Some interval-valued neutrosophic hesitant fuzzy uncertain linguistic
Bonferroni mean aggregation operators and their application in multiple
attribute decision making

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Abstract: Interval-valued neutrosophic hesitant fuzzy uncertain linguistic set (IVNHFULS) has the advantages of both neutrosophic numbers and interval-valued hesitant fuzzy uncertain linguistic variable. In this paper, we firstly introduce the definition, the operational laws, and the score function of IVNHFULS. Then, we combine interval-valued neutrosophic hesitant fuzzy uncertain linguistic set with Bonferroni mean operator and propose some new aggregation operators, such as the interval-valued neutrosophic hesitant fuzzy uncertain linguistic Bonferroni mean (IVNHFULBM) operator, the interval-valued neutrosophic hesitant fuzzy uncertain linguistic weighted Bonferroni mean (IVNHFULWBM) operator, the interval-valued neutrosophic hesitant fuzzy uncertain linguistic geometric Bonferroni mean (IVNHFULGBM) operator, the interval-valued neutrosophic hesitant fuzzy uncertain linguistic weighted geometric Bonferroni mean (IVNHFULWGBM) operator. At the same time, the related properties of these operators are discussed. Furthermore, we propose two multiple attribute decision making methods based on IVNHFULWBM operator and IVNHFULWGBM operator. Finally, we give an illustrative example to demonstrate the practicality and effectiveness of the proposed methods.

Keywords: Multiple attribute decision making; Interval-valued neutrosophic hesitant fuzzy uncertain linguistic set; Bonferroni mean; Interval-valued neutrosophic hesitant fuzzy uncertain linguistic Bonferroni mean aggregation operator.

1. Introduction

Multiple attribute decision making (MADM) has been a hot research topic. In the real decision making, because of the complexity and the uncertainty of decision making problems, how to realistically express the fuzzy and uncertain information is very important. Zadeh [1] firstly proposed the Fuzzy set (FS), which has been successfully applied in various fields. Due to the FS only consider the membership degree, Atanassov [2] further developed the intuitionistic fuzzy set (IFS) based on the FS. In recent years, Smarandache [3] proposed the neutrosophic set (NS) which consisted of membership degree, non-membership degree and indeterminacy-membership degree. In addition, some subsets of neutrosophic set have been proposed. For example, Wang et al. [4][5] further proposed the interval-valued neutrosophic set (IVNS) and the single-valued neutrosophic set (SVNS). Due to the membership degree of FS, the membership degree and non-membership degree of IFS, the membership degree and non-membership degree and indeterminacy-membership degree of NS are a single value ranged from 0 to 1. As we all known, the single values are asymmetric and not comprehensive so that they inappropriately describe the incomplete information. Based on this, Torra and Narukawa [6-7] introduced the hesitant fuzzy sets (HFS). In HFS, the membership degree is a set of several possible values between 0 and 1. The obvious difference between FS and HFS is that the membership degree is
a single value or a set of possible values ranged from 0 to 1. When the decision makers evaluate the alternatives, HFS is a useful tool to describe the evaluation values with hesitance. Chen et al. [8] extended HFS to interval-valued hesitant fuzzy sets (IVHFS), in which the membership degree is a set of possible interval values between 0 and 1. Ye [9] presented a neutrosophic hesitant fuzzy set (NHFS), which is the combination HFS with SVNS. In real decision making, especially for qualitative information, the linguistic variables are suitable to express the evaluation values and it can improve the flexibility and reliability of the decision making information. So far, the linguistic variables have been widely studied [10-12], which could be expressed as single values [13] or interval values [14]. The interval linguistic terms are also known as uncertain linguistic variables. Rodriguez et al. [15-16] originally gave the hesitant fuzzy linguistic term sets (HFTLS) which is used to describe the linguistic variables. Lin et al. [17] developed the hesitant fuzzy linguistic sets (HFLS) and the hesitant fuzzy uncertain linguistic set (HFULS) on the basis of HFS and linguistic variables or uncertain linguistic variables. The characteristics of the HFLS and HFULS are that the membership degree can be expressed as a set of hesitant fuzzy linguistic variables or hesitant fuzzy uncertain linguistic variables. Similar to the HFS, HFLS doesn’t take the non-membership degree, indeterminacy-membership degree into account so that HFLS has the weaknesses in avoiding the information distortion and losing effectively in handling the indeterminate, incomplete and inconsistent problems. In order to overcome the deficiencies, in this paper, we extend NHFS to interval-valued neutrosophic hesitant fuzzy set (IVNHFS) and cobmine IVNHFS with uncertain linguistic variables. Consequently, we finally give the concept of interval-valued neutrosophic hesitant fuzzy uncertain linguistic set (IVNHFULS), which is composed of two main parts: an uncertain linguistic term and an interval-valued neutrosophic hesitant fuzzy element. The two parts respectively express an evaluation value and the hesitancy for the given evaluation value. Certainly, the second part also represents the interval-valued membership degree, interval-valued non-membership degree and interval-valued indeterminacy-membership degree associated with the specific uncertain linguistic term.

Up to now the information aggregation operators are studied deeply, and they have been widely used in various field. All sorts of operators have been proposed to deal with many different situations. For example, Xu [18-19] developed the uncertain linguistic weighted averaging (ULWA) operator, the uncertain linguistic ordered weighted averaging (ULOWA) operator, the uncertain linguistic hybrid averaging (ULHA) operator, the uncertain linguistic geometric averaging (ULGA) operator, uncertain linguistic weighted geometric averaging (ULWGA) operator, uncertain linguistic ordered weighted geometric averaging (ULOWGA) operator, and induced uncertain linguistic ordered weighted geometric averaging (IULOWGA) operator under uncertain linguistic environment. Wei [20] proposed an uncertain linguistic hybrid geometric averaging (ULHGA) operator. In order to more accurately and flexibly deal with the hesitant fuzzy information, Lin et al. [17] developed some hesitant fuzzy uncertain linguistic arithmetic aggregation operators, such as hesitant fuzzy uncertain linguistic weighted averaging (HFULWA) operator, hesitant fuzzy uncertain linguistic ordered weighted averaging (HFULOWA) operator, hesitant fuzzy uncertain linguistic hybrid averaging (HFULHA) operator. Ju and Liu [21] gave the interval-valued hesitant uncertain linguistic weighted averaging (IVHULWA) operator, the interval-valued hesitant uncertain linguistic weighted geometric averaging (IVHULWGA) operator. Liu et al. [22] proposed some interval-valued hesitant uncertain linguistic generalized aggregation operators, including the generalized interval-valued hesitant uncertain linguistic weighted averaging (GIVHULWA) operator, the generalized interval-valued hesitant uncertain linguistic ordered weighted averaging (GIVHULOWA) operator and the generalized
interval-valued hesitant uncertain linguistic hybrid averaging (GIVHULHA) operator. In order to better handle the interrelationship between input arguments, Bonferroni [23] originally introduced the Bonferroni Mean (BM) operator. The extensions of the BM operator have been constantly proposed since it was initially introduced. Yager [24][25] and Beliakov et al. [26] proposed some generalized BM operators. Zhu et al.[27] combined BM operator with geometric operator and finally proposed the geometric Bonferroni mean (GBM) operator. In this paper, we will extend the BM operator to the interval-valued neutrosophic hesitant fuzzy uncertain linguistic environment for the sake of possessing the advantages of both BM operator and IVNHFULS, and flexibly handling the indeterminate, incomplete, inconsistent information and effectively considering the interrelationship of the interval-valued neutrosophic hesitant fuzzy uncertain linguistic information.

The remainder of this paper is shown as follows. In section 2, we mainly introduce the basic concepts of the interval-valued neutrosophic hesitant fuzzy uncertain linguistic set (IVNHFULS), and the operational rules and the characteristics of IVNHFULS. In section 3, we propose some interval-valued neutrosophic hesitant fuzzy uncertain linguistic BM aggregation operators, such as the interval-valued neutrosophic hesitant fuzzy uncertain linguistic Bonferroni mean (IVNHFULBM) operator, the interval-valued neutrosophic hesitant fuzzy uncertain linguistic weighted Bonferroni mean (IVNHFULWBM) operator, the interval-valued neutrosophic hesitant fuzzy uncertain linguistic geometric Bonferroni mean (IVNHFULGBM) operator, the interval-valued neutrosophic hesitant fuzzy uncertain linguistic weighted geometric Bonferroni mean (IVNHFULWGBM) operator. In section 4, two multiple attribute decision making methods based on IVNHFULWBM operator and IVNHFULWGBM operator are proposed. In section 5, we give a numerical example to prove the effectiveness of the proposed methods.

2. Preliminaries

2.1 The uncertain linguistic variables

Suppose that \( S = (s_0, s_1, \ldots, s_{l-1}) \) consists of the finite and complete ordered elements, where \( l \) is an odd value. In general, \( l \) is equal to 3, 5, 7, 9, etc. when \( l = 9 \), \( S = (s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8) \) (extremely poor, very poor, poor, slightly poor, fair, slightly good, good, very good, extremely good). Here, \( s_\alpha (\alpha = 1, 2, \ldots, l - 1) \) can be defined as a linguistic variable.

Let \( s_i \) and \( s_j \) are any two elements in linguistic set \( S \), they must meet the following conditions [28][29]:

1. If \( i > j \), then \( s_i > s_j \) (i.e., \( s_i \) is better than \( s_j \));
2. Negative operator: \( \text{neg}(s_i) = s_{l-1-i} \);
3. If \( s_i \geq s_j \), then \( \text{max}(s_i, s_j) = s_i \);
4. If \( s_i \leq s_j \), then \( \text{min}(s_i, s_j) = s_j \).

For any linguistic set \( S = (s_0, s_1, \ldots, s_{l-1}) \), the element \( s_\alpha \) is a strictly monotonically increasing function of its subscript \( \alpha \) [30]. In order to minimize the loss of linguistic information, the discrete linguistic set \( S = (s_0, s_1, \ldots, s_{l-1}) \) is extended to a continuous linguistic set \( \overline{S} = \{s_\theta | \theta \in [0, q] \} \), \( q \) is sufficiently large number. The operational laws are defined as follows:[28][29]
(1) $\beta s_i = s_{\beta i}$  \hspace{1cm} (1) \\
(2) $s_i \oplus s_j = s_{i,j}$  \hspace{1cm} (2) \\
(3) $s_i / s_j = s_{i/j}$  \hspace{1cm} (3) \\
(4) $(s_j)^\lambda = s_j, \ n \geq 0$  \hspace{1cm} (4)

**Definition 1** [31]. Let $\tilde{s} = [s_a, s_b], s_a, s_b \in S$ and $b \geq a \geq 0, s_a, s_b$ are the lower limit and upper limit of $\tilde{s}$, respectively, then, $\tilde{s}$ is called an uncertain linguistic variable.

Let $\tilde{s}_1 = [s_{a1}, s_{b1}], \tilde{s}_2 = [s_{a2}, s_{b2}]$ be any two uncertain linguistic variables, the operation rules are defined as follows[32][33]:

(1) $\tilde{s}_1 \oplus \tilde{s}_2 = [s_{a1}, s_{a2}][s_{b2}, s_{b2}] = [s_{a1+a2}, s_{b1+b2}]$  \hspace{1cm} (5) \\
(2) $\tilde{s}_1 \otimes \tilde{s}_2 = [s_{a1}s_{b1}][s_{a2}s_{b2}] = [s_{a1+a2}, s_{b1+b2}]$  \hspace{1cm} (6) \\
(3) $\tilde{s}_1 / \tilde{s}_2 = [s_{a1}s_{b1}]/[s_{a2}s_{b2}] = [s_{a1/a2}, s_{b1/b2}]$ if $a2 \neq 0, b2 \neq 0$  \hspace{1cm} (7) \\
(4) $\lambda \tilde{s}_1 = \lambda [s_{a1}, s_{b1}] = [s_{a_{\lambda 1}}, s_{a_{\lambda 1}}], \lambda > 0$  \hspace{1cm} (8)

### 2.2 The neutrosophic set

**Definition 2** [34]. Let $X$ be a universe of discourse, with a generic element in $X$ denoted by $x$. A neutrosophic number $A$ in $X$ is

$$A(x) = \langle x | (T_A(x), I_A(x), F_A(x)) \rangle$$  \hspace{1cm} (9)

where, $T_A(x)$ is the truth-membership function, $I_A(x)$ is the indeterminacy-membership function, and $F_A(x)$ is the falsity-membership function. $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $[0^+, 1^-]$. There is no restriction on the sum of $T_A(x), I_A(x)$ and $F_A(x)$, so

$$0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3^-.$$

**Definition 3** [35]. Let $X$ be a universe of discourse, with a generic element in $X$ denoted by $x$. A single valued neutrosophic number $A$ in $X$ is

$$A(x) = \langle x | (T_A(x), I_A(x), F_A(x)) \rangle$$  \hspace{1cm} (10)

where $T_A(x)$ is the truth-membership function, $I_A(x)$ is the indeterminacy-membership function, and $F_A(x)$ is the falsity-membership function. For each point $x$ in $X$, we have $T_A(x), I_A(x), F_A(x) \in [0, 1]$, and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

**Definition 4** [35]. Let $X$ be a universe of discourse, with a generic element in $X$ denoted by $x$. A interval neutrosophic set $A$ in $X$ is

$$A(x) = \langle x | (T_A(x), I_A(x), F_A(x)) \rangle$$  \hspace{1cm} (11)
where, \( T_x(x) \) is the truth-membership function, \( I_x(x) \) is the indeterminacy-membership function, and \( F_x(x) \) is the falsity-membership function. For each point \( x \in X \), we have

\[
T_x(x), I_x(x), F_x(x) \subseteq [0,1], \text{ and } 0 \leq \sup(T_x(x)) + \sup(I_x(x)) + \sup(F_x(x)) \leq 3.
\]

For convenience, we can use \( x = ([I^L, I^U], [T^L, T^U], [F^L, F^U]) \) to represent an element in INS, and can call an interval neutrosophic number (INN).

### 2.3 Interval-valued hesitant fuzzy set

**Definition 5** [36][37]. A hesitant fuzzy set (HFS) \( E \) on \( X \) is defined in terms of a function that when applied to \( X \), which returns a finite subset of \([0,1]\), to be easily understood, the HFS can be expressed by mathematical symbol as follows:

\[
E = \{< x, h(x) > | x \in X \}
\]  

(12)

where \( h(x) \) is a set of some different values in \([0,1]\), representing the possible membership degrees of the element \( x \in X \) to \( E \). From now on, it will be convenient to call \( h(x) \) a hesitant fuzzy element (HFE) and \( E \) the set of all hesitant fuzzy elements (HFEs).

For any three HFEs \( h_1, h_2 \), Torra [22] defined some basic operations shown as follow:

\[
\begin{align*}
(1) & \quad h^c = \bigcup_{\gamma \in h} [1 - \gamma] \\
(2) & \quad h_1 \cup h_2 = \bigcup_{\gamma_1, \gamma_2 \in h_1 \cup h_2} \max\{\gamma_1, \gamma_2\} \\
(3) & \quad h_1 \cap h_2 = \bigcup_{\gamma_1, \gamma_2 \in h_1 \cap h_2} \min\{\gamma_1, \gamma_2\}
\end{align*}
\]  

(13) (14) (15)

After that, Xia and Xu [23] defined four operations about the HFEs \( h, h_1, h_2 \) with a positive scale \( n \):

\[
\begin{align*}
(1) & \quad h^n = \bigcup_{\gamma \in h} [1 - (1 - \gamma)^n] \\
(2) & \quad nh = \bigcup_{\gamma \in h} [1 - (1 - \gamma)^n] \\
(3) & \quad h_1 \oplus h_2 = \bigcup_{\gamma_1, \gamma_2 \in h_1 \oplus h_2} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\} \\
(4) & \quad h_1 \otimes h_2 = \bigcup_{\gamma_1, \gamma_2 \in h_1 \otimes h_2} \{\gamma_1 \gamma_2\}
\end{align*}
\]  

(16) (17) (18) (19)

**Definition 6** [38]. An interval-valued hesitant fuzzy set (IVHFS) \( \tilde{E} \) on \( X \) is defined in terms of a function that when applied to \( X \), which returns a set of finite closed sub-intervals in \([0,1]\), the IVHFS can be expressed by mathematical symbol as follows:

\[
\tilde{E} = \{< x, \tilde{h}(x) > | x \in X \}
\]  

(20)

where \( \tilde{h}(x) \) is a set of some different interval values in \([0,1]\), representing the possible membership degrees of the element \( x \in X \) to \( \tilde{E} \). From now on, it will be convenient to call \( \tilde{h}(x) \) an interval-valued hesitant fuzzy element (IVHFE) and \( \tilde{E} \) the set of all hesitant fuzzy elements.
2.4 Interval-valued neutrosophic hesitant fuzzy uncertain linguistic set

**Definition 7** [39]. Let \( X \) be a fixed set and \( \tilde{S} \) be an uncertain linguistic set, then the hesitant fuzzy uncertain linguistic set (HFULS) on \( X \) can be defined as follows:

\[
A = \{ < x, [s_{\theta(x)}, s_{\tau(x)}], h(x) > \mid x \in X \} \tag{21}
\]

where \( [s_{\theta(x)}, s_{\tau(x)}] \in \tilde{S} \) and \( h(x) \) is a set of some values in \([0,1]\), denoting the possible membership degree of the element \( x \in X \) to the uncertain linguistic variable \( [s_{\theta(x)}, s_{\tau(x)}] \). From now on, it will be convenient to call \( a = < [s_{\theta(a)}, s_{\tau(a)}], h(a) > \) a hesitant fuzzy uncertain linguistic element (HFULE) and \( A \) the set of all HFULEs.

**Definition 8.** Let \( X \) be a fixed set and \( \tilde{S} \) be an uncertain linguistic set, then an interval-valued neutrosophic hesitant fuzzy uncertain linguistic set (IVNHFULS) on \( X \) can be expressed by:

\[
A = \{ < x, [s_{\theta(x)}, s_{\tau(x)}], (\tilde{t}(x), \tilde{i}(x), \tilde{f}(x)) > \mid x \in X \} \tag{22}
\]

where \( [s_{\theta(x)}, s_{\tau(x)}] \in \tilde{S} \), \( \tilde{t}(x) = \bigcup_{\tilde{y} \in \tilde{t}(x)} \{ \tilde{y} \mid \tilde{y} \in \tilde{t}(x) > \}, \tilde{i}(x) = \bigcup_{\tilde{y} \in \tilde{i}(x)} \{ \tilde{y} \mid \tilde{y} \in \tilde{i}(x) \} \) and \( \tilde{f}(x) = \bigcup_{\tilde{y} \in \tilde{f}(x)} \{ \tilde{y} \mid \tilde{y} \in \tilde{f}(x) \} \) are three sets of some values in \([0,1]\), which represents the possible truth-membership degrees, indeterminacy-membership degrees, and falsity-membership degrees of the element \( x \in X \) to the uncertain linguistic variable \( [s_{\theta(x)}, s_{\tau(x)}] \), and satisfies these limits:

\[
\tilde{t} \in [0,1], \quad \tilde{d} \in [0,1], \quad \tilde{n} \in [0,1] \text{ and } 0 \leq \sup \tilde{t}^+ + \sup \tilde{d}^+ + \sup \tilde{n}^+ \leq 3 \tag{23}
\]

where \( \tilde{t}^+ = \bigcup_{\tilde{y} \in \tilde{t}(x)} \max \{ \tilde{y} \} \), \( \tilde{d}^+ = \bigcup_{\tilde{y} \in \tilde{d}(x)} \max \{ \tilde{y} \} \), and \( \tilde{n}^+ = \bigcup_{\tilde{y} \in \tilde{n}(x)} \max \{ \tilde{y} \} \) for \( x \in X \).

Hence, \( a = < [s_{\theta(a)}, s_{\tau(a)}], (\tilde{t}(a), \tilde{i}(a), \tilde{f}(a)) > \) can be described as an interval-valued neutrosophic hesitant fuzzy uncertain linguistic element (IVNHULE). So, \( A \) is the set of all IVNHFULEs.

Let \( \tilde{a} = < [s_{\theta(a)}, s_{\tau(a)}], (\tilde{t}(a), \tilde{i}(a), \tilde{f}(a)) > \) and \( \tilde{b} = < [s_{\theta(b)}, s_{\tau(b)}], (\tilde{t}(b), \tilde{i}(b), \tilde{f}(b)) > \) be any two IVNHFULEs, the operation rules are defined as follows:

\[
\tilde{a} \oplus \tilde{b} = < [s_{\theta(a)+\theta(b)}, s_{\tau(a)+\tau(b)}], \bigcup_{(\tilde{t}(a) \oplus \tilde{t}(b), \tilde{i}(a) \oplus \tilde{i}(b), \tilde{f}(a) \oplus \tilde{f}(b)) \in \tilde{S}} \{ (\tilde{t}(a) \oplus \tilde{t}(b), \tilde{i}(a) \oplus \tilde{i}(b), \tilde{f}(a) \oplus \tilde{f}(b)) \} \text{ and } \frac{\tilde{a} \oplus \tilde{b} - \tilde{b} \oplus \tilde{a}}{2} \text{ for } x \in X \tag{24}
\]

\[
\tilde{a} \odot \tilde{b} = < [s_{\theta(a) \odot \theta(b)}, s_{\tau(a) \odot \tau(b)}], \bigcup_{(\tilde{t}(a) \odot \tilde{t}(b), \tilde{i}(a) \odot \tilde{i}(b), \tilde{f}(a) \odot \tilde{f}(b)) \in \tilde{S}} \{ (\tilde{t}(a) \odot \tilde{t}(b), \tilde{i}(a) \odot \tilde{i}(b), \tilde{f}(a) \odot \tilde{f}(b)) \} \text{ and } \frac{\tilde{a} \odot \tilde{b} - \tilde{b} \odot \tilde{a}}{2} \text{ for } x \in X \tag{25}
\]
\[
\lambda \tilde{a} = \{s_{\lambda(\varnothing)}, s_{\lambda(\varnothing)}\}, \bigcup_{\bar{\varnothing}(\bar{\varnothing}) \Rightarrow (\bar{\varnothing}, \bar{\varnothing}) \land (\bar{\varnothing}) \Rightarrow (\bar{\varnothing})} \left((1-(1-\tilde{\tau}_a)^2 \cdot 1-(1-\tilde{\tau}_a)^2), \left[(\tilde{\tau}_a^L, \tilde{\tau}_a^U) \cdot \left((\tilde{\tau}_a^L, \tilde{\tau}_a^U) \cdot (\tilde{\tau}_a^L, \tilde{\tau}_a^U)\right)\right] \right)
\]

\[
\tilde{\lambda} \tilde{a} = \{s_{\lambda(\varnothing)}, s_{\lambda(\varnothing)}\}, \bigcup_{\bar{\varnothing}(\bar{\varnothing}) \Rightarrow (\bar{\varnothing}, \bar{\varnothing}) \land (\bar{\varnothing}) \Rightarrow (\bar{\varnothing})} \left((1-(1-\tilde{\tau}_a)^2 \cdot 1-(1-\tilde{\tau}_a)^2), \left[(\tilde{\tau}_a^L, \tilde{\tau}_a^U) \cdot \left((\tilde{\tau}_a^L, \tilde{\tau}_a^U) \cdot (\tilde{\tau}_a^L, \tilde{\tau}_a^U)\right)\right] \right)
\]

Let \( \tilde{a} = [s_{\lambda(\varnothing)}, s_{\lambda(\varnothing)}], (\tilde{i}(\tilde{a}), \tilde{j}(\tilde{a})) \), then we have:

(1) \( \tilde{a} \oplus \tilde{b} = \tilde{b} \oplus \tilde{a} \)

(2) \( \tilde{a} \odot \tilde{b} = \tilde{b} \odot \tilde{a} \)

(3) \( \lambda(\tilde{a} \oplus \tilde{b}) = \lambda\tilde{a} \oplus \lambda\tilde{b} \)

(4) \( \lambda_1 \tilde{a} \oplus \lambda_2 \tilde{a} = (\lambda_1 + \lambda_2)\tilde{a} \)

(5) \( \tilde{a} \oplus \tilde{b} = (\tilde{a} \odot \tilde{b})^2 \)

(6) \( \tilde{a} \odot \tilde{a} = \tilde{a}^{\lambda_1 + \lambda_2} \)

**Proof.**

(1) The formula (27) is obviously right according to the operational rule (1) expressed by (23)

(2) The formula (28) is obviously right according to the operational rule (2) expressed by (24)

(3) For the left of the formula (29), we have

\[
\lambda(\tilde{a} \oplus \tilde{b}) = \lambda <s_{\lambda(\varnothing), \lambda(\varnothing)}), s_{\lambda(\varnothing), \lambda(\varnothing)}>, \bigcup_{\bar{\varnothing}(\bar{\varnothing}) \Rightarrow (\bar{\varnothing}, \bar{\varnothing}) \land (\bar{\varnothing}) \Rightarrow (\bar{\varnothing})} \left((1-(1-\tilde{\tau}_a)^2 \cdot 1-(1-\tilde{\tau}_a)^2), \left[(\tilde{\tau}_a^L, \tilde{\tau}_a^U) \cdot \left((\tilde{\tau}_a^L, \tilde{\tau}_a^U) \cdot (\tilde{\tau}_a^L, \tilde{\tau}_a^U)\right)\right] \right)
\]

and for the right of the formula (29), we have

\[
= <s_{\lambda(\varnothing), \lambda(\varnothing)}), s_{\lambda(\varnothing), \lambda(\varnothing)}>, \bigcup_{\bar{\varnothing}(\bar{\varnothing}) \Rightarrow (\bar{\varnothing}, \bar{\varnothing}) \land (\bar{\varnothing}) \Rightarrow (\bar{\varnothing})} \left((1-(1-\tilde{\tau}_a)^2 \cdot 1-(1-\tilde{\tau}_a)^2), \left[(\tilde{\tau}_a^L, \tilde{\tau}_a^U) \cdot \left((\tilde{\tau}_a^L, \tilde{\tau}_a^U) \cdot (\tilde{\tau}_a^L, \tilde{\tau}_a^U)\right)\right] \right)
\]
So, we get $\lambda(\tilde{a} \oplus \tilde{b}) = \lambda \tilde{a} \oplus \lambda \tilde{b}$, which completes the proof of formula (29).

(4) For formula (30), we have

$$\lambda \tilde{a} \oplus \lambda \tilde{a} = \langle s_{\lambda \tilde{a}}, s_{\lambda \tilde{a}}(\tilde{z}) \rangle \cup \langle (1 - (1 - \tilde{v}_{\tilde{a}}^{L}) \lambda, 1 - (1 - \tilde{v}_{\tilde{a}}^{U}) \lambda), \rangle \cup \langle (\tilde{v}_{\tilde{a}}^{L})^{\lambda}, (\tilde{v}_{\tilde{a}}^{U})^{\lambda} \rangle + \langle (1 - (1 - \tilde{w}_{\tilde{a}}^{L}) \lambda, 1 - (1 - \tilde{w}_{\tilde{a}}^{U}) \lambda) \rangle.$$ 

(5) For the left of the formula (31), we have

$$\tilde{a} \oplus \tilde{b} = \langle s_{\tilde{a}}, s_{\tilde{b}}(\tilde{z}) \rangle \cup \langle (1 - (1 - \tilde{v}_{\tilde{a}}^{L}) \lambda, 1 - (1 - \tilde{v}_{\tilde{b}}^{L}) \lambda) \rangle \cup \langle (\tilde{v}_{\tilde{a}}^{L})^{\lambda}, (\tilde{v}_{\tilde{b}}^{L})^{\lambda} \rangle + \langle (1 - (1 - \tilde{w}_{\tilde{a}}^{L}) \lambda, 1 - (1 - \tilde{w}_{\tilde{b}}^{L}) \lambda) \rangle.$$
can be hesitant fuzzy uncertain linguistic element (IVNHFULE), then the score function 

$$\text{Definition 9}$$

So, we can get the formula (31) is right.

(6) For formula (32), we have

$$a^{\lambda_1} \otimes a^{\lambda_2} = <s_{\theta(\bar{a})} \times s_{\tau(\bar{a})}, \lambda_1, \lambda_2>$$

So, we can get the formula (32) is right.

$$\text{Definition 9} \ [9,15]. \ \text{Let} \ \bar{a} = <s_{\theta(\bar{a})}, s_{\tau(\bar{a})}, \tilde{\tau}(\bar{a}), \tilde{\tau}(\bar{a}), \tilde{f}(\bar{a})> \ \text{be an interval-valued neutrosophic hesitant fuzzy uncertain linguistic element (IVNHFULE), then the score function} \ S(\bar{a}) \ \text{of} \ \bar{a} \ \text{can be defined as follows:}$$

$$S(\bar{a}) = \frac{\frac{1}{# \tilde{\tau}} \sum_{\tilde{\tau}(\bar{a}) \in \tilde{\tau}(\bar{a})} (R_{\bar{a}}^L + R_{\bar{a}}^U)}{2} + \frac{1}{# \tilde{\tau}} \sum_{\tilde{\tau}(\bar{a}) \in \tilde{\tau}(\bar{a})} \left( \frac{(1 - \tilde{\eta}_{\tilde{\tau}}^L) + (1 - \tilde{\eta}_{\tilde{\tau}}^U)}{2} \right) + \frac{1}{# \tilde{f}} \sum_{\tilde{f}(\bar{a}) \in \tilde{f}(\bar{a})} \left( \frac{(1 - \tilde{\eta}_{\tilde{f}}^L) + (1 - \tilde{\eta}_{\tilde{f}}^U)}{2} \right)$$

$$= s_{\theta(\bar{a})} \times s_{\tau(\bar{a})} \ \text{and} \ \tilde{\tau}(\bar{a}) \ \text{are any two IVNHFULEs, the comparison method of IVNHFULEs is expressed as follows:}$$

$$(1) \text{If} \ S(\bar{a}) > S(\tilde{\bar{b}}), \text{then} \ \bar{a} > \tilde{\bar{b}}$$
(2) If \( S(\bar{a}) < S(\tilde{b}) \), then \( \bar{a} < \tilde{b} \)

(3) If \( S(\bar{a}) = S(\tilde{b}) \), then \( \bar{a} = \tilde{b} \)

2.4 The Bonferroni mean (BM) operator

**Definition 10** [40]. Let \( a_i (i = 1, 2, ..., n) \) be a set of non-negative numbers, and \( p, q \geq 0 \), the Bonferroni mean operator is defined as follows:

\[
BM^{p,q}(a_1, a_2, ..., a_n) = \left( \frac{1}{n(n-1)} \sum_{i \neq j} a_i^p a_j^q \right)^{\frac{1}{p+q}}
\]

(34)

In real decision making, the weight of input arguments plays an important part in decision process. But the BM operator doesn’t consider the importance of input argument itself. Hence, Zhou [41] developed a weighted Bonferroni mean (WBM) operator.

**Definition 11** [41]. Let \( a_i (i = 1, 2, ..., n) \) be a set of non-negative numbers, and \( p, q \geq 0 \). The weighted Bonferroni mean operator is defined as follows:

\[
WBM^{p,q}(a_1, a_2, ..., a_n) = \left( \sum_{i \neq j} \frac{w_i w_j}{1-w_i} a_i^p a_j^q \right)^{\frac{1}{p+q}}
\]

(35)

where \( W = (w_1, w_2, ..., w_n)^T \) is the weight vector of \( a_i (i = 1, 2, ..., n) \), satisfying \( 0 \leq w_i \leq 1 \) and \( \sum_{i=1}^{n} w_i = 1 \).

The WBM operator has four desirable properties as follows:

**Theorem 1** [41] (Reducibility).

Let \( W = \left( \frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n} \right)^T \) be the weight vector of \( a_i (i = 1, 2, ..., n) \), then

\[
WBM^{p,q}(a_1, a_2, ..., a_n) = \left( \frac{1}{n(n-1)} \sum_{i \neq j} a_i^p a_j^q \right)^{\frac{1}{p+q}} = BM^{p,q}(a_1, a_2, ..., a_n).
\]

**Theorem 2** [41] (Idempotency).

Let \( a_j = a(j = 1, 2, ..., n) \), then \( WBM^{p,q}(a_1, a_2, ..., a_n) = a \).

**Theorem 3** [41] (Permutation).

Let \( (a_1, a_2, ..., a_n) \) be any permutation of \( (a_1', a_2', ..., a_n') \), then

\[
WBM^{p,q}(a_1', a_2', ..., a_n') = WBM^{p,q}(a_1, a_2, ..., a_n).
\]

**Theorem 4** [41] (Monotonicity).

If \( a_j \geq b_j (j = 1, 2, ..., n) \), then \( WBM^{p,q}(a_1, a_2, ..., a_n) \geq WBM^{p,q}(b_1, b_2, ..., b_n) \).

**Theorem 5** [41] (Boundedness).
The \( WBM^{p,q} \) operator lies between the max and min operators, i.e.,

\[
\min(a_1, a_2, ..., a_n) \leq WBM^{p,q}(a_1, a_2, ..., a_n) \leq \max(a_1, a_2, ..., a_n).
\]

When the WBM operator select different values to the parameters \( p \) and \( q \), some special cases of \( WBM \) can be obtained as follows:

1. If \( q = 0 \), then \( WBM^{p,0}(a_1, a_2, ..., a_n) = \left( \sum_{i=1}^{n} w_i a_i^p \right)^{\frac{1}{p}}. \)
2. If \( p = 1 \) and \( q = 0 \), then \( WBM^{1,0}(a_1, a_2, ..., a_n) = \sum_{i=1}^{n} w_i a_i. \)
3. If \( p = \frac{1}{2} \) and \( q = \frac{1}{2} \), then \( WBM^{\frac{1}{2},\frac{1}{2}}(a_1, a_2, ..., a_n) = \sum_{i,j=1}^{n} \frac{w_i w_j}{1 - w_i} (a_i a_j)^{\frac{1}{2}}. \)
4. If \( p = 1 \) and \( q = 1 \), then \( WBM^{1,1}(a_1, a_2, ..., a_n) = \left( \sum_{i,j=1}^{n} w_i w_j (a_i a_j)^\frac{1}{1} \right)^{\frac{1}{2}}. \)

### 2.5 The geometric Bonferroni mean (GBM) operator

**Definition 12** [42]. Let \( a_i(i=1,2,\ldots,n) \) be a set of non-negative numbers, and \( p,q \geq 0 \), the geometric Bonferroni mean operator is defined as follows:

\[
GBM^{p,q}(a_1, a_2, ..., a_n) = \frac{1}{p+q} \prod_{i,j=1}^{n} (p a_i + q a_j)^{\frac{1}{p+q}}.
\]

Similar to the BM operator, the GBN operator also doesn’t consider the weights of the input arguments. The weighted geometric Bonferroni mean (WGBM) operator was proposed by Sun and Liu [43].

**Definition 13** [43]. Let \( a_i(i=1,2,\ldots,n) \) be a set of non-negative numbers, and \( p,q \geq 0 \). The weighted geometric Bonferroni mean operator is defined as follows:

\[
WGBM^{p,q}(a_1, a_2, ..., a_n) = \frac{1}{p+q} \prod_{i,j=1}^{n} \left( p a_i + q a_j \right)^{\frac{w_i w_j}{1 - w_i}}.
\]

where \( W = (w_1, w_2, ..., w_n)^T \) is the weight vector of \( a_i(i=1,2,\ldots,n) \) satisfying \( 0 \leq w_i \leq 1 \) and \( \sum_{i=1}^{n} w_i = 1. \)

It is easy to demonstrate the WGBM operator has some properties as follows:

**Theorem 6** [43] (Reducibility). Let \( W = \left( \frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n} \right)^T \) be the weight vector of \( a_i(i=1,2,\ldots,n) \) then

\[
WGBM^{p,q}(a_1, a_2, ..., a_n) = GBM^{p,q}(a_1, a_2, ..., a_n).
\]

**Theorem 7** [43] (Idempotency).
Let \( a_j = a(j = 1, 2, ..., n) \), then \( WGBM^{p,q}(a_1, a_2, ..., a_n) = a \).

**Theorem 8 [43] (Permutation).**
Let \( (a_1, a_2, ..., a_n) \) be any permutation of \( (a_1', a_2', ..., a_n') \), then
\[
WGBM^{p,q}(a_1', a_2', ..., a_n') = WGBM^{p,q}(a_1, a_2, ..., a_n).
\]

**Theorem 9 [43] (Monotonicity).**
If \( a_j \geq b_j (j = 1, 2, ..., n) \), then \( WGBM^{p,q}(a_1, a_2, ..., a_n) \geq WGBM^{p,q}(b_1, b_2, ..., b_n) \).

**Theorem 10 [43] (Boundedness).**
The \( WGBM^{p,q} \) operator lies between the max and min operators, i.e.,
\[
\min(a_1, a_2, ..., a_n) \leq WGBM^{p,q}(a_1, a_2, ..., a_n) \leq \max(a_1, a_2, ..., a_n)
\]

When the WGBM operator select different parameters \( p \) and \( q \), some special cases of WGBM operator can be obtained as follows:

1. If \( q = 0 \), then \( WGBM^{p,0}(a_1, a_2, ..., a_n) = \frac{1}{p} \prod_{i=1}^{n} (p a_i)^{w_i} \)

2. If \( q = 0, p = 1 \) then \( WGBM^{1,0}(a_1, a_2, ..., a_n) = \prod_{i=1}^{n} a_i^{w_i} \)

3. If \( p = \frac{1}{2} \) and \( q = \frac{1}{2} \), then \( WGBM^{\frac{1}{2},\frac{1}{2}}(a_1, a_2, ..., a_n) = \prod_{i,j=1 \atop i \neq j}^{n} \left( \frac{1}{2} a_i + \frac{1}{2} a_j \right)^{w_i w_j} \)

4. If \( p = 1 \) and \( q = 1 \), then \( WGBM^{1,1}(a_1, a_2, ..., a_n) = \prod_{i,j=1 \atop i \neq j}^{n} a_i a_j^{w_i w_j} \)

3. Interval-valued neutrosophic hesitant fuzzy uncertain linguistic

**Bonferroni mean aggregation operators**

Interval-valued neutrosophic hesitant fuzzy uncertain linguistic set (IVNHFULS) is an important tool to describe the decision maker’s preference and to handle the hesitant, uncertain, incomplete information. Bonferroni mean (BM) can replace the simple averaging of other mean type operators because of the excellent modeling capability.

In this section, we combine IVNHFULS with BM operator and propose some interval-valued neutrosophic hesitant fuzzy uncertain linguistic Bonferroni mean aggregation operators.

3.1 The interval-valued neutrosophic hesitant fuzzy uncertain linguistic Bonferroni mean operator

**Definition 14.** Let \( \tilde{a}_i = <s_{\tilde{a}_i}, \delta_{\tilde{a}_i}, \alpha_{\tilde{a}_i}, \tilde{r}_{\tilde{a}_i}, \tilde{f}_{\tilde{a}_i}, \tilde{\tilde{r}}_{\tilde{a}_i}, \tilde{\tilde{f}}_{\tilde{a}_i}> (i = 1, 2, ..., n) \) be the set of all IVNHFULEs, and \( p, q \geq 0 \), then the interval-valued neutrosophic hesitant fuzzy uncertain
linguistic Bonferroni mean operator can be defined as follows:

\[
IVNHFULBM^{p,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left\langle \frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^n \tilde{a}_i^p \otimes \tilde{a}_j^q \right\rangle^{1/p+q}
\]

(38)

**Theorem 11.** Let \( \tilde{a}_i = [\{s_{\theta(\tilde{a}_i)}, s_{\tau(\tilde{a}_i)}\}, (\tilde{f}(\tilde{a}_i), \tilde{t}(\tilde{a}_i), \tilde{f}(\tilde{a}_i)) > (i = 1, 2, \ldots, n) \) be the set of IVNFHUEs, and \( p, q \geq 0 \), then, the result aggregated from Definition 14 is still an IVNFULE, and even

\[
IVNHFULBM^{p,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left\langle \frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^n \tilde{a}_i^p \otimes \tilde{a}_j^q \right\rangle^{1/p+q}
\]

(39)

**Proof.**

By the operational rules of the interval-valued neutrosophic hesitant fuzzy uncertain linguistic variables, we have

\[
\tilde{a}_i^p = [s_{\theta(\tilde{a}_i)}^p, s_{\tau(\tilde{a}_i)}^p], \quad \bigcup_{\tilde{f}(\tilde{a}_i) \neq \tilde{t}(\tilde{a}_i)} \{t_{\tilde{a}_i, \tilde{a}_i}^p, t_{\tilde{a}_i, \tilde{a}_i}^p\} >
\]

\[
\tilde{a}_j^q = [s_{\theta(\tilde{a}_j)}^q, s_{\tau(\tilde{a}_j)}^q], \quad \bigcup_{\tilde{f}(\tilde{a}_j) \neq \tilde{t}(\tilde{a}_j)} \{t_{\tilde{a}_j, \tilde{a}_j}^q, t_{\tilde{a}_j, \tilde{a}_j}^q\} >
\]

\[
\tilde{a}_i^p \otimes \tilde{a}_j^q = [s_{\theta(\tilde{a}_i, \tilde{a}_j)}^p, s_{\tau(\tilde{a}_i, \tilde{a}_j)}^p], \quad \bigcup_{\tilde{f}(\tilde{a}_i, \tilde{a}_j) \neq \tilde{t}(\tilde{a}_i, \tilde{a}_j)} \{t_{\tilde{a}_i, \tilde{a}_j}^q, t_{\tilde{a}_i, \tilde{a}_j}^q\} >
\]

then

13
\[
\frac{1}{n(n-1)} \bar{\alpha}_i^p \otimes \bar{\alpha}_j^q = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{n(n-1)} \phi(\bar{\alpha}_i^p \otimes \bar{\alpha}_j^q) \right\}^{\frac{1}{p+q}}
\]

\[
= \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{n(n-1)} \phi(\bar{\alpha}_i^p \otimes \bar{\alpha}_j^q) \right\}^{\frac{1}{p+q}}
\]

\[
= \frac{1}{n(n-1)} \| \bar{\alpha}_i^p \otimes \bar{\alpha}_j^q \|^p \quad \text{and} \quad \| \bar{\alpha}_i^p \otimes \bar{\alpha}_j^q \|^q = \| \bar{\alpha}_i^p \otimes \bar{\alpha}_j^q \|^p \cdot \| \bar{\alpha}_i^p \otimes \bar{\alpha}_j^q \|^q
\]

\[
\bigcup \tilde{\mathcal{F}}(\bar{\alpha}_i) \in \tilde{\mathcal{F}}(\bar{\alpha}_1), \tilde{\mathcal{F}}(\bar{\alpha}_2), \ldots, \tilde{\mathcal{F}}(\bar{\alpha}_n)
\]

\[
\left\{ 1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left( 1 - \left( \bar{\alpha}_i^p \otimes \bar{\alpha}_j^q \right) \right)^{\frac{1}{n(n-1)}} \right\}
\]

\[
\bigcup \tilde{\mathcal{S}}(\bar{\alpha}_i) \in \tilde{\mathcal{S}}(\bar{\alpha}_1), \tilde{\mathcal{S}}(\bar{\alpha}_2), \ldots, \tilde{\mathcal{S}}(\bar{\alpha}_n)
\]

\[
\left\{ \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left( 1 - \left( \bar{\alpha}_i^p \otimes \bar{\alpha}_j^q \right) \right)^{\frac{1}{n(n-1)}} \right\}
\]

Then

\[
\left\{ \frac{1}{n(n-1)} \| \bar{\alpha}_i^p \otimes \bar{\alpha}_j^q \|^p \right\}^{\frac{1}{p+q}} = \left\{ \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{n(n-1)} \phi(\bar{\alpha}_i^p \otimes \bar{\alpha}_j^q) \right\}^{\frac{1}{p+q}} \right\}
\]

\[
\bigcup \tilde{\mathcal{F}}(\bar{\alpha}_i) \in \tilde{\mathcal{F}}(\bar{\alpha}_1), \tilde{\mathcal{F}}(\bar{\alpha}_2), \ldots, \tilde{\mathcal{F}}(\bar{\alpha}_n)
\]

\[
\left\{ 1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left( 1 - \left( \bar{\alpha}_i^p \otimes \bar{\alpha}_j^q \right) \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{p+q}}
\]
Proof.

Theorem 12 (Idempotency).

Let \( \tilde{a}_j = \{ S_{\theta(a)}, s_{\tau(a)}, \{ \{ r^L, r^U \}, \{ \delta^L, \delta^U \}, \{ \eta^L, \eta^U \} \} = \tilde{a} \) \quad (j = 1, 2, \ldots, n),

Then we can get

\[
IVNHFULBM^{P,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{a}
\]

Proof.

Since \( \tilde{a}_j = \{ S_{\theta(a)}, s_{\tau(a)}, \{ \{ r^L, r^U \}, \{ \delta^L, \delta^U \}, \{ \eta^L, \eta^U \} \} = \tilde{a} \) \quad (j = 1, 2, \ldots, n),

we can get

\[
IVNHFULBM^{P,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left( \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \tilde{a}_i \otimes \tilde{a}_j \right)^{1/P+q}
\]

which completes the proof of theorem 11.

Moreover, the IVNHFULBM operator also has the following properties:

**Theorem 13 (Commutativity).**

Let \( (\tilde{a}_1', \tilde{a}_2', \ldots, \tilde{a}_n') \) be any permutation of \( (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \), then we can get

\[
IVNHFULBM^{P,q}(\tilde{a}_1', \tilde{a}_2', \ldots, \tilde{a}_n') = IVNHFULBM^{P,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n).
\]

Proof.
\[ IVNHFULBM^{p,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left( \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \mathbf{\tilde{a}}_i^p \otimes \mathbf{\tilde{a}}_j^q \right)^{\frac{1}{p+q}} \]

\[ \left\{ \begin{array}{l}
\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \theta(\tilde{a}_i)^p \theta(\tilde{a}_j)^q = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \theta(\tilde{a}_i)^p \theta(\tilde{a}_j)^q \\
\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \tau(\tilde{a}_i)^p \tau(\tilde{a}_j)^q = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \tau(\tilde{a}_i)^p \tau(\tilde{a}_j)^q 
\end{array} \right\}^{\frac{1}{p+q}} \]

Since \( \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n \) is any permutation of \( \{a_1, a_2, \ldots, a_n\} \), then we have

\[ \sum_{i=1}^{n} \theta(\tilde{a}_i) = \sum_{i=1}^{n} \theta(a_i), \quad \sum_{i=1}^{n} \theta(\tilde{a}_i)^q = \sum_{i=1}^{n} \theta(a_i)^q, \]

\[ \sum_{i=1}^{n} \tau(\tilde{a}_i) = \sum_{i=1}^{n} \tau(a_i), \quad \sum_{i=1}^{n} \tau(\tilde{a}_i)^q = \sum_{i=1}^{n} \tau(a_i)^q. \]

so, we can get

\[ \left( \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \theta(\tilde{a}_i)^p \theta(\tilde{a}_j)^q \right)^{\frac{1}{p+q}} = \left( \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \theta(a_i)^p \theta(a_j)^q \right)^{\frac{1}{p+q}} \]

\[ = \left( \frac{1}{n(n-1)} \sum_{i=1}^{n} \theta(a_i)^p \sum_{j=1, j \neq i}^{n} \theta(a_j)^q \right)^{\frac{1}{p+q}} \]

and

\[ \left( \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \tau(\tilde{a}_i)^p \tau(\tilde{a}_j)^q \right)^{\frac{1}{p+q}} = \left( \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \tau(a_i)^p \tau(a_j)^q \right)^{\frac{1}{p+q}} \]

\[ = \left( \frac{1}{n(n-1)} \sum_{i=1}^{n} \tau(a_i)^p \sum_{j=1, j \neq i}^{n} \tau(a_j)^q \right)^{\frac{1}{p+q}} \]
\[
\prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} (1 - (r_{a_i}^{-1})^p (r_{a_j}^{-1})^q)^{\frac{1}{n(n-1)}} = \left( \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} (1 - (r_{a_i}^{-1})^p (r_{a_j}^{-1})^q)^{\frac{1}{n(n-1)}} \right)^{n-1} \\
= \left( \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} (1 - (r_{a_i}^{-1})^p (r_{a_j}^{-1})^q)^{\frac{1}{n(n-1)}} \right)^{n-1} = \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} (1 - (r_{a_i}^{-1})^p (r_{a_j}^{-1})^q)^{\frac{1}{n(n-1)}}
\]

Similarly,
\[
\prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} (1 - (r_{a_i}^{-1})^p (r_{a_j}^{-1})^q)^{\frac{1}{n(n-1)}} = \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} (1 - (r_{a_i}^{-1})^p (r_{a_j}^{-1})^q)^{\frac{1}{n(n-1)}}
\]

So, according to the theorem 13, we can get
\[
IVNHFULBM^{p,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = IVNHFULBM^{p,q}(\tilde{a}_1', \tilde{a}_2', \ldots, \tilde{a}_n').
\]

Theorem 14 (Monotonicity).

Suppose \((\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)\) and \((\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n)\) are any two sets of IVNHFULEs. If \(s_{\theta}((\tilde{a})) \leq s_{\theta}((\tilde{b}))\),
\[
s_{\tau}((\tilde{a})) \leq s_{\tau}((\tilde{b})) \leq s_{\theta}((\tilde{b}))\),
\]
then
\[
IVNHFULBM^{p,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq IVNHFULBM^{p,q}(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n).
\]

Proof.

(1) Since \(s_{\theta}((\tilde{a})) \leq s_{\theta}((\tilde{b}))\), then \(\theta((\tilde{a})) \leq \theta((\tilde{b}))\), according to theorem 11, we have
\[
\left( \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{1}{n(n-1)} \theta((\tilde{a}))^p \times \theta((\tilde{a}))^q \right)^{\frac{1}{p+q}} \leq \left( \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{1}{n(n-1)} \theta((\tilde{b}))^p \times \theta((\tilde{b}))^q \right)^{\frac{1}{p+q}}.
\]

(2) Similar to (1), we have
\[
\left( \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{1}{n(n-1)} \tau((\tilde{a}))^p \times \tau((\tilde{a}))^q \right)^{\frac{1}{p+q}} \leq \left( \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{1}{n(n-1)} \tau((\tilde{b}))^p \times \tau((\tilde{b}))^q \right)^{\frac{1}{p+q}}.
\]

(3) Since \(r_{a_i}^{-1} \leq r_{b_i}^{-1}\) for all \(i\) and \(p, q \geq 0\), then \((r_{a_i}^{-1})^p \leq (r_{b_i}^{-1})^p\).
So, similar to (3), we have

\[
(r_{a_i}^{-L})^p (r_{a_j}^{-L})^q \leq (r_{b_i}^{-L})^p (r_{b_j}^{-L})^q, 1 - (r_{a_i}^{-L})^p (r_{a_j}^{-L})^q \geq 1 - (r_{b_i}^{-L})^p (r_{b_j}^{-L})^q,
\]

\[
\prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - (r_{a_i}^{-L})^p (r_{a_j}^{-L})^q\right)^{\frac{1}{m(n-1)}} \geq \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - (r_{b_i}^{-L})^p (r_{b_j}^{-L})^q\right)^{\frac{1}{m(n-1)}}
\]

and

\[
1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - (r_{a_i}^{-L})^p (r_{a_j}^{-L})^q\right)^{\frac{1}{m(n-1)}} \leq 1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - (r_{b_i}^{-L})^p (r_{b_j}^{-L})^q\right)^{\frac{1}{m(n-1)}}
\]

So

\[
\left(1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - (r_{a_i}^{-L})^p (r_{a_j}^{-L})^q\right)^{\frac{1}{m(n-1)}}\right)^{\frac{1}{pq}} \leq \left(1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - (r_{b_i}^{-L})^p (r_{b_j}^{-L})^q\right)^{\frac{1}{m(n-1)}}\right)^{\frac{1}{pq}}
\]

Similar to (3), we have

\[
\left(1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - (r_{a_i}^{-L})^p (r_{a_j}^{-L})^q\right)^{\frac{1}{m(n-1)}}\right)^{\frac{1}{pq}} \leq \left(1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - (r_{b_i}^{-L})^p (r_{b_j}^{-L})^q\right)^{\frac{1}{m(n-1)}}\right)^{\frac{1}{pq}}
\]

(4) since \( \delta_{a_i}^{-L} \geq \delta_{b_i}^{-L} \) for all \( i \) and \( p, q \geq 0 \), then \( 1 - (1 - \delta_{a_i}^{-L})^p \leq (1 - \delta_{b_i}^{-L})^p \),

\[
(1 - \delta_{a_i}^{-L})^p (1 - \delta_{a_j}^{-L})^q \leq (1 - \delta_{b_i}^{-L})^p (1 - \delta_{b_j}^{-L})^q, 1 - (1 - \delta_{a_i}^{-L})^p (1 - \delta_{a_j}^{-L})^q \geq 1 - (1 - \delta_{b_i}^{-L})^p (1 - \delta_{b_j}^{-L})^q
\]

\[
\left(1 - (1 - \delta_{a_i}^{-L})^p (1 - \delta_{a_j}^{-L})^q\right)^{\frac{1}{m(n+1)}} \geq \left(1 - (1 - \delta_{b_i}^{-L})^p (1 - \delta_{b_j}^{-L})^q\right)^{\frac{1}{n(n+1)}}
\]

And

\[
\prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - (1 - \delta_{a_i}^{-L})^p (1 - \delta_{a_j}^{-L})^q\right)^{\frac{1}{m(n-1)}} \geq \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - (1 - \delta_{b_i}^{-L})^p (1 - \delta_{b_j}^{-L})^q\right)^{\frac{1}{m(n-1)}}
\]

Then

\[
\left(1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - (1 - \delta_{a_i}^{-L})^p (1 - \delta_{a_j}^{-L})^q\right)^{\frac{1}{m(n-1)}}\right)^{\frac{1}{pq}} \leq \left(1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - (1 - \delta_{b_i}^{-L})^p (1 - \delta_{b_j}^{-L})^q\right)^{\frac{1}{m(n-1)}}\right)^{\frac{1}{pq}}
\]

So

\[
1 - \left(1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - (1 - \delta_{a_i}^{-L})^p (1 - \delta_{a_j}^{-L})^q\right)^{\frac{1}{m(n-1)}}\right)^{\frac{1}{pq}} \geq 1 - \left(1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - (1 - \delta_{b_i}^{-L})^p (1 - \delta_{b_j}^{-L})^q\right)^{\frac{1}{m(n-1)}}\right)^{\frac{1}{pq}}
\]
Similarly,

$$1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left( 1 - (1 - \delta_{a_i})^p (1 - \delta_{a_j})^q \right)^{\frac{1}{m(n-1)}} \geq 1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left( 1 - (1 - \delta_{a_i})^p (1 - \delta_{a_j})^q \right)^{\frac{1}{m(n-1)}}$$

(5) Similar to (4), we can know

$$1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left( 1 - (1 - \eta_{a_i})^p (1 - \eta_{a_j})^q \right)^{\frac{1}{m(n-1)}} \geq 1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left( 1 - (1 - \eta_{a_i})^p (1 - \eta_{a_j})^q \right)^{\frac{1}{m(n-1)}}$$

$$1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left( 1 - (1 - \eta_{a_i})^p (1 - \eta_{a_j})^q \right)^{\frac{1}{m(n-1)}} \geq 1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left( 1 - (1 - \eta_{a_i})^p (1 - \eta_{a_j})^q \right)^{\frac{1}{m(n-1)}}$$

So, we can get $IVNHFULBM_{\theta}(\vec{a}_1, \vec{a}_2, ..., \vec{a}_n) \leq IVNHFULBM_{\theta}(\vec{b}_1, \vec{b}_2, ..., \vec{b}_n)$, which completes the proof of the theorem 14.

**Theorem 15 (Boundedness).**

Let $\vec{a}_i (i = 1, 2, ..., n)$ be a set of IVNHFULEs. If $s_{\theta-} = \min_{i \in S_{\theta}} \{ s_{\theta(\vec{a}_i)} \}$, $s_{\theta+} = \max_{i \in S_{\theta}} \{ s_{\theta(\vec{a}_i)} \}$, $r_{L-} = \min_{i \in S_{\theta}} \{ r_{L_i} \}$, $r_{U-} = \min_{i \in S_{\theta}} \{ r_{U_i} \}$, $r_{L+} = \max_{i \in S_{\theta}} \{ r_{L_i} \}$, $r_{U+} = \max_{i \in S_{\theta}} \{ r_{U_i} \}$,

$$s_{\theta+} = \max_{i \in S_{\theta}} \{ s_{\theta(\vec{a}_i)} \}, s_{\theta+} = \max_{i \in S_{\theta}} \{ s_{\theta(\vec{a}_i)} \}, r_{L-} = \min_{i \in S_{\theta}} \{ r_{L_i} \}$$

Then

$$\vec{a}_- = (s_{\theta-}, \vec{r}_{L-}, (\delta_{L-}, \delta_{U-}), (\eta_{L-}, \eta_{U-}))$$

$$\vec{a}_+ = (s_{\theta+}, \vec{r}_{U+}, (\delta_{L+}, \delta_{U+}), (\eta_{L+}, \eta_{U+}))$$

for all $i = 1, 2, ..., n$, where

$$\vec{r}_{L-} = \left( \begin{array}{c} r_{L_1} \\ \vdots \\ r_{L_n} \end{array} \right), \vec{r}_{U-} = \left( \begin{array}{c} r_{U_1} \\ \vdots \\ r_{U_n} \end{array} \right)$$

and

$$\vec{r}_{L+} = \left( \begin{array}{c} r_{L_1} \\ \vdots \\ r_{L_n} \end{array} \right), \vec{r}_{U+} = \left( \begin{array}{c} r_{U_1} \\ \vdots \\ r_{U_n} \end{array} \right)$$

for all $i = 1, 2, ..., n$.
Proof.

Based on above terms, we can get

\( \theta_{\cdot}^{p+q} \leq \theta(\tilde{a}_i)^p \theta(\tilde{a}_j)^q \leq \theta_{\cdot}^{p+q} \)

\[
\left( \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{1}{n(n-1)} \theta_{\cdot}^{p+q} \right)^{\frac{1}{p+q}} \leq \left( \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{1}{n(n-1)} \theta(\tilde{a}_i)^p \theta(\tilde{a}_j)^q \right)^{\frac{1}{p+q}} \leq \left( \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{1}{n(n-1)} \theta_{\cdot}^{p+q} \right)^{\frac{1}{p+q}} \]

\( \theta_{\cdot} \leq \left( \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \theta(\tilde{a}_i)^p \theta(\tilde{a}_j)^q \right)^{\frac{1}{p+q}} \leq \theta_{\cdot} \)

Similarly, we can know

\( r_{\cdot} \leq \left( \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} r(\tilde{a}_i)^p r(\tilde{a}_j)^q \right)^{\frac{1}{p+q}} \leq r_{\cdot} \)

\( (r_{L^-})^{p+q} \leq (r_{a_i}^-)^p (r_{a_j}^-)^q \leq (r_{L^-})^{p+q} \)

\[
(1-(r_{L^-})^{p+q})^{\frac{1}{n(n-1)}} \leq (1-(r_{a_i}^-)^p (r_{a_j}^-)^q)^{\frac{1}{n(n-1)}} \leq (1-(r_{L^-})^{p+q})^{\frac{1}{n(n-1)}} \]

\[
\left( \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} (1-(r_{L^-})^{p+q})^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \leq \left( \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} (1-(r_{a_i}^-)^p (r_{a_j}^-)^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \leq \left( \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} (1-(r_{L^-})^{p+q})^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \]

\[ r_{L^-} \leq \left(1- \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} (1-(r_{a_i}^-)^p (r_{a_j}^-)^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \leq r_{L^-} \]

Likewise, we also get

\[ r_{U^+} \leq \left(1- \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} (1-(r_{a_i}^u)^p (r_{a_j}^u)^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \leq r_{U^+} \]

\( (1-\delta_{L^-})^{p+q} \leq (1-\delta_{a_i}^-)^p (1-\delta_{a_j}^-)^q \leq (1-\delta_{L^-})^{p+q} \)

\[
\prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} (1-(1-\delta_{L^-})^{p+q}) \leq \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} (1-(1-\delta_{a_i}^-)^p (1-\delta_{a_j}^-)^q) \leq \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} (1-(1-\delta_{L^-})^{p+q}) \]

\( \alpha_i \leq \text{IVNHFULB}^\cdot (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \tilde{a}_+ \)
\[
1 - \left(1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - (1 - \delta_{U}^{L})^{p} (1 - \delta_{U}^{L})^{q}\right)^{\frac{1}{p+q}}\right)^{\frac{1}{p+q}} \leq 1 - \left(1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - (1 - \delta_{U}^{L})^{p} (1 - \delta_{U}^{L})^{q}\right)^{\frac{1}{p+q}}\right)^{\frac{1}{p+q}}
\]

\[
\delta^{L+} \leq 1 - \left(1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - (1 - \delta_{U}^{L})^{p} (1 - \delta_{U}^{L})^{q}\right)^{\frac{1}{p+q}}\right)^{\frac{1}{p+q}} \leq \delta^{L-}
\]

Similarly,

\[
\delta^{U+} \leq 1 - \left(1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - (1 - \delta_{U}^{L})^{p} (1 - \delta_{U}^{L})^{q}\right)^{\frac{1}{p+q}}\right)^{\frac{1}{p+q}} \leq \delta^{U-}
\]

(4) Similar to the proof of the (3), the results can be got.

\[
\eta^{L+} \leq 1 - \left(1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - (1 - \delta_{U}^{L})^{p} (1 - \delta_{U}^{L})^{q}\right)^{\frac{1}{p+q}}\right)^{\frac{1}{p+q}} \leq \eta^{L-}
\]

\[
\eta^{U+} \leq 1 - \left(1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - (1 - \delta_{U}^{L})^{p} (1 - \delta_{U}^{L})^{q}\right)^{\frac{1}{p+q}}\right)^{\frac{1}{p+q}} \leq \eta^{U-}
\]

So, we can know \( \tilde{a}^- \leq IVNHFULBM^{p,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \tilde{a}^+ \)

which completes the proof of theorem 15.

In the following, we will discuss some special cases of the IVNHFULBM operator with respect to the different parameters \( p \) and \( q \).

(1) If \( q = 0 \), then

\[
IVNHFULBM^{p,0}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left(\frac{1}{n(n-1)} \bigoplus_{i=1}^{n} \bigoplus_{j=1, j \neq i}^{n} \tilde{a}_i^p\right)^{\frac{1}{p}} = \left(\frac{1}{n} \bigoplus_{i=1}^{n} \tilde{a}_i^p\right)^{\frac{1}{p}}
\]

\[
= \left\{\left(\frac{1}{n} \bigoplus_{i=1}^{n} \tilde{a}_i^p\right)^{\frac{1}{p}} \bigcup \tilde{U}_r(\tilde{a}_1), \tilde{U}_r(\tilde{a}_2), \ldots, \tilde{U}_r(\tilde{a}_n) \bigcap \tilde{T}(\tilde{a}_1)\right\}
\]

\[
\left\{\left(\frac{1}{n} \bigoplus_{i=1}^{n} \tilde{a}_i^p\right)^{\frac{1}{p}} \bigcup \tilde{U}_r(\tilde{a}_1), \tilde{U}_r(\tilde{a}_2), \ldots, \tilde{U}_r(\tilde{a}_n) \bigcap \tilde{T}(\tilde{a}_1)\right\}
\]

\[
1 - \left(\frac{1}{n} \bigoplus_{i=1}^{n} \left(1 - (1 - \delta_{U}^{L})^{p}\right)^{\frac{1}{p}}\right)^{\frac{1}{p}} \leq \left(\frac{1}{n} \bigoplus_{i=1}^{n} \left(1 - (1 - \delta_{U}^{L})^{p}\right)^{\frac{1}{p}}\right)^{\frac{1}{p}}
\]

\[
1 - \left(\frac{1}{n} \bigoplus_{i=1}^{n} \left(1 - (1 - \delta_{U}^{L})^{p}\right)^{\frac{1}{p}}\right)^{\frac{1}{p}} \leq \left(\frac{1}{n} \bigoplus_{i=1}^{n} \left(1 - (1 - \delta_{U}^{L})^{p}\right)^{\frac{1}{p}}\right)^{\frac{1}{p}}
\]

\[
1 - \left(\frac{1}{n} \bigoplus_{i=1}^{n} \left(1 - (1 - \delta_{U}^{L})^{p}\right)^{\frac{1}{p}}\right)^{\frac{1}{p}} \leq \left(\frac{1}{n} \bigoplus_{i=1}^{n} \left(1 - (1 - \delta_{U}^{L})^{p}\right)^{\frac{1}{p}}\right)^{\frac{1}{p}}
\]

\[
1 - \left(\frac{1}{n} \bigoplus_{i=1}^{n} \left(1 - (1 - \delta_{U}^{L})^{p}\right)^{\frac{1}{p}}\right)^{\frac{1}{p}} \leq \left(\frac{1}{n} \bigoplus_{i=1}^{n} \left(1 - (1 - \delta_{U}^{L})^{p}\right)^{\frac{1}{p}}\right)^{\frac{1}{p}}
\]

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\[ \bigcup \tilde{\eta}(\tilde{a}_i) \in \tilde{f}(\tilde{a}_i), \tilde{\eta}(\tilde{a}_2) \in \tilde{f}(\tilde{a}_2), \ldots, \tilde{\eta}(\tilde{a}_n) \in \tilde{f}(\tilde{a}_n) \]

\[ 1 - \left( 1 - \left( \prod_{i=1}^{n} (1 - (1 - \eta_n^{-L})^p) \right)^{\frac{1}{2}} \right) \leq 1 - \left( 1 - \left( \prod_{i=1}^{n} (1 - (1 - \eta_n^{-U})^p) \right)^{\frac{1}{2}} \right) \]

(2) If \( p = 1 \) and \( q = 0 \), then

\[ IVNHFULBM^{1,0}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left( \frac{1}{n(n-1)} \bigoplus_{i=1}^{n} \bigoplus_{j=1, j \neq i}^{n} \tilde{a}_i \bigoplus_{j}^{1} \bigoplus_{j}^{1} \right) = \left( \frac{1}{n(n-1)} \bigoplus_{i=1}^{n} \bigoplus_{j=1, j \neq i}^{n} \tilde{a}_i \bigoplus_{j}^{1} \bigoplus_{j}^{1} \right) \]

\[ \left( \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{1}{n(n-1)} \delta(i, j) \right) \left( \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{1}{n(n-1)} \tau(i, j) \right) \]

\[ \bigcup_{\tilde{\eta}(\tilde{a}_i) \in \tilde{f}(\tilde{a}_i)} \left( \prod_{i=1}^{n} (1 - (1 - \eta_n^{-L})^p) \right)^{\frac{1}{2}} (1 - \left( \prod_{i=1}^{n} (1 - (1 - \eta_n^{-U})^p) \right)^{\frac{1}{2}} \right) \]

(3) If \( p = \frac{1}{2} \) and \( q = \frac{1}{2} \), then

\[ IVNHFULBM^{1/2,1/2}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left( \frac{1}{n(n-1)} \bigoplus_{i=1}^{n} \bigoplus_{j=1, j \neq i}^{n} \tilde{a}_i \bigoplus_{j}^{1} \bigoplus_{j}^{1} \right) \]

\[ \left( \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{1}{n(n-1)} \delta(i, j) \right) \left( \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{1}{n(n-1)} \tau(i, j) \right) \]

\[ \bigcup_{\tilde{\eta}(\tilde{a}_i) \in \tilde{f}(\tilde{a}_i)} \left( \prod_{i=1}^{n} (1 - (1 - \eta_n^{-L})^p) \right)^{\frac{1}{2}} (1 - \left( \prod_{i=1}^{n} (1 - (1 - \eta_n^{-U})^p) \right)^{\frac{1}{2}} \right) \]

(4) If \( p = 1 \) and \( q = 1 \), then

\[ IVNHFULBM^{1,1}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left( \frac{1}{n(n-1)} \bigoplus_{i=1}^{n} \bigoplus_{j=1, j \neq i}^{n} \tilde{a}_i \bigoplus_{j}^{1} \bigoplus_{j}^{1} \right)^{\frac{1}{2}} \]
3.2 The interval-valued neutrosophic hesitant fuzzy uncertain linguistic weighted Bonferroni mean operator

Definition 15. Let $\tilde{a}_i = [s_{\delta(i)}, r_{\delta(i)}, \tilde{I}(\tilde{a}_i), \tilde{F}(\tilde{a}_i), \tilde{P}(\tilde{a}_i), \tilde{T}(\tilde{a}_i)] \geq 1, 2, \ldots, n)$ be the set of all IVNHFULEs, and $p, q \geq 0$, then an interval-valued neutrosophic hesitant fuzzy uncertain linguistic weighted Bonferroni mean operator can be expressed as follows:

$$IVNHFULWBM^{p,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left( \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{1}{m(n-1)} (1-r_{\delta(i)} L, r_{\delta(i)} U) \right)^{\frac{1}{p+q}}$$

(40)

Theorem 16. Let $\tilde{a}_i = [s_{\delta(i)}, r_{\delta(i)}, \tilde{I}(\tilde{a}_i), \tilde{F}(\tilde{a}_i), \tilde{P}(\tilde{a}_i), \tilde{T}(\tilde{a}_i)] \geq 1, 2, \ldots, n)$ be the set of all IVNHFULEs, and $p, q \geq 0$, then the result aggregated from Definition 15 is still an IVNHFULE, and even

$$IVNHFULWBM^{p,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left( \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{W_j}{1-w_i} \tilde{a}_i^p \otimes \tilde{a}_j^q \right)^{\frac{1}{p+q}}$$
The proof of Theorem 16 can be easily completed similar to Theorem 11.

The IVNHFULWBM operator satisfies the following properties:

Theorem 17 (Reducibility).

Let \( \mathbf{T} = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right)^T \) be the weight vector of \( (\tilde{a}_1, \ldots, \tilde{a}_n) \), then

\[
\text{IVNHFULWBM}^{p,q}(\tilde{a}_1, \ldots, \tilde{a}_n) = \text{IVNHFULWBM}^{p,q}(\tilde{a}_1, \ldots, \tilde{a}_n)
\]

Proof.

Since \( W = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right)^T \) is the weight vector of \( \tilde{a}_i (i = 1, 2, \ldots, n) \), then according to definition 15, we have

\[
\text{IVNHFULWBM}^{p,q}(\tilde{a}_1, \ldots, \tilde{a}_n) = \left( \sum_{i=1}^{n} \frac{w_i W_j}{1} \tilde{a}_i^p \otimes \tilde{a}_j^q \right)^{1/p+q}
\]

\[
= \left( \sum_{i=1}^{n} \frac{n}{n} \frac{1}{1} \frac{1}{n} \tilde{a}_i^p \otimes \tilde{a}_j^q \right)^{1/p+q}
\]

\[
= \text{IVNHFULWBM}^{p,q}(\tilde{a}_1, \ldots, \tilde{a}_n)
\]

which completes the proof of Theorem 17.

Theorem 18 (Idempotency).

Let \( \tilde{a}_j = [S_{\theta(a)} \cdot s_{\pi(a)}], ([r_a^L, r_a^U], [\delta_a^L, \delta_a^U], [\eta_a^L, \eta_a^U]) \supseteq \tilde{a} \) (\( j = 1, 2, \ldots, n \)), then

\[
\text{IVNHFULWBM}^{p,q}(\tilde{a}_1, \ldots, \tilde{a}_n) = \tilde{a}
\]

The proof of Theorem 18 can be easily completed similar to Theorem 12.

Theorem 19 (Commutativity).

If \( (\tilde{a}_1', \tilde{a}_2', \ldots, \tilde{a}_n') \) be any permutation of \( (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \), then we can get...
The proof of Theorem 19 can be easily completed similar to Theorem 13.

**Theorem 20 (Monotonicity).**

Suppose \( (\vec{a}_1, \ldots, \vec{a}_n) \) and \( (\vec{b}_1, \ldots, \vec{b}_n) \) are two sets of IVNHFULWBM operators. If \( s_{\theta_1}(\vec{a}) \leq s_{\theta_2}(\vec{b}) \), then

\[
IVNHFULWBM^{p,q}(\vec{a}) \leq IVNHFULWBM^{p,q}(\vec{b})
\]

for all \( i = 1, 2, \ldots, n \), then \( IVNHFULWBM^{p,q}(\vec{a}_1, \ldots, \vec{a}_n) \leq IVNHFULWBM^{p,q}(\vec{b}_1, \ldots, \vec{b}_n) \)

The proof of Theorem 20 can be easily completed similar to Theorem 14.

**Theorem 21 (Boundedness).**

Let \( \vec{a}_i (i = 1, 2, \ldots, n) \) be a set of IVNHFULWBM operators. If

\[
s_{\theta_+} = \min_{i \leq n} \{ s_{\theta_+(\vec{a})} \}, s_{\tau_+} = \min_{i \leq n} \{ s_{\tau_+(\vec{a})} \},
\]

\[
r_{L+} = \min_{i \leq n} \{ r_{L_+}(\vec{a}) \} = \min_{i \leq n} \{ r_{L_+}(\vec{a}) \},
\]

\[
r_{U+} = \max_{i \leq n} \{ r_{U_+}(\vec{a}) \} = \max_{i \leq n} \{ r_{U_+}(\vec{a}) \}.
\]

\[
\delta_{L+} = \min_{i \leq n} \{ \delta_{L_+}(\vec{a}) \} = \min_{i \leq n} \{ \delta_{L_+}(\vec{a}) \},
\]

\[
\delta_{U+} = \max_{i \leq n} \{ \delta_{U_+}(\vec{a}) \} = \max_{i \leq n} \{ \delta_{U_+}(\vec{a}) \}.
\]

\[
\eta_{L+} = \min_{i \leq n} \{ \eta_{L_+}(\vec{a}) \} = \min_{i \leq n} \{ \eta_{L_+}(\vec{a}) \},
\]

\[
\eta_{U+} = \max_{i \leq n} \{ \eta_{U_+}(\vec{a}) \} = \max_{i \leq n} \{ \eta_{U_+}(\vec{a}) \}.
\]

\[
\vec{a}^- = \langle s_{\theta_-}, s_{\tau_-}, (r_{L-}, r_{U-}), (\delta_{L-}, \delta_{U-}), (\eta_{L-}, \eta_{U-}) \rangle
\]

\[
\vec{a}^+ = \langle s_{\theta_+}, s_{\tau_+}, (r_{L+}, r_{U+}), (\delta_{L+}, \delta_{U+}), (\eta_{L+}, \eta_{U+}) \rangle
\]

then

\( \vec{a}^- \leq IVNHFULWBM^{p,q}(\vec{a}_1, \ldots, \vec{a}_n) \leq \vec{a}^+ \).

The proof of Theorem 21 can be easily completed similar to Theorem 15.

In the following, we will discuss some specials of the IVNHFULWBM operator with respect to the different parameters \( p \) and \( q \).

(1) If \( q = 0 \), then

\[
IVNHFULWBM^{p,0}(\vec{a}_1, \ldots, \vec{a}_n) = \left( \frac{w_j}{1-w_i} \right)_{i=1, j=1}^n \vec{a}^p = \left( \frac{w_j}{1-w_i} \right)_{i=1}^n \vec{a}^p
\]
\( \bigcup \widehat{\mathcal{I}}(\widehat{a}_1) \subset \overline{\mathcal{I}}(\overline{a}_1). \bigcup \overline{\mathcal{I}}(\overline{a}_1) \subset \overline{\mathcal{I}}(\overline{a}_1). \overline{\mathcal{I}}(\overline{a}_1) = \overline{\mathcal{I}}(\overline{a}_1) \).

\[
\left\{ 1- \prod_{i=1}^{n} (1 - (p_i^{\overline{a}_1})^n) \right\}^{\frac{1}{p}} \prod_{i=1}^{n} \left( \frac{1}{p} \right)^n
\]

(2) If \( p = 1 \) and \( q = 0 \), then

\[
IVNHFULWBM_{1,0}^{1,0}(\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n) = \left( \bigoplus_{i=1}^{n} w_i \overline{a}_i \right)
\]

(3) If \( p = 1 / 2 \) and \( q = 1 / 2 \), then

\[
IVNHFULWBM_{1,1}^{1,1}(\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n) = \left( \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n} \frac{w_i w_j}{w_i - w_j} \overline{a}_i \overline{a}_j \right)^{1/2}
\]

\[
\bigcup \overline{\mathcal{I}}(\overline{a}_1) \subset \bigcup \underline{\mathcal{I}}(\underline{a}_1). \bigcup \underline{\mathcal{I}}(\underline{a}_1) \subset \bigcup \underline{\mathcal{I}}(\underline{a}_1). \underline{\mathcal{I}}(\underline{a}_1) = \underline{\mathcal{I}}(\underline{a}_1)
\]

\[
\left\{ 1- \prod_{i=1}^{n} (1 - (p_i^{\overline{a}_1})^n) \right\}^{\frac{1}{p}} \prod_{i=1}^{n} \left( \frac{1}{p} \right)^n
\]

\[
\left\{ 1- \prod_{i=1}^{n} (1 - (p_i^{\overline{a}_1})^n) \right\}^{\frac{1}{p}} \prod_{i=1}^{n} \left( \frac{1}{p} \right)^n
\]
Theorem 22. Let \( \tilde{a}_i = <[s_{\tilde{a}_i}, r_{\tilde{a}_i}], (\tilde{r}(\tilde{a}_i), \tilde{f}(\tilde{a}_i)) \rangle > \) \((i = 1, 2, ..., n)\) be the set of all IVNHFULEs, and \( p , q \geq 0 \), then an interval-valued neutrosophic hesitant fuzzy uncertain linguistic geometric Bonferroni mean operator can be defined as follows:

\[
IVNHFUBLGBM^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \frac{1}{p+q} \bigoplus_{i=1}^{n} \bigoplus_{j=l}^{n} \bigoplus (p\tilde{a}_i \oplus q\tilde{a}_j)^{\frac{1}{n-1}}
\]

(42)

3.3 The interval-valued neutrosophic hesitant fuzzy uncertain linguistic geometric Bonferroni mean operator

Definition 16. Let \( \tilde{a}_i = <[s_{\tilde{a}_i}, r_{\tilde{a}_i}], (\tilde{r}(\tilde{a}_i), \tilde{f}(\tilde{a}_i)) \rangle > \) \((i = 1, 2, ..., n)\) be the set of all IVNHFULEs, and \( p , q \geq 0 \), then an interval-valued neutrosophic hesitant fuzzy uncertain linguistic geometric Bonferroni mean operator can be defined as follows:
and \( p, q \geq 0 \), then, the result aggregated from Definition 16 is still an IVNHFULE, and even

\[
IVNHFULGBM^{p,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{1}{p+q} \bigotimes_{i=1}^{n} \bigotimes_{j=1,j\neq i}^{n} (p\tilde{a}_i \oplus q\tilde{a}_j) \cdot \frac{1}{(n-1)!}
\]

\[
= \left\{ \begin{array}{l}
\sum_{\tilde{r}(\tilde{a}_1), \ldots, \tilde{r}(\tilde{a}_n)} \left( 1 - \prod_{i=1}^{n} \prod_{j=1,j\neq i}^{n} \left( 1 - (r_{\tilde{a}_i}^L)^p (1-r_{\tilde{a}_j}^L)^q \right)^{p+q} \frac{1}{(n-1)!} \right) \\
\sum_{\tilde{\sigma}(\tilde{a}_1), \ldots, \tilde{\sigma}(\tilde{a}_n)} \left( 1 - \prod_{i=1}^{n} \prod_{j=1,j\neq i}^{n} \left( 1 - (\delta_{\tilde{a}_i}^L)^p (\delta_{\tilde{a}_j}^L)^q \right)^{p+q} \frac{1}{(n-1)!} \right) \\
\sum_{\tilde{\eta}(\tilde{a}_1), \ldots, \tilde{\eta}(\tilde{a}_n)} \left( 1 - \prod_{i=1}^{n} \prod_{j=1,j\neq i}^{n} \left( 1 - (\eta_{\tilde{a}_i}^L)^p (\eta_{\tilde{a}_j}^L)^q \right)^{p+q} \frac{1}{(n-1)!} \right)
\end{array} \right.
\]

(43)

The proof of Theorem 22 can be easily completed similar to Theorem 11.

The IVNHFULGBM operator satisfies the following properties:

**Theorem 23 (Idempotency).**

Let \( \tilde{a}_j = [S_{\theta(a)}, S_{\tau(a)}], ([r_{\tilde{a}_i}^L, r_{\tilde{a}_i}^U], [\delta_{\tilde{a}_i}^L, \delta_{\tilde{a}_i}^U], [\eta_{\tilde{a}_i}^L, \eta_{\tilde{a}_i}^U]) = \tilde{\alpha}_j \), then

\[
IVNHFULGBM^{p,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{\alpha}.
\]

The proof of Theorem 23 can be easily completed similar to Theorem 12.

**Theorem 24 (Commutativity).**

If \( \tilde{a}_1', \tilde{a}_2', \ldots, \tilde{a}_n' \) be any permutation of \( \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n \), then we can get

\[
IVNHFULGBM^{p,q}(\tilde{a}_1', \tilde{a}_2', \ldots, \tilde{a}_n') = IVNHFULGBM^{p,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n).
\]

The proof of Theorem 24 can be easily completed similar to Theorem 13.

**Theorem 25 (Monotonicity).**

Suppose \( \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n \) and \( \tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n \) are two sets of IVNHFULEs. If \( S_{\theta(\tilde{a})} \leq S_{\theta(\tilde{b})} \), then

\[
IVNHFULGBM^{p,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq IVNHFULGBM^{p,q}(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n)
\]

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Theorem 26 (Boundedness).

Let \( \tilde{a}_i (i = 1, 2, ..., n) \) be a set of IVNhFULEs. If \( s_{\theta^-} = \min \{ s_{\theta(h_i)} \} \), \( s_{\theta^+} = \max \{ s_{\theta(h_i)} \} \), \( r_{L^-} = \min \{ r_{a_i}^L \} \), \( r_{L^+} = \max \{ r_{a_i}^L \} \), \( r_{U^-} = \min \{ r_{a_i}^U \} \), \( r_{U^+} = \max \{ r_{a_i}^U \} \), \( \delta_{L^-} = \min \{ \delta_{a_i}^L \} \), \( \delta_{L^+} = \max \{ \delta_{a_i}^L \} \), \( \delta_{U^-} = \min \{ \delta_{a_i}^U \} \), \( \delta_{U^+} = \max \{ \delta_{a_i}^U \} \), then

\[
\begin{align*}
\tilde{a}^- &= \{ [s_{\theta^-}, s_{\theta^+}], (r_{L^-}, r_{L^+}, \delta_{L^-}, \delta_{L^+}, \eta_{L^-}, \eta_{L^+}) \}, \\
\tilde{a}^+ &= \{ [s_{\theta^+}, s_{\theta^-}], (r_{L^+}, r_{L^-}, \delta_{L^+}, \delta_{L^-}, \eta_{L^+}, \eta_{L^-}) \}
\end{align*}
\]

and

\[
\tilde{a}^- \leq \text{IVNhFULGBM}^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \leq \tilde{a}^+.
\]

The proof of Theorem 26 can be easily completed similar to Theorem 15.

In the following, we will discuss some specials of the IVNhFULGBM operator with respect to the different parameters \( p \) and \( q \).

(1) If \( q = 0 \), then

\[
\text{IVNhFULGBM}^{p,0}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \frac{1}{p} \bigotimes (p\tilde{a}_i)^p
\]

(2) If \( p = 1 \) and \( q = 0 \), then

\[
\text{IVNhFULGBM}^{1,0}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n)
\]
IVNHFULGBM \(^{1,0}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \bigotimes_{i=1}^{n} \tilde{a}_i \) \\

\[
= \left\{ \begin{array}{c}
\sum_{\sigma \in S_n} \prod_{i=1}^{n} \tilde{a} \left( \frac{1}{2} \rho(\tilde{a}_i) + \frac{1}{2} \rho(\tilde{a}_j) \right) \\
\sum_{\sigma \in S_n} \prod_{i=1}^{n} \tilde{a} \left( \frac{1}{2} \rho(\tilde{a}_i) + \frac{1}{2} \rho(\tilde{a}_j) \right)
\end{array} \right\},
\]

\[
\bigcup_{\bar{n}(\tilde{a}_i) \in \mathbb{F}(\tilde{a}_i)} \left\{ \prod_{i=1}^{n} \left( 1 - \left( 1 - \tilde{a}_j \right) \tilde{a}_i \right) \prod_{i=1}^{n} \left( 1 - \left( 1 - \tilde{a}_j \right) \tilde{a}_i \right) \right\},
\]

(3) If \( p = \frac{1}{2} \) and \( q = \frac{1}{2} \), then

IVNHFULGBM \(^{1,2}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \bigotimes_{i=1}^{n} \bigotimes_{j=1}^{n} \left( \frac{1}{2} \tilde{a}_i \oplus \frac{1}{2} \tilde{a}_j \right) \frac{1}{n(n-1)}
\]

\[
= \left\{ \begin{array}{c}
\sum_{\sigma \in S_n} \prod_{i=1}^{n} \tilde{a} \left( \frac{1}{2} \rho(\tilde{a}_i) + \frac{1}{2} \rho(\tilde{a}_j) \right) \\
\sum_{\sigma \in S_n} \prod_{i=1}^{n} \tilde{a} \left( \frac{1}{2} \rho(\tilde{a}_i) + \frac{1}{2} \rho(\tilde{a}_j) \right)
\end{array} \right\},
\]

\[
\bigcup_{\bar{n}(\tilde{a}_i) \in \mathbb{F}(\tilde{a}_i)} \left\{ \prod_{i=1}^{n} \left( 1 - \left( 1 - \tilde{a}_j \right) \tilde{a}_i \right) \prod_{i=1}^{n} \left( 1 - \left( 1 - \tilde{a}_j \right) \tilde{a}_i \right) \right\},
\]

(4) If \( p = 1 \) and \( q = 1 \), then

IVNHFULGBM \(^{1,1}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{1}{2} \bigotimes_{i=1}^{n} \bigotimes_{j=1}^{n} \left( \tilde{a}_i \oplus \tilde{a}_j \right) \frac{1}{n(n-1)}
\]

\[
= \left\{ \begin{array}{c}
\sum_{\sigma \in S_n} \prod_{i=1}^{n} \tilde{a} \left( \frac{1}{2} \rho(\tilde{a}_i) + \frac{1}{2} \rho(\tilde{a}_j) \right) \\
\sum_{\sigma \in S_n} \prod_{i=1}^{n} \tilde{a} \left( \frac{1}{2} \rho(\tilde{a}_i) + \frac{1}{2} \rho(\tilde{a}_j) \right)
\end{array} \right\},
\]

\[
\bigcup_{\bar{n}(\tilde{a}_i) \in \mathbb{F}(\tilde{a}_i)} \left\{ \prod_{i=1}^{n} \left( 1 - \left( 1 - \tilde{a}_j \right) \tilde{a}_i \right) \prod_{i=1}^{n} \left( 1 - \left( 1 - \tilde{a}_j \right) \tilde{a}_i \right) \right\}.\]
Theorem 28 (Reducibility).

Let 
\[ \hat{a}_i = \langle s_{\bar{a}_i}, \tau_{\bar{a}_i}, \bar{a}_i \rangle > (i = 1, 2, ..., n) \] be the set of all IVNFULEs, and \( p, q \geq 0 \), then the interval-valued neutrosophic hesitant fuzzy uncertain linguistic weighted geometric Bonferroni mean operator can be defined as follows:

\[
IVNHFULWGBM^{p,q}(\hat{a}_1, \hat{a}_2, ..., \hat{a}_n) = \frac{1}{p + q \sum_{i=1}^{n} j=1, j \neq i} \left( \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left( \eta_{\bar{a}_i}^L \eta_{\bar{a}_j}^L \frac{1}{p+m} \right)^{w_i w_j} \right)^{\frac{1}{p+q}},
\]
Let \( W = \left( \frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n} \right)^T \) be the weight vector of \( \tilde{a}_i (i = 1, 2, ..., n) \), then

\[
IVNHFULWGBM^{\alpha \beta} (\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = IVNHFULGBM^{\alpha \beta} (\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n)
\]

The proof of Theorem 28 can be easily completed similar to Theorem 17.

**Theorem 29 (Idempotency).**

Let \( \tilde{a}_j = \langle [S_{\theta} (a), s_{\tau} (a)], ([r^L_a, r^U_a], [\delta^L_a, \delta^U_a], [\eta^L_a, \eta^U_a]) \rangle \Rightarrow \tilde{a} \quad (j = 1, 2, ..., n) \), then

\[
IVNHFULWGBM^{\alpha \beta} (\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \tilde{a}.
\]

The proof of Theorem 29 can be easily completed similar to Theorem 12.

**Theorem 30 (Commutativity).**

If \( (\tilde{a}_1', \tilde{a}_2', ..., \tilde{a}_n') \) be any permutation of \( (\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \), then we can get

\[
IVNHFULWGBM^{\alpha \beta} (\tilde{a}_1', \tilde{a}_2', ..., \tilde{a}_n') = IVNHFULWGBM^{\alpha \beta} (\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n).
\]

The proof of Theorem 30 can be easily completed similar to Theorem 13.

**Theorem 31 (Monotonicity).**

Suppose \( (\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \) and \( (\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_n) \) are two sets of IVNHFULEs. If \( s_{\theta (\tilde{a})} \leq s_{\theta (\tilde{b})} \),

\[
s_{\tau (\tilde{a})} \leq s_{\tau (\tilde{b})}, \quad r^L_a \leq r^L_b, \quad r^U_a \leq r^U_b, \quad \delta^L_a \geq \delta^L_b, \quad \delta^U_a \geq \delta^U_b \quad \text{and} \quad \eta^L_a \geq \eta^L_b, \quad \eta^U_a \geq \eta^U_b \quad \text{for all} \quad i = 1, 2, ..., n,
\]

then

\[
IVNHFULWGBM^{\alpha \beta} (\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \leq IVNHFULWGBM^{\alpha \beta} (\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_n).
\]

The proof of Theorem 31 can be easily completed similar to Theorem 14.

**Theorem 32 (Boundedness).**

Let \( \tilde{a}_i (i = 1, 2, ..., n) \) be a collection of IVNHFULEs. If \( s_{\theta -} = \min_{1 \leq i \leq n} \{ s_{\theta (\tilde{a}_i)} \}, s_{\tau -} = \min_{1 \leq i \leq n} \{ s_{\tau (\tilde{a}_i)} \}, \)

\[
s_{\theta +} = \max_{1 \leq i \leq n} \{ s_{\theta (\tilde{a}_i)} \}, \quad s_{\tau +} = \max_{1 \leq i \leq n} \{ s_{\tau (\tilde{a}_i)} \}, \quad r^L = \min_{1 \leq i \leq n} \{ r^L_a \mid [r^L_a, r^U_a] \in t(\tilde{a}_i) \},
\]

\[
r^U = \min_{1 \leq i \leq n} \{ r^U_a \mid [r^L_a, r^U_a] \in t(\tilde{a}_i) \}, \quad r^L = \max_{1 \leq i \leq n} \{ r^L_a \mid [r^L_a, r^U_a] \in t(\tilde{a}_i) \},
\]

\[
r^U = \max_{1 \leq i \leq n} \{ r^U_a \mid [r^L_a, r^U_a] \in t(\tilde{a}_i) \}, \quad \delta^L = \min_{1 \leq i \leq n} \{ \delta^L_a \mid [\delta^L_a, \delta^U_a] \in i(\tilde{a}_i) \},
\]

\[
\delta^L = \min_{1 \leq i \leq n} \{ \delta^U_a \mid [\delta^L_a, \delta^U_a] \in i(\tilde{a}_i) \}, \quad \delta^L = \max_{1 \leq i \leq n} \{ \delta^L_a \mid [\delta^L_a, \delta^U_a] \in i(\tilde{a}_i) \},
\]

\[
\delta^L = \max_{1 \leq i \leq n} \{ \delta^U_a \mid [\delta^L_a, \delta^U_a] \in i(\tilde{a}_i) \}, \quad \eta^L = \min_{1 \leq i \leq n} \{ \eta^L_a \mid [\eta^L_a, \eta^U_a] \in f(\tilde{a}_i) \},
\]

\[
\eta^L = \min_{1 \leq i \leq n} \{ \eta^U_a \mid [\eta^L_a, \eta^U_a] \in f(\tilde{a}_i) \}, \quad \eta^L = \max_{1 \leq i \leq n} \{ \eta^L_a \mid [\eta^L_a, \eta^U_a] \in f(\tilde{a}_i) \},
\]

\[
\eta^L = \max_{1 \leq i \leq n} \{ \eta^U_a \mid [\eta^L_a, \eta^U_a] \in f(\tilde{a}_i) \}, \quad \eta^L = \min_{1 \leq i \leq n} \{ \eta^L_a \mid [\eta^L_a, \eta^U_a] \in f(\tilde{a}_i) \}.
\]

\[
\tilde{a} = [s_{\theta -}, s_{\tau -}, [r^L - r^U - \delta^L - \delta^U + \eta^L + \eta^U]] \rangle \quad \text{and}
\]

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\( \tilde{a}^+ = \langle [s_{\theta}, s_{\tau}], ([r^{L*}, r^{U*}],[\delta^{L*}, \delta^{U*}],[\eta^{L*}, \eta^{U*}]) > \) then

\[ \tilde{a}^- \leq IVNFULWGBM^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \leq \tilde{a}^+ . \]

The proof of Theorem 32 can be easily completed similar to Theorem 15.

In the following, we will discuss some specials of the IVNFULWGBM operator with respect to the different parameters \( p \) and \( q \).

1. If \( q = 0 \), then

\[
IVNFULWGBM^{p,0}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \frac{1}{p} \bigotimes_{i=1}^{n} (\tilde{p}_a)^{w_i}
\]

2. If \( p = 1 \) and \( q = 0 \), then

\[
IVNFULWGBM^{1,0}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \bigotimes_{i=1}^{n} \tilde{a}_i^{w_i}
\]

3. If \( p = \frac{1}{2} \) and \( q = \frac{1}{2} \), then

\[
IVNFULWGBM^{\frac{1}{2},\frac{1}{2}}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \bigotimes_{i=1}^{n} \bigotimes_{j=1, j \neq i}^{n} \left( \frac{1}{2} \tilde{a}_i \oplus \frac{1}{2} \tilde{a}_j \right)^{w_i w_j}
\]
be a discrete set of alternatives, \( V \). Let \( \mathcal{A} = \{a_1, a_2, ..., a_m\} \) be a set of attributes. The evaluation values can be represented by the form of \( \mathcal{I} = \mathcal{I}_p \) with \( 0 < p < 1 \) and \( q = 1 \), then

\[
IVNHFULWGBM^{1,1}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \frac{1}{2} \prod_{i=1}^{n} (\tilde{a}_i \oplus \tilde{a}_j)^{w_j}.
\]

Therefore, we can get the decision matrix \( N = (n_{ij})_{mn} \). The weighting vector of attributes is given by \( W = (w_1, w_2, ..., w_n)^T \) with \( w_j \geq 0 \),
\[
\sum_{j=1}^{n} w_j = 1.
\]

Then, the steps of the decision making method are described as follows:

**Step1:** Utilized the IVNHFULWBM operator and IVNHFULWGBM operator to derive the overall assessment values \( N_i(i=1,2,...,n) \) of the alternatives \( A_i(i=1,2,...,n) \).

**Step2:** Calculate the score values of the overall assessment values \( N_i(i=1,2,...,m) \) by the Eq. (33).

**Step3:** Rank all the alternatives \( \{A_1, A_2, ..., A_n\} \) by score values \( S(N_i) \), and select the best alternative.

**Step4:** End.

### 5. A numerical example

In this section, we give a numerical example to demonstrate the MADM methods based on IVNHFULWBM and IVNHFULWGBM operators.

Suppose that there is a company, which wants to invest a sum of money to an industry. There are four companies as alternatives to be chosen, including: (1) \( A_1 \) is a car company; (2) \( A_2 \) is a food company; (3) \( A_3 \) is a computer company; (4) \( A_4 \) is an arms company. There are three evaluation attributes, including: (1) \( C_1 \) is the risk; (2) \( C_2 \) is the growth; (3) \( C_3 \) is the environment. We can know the attributes \( C_1 \) and \( C_2 \) are benefit criteria, and the type of \( C_3 \) is cost criteria. The weight vector of the attribute is \( w = \{0.25, 0.35, 0.4\} \). The final evaluation values are expressed by the IVNHFULEs, and shown in Table 1.

#### Table 1 Interval-valued neutrosophic hesitant fuzzy uncertain linguistic decision matrix

<table>
<thead>
<tr>
<th></th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>([S_{2.90}, S_{4.61}], ([0.3,0.4],[0.4,0.5]), ([0.3088,0.4118],[0.3175,0.3869],[0.3429,0.4118]), ([0.1255,0.2268],[0.2393,0.3431]), ([0.1482,0.2268],[0.2689,0.3431]), ([0.3768,0.4783]) &gt;)</td>
<td>([S_{2.71}, S_{4.72}], ([0.6,0.7]), ([0.1,0.1714]), ([0.1255,0.2268],[0.1623,0.2629]) &gt;)</td>
<td>([S_{2.34}, S_{3.9}], ([0.4258,0.5274],[0.5000,0.6000]), ([0.1623,0.2988],[0.2000,0.3352]), ([0.22,0.3273]) &gt;)</td>
</tr>
<tr>
<td>A2</td>
<td>([S_{2.90}, S_{4.61}], ([0.3,0.4],[0.4,0.5]), ([0.3088,0.4118],[0.3175,0.3869],[0.3429,0.4118]), ([0.1255,0.2268],[0.2393,0.3431]), ([0.1482,0.2268],[0.2689,0.3431]), ([0.3768,0.4783]) &gt;)</td>
<td>([S_{2.71}, S_{4.72}], ([0.6,0.7]), ([0.1,0.1714]), ([0.1255,0.2268],[0.1623,0.2629]) &gt;)</td>
<td>([S_{2.34}, S_{3.9}], ([0.4258,0.5274],[0.5000,0.6000]), ([0.1623,0.2988],[0.2000,0.3352]), ([0.22,0.3273]) &gt;)</td>
</tr>
<tr>
<td>A3</td>
<td>([S_{2.90}, S_{4.61}], ([0.3,0.4],[0.4,0.5]), ([0.3088,0.4118],[0.3175,0.3869],[0.3429,0.4118]), ([0.1255,0.2268],[0.2393,0.3431]), ([0.1482,0.2268],[0.2689,0.3431]), ([0.3768,0.4783]) &gt;)</td>
<td>([S_{2.71}, S_{4.72}], ([0.6,0.7]), ([0.1,0.1714]), ([0.1255,0.2268],[0.1623,0.2629]) &gt;)</td>
<td>([S_{2.34}, S_{3.9}], ([0.4258,0.5274],[0.5000,0.6000]), ([0.1623,0.2988],[0.2000,0.3352]), ([0.22,0.3273]) &gt;)</td>
</tr>
<tr>
<td>A4</td>
<td>([S_{2.90}, S_{4.61}], ([0.3,0.4],[0.4,0.5]), ([0.3088,0.4118],[0.3175,0.3869],[0.3429,0.4118]), ([0.1255,0.2268],[0.2393,0.3431]), ([0.1482,0.2268],[0.2689,0.3431]), ([0.3768,0.4783]) &gt;)</td>
<td>([S_{2.71}, S_{4.72}], ([0.6,0.7]), ([0.1,0.1714]), ([0.1255,0.2268],[0.1623,0.2629]) &gt;)</td>
<td>([S_{2.34}, S_{3.9}], ([0.4258,0.5274],[0.5000,0.6000]), ([0.1623,0.2988],[0.2000,0.3352]), ([0.22,0.3273]) &gt;)</td>
</tr>
</tbody>
</table>

#### 5.1 Procedure of decision making method based on the IVNHFULWBM operator

**Step1:** Utilized the IVNHFULWBM operator to derive the overall assessment values \( N_i(i=1,2,...,n) \) of the alternatives \( A_i(i=1,2,...,n) \). We can get

\[
N_1 = \langle [S_{2.90}, S_{4.61}], ([0.2853,0.3869],[0.3088,0.4118],[0.3175,0.3869],[0.3429,0.4118]), ([0.1255,0.2268],[0.2393,0.3431]), ([0.1482,0.2268],[0.2689,0.3431]), ([0.3768,0.4783]) >
\]

\[
N_2 = \langle [S_{2.71}, S_{4.72}], ([0.6000,0.7000]), ([0.1000,0.1714]), ([0.1255,0.2268],[0.1623,0.2629]) >
\]

\[
N_3 = \langle [S_{2.34}, S_{3.9}], ([0.4258,0.5274],[0.5000,0.6000]), ([0.1623,0.2988],[0.2000,0.3352]), ([0.22,0.3273]) >
\]
Step 2: Calculate the score values of the overall assessment values $N_i (i = 1, 2, \ldots, m)$, which are shown as follows (for convenience, we can use the subscript):

$S(N_1) = 2.1135$, $S(N_2) = 2.8716$, $S(N_3) = 2.0908$, $S(N_4) = 2.0554$

Step 3: Rank all the alternatives $\{A_1, A_2, \ldots, A_4\}$ by score values $S(N_i)$, and select the best alternative.

So, $A_2 > A_1 > A_3 > A_4$, and the best alternative is $A_2$.

Step 4: End.

5.2 Procedure of decision making method based on the IVNHFULWGBM operator

Step 1: Utilized the IVNHFULWGBM operator to derive the overall evaluation values $N_i (i = 1, 2, \ldots, n)$ of the alternatives $A_i (i = 1, 2, \ldots, n)$. We can get

$N_1 = \langle [S_{1.61}, S_{4.24}], ([0.5107, 0.6569], [0.4644, 0.5743], [0.1882, 0.2692], [0.2021, 0.2373]) \rangle$

$N_2 = \langle [S_{1.61}, S_{4.24}], ([0.5107, 0.6569], [0.4644, 0.5743], [0.1882, 0.2692], [0.2021, 0.2373]) \rangle$

$N_3 = \langle [S_{1.61}, S_{4.24}], ([0.5107, 0.6569], [0.4644, 0.5743], [0.1882, 0.2692], [0.2021, 0.2373]) \rangle$

Step 2: Calculate the score values of the overall assessment values $N_i (i = 1, 2, \ldots, m)$, which are shown as follows (for convenience, we can use the subscript):

$S(N_1) = 2.1135$, $S(N_2) = 2.8716$, $S(N_3) = 2.0908$, $S(N_4) = 2.0554$

Step 3: Rank all the alternatives $\{A_1, A_2, \ldots, A_4\}$ by score values $S(N_i)$, and select the best alternative.

So, $A_2 > A_1 > A_3 > A_4$, and the best alternative is $A_2$.

Step 4: End.

5.3 Analysis of the influence of the parameters $p$ and $q$

In order to demonstrate the influence of the parameters $p$ and $q$ on decision making results, we...
use the different values $p$ and $q$ in IVNHFULWBM operator and IVNHFULWGMBM operator to rank the alternatives. The ranking results are shown in Table 2 and Table 3.

**Table 2** Ordering of the alternatives by utilizing the different $p, q$ in IVNHFULWBM operator

<table>
<thead>
<tr>
<th>$p, q$</th>
<th>Score function $S(N_i)$</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p = q = \frac{1}{2}$</td>
<td>$S(N_1) = 2.0983, S(N_2) = 2.8630$</td>
<td>$A_2 &gt; A_1 &gt; A_3 &gt; A_4$</td>
</tr>
<tr>
<td>$p = q = 1$</td>
<td>$S(N_1) = 2.1135, S(N_2) = 2.8716$</td>
<td>$A_2 &gt; A_1 &gt; A_3 &gt; A_4$</td>
</tr>
<tr>
<td>$p = 1, q = 0$</td>
<td>$S(N_1) = 2.1706, S(N_2) = 2.9115$</td>
<td>$A_2 &gt; A_1 &gt; A_4 &gt; A_3$</td>
</tr>
<tr>
<td>$p = 0, q = 1$</td>
<td>$S(N_1) = 2.2300, S(N_2) = 2.8692$</td>
<td>$A_2 &gt; A_1 &gt; A_3 &gt; A_4$</td>
</tr>
<tr>
<td>$p = 1, q = 2$</td>
<td>$S(N_1) = 2.1601, S(N_2) = 2.8801$</td>
<td>$A_2 &gt; A_1 &gt; A_3 &gt; A_4$</td>
</tr>
<tr>
<td>$p = 2, q = 1$</td>
<td>$S(N_1) = 2.1435, S(N_2) = 2.8922$</td>
<td>$A_2 &gt; A_1 &gt; A_3 &gt; A_4$</td>
</tr>
<tr>
<td>$p = 5, q = 5$</td>
<td>$S(N_1) = 2.2531, S(N_2) = 2.9360$</td>
<td>$A_2 &gt; A_3 &gt; A_1 &gt; A_4$</td>
</tr>
<tr>
<td>$p = 10, q = 0$</td>
<td>$S(N_1) = 2.6175, S(N_2) = 3.0393$</td>
<td>$A_2 &gt; A_1 &gt; A_3 &gt; A_4$</td>
</tr>
<tr>
<td>$p = 0, q = 10$</td>
<td>$S(N_1) = 2.6442, S(N_2) = 3.0184$</td>
<td>$A_2 &gt; A_1 &gt; A_3 &gt; A_4$</td>
</tr>
<tr>
<td>$p = 15, q = 15$</td>
<td>$S(N_1) = 2.4632, S(N_2) = 3.0276$</td>
<td>$A_2 &gt; A_4 &gt; A_1 &gt; A_3$</td>
</tr>
</tbody>
</table>

**Table 3** Ordering of the alternatives by utilizing the different $p, q$ in IVNHFULWGMBM operator

<table>
<thead>
<tr>
<th>$p, q$</th>
<th>Score function $S(N_i)$</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p = q = \frac{1}{2}$</td>
<td>$S(N_1) = 2.1727, S(N_2) = 2.9003$</td>
<td>$A_2 &gt; A_1 &gt; A_3 &gt; A_4$</td>
</tr>
<tr>
<td>$p = q = 1$</td>
<td>$S(N_1) = 2.1648, S(N_2) = 2.8975$</td>
<td>$A_2 &gt; A_1 &gt; A_3 &gt; A_4$</td>
</tr>
<tr>
<td>$p = 1, q = 0$</td>
<td>$S(N_1) = 2.0379, S(N_2) = 2.8962$</td>
<td>$A_2 &gt; A_3 &gt; A_1 &gt; A_4$</td>
</tr>
<tr>
<td>$p = 0, q = 1$</td>
<td>$S(N_1) = 2.0995, S(N_2) = 2.8501$</td>
<td>$A_2 &gt; A_1 &gt; A_3 &gt; A_4$</td>
</tr>
<tr>
<td>$p = 1, q = 2$</td>
<td>$S(N_1) = 2.1492, S(N_2) = 2.8819$</td>
<td>$A_2 &gt; A_1 &gt; A_3 &gt; A_4$</td>
</tr>
<tr>
<td>$p = 2, q = 1$</td>
<td>$S(N_1) = 2.1283, S(N_2) = 2.8979$</td>
<td>$A_2 &gt; A_1 &gt; A_3 &gt; A_4$</td>
</tr>
<tr>
<td>$p = 2, q = 2$</td>
<td>$S(N_1) = 2.1477, S(N_2) = 2.8915$</td>
<td>$A_2 &gt; A_1 &gt; A_3 &gt; A_4$</td>
</tr>
</tbody>
</table>
As we can see from Table 2 and Table 3, the ranking of the alternatives may be different for the different values \( p, q \) in IVNHFULWBM operator and IVNHFULWGBM operator. But the best alternative is the same although the different values \( p, q \). Thus, the organization can properly select the desirable alternative according to his interest and the actual needs. In general, we can get \( p = q = 1 \) or \( p = q = \frac{1}{2} \).

6. Conclusions

Interval-valued neutrosophic hesitant fuzzy uncertain linguistic set (IVNHFULS) can be as an important tool of describing the decision maker’s preference and handling the hesitant, uncertain, incomplete information, which plays a vital role in decision making. Bonferroni mean (BM) can replace the simple averaging of other mean type operators because of the excellent modeling capability. In real decision making, how to flexibly and rationally deal with input arguments is a vital problem. In order to have the advantages of both IVNHFULS and BM, we extend BM operator to interval-valued neutrosophic hesitant fuzzy uncertain linguistic environment. Hence, we finally give some interval-valued neutrosophic hesitant fuzzy uncertain linguistic BM aggregation operators, such as interval-valued neutrosophic hesitant fuzzy uncertain linguistic Bonferroni mean (IVNHFULBM) operator, interval-valued neutrosophic hesitant fuzzy uncertain linguistic weighted Bonferroni mean (IVNHFULWBM) operator, interval-valued neutrosophic hesitant fuzzy uncertain linguistic geometric Bonferroni mean (IVNHFULGBM) operator, interval-valued neutrosophic hesitant fuzzy uncertain linguistic weighted geometric Bonferroni mean (IVNHFULWGBM) operator. It is essential to verify the effectiveness of IVNHFULWBM operator and IVNHFULWGBM operator, so we apply the two operators to deal with a numerical example. To get the desirable alternative decision maker wanted, we change the values of \( p, q \) according to their interest and actual need.

In the further research, we should dug deeper into the applications of the proposed methods and constantly improve the multiple attribute decision making method.

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Compliance with Ethical Standards

(1) Disclosure of potential conflicts of interest
We declare that we do have no commercial or associative interests that represent a conflict of interests in connection with this manuscript. There are no professional or other personal interests that can inappropriately influence our submitted work.

(2) Research involving human participants and/or animals
This article does not contain any studies with human participants or animals performed by any of the authors.

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