$$\text{card}(S^{-1}(n)) = d(n!) - d((n-1)!) \quad (16)$$

*Example.* In [6] A. Stuparu and D. W. Sharpe has proved that if $p$ is a given prime, the equation

$$S(x) = p$$

has just $d((p-1)!)$ solutions (all of them in between $p$ and $p!$). Let us observe that $e_p(p!) = 1$ and $e_p((p-1)!) = 0$, so because

$$d(p!) = (e_p(p!)+1)(f_1+1)(f_2+1)...(f_k+1) = 2(f_1+1)(f_2+1)...(f_k+1)$$
$$d((p-1)!) = (f_1+1)(f_2+1)...(f_k+1)$$

it results

$$\text{card}(S^{-1}(p!)) = d(p!) - d((p-1)! = d((p-1)!)$$

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## THE FIRST CONSTANT OF SMARANDACHE

by


**Ion Cojocaru and Sorin Cojocaru**


In this note we prove that the series $\sum_{n=2}^{\infty} \frac{1}{S(n)!}$ is convergent to a real number $s \in (0.717, 1.253)$ that we call the first constant constant of Smarandache.

It appears as an open problem, in [1], the study of the nature of the series $\sum_{n=2}^{\infty} \frac{1}{S(n)!}$. We can write it as it follows :

$$\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{3!} + \cdots = \frac{1}{2!} + \frac{2}{3!} + \frac{4}{4!} + \frac{8}{5!} + \frac{14}{6!} + \cdots =$$

$= \sum_{n=2}^{\infty} \frac{a(n)}{n!}$, where $a(n)$ is the number of the equation $S(x) = n$, $n \in \mathbf{N}$, $n \geq 2$ solutions.

It results from the equality $S(x) = n$ that $x$ is a divisor of $n!$, so $a(n)$ is smaller than $d(n!)$.

So, $\quad a(n) < d(n!)$. $\quad$ (1)

**Lemma 1.** We have the inequality :

$d(n) \leq n - 2$, for each $n \in \mathbf{N}$, $n \geq 7$. $\quad$ (2)

**Proof.** Be $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ with $p_1, p_2, \ldots, p_k$ prime numbers, and $a_i \geq 1$ for each $i \in \{1, 2, \ldots, k\}$. We consider the function $f : [1, \infty) \rightarrow \mathbf{R}$, $f(x) = a^x - x - 2$, $a \geq 2$, fixed. It is derivable on $[1, \infty)$ and $f'(x) = a^x \ln a - 1$. Because $a \geq 2$, and $x \geq 1$ it results that $a^x \geq 2$, so $a^x \ln a \geq 2 \ln a = \ln a^2 \geq \ln 4 > \ln e = 1$, i.e., $f'(x) > 0$ for each $x \in [1, \infty)$ and $a \geq 2$, fixed. But $f(1) = a - 3$. It results that for $a \geq 3$ we have $f(x) \geq 0$, that means $a^x \geq x + 2$.

Particularly, for $a = p_i$, $i \in \{1, 2, \ldots, k\}$, we obtain $p_i^{a_i} \geq a_i + 2$ for each $p_i \geq 3$.

If $n = 2^s$, $s \in \mathbf{N}^*$, then $d(n) = s + 1 < 2^s - 2 = n - 2$ for $s \geq 3$.

So we can assume $k \geq 2$, i.e. $p_2 \geq 3$. It results the inequalities :

$p_1^{a_1} \geq a_1 + 1$

$p_2^{a_2} \geq a_2 + 2$

......................

$p_k^{a_k} \geq a_k + 2,$

equivalent with

$$p_1^{a_1} \geq a_1 + 1,\ p_2^{a_2} - 1 \geq a_2 + 1, \ldots, p_k^{a_k} - 1 \geq a_k + 1. \qquad (3)$$

Multiplying, member with member, the inequalities (3) we obtain :

$$p_1^{a_1}(p_2^{a_2} - 1)\cdots(p_k^{a_k} - 1) \geq (a_1 + 1)(a_2 + 1)\cdots(a_k + 1) = d(n). \qquad (4)$$

Considering the obvious inequality :

$$n - 2 \geq p_1^{a_1}(p_2^{a_2} - 1)\cdots(p_k^{a_k} - 1) \qquad (5)$$

and using (4) it results that :

$n - 2 \geq d(n)$ for each $n \geq 7$.

**Lemma 2.** $d(n!) < (n - 2)!$ for each $n \in \mathbf{N}, n \geq 7$. (6)

**Proof.** We ration trough induction after n. So, for $n = 7$,

$d(7!) = d(2^4 \cdot 3^2 \cdot 5 \cdot 7) = 60 < 120 = 5!$.

We assume that $d(n!) < (n - 2)!$.

$d((n+1)!) = d(n!(n + 1)) \leq d(n!) \cdot d(n + 1) < (n - 2)!\, d(n + 1) < (n-2)!\,(n - 1) = (n - 1)!$ ,

because in accordance with Lemma 1, $d(n + 1) < n - 1$.

**Proposition.** The series $\sum_{n=2}^{\infty} \frac{1}{S(n)!}$ is convergent to a number $s \in (0.717, 1.253)$, that we call the first constant constant of Smarandache.

**Proof.** From Lemma 2 it results that $a(n) < (n-2)!$, so $\frac{a(n)}{n!} < \frac{1}{n(n-1)}$ for every $n \in \mathbf{N}$, $n \geq 7$ and $\sum_{n=2}^{\infty} \frac{1}{S(n)!} = \sum_{n=2}^{6} \frac{a(n)}{n!} + \sum_{n=7}^{\infty} \frac{1}{(n-1)}$.

Therefore $\sum_{n=2}^{\infty} \frac{1}{S(n)!} < \frac{1}{2!} + \frac{2}{3!} + \frac{4}{4!} + \frac{8}{5!} + \frac{14}{6!} + \sum_{n=7}^{\infty} \frac{1}{n^2 - n}$. (7)

Because $\sum_{n=2}^{\infty} \frac{1}{n^2 - n} = 1$ we have : it exists the number $s > 0$, that we call the Smarandache constant, $s = \sum_{n=2}^{\infty} \frac{1}{S(n)!}$.

From (7) we obtain :

$$\sum_{n=2}^{\infty} \frac{1}{S(n)!} < \frac{391}{360} + 1 - \frac{1}{2^2 - 2} - \frac{1}{3^2 - 3} - \frac{1}{4^2 - 4} +$$

$$+ \frac{1}{5^2 - 5} + \frac{1}{6^2 - 6} = \frac{751}{360} - \frac{5}{6} = \frac{451}{360} < 1,253.$$

But, because $S(n) \leq n$ for every $n \in \mathbf{N}^*$, it results :

$$\sum_{n=2}^{\infty} \frac{1}{S(n)!} \geq \sum_{n=2}^{\infty} \frac{1}{n!} = e - 2.$$

Consequently, for this first constant we obtain the framing $e - 2 < s < 1,253$, i.e., $0,717 < s < 1,253$.

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# THE SECOND CONSTANT OF SMARANDACHE

by

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In the present note we prove that the sum of remarcable series $\sum_{n\geq 2} \frac{S(n)}{n!}$, which implies the Smarandache function is an irrational number (second constant of Smarandache).

Because $S(n) \leq n$, it results $\sum_{n\geq 2} \frac{S(n)}{n!} \leq \sum_{n\geq 2} \frac{1}{(n-1)!}$. Therefore the serie $\sum_{n\geq 2} \frac{S(n)}{n!}$ is convergent to a number f.

**Proposition**. The sum f of the series $\sum_{n\geq 2} \frac{S(n)}{n!}$ is an irrational number.

**Proof**. From the precedent lines it results that $\lim_{n\to\infty} \sum_{i=2}^{n} \frac{S(n)}{n!} = f$. Against all reson we assume that $f \in Q$, $f > 0$. Therefore it exists $a,b \in N$, $(a,b) = 1$, so that $f = \frac{a}{b}$.

Let p be a fixed prime number, $p > b$, $p \geq 3$. Obviously, $\frac{a}{b} = \sum_{i=2}^{p-1} \frac{S(i)}{i!} + \sum_{i\geq p} \frac{S(i)}{i!}$ which leads to:

$$\frac{(p-1)!a}{b} = \sum_{i=2}^{p-1} \frac{(p-1)!S(i)}{i!} + \sum_{i\geq p} \frac{(p-1)!S(i)}{i!}$$

Because $p > b$ it results that $\frac{(p-1)!a}{b} \in N$ and $\sum_{i=2}^{p-1} \frac{(p-1)!S(i)}{i!} \in N$. Consequently we have $\sum_{i\geq p} \frac{(p-1)!S(i)}{i!} \in N$ too.

Be $\alpha = \sum_{i\geq p} \frac{(p-1)!S(i)}{i!} \in N$. So we have the relation

$$\alpha = \frac{(p-1)!S(p)}{p!} + \frac{(p-1)!S(p+1)}{(p+1)!} + \frac{(p-1)!S(p+2)}{(p+2)!} + \dots .$$

Because p is a prime number it results $S(p) = p$.

So

$$\alpha = 1 + \frac{S(p+1)}{p(p+1)} + \frac{S(p+2)}{p(p+1)(p+2)} + \dots > 1 \qquad (1)$$

We know that $S(p+i) \leq p+i \ (\forall) i \geq 1$, with equality only if the number $p+i$ is prime. Consequently, we have

$$\alpha < 1 + \frac{1}{p} + \frac{1}{p(p+1)} + \frac{1}{p(p+1)(p+2)} + \ldots < 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} \ldots = \frac{p}{p-1} < 2 \qquad (2)$$

From the inequalities (1) and (2) it results that $1 < \alpha < 2$, impossible, because $\alpha \in N$. The proposition is proved.

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# THE THIRD AND FOURTH CONSTANTS OF SMARANDACHE

by

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In the present note we prove the divergence of some series involving the Smarandache function, using an unitary method, and then we prove that the series

$$\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\dots S(n)}$$

is convergent to a number $s \in (71/100, 101/100)$ and we study some applications of this series in the Number Theory (third constant of Smarandache).

The Smarandache Function $S : N^* \to N$ is defined [1] such that $S(n)$ is the smallest integer $k$ with the property that $k!$ is divisible by $n$.

**Proposition 1.** If $(x_n)_{n \geq 1}$ is a strict increasing sequence of natural numbers, then the series :

$$\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}, \tag{1}$$

where S is the Smarandache function, is divergent.

**Proof.** We consider the function $f : [x_n, x_{n+1}] \to \mathbf{R}$, defined by $f(x) = \ln \ln x$. It fulfils the conditions of the Lagrange's theorem of finite increases. Therefore there is $c_n \in (x_n, x_{n+1})$ such that :

$$\ln \ln x_{n+1} - \ln \ln x_n = \frac{1}{c_n \ln c_n}(x_{n+1} - x_n). \tag{2}$$

Because $x_n < c_n < x_{n+1}$, we have :

$$\frac{x_{n+1} - x_n}{x_{n+1} \ln x_{n+1}} < \ln \ln x_{n+1} - \ln \ln x_n < \frac{x_{n+1} - x_n}{x_n \ln x_n}, \quad (\forall) n \in N, \tag{3}$$

if $x_n \neq 1$.

We know that for each $n \in N^* \setminus \{1\}$, $\frac{S(n)}{n} \le 1$, i.e.

$$0 < \frac{S(n)}{n \ln n} \le \frac{1}{\ln n}, \tag{4}$$

from where it results that $\lim_{n\to\infty} \frac{S(n)}{n \ln n} = 0$. Hence there exists $k > 0$ such that $\frac{S(n)}{n \ln n} < k$, i.e., $n \ln n > \frac{S(n)}{k}$ for any $n \in N^*$, so

$$\frac{1}{x_n \ln x_n} < \frac{k}{S(x_n)}. \tag{5}$$

Introducing (5) in (3) we obtain :

$$\ln\ln x_{n+1} - \ln\ln x_n < k\frac{x_{n+1} - x_n}{S(x_n)}, \ (\forall)n \in N^*\setminus\{1\}. \tag{6}$$

Summing up after n it results :

$$\sum_{n=1}^{m} \frac{x_{n+1} - x_n}{S(x_n)} > \frac{1}{k}(\ln\ln x_{m+1} - \ln\ln x_1).$$

Because $\lim_{m\to\infty} x_m = \infty$ we have $\lim_{m\to\infty} \ln\ln x_m = \infty$, i.e., the series :

$$\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}$$

is divergent. The Proposition 1 is proved.

**Proposition 2.** Series $\sum_{n=2}^{\infty} \frac{1}{S(n)}$, where S is the Smarandache function, is divergent.

**Proof.** We use Proposition 1 for $x_n = n$.

**Remarks.** 1) If $x_n$ is the n - th prime number, then the series $\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}$ is divergent.

2) If the sequence $(x_n)_{n\ge 1}$ forms an arithmetical progression of natural numbers, then the series $\sum_{n=1}^{\infty} \frac{1}{S(x_n)}$ is divergent.

3) The series $\sum_{n=1}^{\infty} \frac{1}{S(2n+1)}$, $\sum_{n=1}^{\infty} \frac{1}{S(4n+1)}$ etc., are all divergent.

In conclusion, Proposition 1 offers us an unitary method to prove that the series having one of the precedent aspects are divergent.

**Proposition 3.** The series :

$$\sum_{n=2}^{\infty} \frac{1}{S(2) \cdot S(3) \cdots S(n)},$$

where S is the Smarandache function, is convergent to a number $s \in (71/100, 101/100)$.

**Proof.** From the definition of the Smarandache function it results $S(n) \leq n!$, $(\forall) n \in N^* \setminus \{1\}$, so $\frac{1}{S(n)} \geq \frac{1}{n!}$.

Summing up, begining with n = 2 we obtain :

$$\sum_{n=2}^{\infty} \frac{1}{S(2) \cdot S(3) \cdots S(n)} \geq \sum_{n=2}^{\infty} \frac{1}{n!} = e - 2.$$

The product $S(2) \cdot S(3) \ldots S(n)$ is greater than the product of prime numbers from the set $\{1, 2, \ldots, n\}$, because $S(p) = p$, for p = prime number. Therefore :

$$\frac{1}{\prod_{i=2}^{n} S(i)} < \frac{1}{\prod_{i=1}^{k} p_i}, \tag{7}$$

where $p_k$ is the biggest prime number smaller or equal to n.

There are the inequalities :

$$S = \sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\cdots S(n)} = \frac{1}{S(2)} + \frac{1}{S(2)S(3)} + \frac{1}{S(2)S(3)S(4)} + \cdots +$$

$$+ \frac{1}{S(2)S(3)\cdots S(k)} + \cdots < \frac{1}{2} + \frac{2}{2 \cdot 3} + \frac{2}{2 \cdot 3 \cdot 5} + \frac{4}{2 \cdot 3 \cdot 5 \cdot 7} +$$

$$+ \frac{2}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11} + \cdots + \frac{p_{k+1} - p_k}{p_1 p_2 \cdots p_k} + \cdots. \tag{8}$$

Using the inequality $p_1 p_2 \cdots p_k > p_{k+1}^3$, $(\forall) k \geq 5$ [2], we obtain :

$$S < \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{2}{105} + \frac{1}{p_6^2} + \frac{1}{p_7^2} + \cdots + \frac{1}{p_{k+1}^2} + \cdots. \qquad (9)$$

We note $P = \frac{1}{p_6^2} + \frac{1}{p_7^2} + \cdots$ and observe that $P < \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{15^2} + \cdots$.

It results :

$$P < \frac{\pi^2}{6} - \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{12^2}\right),$$

where

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad \text{(EULER)}.$$

Introducing in (9) we obtain :

$$S < \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{2}{105} + \frac{\pi^2}{6} - 1 - \frac{1}{2^2} - \frac{1}{3^2} - \cdots - \frac{1}{12^2}.$$

Estimating with an approximation of an order not more than $\frac{1}{10^2}$, we find :

$$0,71 < \sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\cdots S(n)} < 1,01. \qquad (10)$$

The Proposition 3 is proved.

**Remark.** Giving up at the right increase from the first terms in the inequality (8) we can obtain a better right framing :

$$\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\cdots S(n)} < 0,97. \qquad (11)$$

**Proposition 4.** Let $\alpha$ be a fixed real number, $\alpha \geq 1$. Then the series $\sum_{n=2}^{\infty} \frac{n^\alpha}{S(2)S(3)\cdots S(n)}$ is convergent (fourth constant of Smarandache).

**Proof.** Be $(p_k)_{k\geq 1}$ the sequence of prime numbers. We can write :

$$\frac{2^\alpha}{S(2)} = \frac{2^\alpha}{2} = 2^{\alpha-1}$$

$$\frac{3^\alpha}{S(2)S(3)} = \frac{3^\alpha}{p_1p_2}$$

$$\frac{4^\alpha}{S(2)S(3)S(4)} < \frac{4^\alpha}{p_1p_2} < \frac{p_3^\alpha}{p_1p_2}$$

$$\frac{5^\alpha}{S(2)S(3)S(4)S(5)} < \frac{5^\alpha}{p_1p_2p_3} < \frac{p_4^\alpha}{p_1p_2p_3}$$

$$\frac{6^\alpha}{S(2)S(3)S(4)S(5)S(6)} < \frac{6^\alpha}{p_1p_2p_3} < \frac{p_4^\alpha}{p_1p_2p_3}$$

..............................................................................

$$\frac{n^\alpha}{S(2)S(3)\cdots S(n)} < \frac{n^\alpha}{p_1p_2\cdots p_k} < \frac{p_{k+1}^\alpha}{p_1p_2\cdots p_k},$$

where $p_i \le n$, $i \in \{1, \dots, k\}$, $p_{k+1} > n$.

Therefore

$$\sum_{n=2}^{\infty} \frac{n^\alpha}{S(2)S(3)\cdots S(n)} < 2^{\alpha-1} + \sum_{k=1}^{\infty} \frac{(p_{k+1} - p_k)\cdot p_{k+1}^\alpha}{p_1p_2\cdots p_k} <$$

$$< 2^{\alpha-1} + \sum_{k=1}^{\infty} \frac{p_{k+1}^{\alpha+1}}{p_1p_2\cdots p_k}.$$

Then it exists $k_0 \in N$ such that for any $k \ge k_0$ we have :

$$p_1p_2\cdots p_k > p_{k+1}^{\alpha+3}.$$

Therefore

$$\sum_{n=2}^{\infty} \frac{n^\alpha}{S(2)S(3)\cdots S(n)} < 2^{\alpha-1} + \sum_{k=1}^{k_0-1} \frac{p_{k+1}^{\alpha+1}}{p_1p_2\cdots p_k} + \sum_{k\ge k_0} \frac{1}{p_{k+1}^2}$$

Because the series $\sum_{k \geq k_0} \frac{1}{p_{k+1}^2}$ is convergent it results that the given series is convergent too.

**Consequence 1.** It exists $n_0 \in N$ so that for each $n \geq n_0$ we have $S(2)S(3) \dots S(n) > n^\alpha$.

**Proof.** Because $\lim_{n \to \infty} \frac{n^\alpha}{S(2)S(3)\cdots S(n)} = 0$, there is $n_0 \in N$ so that

$$\frac{n^\alpha}{S(2)S(3)\cdots S(n)} < 1 \text{ for each } n \geq n_0.$$

**Consequence 2.** It exists $n_0 \in N$ so that :

$$S(2) + S(3) + \cdots + S(n) > (n-1) \cdot n^{\frac{\alpha}{n-1}} \text{ for each } n \geq n_0.$$

**Proof.** We apply the inequality of averages to the numbers S(2), S(3), ..., S(n) :

$$S(2) + S(3) + \cdots + S(n) > (n-1) \sqrt[n-1]{S(2)S(3)\cdots S(n)} > (n-1) n^{\frac{\alpha}{n-1}}, \quad \forall n \geq n_0.$$

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# A PARADOXICAL MATHEMATICIAN: HIS FUNCTION, PARADOXIST GEOMETRY, AND CLASS OF PARADOXES

by Michael R. Mudge

Described by Charles T. Le, Bulletin of Number Theory, vol.3, No.1, March 1995,as "The most paradoxist mathematician of the world" FLORENTIN SMARANDACHE was born on December $10^{th}$, 1954, in Balcesti (a large village), Valcea, Romania of peasant stock. A very hard and socially deprived childhood led to a period of eccentric teenage behaviour, he was close to being expelled from his high school in Craiova for disciplinary reasons. Eventually, however, a period of university studies, 1974-79, resulted in the recognition of mathematical brilliance by the professor of algebra, Alexandru Dinca. Florentin generalised Euler's Theorem from:

If $(a,m) = 1$, then $a^{\phi(m)} \equiv 1 \pmod m$ to,

If $(a,m) = d_s$, then $a^{\phi(m_s + s)} \equiv a^s \pmod m$ where $m_s$ divides $m$ and $s$ is the number of steps to get $m_s$.

An industrial appointment from 1979 to 1981 was disastrous, ending in dismissal for disciplinary reasons. In 1986 an apparently successful teaching appointment was terminated by the Ceausescu dictatorship and two years of unemployment followed. In 1988 an illegal escape from Romania through Bulgaria resulted in two years in a Turkish refugee camp, where much time was spent as a drunken vagrant.

Many mathematicians and writers lobbied the United Nations Commision for Refugees,based in Rome, and exile to the United States followed in 1990. As a member of the American Mathematical Society since 1992 and of the Romanian Scientists Association (Bucharest) since 1993 and a reviewer for the Number Theory to Zentralblatt für Mathematik scores of publications and four books bear the name of Smarandache, publishing in English, French and his native Romanian.

The Smarandache Function, S(n), is defined for positive integer argument only as the smallest integer such that S(n)! is divisible by n: (the extension to other real/complex argument has not, yet, been investigated).

The Smarandache Quotient, Q(n), is defined to be S(n)!/n.

Limited tabulation of both functions appears in The Encyclopedia of Integer Sequences by N.J.A.Sloane & Simon Plouffe, Academic Press, 1995. M1669 & M0453.

There exists an extensive literature dealing with properties of these functions; "An Infinity of Unsolved Problems Concerning a Function in Number Theory", Smarandache Function Journal, vol.1., No.1, December 1990, pp12 - 55, ISSN 1053-4792, Number Theory Publishing Company, P.O. Box 42561, Phoenix, Arizona 85080, USA providing an ideal starting point for interested readers.

A recent paper by Charles Ashbacher, Mathematical Spectrum, 1995/96, vol.28., No.1, pp20-21 addresses the question of when the Smarandache Function satisfies a Fibonacci recurrence relation, i.e. S(n) = S(n-1) + S(n-2). Empirical evidence is for 'few' occasions the largest known being n = 415664. Are there infitely many?

A Paradoxist Geometry. In 1969, at the age of 15, fascinated by geometry, Florentin Smarandache constructed a partially Euclidean and partially non-Euclidean geometry in the same space by a strange replacement of the Euclid's fifth postulate (the axiom of paralells) with the following five-statement proposition:

a)there are at least a straight line and an exterior point to it in this space for which only one line passes through the point and does not intersect the initial line;

b)there are at least a straight line and an exterior point to it in this space for which only a finite number of lines, say $k \geq 2$, pass through the point and do not intersect the initial line;

c)there are at least a straight line and an exterior point to it in this space for which any line that passes through the point intersects the initial line;

d)there are at least a straight line and an exterior point to it in this space for which an infinite number of lines that pass through the point (but not all of them) do not intersect the initial line;

e)there are at least a straight line and an exterior point to it in this space for which any line that passess through the point does not intersect the initial line.

Does there exist a model for this PARADOXIST GEOMETRY? If not can a contradiction be found using the above set of propositions together with Euclid's remaining Axioms?

Smarandache Classes of Paradoxes. Contributed by Dr.Charles T.Le, Erhus University, Box 10163, Glendale, ARIZONA 85318. USA.

Let @ be an attribute and non-@ its negation.

Thus if @ means 'possible' then non-@ means 'impossible'.

The original set of Smarandache Paradoxes are:

ALL is "@", THE "NON-@" TOO.

ALL IS "NON-@", THE "@" TOO.

NOTHING IS "@", NOT EVEN "@".

These three kinds of paradox are mutually equivalent and reduce to:

PARADOX: ALL (verb) "@", THE "NON-@" TOO.

See Florentin Smarandache, "Mathematical Fancies & Paradoxes", paper presented at the Eugene Strens Memorial on Intuitive and Recreational Mathematics and its History, University of Calgary, Alberta, Canada, July 27 - August 2, 1986.

8/10/95

Further Reading:

Only Problems, Not Solutions!, Florentin Smarandache, Xiquan Publishing House, 1993 (fourth edition), ISBN- 1-879585-00-6.

Some Notions and Questions in Number Theory, C.Dumitrescu & V.Seleacu, Ehrus University Press, Glendale,1994, ISBN 1-879585-48-0.

# Smarandache - Fibonacci Triplets

H. Ibstedt

We recall the definition of the Smarandache Function S(n):

**S(n) = the smallest positive integer such that S(n)! is divisible by n.**

and the Fibonacci recurrence formula:

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2)$$

which for $F_0 = F_1 = 1$ defines the Fibonacci series.

This article is concerned with isolated occurrencies of triplets n, n-1, n-2 for which S(n)=S(n-1)+S(n-2). Are there infinitly many such triplets? Is there a method of finding such triplets that would indicate that there are in fact infinitely many of them.

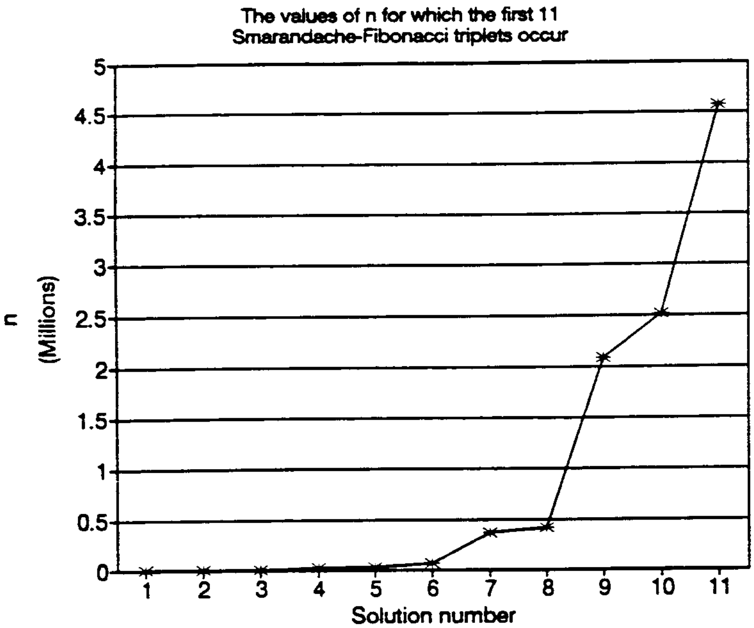
A straight forward search by applying the definition of the Smarandache Function to consecutive integers was used to identify the first eleven triplets which are listed in table 1. As often in empirical number theory this merely scratched the surface of the ocean of integers. As can be seen from diagram 1 the next triplet may occur for a value of n so large that a sequential search may be impractical and will not make us much wiser.

Table 1. The first 11 Smarandache-Fibonacci Triplets

| # | n | S(n) | S(n-1) | S(n-2) |
|---|---|---|---|---|
| 1 | 11 | 11 | 5 | 2*3 |
| 2 | 121 | 2*11 | 5 | 17 |
| 3 | 4902 | 43 | 29 | 2*7 |
| 4 | 26245 | 181 | 18 | 163 |
| 5 | 32112 | 223 | 197 | 2*13 |
| 6 | 64010 | 173 | 2*23 | 127 |
| 7 | 368140 | 233 | 2*41 | 151 |
| 8 | 415664 | 313 | 2*73 | 167 |
| 9 | 2091206 | 269 | 2*101 | 67 |
| 10 | 2519648 | 1109 | 2*101 | 907 |
| 11 | 4573053 | 569 | 2*53 | 463 |

However, an interesting observation can be made from the triplets already found. Apart from n=26245 the Smarandache-Fibonacci triplets have in common that one member is two times a prime number while the other two members are prime numbers. This observation

Diagram1.




leads to a method to search for Smarandache-Fibonacci triplets in which the following two theorems play a rôle:

I. If $n=ab$ with $(a,b)=1$ and $S(a)<S(b)$ then $S(n)=S(b)$.

II. If $n = p^a$ where p is a prime number and $a \leq p$ then $S(p^a)=ap$.

The search for Smarandache-Fibonacci triplets will be restricted to integers which meet the following requirements:

$$n = xp^a \text{ with } a \leq p \text{ and } S(x)<ap \quad (1)$$

$$n-1 = yq^b \text{ with } b \leq q \text{ and } S(y)<bq \quad (2)$$

$$n-2 = zr^c \text{ with } c \leq r \text{ and } S(z)<cr \quad (3)$$

p,q and r are primes. We then have $S(n)=ap$, $S(n-1)=bq$ and $S(n-2)=cr$. From this and by subtracting (2) from (1) and (3) from (2) we get

$$ap = bq + cr \quad (4)$$

$$xp^a - yq^b = 1 \quad (5)$$

$$yq^b - zr^c = 1 \quad (6)$$

TABLE 2. Smarandache - Fibonacci Triplets.

| # | N | S(N) | S(N-1) | S(N-2) | t |
|---|---|---|---|---|---|
| 1 | 4 | 4 * | 3 | 2 * | 0 |
| 2 | 11 | 11 | 5 | 6 * | 0 |
| 3 | 121 | 22 * | 5 | 17 | 0 |
| 4 | 4902 | 43 | 29 | 14 * | -4 |
| 5 | 32112 | 223 | 197 | 26 * | -1 |
| 6 | 64010 | 173 | 46 * | 127 | -1 |
| 7 | 368140 | 233 | 82 * | 151 | -1 |
| 8 | 415664 | 313 | 167 | 146 * | -8 |
| 9 | 2091206 | 269 | 202 * | 67 | -1 |
| 10 | 2519648 | 1109 | 202 * | 907 | 0 |
| 11 | 4573053 | 569 | 106 * | 463 | -3 |
| 12 | 7783364 | 2591 | 202 * | 2389 | 0 |
| 13 | 79269727 | 2861 | 2719 | 142 * | 10 |
| 14 | 136193976 | 3433 | 554 * | 2879 | -1 |
| 15 | 321022289 | 7589 | 178 * | 7411 | 5 |
| 16 | 445810543 | 1714 * | 761 | 953 | -1 |
| 17 | 559199345 | 1129 | 662 * | 467 | -5 |
| 18 | 670994143 | 6491 | 838 * | 5653 | -1 |
| 19 | 836250239 | 9859 | 482 * | 9377 | 1 |
| 20 | 893950202 | 2213 | 2062 * | 151 | 0 |
| 21 | 937203749 | 10501 | 10223 | 278 * | -9 |
| 22 | 1041478032 | 2647 | 1286 * | 1361 | -1 |
| 23 | 1148788154 | 2467 | 746 * | 1721 | 3 |
| 24 | 1305978672 | 5653 | 1514 * | 4139 | 0 |
| 25 | 1834527185 | 3671 | 634 * | 3037 | -5 |
| 26 | 2390706171 | 6661 | 2642 * | 4019 | 0 |
| 27 | 2502250627 | 2861 | 2578 * | 283 | -1 |
| 28 | 3969415464 | 5801 | 1198 * | 4603 | -2 |
| 29 | 3970638169 | 2066 * | 643 | 1423 | -6 |
| 30 | 4652535626 | 3506 * | 3307 | 199 | 0 |
| 31 | 6079276799 | 3394 * | 2837 | 557 | -1 |
| 32 | 6493607750 | 3049 | 1262 * | 1787 | 5 |
| 33 | 6964546435 | 2161 | 1814 * | 347 | -4 |
| 34 | 11329931930 | 3023 | 2026 * | 997 | -4 |
| 35 | 11695098243 | 12821 | 1294 * | 11527 | 2 |
| 36 | 11777879792 | 2174 * | 1597 | 577 | 6 |
| 37 | 13429326313 | 4778 * | 1597 | 3181 | 1 |
| 38 | 13849559620 | 6883 | 2474 * | 4409 | 1 |
| 39 | 14298230970 | 2038 * | 1847 | 191 | 7 |
| 40 | 14988125477 | 3209 | 2986 * | 223 | 2 |
| 41 | 17560225226 | 4241 | 3118 * | 1123 | -2 |
| 42 | 18704681856 | 3046 * | 1823 | 1223 | 4 |
| 43 | 23283250475 | 4562 * | 463 | 4099 | -10 |
| 44 | 25184038673 | 5582 * | 1951 | 3631 | -2 |
| 45 | 29795026777 | 11278 * | 8819 | 2459 | 0 |
| 46 | 69481145903 | 6301 | 3722 * | 2579 | 3 |
| 47 | 107456166733 | 10562 * | 6043 | 4519 | -1 |
| 48 | 107722646054 | 8222 * | 6673 | 1549 | -1 |
| 49 | 122311664350 | 20626 * | 10463 | 10163 | 0 |
| 50 | 126460024832 | 6917 | 2578 * | 4339 | 11 |
| 51 | 155205225351 | 8317 | 4034 * | 4283 | -5 |
| 52 | 196209376292 | 7246 * | 3257 | 3989 | -5 |
| 53 | 210621762776 | 6914 * | 1567 | 5347 | 11 |
| 54 | 211939749997 | 16774 * | 11273 | 5501 | 0 |
| 55 | 344645609138 | 7226 * | 2803 | 4423 | 9 |
| 56 | 484400122414 | 16811 | 12658 * | 4153 | -1 |
| 57 | 533671822944 | 21089 | 18118 * | 2971 | 0 |
| 58 | 620317662021 | 21929 | 20302 * | 1627 | 0 |
| 59 | 703403257356 | 13147 | 10874 * | 2273 | -2 |
| 60 | 859525157632 | 14158 * | 3557 | 10601 | -5 |
| 61 | 898606860813 | 19973 | 13402 * | 6571 | 1 |
| 62 | 972733721905 | 10267 | 10214 * | 53 | -4 |
| 63 | 1185892343342 | 18251 | 12022 * | 6229 | -2 |
| 64 | 1225392079121 | 12202 * | 9293 | 2909 | -4 |
| 65 | 1294530625810 | 17614 * | 5807 | 11807 | -3 |
| 66 | 1517767218627 | 11617 | 8318 * | 3299 | -8 |
| 67 | 1905302845042 | 22079 | 21478 * | 601 | -1 |
| 68 | 2679220490034 | 11402 * | 7459 | 3943 | 11 |
| 69 | 3043063820555 | 14951 | 12202 * | 2749 | 5 |
| 70 | 6098616817142 | 24767 | 20206 * | 4561 | 2 |
| 71 | 6505091986039 | 31729 | 19862 * | 11867 | 2 |
| 72 | 13666465868293 | 28099 | 16442 * | 11657 | 7 |

The greatest common divisor $(p^a, q^b)=1$ obviously divides the right hand side of (5). This is the condition for (5) to have infinitely many solutions for each solution (p,q) to (4). These are found using Euclid's algorithm and can be written in the form:

$$x = x_0 + q^b t, \qquad y = y_0 - p^a t \tag{5'}$$

where t is an integer and $(x_0, y_0)$ is the principal solution.

Solutions to (5') are substituted in (6') in order to obtain integer solutions for z.

$$z = (yq^b - 1)/r^c \tag{6'}$$

Solutions were generated for (a,b,c)=(2,1,1), (a,b,c)=(1,2,1) and (a,b,c)=(1,1,2) with the parameter t restricted to the interval $-11 \leq t \leq 11$. The result is shown in table 2. Since the correctness of these calculations are easily verfied from factorisations of S(n), S(n-1), and S(n-2) these are given in table 3 for two large solutions taken from an extension of table 2.

Table 3. Factorisation of two Smarandache-Fibonacci Triplets.

| | | | |
|---|---|---|---|
| n= | $16{,}738{,}688{,}950{,}356 = 2^2 \cdot 3 \cdot 31 \cdot 193 \cdot \underline{15{,}269}^2$ | S(n)= | $\underline{2 \cdot 15{,}269}$ |
| n-1= | $16{,}738{,}688{,}950{,}355 = 5 \cdot 197 \cdot 1{,}399 \cdot 1{,}741 \cdot \underline{6{,}977}$ | S(n-1)= | $\underline{6{,}977}$ |
| n-2= | $16{,}738{,}688{,}950{,}354 = 2 \cdot 7^2 \cdot 19 \cdot 23 \cdot 53 \cdot 313 \cdot \underline{23{,}561}$ | S(n-2)= | $\underline{23{,}561}$ |
| n= | $19{,}448{,}047{,}080{,}036 = 2^2 \cdot 3^2 \cdot 43^2 \cdot \underline{17{,}093}^2$ | S(n)= | $\underline{2 \cdot 17{,}093}$ |
| n-1= | $19{,}448{,}047{,}080{,}035 = 5 \cdot 7 \cdot 19 \cdot 37 \cdot 61 \cdot 761 \cdot \underline{17{,}027}$ | S(n-1)= | $\underline{17{,}027}$ |
| n-2= | $19{,}448{,}047{,}080{,}034 = 2 \cdot 97 \cdot 1{,}609 \cdot 3{,}631 \cdot \underline{17{,}159}$ | S(n-1)= | $\underline{17{,}159}$ |

**Conjecture.** There are infinitely many triplets n, n-1, n-2 such that S(n)=S(n-1)+S(n-2).

**Questions:**

**1.** It is interesting to note that there are only 7 cases in table 2 where S(n-2) is two times a prime number and that they all occur for relatively small values of n. Which is the next one?

**2.** The solution for n=26245 stands out as a very interesting one. Is it a unique case or is it a member of family of Smarandache-Fibonacci triplets different from those studied in this article?

**References:**

C. Ashbacher and M. Mudge, *Personal Computer World*, October 1995, page 302.

M. Mudge, in a Letter to R. Muller (05/14/96), states that: "John Humphries of Hulse Ground Farm, Little Faringdo, Lechlade, Glovcester, GL7 3QR, U.K., has found a set of three numbers, greater than 415662, whose Smarandache Function satisfies the Fibonacci Recurrence, i.e. S(2091204) = 67, S(2091205) = 202, S(2091206) = 269, and 67 + 202 = 269.

# THE SOLUTION OF SOME DIOPHANTINE EQUATIONS RELATED TO SMARANDACHE FUNCTION

by

Ion Cojocaru and Sorin Cojocaru

In the present note wesolve two diophantine eqations concerning the Smarandache function.

First, we try to solve the diophantine eqation :

$$S(x^y) = y^x \qquad (1)$$

It is porposed as an open problem by F. Smarandache in the work [1], pp. 38 (the problem 2087).

Because S(1) = 0, the couple (1,0) is a solution of eqation (1). If x = 1 and y ≥1, the eqation there are no (1,y) solutions. So, we can assume that x ≥ 2. It is obvious that the couple (2,2) is a solution for the eqation (1).

If we fix y = 2 we obtain the equation $S(x^2) = 2^x$. It is easy to verify that this eqation has no solution for x ∈{3,4}, and for x ≥5 we have $2^x > x^2 \geq S(x^2)$, so $2^x > S(x^2)$. Consequently for every $x \in N^* \setminus \{2\}$, the couple (x,2) isn't a solution for the eqation (1).

We obtein the equation $S(2^y) = y^2$, y ≥3, fixing x = 2. It is know that for p = prime number we have the ineqality:

$$S(p^r) \leq p \cdot r \qquad (2)$$

Using the inequality (2) we obtein the inequality $S(2^y) \leq 2 \cdot y$. Because y ≥3 implies $y^2 > 2y$, it results $y^2 > S(2^y)$ and we can assume that x ≥3 and y ≥3.

We consider the function $f: [3,\infty] \to R_+^*$ defined by $f(x) = \frac{y^x}{x^y}$, where y ≥ 3 is fixed.

This function is derivable on the considered interval, and $f'(x) = \frac{y^x x^{y-1}(x \ln y - y)}{x^{2y}}$. In the point $x_o = \frac{y}{\ln y}$ it is equal to zero, and $f(x_o) = f(\frac{y}{\ln y}) = y^{\frac{1}{\ln y}} (\ln y)^y$.

Because y ≥ 3 it results that lny > 1 and $y^{\frac{1}{\ln y}} > 1$, so $f(x_o) > 1$. For $x > x_o$, the function f is strict incresing, so $f(x) > f(x_o) > 1$, that leads to $y^x > x^y \geq S(x^y)$, respectively $y^x > S(x^y)$. For $x < x_o$, the function f is strict decreasing, so $f(x) > f(x_o) > 1$ that lands to $y^x > S(x^y)$. There fore, the only solution of the eqaution (1) are the couples (1,0) and (2,2).

# SOLVING THE DIOPHANTINE EQUATION

$$x^y - y^x = S(x) \qquad (3)$$

It is obvious that the couples (1,1) is a solution of the eqaution (3).

Because $x^y - y^x = S(x)$ it results $x \neq y$ (otherwise we have $S(x)=0$, i.e., $x = 1 = y$). We prove that the equation (3) has an unique solution.

Case I : $x > y$. Therefore it exists $a \in N^*$ so $x = y + a$, $(y + a)^y - y^{y+a} = S(y+a)$ or $(1 + \frac{a}{y})^y - y^a = \frac{S(y+a)}{y^y}$. But $(1+\frac{a}{y})^y < e^a$. It results $e^a - y^a > \frac{S(y+a)}{y^y}$, false inequality for $y > e$ ($e^a - y^a < 0$ for $y > e$). So we have $y = 1$ or $y = 2$. If $y = 1$ we have $x-1 = S(x)$. In this situation it is obvious that x is a compound number. If $x = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n}$ is the factorization of x into prims wich $p_i \neq p_j$ , $a_i \neq 0$ , $i,j = \overline{1,n}$, then we have $S(x) = \max_{1 \le i \le n} S(p_i^{a_i}) = S(p_e^{a_e})$, $1 \le e \le r$. But, because $S(x) = S(p_e^{a_e}) < p_e a_e < x - 1$ it results that $S(x) < x - 1$, that is fals.

If $y = 2$, we have $x^2 - 2^x = S(x)$. For $x \ge 4$ we obtein $x^2 - 2^x < 0$, and for $x \in \{2,3\}$ there is no solution.

Case II: $x < y$. Therefore it exists $b > 0$ such that $y = x + b$. Then we have $x^{x+b} - (x+b)x = S(x)$, so $x^b - (1 + \frac{b}{x})^x = \frac{S(x)}{x^x} \le \frac{x}{x^x} \le 1$.

But, because $(1 + \frac{b}{x})^x < e^b$ we obtain $x^b - e^b < 1$, which is a false inequality for $x \ge 4$. If $x = 2$, then $2^y - y^2 = 2$, an equation which fas no solution because $2^y - y^2 \ge 7$ for $y \ge 5$.

If $x = 3$, then $3^y - y^3 = 3$, an equation which has no solutions for $y \in \{1,2,3\}$, because, if $y \ge 4$ it results $3^y - y^3 \ge 17$.

Therefore the equation (3) admits an unique solution (1,1).

## REFERENCES

[1] **F. Smarandache** : *An infinity of unsolved problems concerning a Function in the Number Theory* ( Presented at the 14th American Romanian Academy Anual cOnvention, hold in Los Angeles, California, hosted by the University of Southern California, from April 20 to April 22, 1989 ).

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## Problems

Edited by

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Welcome to the inaugural version of what is to be a regular feature in **Smarandache Notions**! Our goal is to present interesting and challenging problems in all areas and at all levels of difficulty with the only limits being good taste. Readers are encouraged to submit new problems and solutions to the editor at one of the addresses given above. All solvers will be acknowledged in a future issue. Please submit a solution along with your proposals if you have one. If there is no solution and the editor deems it appropriate, that problem may appear in the companion column of unsolved problems. Feel free to submit computer related problems and use computers in your work. Programs can also be submitted as part of the solution. While the editor is fluent in many programming languages, be cautious in submitting programs as solutions. Wading through several pages of obtuse program to determine if the submitter has done it right is not the editors idea of a good time. Make sure you explain things in detail.

If no solution is currently available, the problem will be flagged with an asterisk*. The deadline for submission of solutions will generally be six months after the date appearing on that issue. Regardless of deadline, no problem is ever officially closed in the sense that new insights or approaches are always welcome. If you submit a problem or solution and wish to guarantee a reply, please include a self-addressed envelope or postcard with appropriate stamps attached. Suggestions for improvement or modification are also welcome at any time. All proposals in this initial offering are by the editor.

The Smarandache function S(n) is defined in the following way

For $n \geq 1$, $S(n) = m$ is the smallest nonnegative integer such that n evenly divides m factorial.

### New Problems

1) The Euler phi function $\phi(n)$ is defined to the number of positive integers not exceeding n that are relatively prime to n.

a) Prove that there are no solutions to the equation

$$\phi(S(n)) = n$$

b) Prove that there are no solutions to the equation

$$S(\phi(n)) = n$$

c) Prove that there are an infinite number of solutions to the equation

$$n - \phi(S(n)) = 1$$

d) Prove that for every odd prime p, there is a number n such that

$$n - \phi(S(n)) = p+1$$

2) This problem was proposed in **Canadian Mathematical Bulletin** by P. Erdos and was listed as unsolved in the book **Index to Mathematical Problems 1980-1984** edited by Stanley Rabinowitz and published by MathPro Press.

Prove that for infinitely many n

$$\phi(n) < \phi(n - \phi(n)).$$

3) The following appeared as unsolved problem (21) in **Unsolved Problems Related To Smarandache Function**, edited by R. Muller and published by Number Theory Publishing Company.

Are there m,n,k non-null positive integers, $m,n \neq 1$ for which

$$S(mn) = m^k * S(n)?$$

Find a solution.

4) The following appeared as unsolved problem (22) in **Unsolved Problems Related to Smarandache Function**, edited by R. Muller and published by Number Theory Publishing Company.

Is it possible to find two distinct numbers k and n such that

$$\log_{S(k^n)} S(n^k)$$

is an integer?

Find two integers n and k that satisfy these conditions.

5) Solve the following doubly true Russian alphametic

| | |
|---|---|
| ДВА | 2 |
| ДВА | 2 |
| ТРИ | 3 |
| ------- | -- |
| СЕМЬ | 7 |

where 2 divides ДВА, 3 divides ТРИ and 7 divides СЕМЬ.

Can anyone come up with a similar Romanian alphametic?

6) Prove the Smarandache Divisibility Theorem

If a and m are integers and $m > 0$, then

$$(a^m - a)(m - 1)!$$

is divisible by m.

Which was problem (126) in **Some Notions and Questions in Number Theory**, published by Erhus University Press.

7) Let $D = \{0,1,2,3,4,5,6,7,8,9\}$. For any number $1 \leq n \leq 10$, we can take n unique digits from D and form a number, leading zero not allowed. Let $P_n$ be the set of all numbers that can be formed by choosing n unique digits from D. If 1 is not considered prime, which of the sets $P_n$ contains the largest percentage of primes?

This problem is similar to unsolved problem 3 part (a) that appeared in **Only Problems, Not Solutions**, by Florentin Smarandache.

*8) The following four problems are all motivated by unsolved problem 3 part (b) that appeared in **Only Problems, Not Solutions**, by Florentin Smarandache.

a) Find the smallest integer n such that n! contains all 10 decimal digits.

b) Find the smallest integer n such that the n-th prime contains all 10 decimal digits.

c) Find the smallest integer n such that $n^n$ contains all 10 decimal digits.

d) Find the smallest integer n such that n! contains one digit 10 times. What is that digit?

# PROPOSED PROBLEMS

by

M. Bencze

( i ) Solve the following equations:

1) $S^k(x) + S^k(y) = S^k(z)$ , $k \in Z$, $x, y, z \in Z$

where S' is the Smarandache function and $S(-n) = -S(n)$

2) $\frac{4}{n} = \frac{1}{S(x)} + \frac{1}{S(y)} + \frac{1}{S(z)}$ , $n > 4$

3) $\frac{5}{n} = \frac{1}{S(x)} + \frac{1}{S(y)} + \frac{1}{S(z)}$ , $n > 5$

4) $S^{S(y)}(x) = S^{S(x)}(y)$

5) $S\left(\sum_{k=1}^{n} x_k^u\right) = S^u\left(\sum_{k=1}^{n} x_k\right)$ , $u \in Z$

6) $S^y(x) - S^t(z) = S^{y-t}(x-z)$

7) $\sum_{k=1}^{n} S^m(x_k) = \sum_{k=n}^{2n} S^m(x_k)$

8) $2S(x^4) - S^2(y) = 1$

9) $S\left(\frac{x+y+z}{3}\right) + \frac{S(x)+S(y)+S(z)}{3} = \frac{2}{3}\left[S\left(\frac{x+y}{2}\right) + S\left(\frac{y+z}{2}\right) + S\left(\frac{z+x}{2}\right)\right]$

10) $S(x_1^{x_1}) \cdot S(x_2^{x_2}) \ldots S(x_n^{x_n}) = S(x_{n+1}^{x_{n+1}})$

11) $S(x_1^{x_2}) \cdot S(x_2^{x_3}) \ldots S(x_{n-1}^{x_n}) = S(x_n^{x_1})$

12) $S(x) = \mu(y)$ , where $\mu$ is the Möbius function

13) $S^2(Q_n) = \sum_{Q_{n-1} | Q_n} \ldots \sum_{Q_2 | Q_3} \sum_{Q_1 | Q_2} \mu^2(Q_1)$

14) $S(x) = B_y$ , where $B_y$ is a Bernoulli number

15) $S(x+y)\,(S(x) - S(y)) = S(x-y)\,(S(x) + S(y))$

16) $S(x) = F_y$ ,  where $F_y$ is a Fibonacci number

17) $\sum_{k=1}^{n} S(k^p) = \sum_{k=1}^{n} S^p(k)$

18) $\sum_{k=1}^{n} S(k) = S\left(\frac{n(n+1)}{2}\right)$

19) $\sum_{k=1}^{n} S(k^2) = S\left(\frac{n(n+1)(2n+1)}{6}\right)$

20) $\sum_{k=1}^{n} S(k^3) = S\left(\frac{n^2(n+1)^2}{4}\right)$

21) $\sum_{k=1}^{n} k(S(k)!) = (S(n+1))! - 1$

22) $\sum_{k=1}^{n} \frac{1}{S(k)S(k+1)} = \frac{S(n)}{S(n+1)}$

( ii ) Solve the system

$$\begin{cases} S(x) + S(y) = 2S(z) \\ S(x) \cdot S(y) = S^2(z) \end{cases}$$

( iii ) Find n such that n divides the sum

$$1^{S(n-1)} + 2^{S(n-1)} + \ldots (n-1)^{S(n-1)} + 1$$

( iv ) May be writen every positive integer n as

$$n = S^3(x) + 2\,S^3(y)\,3\,S^3(z) \quad ?$$

( v ) Prove that

$$|S(x) + S(y) + S(z)| + |S(x)| + |S(y)| + |S(z)| \geq$$

$$\geq |S(x) + S(y)| + |S(y) + S(z)| + |S(z) + S(x)|$$

for all $x, y, z \in Z$

( vi ) Find all the positive integers x, y, z for which

$$(x+y+z) + S(x) + S(y) + S(z) \geq S(x+y) + S(y+z) + S(z+x)$$

( vii ) There exists an infinity of prime numbers which may be writen under the form

$$P = S^3(x) + S^3(y) + S^3(z) + S^3(t) \quad ?$$

( viii ) Let $M_1, M_2, \ldots, M_n$ be finite sets and $a_{ij} = \text{card}(M_i \cap M_j)$ , $b_{ij} = S(a_{ij})$. Prove that $\det(a_{ij}) \geq 0$ and $\det(b_{ij}) \geq 0$ .

( ix ) Find the sum

$$\sum_{Q_{n-1}|Q_n} \cdots \sum_{Q_2|Q_3} \sum_{Q_1|Q_2} \frac{1}{S^2(Q_1)}$$

( x ) Prove that

$$\sum_{k=1}^{\infty} \frac{1}{S^2(k) - S(k) + 1} \quad \text{is irational}$$

( xi ) Find all the positive integers x for which

$$S\left(\left[\frac{x^{n+1} - 1}{(n+1)(x-1)}\right]\right) \geq S\left(\left[x^{\frac{n}{2}}\right]\right)$$

where [x] is the integer part of x.

( xii ) There exists at lest a prime between S(n)!, and S(n+1)! ?

(xiii) If $\sigma \in S_n$ is a permutation, prove that

$$\sum_{k=1}^{n} \frac{\sigma(k)}{S^{m+1}(k)} \geq \sum_{k=1}^{n} \frac{1}{k^m}$$

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# PROPOSED PROBLEM*

by

I. M. Radu

Show that (except for a finite set of numbers) between S(n) and S(n+1) there exist at least a prime number. (One notes by S(n) the Smarandache Function: the smallest integer such that S(n)! is divisible by n.)

If $N_s(n)$ denotes the number of primes between S(n) and S(n+1), calculate an asymptotic formula for $N_s(n)$.

**Some comments:**

If n or n+1 is prime, then S(n) or S(n+1) respectively is prime. And the above conjuncture is solved.

But I was not able to find a general proof. It might be a useful thing to apply the Brensch Theorem (if $n \geq 48$, then there exist at least a prime between n and $\frac{9}{8}$n), in stead of Bertrand Postulate / Tchebychev Theorem (between *n* and *2n* there exist at least a prime)

The last question may be writing as

$$N_s(n) = |\Pi(S(n+1)) - \Pi(S(n))| ,$$

where $\Pi(x)$ is the number of primes $\leq x$,
but how can we compose the function $\Pi$ and S ?

**References:**

[1] Dumitrescu Constantin, "The Smarandache Function", in "Mathematical Spectrum", Sheffield, Vol. 29, No. 2, 1993, pp. 39-40.

[2] Ibstedt Henry, "The F. Smarandache Function S(n): programs, tables, graphs, comments", in "Smarandache Function Journal", Vol. 2-3, No. 1, 1993, pp. 38-71.

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*Charles Ashbacher (U.S.A.), using a computer program that computes the values of S(n) conducted a search up through n<1,033,197 and found where there is no prime p, where $S(n) \leq p \leq S(n+1)$. They are as follows:

| | |
|---|---|
| n = 224 | and S(n) = 8, S(n+1) = 10 |
| n = 2057 | and S(n) = 22, S(n+1) = 21 |
| n = 265225 | and S(n) = 206, S(n+1) = 202 |
| n = 843637 | and S(n) = 302, S(n+1) = 298 |

# PROPOSED PROBLEMS

by

M. R. Mudge

Problem 1:

The Smarandache no prime digits sequence is defined as follows:

1,4,6,8,9,10,11,1,1,14,1,16,1,18,19,0,1,4,6,8,9,0,1,4,6,8,9,40,41,
42,4,44,4,46,4,48,49,0,...

(Take out all prime digits of n.)

Is it any number that occurs infinetely many times in this sequence?
(for example 1, or 4, or 6, or 11, etc.).

Solution by Dr. Igor Shparlinski,
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It seems that: if, say n has already occured, then for example n3, n33, n333, etc. gives an infinitely many repetitions of this number.

Problem 2:

The Smarandache no square digits sequence is defined as follows:

2,3,5,6,7,8,2,3,5,6,7,8,2,2,22,23,2,25,26,27,28,2,3,3,32,33,3,35,
36,37,38,3,2,3,5,6,7,8,5,5,52,52,5,55,56,57,58,5,6,6,62,...

(Take out all square digits of n.)

Is it any number that occurs infinetely many times in this sequence?
(for example 2, or 3, or 6, or 22, etc. ?)

Solution by Carl Pomerance (E-mail: carl@alpha.math.uga.edu):

If any number appears in the sequence, then clearly it occurs infinitely often, since if the number that appears is k, and it comes from n by deleting square digits, then k also comes from 10n.

Problem 3:

How many regions are formed by joining, with straight chords, n point situated regularly on the circumference of a circle ?

The degeneracy from the maximum possible number of regions for n points on the circumference of a circle seems almost intractable in general.

Perhaps the use of regularly distributed point, i.e. separated by $\frac{2\pi}{n}$ radians, produces the Smarandache Portions of Pi (e) !!

## Unsolved Problems

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Welcome to the first installment of what is to be a regular feature in **Smarandache Notions**! In this column, we will present problems where the solution is either unknown or incomplete. This is meant to be an interactive endeavor, so input from readers is strongly encouraged. Always feel free to contact the editor at any of the addresses given above. It is hoped that we can together advance the flow of mathematics in some small way. There will be no deadlines here, and even if a problem is completely solved, new insights or more elegant proofs are always welcome. All correspondents who are the first to resolve any issue appearing here will have their efforts acknowledged in a future column.

While there will almost certainly be an emphasis on problems related to Smarandache notions, it will not be exclusive. Our goal here is to be interesting, challenging and maybe at times even profound. In modern times computers are an integral part of mathematics and this column is no exception. Feel free to include computer programs with your submissions, but please make sure that adequate documentation is included. If you are someone with significant computer resources and would like to be part of a collective effort to resolve outstanding problems, please contact the editor. If such a group can be formed, then sections of a problem can be parceled out and all those who participated will be given credit for the solution.

And now, it is time to stop chatting and get to work!

Definition of the Smarandache function S(n):

S(n) = m, smallest positive integer such that m! is evenly divisible by n.

In [1], T. Yau posed the following question:

For what triplets n, n+1, and n+2 does the Smarandache function satisfy the Fibonacci relationship

$$S(n) + S(n+1) = S(n+2)?$$

And two solutions

$S(9) + S(10) = S(11)$;     $S(119) + S(120) = S(121)$

were given.

In [2], C. Ashbacher listed the additional solutions

$S(4900) + S(4901) = S(4902)$; $S(26243) + S(26244) = S(26245)$;
$S(32110) + S(32111) = S(32112)$; $S(64008) + S(64009) = S(64010)$;
$S(368138) + S(368139) = S(368140)$; $S(415662) + S(415663) = S(415664)$

discovered in a computer search up through n = 1,000,000. He then presented arguments to support the conjecture that the number of solutions is in fact infinite.

Recently, Henry Ibstedt from Sweden sent a letter in response to this same problem appearing in the October issue of **Personal Computer World**. He has conducted a more extensive computer search, finding many other solutions. His conclusion was, "This study strongly indicates that the set of solutions is infinite." The complete report has been submitted to **PCW** for publication.

Another problem dealing with the Smarandache function has been given the name Radu's problem, having been first proposed by I.M. Radu[3].

Show that, except for a finite set of numbers, there exist at least one prime number between S(n) and S(n+1).

Ashbacher also dealt with this problem in [2] and conducted another computer search up through n = 1,000,000. Four instances where there are no primes between S(n) and S(n+1) were found.

n = 224 = 2*2*2*2*2*7  S(n) = 8     n+1 = 225 = 3*3*5*5  S(225) = 10
n = 2057 = 11*11*17  S(n) = 22  n+1 = 2058 = 2*3*7*7*7  S(2058) = 21
n = 265225 = 5*5*103*103  S(n) = 206  n+1 = 265226 = 2*13*101*101
S(265226) = 202
n = 843637 = 37*151*151  S(n) = 302  n+1 = 843638 = 2*19*149*149
S(843638) = 298

The fact that the last two solutions involve the pairs of twin primes (101,103) and (149,151) was one point used to justify the conjecture that there is an infinite set of numbers such that there is no prime between S(n) and S(n+1).

Ibstedt also extended the computer search for solutions and found many other cases where there is no prime between S(n) and S(n+1). His conclusion is quoted below.

"A very large set of solutions was obtained. There is no indication that the set would be finite."

This conclusion is also due to appear in a future issue of **Personal Computer World**.

The following statement appears in [4].

$$1141^6 = 74^6 + 234^6 + 402^6 + 474^6 + 702^6 + 894^6 + 1077^6$$

This is the smallest known solution for 6th power as the sum of 7 other 6th powers.

Is this indeed the smallest such solution? No one seems to know. The editor would be interested in any information about this problem. Clearly, given enough computer time, it can be resolved. This simple problem is also a prime candidate for a group effort at resolution.

Another related problem that would be also be a prime candidate for a group effort at computer resolution appeared as problem 1223 in **Journal of Recreational Mathematics**.

Find the smallest integer that is the sum of two unequal fifth powers in two different ways, or prove that there is none.

The case of third powers is well known as a result of the famous story concerning the number of a taxicab

$$1729 = 1^3 + 12^3 = 9^3 + 10^3$$

as related by Hardy[4].

It was once conjectured that there might be a solution for the fifth power case where the sum had about 25 decimal digits, but a computer search for a solution with sum $< 1.02 \times 10^{26}$ yielded no solutions[5].

Problem (24) in [6] involves the Smarandache Pierced Chain(SPC) sequence.

{ 101, 1010101, 10101010101, 101010101010101, . . . }

or

```
SPC(n) = 101 * 1 0001  0001 . . . 0001
                 | ---- |
```

where the section in | ---- | appears n-1 times.

And the question is, how many of the numbers

SPC(n) / 101 are prime?

It is easy to verify that if n is evenly divisible by 3, then the number of 1's in SPC(n) is evenly divisible by 3. Therefore, so is SPC(n). And since 101 is not divisible by 3, it follows that

SPC(n) / 101

must be divisible by 3.

A simple induction proof verifies that SPC(2k)/101 is evenly divisible by 73 for k = 1,2,3, . . .

Basis step:

SPC(2)/101 = 73*137

Inductive step:

Assume that SPC(2k)/101 is evenly divisible by 73. From this, it is obvious that 73 divides SPC(2k). Following the rules of the sequence, SPC(2(k+1)) is formed by appending 01010101 to the end of SPC(2k). Since

01010101 / 73 = 13837

it follows that SPC(2(k+1)) must also be divisible by 73.

Therefore, SPC(2k) is divisible by 73 for all k > 0. Since 73 does not divide 101, it follows that SPC(2k) / 101 is also divisible by 73.

Similar reasoning can be used to obtain the companion result.

SPC(3 + 4k) is evenly divisible by 37 for all k > 0.

With these restrictions, the first element in the sequence that can possibly be prime when divided by 101 is

SPC(5) = 1010101010101010101.

However, this does not yield a prime as

SPC(5) = 41 * 101 * 271 * 3541 * 9091 * 27961.

Furthermore, since the elements of the sequence SPC(5k), k > 0 are made by appending the string

0101010101010101010101 = 41 * 101 * 271 * 3541 * 9091 * 27961

to the previous element, it is also clear that every number SPC(5k) is evenly divisible by 271 and therefore so is SPC(5k)/101.

Using these results to reduce the field of search, the first one that can possibly be prime is SPC(13)/101. However,

SPC(13)/101 = 53 * 79 * 521 * 859 * ......

SPC(17)/101 is the next not yet been filtered out. But it is also not prime as

SPC(17)/101 = 103 * 4013 * .....

The next one to check is SPC(29)/101, which is also not prime as

SPC(29)/101 = 59 * 349 * 3191 * 16763 * 38861 * 43037 * 62003 * ....

SPC(31)/101 is also not prime as

SPC(31)/101 = 2791 * ....

At this point we can stop and argue that the numerical evidence strongly indicates that there are no primes in this sequence. The problem is now passed on to the readership to perform additional testing or perhaps come up with a proof that there are no primes in this sequence.

### References

1. T. Yau: `A Problem Concerning the Fibonacci Series', Smarandache Function Journal, V. 4-5, No. 1, (1994), page 42.

2. C. Ashbacher, **An Introduction To the Smarandache Function**, Erhus University Press, 1995.

3. I. M. Radu, Letter to the Editor, Math. Spectrum, Vol. 27, No. 2, (1994/1995), page 44.

4. D. Wells, **The Penguin Dictionary of Curious and Interesting Numbers**, Penguin Books, 1987.

5. R. Guy, **Unsolved Problems in Number Theory 2nd. Ed.**,Springer Verlag, 1994.

6. F. Smarandache, **Only Problems, Not Solutions!**, Xiquan Publishing House, 1993.

Review of

**Have Some Sums To Solve: The Compleat Alphametics Book**, by Steven Kahan, Baywood Publishing Company, Amityville, NY, 1978. 114 pp.(paper), $12.45 including poastage, ISBN 0-89503-007-1.

**At Last!! Encoded Totals Second Addition**, by Steven Kahan, Baywood Publishing Company, Amityville, NY, 1994. 122 pp.,(paper), $12.45 including postage, ISBN 0-89503-171-X.

To many people, alphametics, problems where letters replace digits and those letters form the words of a message, are enjoyable to do, but clearly restricted to the area known as recreational mathematics. However, such an approach is simplistic. Solving a properly constructed alphametic is an exercise in logic and basic number theory that forces the solver to use many elementary rules of arithmetic and algebra if the solution is to be found in a reasonable length of time.

Steven Kahan, the longtime editor of the Alphametics Column of **Journal of Recreational Mathematics**, is clearly the leading expert on this form of problem and these two books present many of his best efforts. The problems and messages are quite good and detailed solutions to all problems are included.

For example, replace the letters of the following message with digits so that the addition is correct

```
  ROMANS
+   ALSO
+   MORE
+     OR
+   LESS
+  ADDED
---------
 LETTERS
```

If you like logic puzzles or are a teacher looking for extra credit problems that involve more complex, yet elementary mathematics, either or both of these books would be an excellent solution to your problem.

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Review of

**Circles: A Mathematical View**, by Dan Pedoe, The Mathematical Association of America, Washington, D. C., 1995. 144 pp. $18.95(paper). ISBN 0-88385-518-6.

Although it is the simplest of all nonlinear geometric forms, the circle is far from trivial. It is indeed a pleasure that The Mathematical Association of America chose to reprint an update of this classic first printed in 1957. Geometry teaching has been in retreat for many years in the US and that has been a sad (and very bad) thing. It is also puzzling as so many people say that the reason why they cannot do mathematics (i.e. algebra) is that they need to see something in order to understand it. Furthermore, the first mathematical education most children receive contains the differentiation of shapes and their different properties.

Circles and lines as used in geometry are abstractions that are easily grasped, much simpler to many than the abstract generalizations of algebra. One can only hope that this book signals a rebirth in interest in geometry education. Without question, it can be used as a text for that education and would help parent a rebirth. To remedy this modern affliction and make the material available to the current readership, a chapter zero was included. This new chapter is used to introduce the background concepts and terminology that could be assumed when it was first published.

No one can truly appreciate the intellectual achievements of the ancients as summarized by Euclid without doing some of the problems. There is also a stark beauty to a form of mathematics where the tools are a compass, straightedge and a mind. Particularly in the age of calculators and computers. All of the basic, ancient, results concerning circles are covered as well as some very recent ones. The theorems are well presented and complete without being overdone. In keeping with the ancient traditions, pencil, paper, compass and straightedge are the only tools used. A short collection of solved exercises is also included.

Like the books of Euclid, this work will grow old but never dated. It was destined to be a classic when it was first printed and remains so today.

**From Erdos to Kiev: Problems of Olympiad Caliber**, edited by Ross Honsberger, The Mathematical Association of America, 1995. 250 pp., $31.00(paper). ISBN 0-88385-324-8.

Mathematicians by definition have a love affair with good problems, and this is a collection of the best. While designed to be at a level for mathematical olympiad use, all mathematicians will find something in here that will stretch them. Some are at the level where the solution requires a simple insight, but others may require reaching for your thinking cap. However, all can be solved using arguments considered within the reach of an olympic mathlete. Which is encouraging. It is nice to know that there are young people who can do problems that force me to strain a few neurons. Solutions are included, most of which were created by the editor. The problems are taken from geometry, number theory, probability and combinatorics.

Another high quality entry in the series of problem books by Ross Honsberger, this is a book for all mathematicians, potential olympiads to professionals.

Reviewed by
Charles Ashbacher

Review of

*An Introduction to the Smarandache Function*, by Charles Ashbacher, Erhus University Press, 1995, 60 pp. (paper), $7.95.
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This slim volume patently lives up to its title. It does give an introduction to the Smarandache function reaching from its definition all the way to an enumeration and brief discussion of several unsolved problems. Theorems are clearly stated and proofs are always supplied. However, the exposition is relatively lively and informal, lending to this book's readability and brevity. One could get an overview of the topic by skimming this book in an hour or two, skipping the proofs and algorithms. The more diligent reader will spend considerably more time constructing his own examples to illustrate the proofs and test the algorithms.

Chapter one covers basics of the number theoretic Smarandache function, $S(n)$, where $n$ is a positive integer. Included are its definition, 16 theorems and a ready-to-use C++ program for computing values of this function. A background in Number Theory is certainly helpful for approaching this topic, but not absolutely necessary. Just in case, the chapter begins with a one page summary of the idea of divisibility and definitions of the standard arithmetic functions f, s and t. It culminates with a theorem characterizing the range of $S(n)$. The author has considerable experience in computer investigations of this and other topics in number theory and recreational mathematics. In addition to the C++ implementation, he has supplied a UBASIC program, useful for handling extremely large numbers which surpass the maximum allowable integer size of C++.

Chapter two takes up some deeper questions. Topics include iteration and fixed points of the Smarandache function as well as solutions of numerous equations such as the Fibonacci-like relation $S(n+2) = S(n+1) + S(n)$ . Various problems are presented and solved. Many other, as yet unsolved, problems are presented. In the latter case the author often furnishes a conjecture along with helpful rationale. The reader is led to the jumping off place, ready for his own foray into unresolved areas of investigation. These conjectures and plausibility arguments are clearly labelled as such and hence distinguishable from the theorems and proofs with which they are interspersed.

This book is not without its niggling errors, mostly typographical and obvious enough as to cause no serious confusion. A few discrepancies in terminology and notation were also noted, probably not uncommon in the literature pertaining to a mathematical topic which is less than 20 years old. As Ashbacher notes in his introductory material, the Smarandache function was created in the 1970's and first published in 1980. In this work, he has given us a bibliography guiding us to works published in the intervening years and provided a good roadmap taking us from the beginnings to the current state of knowledge of his topic.

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