THE PROBLEM OF LIPSCHITZ CONDITION

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In our paper we prove that the Smarandache function S does not verify the Lipschitz condition, giving an answer to a problem proposed in [2] and we investigate also the possibility that some other functions, which involve the function S, verify or not verify the Lipschitz condition.

Proposition 1 *The function $\{n \to S(n)\}$ does not verify the Lipschitz condition, where $S(n)$ is the smallest integer m such that $m!$ is divisible by n. (S is called the Smarandache function.)*

Proof. A function $f : M \subseteq R \to R$ is Lipschitz iff the following condition holds:

$$(\exists)K > 0, (\forall)x, y \in M \Rightarrow | f(x) - f(y) | \leq K \mid x - y \mid$$

(K is called a Lipschitz constant).

We have to prove that for every real $K > 0$ there existe $x, y \in N^*$ such that $| f(x) - f(y) | > K \mid x - y \mid$.

Let $K > 0$ be a given real number. Let $x = p > 3K + 2$ be a prime number and consider $y = p + 1$ which is a composite number, beeig even. Since $x = p$ is a prime number we have $S(p) = p$. Using [1] we have $\max\limits_{n \in N^*, n \neq 4} \{S(n)/n\} = 2/3$, then $\frac{S(y)}{y} = \frac{S(p+1)}{p+1} \leq \frac{2}{3}$ which implies that $S(p+1) \leq \frac{2}{3}(p+1) < p = S(p)$. We have

$$|S(p) - S(p+1)| = p - S(p+1) \geq p - \frac{2}{3}(p+1) > \frac{3K + 2 - 2}{3} = K$$

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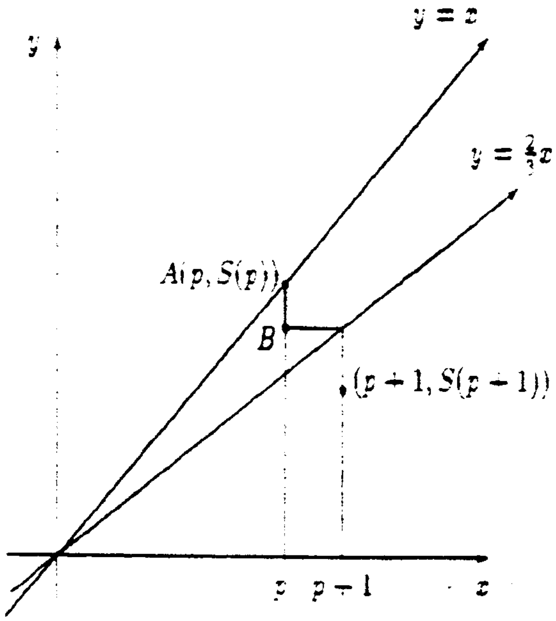
Remark 1. The ideea of the proof is based on the following observations:

If p is a prime number, then $S(p) = p$, thus the point $(p, S(p))$ belongs to the line of equation $y = x$;

If q is a composite integer, $q \neq 4$, then $\frac{S(q)}{q} \leq \frac{2}{3}$ which means that the point $(q, S(q))$ is under the graphic of the line of equation $y = \frac{2}{3}x$ and above the axe $\overrightarrow{Ox}$.



Thus, for every consecutive integer numbers x, y where $x = p$ is a prime number and $y = p-1$, the leught AB can be made as great as we need, for x, y sufficiently great.

Remark 2. In fact we have proved that the function $f : N^* \to N$ defined by $f(n) = |S(n) - S(n-1)|$ is unbounded, which imply that the Smarandache's function is not Lipschitz.

In the sequel we study the Lipschitz condition for other functions which involve the Smarandache's function.

Proposition 2 *The function $S_1 : N \setminus \{0,1\} \to N$, $S_1(n) = \frac{1}{S(n)}$ verify the Lipschitz condition.*

Proof. For every $x \geq 2$ we have $S(x) \geq 2$, therefore $0 < \frac{1}{S(x)} \leq \frac{1}{2}$. If we take $x \neq y$ in $N \setminus \{0,1\}$, we have

$$\left|\frac{1}{S(x)} - \frac{1}{S(y)}\right| \leq \frac{1}{2} \leq \frac{1}{2}|x - y| \ .$$

For $x = y$ we have an equality in the relation above, therefore S_1 is a function which verify the Lipschitz condition with $K = \frac{1}{2}$ and more, it is a contractant function.

Remark 3. In [2] it is proved that $\sum_{n \geq 1} \frac{1}{S(n)}$ is divergent .

Proposition 3 *The function $S_2 : N^* \to N$, $S_2(n) = \frac{S(n)}{n}$ verify the Lipschitz condition.*

Proof. For every $x, y \in N$, $1 < x < y$ we have $x = n$ and $y = n + m$ where $m \in N^*$. In [2] is proved that

$$\frac{1}{(n-1)!} \leq \frac{S(n)}{n} \leq 1, \ (\forall) n \in N \setminus \{0, 1\} .$$

Using this we have

$$\left| \frac{S(x)}{x} - \frac{S(y)}{y} \right| = \left| \frac{S(n)}{n} - \frac{S(n+m)}{n+m} \right| \leq 1 - \frac{1}{(n+m-1)!} < 1 \leq |x - y|$$

therefore

$$\left| \frac{S(x)}{x} - \frac{S(y)}{y} \right| \leq |x - y|$$

for x and y as above. For $x = y$ we have an equality in the relation above. It follows that S_2 is verify the Lipschitz condition with $K = 1$.

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Remark 4. Using the proof of *Proposition 5* proved below, it can be shown that the Lipschitz constant $K = 1$ is the best possible. Indeed, take $x = n = p - 1$, $m = 1$ and therefore $y = p$ (with the notations from the proof of *Proposition 3*), with p a primenumber. From the proof of *Proposition 5*, there is a subsequence of prime numbers $\{p_{n_k}\}_{k \geq 1}$ such that $\frac{S(p_{n_k} - 1)}{p_{n_k} - 1} \xrightarrow{k \to \infty} 0$. For $k \geq 1$ we have, for a Lipschitz constant K of S_2 :

$$K \geq \left| \frac{S(p_{n_k})}{p_{n_k}} - \frac{S(p_{n_k} - 1)}{p_{n_k} - 1} \right| = 1 - \frac{S(p_{n_k} - 1)}{p_{n_k} - 1} \xrightarrow{k \to \infty} 1$$

Thus, $K \geq 1$

Proposition 4 *The function $S_3 : N \setminus \{0,1\} \to N$, $S_3(n) = \frac{n}{S(n)}$ does not verify the Lipschitz condition.*

Proof. (Compare with the proof of Proposition 1.)

We have to prove that for every real $K > 0$ there exists $x, y \in N^*$ such that $|S_3(x) - S_3(y)| > K|x - y|$.

Let $K > 0$ be a given real number, $x = p$ be a prime number and $y = x - 1$. Using the Proposition 5 proved below, which asserts that the sequence $\left\{\frac{p_n - 1}{S(p_n - 1)}\right\}_{n\geq 2}$ is unbunded (where $\{p\}_{n\geq 1}$ is the prime numbers sequence), we have, for a prime number p such that $\frac{p-1}{S(p-1)} > K + 1$:

$$\left|\frac{x}{S(x)} - \frac{y}{S(y)}\right| = \left|\frac{p}{S(p)} - \frac{p-1}{S(p-1)}\right| = \frac{p-1}{S(p-1)} - 1 > K + 1 - 1 = K = K|x - y|$$

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Proposition 5 *If $\{p_n\}_{n\geq 1}$ is the prime numbers sequence, then the sequence $\left\{\frac{p_n-1}{S(p_n-1)}\right\}_{n\geq 2}$ is unbounded.*

Proof. Denote $q_n = p_n - 1$ and let r_n be the number of the distinct prime numbers which appear in the prime factor decomposition of q_n, for $n \geq 2$. We show below that $\{r_n\}_{n\geq 2}$ is an unbounded sequence.

For a fixed $k \in N^*$, consider $\pi_k \overset{def}{=} p_1 \cdots p_k$ and the arithmetic progression $\{1 + \pi_k \cdot m\}_{m\geq 1}$. From the Dirichlet Theorem [3, pg.194], it follows that this sequence contains a subsequence $\{1 + \pi_k \cdot m_l\}_{l\geq 1}$ of prime numbers: $p_{n_l} = 1 + \pi_k \cdot m_l$, therefore $\pi_k \cdot m_l = p_{n_l} - 1 = q_{n_l}$ which implies that $r_{n_l} \geq k$. It shows that the sequence $\{r_n\}_{n\geq 2}$ is an unbounded sequence.

If $q_n = \prod_{i=1}^{r_n} p_{\beta_i}^{\alpha_i}$ then it is known (see [4]) that:

$$S(q_n) = \max_{i=\overline{1,r_n}} \left\{S\left(p_{\beta_i}^{\alpha_i}\right)\right\} = S\left(p_{\beta_j}^{\alpha_j}\right) \leq \alpha_j p_{\beta_j}$$

thus

$$\frac{q_n}{S(q_n)} = \frac{\prod_{i=1}^{r_n} p_{\beta_i}^{\alpha_i}}{S\left(p_{\beta_j}^{\alpha_j}\right)} \geq \left(\prod_{i=1, i\neq j}^{r_n} p_{\beta_i}^{\alpha_i}\right) \frac{p_{\beta_j}^{\alpha_j - 1}}{\alpha_j}. \qquad (1)$$

We have:

$$u_j = \frac{p_{\beta_j}^{\alpha_j - 1}}{\alpha_j} \geq 2 \tag{2}$$

Indeed, if $\alpha_j = 1$, then $u_j = 1$. If $\alpha_j > 1$, then

$$u_j \geq \frac{(p_j - 1)(\alpha_j - 1)}{\alpha_j} \geq \frac{p_j - 1}{2} \geq \frac{1}{2}.$$

But $v_n = \prod_{i=1, i \neq j}^{r_n} p_{\beta_i}^{\alpha_i}$ has $r_n - 1$ prime factors and $\{r_n\}_{n \geq 2}$ is unbounded, then it follows that $\{v_n\}_{n \geq 2}$ is unbounded. Using this, (1) and (2), it follows that the sequence $\left\{\frac{q_n}{S(q_n)}\right\}_{n \geq 2}$ is unbounded.

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Remark 5. Using the same ideea, the *Proposition 5* is true in a more general form:

For $a \in Z$, the sequence $\left\{\frac{p_n + a}{S(p_n + a)}\right\}_{p_n + a \geq 2}$ is unbounded, where $\{p_n\}_{n \geq 1}$ is the prime numbers sequence.

References

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[3] C Popovici *Teoria numerelor*, Editura didactică şi pedagogică, Bucureşti, 1973.

[4] Pål Gronås *A proof of the non-existence of SAMMA*, Smarandache Function Journal, vol.4-5, No.1, September 1994, pg.22-23.