Neutrosophic Set and Neutrosophic Topological Spaces

1A.A.Salama, 2S.A.Alblowi
1Egypt, Port Said University, Faculty of Sciences Department of Mathematics and Computer Science
2Department of Mathematics, King Abdulaziz University, Saudi Arabia

Abstract: Neutrosophy has been introduced by Smarandache [7, 8] as a new branch of philosophy. The purpose of this paper is to construct a new set theory called the neutrosophic set. After given the fundamental definitions of neutrosophic set operations, we obtain several properties, and discussed the relationship between neutrosophic sets and others. Finally, we extend the concept of fuzzy topological space [4], and intuitionistic fuzzy topological space [5, 6] to the case of neutrosophic sets. Possible application to superstrings and space-time are touched upon.

Keywords: Fuzzy topology; fuzzy set; neutrosophic set; neutrosophic topology

I. Introduction
The fuzzy set was introduced by Zadeh [9] in 1965, where each element had a degree of membership. The intuitionstic fuzzy set (Ifs for short) on a universe X was introduced by K. Atanassov [1, 2, 3] in 1983 as a generalization of fuzzy set, where besides the degree of membership and the degree of non-membership of each element. After the introduction of the neutrosophic set concept [7, 8]. In recent years neutrosophic algebraic structures have been investigated. Neutrosophy has laid the foundation for a whole family of new mathematical theories generalizing both their classical and fuzzy counterparts, such as a neutrosophic set theory.

II. Terminologies
We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [7, 8], and Atanassov in [1, 2, 3]. Smarandache introduced the neutrosophic components T, I, F which represent the membership, indeterminacy, and non-membership values respectively, where $[0^-,1^+]$ is nonstandard unit interval.

2.1 Definition. [3,4]
Let T, I, F be real standard or nonstandard subsets of $[0^-,1^+]$, with
Sup_T=t_sup, inf_T=t_inf
Sup_I=i_sup, inf_I=i_inf
Sup_F=f_sup, inf_F=f_inf

n_sup=t_sup+i_sup+f_sup
n_inf=t_inf+i_inf+f_inf.

T, I, F are called neutrosophic components

III. Neutrosophic Sets and Its Operations
We shall now consider some possible definitions for basic concepts of the neutrosophic set and its operations.

3.1 Definition
Let $X$ be a non-empty fixed set. A neutrosophic set ($NS$ for short) $A$ is an object having the form $A = \{x, \mu_A(x), \sigma_A(x), \gamma_A(x) : x \in X\}$ Where $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x)$ which represent the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$), and the degree of non-membership (namely $\gamma_A(x)$) respectively of each element $x \in X$ to the set $A$.

3.1 Remark
A neutrosophic $A = \{x, \mu_A(x), \sigma_A(x), \gamma_A(x) : x \in X\}$ can be identified to an ordered triple $<\mu_A, \sigma_A, \gamma_A>$ in $[0^-,1^+]$ on $X$. 

www.iosrjournals.org
3.2 Remark
For the sake of simplicity, we shall use the symbol $A = \{x, \mu_A(x), \sigma_A(x), \gamma_A(x)\}$ for the $NS$ $A = \{x, \mu_A(x), \sigma_A(x), \gamma_A(x)\} \times x \in \mathbb{X}$

3.1 Example
Every IFS $A$ a non-empty set $X$ is obviously on $NS$ having the form

$A = \{x, \mu_A(x), 1 - (\mu_A(x) + \sigma_A(x)), \gamma_A(x)\} \times x \in \mathbb{X}$

Since our main purpose is to construct the tools for developing neutrosophic set and neutrosophic topology, we must introduce the $NSS$ $0_v$ and $1_v$ in $X$ as follows:

$0_v$ may be defined as:

$(0_1) 0_v = \{(x, 0, 0, 1): x \in \mathbb{X}\}$
$(0_2) 0_v = \{(x, 0, 1, 1): x \in \mathbb{X}\}$
$(0_3) 0_v = \{(x, 0, 1, 0): x \in \mathbb{X}\}$
$(0_4) 0_v = \{(x, 0, 0, 0): x \in \mathbb{X}\}$

$1_v$ may be defined as:

$(1_1) 1_v = \{(x, 1, 0, 0): x \in \mathbb{X}\}$
$(1_2) 1_v = \{(x, 1, 0, 1): x \in \mathbb{X}\}$
$(1_3) 1_v = \{(x, 1, 1, 0): x \in \mathbb{X}\}$
$(1_4) 1_v = \{(x, 1, 1, 1): x \in \mathbb{X}\}$

3.2 Definition
Let $A = \{\mu_A, \sigma_A, \gamma_A\}$ a $NS$ on $X$, then the complement of the set $A$ $(C(A)$, for short) maybe defined as three kinds of complements

$(C_1) C(A) = \{(x, 1 - \mu_A(x), 1 - \sigma_A(x), 1 - \gamma_A(x)): x \in \mathbb{X}\}$
$(C_2) C(A) = \{(x, \gamma_A(x), \sigma_A(x), \mu_A(x)): x \in \mathbb{X}\}$
$(C_3) C(A) = \{(x, \gamma_A(x), 1 - \sigma_A(x), 1 - \mu_A(x)): x \in \mathbb{X}\}$

One can define several relations and operations between $NSS$ follows:

3.3 Definition
Let $x$ be a non-empty set, and $NSS A$ and $B$ in the form $A = \{x, \mu_A(x), \sigma_A(x), \gamma_A(x)\}$, $B = \{x, \mu_B(x), \sigma_B(x), \gamma_B(x)\}$, then we may consider two possible definitions for subsets $(A \subseteq B)$

$(A \subseteq B)$ may be defined as

$(1) A \subseteq B \iff \mu_A(x) \leq \mu_B(x), \gamma_A(x) \geq \gamma_B(x)$ and $\sigma_A(x) \leq \sigma_B(x) \forall x \in \mathbb{X}$

$(2) A \subseteq B \Rightarrow \mu_A(x) \leq \mu_B(x), \gamma_A(x) \geq \gamma_B(x)$ and $\sigma_A(x) \geq \sigma_B(x)$

3.1 Proposition
For any neutrosophic set $A$ the following are holds

$(1) 0_v \subseteq A \Rightarrow 0_v \subseteq 0_v$

$(2) A \subseteq 1_v \Rightarrow 1_v \subseteq 1_v$

3.4. Definition
Let $X$ be a non-empty set, and $A = \{x, \mu_A(x), \gamma_A(x), \sigma_A(x)\}$, $B = \{x, \mu_B(x), \gamma_B(x), \sigma_B(x)\}$ are $NSS$. Then

$(1) A \cap B$ maybe defined as:

$(I_1) A \cap B = \{x, \mu_A(x) \land \mu_B(x), \sigma_A(x) \land \sigma_B(x), \gamma_A(x) \lor \gamma_B(x)\}$

$(I_2) A \cap B = \{x, \mu_A(x) \lor \mu_B(x), \sigma_A(x) \lor \sigma_B(x), \gamma_A(x) \land \gamma_B(x)\}$

$(I_3) A \cap B = \{x, \mu_A(x) \lor \mu_B(x), \sigma_A(x) \lor \sigma_B(x), \gamma_A(x) \lor \gamma_B(x)\}$
Neutrosophic Set And Neutrosophic Topological Spaces

(2) \( A \cup B \) may be defined as:

\[
\gamma_a(x) \wedge \gamma_b(x) \bigg) \\
\gamma_a(x) < x, \mu_a(x) \vee \mu_b(x), \sigma_a(x) \vee \sigma_b(x) >
\]

\[
\gamma_a(x) \vee \gamma_b(x) \bigg) \\
\gamma_a(x) < x, \mu_a(x) \vee \mu_b(x), \sigma_a(x) \wedge \sigma_b(x) >
\]

\[
[ ] \gamma_a(x) < x, \mu_a(x), \sigma_a(x), 1 - \mu_a(x) >
\]

\[
\gamma_a(x) > x, \sigma_a(x), \gamma_a(x) >
\]

We can easily generalize the operations of intersection and union in definition 3.4 to arbitrary family of NSS as follow:

3.5 Definition

Let \( \{ Aj : j \in J \} \) be a arbitrary family of NSS in \( X \), then

\[\big( i\big) \bigcap A_j = \left\{ x, \wedge \mu_j(x), \wedge \sigma_j(x), \vee \gamma_j(x) \right\} \]

\[\big( ii\big) \bigcap A_j = \left\{ x, \wedge \mu_j(x), \vee \sigma_j(x), \vee \gamma_j(x) \right\} \]

\[\big( i\big) \bigcup A_j = \{ x, \vee, \wedge, \wedge \} \]

\[\big( ii\big) \bigcup A_j = \{ x, \vee, \wedge, \wedge \} \]

3.6. Definition

Let \( A \) and \( B \) are neutrosophic sets then \( AB \) may be defined as

\[\gamma\gamma = \left\{ x, \mu_u(x) \wedge \gamma_a(x), \sigma_a(x) \vee \gamma_b(x) \right\} \]

3.2. Proposition

For all \( A, B \) two neutrosophic sets then the following are true

\[\big( 1\big) C(A \cap B) = C(A) \cup C(B) \]

\[\big( 2\big) C(A \cup B) = C(A) \cap C(B) \]

IV. Neutrosophic Topological Spaces

Here we extend the concepts of fuzzy topological space [4], and intuitionistic fuzzy topological space [5, 7] to the case of neutrosophic sets.

4.1 Definition

A neutrosophic topology (NT for short) an a non empty set \( X \) is a family \( \tau \) of neutrosophic subsets in \( X \) satisfying the following axioms

\[
\big( NT_1 \big) O_y, y_k \in \tau , \\
\big( NT_2 \big) G_1 \cap G_2 \in \tau \quad \text{for any } G_1, G_2 \in \tau , \\
\big( NT_3 \big) \bigcup G_i \in \tau \quad \forall \{ G_i : i \in J \} \subseteq \tau
\]

In this case the pair \( (X, \tau) \) is called a neutrosophic topological space (NTS for short) and any neutrosophic set in \( \tau \) is known as neutrosophic open set (NOS for short) in \( X \). The elements of \( \tau \) are called open neutrosophic sets, A neutrosophic set \( F \) is closed if and only if it \( C(F) \) is neutrosophic open.

4.1 Example

Any fuzzy topological space \( (X, \tau) \) in the sense of Chang is obviously a \( NTS \) in the form \( \tau = \{ A : \mu_x \in \tau \} \)

wherever we identify a fuzzy set in \( X \) whose membership function is \( \mu_x \) with its counterpart.

4.1. Remark Neutrosophic topological spaces are very natural generalizations of fuzzy topological spaces allow more general functions to be members of fuzzy topology.

4.3 Example

Let \( X = \{ x \} \) and

\[\{ x, 0.5, 0.5, 0.4 \} : x \in X \}

\[\{ x, 0.4, 0.6, 0.8 \} : x \in X \}

\[\{ x, 0.5, 0.6, 0.4 \} : x \in X \}

\[\{ x, 0.5, 0.6, 0.4 \} : x \in X \}
Neutrosophic Set And Neutrosophic Topological Spaces

\[ C = \{x \mid 0.4, 0.5, 0.8 \} : x \in X \]

Then the family \( \tau = \{O_x, 1.0, A, B, C, D\} \) of NSs in X is neutrosophic topology on X

4.4 Example
Let \((X, \tau_0)\) be a fuzzy topological space in changes sense such that \( \tau_0 \) is not indiscrete suppose now that \( \tau_0 = \{O_{N, 1.0} \cup V_j : j \in J\} \) then we can construct two NTSS on X as follows

a) \( \tau_0 = \{O_{N, 1.0} \cup \langle x, V_j, \sigma(x), 0 > : j \in J\} \).

b) \( \tau_0 = \{O_{N, 1.0} \cup \langle x, V_j, O, \sigma(x), 1 - V_j > : j \in J\} \).

4.1 Proposition
Let \((X, \tau)\) be a NTS on X, then we can also construct several NTSS on X in the following way:

a) \( \tau_{a, 1} = \{\} G : G \in \tau \}, \)

b) \( \tau_{a, 2} = \langle\rangle G : G \in \tau \}.

Proof
a) \((NT, _1)\) and \((NT, _2)\) are easy.

\(NT, _1\) Let \( \{G_j' : j \in J, G_j' \in \tau \} \subseteq \tau_{0, 1} \). Since

\[ \cup G_j' = \left\{ \left\{ x, \cup \mu_{G_j'}, \sigma_{G_j'}, \gamma_{G_j'} \right\} \mid \left\{ x, \cup \mu_{G_j'}, \sigma_{G_j'}, \gamma_{G_j'} \right\} \in \tau \right\} \],

we have

\[ \cup \{G_j' \} = \left\{ x, \cup \mu_{G_j'}, \sigma_{G_j'}, \gamma_{G_j'} \right\} \mid \left\{ x, \cup \mu_{G_j'}, \sigma_{G_j'}, \gamma_{G_j'} \right\} \in \tau_{0, 1} \].

Proof similar to (a)

4.2 Definition
Let \((X, \tau_1), (X, \tau_2)\) be two neutrosophic topological spaces on X. Then \( \tau_1 \) is said be contained in \( \tau_2 \) (in symbols \( \tau_1 \subseteq \tau_2 \)) if \( G \in \tau_2 \) for each \( G \in \tau_1 \). In this case, we also say that \( \tau_1 \) is coarser than \( \tau_2 \).

4.2 Proposition
Let \( \{\tau_j : j \in J\} \) be a family of NTSS on X. Then \( \sqcap \tau_j \) is a neutrosophic topology on X. Furthermore, \( \sqcap \tau_j \) is the coarsest NT on X containing all \( \tau_j \).

Proof. Obvious.

4.3 Definition
The complement of A (C(A) for short) of NOSs A is called a neutrosophic closed set (NCS for short) in X.

Now, we define neutrosophic closure and interior operations in neutrosophic topological spaces:

4.4 Definition
Let \((X, \tau)\) be NTS and \( A = \langle x, \mu_A(x), \gamma_A(x), \sigma_A(x) \rangle \) be a NS in X.

Then the neutrosophic closer and neutrosophic interior of A are defined by

\( \text{NCI}(A) = \sqcap \{K : K \text{ is an NCS in X and } A \subseteq K\} \)

\( \text{NInt}(A) = \cup \{G : G \text{ is an NOS in X and } G \subseteq A\} \). It can be also shown that

It can be also shown that \( \text{NCI}(A) \) is NCS and \( \text{NInt}(A) \) is a NOS in X.

a) \( A \) is in X if and only if \( \text{NCI}(A) \).

b) \( A \) is NCS in X if and only if \( \text{NInt}(A) = A \).

4.2 Proposition
For any neutrosophic set \( \bar{A} \) in \((X, \tau)\) we have

(a) \( \text{NCI}(C(A)) = C(\text{NInt}(A)) \).

(b) \( \text{NInt}(C(A)) = C(\text{NCI}(A)) \).

Proof.

a) Let \( A = \{x, \mu_A(x), \sigma_A(x), \nu_A : x \in X\} \) and suppose that the family of neutrosophic subsets contained in \( A \) are indexed by the family if NSS contained in \( \bar{A} \) are indexed by the
family $A = \{ \langle x, \mu_{G_i}, \sigma_{G_i}, \sigma_{G_i} \rangle : i \in J \}$. Then we see that $\text{NInt}(A) = \{ \langle x, \sigma_{G_i}, \sigma_{G_i} \rangle : i \in J \}$ and hence $C(\text{NInt}(A)) = \{ \langle x, \sigma_{G_i}, \sigma_{G_i} \rangle : i \in J \}$. Since $C(A)$ and $\mu_{G_i} \leq \mu_A$ and $\sigma_{G_i} \geq \sigma_A$ for each $i \in J$, we obtain $\text{NInt}(C(A)) = \{ \langle x, \sigma_{G_i}, \sigma_{G_i} \rangle : i \in J \}$. Hence $\text{NCl}(C(A)) = C(\text{NInt}(A))$, follows immediately

b) This is analogous to (a).

4.3 Proposition

Let $(x, \tau)$ be a $\text{NTS}$ and $A, B$ be two neutrosophic sets in $X$. Then the following properties hold:

(a) $\text{NInt}(A) \subseteq A$,
(b) $A \subseteq \text{NCl}(A)$,
(c) $A \subseteq B \Rightarrow \text{NInt}(A) \subseteq \text{NInt}(B)$,
(d) $A \subseteq B \Rightarrow \text{NCl}(A) \subseteq \text{NCl}(B)$,
(e) $\text{NInt}(\text{NInt}(A)) = \text{NInt}(A) \cap \text{NInt}(B)$,
(f) $\text{NCl}(A \cup B) = \text{NCl}(A) \cup \text{NCl}(B)$,
(g) $\text{NInt}(1_N) = 1_N$,
(h) $\text{NCl}(O_N) = O_N$.

Proof (a), (b) and (e) are obvious (c) follows from (a) and Definitions.

References

[7] Florentin Smarandache, Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability, and Statistics University of New Mexico, Gallup, NM 87301, USA, 2002, smarand@unm.edu