

Formula to obtain the next prime in an arithmetic progression

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Abstract: In this article, a formula is given to obtain the next prime in an arithmetic progression.

Theorem: We consider the arithmetic progression $a + di \quad i \geq 0$ of positive integers with $GCD(a, d) = 1$ and considering that the final term is $a + dM$ is to say $0 \leq i \leq M$. Let p a term in the arithmetic progression (it doesn't have to be prime). Then the next prime in the arithmetic progression is:

$$nxt(a, d)(p) = p + d + d \cdot \sum_{k=1+(p-a)/d}^M \prod_{j=1+(p-a)/d}^k \left[\frac{2 - \sum_{s=1}^{a+jd} (\lfloor (a+jd)/s \rfloor - \lfloor (a+jd-1)/s \rfloor)}{a+jd} \right]$$

and the improved formula:

$$nxt(a, d)(p) = p + d + d \cdot \sum_{k=1+(p-a)/d}^M \prod_{j=1+(p-a)/d}^k \left[- \left(\left(2 + 2 \sum_{s=1}^{\sqrt{a+jd}} ((a+jd-1)/s - (a+jd)/s) \right) / (a+jd) \right) \right]$$

Where $\lfloor x \rfloor$ is the floor function. And where x/y is the integer division in the improved formula.

Proof:

By a past article [1] we have that the next prime function is:

$$nxt(p) = p + 1 + \sum_{k=p+1}^{2p} \prod_{i=p+1}^k \left[\frac{2 - \sum_{s=1}^i \left(\left\lfloor \frac{i}{s} \right\rfloor - \left\lfloor \frac{i-1}{s} \right\rfloor \right)}{i} \right]$$

Where the expression of the product is the Smarandache Prime Function:

$$G(i) = \begin{cases} 1 & \text{if } i \text{ is composite} \\ 0 & \text{if } i \text{ is prime} \end{cases}$$

We consider that $a + j_0 d$ is the next prime of a number p in an arithmetic progression $a + jd$. We have that $G(a + j_0 d) = 0$.

And for all j such that $p < a + jd < a + j_0 d$ we have that $G(a + jd) = 1$.

It is deduced that:

$$\prod_{j=1+(p-a)/d}^k G(a + jd) = \prod_{j=1+(p-a)/d}^{j_0-1} G(a + jd) \cdot \prod_{j \geq j_0}^k G(a + jd) = \begin{cases} 0 & k > j_0 - 1 \\ 1 & k \leq j_0 - 1 \end{cases}$$

since the first product has the value of 1, and the second product is zero since it has the factor $G(a + j_0 d) = 0$.

As a result in the formula $nxt(a, d)$ the non zero terms are summed until $j_0 - 1$ and has the value of 1.

$$nxt(a, d)(p) = p + d + d \cdot \sum_{k=1+(p-a)/d}^{j_0-1} 1 = p + d + d \cdot (j_0 - 1 + 1 - 1 - (p-a)/d) =$$

$$= p + d + j_0 d - d - p + a = a + j_0 d$$

And the result is proven .

The improved formula [2] is obtained by considering that the sum, in the Smarandache prime function, until the integer part of the square root and multiplied by 2 the result. Also the floor function is changed $\lfloor x \rfloor$ for the integer division operator x/y that it faster for the computation.

Let us see an example made in MATHEMATICA:

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a=5
5
dd=4
4
M=20
20
p=5
5
DD[i_]:=Sum[Quotient[(a+i*dd),j]-Quotient[a+i*dd-1,j],
{j,1,Sqrt[a+i*dd]}]
G[i_]:=-Quotient[(2-2*DD[i]),(a+i*dd)]
F[m_]:=Product[G[i],{i,(p-a)/dd+1,m}]
S[n_]:=Sum[F[m],{m,(p-a)/dd+1,M}]
While[p<a+(M-1)*dd+1,Print["nxt(",p,")=",p+dd+dd*S[p]];
p=p+dd+dd*S[p]]
nxt(5)=13
nxt(13)=17
nxt(17)=29
nxt(29)=37
nxt(37)=41

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nxt (41) =53

nxt (53) =61

nxt (61) =73

nxt (73) =89

The question is that if these formulas can be applied to prove the Dirichlet's Theorem [3]for arithmetic progressions.

That is to say: does any arithmetic progression $a + jd$ such that $GCD(a, d) = 1$ have infinite primes?

REFERENCES:

- [1] **S M Ruiz**. A functional recurrence to obtain the prime numbers using the Smarandache prime function. Smarandache Notions J., Vol. 11, No. 1-2-3, Spring 2000, p. 56.
<http://personal.telefonica.terra.es/web/smruiz/>
- [2] **Carlos Rivera**. The Prime Puzzles & Problems Connection. Problem 38.
www.primepuzzles.net
- [3] **Thomas Koshy**. Elementary Number Theory with applications. Page 178.