AN APPLICATION OF THE GENERALIZATION OF CEVA'S THEOREM

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Theorem: Let us consider a polygon $A_1A_2...A_n$ inserted in a circle. Let s and t be two non zero natural numbers such that $2s+t=n$. By each vertex A_i passes a line d_i which intersects the lines $A_{i+s}A_{i+s+1},....,A_{i+s+t-1}A_{i+s+t}$ at the points $M_{i,i+s},...,M_{i+s+t-1}$ respectively and the circle at the point $M_i^{'}$. Then one has:

$$\prod_{i=1}^{n}\prod_{j=i+s}^{i+s+t-1}\frac{\overline{M_{ij}A_j}}{\overline{M_{ij}A_{j+1}}}=\prod_{i=1}^{n}\frac{\overline{M_i^{'}A_{i+s}}}{\overline{M_i^{'}A_{i+s+t}}}.$$

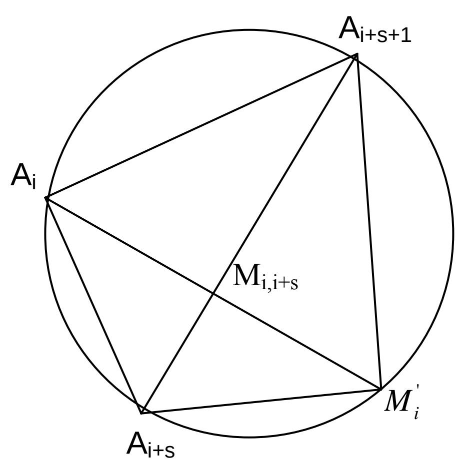
Proof:

Let i be fixed.

1) The case where the point $M_{i,i+s}$ is inside the circle.

There are the triangles $A_iM_{i,i+s}A_{i+s}$ and $M_i^{'}M_{i,i+s}A_{i+s+1}$ similar, since the angles $M_{i,i+s}A_iA_{i+s}$ and $M_{i,i+s}A_{i+s+1}M_i^{'}$ on one side, and $A_iM_{i,i+s}A_{i+s}$ and $A_{i+s+1}M_{i,i+s}M_i^{'}$ are equal. It results from it that:

(1) $$\frac{\overline{M_{i,i+s}A_i}}{\overline{M_{i,i+s}A_{i+s+1}}}=\frac{\overline{A_iA_{i+s}}}{\overline{M_i^{'}A_{i+s+1}}}$$

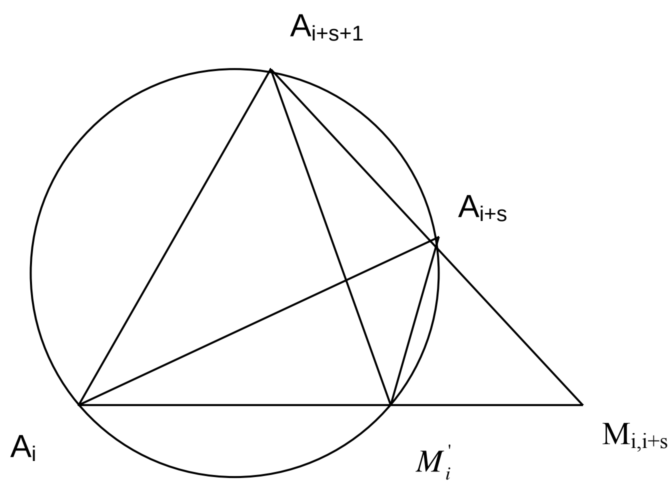


In a similar manner, one shows that the triangles $M_{i,i+s}A_iA_{i+s+1}$ and $M_{i,i+s}A_{i+s}M_i^{'}$ are similar, from which:

(2) $\dfrac{\overline{M_{i,i+s}A_i}}{\overline{M_{i,i+s}A_{i+s}}} = \dfrac{\overline{A_iA_{i+s+1}}}{\overline{M_i^{'}A_{i+s}}}$. Dividing (1) by (2) we obtain:

(3) $$\frac{\overline{M_{i,i+s}A_{i+s}}}{\overline{M_{i,i+s}A_{i+s+1}}} = \frac{\overline{M_i^{'}A_{i+s}}}{\overline{M_i^{'}A_{i+s+1}}} \cdot \frac{\overline{A_iA_{i+s}}}{\overline{A_iA_{i+s+1}}}.$$

2) The case where $M_{i,i+s}$ is exterior to the circle is similar to the first, because the triangles (notations as in 1) are similar also in this new case. There are the same interpretations and the same ratios; therefore one has also the relation (3).



Let us calculate the product:

$$\prod_{j=i+s}^{i+s+t-1} \frac{\overline{M_{ij}A_j}}{\overline{M_{ij}A_{j+1}}} = \prod_{j=i+s}^{i+s+t-1} \left(\frac{\overline{M_i^{'}A_j}}{\overline{M_i^{'}A_{j+1}}} \cdot \frac{\overline{A_iA_j}}{\overline{A_iA_{j+1}}} \right) =$$

$$= \frac{\overline{M_i^{'}A_{i+s}}}{\overline{M_i^{'}A_{i+s+1}}} \cdot \frac{\overline{M_i^{'}A_{i+s+1}}}{\overline{M_i^{'}A_{i+s+2}}} \cdots \frac{\overline{M_i^{'}A_{i+s+t-1}}}{\overline{M_i^{'}A_{i+s+t}}} \cdot$$

$$\cdot \frac{\overline{A_iA_{i+s}}}{\overline{A_iA_{i+s+1}}} \cdot \frac{\overline{A_iA_{i+s+1}}}{\overline{A_iA_{i+s+2}}} \cdots \frac{\overline{A_iA_{i+s+t-1}}}{\overline{A_iA_{i+s+t}}} = \frac{\overline{M_i^{'}A_{i+s}}}{\overline{M_i^{'}A_{i+s+t}}} \cdot \frac{\overline{A_iA_{i+s}}}{\overline{A_iA_{i+s+t}}}$$

Therefore the initial product is equal to:

$$\prod_{i=1}^{n}\left(\frac{\overline{M_i^{'}A_{i+s}}}{\overline{M_i^{'}A_{i+s+t}}}\cdot\frac{\overline{A_iA_{i+s}}}{\overline{A_iA_{i+s+t}}}\right)=\prod_{i=1}^{n}\frac{\overline{M_i^{'}A_{i+s}}}{\overline{M_i^{'}A_{i+s+t}}}$$

since:

$$\prod_{i=1}^{n}\frac{\overline{A_iA_{i+s}}}{\overline{A_iA_{i+s+t}}}=\frac{\overline{A_1A_{1+s}}}{\overline{A_1A_{1+s+t}}}\cdot\frac{\overline{A_2A_{2+s}}}{\overline{A_2A_{2+s+t}}}\cdots\frac{\overline{A_sA_{2s}}}{\overline{A_{s+1}A_1}}\cdot$$

$$\cdot\frac{\overline{A_{s+2}A_{2s+2}}}{\overline{A_{s+2}A_2}}\cdots\frac{\overline{A_{s+t}A_n}}{\overline{A_{s+t}A_t}}\cdot\frac{\overline{A_{s+t+1}A_1}}{\overline{A_{s+t+1}A_{t+1}}}\cdot\frac{\overline{A_{s+t+2}A_2}}{\overline{A_{s+t+2}A_{t+2}}}\cdots\frac{\overline{A_nA_s}}{\overline{A_nA_{s+t}}}=1$$

(by taking into account the fact that $2s+t=n$).

Consequence 1: If there is a polygon $A_1A_2,....,A_{2s-1}$ inscribed in a circle, and from each vertex A_i one traces a line d_i which intersects the opposite side $A_{i+s-1}A_{i+s}$ in M_i and the circle in $M_i^{'}$ then:

$$\prod_{i=1}^{n}\frac{\overline{M_iA_{i+s-1}}}{\overline{M_iA_{i+s}}}=\prod_{i=1}^{n}\frac{\overline{M_i^{'}A_{i+s-1}}}{\overline{M_i^{'}A_{i+s}}}$$

In fact for $t=1$, one has n odd and $s=\frac{n+1}{2}$.

If one makes $s=1$ in this consequence, one finds the mathematical note from [1], pages 35-37.

Application: If in the theorem, the lines d_i are concurrent, one obtains:

$$\prod_{i=1}^{n}\frac{\overline{M_i^{'}A_{i+s}}}{\overline{M_i^{'}A_{i+s+t}}}=(-1)^n$$ (For this, see [2]).

Bibliography:

[1] Dan Barbilian, Ion Barbu – "Pagini inedite", Editura Albatros, Bucharest, 1981 (Ediție îngrijită de Gerda Barbilian, V. Protopopescu, Viorel Gh. Vodă).

[2] Florentin Smarandache – "Généralisation du théorème de Ceva".