

# ECONOMETRIC ANALYSIS ON EFFICIENCY OF ESTIMATOR

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## ABSTRACT

This paper investigates the efficiency of an alternative to ratio estimator under the super population model with uncorrelated errors and a gamma-distributed auxiliary variable. Comparisons with usual ratio and unbiased estimators are also made.

Key words: Bias, Mean Square Error, Ratio Estimator Super Population.

## 1. INTRODUCTION

It is well known that the ratio method of estimation occupies an important place in sample surveys. When the study variate  $y$  and the auxiliary variate  $x$  is positively (high) correlated, the ratio method of estimation is quite effective in estimating the population mean of the study variate  $y$  utilizing the information on auxiliary variate  $x$ .

Consider a finite population with  $N$  units and let  $x_i$  and  $y_i$  denote the values for two positively correlated variates  $x$  and  $y$  respectively for the  $i$ th unit in this population,  $i=1,2,\dots,N$ . Assume that the population mean  $\bar{X}$  of  $x$  is known. Let  $\bar{x}$  and  $\bar{y}$  be the sample means of  $x$  and  $y$  respectively based on a simple random sample of size  $n$  ( $n < N$ ) units drawn without replacement scheme. Then the classical ratio estimator for  $\bar{Y}$  is defined by

$$\bar{y}_r = \bar{y}(\bar{X} / \bar{x}) \quad (1.1)$$

The bias and mean square error (MSE) of  $\bar{y}_r$  are, up to second order moments,

$$B(\bar{y}_r) = \lambda(R S_x^2 - S_{yx}) / \bar{X} \quad (1.2)$$

$$M(\bar{y}_r) = \lambda(S_y^2 + R^2 S_x^2 - 2R S_{yx}), \quad (1.3)$$

where  $\lambda = (N - n)/(nN)$ ,

$$R = \bar{Y}/\bar{X}, \quad S_y^2 = (N-1)^{-1} \sum_{i=1}^N (y_i - \bar{Y})^2, \quad S_x^2 = (N-1)^{-1} \sum_{i=1}^N (x_i - \bar{X})^2,$$

$$\text{and } S_{yx} = (N-1)^{-1} \sum_{i=1}^N (y_i - \bar{Y})(x_i - \bar{X}).$$

It is clear from (1.3) that  $M(\bar{y}_r)$  will be minimum when

$$R = S_{yx}/S_x^2 = \beta, \quad (1.4)$$

where  $\beta$  is the regression coefficient of  $y$  on  $x$ . Also for  $R = \beta$ ,

the bias of  $\bar{y}_r$  in (1.2) is zero. That is,  $\bar{y}_r$  is almost unbiased for  $\bar{Y}$ .

Let  $E(\bar{y}|\bar{x}) = \alpha + \beta \bar{x}$  be the line of regression of  $\bar{y}$  on  $\bar{x}$ , where  $E$  denotes averaging over all possible sample design simple random sampling without replacement (SRSWOR). Then  $\beta = S_{yx}/S_x^2$  and  $\bar{Y} = \alpha + \beta \bar{X}$  so that, in general,

$$R = (\alpha/\bar{X}) + \beta \quad (1.5)$$

It is obvious from (1.4) and (1.5) that any transformation that brings the ratio of population means closer to  $\beta$  will be helpful in reducing the mean square error (MSE) as well as the bias of the ratio estimator  $\bar{y}_r$ . This led Srivenkataramana and Tracy (1986) to suggest an alternative to ratio estimator  $\bar{y}_r$  as

$$\bar{y}_a = \bar{z}(\bar{X}/\bar{x}) + A = \bar{y}_r - A\{(\bar{X}/\bar{x}) - 1\} \quad (1.6)$$

which is based on the transformation

$$\bar{z} = \bar{y} - A, \quad (1.7)$$

where  $E(\bar{z}) = \bar{Z} (= \bar{Y} - A)$  and  $A$  is a suitably chosen scalar.

In this paper exact expressions of bias and MSE of  $\bar{y}_a$  are worked out under a super population model and compared with the usual ratio estimator.

## 2. THE SUPER POPULATION MODEL

Following Durbin (1959) and Rao (1968) it is assumed that the finite population under consideration is itself a random sample from a super population and the relation between  $x$  and  $y$  is of the form:

$$y_i = \alpha + \beta x_i + u_i ; \quad (i = 1, 2, \dots, N)$$

where  $\alpha$  and  $\beta$  are unknown real constants;  $u_i$ 's are uncorrelated random errors with conditional (given  $x_i$ ) expectations

$$E(u_i | x_i) = 0$$

$$E(u_i^2 | x_i) = \delta x_i^g$$

( $i=1, 2, \dots, N$ ),  $0 < \delta < \infty$ ,  $0 \leq g \leq 2$  and  $x_i$  are independently identically

distributed (i.i.d.) with a common gamma density

$$G(\theta) = e^{-x} x^{\theta-1} / \Gamma \theta, \quad x > 0, \quad 2 < \theta < \infty \quad (2.1)$$

We will write  $E_x$  to denote expectation operator with respect to the common distribution of  $x_i$  ( $i=1, 2, 3, \dots, N$ ) and  $E_x E_c$ , as the over all expectation operator for the model. We denote a design by  $p$  and the design expectation  $E_p$ , for instance, see Chaudhuri and Adhikary (1983,89) and Shah and Gupta (1987). Let 's' denote a simple random sample of  $N$  distinct labels chosen without replacement out of  $i=1, 2, 3, \dots, N$ . Then

$$X(=N \bar{X}) = \sum_{i \in s} x_i + \sum_{i \notin s} x_i$$

Following Rao and Webster (1966) we will utilize the distributional properties of  $x_j / x_i$ ,  $\sum_{i \in s} x_i$ ,  $\sum_{i \notin s} x_i$ ,  $\sum_{i \in s} x_i / \sum_{i \notin s} x_i$  in our subsequent derivations.

### 3. THE BIAS AND MEAN SQUARE ERROR

The estimator  $\bar{y}_a$  in (1.6) can be written as

$$\bar{y}_a = \left[ (1/n) \left( \sum_{i \in S} y_i \right) \frac{\binom{n \sum_{i=1}^N x_i}{i=1}}{\binom{N \sum_{i \in S} x_i}{i \in S}} - A \left\{ \frac{\binom{n \sum_{i=1}^N x_i}{i=1}}{\binom{N \sum_{i \in S} x_i}{i \in S}} - 1 \right\} \right] \quad (3.1)$$

based on a simple random sample of n distinct labels chosen without replacement out of  $i = 1, 2, \dots, N$ .

The bias

$$B = E_p (\bar{y}_a - \bar{Y}) \quad (3.2)$$

of  $\bar{y}_a$  has model expectation  $E_m(B)$  which works out as follows:

$$\begin{aligned} E_m (B (\bar{y}_a)) &= E_p E_x E_c \left[ \left\{ \alpha + \beta (1/n) \left( \sum_{i \in S} x_i \right) + \bar{u} \right\} \frac{n \sum_{i=1}^N x_i}{n \sum_{i \in S} x_i} \right. \\ &\quad - A \left\{ \frac{n \binom{\sum_{i=1}^N x_i}{i=1} - 1}{N \binom{\sum_{i \in S} x_i}{i \in S}} \right\} - \\ &\quad \left. - E_x E_c (\alpha + \beta \bar{x} + \bar{U}) \right] \\ &= E_p E_x E_c \\ &\quad \left[ \alpha \left( n \sum_{i=1}^N x_i / N \sum_{i \in S} x_i \right) + \beta (1/N) \left( \sum_{i=1}^N x_i \right) + \left( \sum_{i \in S} u_i \right) \left( \sum_{i=1}^N x_i / N \sum_{i \in S} x_i \right) - A \left\{ \left( \binom{n \sum_{i=1}^N x_i}{i=1} \right) / \left( \binom{N \sum_{i \in S} x_i}{i \in S} \right) - 1 \right\} \right] \\ &\quad - E_x E_c (\alpha + \beta \bar{X}) \\ &= E_p E_x \left[ \alpha \left( n \sum_{i=1}^N x_i \right) / \left( N \sum_{i \in S} x_i \right) + \beta \bar{X} - A \left\{ \left( n \sum_{i=1}^N x_i \right) / \left( N \sum_{i \in S} x_i \right) - 1 \right\} \right] - \alpha - \beta E_x (\bar{X}) \\ &= E_x \left[ \alpha (n/N) \left( 1 + \sum_{i \notin S} x_i / \sum_{i \in S} x_i \right) - A \left\{ (n/N) \left( 1 + \sum_{i \notin S} x_i / \sum_{i \in S} x_i \right) - 1 \right\} \right] - \alpha \\ &= \alpha (n/N) \{ 1 + (N-n)\theta / (n\theta - 1) \} \\ &\quad - A \{ (n/N) (1 + (N-n)\theta / (n\theta - 1)) - 1 \} - \alpha \\ &= \alpha \{ (n/N - 1) + \{ n(N-n)\theta / N(n\theta - 1) \} \} \end{aligned}$$

$$\begin{aligned}
& -A \left[ -(N-n)/N + \{(N-n)n\theta / N(n\theta - 1)\} \right] \\
= & (N-n) (\alpha - A) / N(n\theta - 1) \tag{3.3}
\end{aligned}$$

For SRSWOR sampling scheme , the mean square error

$$M(\bar{y}_a) = E_p (\bar{y}_a - \bar{Y})^2 \tag{3.4}$$

of  $\bar{y}_a$  has the following formula for model expectations

$E_m ( M(\bar{y}_a) ) :$

$$E_m(M(\bar{y}_a)) = [E_m(M(\bar{y}_r)) + (N-n)(Nn\theta + 2N - 2n)(A^2 - 2A\alpha) / N^2(n\theta - 1)(n\theta - 2)] \tag{3.5}$$

where

$$M(\bar{y}_r) = E_p (\bar{y}_r - \bar{Y})^2 \tag{3.6}$$

is the MSE of  $\bar{y}_r$  under SRSWOR scheme has the model expectation

$$\begin{aligned}
E_m(M(\bar{y}_r)) = & \left\{ (N-n) / N^2 \right\} \\
& \left[ \left\{ \frac{(Nn\theta + 2N - 2n)\alpha^2}{(n\theta - 1)(n\theta - 2)} \right\} + \frac{\delta \left\{ (n\theta + g - 1)(n\theta + g - 2) + n\theta(N\theta - n\theta + 1) \right\}}{(n\theta + g - 1)(n\theta + g - 2)} \frac{\Gamma(\theta + g)}{\Gamma\theta} \right] \tag{3.7}
\end{aligned}$$

[See, Rao(1968, p.439)]

Further, we note that for SRSWOR sampling scheme, the bias

$$B(\bar{y}_r) = E_p (\bar{y}_r - \bar{Y}) \tag{3.8}$$

of usual ratio estimator has the model expectation

$$E_m(B(\bar{y}_r)) = (N-n)\alpha / (n\theta - 1) \tag{3.9}$$

We note from (3.3) and (3.9) that

$$|E_m(B(\bar{y}_a))| < |E_m(B(\bar{y}_r))|$$

if

$$\begin{aligned}
& |(\alpha - A)| < |\alpha| \\
\text{or if} & (\alpha - A)^2 < \alpha^2 \\
\text{or if} & o < A < 2\alpha \tag{3.10}
\end{aligned}$$

Further we have from (3.5) that

$$E_m(M(\bar{y}_a)) - E_m(M(\bar{y}_r)) < o$$

if

$$(A^2 - 2A\alpha) < o$$

or if

$$o < A < 2\alpha \tag{3.11}$$

which is the same as in (3.10).

Thus we state the following theorem:

**Theorem 3.1 :** The estimator  $\bar{y}_a$  is less biased as well as more efficient than usual ratio estimator  $\bar{y}_r$  if

$$o < A < 2\alpha \quad (\alpha \neq o)$$

i . e . when A lies between  $o$  and  $2\alpha$  .

Therefore , when intercept term  $\alpha (\neq o)$  in the model (2.1) is sizable , there will be sufficient flexibility in picking A.

It is to be noted that for  $\alpha = o, \bar{y}_r$  is unbiased and efficient than  $\bar{y}_a$  .

The minimization of (3.5) with respect to A leads to

$$A = \alpha = A_{\text{opt}} \text{ (say)} \tag{3.12}$$

Substitution of (3.12) in (3.5) yields the minimum value of

$$\begin{aligned}
& E_m(M(\bar{y}_a)) \text{ as} \\
\text{min. } E_m(M(\bar{y}_a)) &= \frac{(N-1) \delta [(n\theta + g - 1)(n\theta + g - 2) + n\theta(N\theta - n\theta + 1)] \Gamma(\theta + g)}{N^2 (n\theta + g - 1)(n\theta + g - 2) \Gamma\theta} \tag{3.13}
\end{aligned}$$

which equals to  $E_m(M(\bar{y}_r))$  when  $\alpha = o$ .

It is interesting to note that when  $A = \alpha, \bar{y}_a$  is unbiased and attained its minimum average MSE in model (2.1).

In practice the value of  $\alpha$  will have to be assessed, at the estimation stage, to be used as A. To assess  $\alpha$ , we may use scatter diagram of y versus x for data from a pilot study, or a part of the data from the actual study and judge the y-intercept of the best fitting line.

From (3.7) and (3.13) we have

$$E_m(M(\bar{y}_r)) - \min E_m(M(\bar{y}_a)) = \frac{\{(N-n)(Nn\theta + 2N - 2n)\alpha^2\}}{\{N^2(n\theta - 1)(n\theta - 2)\}} > 0 \quad (3.14)$$

which shows that  $\bar{y}_a$  is more efficient than ratio estimator when  $A = \alpha$  is known exactly. For  $\alpha = 0$

$$\min E_m(M(\bar{y}_a)) = E_m(M(\bar{y}_r)) \quad (3.15)$$

For SRSWOR, the variance

$$V(\bar{y}) = E_p(\bar{y} - \bar{Y})^2 \quad (3.16)$$

of usual unbiased estimator has the model expectation:

$$E_m(V(\bar{y})) = (N-n) \left[ \beta^2\theta + \frac{\delta\Gamma(\theta+g)}{\Gamma\theta} \right] / nN \quad (3.17)$$

The expressions of  $E_m(M(\bar{y}_a))$  and  $E_m(V(\bar{y}))$  are not easy task to compare algebraically. Therefore in order to facilitate the comparison, denoting

$$E_1 = 100E_m(V(\bar{y})) / E_m(M(\bar{y}_a)) \quad \text{and} \quad E_2 = 100E_m(V(\bar{y}_r)) / E_m(M(\bar{y}_a)),$$

we present below in tables 1,2,3, the values of the relative efficiencies of

$\bar{y}_a$  with respect to  $\bar{y}$  and  $\bar{y}_r$  for a few combination of the parametric values under the model (2.1). Values are given for  $N = 60$ ,  $\delta = 2.0, \theta = 8, \alpha = 0.5, 1.0, 1.5$ ,  $\beta = 0.5, 1.0, 1.5$  and  $g = 0.0, 0.5, 1.0, 1.5, 2.0$ . The ranges of A, for  $\bar{y}_a$  to be better than  $\bar{y}_r$  for given  $\alpha = 0.5, 1.0, 1.5$  are respectively (0,1), (0,2), (0,3). This clearly indicates that as the size of  $\alpha$  increases the range of A for  $\bar{y}_a$  to be better than  $\bar{y}_r$  increases i.e. flexibility of choosing A increases.

We have made the following observations from the tables 1,2 and 3 :

- (i) As g increases both  $E_1$  and  $E_2$  decrease. When n increases  $E_1$  increases while  $E_2$  decreases.

- (ii) As  $\alpha$  increases ( i.e. if the intercept term  $\alpha$  departs from origin in positive direction) relative efficiency of  $\bar{y}_a$  with respect to  $\bar{y}$  decreases while  $E_2$  increases.
- (iii) As  $\beta$  increases  $E_1$  increases for fixed  $g$  while  $E_2$  is unaffected.
- (iv) The maximum gain in efficiency is observed over  $\bar{y}$  as well as over  $\bar{y}_r$  if  $A$  coincide with the value of  $\alpha$ . Finally, the estimator  $\bar{y}_a$  is to be preferred when the intercept term  $\alpha$  departs substantially from origin.

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Table 1: Relative efficiencies of  $\bar{y}_a$  with respect to  $\bar{y}$  and  $\bar{y}_\Gamma$

$\alpha = 0.5$							
g	$\beta$	n = 10					
		E <sub>1</sub>			E <sub>2</sub>		
		A			A		
		0.30	0.60	0.90	0.30	0.60	0.90
0.0	0.5	192.86	193.23	191.40	101.34	101.54	100.57
	1.0	482.16	483.16	478.09	101.34	101.54	100.57
	1.5	964.32	966.17	956.98	101.34	101.54	100.57
0.5	0.5	132.67	132.77	132.30	100.49	100.56	100.21
	1.0	237.82	237.99	237.16	100.49	100.56	100.21
	1.5	413.08	413.36	411.93	100.49	100.56	100.21
1.0	0.5	111.06	111.08	110.95	10.17	100.19	100.07
	1.0	148.08	148.11	147.93	10.17	100.19	100.07
	1.5	209.78	209.83	209.57	10.17	100.19	100.07
1.5	0.5	103.99	104.00	103.96	100.06	100.07	100.03
	1.0	116.64	116.65	116.60	100.06	100.07	100.03
	1.5	137.71	137.72	137.66	100.06	100.07	100.03
2.0	0.5	102.23	102.23	102.22	100.02	100.02	100.01
	1.0	106.43	106.43	106.42	100.02	100.02	100.01
	1.5	113.43	113.43	113.42	100.02	100.02	100.01

$\alpha = 0.5$							
g	$\beta$	n = 20					
		E <sub>1</sub>			E <sub>2</sub>		
		A			A		
		0.30	0.60	0.90	0.30	0.60	0.90
0.0	0.5	196.58	196.96	195.11	103.33	101.52	100.56
	1.0	491.46	492.39	487.77	103.33	101.52	100.56
	1.5	982.92	984.39	975.53	103.33	101.52	100.56
0.5	0.5	134.37	134.46	134.46	100.48	100.55	100.20
	1.0	240.86	241.02	240.02	100.48	100.55	100.20
	1.5	418.35	418.63	417.20	100.48	100.55	100.20
1.0	0.5	111.76	111.79	111.65	100.17	100.19	100.07
	1.0	149.01	149.05	148.87	100.17	100.19	100.07
	1.5	211.10	211.16	210.90	100.17	100.19	100.07
1.5	0.5	104.00	104.00	103.96	100.06	100.07	100.02
	1.0	116.64	116.65	116.60	100.06	100.07	100.02
	1.5	137.71	137.73	137.67	100.06	100.07	100.02
2.0	0.5	101.60	101.60	101.58	100.02	100.02	100.01
	1.0	105.77	105.77	105.76	100.02	100.02	100.01
	1.5	112.73	112.73	112.73	100.02	100.02	100.01

Table 2: Relative efficiencies of  $\bar{y}_a$  with respect to  $\bar{y}$  and  $\bar{y}_r$ .

$\alpha = 1.0$									
g	$\beta$	n = 10							
		E <sub>1</sub> A				E <sub>2</sub> A			
		0.50	1.0	1.50	1.90	0.50	1.0	1.50	1.90
0.0	0.5	190.31	193.36	190.31	183.82	104.73	106.41	104.73	101.16
	1.0	475.78	483.40	475.78	459.55	104.73	106.41	104.73	101.16
	1.5	951.55	966.79	951.55	919.10	104.73	106.41	104.73	101.16
0.5	0.5	132.03	132.80	132.03	130.34	101.73	102.32	101.73	100.43
	1.0	236.67	238.05	236.67	233.65	101.73	102.32	101.73	100.43
	1.5	411.07	413.46	411.07	405.82	101.73	102.32	101.73	100.43
1.0	0.5	110.87	111.09	110.87	110.36	100.61	100.82	100.61	100.15
	1.0	147.82	148.12	147.82	147.15	100.61	100.82	100.61	100.15
	1.5	209.42	209.84	209.42	208.46	100.61	100.82	100.61	100.15
1.5	0.5	103.93	104.00	103.93	103.77	100.21	100.28	100.21	100.05
	1.0	116.57	116.65	116.57	116.39	100.21	100.28	100.21	100.05
	1.5	137.63	137.73	137.63	137.41	100.21	100.28	100.21	100.05
2.0	0.5	102.21	102.23	102.21	102.15	100.67	100.09	100.07	100.01
	1.0	106.41	106.43	106.41	106.3	100.67	100.09	100.07	100.01
	1.5	113.41	113.43	113.41	113.35	100.67	100.09	100.07	100.01

$\alpha = 1.0$									
g	$\beta$	n = 20							
		E <sub>1</sub> A				E <sub>2</sub> A			
		0.50	1.0	1.50	1.90	0.50	1.0	1.50	1.90
0.0	0.5	194.01	197.08	194.01	187.47	104.67	106.33	104.67	101.14
	1.0	485.03	492.70	485.03	468.68	104.67	106.33	104.67	101.14
	1.5	970.06	985.40	970.06	937.36	104.67	106.33	104.67	101.14
0.5	0.5	133.73	134.49	133.73	132.05	101.70	102.28	101.70	100.08
	1.0	239.71	241.08	239.71	236.71	101.70	102.28	101.70	100.08
	1.5	416.35	418.73	416.35	411.13	101.70	102.28	101.70	100.08
1.0	0.5	111.07	111.08	111.07	111.08	100.60	100.80	100.60	100.15
	1.0	148.77	149.06	148.77	148.11	100.60	100.80	100.60	100.15
	1.5	210.75	211.17	210.75	209.82	100.60	100.80	100.60	100.15
1.5	0.5	103.94	104.01	103.94	103.78	100.20	100.27	100.20	100.05
	1.0	116.57	116.65	116.57	116.40	100.20	100.27	100.20	100.05
	1.5	137.64	137.73	137.64	137.42	100.20	100.27	100.20	100.05
2.0	0.5	101.58	101.60	101.58	101.52	100.07	100.09	100.07	100.01
	1.0	105.75	105.77	105.75	105.70	100.07	100.09	100.07	100.01
	1.5	112.71	112.73	112.71	112.65	100.07	100.09	100.07	100.01

Table 3: Relative efficiencies of  $\bar{y}_a$  with respect to  $\bar{y}$  and  $\bar{y}_r$ .

		$\alpha = 1.5$									
g	$\beta$	n = 10									
		E <sub>1</sub> A					E <sub>2</sub> A				
		0.60	1.20	1.80	2.40	2.90	0.60	1.20	1.80	2.40	2.90
0.0	0.5	183.82	192.25	192.25	183.82	171.79	108.77	113.76	113.76	108.77	101.65
	1.0	459.55	480.62	480.62	459.55	429.47	108.77	113.76	113.76	108.77	101.65
	1.5	919.10	961.25	961.25	919.10	858.94	108.77	113.76	113.76	108.77	101.65
0.5	0.5	130.34	132.52	132.52	130.34	127.01	103.29	105.01	105.01	103.29	100.64
	1.0	233.64	237.55	237.55	233.65	227.67	103.29	105.01	105.01	103.29	100.64
	1.5	405.82	412.60	412.60	405.82	395.44	103.29	105.01	105.01	103.29	100.64
1.0	0.5	110.36	111.01	111.01	110.36	109.34	101.17	101.77	101.77	101.17	100.23
	1.0	147.15	148.02	148.02	147.15	147.79	101.17	101.77	101.77	101.17	100.23
	1.5	208.46	209.69	209.69	208.46	206.53	101.17	101.77	101.77	101.17	100.23
1.5	0.5	103.77	103.98	103.98	103.77	103.44	100.40	100.60	100.60	100.40	100.08
	1.0	116.39	116.62	116.62	116.39	116.01	100.40	100.60	100.60	100.40	100.08
	1.5	137.41	137.69	137.69	137.41	139.68	100.40	100.60	100.60	100.40	100.08
2.0	0.5	102.15	102.22	102.22	102.15	102.04	100.13	100.20	100.20	100.13	100.03
	1.0	106.35	106.42	106.42	106.35	106.24	100.13	100.20	100.20	100.13	100.03
	1.5	113.35	113.42	113.42	113.35	113.23	100.13	100.20	100.20	100.13	100.03

		$\alpha = 1.5$									
G	$\beta$	n = 20									
		E <sub>1</sub> A					E <sub>2</sub>				
		0.60	1.20	1.80	2.40	2.90	0.60	1.20	1.80	2.40	2.90
0.0	0.5	187.47	196.97	195.97	187.47	175.33	108.67	113.59	113.59	108.67	101.63
	1.0	468.68	489.91	489.91	468.68	438.34	108.67	113.59	113.59	108.67	101.63
	1.5	937.36	979.83	979.83	937.36	876.67	108.67	113.59	113.59	108.67	101.63
0.5	0.5	132.05	134.21	134.21	132.05	128.73	103.23	104.92	104.92	103.23	100.63
	1.0	236.70	240.58	240.58	236.70	230.76	103.23	104.92	104.92	103.23	100.63
	1.5	411.13	417.87	417.87	411.13	400.80	103.23	104.92	104.92	103.23	100.63
1.0	0.5	111.08	111.72	111.72	111.08	110.08	101.14	101.72	101.72	101.14	100.23
	1.0	148.11	148.96	148.96	148.11	146.77	101.14	101.72	101.72	101.14	100.23
	1.5	209.82	211.02	211.02	209.82	207.92	101.14	101.72	101.72	101.14	100.23
1.5	0.5	103.78	103.98	103.98	103.78	103.46	100.39	100.58	100.58	100.39	100.08
	1.0	116.40	116.62	116.62	116.40	116.40	100.39	100.58	100.58	100.39	100.08
	1.5	137.43	137.70	137.70	137.43	137.00	100.39	100.58	100.58	100.39	100.08
2.0	0.5	101.53	101.59	101.59	101.53	101.42	100.13	100.19	100.19	100.03	100.03
	1.0	105.70	105.77	105.77	105.70	105.59	100.13	100.19	100.19	100.03	100.03
	1.5	112.65	112.72	112.72	112.65	112.54	100.13	100.19	100.19	100.03	100.03