A GENERALIZATION OF A PROBLEM OF STUPARU

by L. Seagull, Glendale Community College

Let n be a composite integer >= 48. Prove that between n and S(n) there exist at least 5 prime numbers.

Solution:

T. Yau proved that Smarandache function has the following property:
S(n) <= n/2 for any composite number n >= 10,
because:
if n = pq, with p < q and (p, q) = 1, then:
S(n) = max {S(p), S(q)} = S(q) <= q = n/p <= n/2;
if n = p^r, with p prime and r integer >= 2, then:
S(n) <= pr <= (p^r)/2 = n/2.

(Inequation pr <= (p^r)/2 doesn't hold:
for p = 2 and r = 2, 3;
as well as for p = 3 and r = 2;
but in either case n = p^r is less than 10.
For p = 2 and r = 4, we have 8 <= 16/2;
therefore for p = 2 and r >= 5, inequality holds because the right side is exponentially increasing while the left side is only linearly increasing,
i.e. 2r <= (2^r)/2 for r >= 4 (1)
Similarly for p = 3 and r >= 3,
i.e. 3r <= (3^r)/2 for r >= 3. (2)
Both of these inequalities can be easily proved by induction.
For p = 5 and r = 2, we have 10 <= 25/2;
and of course for r >= 3 inequality 5r <= (5^r)/2 will hold.
If p >= 7 and r = 2, then p2 <= (p^2)/2,
which can be also proved by induction.)

Stuparu proved, using Bertrand/Tchebychev postulate/theorem, that there exists at least one prime between n and n/2 {i.e. between n and S(n)}.
But we improve this if we apply Breusch's Theorem,
which says that between n and (9/8)n there exists at least one prime.
Therefore, between n and 2n there exist at least 5 primes,
because (9/8)^5 = 1.802032470703125... < 2,
while (9/8)^6 = 2.027286529541016... > 2.

References:

I. M. Radu, "Mathematical Spectrum", Vol. 27, No. 2, p. 43, 1994/5.
D. W. Sharpe, Letters to the Author, 24 February & 16 March, 1995.