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On Smarandache Multiplicative Sequence and Its Generation

XU Zhe-feng

(*Department of Mathematics, Northwest University, Xi'an* 710069, *China*)

Abstract: The main purpose of this paper is to introduce the general Smarandache multiplicative sequence based on the Smarandache multiplicative sequence, and calculate the value of some infinite series involving these sequences.

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§1. Introduction

Let m_1 and m_2 are any fixed positive integers with $m_1 < m_2$. The Smarandache multiplicative sequence is defined as follows (see [1]): if m_1 and m_2 are the first two terms of the sequence, then m_k, for $k \geq 3$, is the smallest positive integer equal to the product of two previous distinct terms. It is obviously that all terms of rank≥ 3 are divisible by $m_1 \cdot m_2$. Set $m_1 = 2$ and $m_2 = 3$, the Smarandache multiplicative sequence is: 2, 3, 6, 12, 18, 24, 36, 48, 54, $\cdots$. Similarly, we can also define the general Smarandache multiplicative sequence: let integer $k \geq 2$, if positive integers $m_1, m_2, \cdots, m_k$ are the first k terms of the sequence, then m_l, for $l \geq k+1$, is the smallest positive integer equal to the product of k previous distinct terms. Of course, all terms of rank$\geq k+1$ can be divided by the product $m_1 m_2 \cdots m_k$. In this paper, we shall study some infinite series involving the Smarandache multiplicative sequence generated by $m_1, m_2, \cdots, m_k$ relatively prime in pairs, and give some interesting identities for these series. That is, we shall prove the following:

Theorem 1 Let A denotes the set of all numbers in Smarandache multiplicative sequence generated by any positive integers m_1 and m_2 with $(m_1, m_2) = 1$. Then for any real number

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Biography: XU Zhe-feng(1980-), male, native of Xi'an, Shaanxi, a graduate student for Ph.D. of Northwest University, engages in number theory.

$\alpha > 0$, we have

$$\sum_{n\in A}\frac{1}{n^\alpha}=\frac{1}{m_1^\alpha}+\frac{1}{m_2^\alpha}+\frac{1}{(m_1^\alpha-1)(m_2^\alpha-1)}\ .$$

Specially, taking $m_1 = 2, m_2 = 3$ and $\alpha = 1,\ 2,\ 3,\ 4$, we may immediately obtain

Corollary 1 Let B denotes the set of all numbers in Smarandache multiplicative sequence generated by 2 and 3. Then we have the identities

$$\sum_{n\in B}\frac{1}{n}=\frac{4}{3};\qquad \sum_{n\in B}\frac{1}{n^2}=\frac{29}{72};\qquad \sum_{n\in B}\frac{1}{n^3}=\frac{3293}{19656};\qquad \sum_{n\in B}\frac{1}{n^4}=\frac{409}{5400}.$$

Theorem 2 Let C denotes the set of all numbers in Smarandache multiplicative sequence generated by any positive integers $m_1, m_2, \cdots, m_k$ relatively prime in pairs: $(m_i, m_j) = 1 (i \neq j)$. For any real number $\alpha > 0$, we have

$$\sum_{n\in C}\frac{1}{n^\alpha}=\sum_{i=1}^{k}\frac{1}{m_i^\alpha}+\prod_{i=1}^{k}\frac{1}{m_i^\alpha-1}.$$

Theorem 3 Let D denotes the set of all numbers in Smarandache multiplicative sequence generated by any primes $p_1, p_2, \cdots, p_k$. Then we have

$$\sum_{n\in D}\frac{d(n)}{n}=\sum_{i=1}^{k}\frac{2}{p_i}+\prod_{i=1}^{k}\frac{2p_i-1}{(p_i-1)^2},$$

where $d(n)$ is the Dirichlet divisor function.

Let $k = 2$, we can get the following

Corollary 2 Let D_{p_1,p_2} denotes the set of all numbers in Smarandache multiplicative sequence generated by primes p_1 and p_2. Then we have

$$\sum_{n\in D_{2,3}}\frac{d(n)}{n}=\frac{65}{12};\qquad \sum_{n\in D_{2,5}}\frac{d(n)}{n}=\frac{247}{80};\qquad \sum_{n\in D_{3,5}}\frac{d(n)}{n}=\frac{1699}{960}.$$

§2. Proof of the Theorems

In this section, we will complete the proof of the theorem. First, we prove Theorem 1. From the definition of Smarandache multiplicative sequence, we know that every terms of rank≥ 3 have the form $m_1^i m_2^j \quad (i, j \geq 1)$, and all the numbers of this form must be in set A. Noting that $(m_1, m_2) = 1$, the reciprocal sum of the terms of rank≥ 3 can be write as

$$\left(\frac{1}{m_1}+\frac{1}{m_1^2}+\frac{1}{m_1^3}+\cdots\right)\left(\frac{1}{m_2}+\frac{1}{m_2^2}+\frac{1}{m_2^3}+\cdots\right).$$

So we have

$$\begin{aligned}\sum_{n\in A}\frac{1}{n^\alpha} &= \frac{1}{m_1^\alpha}+\frac{1}{m_2^\alpha}+\frac{1}{m_1^\alpha m_2^\alpha}+\frac{1}{m_1^{2\alpha}m_2^\alpha}+\cdots\\ &= \frac{1}{m_1^\alpha}+\frac{1}{m_2^\alpha}+\left(\frac{1}{m_1^\alpha}+\frac{1}{m_1^{2\alpha}}+\frac{1}{m_1^{3\alpha}}+\cdots\right)\left(\frac{1}{m_2^\alpha}+\frac{1}{m_2^{2\alpha}}+\frac{1}{m_2^{3\alpha}}+\cdots\right)\\ &= \frac{1}{m_1^\alpha}+\frac{1}{m_2^\alpha}+\frac{\frac{1}{m_1^\alpha}}{1-\frac{1}{m_1^\alpha}}\frac{\frac{1}{m_2^\alpha}}{1-\frac{1}{m_2^\alpha}}\\ &= \frac{1}{m_1^\alpha}+\frac{1}{m_2^\alpha}+\frac{1}{(m_1^\alpha-1)(m_2^\alpha-1)}.\end{aligned}$$

This completes the proof of Theorem 1. By the same method of proving Theorem 1, we can also prove Theorem 2. Now we prove Theorem 3. From the definition of the general Smarandache multiplicative sequence and note that $d(p)=2$ (for any prime p), we have

$$\begin{aligned}\sum_{n\in D}\frac{d(n)}{n} &= \sum_{i=1}^{k}\frac{2}{p_i}+\prod_{i=1}^{k}\left(\frac{2}{p_i}+\frac{3}{p_i^2}+\cdots\right)\\ &= \sum_{i=1}^{k}\frac{2}{p_i}+\prod_{i=1}^{k}\left(\frac{1}{1-\frac{1}{p_i}}\right)\left(\frac{2}{p_i}+\frac{1}{p_i^2}+\frac{1}{p_i^3}+\cdots\right)\\ &= \sum_{i=1}^{k}\frac{2}{p_i}+\prod_{i=1}^{k}\left(\frac{1}{p_i-1}\right)\left(1+\frac{1}{1-\frac{1}{p_i}}\right)\\ &= \sum_{i=1}^{k}\frac{2}{p_i}+\prod_{i=1}^{k}\frac{2p_i-1}{(p_i-1)^2}.\end{aligned}$$

This completes the proof of the theorems.

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[References]

[1] Mihaly Bencze , Lucian Tutescu. Some Notions and Questions In Number Theory(vol.II)[M]. New Mexico: American Research Press, 1999.

[2] TOM M. Apostil Introduction to Analytic Number Theory[M]. New York: Springer-Verlag, 1976.