ON THE SUMATORY FUNCTION ASSOCIATED TO THE SMARANDACHE FUNCTION

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It is said that for every numerical function f it can be attashed the sumatory function :

$$F(n) = \sum_{d \mid n} f(d) \tag{1}$$

The function f is expressed as :

$$f(n) = \sum_{uv=n} \mu(u) . F_f(v) \tag{2}$$

Where μ is the Möbius function ($\mu(1)=1$, $\mu(n)=0$ if n is divisible by the square of a prime number , $\mu(n)=(-1)^k$ if n is the product of k different prime numbers)

If f is the Smarandache function and $n = p^\alpha$ then :

$$F_S(p^\alpha) = \sum_{j=1}^{\alpha} S(p^j)$$

In [2] it is proved that

$$S(p^j) = (p - 1) \cdot j + \alpha_{[p]}(j) \tag{3}$$

Where $\alpha_{[p]}(j)$ is the sum of the digits of the integer j,written in the generalised scale

$$[p] = a_1(p) \ , \ a_2(p) \ , \ \ldots \ , \ a_k(p) \ , \ \ldots$$

with $a_i(p) = (p^n - 1)/(p - 1)$

So

$$F_s(p^\alpha) = \sum_{j=1}^{\alpha} S(p^j) = (p - 1) \frac{\alpha(\alpha + 1)}{2} + \sum_{j=1}^{\alpha} \alpha_{[p]}(j) \tag{4}$$

Using the expresion of α given by (3) it results

$$(\alpha + 1)(S(p^\alpha) - \alpha_{[p]}(\alpha)) = 2(F_S(p^\alpha) - \sum_{j=1}^{\alpha} \alpha_{[p]}(j))$$

In the following we give an algorithm to calculate the sum in the right hand of (4). For this, let $\alpha_{(p)} = \overline{k_s . k_{s-1} . \ ... \ . k_1}$ the expresion of α in the scale [p] and $j_{(p)} = \overline{k_{s_j} . k_{s_j - 1} . \ ... \ . k_1}$. We shall say that k_{i_j} are the digits of order i, for j = 1,2, ... ,α.
To calculate the sum of all the digits of order i, let $\nu_i = \alpha - a_i(p) + 1$.
Now we consider two cases :

(i) if $k_i \neq 0$, let :

$z_i(\alpha) = (\overline{k_s k_{s-1} \ldots k_{i+1}})_{u=\alpha(p)}$, the equality $u = a_i(p)$ denothig that for the number writen between parantheses, the classe of units is $a_i(p)$.

Then $z_i(\alpha)$ is the number of all zeros of order i for the integers $j \leq \alpha$ and $\alpha_i = \nu_i(\alpha) - z_i(\alpha)$ is the number of the non-null digits.

(ii) if $k_i = 0$, let β the greatest number, less then α, having a non-null digit of order i. Then β is of the form :

$\beta_{(p)} = \overline{k_s k_{s-1} \ldots k_{i+2}(k_{i+1} - 1)p00\ldots 0}$ and of course $s_i(\alpha) = s_i(\beta)$. It results that there exist $\alpha_i(\beta)$ non-null digits of order i.
Let A_i, B_i, r_i, ρ_i given by equalities :

$\alpha_i = A_i((p-1)a_i(p) + 1) + r_i = A_i(a_{i+1}(p) - a_i(p)) + r_i$

$r_i = B_i a_i(p) + \rho_i$

Then

$$s_i(\alpha) = A_i a_i(p) \frac{p(p-1)}{2} + A_i p + a_i(p) \frac{B_i(B_i + 1)}{2} + \rho_i(B_i + 1)$$

and

$$\sum_{j=1}^{\alpha} \sigma_{(p)}(j) = \sum_{i=1}^{\alpha} s_i(\alpha) = \frac{p(p-1)}{2} \sum_{i\geq 1} A_i a_i(p) + p \sum_{i\geq 1} A_i +$$

$$\frac{1}{2} \sum_{i\geq 1} a_i(p) B_i(B_i + 1) + \sum_{i\geq 1} \rho_i(B_i + 1)$$

For example if α = 149 and p = 3 it results :

[3] 1 , 4 , 13 , 40 , 121 , ...

$\alpha_{(3)} = 10202$, $\nu_1(\alpha) = (1020)_{u=\alpha_1(3)} = 48$ $\alpha_1 = \nu_1(\alpha) - z_1(\alpha) = 101$

For $\beta_{(3)} = 10130 = 146$ it results $\nu_2(\beta) = 143$, $z_2(\beta) =$

$(101)_{u=\alpha_2(3)} = u_3 + u = 3u_2 + 1 + u = 3(3u + 1) + 1 + u = 44$,

$\alpha_2 = 99$, $\nu_3(\alpha) = 137$, $z_3(\alpha) = (10)_{u=\alpha_3(3)} = 40$. $\alpha_3 = 97$.

For $\beta_{(3)} = 3000 = 120$ it results $\nu_4(\beta) = 81$, $z_4(\beta) = 0$, $\alpha_4 = 108$.

$\nu_5(\alpha) = 29$, $z_5(\alpha) = 0$, $\alpha_5 = 29$, and

$$A_1 = \left[\frac{\alpha_1}{\alpha_2 - \alpha_1}\right] = 33 \;,\; r_1 = 2 \;,\; B_1 = \left[\frac{z}{\alpha_1}\right] \;,\; \rho_1 = 0 \;,\; s_1 = 201$$

Analogously $s_2 = 165$, $s_3 = 145$, $s_4 = 123$ and $s_5 = 129$, so

$\sum_{i=1}^{140} \alpha_{(3)}(i) = 633$, $F_s(3^{140}) = 22983$.

Now let us consider $n = p_1 \cdot p_2 \cdot \ \ldots \ \cdot p_k$, with $p_1 < p_2 < \ldots < p_k$ prime numbers. Of course, $S(n) = p_k$ and from

$F_s(1) = S(1) = 0$

$F_s(p_1) = S(1) + S(p_1) = p_1$

$F_s(p_1 \cdot p_2) = p_1 + 2p_2 = F(p_1) + 2p_2$

$F_s(p_1 \cdot p_2 \cdot p_3) = p_1 + 2p_2 + 2^2 p_3 = F(p_1 \cdot p_2) + 2^2 p_3$

it results :

$$F_s(p_1 \cdot p_2 \cdot \ \ldots \ \cdot p_k) = F(p_1 \cdot p_2 \cdot \ \ldots \ p_{k-1}) + 2^{k-1} p^k$$

That is :

$$F(p_1 \cdot p_2 \cdot \ \ldots \ \cdot p_k) = \sum_{i=1}^{k} 2^{i-1} p_i$$

The equality (2) becomes :

$$p_k = S(n) = \sum_{uv=n} \mu(n) \; F_s(v) =$$

$$= F(n) - \sum_i F\left(\frac{n}{p_i}\right) + \sum_{i,j} F\left(\frac{n}{p_i p_j}\right) + \ldots + \sum_{i=1}^{k} F(p_i)$$

and became $F(p_i) = p_i$, it results :

$$F\left(\frac{n}{p_i}\right) = F(p_1 \cdot p_2 \cdot \ \dots \ \cdot p_{i-1} \cdot p_{i+1} \cdot \ \dots \ \cdot p_k) = \sum_{j=1}^{i-1} 2^{j-1} p_j + \sum_{j=i+1}^{k} 2^{j-1} p_j =$$

$$= F(p_1 \cdot p_2 \cdot \ \dots \ \cdot p_{i-1}) + 2^{i-1} F(p_{i+1} \cdot p_{i+2} \cdot \ \dots \ \cdot p_k).$$

Analogously,

$$F\left(\frac{n}{p_i p_j}\right) = F(p_1 \cdot p_2 \cdot \ \dots \ \cdot p_{i-1}^{j}) + 2^{i-1} F(p_{i+1} \cdot p_{i+2} \cdot \ \dots \ \cdot p_{j-1}) +$$
$$+ 2^{j-1} F(p_{j+1} \ \dots \ \cdot p_k)$$

Finaly, we point out as an open problem that, by the Shapiro's theorem, if it exist a numerical function $g : \mathbb{N} \longrightarrow \mathbb{R}$ such that

$$g(n) = \sum_{d|n} P(n) \ S\left(\frac{n}{d}\right)$$

were P is a totaly multiplicative function and $P(1) = 1$, then

$$S(n) = \sum_{d|n} \mu(d) \ P(d) \ g\left(\frac{n}{d}\right)$$

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