On Neutro-BE-algebras and Anti-BE-algebras

(revisited)

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Abstract

In this paper, the concepts of Neutro-BE-algebra and Anti-BE-algebra are introduced, and some related properties and four theorems are investigated. We show that the classes of Neutro-BE-algebra and Anti-BE-algebras are alternatives of the class of BE-algebras.

Keywords: BE-algebra; Neutro-sophication; Neutro-BE-algebra; Anti-sophication; Anti-BE-algebra.

1. Introduction

Neutrosophy, introduced by F. Smarandache in 1998, is a new branch of philosophy that generalized the dialectics and took into consideration not only the dynamics of opposites, but the dynamics of opposites and their neutrals [8]. Neutrosophic Logic / Set / Probability / Statistics / Measure / Algebraic Structures etc. are all based on it. One of the most striking trends in the neutrosophic theory is the hybridization of neutrosophic set with other potential sets such as rough set, bipolar set, soft set, vague set, etc. The different hybrid structures such as rough neutrosophic set, single valued neutrosophic rough set, bipolar neutrosophic set, single valued neutrosophic vague set, etc. are proposed in the literature in a short period of time. Neutrosophic set has been a very important tool in all various areas of data mining, decision making, e-learning, engineering, computer science, graph theory, medical diagnosis, probability theory, topology, social science, etc.

A classical Algebra may be transformed into a NeutroAlgebra by a process called neutro-sophication, and into an AntiAlgebra by a process called anti-sophication.

In [2], H.S. Kim et al. introduced the notion of a BE-algebra as a generalization of a BCK-algebra. S.S. Ahn et al. introduced the notion of ideals in BE-algebras, and they stated and proved several properties of such ideals [1]. A. Borumand Saeid et al defined some filters in BE-algebras and investigated relation between them [3]. A. Rezaei et al. investigated the relationship between Hilbert algebras and BE-algebras and showed that commutative self-distributive BE-algebras and Hilbert algebras are equivalent [4]. In this paper, the concepts of a Neutro-BE-algebra and Anti-BE-algebra are introduced, and some related properties are investigated. We show that the class of Neutro-BE-algebra is an alternative of the class of BE-algebras.

2. NeutroLaw, NeutroOperation, NeutroAxiom, and NeutroAlgebra

In this section, we review the basic definitions and some elementary aspects that are necessary for this paper.

The Neutrosophy's Triplet is (<A>, <neutroA>, <antiA>), where <A> may be an item (concept, idea, proposition, theory, structure, algebra, etc.), <antiA> the opposite of <A>, while <neutroA> {also the notation <neutA> was employed before} the neutral between these opposites. Based on the above triplet the following Neutrosophic Principle one has: a law of composition defined on a given set may be true (T) for some set elements, indeterminate (I) for other set's elements, and false (F) for the remainder of the set's elements; we call it NeutroLaw. A law of composition defined on a given sets, such that the law is false (F) for all set's elements is called AntiLaw. Similarly, an operation defined on a given set may be well-defined for some set elements, indeterminate for other set's elements, and undefined for the remainder of the set's elements; we call it NeutroOperation. While, an operation defined on a given set that is undefined for all set's elements is called AntiOperation.

In classical algebraic structures, the laws of compositions or operations defined on a given set are automatically well-defined [i.e. true (T) for all set's elements], but this is idealistic. Consequently, an axiom (let's say Commutativity, or Associativity, etc.) defined on a given set, may be true (T) for some set's elements, indeterminate (I) for other set's elements, and false (F) for the remainder of the set's elements; we call it NeutroAxiom. In classical algebraic structures, similarly an axiom defined on a given set is automatically true (T) for all set's elements, but this is idealistic too. A NeutroAlgebra is a set endowed with some NeutroLaw (NeutroOperation) or some NeutroAxiom. The NeutroLaw, NeutroOperation, NeutroAxiom, NeutroAlgebra and respectively AntiLaw, AntiOperation, AntiAxiom and AntiAlgebra were introduced by Smarandache in 2019 [6] and afterwards he recalled, improved and extended them in 2020 [7]. Recently, the concept of a Neutrosophic Triplet of BI-algebra was defined [5].

3. Neutro-BE-algebras, Anti-BE-Algebras

Definition 3.1. (Definition of classical BE-algebras [1])
An algebra $(X,*,0)$ of type $(2,0)$ (i.e. X is a nonempty set, $*$ is a binary operation and 0 is a constant element of X) is said to be a BE-*algebra* if:

(L) **The law $*$ is well-defined**, i.e. $(\forall x, y \in X)(x * y \in X)$.

And the following axioms are totally true on X:

$(BE1)$ $(\forall x \in X)(x * x = 0)$,

$(BE2)$ $(\forall x \in X)(0 * x = x)$,

$(BE3)$ $(\forall x \in X)(x * 0 = 0)$,

$(BE4)$ $(\forall x, y, z \in X, with\ x \neq y)(x * (y * z) = y * (x * z))$.

Example 3.2.

(i) Let $\mathbb{N}$ be the set of all natural numbers and $*$ be the binary operation on $\mathbb{N}$ defined by

$$x * y = \begin{cases} y & if\ x = 1; \\ 1 & if\ x \neq 1. \end{cases}$$

Then $(\mathbb{N},*,1)$ is a BE-algebra.

(ii) Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let $*$ be the binary operation on $\mathbb{N}_0$ defined by

$$x * y = \begin{cases} 0 & if\ x \geq y; \\ y - x & otherwise. \end{cases}$$

Then $(\mathbb{N}_0, *, 0)$ is a BE-algebra.

Definition 3.3. (Neutro-sophications)

The Neutro-sophication of the Law (degree of well-defined, degree of indeterminacy, degree of outer-defined)

(NL) $(\exists x, y \in X)(x * y \in X)$ and $(\exists x, y \in X)(x * y = \ indeterminate$ or $* y \notin X)$,

The Neutro-sophication of the Axioms (degree of truth, degree of indeterminacy, degree of falsehood)

$(NBE1)$ $(\exists x \in X)(x * x = 0)$ and $(\exists x \in X)(x * x = indeterminate$ or $x * x \neq 0)$,

$(NBE2)$ $(\exists x \in X)(0 * x = x)$ and $(\exists x \in X)(0 * x = indeterminate$ or $0 * x \neq x)$,

$(NBE3)$ $(\exists x \in X)(x * 0 = 0)$ and $(\exists x \in X)(x * 0 = \ indeterminate$ or $x * 0 \neq 0)$,

$(NBE4)$ $(\exists x, y, z \in X, with\ x \neq y)(x * (y * z) = y * (x * z))$ and

$(\exists x, y, z \in X, with\ x \neq y)\big(x * (y * z) = indeterminate$ or $x * (y * z) \neq y * (x * z)\big)$.

Definition 3.4. (Anti-sophications)

The Anti-sophication of the Law (totally outer-defined)

(AL) $(\forall x, y \in X)(x * y \notin X)$.

The Anti-sophication of the Axioms (totally false)

$(ABE1)$ $(\forall x \in X)(x * x \neq 0)$,

$(ABE2)$ $(\forall x \in X)(0 * x \neq x)$,

$(ABE3)$ $(\forall x \in X)(x * 0 \neq 0)$,

$(ABE4)$ $(\forall x, y, z \in X$, with $x \neq y)(x * (y * z) \neq y * (x * z))$.

Definition 3.5. (Neutro-*BE*-algebras)

A Neutro-BE-algebra is an alternative of BE-algebra that has at least a (NL) or at least one $(NBEi)$, $i \in \{1, 2, 3, 4\}$, with no anti-law and no anti-axiom.

Example 3.6.

(i) Let $\mathbb{N}$ be the set of all natural numbers and $*$ be the Neutro-sophication of the Law $*$ on $\mathbb{N}$ from Example 2.2. (i) defined by

$$x * y = \begin{cases} y & if\ x = 1; \\ \frac{1}{2} & if\ x \in \{3,5,7\}; \\ 1 & otherwise. \end{cases}$$

Then $(\mathbb{N},*,1)$ is a Neutro-BE-algebra. Since

(NL) if $x \in \{3,5,7\}$, then $x * y = \frac{1}{2} \notin \mathbb{N}$, for all $y \in \mathbb{N}$, while if $x \notin \{3,5,7\}$ and $x \in \mathbb{N}$, then $x * y \in \{1, y\} \subseteq \mathbb{N}$, for all $y \in \mathbb{N}$.

(NBE1) $1 * 1 = 1 \in \mathbb{N}$ and $3 * 3 = \frac{1}{2} \notin \mathbb{N}$,

(BE2) holds always since $1 * x = x$, for all $x \in \mathbb{N}$.

(NBE3) $5 * 1 = \frac{1}{2} \neq 1$ and if $x \notin \{3,5,7\}$, then $x * 1 = 1$,

(NBE4) $5 * (3 * 4) = 5 * \frac{1}{2} =?\,(indeterminate)$ and $3 * (5 * 4) = 3 * \frac{1}{2} = ?\,(indeterminate)$

Also, $2 * (3 * 4) = 2 * \frac{1}{2} = \ ?\,(indeterminate)$, but $3 * (2 * 4) = 3 * 1 = \frac{1}{2}$.

Further, $4 * (8 * 2) = 4 * 1 = 1 = 8 * (4 * 2)$.

(ii) Let S be a nonempty set and $\mathcal{P}(S)$ be the power set of S. Then $(\mathcal{P}(S),\cap,\emptyset)$ is a Neutro-BE-algebra.

$\cap$ is the binary set intersection operation, but

(NBE1) is valid, since $\emptyset \cap \emptyset = \emptyset$ and for all $\emptyset \neq A \in \mathcal{P}(S)$, $A \cap A = A \neq \emptyset$.

(NBE2) $\emptyset \cap \emptyset = \emptyset$ and if $\emptyset \neq A$, then $\emptyset \cap A = \emptyset \neq A$,

(BE3) holds, since $A \cap \emptyset = \emptyset$,

(BE4) holds, since $A \cap (B \cap C) = B \cap (A \cap C)$.

(iii) Similarly, $(\mathcal{P}(S),\cup,\emptyset)$, $(\mathcal{P}(S),\cap,S)$, $(\mathcal{P}(S),\cup,S)$, where $\cup$ is the binary set union operation, are Neutro-BE-algebras.

(iv) Let $X := \{0, a, b, c, d\}$ be a set with the following table.

Table 1

*	0	a	b	c	d
0	c	a	b	c	a
a	b	0	b	c	d
b	0	a	0	c	c
c	?	0	b	0	b
d	0	0	0	0	?

Then $(X,*,0)$ is a Neutro-BE-algebra.

(NL) $c * 0 =?\,(indeterminate)$, and $d * d = ?\,(indeterminate)$, and for all $x, y \in \{0, a, b\}$, then $x * y \in X$.

(NBE1) $a * a = 0$ and $0 * 0 = c \neq 0$ $or\, d * d =?\,(indeterminate)$.

(NBE2) holds since $0 * b = b$, and $0 * d = a \neq d$.

(NBE3) $c * 0 =? (indeterminate) \neq 0$ and if $x \in \{b, d\}$, then $x * 0 = 0$,

(NBE4) $d * (c * b) = d * b = 0 \neq c * (d * b) = c * 0 =? (indeterminate)$ and

$a * (b * c) = a * c = c = b * (a * c)$.

(v) Let S be a nonempty set and $\mathcal{P}(S)$ be the power set of S. Then $(\mathcal{P}(S), -, \emptyset)$ is an Anti-BE-algebra, where $-$ is the binary operation of set subtraction, because:

(BE1) is valid, since $A - A = \emptyset$,

(NBE2) holds, since $\emptyset - A = \emptyset \neq A$ and $\emptyset - \emptyset = \emptyset$,

(NBE3) holds, since $A - \emptyset = A \neq \emptyset$ and $\emptyset - \emptyset = \emptyset$

(ABE4) is valid, since for A ≠ B, one has $A - (B - C) \neq B - (A - C)$, because:

x ∈ $A - (B - C)$ means (x ∈ A and x ∉ B-C), or {x ∈ A and (x ∉ B or x ∈ C) }, or {(x ∈ A and x ∉ B) or (x ∈ A and x ∈ C)}; while x ∈ $B - (A - C)$ means {(x ∈ B and x ∉ A) or (x ∈ B and x ∈ C)}.

(vi) Let ℝ be the set of all real numbers and $*$ be a binary operation on ℝ defined by $x * y = |x - y|$. Then (ℝ,$*$,0) is a Neutro-BE-algebra.

(BE1) holds, since $x * x = |x - x| = 0$, for all $x \in \mathbb{R}$.

(NBE2) is valid, since if $x \geq 0$, then $x * 0 = |x - 0| = |x| = x$, and if $x < 0$, then $x * 0 = |x - 0| = |x| = -x \neq x$.

(NBE3) is valid, since if $x \neq 0$, then $0 * x = |0 - x| = |-x| \neq 0$, and if $x = 0$, then $0 * 0 = 0$.

(NBE4) holds, if x = 2, y = 3, z = 4 we get |2-|3-4|| = |2 – 1| = 1 and |3-|2-4|| = |3-2| = 1;

while for x = 4, y =8, z =3 we get |4 -|8-3|| = |4-5| = 1 and |8-|4-3|| = |8-1| = 7 ≠ 1.

Theorem 3.7.

The total number of Neutro-BE-algebras is 31.

Proof.

The classical BE-algebra has: 1 classical Law and 4 classical Axioms:

1 + 4 = 5 classical mathematical propositions.

Let C_n^m mean combinations of n elements taken by m, where n, m are positive integers, n ≥ m ≥ 0.

We transform (neutro-sophicate) the classical BE-algebra, by neutro-sophicating some of the 5 classical mathematical propositions, while the others remain classical (unchanged) mathematical propositions:

either only 1 of the 5 classical mathematical propositions (hence we have $C_5^1 = 5$ possibilities) – so 4 classical mathematical propositions remain unchanged,

or only 2 of the 5 classical mathematical propositions (hence we have $C_5^2 = 10$ possibilities) – so 3 classical mathematical propositions remain unchanged, or only 3 of the 5 classical mathematical propositions (hence we have $C_5^3 = 10$ possibilities) – so 2 classical mathematical propositions remain unchanged,

or only 4 of the 5 classical mathematical propositions (hence we have $C_5^4 = 5$ possibilities) – so 1 classical mathematical proposition remainsnchanged,

or all 5 of the 5 classical mathematical propositions (hence we have $C_5^1 = 1$ possibilities).

Whence the total number of possibilities will be:

$$C_5^1 + C_5^2 + C_5^3 + C_5^4 + C_5^5 = (1+1)^5 - C_5^0 = 2^5 - 1 = 31.$$

Definition 3.8. (Anti-BE-algebras)

An Anti-BE-algebra is an alternative of BE-algebra that has at least an (AL) or at least one $(ABEi), i \in \{1,2,3,4\}$.

Example 3.9.

(i) Let $\mathbb{N}$ be the natural number set and $X := \mathbb{N} \cup \{0\}$. Define a binary operation $*$ on X by $x *_A y = x^2 + y^2 + 1$. Then $(X,*,0)$ is not a BE-algebra, nor a Neutro-BE-algebra, but an Anti-BE-algebra.

Since $x *_A x = x^2 + x^2 + 1 \neq 0$, for all $x \in X$, and so $(ABE1)$ holds.

For all $x \in \mathbb{N}$, we have $x * 0 = x^2 + 1 \neq 0$, so $(ABE2)$ is valid. By a similar argument $(ABE3)$ is valid.

Since for $x \neq y$, one has $x *_A (y *_A z) = x^2 + (y^2 + z^2 + 1)^2 + 1 \neq y *_A (x *_A z) = y^2 + (x^2 + z^2 + 1)^2 + 1$,

thus $(ABE4)$ is valid.

(ii) Let S be a nonempty set and $\mathcal{P}(S)$ be the power set of S. Define the binary operation Δ (i.e. symmetric difference) by $A\Delta B = (A \cup B) - (A \cap B)$ for every $A, B \in \mathcal{P}(S)$. Then $(\mathcal{P}(S), \Delta, S)$ is not a BE-algebra, nor Neutro-BE-algebra, but it is an Anti-BE-algebra.

Since $A\Delta A = \emptyset \neq S$ for every $A \in \mathcal{P}(S)$ we get $(ABE1)$ holds, and so $(BE1)$ and $(NBE1)$ are not valid.

Also, for all $A, B, C \in \mathcal{P}(S)$ one has $A\Delta(B\Delta C) = B\Delta(A\Delta C)$. Thus, $(BE4)$ is valid.

Since there is at least one anti-axiom (ABE1), then $(\mathcal{P}(S), \Delta, S)$ is an Anti-BE-algebra.

(iii) Let $\mathcal{U} = \{0, a, b, c, d\}$ be a universe of discourse, and a subset $S = \{0, c\}$, and the below binary well-defined Law * with the following Cayley table.

Table 2

*	0	c
0	c	0
c	c	c

Then $(S,*,0)$ is an Anti-BE-algebra, since (ABE1) is valid, because: 0*0 = c ≠ 0 and c*c = c ≠ 0, and it is sufficient to have a single anti-axiom.

Theorem 3.10.

The total number of Anti-BE-algebras is 211.

Proof.

The classical BE-algebra has: 1 classical Law and 4 classical Axioms:

$1 + 4 = 5$ classical mathematical propositions.

Let C_n^m mean combinations of n elements taken by m, where n, m are positive integers, n ≥ m ≥ 0.

We transform (anti-sophicate) the classical BE-algebra, by anti-sophicating some of the 5 classical mathematical propositions, while the others remain classical (unchanged) or neutro-mathematical propositions:

either only 1 of the 5 classical mathematical propositions (hence we have $C_5^1 = 5$ subpossibilities) – so 4 classical mathematical propositions remain some unchanged others neutro-sophicated or $2^4 = 16$ subpossibilities; hence total number of possibilities in this case is: $5 \cdot 16 = 80$;

or 2 of the 5 classical mathematical propositions (hence we have $C_5^2 = 10$ subpossibilities) – so 3 classical mathematical propositions remain some unchanged other neutro-sophicated or $2^3 = 8$ subpossibilities; hence total number of possibilities in this case is: $10 \cdot 8 = 80$;

or 3 of the 5 classical mathematical propositions (hence we have $C_5^3 = 10$ subpossibilities) – so 2 classical mathematical propositions remain some unchanged other neutro-sophicated or $2^2 = 4$ subpossibilities; hence total number of possibilities in this case is: $10 \cdot 4 = 40$;

or 4 of the 5 classical mathematical propositions (hence we have $C_5^4 = 5$ subpossibilities) – so 1 classical mathematical propositions remain either unchanged other neutro-sophicated or $2^1 = 2$ subpossibilities; hence total number of possibilities in this case is: $5 \cdot 2 = 10$;

or all 5 of the 5 classical mathematical propositions (hence we have $C_5^5 = 1$ subpossibility) – so no classical mathematical propositions remain.

Hence, the total number of Anti-BE-algebras is:

$$C_5^1 . 2^{5-1} + C_5^2 . 2^{5-2} + C_5^3 . 2^{5-3} + C_5^4 . 2^{5-4} + C_5^5 . 2^{5-5} = 5 \cdot 16 + 10 \cdot 8 + 10 \cdot 4 + 5 \cdot 2 + 1 \cdot 1 = 211.$$

Theorem 3.11.

As a particular case, for BE-algebras, we have:

1 (classical) BE-algebra + 31 Neutro-BE-algebras + 211 Anti-BE-algebras = $243 = 3^5$ algebras.

Where, $31 = 2^5 - 1$, and $211 = 3^5 - 2^5$.

Proof.

It results from the previous Theorem 3.10 and 3.11.

Theorem 3.12.

Let U be a nonempty finite or infinite universe of discourse, and S a nonempty finite or infinite subset of U. A classical Algebra is defined on S.

In general, for a given classical Algebra, having n operations (laws) and axioms altogether, for integer $n \geq 1$, there are 3^n total number of Algebra / NeutroAlgebras / AntiAlgebras as below:

1 (classical) Algebra, $(2^n - 1)$ Neutro-Algebras, and $(3^n - 2^n)$ Anti-Algebras.

The finite or infinite cardinal of set the classical algebra is defined upon, does not influence the numbers of Neutro-BE-algebras and Anti-BE-algebras.

Proof.

It is similar to Theorem 3.11, and based on Theorems 3.10 and 3.11.

Where 5 (total number of classical laws and axioms altogether) is extended/replaced by n.

5. Conclusion.

We have studied and presented the neutrosophic triplet (BE-algebra, Neutro-BE-algebra, Anti-BE-algebra) together with many examples, several properties and four theorems.

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