DE LONGCHAMPS' POINT, LINE AND CIRCLE

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The purpose of this article is to familiarize the reader with these notions, emphasizing on connections between them.

Lemma

The circles drawn on the sides of an obtuse triangle ABC, as diameters, and on the medians of this triangle, as diameters, have the same radical circle and that it is the circle with the center in the orthocenter of the triangle ABC and the ray $\rho = 2R\sqrt{-\cos A \cdot \cos B \cdot \cos C}$ (this circle is called the polar circle of the triangle ABC).

Proof

Let's consider A', B', C' the altitudes' base points of the obtuse triangle ABC and A_1, B_1, C_1 its sides centers (see fig. 1).

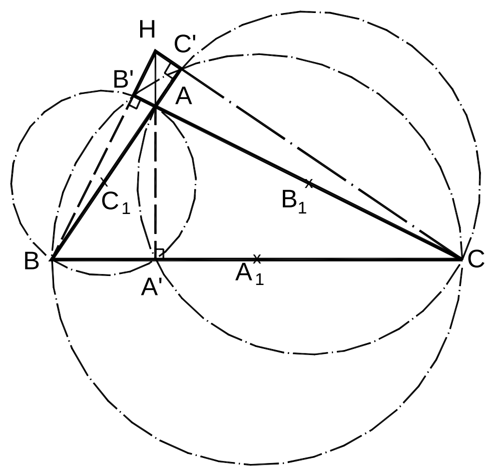

Fig. 1

The circle drawn on [AB] as diameter passes through A' and B', the circle drawn on [BC] as diameter passes through B' and C', and the circle drawn on [AC] as diameter passes through A' and C'. We notice that these circles have as common chords (radical axes) the altitudes of the triangle and because these are concurrent in H, it results that the orthocenter is the radical center of the considered circles.

On the other side, the circle which has the median $[AA_1]$ as diameter passes through A'. We have:

$$HA \cdot HA' = HB \cdot HB' = HC \cdot HC',$$

which shows that the power of H in respect to the constructed circle on $[AA_1]$ as diameter is equal to the power of H in respect to the constructed circles on the sides of the triangle ABC as diameters. Therefore, H is the radical center of the constructed circles on medians as diameters.

From the relation: $HA \cdot HA' = \rho^2$ we'll determine the ray ρ of the polar circle.

It is known that $AH = -2R\cos A$; from the triangle BHA' we obtain:

$$HA' = BH \cdot \cos\left(\widehat{BHA'}\right) = BH \cdot \cos C.$$

Taking into account that $BH = 2R\cos B$,
we obtain:

$$\rho = 2R\sqrt{-\cos A \cdot \cos B \cdot \cos C}$$

Definition 1.

If ABC is an obtuse triangle we say that the **De Longchamps' circle** of the triangle ABC is the circle that is orthogonal to the circles that have their centers in the triangle's vertexes and as rays the opposed sides of these vertexes (Casey – 1886).

Theorem 1.

The De Longchamps' circle of the obtuse triangle ABC has the ray given by the formula:

$$R_L = 4R\sqrt{-\cos A \cdot \cos B \cdot \cos C}$$

Proof

The circle $\mathcal{C}(A;BC)$ intersects the circle $\mathcal{C}(B;AC)$ in the points P and M and the circle $\mathcal{C}(A;BC)$ intersects the circle $\mathcal{C}(C;AB)$ in the points N and M.

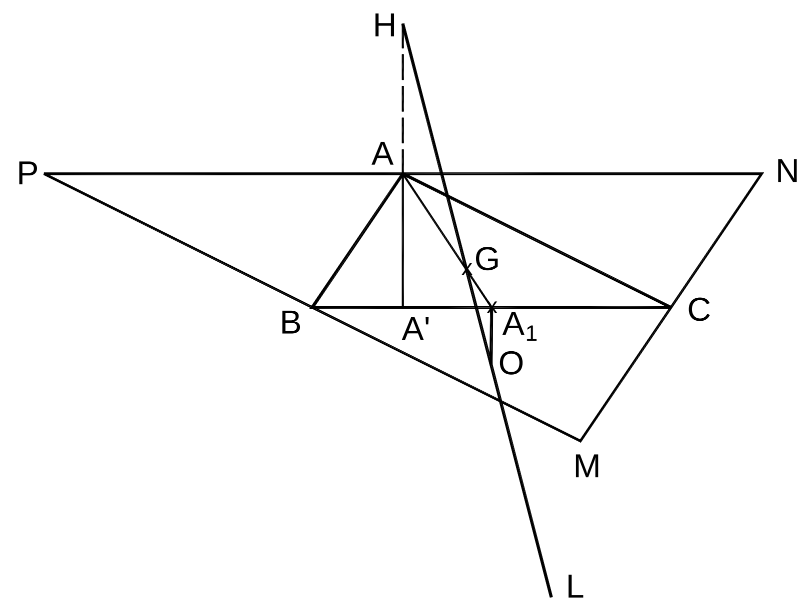

Fig. 2

The M, N and P are the vertex of the triangle **anti-complementary** of the triangle ABC (the triangle with the sides parallel to the sides of the given triangle, drawn through the vertexes of the given triangle).

We observe that the quadrilaterals ACBP, ACMB and ABCN are parallelograms and that the circles from the theorem's enunciation are the circles drawn on the sides of the anti-complementary triangle as diameters.

Applying the lemma it results that it exists an orthogonal circle to these circles, which this has as center the orthocenter L of the triangle MNP. Because the triangle MNP is the anti-complementary triangle of the triangle ABC and is similar to it, the similarity rapport being equal to 2, it results that the ray of the De Longchamps' circle will be the double of the polar circle's ray of the triangle ABC, therefore, $R_L = 2\rho$, thus:

$$R_L = 4R\sqrt{-\cos A \cdot \cos B \cdot \cos C}$$

Definition 2.

We call **power circles** of a triangle ABC the three circles with the centers in the middle points A_1, B_1, C_1 of the triangle's sides and which pass, respectively, through the opposite vertexes A, B and C.

Theorem 2.

If ABC is an obtuse triangle, the De Longchamps's circle is the radical circle of the power circles of the triangle ABC.

Proof

Let MNP the anti-complementary triangle of the triangle ABC, the power circle with the center A_1 and the ray A_1A passes through M (see fig. 3), similarly, this circle intersects for the second time NP in the altitude's base point from M of the anti-complementary triangle.

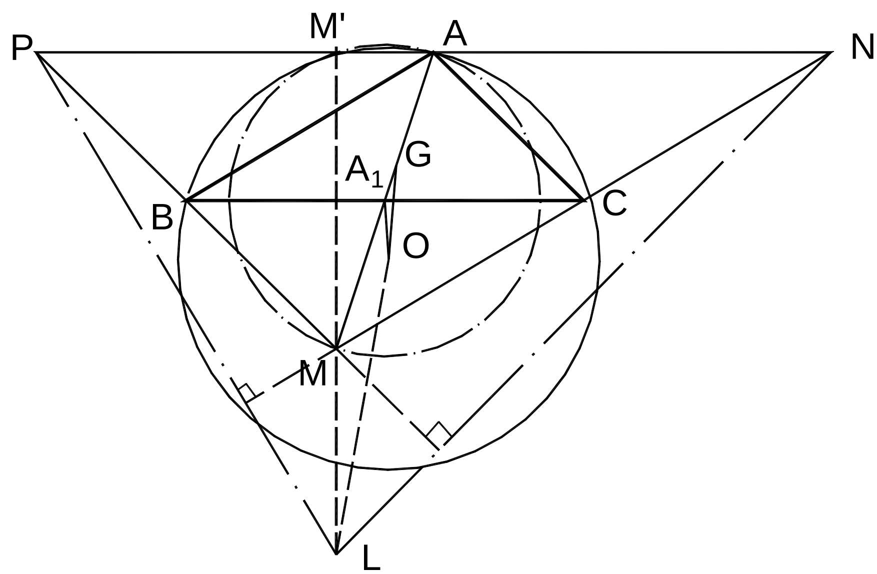

Fig. 3

The circle constructed on [MP] as diameter intersects with the above mentioned power circle on the altitude MM′ and also trough the points M and M′ passes the circle constructed on [MN] as diameter. These circles have as ortho-central radical center L of the triangle MNP. Repeating this reasoning we obtain that L is the radical center of the power circles of the triangle ABC.

Observation 1.

The De Longchamps's circle of a triangle is defined only if the triangle is obtuse.

Definition 3.

The **De Longchamps' point** of a triangle is the radical center of the power circles of the triangle.

Theorem 3.

The De Longchamps' point L, of the triangle ABC, is the symmetric of the orthocenter H of the triangle in rapport to the center O of the circumscribed circle of the triangle.

Proof

The anti-complementary triangle MNP of the triangle ABC and the triangle ABC are homothetic through the homotopy of the pole G and of rapport 2; the same are the De Longchamps's circles and the polar circle of the triangle ABC, it follows that the points L, G, H are collinear and $LG = 2GH$. On the other side the points O, G, H are collinear (the Euler's line) and $GH = 2GO$.

We have:

$$LG = LO + OG;\ OG = \frac{1}{2}GH = \frac{1}{3}OH.$$

$$LO = LG - OG = 2GH - \frac{1}{2}GH = \frac{3}{2}GH.$$

We obtain:

$$LO = OH.$$

Definition 4.

The **De Longchamps'** line is defined as the radical axes of the De Longchamps' circle and of the circumscribed circle of a triangle.

Theorem 4.

The De Longchamps's line of a triangle is the radical axes of the circumscribed circle to the triangle and of the circle circumscribed to the anti-complementary triangle of the given triangle.

Proof

The center of the circle circumscribed to the anti-complementary triangle of the triangle ABC is the orthocenter H of the triangle ABC and its ray is 2R. We'll denote with Q the intersection between the De Longchamps' line and the Euler's line (see fig. 4).

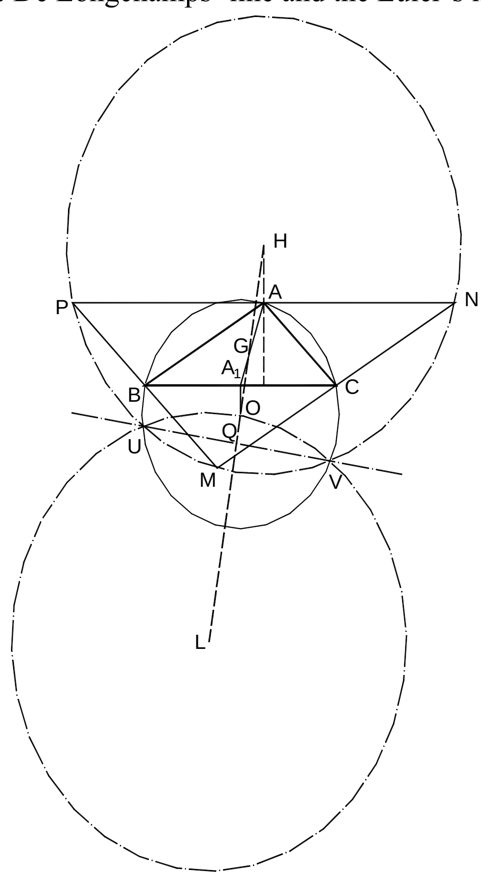

Fig. 4

We have:

$$R^2 - OQ^2 = R_L^2 - LQ^2$$

$$OQ = LO - LQ = HO - LQ$$

$$R^2 - (HO - LQ)^2 = 4\rho^2 - LQ^2,$$

it results:

$$LQ = \frac{4\rho^2 - R^2 + OH^2}{2HO}.$$

Because

$$HO^2 = R^2 \cdot (1 - 8\cos A \cdot \cos B \cdot \cos C) = R^2 + 2\rho^2,$$

we obtain:

$$LQ = \frac{3\rho^2}{HO} \text{ şi } R^2 - OQ^2 = 4\rho^2 - LQ^2 = \rho^2\left(4 - \frac{9\rho^2}{HO^2}\right)$$

$$4R^2 - HQ^2 = 4R^2 - (HL - LQ)^2 = 4R^2 - 2(HO - LQ)^2 = \rho^2\left(4 - \frac{9\rho^2}{HO^2}\right).$$

Therefore

$$4R^2 - HQ^2 = R^2 - OQ^2,$$

thus the radical axes of the circumscribed circles to the anti-complementary MNP and ABC is the De Longchamps' line.

Theorem 5.

The De Longchamps' line of a triangle is the polar of the triangle's we sight center in rapport to the De Longchamps' circle.

Proof

We have $LG = \frac{4}{3}HO$ and $LQ \cdot LG = 4\rho^2$, then $LQ \cdot LG = R_L^2$. It results that GV (see fig. 4) is tangent to the De Longchamps' circle. Therefore, the line UV (the polar of G) is the De Longchamps' line.

Definition 5.

It is called **reciprocal transversal** of a transversal M, N, P in the triangle ABC the line M′, N′, P′ formed by the symmetric points of the points M, N, P in rapport to the centers of the sides BC, and AB.

Observation 2.

a) In figure 5 the sides M, N, P and M′, N′, P′ are reciprocal transversals.

b) The notion of reciprocal transversal was **introduced by G. De Longchamps** in 1866.

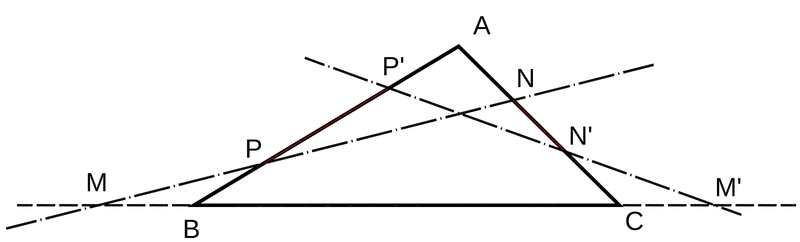

Fig. 5

Definition 6.

The Lemoine's line of a triangle ABC is the line that contains the intersections with the opposite sides of the triangle of the tangents constructed on the triangle's vertexes to its circumscribed circle.

Theorem 6.

The De Longchamps' line is the reciprocal transversal of the Lemoine's line.

Proof

Let S be the intersection of the tangent constructed from A to the circumscribed circle of the triangle ABC with the side BC (see fig. 6).

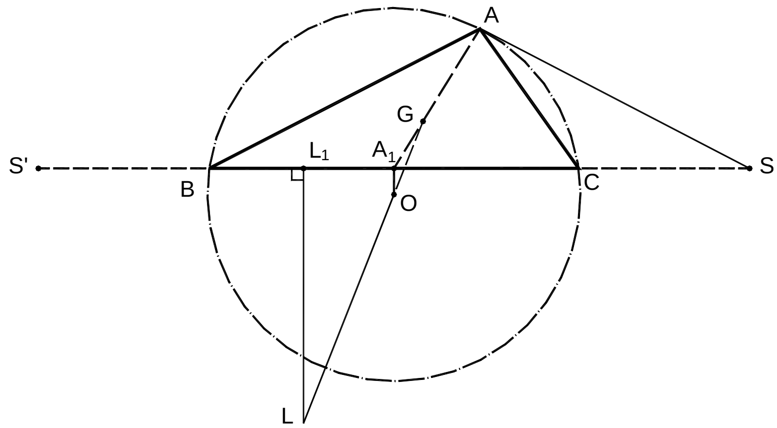

Fig. 6

The point S is, practically, the base of the external simediane from A. It is known that $\frac{SC}{SB}=\frac{b^2}{c^2}$, and we find $SC=\frac{ab^2}{c^2-b^2}$.

We will proof that the symmetrical point of S in rapport to the middle of BC, S′, belongs to the radical axes of the De Longchamps's circle and of the circumscribed circle (the De Longchamps' line).

We have that $S'B=\frac{a\,b^2}{c^2-b^2}$.

Let L_1 be the orthogonal projection of L on BC. We'll proof that

$$S'L^2-R_L^2=S'O^2-R^2,$$

L is the radical center, then

$$LB^2-b^2=LC^2-c^2=R_L^2.$$

We obtain that $LB^2-LC^2=b^2-c^2$

and also

$$L_1B^2-L_1C^2=b^2-c^2.$$

$$S'L^2=LL_1^2+S'L_1^2 \text{ şi } S'O^2=S'A_1^2+OA_1^2 \text{ (}A_1\text{ the middle of (BC))}$$

$$S'L^2-S'O^2=LL_1^2+(S'B+BL_1)^2-S'A_1^2-OA_1^2=LL_1^2+BL_1^2+S'B^2+2S'B\cdot BL_1-$$
$$-(S'B+BA_1)^2-OA_1^2$$

We find that

$$S'L^2-S'O^2=LL_1^2+BL_1^2+2S'B\cdot BL_1-2S'B\cdot BA_1-R^2=$$
$$=LB^2+2S'B(BL_1-BA_1)-R^2.$$

We substitute $S'B=\frac{a\,b^2}{c^2-b^2}$ and $L_1B=\frac{a^2+b^2-c^2}{2a}$ and we obtain

$$S'L^2-S'O^2=R_L^2-R^2.$$

Similarly it can be shown that the symmetric points in rapport to the middle points of the sides (AC) and (AB) of the base of the exterior simediane constructed from B and C, belong to the De Longchamps' line.

Application

Let ABC be an acute triangle and A_1 the middle of (BC). The circles $\mathcal{C}(A;BC)$ and $\mathcal{C}(A_1;AA_1)$ have a common chord A′A". Similarly, we define the line segments B′B" and C′C". Prove that the line segments A′A", B′B", C′C" are concurrent.

(Problem given at the test for training the 2008 team)

Solution

We denote $\mathcal{C}(A;BC) \cap \mathcal{C}(B;AC) = \{P, P'\}$

$\mathcal{C}(A;BC) \cap \mathcal{C}(C;AB) = \{N, N'\}$

$\mathcal{C}(B;AC) \cap \mathcal{C}(C;AB) = \{M, M'\}$

The triangle MNP will be the anti-complementary triangle of the triangle ABC while the triangle M′N′P′ will be the orthic triangle of the triangle MNP.

It results that $\mathcal{C}(A;BC)$, $\mathcal{C}(B;AC)$ and $\mathcal{C}(C_1;CC_1)$ have as the radical axis the altitude PP′ of the triangle MNP.

The circles $\mathcal{C}(A;BC)$, $\mathcal{C}(C;AB)$ and $\mathcal{C}(B_1;BB_1)$ have as radical axis the altitude NN′, while the circles $\mathcal{C}(B;AC)$, $\mathcal{C}(C;AB)$ and $\mathcal{C}(A_1;AA_1)$ have as radical axis the altitude MM′ of the triangle MNP.

Let $\{L\} = MM' \cap NN' \cap PP'$ be the orthocenter of the triangle MNP.

We have that L is the radical center of the circles $\mathcal{C}(A;BC)$, $\mathcal{C}(B;AC)$ and $\mathcal{C}(C;AB)$. L is the radical center of the circles $\mathcal{C}(A_1;AA_1)$, $\mathcal{C}(B_1;BB_1)$ and $\mathcal{C}(C_1;CC_1)$.

Also, L is the radical center of the circles $\mathcal{C}(A;BC)$, $\mathcal{C}(A_1;AA_1)$ and $\mathcal{C}(C;AB)$.

Indeed, the radical axis of the circles $\mathcal{C}(A;BC)$ and $\mathcal{C}(C;AB)$ is the altitude NN′, and the radical axis of the circles $\mathcal{C}(A_1;AA_1)$ and $\mathcal{C}(C;AB)$ is the altitude MM′.

It will result that the radical axis of the circles $\mathcal{C}(A;BC)$ and $\mathcal{C}(A_1;AA_1)$, that is the chord A′A" passes through L.

Similarly, it results that B′B" and C′C" pass through L. The concurrence point is L, the orthocenter of the anti-complementary triangle of the triangle ABC, therefore the De Longchamps' point of the triangle ABC.

Reference

[1]. D. Efremov, Noua geometrie a triunghiului (The New Geometry of the Triangle), Odessa, 1902.