K-Divisibility and K-Strong Divisibility Sequences

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A sequence of rational integers g is called a **divisibility sequence** if and only if

$$n \mid m \Rightarrow g(n) \mid g(m)$$

for all positive integers n, m. [See [3] and [4]].

Also, g is called a **strong divisibility sequence** if and only if

$$\left(g(n), g(m)\right) = g\left(\left(n, m\right)\right)$$

for all positive integers n, m. [See [1], [2], [3], [4], and [5]].

Of course, it is easy to show that the results of the Smarandache function $S(n)$ is neither a divisibility nor a strong divisibility sequence, because $4 \mid 20$ but $S(4) = 4$ does not divide $5 = S(20)$, and $\left(S(4), S(20)\right) = (4,5) = 1 \neq 4 = S(4) = S\left(\left(4,20\right)\right)$.

a) However, is there an infinite subsequence of integers $M = \{m_1, m_2, ...\}$ such that S is a divisibility sequence on M ?

b) If $P\{p_1, p_2, ...\}$ is the set of prime numbers, the S is not a strong divisibility sequence on P, because for $i \neq j$ we have

$$\left(S(p_i), S(p_j)\right) = \left(p_i, p_j\right) = 1 \neq 0 = S(1) = S\left(\left(p_i, p_j\right)\right).$$

And the same question can be asked about P as it was asked in part a).

We introduce the following two notions, which are generalizations of a "divisibility sequence" and "strong divisibility sequence" respectively.

1) A k-divisibility sequence, where $l \geq 1$ is an integer, is defined in the following way:

If

$$n \mid m \Rightarrow g(n) \mid g(m) \Rightarrow g\left(g(n)\right) \mid g\left(g(m)\right) \Rightarrow ... \Rightarrow \underbrace{g\left(...\left(g(n)\right)...\right)}_{k\ times} \mid \underbrace{g\left(...\left(g(m)\right)...\right)}_{k\ times}$$

for all positive integers n, m.

For example, $g(n) = n!$ is a k-divisibility sequence.

Also, any constant sequence is a k-divisibility sequence.

2) A k-strong divisibility sequence, where $k \geq 1$ is an integer, is defined in the following way:

If $\left(g(n_1), g(n_2), ..., g(n_k)\right) = g\left(\left(n_1, n_2, ..., n_k\right)\right)$ for all positive integers $n_1, n_2, ..., n_k$.

For example, $g(n)=2n$ is a k-strong divisibility sequence, because

$$\left(2n_1, 2n_2, ..., 2n_k\right) = 2 * \left(n_1, n_2, ..., n_k\right) = g\left(\left(n_1, n_2, ..., n_k\right)\right).$$

Remarks: If g is a divisibility sequence and we apply its definition k-times, we obtain that g is a k-divisibility sequence for any $k \geq 1$. The converse is also true. If g is k-strong divisibility sequence, $k \geq 2$, then g is a strong divisibility sequence. This can be seen by taking the definition of a k-strong divisibility sequence and replacing n by n_1 and all $n_2, ..., n_k$ by m to obtain

$\left(g(n), g(m), ..., g(m)\right) = g\left(\left(n, m, ..., m\right)\right)$ or $\left(g(n), g(m)\right) = g\left(\left(n, m\right)\right)$.

The converse is also true, as

$$\left(n_1, n_2, ..., n_k\right) = \left(\left(...\left(\left(n_1, n_2\right), n_3\right), ...\right), n_k\right).$$

Therefore, we found that:

a) The divisibility sequence notion is equivalent to a k-divisibility sequence, or a generalization of a notion is equivalent to itself.

Is there any paradox or dilemma?

b) The strong divisibility sequence is equivalent to the k-strong divisibility sequence notion

As before, a generalization of a notion is equivalent to itself.

Again, is there any paradox or dilemma?

REFERENCES

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