Applications of Smarandache Function, and Prime and Coprime Functions

\[ C_k(n_1, n_2, \ldots, n_k) = \begin{cases} 0 & \text{if } n_1, n_2, \ldots, n_k \text{ are coprime numbers} \\ 1 & \text{otherwise} \end{cases} \]

American Research Press
Rehoboth
2002
Applications of Smarandache Function, and Prime and Coprime Functions
Contents:

Chapter 1: Smarandache Function applied to perfect numbers
Chapter 2: A result obtained using the Smarandache Function
Chapter 3: A Congruence with the Smarandache Function
Chapter 4: A functional recurrence to obtain the prime numbers using the Smarandache prime function
Chapter 5: The general term of the prime number sequence and the Smarandache prime function
Chapter 6: Expressions of the Smarandache Coprime Function
Chapter 7: New Prime Numbers
Chapter 1: Smarandache function applied to perfect numbers

The Smarandache function is defined as follows:

\[ S(n) = \text{the smallest positive integer such that } S(n)! \text{ is divisible by } n. \] [1]

In this article we are going to see that the value this function takes when \( n \) is a perfect number of the form \( n = 2^{k-1} \cdot (2^k - 1) \), \( p = 2^k - 1 \) being a prime number.

Lemma 1: Let \( n = 2^i \cdot p \) when \( p \) is an odd prime number and \( i \) an integer such that:

\[
0 \leq i \leq E\left(\frac{p}{2}\right) + E\left(\frac{p}{2^2}\right) + E\left(\frac{p}{2^3}\right) + \cdots + E\left(\frac{p}{2^{E\left(\log_2 p\right)}}\right) = e_2(p!)
\]

where \( e_2(p!) \) is the exponent of 2 in the prime number decomposition of \( p! \).

\( E(x) \) is the greatest integer less than or equal to \( x \).

One has that \( S(n) = p \).

Demonstration:
Given that \( \text{GCD}(2^i, p) = 1 \) (GCD= greatest common divisor) one has that \( S(n) = \max\{S(2^i), S(p)\} \geq S(p) = p \). Therefore \( S(n) \geq p \).

If we prove that \( p! \) is divisible by \( n \) then one would have the equality.

\[ p! = p_1^{e_{p_1}(p)} \cdot p_2^{e_{p_2}(p)} \cdot \cdots \cdot p_s^{e_{p_s}(p)} \]

where \( p_i \) is the \( i \)-th prime of the prime number decomposition of \( p! \). It is clear that \( p_1 = 2, \ p_s = p, \ e_{p_i}(p!) = 1 \) for which:

\[ p! = 2^{e_2(p)} \cdot p_2^{e_{p_2}(p)} \cdot \cdots \cdot p_{s-1}^{e_{p_{s-1}}(p)} \cdot p \]
From where one can deduce that:

\[
\frac{p_i}{n} = 2^{e_2(p_i)^{l-i}} \cdot p_2^{e_2(p_2)^{l-2}} \cdots p_{i-1}^{e_2(p_{i-1})^{l-(i-1)}}
\]

is a positive integer since \(e_2(p!) - i \geq 0\).

Therefore one has that \(S(n) = p\)

Proposition 1: If \(n\) is a perfect number of the form \(n = 2^{k-1} \cdot (2^k - 1)\) with \(k\) is a positive integer, \(2^k - 1 = p\) prime, one has that \(S(n) = p\).

Demonstration:

For the Lemma it is sufficient to prove that \(k - 1 \leq e_2(p!)\).

If we can prove that:

\[
k - 1 \leq 2^{k-1} - \frac{1}{2}
\]

we will have proof of the proposition since:

\[
k - 1 \leq 2^{k-1} - \frac{1}{2} = \frac{2^k - 1}{2} = \frac{p}{2}
\]

As \(k - 1\) is an integer one has that \(k - 1 \leq E\left(\frac{p}{2}\right) \leq e_2(p!)\).

Proving (1) is the same as proving \(k \leq 2^{k-1} + \frac{1}{2}\) at the same time, since \(k\) is integer, is equivalent to proving \(k \leq 2^{k-1}\) (2).

In order to prove (2) we may consider the function: \(f(x) = 2^{x-1} - x\) \(x\) real number.

This function may be derived and its derivate is \(f'(x) = 2^{x-1} \ln 2 - 1\).

\(f\) will be increasing when \(2^{x-1} \ln 2 - 1 > 0\) resolving \(x\):

\[
x > 1 - \frac{\ln(\ln 2)}{\ln 2} \approx 1.5287
\]

In particular \(f\) will be increasing \(\forall x \geq 2\).

Therefore \(\forall x \geq 2\) \(f(x) \geq f(2) = 0\) that is to say \(2^{x-1} - x \geq 0\) \(\forall x \geq 2\).
Therefore: $2^{k-1} \geq k \quad \forall k \geq 2$ integer.

And thus is proved the proposition.

**EXAMPLES:**

<table>
<thead>
<tr>
<th>$n$</th>
<th>Factorization</th>
<th>$S(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$2 \cdot 3$</td>
<td>$S(6)=3$</td>
</tr>
<tr>
<td>28</td>
<td>$2^2 \cdot 7$</td>
<td>$S(28)=7$</td>
</tr>
<tr>
<td>496</td>
<td>$2^4 \cdot 31$</td>
<td>$S(496)=31$</td>
</tr>
<tr>
<td>8128</td>
<td>$2^6 \cdot 127$</td>
<td>$S(8128)=127$</td>
</tr>
</tbody>
</table>

References:

Chapter 2: A result obtained using the Smarandache Function

Smarandache Function is defined as followed: S(m)=The smallest positive integer so that S(m)! is divisible by m. [1]

Let’s see the value which such function takes for \( m = p^n \) with n integer, \( n \geq 2 \) and p prime number. To do so a Lemma required.

**Lemma 1** \( \forall \ m,n \in \mathbb{N} \ \ m,n \geq 2 \)

\[
m^n = E\left[\frac{m^{n+1} - m^n + m}{m} \right] + E\left[\frac{m^{n+1} - m^n + m}{m^2} \right] + \cdots + E\left[\frac{m^{n+1} - m^n + m}{m^{E\left[\log_m(m^{n+1} - m^n + m)\right]}} \right]
\]

where \( E(x) \) gives the greatest integer less than or equal to \( x \).

**Proof:**

Let’s see in the first place the value taken by \( E\left[\log_m(m^{n+1} - m^n + m)\right] \).

If \( n \geq 2 \):

\[
m^{n+1} - m^n + m < m^{n+1}
\]

and therefore

\[
\log_m(m^{n+1} - m^n + m) < \log_m(m^{n+1}) = n + 1.
\]

And if \( m \geq 2 \):

\[
mm^n \geq 2m^n \Rightarrow m^{n+1} \geq 2m^n \Rightarrow m^{n+1} + m \geq 2m^n \Rightarrow m^{n+1} - m^n + m \geq m^n
\]

\[
\log_m(m^{n+1} - m^n + m) \geq \log_m(m^n) = n \Rightarrow E\left[\log_m(m^{n+1} - m^n + m)\right] \geq n
\]

As a result: \( n \leq E\left[\log_m(m^{n+1} - m^n + m)\right] < n + 1 \) therefore:

\[
E\left[\log_m(m^{n+1} - m^n + m)\right] = n \quad \text{if} \quad n,m \geq 2
\]

Now let’s see the value which it takes for \( 1 \leq k \leq n \):

\[
E\left[\frac{m^{n+1} - m^n + m}{m^k}\right] = E\left[\frac{m^{n+1-k} - m^{n-k} + \frac{1}{m^{k-1}}}{m^k}\right]
\]
If \( k = 1 \):
\[
E\left[ \frac{m^{n+1} - m^n + m}{m^k} \right] = m^n - m^{n-1} + 1
\]

If \( 1 < k \leq n \):
\[
E\left[ \frac{m^{n+1} - m^n + m}{m^k} \right] = m^{n-1} - m^{n-k}
\]

Let’s see what is the value of the sum:

\[
\begin{align*}
&\text{k=1} & &m^n & -m^{n-1} & &\ldots & &\ldots & &\ldots & &\ldots & &+1 \\
&\text{k=2} & &m^{n-1} & -m^{n-2} \\
&\text{k=3} & &m^{n-2} & -m^{n-3} \\
&\ldots & &\ldots & &\ldots & &\ldots & &\ldots \\
&\text{k=n-1} & &m^2 & -m \\
&\text{k=n} & &m & -1
\end{align*}
\]

Therefore:
\[
\sum_{k=1}^{n} E\left[ \frac{m^{n+1} - m^n + m}{m^k} \right] = m^n \quad m, n \geq 2
\]

**Proposition:** \( \forall \ p \ \text{prime number} \ \forall n \geq 2 : \)
\[
S(p^{p^n}) = p^{n+1} - p^n + p
\]

**Proof:**

Having \( e_p(k) = \) exponent of the prime number \( p \) in the prime decomposition of \( k \).

We get:
\[
e_p(k) = E\left( \frac{k}{p} \right) + E\left( \frac{k}{p^2} \right) + E\left( \frac{k}{p^3} \right) + \ldots + E\left( \frac{k}{p^{E(\log_p k)}} \right)
\]
And using the lemma we have

\[ e_p[(p^{n+1} - p^n + p)] = E\left[p^{n+1} - p^n + \frac{p^n}{p}\right] + \cdot \cdot \cdot + E\left[p^{n+1} - p^n + \frac{p^n}{p}\right] = p^n \]

Therefore:

\( \frac{(p^{n+1} - p^n + p)}{p^n} \in \mathbb{N} \) and \( \frac{(p^{n+1} - p^n + p - 1)}{p^n} \notin \mathbb{N} \)

And:

\[ S(p^n) = p^{n+1} - p^n + p \]

References:

Smarandache’s function is defined thus:

\[ S(n) = \text{the smallest integer such that } S(n)! \text{ is divisible by } n. \] \[ [1] \]

In this article we are going to look at the value that has \( S(2^k - 1) \pmod{k} \)

For all integer, \( 2 \leq k \leq 97 \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>( S(2^k-1) )</th>
<th>( S(2^k-1) \pmod{k} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>31</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>127</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>17</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>73</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>31</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>89</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>13</td>
<td>1</td>
</tr>
<tr>
<td>13</td>
<td>8191</td>
<td>1</td>
</tr>
<tr>
<td>14</td>
<td>127</td>
<td>1</td>
</tr>
<tr>
<td>15</td>
<td>151</td>
<td>1</td>
</tr>
<tr>
<td>16</td>
<td>257</td>
<td>1</td>
</tr>
<tr>
<td>17</td>
<td>131071</td>
<td>1</td>
</tr>
<tr>
<td>18</td>
<td>73</td>
<td>1</td>
</tr>
<tr>
<td>19</td>
<td>524287</td>
<td>1</td>
</tr>
<tr>
<td>20</td>
<td>41</td>
<td>1</td>
</tr>
<tr>
<td>21</td>
<td>337</td>
<td>1</td>
</tr>
<tr>
<td>22</td>
<td>683</td>
<td>1</td>
</tr>
<tr>
<td>23</td>
<td>178481</td>
<td>1</td>
</tr>
<tr>
<td>24</td>
<td>241</td>
<td>1</td>
</tr>
<tr>
<td>25</td>
<td>1801</td>
<td>1</td>
</tr>
<tr>
<td>26</td>
<td>8191</td>
<td>1</td>
</tr>
<tr>
<td>27</td>
<td>262657</td>
<td>1</td>
</tr>
<tr>
<td>28</td>
<td>127</td>
<td>15</td>
</tr>
<tr>
<td>29</td>
<td>2089</td>
<td>1</td>
</tr>
<tr>
<td>30</td>
<td>331</td>
<td>1</td>
</tr>
<tr>
<td>k</td>
<td>S($2^k$-1)</td>
<td>S($2^k$-1) (mod k)</td>
</tr>
<tr>
<td>----</td>
<td>--------------</td>
<td>-------------------</td>
</tr>
<tr>
<td>31</td>
<td>2147483647</td>
<td>1</td>
</tr>
<tr>
<td>32</td>
<td>65537</td>
<td>1</td>
</tr>
<tr>
<td>33</td>
<td>599479</td>
<td>1</td>
</tr>
<tr>
<td>34</td>
<td>131071</td>
<td>1</td>
</tr>
<tr>
<td>35</td>
<td>122921</td>
<td>1</td>
</tr>
<tr>
<td>36</td>
<td>109</td>
<td>1</td>
</tr>
<tr>
<td>37</td>
<td>616318177</td>
<td>1</td>
</tr>
<tr>
<td>38</td>
<td>524287</td>
<td>1</td>
</tr>
<tr>
<td>39</td>
<td>121369</td>
<td>1</td>
</tr>
<tr>
<td>40</td>
<td>61681</td>
<td>1</td>
</tr>
<tr>
<td>41</td>
<td>164511353</td>
<td>1</td>
</tr>
<tr>
<td>42</td>
<td>5419</td>
<td>1</td>
</tr>
<tr>
<td>43</td>
<td>2099863</td>
<td>1</td>
</tr>
<tr>
<td>44</td>
<td>2113</td>
<td>1</td>
</tr>
<tr>
<td>45</td>
<td>23311</td>
<td>1</td>
</tr>
<tr>
<td>46</td>
<td>2796203</td>
<td>1</td>
</tr>
<tr>
<td>47</td>
<td>13264529</td>
<td>1</td>
</tr>
<tr>
<td>48</td>
<td>673</td>
<td>1</td>
</tr>
<tr>
<td>49</td>
<td>4432676798593</td>
<td>1</td>
</tr>
<tr>
<td>50</td>
<td>4051</td>
<td>1</td>
</tr>
<tr>
<td>51</td>
<td>131071</td>
<td>1</td>
</tr>
<tr>
<td>52</td>
<td>8191</td>
<td>27</td>
</tr>
<tr>
<td>53</td>
<td>20394401</td>
<td>1</td>
</tr>
<tr>
<td>54</td>
<td>262657</td>
<td>1</td>
</tr>
<tr>
<td>55</td>
<td>201961</td>
<td>1</td>
</tr>
<tr>
<td>56</td>
<td>15790321</td>
<td>1</td>
</tr>
<tr>
<td>57</td>
<td>1212847</td>
<td>1</td>
</tr>
<tr>
<td>58</td>
<td>3033169</td>
<td>1</td>
</tr>
<tr>
<td>59</td>
<td>3203431780337</td>
<td>1</td>
</tr>
<tr>
<td>60</td>
<td>1321</td>
<td>1</td>
</tr>
<tr>
<td>61</td>
<td>2305843009213693951</td>
<td>1</td>
</tr>
<tr>
<td>62</td>
<td>2147483647</td>
<td>1</td>
</tr>
<tr>
<td>63</td>
<td>649657</td>
<td>1</td>
</tr>
<tr>
<td>64</td>
<td>6700417</td>
<td>1</td>
</tr>
<tr>
<td>65</td>
<td>145295143558111</td>
<td>1</td>
</tr>
<tr>
<td>66</td>
<td>599479</td>
<td>1</td>
</tr>
<tr>
<td>67</td>
<td>761838257287</td>
<td>1</td>
</tr>
<tr>
<td>68</td>
<td>131071</td>
<td>35</td>
</tr>
<tr>
<td>k</td>
<td>$S(2^k-1)$</td>
<td>$S(2^k-1) \mod k$</td>
</tr>
<tr>
<td>------</td>
<td>--------------------</td>
<td>--------------------</td>
</tr>
<tr>
<td>69</td>
<td>10052678938039</td>
<td>1</td>
</tr>
<tr>
<td>70</td>
<td>122921</td>
<td>1</td>
</tr>
<tr>
<td>71</td>
<td>212885833</td>
<td>1</td>
</tr>
<tr>
<td>72</td>
<td>38737</td>
<td>1</td>
</tr>
<tr>
<td>73</td>
<td>9361973132609</td>
<td>1</td>
</tr>
<tr>
<td>74</td>
<td>616318177</td>
<td>1</td>
</tr>
<tr>
<td>75</td>
<td>10567201</td>
<td>1</td>
</tr>
<tr>
<td>76</td>
<td>525313</td>
<td>1</td>
</tr>
<tr>
<td>77</td>
<td>581283643249112959</td>
<td>1</td>
</tr>
<tr>
<td>78</td>
<td>22366891</td>
<td>1</td>
</tr>
<tr>
<td>79</td>
<td>1113491139767</td>
<td>1</td>
</tr>
<tr>
<td>80</td>
<td>4278255361</td>
<td>1</td>
</tr>
<tr>
<td>81</td>
<td>97685839</td>
<td>1</td>
</tr>
<tr>
<td>82</td>
<td>8831418697</td>
<td>1</td>
</tr>
<tr>
<td>83</td>
<td>57912614113275649087721</td>
<td>1</td>
</tr>
<tr>
<td>84</td>
<td>14449</td>
<td>1</td>
</tr>
<tr>
<td>85</td>
<td>9520972806333758431</td>
<td>1</td>
</tr>
<tr>
<td>86</td>
<td>2932031007403</td>
<td>1</td>
</tr>
<tr>
<td>87</td>
<td>9857737155463</td>
<td>1</td>
</tr>
<tr>
<td>88</td>
<td>2931542417</td>
<td>1</td>
</tr>
<tr>
<td>89</td>
<td>618970019642690137449562111</td>
<td>1</td>
</tr>
<tr>
<td>90</td>
<td>18837001</td>
<td>1</td>
</tr>
<tr>
<td>91</td>
<td>23140471537</td>
<td>1</td>
</tr>
<tr>
<td>92</td>
<td>2796203</td>
<td>47</td>
</tr>
<tr>
<td>93</td>
<td>658812288653553079</td>
<td>1</td>
</tr>
<tr>
<td>94</td>
<td>165768537521</td>
<td>1</td>
</tr>
<tr>
<td>95</td>
<td>30327152671</td>
<td>1</td>
</tr>
<tr>
<td>96</td>
<td>22253377</td>
<td>1</td>
</tr>
<tr>
<td>97</td>
<td>13842607235828485645766393</td>
<td>1</td>
</tr>
</tbody>
</table>

One can see from the table that there are only 4 exceptions for $2 \leq k \leq 97$.
We can see in detail the 4 exceptions in a table:

\[
\begin{array}{ccc}
  k=28=2^2 \cdot 7 & S(2^{28}-1) \equiv 15 \pmod{28} \\
  k=52=2^2 \cdot 13 & S(2^{52}-1) \equiv 27 \pmod{52} \\
  k=68=2^2 \cdot 17 & S(2^{68}-1) \equiv 35 \pmod{68} \\
  k=92=2^2 \cdot 23 & S(2^{92}-1) \equiv 47 \pmod{92}
\end{array}
\]

One can observe in these 4 cases that \( k=2^p \) with \( p \) is a prime and more over \( S(2^k-1) \equiv \frac{k}{2} + 1 \pmod{k} \)

**UNSOLVED QUESTION:**

One can obtain a general formula that gives us, in function of \( k \) the value \( S(2^k-1) \pmod{k} \) for all positive integer values of \( k \).

**Reference:**

Chapter 4: A functional recurrence to obtain the prime numbers using the Smarandache prime function.

Theorem: We are considering the function:

For \( n \) integer:

\[
F(n) = n + 1 + \sum_{m=1}^{2n} \prod_{j=1}^m - \left[ \sum_{j=1}^{m} \left( \left\lfloor \frac{i}{j} \right\rfloor - \left\lfloor \frac{i-1}{j} \right\rfloor \right) - 2 \right]
\]

one has: \( p_{k+1} = F(p_k) \) for all \( k \geq 1 \) where \( \{p_k\}_{k \geq 1} \) are the prime numbers and \( \lfloor x \rfloor \) is the greatest integer less than or equal to \( x \).

Observe that the knowledge of \( p_{k+1} \) only depends on knowledge of \( p_k \) and the knowledge of the fore primes is unnecessary.

Proof:

Suppose that we have found a function \( P(i) \) with the following property:

\[
P(i) = \begin{cases} 
1 \text{ if } i \text{ is composite} \\
0 \text{ if } i \text{ is prime} 
\end{cases}
\]

This function is called Smarandache prime function. (Ref.)

Consider the following product:

\[
\prod_{i=p_k+1}^{m} P(i)
\]

If \( p_k < m < p_{k+1} \) \( \prod_{i=p_k+1}^{m} P(i) = 1 \) since \( i : p_k + 1 \leq i \leq m \) are all composites.
If \( m \geq p_{k+1} \):

\[
\prod_{i = p_k + 1}^{m} P(i) = 0 \quad \text{since} \quad P(p_{k+1}) = 0
\]

Here is the sum:

\[
\sum_{m=p_k+1}^{p_{k+1}} \prod_{i=p_k+1}^{m} P(i) + \sum_{m=p_k+1}^{p_{k+1}} \prod_{i=p_k+1}^{m} P(i) = \sum_{m=p_k+1}^{p_{k+1}} 1
\]

\[
= p_{k+1} - 1 - (p_k + 1) + 1 = p_{k+1} - p_k - 1
\]

The second sum is zero since all products have the factor \( P(p_{k+1}) = 0 \).

Therefore we have the following recurrence relation:

\[
p_{k+1} = p_k + 1 + \sum_{m=p_k+1}^{p_{k+1}} \prod_{i=p_k+1}^{m} P(i)
\]

Let’s now see we can find \( P(i) \) with the asked property.

Consider:

\[
\left\lfloor \frac{i}{j} \right\rfloor - \left\lfloor \frac{i-1}{j} \right\rfloor = \begin{cases} 
1 & \text{if } j \mid i \\
0 & \text{if } j \notmid i
\end{cases} \quad j = 1, 2, \ldots, i \quad i \geq 1
\]

We deduce of this relation:

\[
d(i) = \sum_{j=1}^{i} \left\lfloor \frac{i}{j} \right\rfloor - \left\lfloor \frac{i-1}{j} \right\rfloor
\]

where \( d(i) \) is the number of divisors of \( i \).
If \( i \) is prime \( d(i) = 2 \) therefore:

\[
- \left[ -\frac{d(i) - 2}{i} \right] = 0
\]

If \( i \) is composite \( d(i) > 2 \) therefore:

\[
0 < \frac{d(i) - 2}{i} < 1 \Rightarrow - \left[ -\frac{d(i) - 2}{i} \right] = 1
\]

Therefore we have obtained the Smarandache Prime Function \( P(i) \) which is:

\[
P(i) = \left[ -\sum_{j=1}^{i} \left( \left\lfloor \frac{i}{j} \right\rfloor - \left\lfloor \frac{i-1}{j} \right\rfloor \right) \right]^{-2} \quad i \geq 2 \quad \text{integer}
\]

With this, the theorem is already proved.

References:

www.gallup.unm.edu/~Smarandache/primfnct.txt

Chapter 5: The general term of the prime number sequence and the Smarandache prime function.

Let us consider the function \( d(i) = \text{number of divisors of the positive integer number } i \). We have found the following expression for this function:

\[
d(i) = \sum_{k=1}^{i} E\left(\frac{i}{k}\right) - E\left(\frac{i-1}{k}\right)
\]

“\( E(x) = \text{Floor}[x] \)”

We proved this expression in the article “A functional recurrence to obtain the prime numbers using the Smarandache Prime Function”.

We deduce that the following function:

\[
G(i) = -E\left[-\frac{d(i) - 2}{i}\right]
\]

This function is called the Smarandache Prime Function (Reference). It takes the next values:

\[
G(i) = \begin{cases} 
0 & \text{if } i \text{ is prime} \\
1 & \text{if } i \text{ is composite}
\end{cases}
\]

Let us consider now \( \pi(n) = \text{number of prime numbers smaller or equal than } n \).

It is simple to prove that:

\[
\pi(n) = \sum_{i=2}^{n} (1 - G(i))
\]
Let is have too:

\[ \text{If } 1 \leq k \leq p_n - 1 \Rightarrow E\left( \frac{\pi(k)}{n} \right) = 0 \]

\[ \text{If } C_n \geq k \geq p_n \Rightarrow E\left( \frac{\pi(k)}{n} \right) = 1 \]

We will see what conditions have to carry \( C_n \).

Therefore we have the following expression for \( p_n \) n-th prime number:

\[ p_n = 1 + \sum_{k=1}^{C_n} \left( 1 - E\left( \frac{\pi(k)}{n} \right) \right) \]

If we obtain \( C_n \) that only depends on \( n \), this expression will be the general term of the prime numbers sequence, since \( \pi \) is in function with \( G \) and \( G \) does with \( d(i) \) that is expressed in function with \( i \) too. Therefore the expression only depends on \( n \).

Let is consider \( C_n = 2(E(n \log n) + 1) \)

Since \( p_n \approx n \log n \) from of a certain \( n_0 \) it will be true that

\[ (1) \quad p_n \leq 2(E(n \log n) + 1) \]

If \( n_0 \) it is not too big, we can prove that the inequality is true for smaller or equal values than \( n_0 \).

It is necessary to that:

\[ E\left( \frac{\pi(2(E(n \log n) + 1))}{n} \right) = 1 \]

If we check the inequality:

\[ (2) \quad \pi(2(E(n \log n) + 1)) < 2n \]
We will obtain that:

\[
\frac{\pi(C_n)}{n} < 2 \Rightarrow E\left[ \frac{\pi(C_n)}{n} \right] \leq 1 \quad ; \quad C_n \geq p_n \Rightarrow E\left[ \frac{\pi(C_n)}{n} \right] = 1
\]

We can experimentally check this last inequality saying that it checks for a lot of values and the difference tends to increase, which makes to think that it is true for all \( n \).

Therefore if we prove that the (1) and (2) inequalities are true for all \( n \) which seems to be very probable; we will have that the general term of the prime numbers sequence is:

\[
p_n = 1 + \sum_{k=1}^{2(E(n \log n) + 1)} \left[ 1 - E\left[ \frac{j}{n} \left( \sum_{j=2}^{k} \frac{\sum_{s=1}^{j} (E(j/s) - E((j-1)/s)) - 2}{j} \right) \right] \right]
\]

Reference:
Http://www.gallup.unm.edu/~Smarandache/primfnct.txt
Chapter 6: Expressions of the Smarandache Coprime Function

Smarandache Coprime function is defined this way:

\[
C_k (n_1, n_2, \ldots, n_k) = \begin{cases} 
0 & \text{if } n_1, n_2, \ldots, n_k \text{ are coprime numbers} \\
1 & \text{otherwise}
\end{cases}
\]

We see two expressions of the Smarandache Coprime Function for k=2.

EXPRESSION 1:

\[
C_2 (n_1, n_2) = \left\lfloor -\frac{n_1 n_2 - \text{lcm}(n_1, n_2)}{n_1 n_2} \right\rfloor
\]

\[\lfloor x \rfloor \text{ is the biggest integer number smaller or equal than } x.\]

If \(n_1, n_2\) are coprime numbers:

\[\text{lcm}(n_1, n_2) = n_1 n_2 \text{ therefore: } C_2 (n_1, n_2) = - \left\lfloor \frac{0}{n_1 n_2} \right\rfloor = 0\]

If \(n_1, n_2\) aren’t coprime numbers:

\[\text{lcm}(n_1, n_2) < n_1 n_2 \Rightarrow 0 < \frac{n_1 n_2 - \text{lcm}(n_1, n_2)}{n_1 n_2} < 1 \Rightarrow C_2 (n_1, n_2) = 1\]

EXPRESSION 2:

\[
C_2 (n_1, n_2) = 1 + \left\lfloor \prod_{d|n_1} \prod_{d'|n_2} \frac{|d - d'|}{\prod_{d \neq d'} (d + d')} \right\rfloor
\]
If \( n_1, n_2 \) are coprime numbers then \( d \neq d' \quad \forall d, d' \neq 1 \)

\[
\prod_{d \mid n_1} \prod_{d' \mid n_2} |d - d'|
\]
\[
\Rightarrow 0 < \frac{d}{d'} < 1 \Rightarrow C_2(n_1, n_2) = 0
\]

If \( n_1, n_2 \) aren’t coprime numbers \( \exists d = d' \quad d > 1, d' > 1 \Rightarrow C_2(n_1, n_2) = 1 \)

**EXPRESSION 3:**

Smarandache Coprime Function for \( k \geq 2 \):

\[
C_k(n_1, n_2, \ldots, n_k) = \left\lfloor \frac{1}{GCD(n_1, n_2, \ldots, n_k)} - 1 \right\rfloor
\]

If \( n_1, n_2, \ldots, n_k \) are coprime numbers:

\[
GCD(n_1, n_2, \ldots, n_k) = 1 \Rightarrow C_k(n_1, n_2, \ldots, n_k) = 0
\]

If \( n_1, n_2, \ldots, n_k \) aren’t coprime numbers: \( GCD(n_1, n_2, \ldots, n_k) > 1 \)

\[
0 < \frac{1}{GCD} < 1 \Rightarrow \left\lfloor \frac{1}{GCD} - 1 \right\rfloor = 1 = C_k(n_1, n_2, \ldots, n_k)
\]

References:

1. E. Burton, “Smarandache Prime and Coprime Function”
Chapter 7: New Prime Numbers

I have found some new prime numbers using the PROTH program of Yves Gallot.
This program in based on the following theorem:

**Proth Theorem (1878):**
Let \( N = k \cdot 2^n + 1 \) where \( k < 2^n \). If there is an integer number \( a \) so that
\[
\frac{N-1}{2} \equiv -1 \pmod{N}
\]
therefore \( N \) is prime.

The Proth program is a test for primality of greater numbers defined as
\( k \cdot b^n + 1 \) or \( k \cdot b^n - 1 \). The program is made to look for numbers of less
than 5.000000 digits and it is optimized for numbers of more than 1000
digits.

Using this Program, I have found the following prime numbers:

- \( 3239 \cdot 2^{12345} + 1 \) with 3720 digits \( a = 3, \ a = 7 \)
- \( 7551 \cdot 2^{12345} + 1 \) with 3721 digits \( a = 5, \ a = 7 \)
- \( 7595 \cdot 2^{12345} + 1 \) with 3721 digits \( a = 3, \ a = 11 \)
- \( 9363 \cdot 2^{12321} + 1 \) with 3713 digits \( a = 5, \ a = 7 \)

Since the exponents of the first three numbers are Smarandache number
\( Sm(5)=12345 \) we can call this type of prime numbers, prime numbers
of Smarandache.

Helped by the MATHEMATICA program, I have also found new prime
numbers which are a variant of prime numbers of Fermat. They are the
following:

\[
2^n \cdot 3^{2^n} - 2^{2^n} - 3^{2^n} \quad \text{for } n=1, 4, 5, 7.
\]

It is important to mention that for \( n=7 \) the number which is obtained has
100 digits.
Chris Nash has verified the values n=8 to n=20, this last one being a number of 815.951 digits, obtaining that they are all composite. All of them have a tiny factor except n=13.

References:

A book for people who love numbers:
Smarandache Function applied to perfect numbers, congruences.
Also, the Smarandache Prime and Coprime functions in connection with the expressions of the prime numbers.