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TWO TRIANGLES WITH THE SAME ORTHOCENTER AND A VECTORIAL PROOF OF STEVANOVIC'S THEOREM

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Abstract. In this article we'll emphasize on two triangles and provide a vectorial proof of the fact that these triangles have the same orthocenter. This proof will, further allow us to develop a vectorial proof of the Stevanovic's theorem relative to the orthocenter of the Fuhrmann's triangle.

Lemma 1. Let ABC an acute angle triangle, H its orthocenter, and A',B',C' the symmetrical points of H in rapport to the sides BC,CA,AB. We denote by X,Y,Z the symmetrical points of A,B,C in rapport to $B'C',C'A',A'B'$. The orthocenter of the triangle XYZ is H.

Proof. We will prove that $XH \perp YZ$, by showing that $\overrightarrow{XH}\cdot\overrightarrow{YZ}=0$.
We have (see Figure1) $\overrightarrow{VH}=\overrightarrow{AH}-\overrightarrow{AX}$, $\overrightarrow{BC}=\overrightarrow{BY}+\overrightarrow{YZ}+\overrightarrow{ZC}$,
from here $\overrightarrow{YZ}=\overrightarrow{BC}-\overrightarrow{BY}-\overrightarrow{ZC}$.

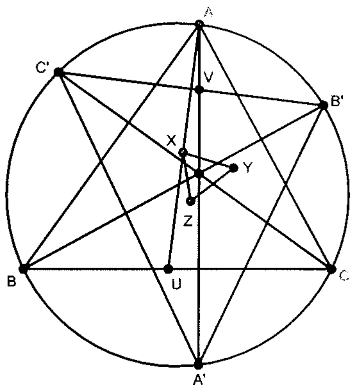

Figure 1

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Because Y is the symmetric of B in rapport to $A'C'$ and Z is the symmetric of C in rapport to $A'B'$, the parallelogram's rule gives us that: $\overrightarrow{BY} = \overrightarrow{BC'} + \overrightarrow{BA'}$, $\overrightarrow{CZ} = \overrightarrow{CB'} + \overrightarrow{CA'}$. Therefore

$$\overrightarrow{YZ} = \overrightarrow{BC} - (\overrightarrow{BC'} + \overrightarrow{BA'}) + \overrightarrow{B'C} + \overrightarrow{A'C}$$

But, $\overrightarrow{BC'} = \overrightarrow{BH} + \overrightarrow{HC'}$, $\overrightarrow{BA'} = \overrightarrow{BH} + \overrightarrow{HA'}$, $\overrightarrow{CB'} = \overrightarrow{CH} + \overrightarrow{HB'}$, $\overrightarrow{CA'} = \overrightarrow{CH} + \overrightarrow{HA'}$. By substituting these relations in the $\overrightarrow{YZ}$, we find:

$$\overrightarrow{YZ} = \overrightarrow{BC} + \overrightarrow{C'B'}$$

We compute

$$\overrightarrow{XH} \cdot \overrightarrow{YZ} = (\overrightarrow{AH} - \overrightarrow{AX}) \cdot (\overrightarrow{BC} + \overrightarrow{C'B'}) =$$

$$\overrightarrow{AX} \cdot \overrightarrow{BC} + \overrightarrow{AH} \cdot \overrightarrow{C'B'} - \overrightarrow{AX} \cdot \overrightarrow{BC} - \overrightarrow{AX} \cdot \overrightarrow{C'B'}$$

Because $AH \perp BC$ we have $\overrightarrow{AH} \cdot \overrightarrow{BC} = 0$, also $AX \perp B'C'$ and therefore $\overrightarrow{AX} \cdot \overrightarrow{B'C'} = 0$. We need to prove also that $\overrightarrow{XH} \cdot \overrightarrow{YZ} = \overrightarrow{AH} \cdot \overrightarrow{C'B'} - \overrightarrow{AX} \cdot \overrightarrow{BC}$. We note: $\{U\} = AX \cap BC$ and $\{V\} = AH \cap B'C'$. Then

$$\overrightarrow{AX} \cdot \overrightarrow{BC} = AX \cdot BC \cdot \cos(\sphericalangle AUC)$$

$$\overrightarrow{AH} \cdot \overrightarrow{C'B'} = AH \cdot C'A' \cdot \cos(\sphericalangle AVC')$$

We observe that $\sphericalangle AUC \equiv \sphericalangle AVC'$ (angles with the sides respectively perpendicular). The point B' is the symmetric of H in rapport to AC, consequently $\sphericalangle HAC \equiv \sphericalangle CAB'$, also the point C' is the symmetric of the point H in rapport to AB, and therefore $\sphericalangle HAB \equiv \sphericalangle BAC'$.

From these last two relations we find that $\sphericalangle B'AC' = 2\sphericalangle A$. The sinus theorem applied in the triangles $AB'C'$ and ABC gives:

$$B'C' = 2R \cdot \sin 2A$$

$$BC = 2R \sin A$$

We'll show that

$$AX \cdot BC = AH \cdot C'B',$$

and from here

$$AX \cdot 2R \sin A = AH \cdot 2R \cdot \sin 2A$$

which is equivalent to

$$AX = 2AH\cos A$$

We noticed that $\sphericalangle B'AC' = 2A$. Because $AX \perp B'C'$, it results that $\sphericalangle TAB \equiv \sphericalangle A$, we noted $\{T\} = AX \cap B'C'$. On the other side $AC' = AH$, $AT = \frac{1}{2}AY$, and $AT = AC'\cos A = AH\cos A$, therefore $\overrightarrow{XH} \cdot \overrightarrow{YZ} = 0$.

Similarly, we prove that $YH \perp XZ$, and therefore H is the orthocenter of triangle XYZ.

Lemma 2. Let ABC a triangle inscribed in a circle, I the intersection of its bisector lines, and A', B', C' the intersections of the circumscribed circle with the bisectors AI, BI, CI respectively. The orthocenter of the triangle $A'B'C'$ is I.

Proof.

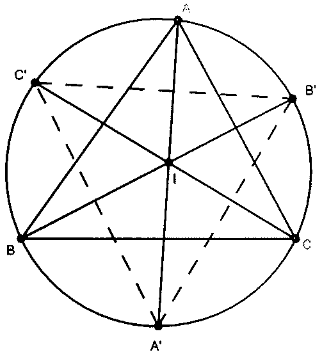

Figure 2

We'll prove that $A'I \perp B'C'$ (see Figure 2). Let $\alpha = m(\overset{\frown}{A'C}) = m(\overset{\frown}{A'B})$, $\beta = m(\overset{\frown}{B'C}) = m(\overset{\frown}{B'A})$, $\gamma = m(\overset{\frown}{C'A}) = m(\overset{\frown}{C'B})$. Then $m(\sphericalangle A'IC') = \frac{1}{2}(\alpha + \beta + \gamma)$. Because $2(\alpha + \beta + \gamma) = 360°$ it results $m(\sphericalangle A'IC') = 90°$, therefore $A'I \perp B'C'$.

Similarly, we prove that $B'I \perp A'C'$, and consequently the orthocenter of the triangle $A'B'C'$ is I.

Definition. Let ABC a triangle inscribed in a circle with the center in O and A', B', C' the middle of the arcs $\overset{\frown}{BC}$, $\overset{\frown}{CA}$, $\overset{\frown}{AB}$ respectively. The triangle XYZ formed by the symmetric of the points A', B', C' respectively in rapport to BC, CA, AB is called the *Fuhrmann triangle* of the triangle ABC.

Note. In 2002 the mathematician Milorad Stevanovic proved the following theorem:

Theorem (M. Stevanovic). *In an acute angle triangle the orthocenter of the Fuhrmann's triangle coincides with the center of the circle inscribed in the given triangle.*

Proof. We note $A'B'C'$ the given triangle and let A, B, C respectively the middle of the arcs $\overset{\frown}{B'C'}$, $\overset{\frown}{C'A'}$, $\overset{\frown}{A'B'}$ (see Figure 1). The lines AA', BB', CC' being bisectors in the triangle $A'B'C'$ are concurrent in the center of the circle inscribed in this triangle, which will note H, and which, in conformity with Lemma 2 is the orthocenter of the triangle ABC. Let XYZ the Fuhrmann triangle of the triangle $A'B'C'$, in conformity with Lemma 1, the orthocenter of XYZ coincides with H the orthocenter of ABC, therefore with the center of the inscribed circle in the given triangle $A'B'C'$.

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