Chinese Journal of Aeronautics 24 (2011) 449-460

Contents lists available at ScienceDirect

Chinese Journal of Aeronautics

journal homepage: www.elsevier.com/locate/cja

A New Probabilistic Transformation in Generalized Power Space

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Received 23 June 2010; revised 12 August 2010; accepted 11 February 2011

Abstract

The mapping from the belief to the probability domain is a controversial issue, whose original purpose is to make (hard) decision, but for contrariwise to erroneous widespread idea/claim, this is not the only interest for using such mappings nowadays. Actually the probabilistic transformations of belief mass assignments are very useful in modern multitarget multisensor tracking systems where one deals with soft decisions, especially when precise belief structures are not always available due to the existence of uncertainty in human being's subjective judgments. Therefore, a new probabilistic transformation of interval-valued belief structure is put forward in the generalized power space, in order to build a subjective probability measure from any basic belief assignment defined on any model of the frame of discernment. Several examples are given to show how the new transformation works and we compare it to the main existing transformations proposed in the literature so far. Results are provided to illustrate the rationality and efficiency of this new proposed method making the decision problem simpler.

Keywords: Dempster-Shafer theory; generalized power space; information fusion; interval value; uncertainty

1. Introduction

The information fusion technology originating from the end of 1970s results from the development of information science. Especially, since ten years or so ago, with the transfer of information fusion technology from the military applications to civil ones, the control architectures or the theories of belief functions have been developed very rapidly [1] for dealing with imperfect information (incomplete, imprecise, uncertain, inconsistent). Among the theories of belief functions such as Dempster-Shafer theory (DST) [2-3], transferable belief model (TBM) [4-6] and Dezert-Smarandache theory (DSmT) [7-9], the mapping from the belief to the probability offers interesting issues to combine uncertain sources of information expressed in terms of belief functions.

And, it is more often that time critical decisions must be made with incomplete information for many real time information fusion systems, which means the elements in object set cannot get accurate evaluations using Dempster rule of combination thus adding greater complexity to decision-making. The belief function (or basic probability assignment (BPA), plausibility function) should be transformed to the probability measure, when the decision is to be made based on the classical probabilistic theories and methods.

Moreover, in many decision situations, precise belief structures are not always available due to the existence of uncertainty in human being's subjective judgments.

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Foundation items: State Key Development Program for Basic Research of China (2007CB311006); National Natural Science Foundation of China (60572161, 60874105, 60904099); Excellent Ph.D. Paper Author Foundation of China (200443)

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doi: 10.1016/S1000-9361(11)60052-6

In this case, an interval-valued belief degree rather than a precise one may be provided.

However, most current probability transforms are based on single belief function [10-16] in DST, which will lead to incomplete use of belief functions and fragility to immature information sets. Especially, when there exist complex static or dynamic fusion problems beyond the limits of the DST framework, and when the refinement of the frame of the problem under consideration, denoted as Θ, becomes inaccessible because of the vague, relative and imprecise nature of elements of Θ, the generalized power space including DSmT may be considered in the information fusion.

A classical transformation is the so-called pignistic probability [4-6], denoted as BetP, which offers a good compromise between the maximum of credibility Bel and the maximum of plausibility (PI) for decision-support. Unfortunately, BetP does not provide the highest probabilistic information content (PIC) [10-17].

In this paper, we aim to design a new probability transformation approach which provides a low uncertainty and a high PIC for expecting better performances.

The problem of probability transformation is converted to an optimization problem with constraints in the generalized power space. The objective function is established based on the maximization of distance and the constraints are related to the given belief and plausibility functions. Numerical examples show that the probability measure generated based on our approach has less uncertainty and more stability when compared with other available probability transformation approaches of belief function.

2. Background Material

2.1. DST

A brief review of DST is as follows:

In DST [2-3], the elements in the frame of discernment (FOD) Θ are mutually exclusive. Define the function m:$2^\Theta\to[0,1]$ as the BPA (also called a belief structure or a basic belief assignment), which satisfies

$$\sum m(A)=1, A\subseteq U, m(\varnothing)=0 \quad (1)$$

then $m(A)$ is defined as the BPA of A, representing the strength of all the incomplete information set for A.

The degree of one's belief to a given proposition is represented by a two-level probabilistic portrayal of the information set: the belief level and the plausibility level (see Fig. 1). They are defined respectively as follows:

$$\text{Bel}(A)=\sum_{B\subseteq A} m(B) \quad (2)$$

$$\text{Pl}(A)=1-\text{Bel}(\overline{A})=\sum_{A\cap B\neq\varnothing} m(B), \forall A\subseteq\Theta \quad (3)$$

with $\overline{A}=\Theta-A$. Bel(A) is the sum of $m(B)$ for all subsets B contained in A, representing all evidences that support the given proposition A. The Pl(A) is the sum of $m(B)$ for all subsets B that have a non null intersection of A, representing all evidences that do not rule out the given proposition A. Absolutely, Pl(A)≥Bel(A). The belief interval [Bel(A), Pl(A)] represents the uncertainty of A. Bel(A)= Pl(A) means absolute confirmation to A.

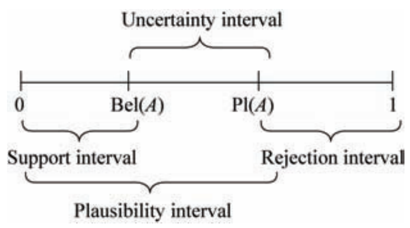

Fig. 1 Description of evidence intervals of DST.

2.2. Pignistic probability in TBM and DSmT

Pignistic probability was firstly proposed by Smets[4-6] to solve the decision problem under uncertainty. Smets analyzed the rationale more deeply and proved the decision efficiency of pignistic probability in the field of incomplete information fusion.

Suppose Θ is the FOD. The classical pignistic probability transformation in TBM framework is given by BetP(∅)=0 and $\forall X\in 2^\Theta/\{\varnothing\}$:

$$\text{BetP}(\theta_i)=\sum_{B\subseteq 2^\Theta, B\neq\varnothing}\frac{|\theta_i\cap B|}{|B|}\frac{m(B)}{1-m(\varnothing)} \quad (4)$$

where 2^Θ is the power set of the finite and discrete frame Θ if Shafer's model is applied, i.e. all elements of Θ are assumed truly exclusive. In Shafer's approach, $m(\varnothing)$=0 and Eq. (4) can be rewritten for any singleton $\theta_i\in\Theta$ as

$$\text{BetP}(\theta_i)=\sum_{Y\in 2^\Theta,\theta_i\subseteq Y}\frac{1}{|Y|}m(Y)=m(\theta_i)+\sum_{Y\in 2^\Theta,\theta_i\subset Y}\frac{m(Y)}{|Y|} \quad (5)$$

This transformation has been generalized in DSmT for any regular normalized basic belief assignments (BBAS) $m(\cdot)$:$G^\Theta\to[0,1]$ (i.e. such that $m(\varnothing)$=0 and $\sum_{X\in G^\Theta} m(X)=1$) and for any model of the frame (free DSm model, hybrid DSm model and Shafer's model as well) [17]. It is given by BetP(∅)=0 and $\forall X\in 2^\Theta/\{\varnothing\}$:

$$\text{BetP}(\theta_i)=\sum_{Y\subseteq G^\Theta}\frac{C_\text{M}(\theta_i\cap Y)}{C_\text{M}(Y)}m(Y) \quad (6)$$

where G^Θ corresponds to the hyper-power set including all the integrity constraints of the model (if any), and $C_\text{M}(Y)$ the DSm cardinal of the DSm cardinal of the set Y. Eq. (6) reduces to Eq. (4) when G^Θ reduces to classical power set 2^Θ as Shafer's model is adopted.

3. Previous Works

Several pignistic probabilities of precise belief degree in DST are recalled in this section [10-17].

3.1. Sudano's probabilities

Sudano has proposed interesting alternatives similar to BetP, which are called PrPl, PrNPl, PraPl, PrBel and PrHyb and are all defined in DST framework [10-13].

PrPl and PrBel are defined for all $\forall X \neq \varnothing \in \Theta$ by

$$\mathrm{PrPl}(\theta_i) = \mathrm{Pl}(\{\theta_i\}) \sum_{\substack{B \in 2^\Theta \\ \theta_i \subseteq B}} \frac{m(B)}{\sum\limits_{\theta_j \in B, \bigcup_j \theta_j = B} \mathrm{Pl}(\{\theta_j\})} \tag{7}$$

$$\mathrm{PrBel}(\theta_i) = \mathrm{Bel}(\{\theta_i\}) \sum_{\substack{B \in 2^\Theta \\ \theta_i \subseteq B}} \frac{m(B)}{\sum\limits_{\theta_j \in B, \bigcup_j \theta_j = B} \mathrm{Bel}(\{\theta_j\})} \tag{8}$$

PrNPl, PraPl and PrHyb are, respectively, in fact a mapping proportional to the normalized plausibility function, a mapping proportional to all plausibilities and a hybrid transformation defined by

$$\mathrm{PrNPl}(\theta_i) = \mathrm{Pl}(\{\theta_i\}) \Big/ \sum_{\theta_j \in \Theta} \mathrm{Pl}(\{\theta_j\}) \tag{9}$$

$$\mathrm{PraPl}(\theta_i) = \mathrm{Bel}(\{\theta_i\}) + \varepsilon \mathrm{Pl}(\{\theta_i\}) \tag{10}$$

with

$$\varepsilon = (1 - \sum_{\theta_i \in 2^\Theta} \mathrm{Bel}(\{\theta_i\})) \Big/ \sum_{\theta_i \in 2^\Theta} \mathrm{Pl}(\{\theta_i\}) \tag{11}$$

$$\mathrm{PrHyb}(\theta_i) = \mathrm{PraPl}(\{\theta_i\}) \sum_{\substack{B \in 2^\Theta \\ \theta_i \subseteq B}} \frac{m(B)}{\sum\limits_{\theta_j \in B, \bigcup_j \theta_j = B} \mathrm{PraPl}(\{\theta_i\})} \tag{12}$$

3.2. Cuzzolin's intersection probability

A new transformation denoted as CuzzP has been proposed in Ref. [14] by Cuzzolin in the framework of DST in 2007, which is defined on any finite and discrete frame $\Theta=\{\theta_1, \theta_2, \cdots, \theta_n\}$, $n \geq 2$ satisfying Shafer's model, by

$$\mathrm{CuzzP}(\theta_i) = m(\theta_i) + \frac{\Delta(\theta_i)}{\sum\limits_{j=1}^{n} \Delta(\theta_j)} \cdot \mathrm{TNSM} \tag{13}$$

with $\Delta(\theta_i) = \mathrm{Pl}(\theta_i) - m(\theta_i)$ and

$$\mathrm{TNSM} = 1 - \sum_{j=1}^{n} m(\theta_j) = \sum_{A \in 2^\Theta, |A|>1} m(A) \tag{14}$$

3.3. B&P1, B&P2 and B&P3 algorithms

Three new pignistic probability transformations based on multiple belief functions were proposed by Pan [15].

The B&P1 proportional transformation hypothesis assumes that the BPA is proportional to the product of $\mathrm{Bel}(\theta_i)$ and $\mathrm{Pl}(\theta_i)$ among each singleton element of $\theta_i \subseteq Y$ with $Y \subseteq \Theta$ for all $Y \in 2^\Theta$.

$$\mathrm{PrBP1}(\theta_i) = \sum_{\theta_i \subseteq Y} \left(\frac{\mathrm{Bel}(\theta_i)\mathrm{Pl}(\theta_i)}{\sum\limits_{\theta_j \subseteq Y} \mathrm{Bel}(\theta_j)\mathrm{Pl}(\theta_j)} \right) m(Y) \tag{15}$$

The B&P2 transformation hypothesis assumes that the BPA is distributed proportionally to the parameter $s_i = \dfrac{\mathrm{Bel}(\theta_i)}{1-\mathrm{Pl}(\theta_i)}$ among each singleton element of $\theta_i \subseteq Y$ with $Y \subseteq \Theta$ for all $Y \in 2^\Theta$.

$$\mathrm{PrBP2}(\theta_i) = \sum_{\theta_i \subseteq Y} \left(\frac{s_i}{\sum\limits_j s_j} \right) m(Y) \tag{16}$$

The B&P3 transformation hypothesis assumes that the BPA is distributed proportionally to the parameter $s_i = \dfrac{\mathrm{PrBP3}(\theta_i)}{1-\mathrm{PrBP3}(\theta_i)}$ among each singleton element of $\theta_i \subseteq Y$ with $Y \subseteq \Theta$ for all $Y \in 2^\Theta$, where s can be initiated by $\mathrm{Bel}(\theta_i)$ and $\mathrm{Pl}(\theta_i)$, that is, $s_i = \dfrac{\mathrm{Bel}(\theta_i)}{1-\mathrm{Pl}(\theta_i)}$.

$$\mathrm{PrBP3}(\theta_i) = \sum_{\theta_i \subseteq Y} \left(\frac{s_i}{\sum\limits_j s_j} \right) m(Y) \tag{17}$$

For each singleton element θ_i, these pignistic probability transforms are bound between the belief and the plausibility, and these pignistic probability transforms are all normalized to 1.

3.4. Dezert and Smarandache's probability

Another new transformation was proposed in Ref. [16] by Dezert and Smarandache in the framework of DSmT (free DSm model, hybrid DSm model or Shafer's model), which is called DSmP formula.

Let us consider a discrete frame Θ with a given model (free DSm model, hybrid DSm model or Shafer's model), and the DSmP mapping is defined by $\mathrm{DSmP}_\varepsilon(\varnothing)=0$ and $\forall X \in G^\Theta / \{\varnothing\}$:

$$\mathrm{DSmP}_\varepsilon(X) = \frac{\sum\limits_{\substack{Z \subseteq X \cap Y \\ C(Z)=1}} m(Z) + \varepsilon C(X \cap Y)}{\sum\limits_{\substack{Z \subseteq Y \\ C(Z)=1}} m(Z) + \varepsilon C(Y)} m(Y) \tag{18}$$

where $\varepsilon \geq 0$ is a tuning parameter and G^Θ corresponds to the hyper-power set including all the integrity constraints (if any) of the model M; $C(X \cap Y)$ and $C(Y)$ denote the DSm cardinals of the sets $X \cap Y$ and Y respectively·

Deng proposed a modified probability transformation based on fractal theory, called FPT in Ref. [17].

Although there exist many different expressions among the probability transforms, these transformations aim to enlarge the belief differences among all the propositions to derive a more reliable decision result.

4. New Probability Transformation Based on Distance Maximization

4.1. Interval evidence

The interval operations defined here about imprecision are similar to the rational interval extension through the interval arithmetics [7]. Then, the interval operations to any set operation are generalized, where real sub-unitary sets are needed and the defined set operation can be used for any kind of sets.

Let s_1 and s_2 be two (unidimensional) real standard subsets of the unit interval [0,1], and a number $k\in[0,1]$, then one defines

Addition of sets:

$$s_1 \oplus s_2 = s_2 \oplus s_1 = \{x \mid x = s_1 + s_2, s_1 \in S_1, s_2 \in S_2\} \tag{19}$$

with

$$\begin{cases}\inf(s_1 \oplus s_2) = \inf(s_1) + \inf(s_2) \\ \sup(s_1 \oplus s_2) = \sup(s_1) + \sup(s_2)\end{cases} \tag{20}$$

and, as a particular case, it is defined as

$$k \oplus s_2 = s_2 \oplus k = \{x \mid x = k + s_2, s_2 \in S_2\} \tag{21}$$

with

$$\begin{cases}\inf(\{k\} \oplus s_2) = k + \inf(s_2) \\ \sup(\{k\} \oplus s_2) = k + \sup(s_2)\end{cases} \tag{22}$$

Let us now consider some given sources of information which are not able to provide us a specific/precise mass $m_j\in[0,1]$, but only an interval centered in $m_j\in[0,1]$, i.e. $I_j = [m_j-\varepsilon_j, m_j+\varepsilon_j]$ where $\varepsilon_j\in[0,1]$ and $I_j\in[0,1]$ for all $1 \le j \le n$. The cases when I_j are half-closed and open are similarly treated.

A scalar α can be regarded as a particular interval $[\alpha,\alpha]$, thus the mass matrix $\boldsymbol{m}$ is extended to

$$\inf(\boldsymbol{m}) = [m_1 - \varepsilon_1 \quad m_2 - \varepsilon_2 \quad \cdots \quad m_n - \varepsilon_n] \tag{23}$$

$$\sup(\boldsymbol{m}) = [m_1 + \varepsilon_1 \quad m_2 + \varepsilon_2 \quad \cdots \quad m_n + \varepsilon_n] \tag{24}$$

Of course, the closeness of this interval to the left and/or to the right depends on the closeness of the interval I_j. If all of them are closed to the left, then $m(A)$ is also closed to the left. But, if at least one is open to the left, then $m(A)$ is open to the left. The same is true with the closeness to the right. Because one has $\forall j$=1, 2, $\cdots$, k:

$$\lim_{\varepsilon_j \to 0}(\inf(\boldsymbol{m})) = \lim_{\varepsilon_j \to 0}(\sup(\boldsymbol{m})) = \boldsymbol{m} \tag{25}$$

from which the following theorem can be obtained.

Theorem 1: $\forall A\in G^{\Theta}$, $\forall j$=1, 2, $\cdots$, k, one has $\lim_{\varepsilon_j \to 0} m(A)=[\lim_{\inf_j} m(A), \lim_{\sup_j} m(A)]$ with

$$\begin{cases}\lim_{\inf_j} m(A) = \lim_{\varepsilon_j \to 0}(\inf(m(A))) \\ \lim_{\inf_j} m(A) = \lim_{\varepsilon_j \to 0}(\sup(m(A)))\end{cases} \tag{26}$$

In other words, if all centered sub-unitary intervals converge to their corresponding mid points (the imprecision becomes zero), then the intervals converge towards precise values for scalars. In what follows, we assume that interval-valued belief structures are all normalized.

4.2. New method

Our new mapping is straight, and can make decision more quickly than DSmP. It is different from Sudano's, Pan's and Cuzzolin's mappings which are more refined but less interesting in our opinions than what we present here. The basic idea of the new method consists in an optimization with constraints. This new transformation takes into account the values of the masses and the corresponding belief structures in the optimization process.

Before putting forward the new transformation, we first recall the characteristic of probability distributions $\langle p_\theta | \theta \in \Theta \rangle$, which must meet the usual requirements for probability distributions, i.e.

$$\begin{cases}0 \le p_\theta \le 1, \forall \theta \in \Theta \\ \sum_{\theta \in \Theta} p_\theta = 1\end{cases} \tag{27}$$

Definition 1 Let m be a normalized interval belief structure with interval probability mass $\inf(m(A_i)) \le m(A_i) \le \sup(m(A_i))$ for i=1,2,$\cdots$, n and A be a subset of Θ. The belief measure (Bel) and the plausibility measure (Pl) of A are the closed intervals respectively defined by

$$\mathrm{Bel}_m(A) = [\inf(\mathrm{Bel}_m(A)), \sup(\mathrm{Bel}_m(A))] \tag{28}$$

$$\mathrm{Pl}_m(A) = [\inf(\mathrm{Pl}_m(A)), \sup(\mathrm{Pl}_m(A))] \tag{29}$$

A reasonable probability distributions should not only satisfy the less uncertainty especially when it is difficult to make decision only with the BPA, but also make a reasonable decision, that is, if the evidence shows more possibility of some elements, the element should obtain more support. Moreover, due to the interval characteristic of belief structure, the probability distribution is also an interval-value.

Definition 2 Let m be an interval belief structure with interval probability masses $\inf(m(A_i)) \le m(A_i) \le \sup(m(A_i))$ for i=1,2,$\cdots$, n and A be a subset of the generalized power space G^{Θ}. Its probability distribution, denoted by p_{θ_i}, is also an interval belief structure defined by

$$p_{\theta_i} \in [\inf(p_{\theta_i}), \sup(p_{\theta_i})] \tag{30}$$

The probability matrix $\boldsymbol{P}$ is extended to

$$\inf(\boldsymbol{P})=[p_{\theta_1}-\beta_1 \quad p_{\theta_2}-\beta_2 \quad \cdots \quad p_{\theta_n}-\beta_n] \tag{31}$$

$$\sup(\boldsymbol{P})=[p_{\theta_1}+\beta_1 \quad p_{\theta_2}+\beta_2 \quad \cdots \quad p_{\theta_n}+\beta_n] \tag{32}$$

where $\beta_i\in[0,1]$.

Theorem 2 $\forall A\in G^{\Theta}, \forall i=1,2,\cdots,n$, one has

$$\lim_{\beta_i\to 0} p_{\theta_i}=[\lim_{\inf_i} p_{\theta_i}, \lim_{\sup_i} p_{\theta_i}] \tag{33}$$

with

$$\begin{cases} \lim\limits_{\inf_i} p_{\theta_i}=\lim\limits_{\beta_i\to 0}(\inf(p_{\theta_i})) \\ \lim\limits_{\sup_i} p_{\theta_i}=\lim\limits_{\beta_i\to 0}(\sup(p_{\theta_i})) \end{cases} \tag{34}$$

Definition 3 Let m be an interval belief structure with interval probability masses $\inf(m(A_i))\le m(A_i)\le \sup(m(A_i))$ for $i=1,2,\cdots,n$ and A be a subset of the generalized power space G^{Θ}. Its distance measure denoted by d_{P} is also an interval belief structure defined by

$$d_{\mathrm{P}}=[\inf(d_{\mathrm{P}}),\sup(d_{\mathrm{P}})] \tag{35}$$

where $\inf(d_{\mathrm{P}})$ and $\sup(d_{\mathrm{P}})$ are respectively the maximum of the following optimization problem:

$$\underset{\{p_{\theta_i}|\theta_i\in\Theta\}}{\mathrm{Max}} \; d_{\mathrm{P}}=\sqrt{\sum_{\theta_i\in\Theta}(p_{\theta_i}-m(\theta_i))^2} \tag{36}$$

where there exist four kinds of constraints which are written as follows:

Constraint 1

s.t.

$$\sum_{A\in G^{\Theta}} m(A)=1 \tag{37}$$

$$\inf(m(A))\le m(A)\le \sup(m(A)), \quad \forall A\in G^{\Theta} \tag{38}$$

$$\begin{cases} \inf(\mathrm{Bel}_m(A))\le p_{\theta_i}\le \\ \quad \inf(\mathrm{Pl}_m(A)) & A=\theta_i \\ \inf(\mathrm{Bel}_m(A))\le p_{\theta_i}+\cdots+p_{\theta_j}\le \\ \quad \inf(\mathrm{Pl}_m(A)) & A=\theta_i\cup\cdots\cup\theta_j \\ \inf(\mathrm{Bel}_m(A))\le \sum\limits_{\theta_i\in\Theta} p_{\theta_i}\le \\ \quad \inf(\mathrm{Pl}_m(A)) & B=\Theta \\ p_{\theta_i}\ge p_{\theta_j} & \inf(\mathrm{Pl}_m(\theta_i))\ge\inf(\mathrm{Pl}_m(\theta_j)) \end{cases} \tag{39}$$

$$0\le p_{\theta_i}\le 1, \quad \forall\theta_i\in G^{\Theta} \tag{40}$$

$$\sum_{\theta_i\in\Theta} p_{\theta_i}=1 \tag{41}$$

Constraint 2

s.t.

$$\sum_{A\in G^{\Theta}} m(A)=1 \tag{42}$$

$$\inf(m(A))\le m(A)\le \sup(m(A)), \quad \forall A\in G^{\Theta} \tag{43}$$

$$\begin{cases} \sup(\mathrm{Bel}_m(A))\le p_{\theta_i}\le \\ \quad \sup(\mathrm{Pl}_m(A)) & A=\theta_i \\ \sup(\mathrm{Bel}_m(A))\le p_{\theta_i}+\cdots+ \\ \quad p_{\theta_j}\le \sup(\mathrm{Pl}_m(A)) & A=\theta_i\cup\cdots\cup\theta_j \\ \sup(\mathrm{Bel}_m(A))\le \sum\limits_{\theta_i\in\Theta} p_{\theta_i}\le \\ \quad \sup(\mathrm{Pl}_m(A)) & A=\Theta \\ p_{\theta_i}\ge p_{\theta_j} & \sup(\mathrm{Pl}_m(\theta_i))\ge\sup(\mathrm{Pl}_m(\theta_j)) \end{cases} \tag{44}$$

$$0\le p_{\theta_i}\le 1, \quad \forall\theta_i\in G^{\Theta} \tag{45}$$

$$\sum_{\theta_i\in\Theta} p_{\theta_i}=1 \tag{46}$$

Constraint 3

s.t.

$$\sum_{A\in G^{\Theta}} m(A)=1 \tag{47}$$

$$\inf(m(A))\le m(A)\le \sup(m(A)), \quad \forall A\in G^{\Theta} \tag{48}$$

$$\begin{cases} \inf(\mathrm{Bel}_m(A))\le p_{\theta_i}\le \\ \quad \sup(\mathrm{Pl}_m(A)) & A=\theta_i \\ \inf(\mathrm{Bel}_m(A))\le p_{\theta_i}+\cdots+p_{\theta_j}\le \\ \quad \sup(\mathrm{Pl}_m(A)) & A=\theta_i\cup\cdots\cup\theta_j \\ \inf(\mathrm{Bel}_m(A))\le \sum\limits_{\theta_i\in\Theta} p_{\theta_i}\le \\ \quad \sup(\mathrm{Pl}_m(A)) & A=\Theta \\ p_{\theta_i}\ge p_{\theta_j} & \sup(\mathrm{Pl}_m(\theta_i))\ge\sup(\mathrm{Pl}_m(\theta_j)) \end{cases} \tag{49}$$

$$0\le p_{\theta_i}\le 1, \quad \forall\theta_i\in G^{\Theta} \tag{50}$$

$$\sum_{\theta_i\in\Theta} p_{\theta_i}=1 \tag{51}$$

Constraint 4

s.t.

$$\sum_{A\in G^{\Theta}} m(A)=1 \tag{52}$$

$$\inf(m(A))\le m(A)\le \sup(m(A)), \quad \forall A\in G^{\Theta} \tag{53}$$

$$\begin{cases} \sup(\mathrm{Bel}_m(A))\le p_{\theta_i}\le \\ \quad \inf(\mathrm{Pl}_m(A)) & A=\theta_i \\ \sup(\mathrm{Bel}_m(A))\le p_{\theta_i}+\cdots+p_{\theta_j}\le \\ \quad \inf(\mathrm{Pl}_m(A)) & A=\theta_i\cup\cdots\cup\theta_j \\ \sup(\mathrm{Bel}_m(A))\le \sum\limits_{\theta_i\in\Theta} p_{\theta_i}\le \\ \quad \inf(\mathrm{Pl}_m(A)) & A=\Theta \\ p_{\theta_i}\ge p_{\theta_j} & \inf(\mathrm{Pl}_m(\theta_i))\ge\inf(\mathrm{Pl}_m(\theta_j)) \end{cases} \tag{54}$$

$$0\le p_{\theta_i}\le 1, \quad \forall\theta_i\in G^{\Theta} \tag{55}$$

$$\sum_{\theta_i\in\Theta} p_{\theta_i}=1 \tag{56}$$

Thus the final probability distributions $p_{\theta_i} \in [p_{\theta_i}^-, p_{\theta_i}^+]$ must meet $p_{\theta_i} \in [\min(p_{\theta_i}), \max(p_{\theta_i})]$ for i=1,2,$\cdots$,n.

When the interval-valued belief structure is changed to precise belief structure, that is, inf($m(A)$)=sup($m(A)$), inf(Bel$_m(A)$)=sup(Bel$_m(A)$) and inf(Pl$_m(A)$)=sup(Pl$_m(A)$), and the optimality approach is

$$\underset{\{p_{\theta_i}|\theta_i \in \Theta\}}{\mathrm{Max}} d_\mathrm{P} = \sqrt{\sum_{\theta_i \in \Theta} (p_{\theta_i} - m(\theta_i))^2} \quad (57)$$

s.t.

$$\sum_{A \in G^\Theta} m(A) = 1 \quad (58)$$

$$\inf(m(A)) \le m(A) \le \sup(m(A)) \quad \forall A \in G^\Theta \quad (59)$$

$$\begin{cases} \mathrm{Bel}_m(A) \le p_{\theta_i} \le \\ \quad \mathrm{Pl}_m(A) & A = \theta_i \\ \mathrm{Bel}_m(A) \le p_{\theta_i} + \cdots + \\ \quad p_{\theta_j} \le \mathrm{Pl}_m(A) & A = \theta_i \cup \cdots \cup \theta_j \\ \mathrm{Bel}_m(A) \le \sum_{\theta_i \in \Theta} p_{\theta_i} \le \\ \quad \mathrm{Pl}_m(A) & A = \Theta \\ p_{\theta_i} \ge p_{\theta_j} & \mathrm{Pl}_m(\theta_i) \ge \mathrm{Pl}_m(\theta_j) \end{cases} \quad (60)$$

$$0 \le p_{\theta_i} \le 1, \quad \forall \theta_i \in G^\Theta \quad (61)$$

$$\sum_{\theta_i \in \Theta} p_{\theta_i} = 1 \quad (62)$$

If we do not consider the relationship between p_{θ_i} and p_{θ_j} in all the four constraints, some incredible results may be obtained, that is to say, Eq. (60) should be defined as follows:

$$\begin{cases} \mathrm{Bel}_m(A) \le p_{\theta_i} \le \mathrm{Pl}_m(A) & A = \theta_i \\ \mathrm{Bel}_m(A) \le p_{\theta_i} + \cdots + p_{\theta_j} \le \mathrm{Pl}_m(A) & A = \theta_i \cup \cdots \cup \theta_j \\ \mathrm{Bel}_m(A) \le \sum_{\theta_i \in \Theta} p_{\theta_i} \le \mathrm{Pl}_m(A) & A = \Theta \end{cases} \quad (63)$$

Example 1 Let one BPA from a distinct source on frame Θ={θ_1, θ_2} be

$$m(\{\theta_1\}) = 0.01, \quad m(\{\theta_2\}) = 0.4, \quad m(\{\theta_1 \cup \theta_2\}) = 0.59$$

If this new method chooses the constraint condition given in Eq. (63), different results may be obtained. The final result after probability transformation may be p_1=P({θ_1})=0.6, p_2=P({θ_2})=0.4 or p_1=P({θ_1})=0.2, p_2=P({θ_2})=0.8. The reason is that the objective function is not convex. So, the proposed approach in this paper is of statistical significance. Thus, if we use the proposed approach whose constraint condition is given in Eq. (60), the final result is p_1=P({θ_1})=0.01, p_2=P({θ_2})=0.99. The difference among the two propositions can be further enlarged, which is helpful for the more consolidated and reliable decision.

Example 2 Let one BPA from a distinct source on frame Θ={θ_1, θ_2, θ_3} be

$$m(\{\theta_1 \cup \theta_2\}) = m(\{\theta_2 \cup \theta_3\}) = m(\{\theta_1 \cup \theta_3\}) = 1/3$$

If this new method chooses the constraint condition given in Eq. (63), we can derive six different probability distributions yielding the same maximal distance, which are listed as follows:

$$p_1 = 1/3, \quad p_2 = 2/3, \quad p_3 = 0$$
$$p_1 = 1/3, \quad p_2 = 0, \quad p_3 = 2/3$$
$$p_1 = 0, \quad p_2 = 1/3, \quad p_3 = 2/3$$
$$p_1 = 0, \quad p_2 = 2/3, \quad p_3 = 1/3$$
$$p_1 = 2/3, \quad p_2 = 1/3, \quad p_3 = 0$$
$$p_1 = 2/3, \quad p_2 = 0, \quad p_3 = 1/3$$

It is clear that the problem of finding a probability distribution with maximal distance or even minimal entropy does not admit a unique solution in general [18].

So if we use the constraint conditions given in Eq. (60), the decision result is p_1=1/3, p_2=1/3, p_3=1/3, which is the same as the result by the classical pignistic probability transformation in TBM framework.

Once m({θ_1})> m({θ_2}), even in very special situations where the difference between masses of singletons is very small, the mass of belief m({$\theta_1 \cup \theta_2$})>0 is always fully distributed back to θ_1. The following example [18] illustrates this:

$$m(\{\theta_1\}) = 0.100\,000\,1, \quad m(\{\theta_2\}) = 0.1,$$
$$m(\{\theta_1 \cup \theta_2\}) = 0.799\,999\,9$$

So if we use the constraint conditions given in Eq. (60), the decision result is p_1=1/2, p_2=1/2. Although m({θ_1})>m({θ_2}), m({θ_1}) is almost the same as m({θ_2}) and so there is no solid reason to obtain a very high probability for θ_1 and a small probability for θ_2. Therefore, the decision based on the result derived from the new method is reasonable.

From our analysis, it can be concluded that the maximization of distance without considering the essential relationship of BPA is not sufficient for evaluating the quality of a probability transformation and the maximization of distance with the essential relationship of BPA is useful to give more acceptable probability distribution from belief functions.

Therefore, these constraints make sure that the proposed approach is more reasonable, and more fit for the real world, that is, the more support one gets from its belief function, the more possibility one can obtain. Moreover, the optimization of the distance means once the probabilistic transformation satisfies these constraints, the more distance there exists, the larger possibility differences of certain hypotheses concerning the class membership of those patterns are, and thus more consolidated and reliable decision can be made.

Furthermore, this new method works for all models (free, hybrid and Shafer's). In order to apply classical BetP, CuzzP, DSmP, Pan's or Sudano's mappings, we need at first to refine the frame in order to work with Shafer's model, and then apply their formulas [16]. In the case where refinement makes sense, one can apply other subjective probabilities on the refined frame. This new method works on the refined frame as well and gives the same result as it does on the non-refined frame.

5. Measuring Information and Uncertainty

In probability theory a well-known concept is the Shannon entropy measure, which is most widely used in Ref. [19].

If P is a probability distribution on $\Theta=\{\theta_1, \theta_2, \cdots \theta_n\}$, then the entropy of P is expressed as

$$H(P) = -\sum_{i=1}^{n} p_{\theta_i} \log_2 p_{\theta_i} \tag{64}$$

where p_{θ_i} is the probability of θ_i.

It is well known that entropy measures the uncertainty associated with the probability distribution P. When total information is available and there is no ambiguity for decision-making, $H_{\min}(P) = 0$. When $p_{\theta_i} = 1/n$,

$$H_{\max}(P) = -\sum_{i=1}^{n} \frac{1}{n} \log_2 \frac{1}{n} = \log_2 n \tag{65}$$

So the normalized measure evaluation of probability distribution EH is

$$\text{EH}(P) = \frac{-\sum_{i=1}^{n} p_{\theta_i} \log_2 p_{\theta_i}}{\log_2 n} \tag{66}$$

The less EH, the less uncertainty of information, and the more accurate for decision-making. When EH(P)=0, there is only one hypothesis with a probability value of 1 and the rest has zero value, and the system can make decision unambiguously. When EH(P)= 1, it is impossible to make a correct decision.

The probabilistic information content (PIC) of a probability measure P associated with a probabilistic source over a discrete finite set $\Theta=\{\theta_1, \theta_2, \cdots \theta_n\}$ is defined as [11]:

$$\text{PIC}(P) = 1 + \frac{1}{\log_2 n} \sum_{i=1}^{n} p_{\theta_i} \log_2 p_{\theta_i} \tag{67}$$

The PIC is nothing but the dual of the normalized Shannon entropy and thus is actually unit less. PIC(P) takes its values in $[0, 1]$. PIC(P) is maximum, i.e. $\text{PIC}_{\max}$=1 with any deterministic probability; it is minimum, i.e. $\text{PIC}_{\min}$=0, with the uniform probability over the frame Θ. The simple relationships between these measures are

$$\text{PIC}(P) = 1 - (H(P)/H_{\max}) = 1 - \text{EH}(P) \tag{68}$$

$$H(P) = H_{\max}(P)(1 - \text{PIC}(P)) = H_{\max}(P) \cdot \text{EH}(P) \tag{69}$$

$$\text{EH}(P) = 1 - \text{PIC}(P) = H(P)/H_{\max}(P) \tag{70}$$

For information fusion at decision level, the uncertainty should be reduced as much as possible. The less the uncertainty in probability measure is, the more consolidated and reliable decision can be made. The larger d_P, the better/bigger PIC(P) value, the worse/smaller $H(P)$ value, and the worse/smaller EH(P) value. Given belief function (or the BPA, the plausibility), by the above new method, a probability distribution can be derived, which has less uncertainty measured by Shannon entropy or larger stability measured by PIC(P) and thus is more proper to be used in decision procedure.

6. Examples with Precise BPA

The following numerical examples and comparisons with respect to other transformations illustrate some design concepts presented in this paper. To make the results more comparable, we use the data provided in Ref. [16] directly.

6.1. Example 3 (Shafer's model)

Let us define Shafer's model and the vacuous BPA characterizing the totally ignorant source, i.e. $m(\theta_1 \cup \theta_2) = 1$. It can be verified that all mappings coincide with the uniform probability measure over singletons of Θ, except PrBel, PrBPl and Deng's method which are mathematically not defined in that case. This result can be easily proved for any size of the frame Θ with $|\Theta|>2$.

6.2. Example 4 (Shafer's model and a probabilistic-source)

Let us still apply Shafer's model and see what happens when applying all the transformations on a probabilistic source which commits a belief mass only to singletons of 2^{Θ}. If we consider for example the uniform Bayesian mass defined by $m(\theta_1)=m(\theta_2)=1/2$, all transformations coincide with the probabilistic input mass as expected, so that the idempotency property is satisfied. Only Cuzzolin's transformation fails which is mathematically not defined in that case because one gets 0/0 indetermination. The result is important only from the mathematical point of view.

6.3. Example 5 (Shafer's model and non-Bayesian mass)

Assume that Shafer's model and the non-Bayesian mass (more precisely the simple support mass) have been given in Table 1. We summarize the results obtained with all transformations in Table 2. We use NaN acronym here standing for Not a Number due to zero assignment to singletons.

Table 1 Input of precise BPA for Example 5

Subsets	$\{\theta_1\}$	$\{\theta_2\}$	$\{\theta_1 \cup \theta_2\}$
$m(\cdot)$	0.4	0	0.6

Table 2 Probability transformation results of Example 5 based on different approaches

Result	θ_1	θ_2	d_P	EH(·)	PIC(·)
PrBel(·)	1.000 00	0	0.600 00	NaN	NaN
PrPl(·)	0.775 00	0.225 00	0.437 32	0.769 19	0.230 81
PrNPl(·)	0.625 00	0.375 00	0.437 32	0.954 43	0.045 57
BetP(·)	0.700 00	0.300 00	0.424 26	0.881 29	0.118 71
CuzzP(·)	0.700 00	0.300 00	0.424 26	0.881 29	0.118 71
PraPl(·)	0.775 00	0.225 00	0.437 32	0.769 19	0.230 81
PrHyb(·)	0.865 00	0.135 00	0.484 20	0.570 99	0.429 01
$\text{DSmP}_{\varepsilon=0.001}(\cdot)$	0.998 51	0.001 49	0.598 51	0.016 16	0.983 84
PrBPl(·)	1.000 00	0	0.600 00	NaN	NaN
PrBP2(·)	0.841 18	0.158 82	0.468 89	0.631 49	0.368 51
PrBP3(·)	0.999 73	0.000 27	0.599 73	0.003 55	0.996 45
Ref. [17]	1.000 00	0	0.600 00	NaN	NaN
This paper	1.000 00	$\to 0$	0.600 00	0	1.000 00

One sees that PrBel, PrBPl and Deng's method do not work correctly since they cannot have a division by zero. That is to say, they do not work when the masses of all singletons involved in an ignorance are null since they give the indetermination 0/0. In the case when at least one singleton mass involved in an ignorance is zero, that singleton does not receive any mass from the distribution even if it is involved in an ignorance, which is not fair/good. So, the new method solves their problem by doing a redistribution of the ignorance mass with an optimization process, whether all masses of singletons involved in all ignorances are different from zero or at least one singleton mass involved in ignorance is zero, which has the best PIC value and smallest EH value.

6.4. *Example 6 (free DSm model)*

Let us assume the free DSm model (i.e. $\theta_1 \cap \theta_2 \neq \varnothing$) and the generalized mass given in Table 3.

Table 3 Input of precise BPA for Example 6

Subset	$\{\theta_1 \cap \theta_2\}$	$\{\theta_1\}$	$\{\theta_2\}$	$\{\theta_1 \cup \theta_2\}$
$m(\cdot)$	0.4	0.2	0.1	0.3

In the case of free-DSm (or hybrid DSm) models, almost all methods cannot be derived directly for such models, so it needs to refine the frame Θ into Θ^{ref} which satisfies Shafer's model, that is, the original 2D frame $\Theta=\{\theta_1, \theta_2\}$ with $m(\cdot)$ given in Table 3 is changed into a refined 3D frame $\Theta^{\text{ref}} = \{\theta_1'=\theta_1/\{\theta_1 \cap \theta_2\}, \theta_2'=\theta_2/\{\theta_1 \cap \theta_2\}, \theta_3'=\theta_1 \cap \theta_2\}$, which is considered to satisfy Shafer's model with the equivalent BPA $m(\cdot)$ defined in Table 4.

Table 4 Input of precise BPA on Θ^{ref} for Example 6

Subset	$\{\theta_3'\}$	$\{\theta_1' \cup \theta_3'\}$	$\{\theta_1' \cup \theta_3'\}$	$\{\theta_1' \cup \theta_2' \cup \theta_3'\}$
$m(\cdot)$	0.4	0.2	0.1	0.3

The results are then given in Table 5. One sees that PIC(P) of the new method is the maximum value. And EH(P) of the new method is minimum. PrBel, PrBPl and Deng's method still do not work correctly because they cannot be directly evaluated for θ_1 and θ_2 since the underlying probabilities are mathematically undefined in such case.

Table 5 Probability transformation results of Example 6 based on different approaches

Result	θ_1'	θ_2'	θ_3'	d_P	EH(·)	PIC(·)
PrBel(·)	NaN	NaN	1.000 00	NaN	NaN	NaN
PrBl(·)	0.145 61	0.091 73	0.762 66	0.401 42	0.642 94	0.357 06
PrNPl(·)	0.263 16	0.210 53	0.526 32	0.359 90	0.925 86	0.074 14
BetP(·)	0.200 00	0.150 00	0.650 00	0.353 55	0.806 90	0.193 11
CuzzP(·)	0.200 00	0.160 00	0.640 00	0.351 00	0.819 88	0.180 13
PraPl(·)	0.157 90	0.126 32	0.715 79	0.374 98	0.721 03	0.278 97
PrHyb(·)	0.083 51	0.052 90	0.863 59	0.474 02	0.445 54	0.554 46
$\text{DSmP}_{\varepsilon=0.001}(\cdot)$	0.001 24	0.000 99	0.997 77	0.597 78	0.015 85	0.984 15
PrBPl(·)	NaN	NaN	1.000 00	NaN	NaN	NaN
PrBP2(·)	0.104 60	0.066 11	0.829 30	0.446 77	0.519 70	0.480 30
PrBP3(·)	0.000 10	0.000 04	0.999 86	0.599 86	0.001 36	0.998 64
Ref. [17]	NaN	NaN	1.000 00	NaN	NaN	NaN
This paper	$\to 0$	$\to 0$	1.000 00	0.600 00	0	1.000 00

6.5. Example 7 (Shafer's model and non-Bayesian mass)

This example is selected from Ref. [13]. Let us apply Shafer's model and the non-Bayesian belief mass given by $m(\theta_1)=0.35$, $m(\theta_2)=0.25$, $m(\theta_3)=0.02$, $m(\theta_1\cup\theta_2)=0.20$, $m(\theta_1\cup\theta_3)=0.20$, $m(\theta_2\cup\theta_3)=0.20$ and $m(\theta_1\cup\theta_2\cup\theta_3)=0.06$. The results of the mappings are given in Table 6. One sees that although all transformation methods can get reasonable results when the masses of all singletons involved in an ignorance are not null, the new method provides the best performance.

6.6. Example 8 (Shafer's model with non-Bayesian mass)

Let us apply Shafer's model and change a bit the non-Bayesian input mass by taking $m(\theta_1)=0.10$, $m(\theta_2)=0$, $m(\theta_3)=0.20$, $m(\theta_1\cup\theta_2)=0.30$, $m(\theta_1\cup\theta_3)=0.10$, $m(\theta_2\cup\theta_3)=0$ and $m(\theta_1\cup\theta_2\cup\theta_3)=0.30$. The results of the mappings are given in Table 7.

Table 7 shows that although all transformation methods can provide the better performance than the original BPA in decision-making, the new method achieves the best performance which has the largest PIC value and smallest EH value.

Table 6 Probability transformation results of Example 7 based on different approaches

Result	θ_1	θ_2	θ_3	d_P	EH(·)	PIC(·)
PrBel(·)	0.566 75	0.403 82	0.029 42	0.265 96	0.720 68	0.279 33
PrPl(·)	0.542 10	0.400 50	0.057 40	0.246 88	0.785 02	0.214 98
PrNPl(·)	0.472 22	0.388 89	0.138 89	0.219 92	0.906 40	0.093 60
BetP(·)	0.505 00	0.395 00	0.100 00	0.226 83	0.857 61	0.142 39
CuzzP(·)	0.502 93	0.393 66	0.103 42	0.225 79	0.862 28	0.137 72
PraPl(·)	0.529 44	0.397 78	0.072 78	0.238 38	0.813 83	0.186 17
PrHyb(·)	0.557 51	0.401 93	0.040 56	0.258 00	0.748 29	0.251 71
$\text{DSmP}_{\varepsilon=0.001}(\cdot)$	0.566 46	0.403 70	0.029 83	0.265 66	0.721 74	0.278 26
PrBPl(·)	0.582 15	0.394 68	0.023 17	0.273 56	0.700 09	0.299 91
PrBP2(·)	0.588 96	0.386 48	0.024 56	0.275 23	0.701 11	0.298 89
PrBP3(·)	0.603 34	0.372 91	0.023 76	0.281 60	0.693 19	0.306 81
Ref. [17]	0.566 75	0.403 82	0.029 42	0.265 96	0.720 68	0.279 33
This paper	0.680 00	0.300 00	0.020 00	0.333 77	0.638 70	0.361 30

Table 7 Probability transformation results of Example 8 based on different approaches

Result	θ_1	θ_2	θ_3	d_P	EH(·)	PIC(·)
PrBel(·)	0.533 33	0	0.466 67	0.508 81	NaN	NaN
PrPl(·)	0.448 57	0.218 57	0.332 86	0.432 35	0.963 16	0.036 84
PrNPl(·)	0.400 00	0.300 00	0.300 00	0.435 89	0.991 16	0.008 84
BetP(·)	0.400 00	0.250 00	0.350 00	0.418 33	0.983 54	0.016 46
CuzzP(·)	0.388 24	0.247 06	0.364 71	0.413 82	0.983 62	0.016 38
PraPl(·)	0.380 00	0.210 00	0.410 00	0.408 17	0.965 74	0.034 26
PrHyb(·)	0.455 32	0.169 78	0.374 90	0.430 89	0.934 91	0.065 09
$\text{DSmP}_{\varepsilon=0.001}(\cdot)$	0.530 50	0.003 93	0.465 57	0.505 84	0.649 91	0.350 09
PrBPl(·)	0.560 00	0.000 00	0.440 00	0.518 85	NaN	NaN
PrBP2(·)	0.452 45	0.201 29	0.346 26	0.431 43	0.954 60	0.045 40
PrBP3(·)	0.635 42	0.051 54	0.313 04	0.549 64	0.732 34	0.267 66
Ref. [17]	0.533 33	0	0.466 67	0.508 81	NaN	NaN
This paper	0.800 00	$\to 0$	0.200 00	0.700 00	0.455 49	0.544 51

6.7. Example 9 (hybrid DSm model)

Consider the hybrid DSm model in which all intersections of elements of Θ are empty except $\theta_1 \cap \theta_2$. In this case, G^Θ reduces to nine elements $\{\varnothing, \theta_1 \cap \theta_2, \theta_1, \theta_2, \theta_3, \theta_1 \cup \theta_2, \theta_1 \cup \theta_3, \theta_2 \cup \theta_3, \theta_1 \cup \theta_2 \cup \theta_3\}$. The input masses of focal elements are given by $m(\theta_1 \cap \theta_2)$=0.20, $m(\theta_1)$=0.10, $m(\theta_3)$=0.20, $m(\theta_1 \cap \theta_2)$ = 0.30, $m(\theta_1 \cup \theta_3)$=0.10 and $m(\theta_1 \cup \theta_2 \cup \theta_3)$= 0.10.

In order to apply all methods, the refined frame $\Theta^{\text{ref}}=\{\theta_1' = \theta_1/(\theta_1 \cap \theta_2), \theta_2' = \theta_2/(\theta_1 \cap \theta_2), \theta_3' = \theta_3, \theta_4' = \theta_1 \cap \theta_2\}$ with Shafer's model as depicted in Fig. 2 is given in Table 8.

As shown in Table 9, the new method provides the best results in terms of PIC and EH metric. Moreover, in the refined frame Θ^{ref}, the masses of θ_1' and θ_2' involved in the ignorance are null, so PrBel, PrBPl and Deng's method do not work correctly because they cannot be directly evaluated for θ_1' and θ_2' in such case.

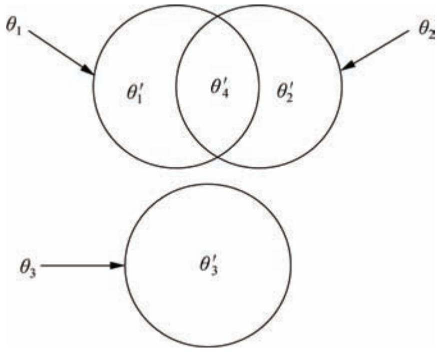

Fig. 2 Refined 3D frame for Example 9.

Table 8 Input of precise BPA for Example 9

Subset	$m(\cdot)$
$\{\theta_3'\}$	0.2
$\{\theta_4'\}$	0.2
$\{\theta_1' \cup \theta_4'\}$	0.1
$\{\theta_1' \cup \theta_2' \cup \theta_4'\}$	0.3
$\{\theta_1' \cup \theta_3' \cup \theta_4'\}$	0.1
$\{\theta_1' \cup \theta_2' \cup \theta_3' \cup \theta_4'\}$	0.1

Table 9 Probability transformation results of Example 9 based on different approaches

Result	θ_1'	θ_2'	θ_3'	θ_4'	d_P	EH(·)	PIC(·)
PrBel(·)	NaN	NaN	0.300 00	0.700 00	NaN	NaN	NaN
PrPl(·)	0.203 46	0.084 85	0.240 40	0.471 28	0.351 89	0.887 62	0.112 38
PrNPl(·)	0.272 73	0.181 82	0.181 82	0.363 64	0.366 80	0.968 13	0.031 87
BetP(·)	0.208 33	0.125 00	0.258 33	0.408 33	0.325 32	0.939 28	0.060 72
CuzzP(·)	0.200 00	0.133 33	0.266 67	0.400 00	0.319 72	0.944 62	0.055 38
PraPl(·)	0.163 64	0.109 09	0.309 09	0.418 18	0.313 34	0.912 79	0.087 21
PrHyb(·)	0.133 91	0.058 28	0.265 60	0.542 21	0.377 81	0.807 12	0.192 88
$\text{DSmP}_{\varepsilon=0.001}(\cdot)$	0.002 47	0.001 73	0.299 63	0.696 18	0.506 09	0.460 97	0.539 03
PrBPl(·)	NaN	NaN	0.266 67	0.733 33	NaN	NaN	NaN
PrBP2(·)	0.180 07	0.076 00	0.243 81	0.500 12	0.360 82	0.862 16	0.137 84
PrBP3(·)	0.032 18	0.004 99	0.228 76	0.734 07	0.535 83	0.505 97	0.494 03
Ref. [17]	NaN	NaN	0.300 00	0.700 00	NaN	NaN	NaN
This paper	→0	→0	0.200 00	→0.8	0.600 00	0.360 96	0.639 04

6.8. Example 10 (free DSm model)

Consider the free DSm model depicted in Fig. 3 with the input masses given in Table 10. One works on the refined frame $\Theta^{\text{ref}}=\{\theta_1', \theta_2', \theta_3', \theta_4', \theta_5', \theta_6', \theta_7'\}$ where the elements of Θ^{ref} are exclusive (assuming such refinement has a physical sense) according to Fig. 3. The PIC values obtained with different mappings are given in Table 11. One sees that the new method gets

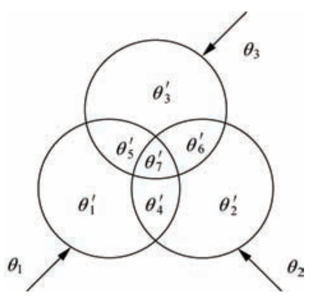

Fig. 3 Free DSm models for 3D frame for Example 10.

Table 10 Input of precise BPA for Example 10

Subset	$m(A)$
$\{\theta_1 \cap \theta_2 \cap \theta_3\}$	0.1
$\{\theta_1 \cap \theta_2\}$	0.2
$\{\theta_1\}$	0.3
$\{\theta_1 \cup \theta_2\}$	0.1
$\{\theta_1 \cup \theta_2 \cup \theta_3\}$	0.3

Table 11 Probability transformation results of Example 10 based on different approaches

Result	θ_1'	θ_2'	θ_3'	θ_4'	θ_5'	θ_6'	θ_7'	d_P	EH(·)	PIC(·)
PrBel(·)	0	0	0	0	0	0	1.000 00	0.900 00	NaN	NaN
PrPl(·)	0.128 44	0.037 03	0.020 46	0.259 87	0.128 44	0.037 03	0.388 74	0.432 50	0.963 16	0.194 04
PrNPl(·)	0.159 09	0.090 91	0.068 18	0.204 55	0.159 09	0.090 91	0.327 27	0.406 56	0.991 16	0.026 60
BetP(·)	0.134 52	0.059 52	0.042 86	0.234 52	0.134 52	0.059 52	0.334 52	0.393 85	0.983 54	0.117 63
CuzzP(·)	0.146 51	0.083 72	0.062 79	0.188 37	0.146 51	0.083 72	0.288 37	0.363 13	0.983 62	0.062 16
PraPl(·)	0.143 18	0.081 82	0.061 36	0.184 09	0.143 18	0.081 82	0.304 55	0.365 90	0.965 74	0.069 28
PrHyb(·)	0.113 63	0.033 26	0.018 41	0.221 45	0.113 63	0.033 26	0.466 35	0.460 03	0.934 91	0.237 48
$\text{DSmP}_{\varepsilon=0.001}(\cdot)$	0.006 63	0.003 75	0.002 80	0.008 59	0.006 63	0.003 75	0.967 85	0.867 96	0.649 91	0.898 57
PrBPl(·)	0	0	0	0	0	0	1.000 00	0.900 00	NaN	NaN
PrBP2(·)	0.124 74	0.036 10	0.019 95	0.249 88	0.124 74	0.036 10	0.408 49	0.437 87	0.954 60	0.203 74
PrBP3(·)	0.028 80	0.002 24	0.000 68	0.148 08	0.028 80	0.002 24	0.789 17	0.706 08	0.732 34	0.637 06
Ref. [17]	0	0	0	0	0	0	1.000 00	0.900 00	NaN	NaN
This paper	→0	→0	→0	→0	→0	→0	→1	0.900 00	0	1.000 00

here again the best results in terms of PIC and EH. Although PrBel, PrBPl and Deng's method can get the largest distance, values of PIC and EH do not work due to zero assignment to singletons.

This new method is complicated and indeed results in a nonlinear problem. However, those above examples reveal the rationality and usefulness of the new method. So, when the size of the frame of discernment is not too large and the high computational complexity due to the nonlinearity of the maximization problem can be ignored, we advice to apply this new method. Moreover, it can solve the probability transformation of interval-valued belief structure.

7. Example with Imprecise BPA

The above examples are all precise belief structures. This section gives an interval-valued belief in Table 12, which shows an illustrative example and the results of belief and plausibility measures in an interval-valued belief environment.

Table 12 An interval-valued belief structure and corresponding belief and plausibility measures in Example 11

Result	$m(A)$	$\text{Bel}_m(A)$	$\text{Pl}_m(A)$
$\{\theta_1\}$	[0.15,0.16]	[0.15,0.16]	[0.51,0.52]
$\{\theta_2\}$	[0.22,0.23]	[0.22,0.23]	[0.60,0.61]
$\{\theta_3\}$	[0.06,0.07]	[0.06,0.07]	[0.49,0.50]
$\{\theta_1\cup\theta_2\}$	[0.13,0.14]	[0.50,0.51]	[0.93,0.94]
$\{\theta_1\cup\theta_3\}$	[0.18,0.19]	[0.39,0.40]	[0.77,0.78]
$\{\theta_2\cup\theta_3\}$	[0.20,0.21]	[0.48,0.49]	[0.84,0.85]
$\{\theta_1\cup\theta_2\cup\theta_3\}$	[0.05,0.06]	[1.00,1.00]	[1.00,1.00]

Thus the final probability transform is

$$p_A\in[0.329\,66,0.330\,00],\quad p_B\in[0.600\,00,0.610\,00]$$

$$p_C\in[0.060\,34,0.070\,07]$$

where the imprecise probability obtained by this new probability transform is compatible with its lower and upper bounds provided by imprecise Bel and Pl given in Table 13.

Table 13 Probability transformations in four constraints in Example 11

Constraint	p_A	p_B	p_C
Constraint 1	0.330 00	0.600 00	0.070 00
Constraint 2	0.329 96	0.600 00	0.070 04
Constraint 3	0.329 66	0.610 00	0.060 34
Constraint 4	0.329 94	0.600 00	0.070 07

8. Conclusions

Decision rules play an important role in complex and real time information fusion systems. Probability transformation of belief function can be considered as a probabilistic approximation of belief assignment, which aims to gain more reliable decision results.

This paper proposes a novel probability transformation of belief function based on distance maximization, and gives examples in all models including Shafer's model, free DSm model and hybrid DSm model in precise belief and interval-valued belief environments.

The experimental results based on these provided numerical examples show that the probability measure generated based on the proposed approach has less uncertainty and more stability when compared with other available probability transformation approaches of belief function. It can be concluded that the proposed approach is rational and effective. Significant differences in all the propositions can be further enlarged no matter the evidence is precise or imprecise, which is helpful for more consolidated and reliable decision.

Also, the proposed approach in this paper is more robust. But it should be noted that, the probability distribution derived based on the proposed approach is not definitely the optimal result, which relies on the optimization algorithms chosen. Therefore, the design of more reasonable objective function and the design of more powerful global optimization algorithm are important works in future.

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