Heliyon

Received:
14 June 2018
Revised:
3 September 2018
Accepted:
11 October 2018

Cite as: Pairote Yiarayong, Piyada Wachirawongsakorn. A new generalization of BE-algebras. Heliyon 4 (2018) e00863. doi: 10.1016/j.heliyon.2018.e00863

A new generalization of BE-algebras

Pairote Yiarayong [a,*], **Piyada Wachirawongsakorn** [b]

[a] *Department of Mathematics, Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanulok 65000, Thailand*

[b] *Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanulok 65000, Thailand*

* Corresponding author.
E-mail address: pairote0027@hotmail.com (P. Yiarayong).

Abstract

In this paper we introduce the notion of $PSRU$-algebras as a generalization of BE-algebras, we investigate its elementary properties. The aim of this paper is to investigate the concept of filters, left ideals (right ideal, ideal) and fuzzy filters in $PSRU$-algebras. Moreover, we investigate relationships between left ideals and filters in $PSRU$-algebras. Furthermore, we characterize filters in terms of fuzzy filters. It is shown that if I be a left ideal of a $PSRU$-algebra X, then I_x is a left ideal of X for all $x \in X$.

Keywords: Mathematics

1. Introduction

In 2007, Kim and Kim [1] introduced the notion of BE-algebras which is another generalization of BCK-algebras. According to their definition, an algebra $(X; *, 1)$ of type $(2, 0)$ is called a BE-algebra if

$$x * x = 1, \tag{1}$$

$$x * 1 = 1, \tag{2}$$

$$1 * x = x, \tag{3}$$

$$x * (y * z) = y * (x * z) \tag{4}$$

for all $x, y, z \in X$. In 2008, Ahn and So [2] introduced the notion of ideals in BE-algebras and then stated and proved several characterizations of such ideals.

They have shown that if $(X; *, 1)$ is a self distributive BE-algebra, then $A(x, y)$ is an ideal of X for all $x, y \in X$. In 2009, Walendziak [3] introduced the notion of commutative BE-algebras. He defined $(X; *, 1)$ as a commutative BE-algebra with the following properties: $(X; *, 1)$ is a BE-algebra and $(x * y) * y = (y * x) * x$ for all $x, y \in X$. In [4], Ahn and So has studied the generalized upper set in BE-algebras. In 2010, Kim, [5] introduced the notion of essence in BE-algebras. In 2012, Ahn et al. [6] introduced the notion of terminal sections in BE-algebras. In [7], Rezaei and Saeid studied the concept of quotient in commutative BE-algebras. Saeid [8] introduced the notion of Smarandache weak BE-algebra, Q-Smarandache filters and Q-Smarandache ideals. He proved that a non empty subset F of a BE-algebra X is a Q-Smarandache filter if and only if $A(u, v) \subseteq F$ which $A(u, v)$ is a Q-Smarandache upper set. In [9], Jun and Kang introduced the notion of $\mathcal{N}$-ideals in BE-algebras. Rezaei and Saeid [10] studied the concept of homomorphisms in BE-algebras. In 2013, Saeid et al. [11] introduced and studied implicative filters of BE-algebras. In 2014, Radfar et al. [12] introduced the notion of hyper BE-algebras which is another generalization of BE-algebras. Borzooei et al. [13] studied the concept of bounded and involutory BE-algebras. In 2017, Jun and Ahn [14] introduced the notion of I_{BE}-energetic subsets in BE-algebras. In 2018, Hamidi and Saeid [15] studied the concept of (BE-algebras) dual BCK-algebras and hyper (BE-algebras) dual K-algebras. They proved that the (BE-algebra) dual BCK-algebra is a fundamental (BE-algebra) dual BCK-algebra.

In 2010, Song et al. [16] introduced concepts of fuzzy ideals of BE-algebras. In 2011, Rezaei and Saeid [17] introduced and studied fuzzy subalgebras of BE-algebras. In 2013, Dymek and Walendziak [18] introduced concepts of fuzzy filters of BE-algebras. In 2015, Abdullaha and Alib [19] introduced concepts of $\mathcal{N}$-fuzzy filters in BE-algebras. In 2017, Ahn and Kim [20] introduced concepts of rough fuzzy filters in BE-algebras. In 2018, Bandaru et al. [21] introduced concepts of falling fuzzy implicative filters of BE-algebras.

The aim of this paper is to extend the concept of BE-algebras given by Kim and Kim [1] to the context of $PSRU$-algebras. We introduce the notions of filters, left ideals (right ideal, ideal) and fuzzy filters in $PSRU$-algebras. Some characterizations of left ideals and filters are obtained. The relationship between these notions are stated and proved. Furthermore, we characterize filters in terms of fuzzy filters. It is shown that if I be a left ideal of a $PSRU$-algebra X, then I_x is a left ideal of X for all $x \in X$.

 https://doi.org/10.1016/j.heliyon.2018.e00863
2405-8440/

2. Results

2.1. The notion and elementary properties of $PSRU$-algebras

In this section, we mainly concentrate on the $PSRU$-algebra $(X;*,1)$. We need to extend the BE-algebras to the $PSRU$-algebra.

Definition 1. Let X be non empty set with a constant 1. A $PSRU$**-algebra** is a set $(X;*,1)$ and a binary operation "$*$" satisfying the following axioms:

$$x * 1 = 1, \tag{5}$$

$$x * (y * z) = y * (x * z), \tag{6}$$

for all $x, y \in X$.

Example 1. Let $X = \{1, b, c, d, e\}$ be a set with the following table:

$*$	1	b	c	d	e
1	1	1	1	1	1
b	1	b	b	b	b
c	1	b	d	e	c
d	1	b	c	d	e
e	1	b	e	c	d

By Equations (5) and (6), it is easy to know that $(X;*,1)$ is a $PSRU$-algebra.

Remark 1. By Equations (1), (2), (3) and (4), it is easy to see that every BE-algebra is $PSRU$-algebra.

The following example shows that the converse of Remark 1 is not true.

Example 2. Let $X = \mathbf{R}$ and the operation "$*$" be defined as follows:

$$x * y = \begin{cases} \frac{y}{x} & ; x, y \notin \{0\} \\ 0 & ; \text{otherwise.} \end{cases} \tag{7}$$

By Equation (7), it is easy to check that $(X;*,1)$ is a $PSRU$-algebra. Indeed, if $x, y, z \notin \{0\}$, then

$$\begin{aligned} x * (y * z) &= x * \left(\frac{z}{y}\right) \\ &= \frac{\left(\frac{z}{y}\right)}{x} \\ &= \frac{z}{xy} \end{aligned} \tag{8}$$

https://doi.org/10.1016/j.heliyon.2018.e00863

$$
\begin{aligned}
&= \frac{\left(\frac{z}{x}\right)}{y} \\
&= y * \frac{z}{x} \\
&= y * (x * z).
\end{aligned}
$$

If $x = 0$ or $y = 0$ or $z = 0$, then $x * (y * z) = 0 = y * (x * z)$. Notice that $2 * 2 = 1 \neq 0$. By Equation (8), $(X; *, 0)$ is not a BE-algebra.

For any $PSRU$-algebra $(X; *, 1)$, denote $B(X) = \{x \in X : 1 * x = x\}$ and $E(X) = \{x \in X : x * x = 1\}$. Hence a $PSRU$-algebra is a BE-algebra if and only if $X = B(X) \cap E(X)$. We give the definition of a filter in a $PSRU$-algebra.

Definition 2. Let $(X; *, 1)$ be a $PSRU$-algebra. A non empty subset F of X is called a **filter** if it satisfies:

- $1 \in F$,
- if $x * y \in F$ and $x \in F$, then $y \in F$.

Example 3. 1. Let $X = \{1, x\}$ be a set with the following table:

$*$	1	x
1	1	x
x	1	x

It is easy to know that $(X; *, 1)$ is a $PSRU$-algebra. Then $\{1\}$ is filter of X.

2. In Example 1, X is a $PSRU$-algebra X. Then $\{1\}$ is not a filter of X, since $c * 1 = 1 \in \{1\}$ and $1 \in \{1, c\}$, but $c \notin \{1\}$.

Let $(X; *, 1)$ be a $PSRU$-algebra. For any elements $x, y \in X$ and $n \in \mathbf{Z}^+$, we use the notation $x^n * y$ instead of $x * (\dots (x * (x * y))) \dots)$ in which x occurs n times. We give the definition of a upper set (generalized upper set) in a $PSRU$-algebra.

Definition 3. Let $(X; *, 1)$ be a $PSRU$-algebra and let $x, y \in X$. Define

$$U(x, y) := \{z \in X : x * (y * z) = 1\} \tag{9}$$

$$(U^n(x, y) := \{z \in X : x^n * (y * z) = 1\}). \tag{10}$$

We call $U(x, y)$ $(U^n(x, y))$ an **upper set (generalized upper set)** of x and y.

Remark 2. Let $(X; *, 1)$ be a $PSRU$-algebra and let $x, y \in X$. By Equations (9) and (10), respectively, it is easy to see that $1 \in U(x, y)$ and $1 \in U^n(x, y)$.

https://doi.org/10.1016/j.heliyon.2018.e00863

In what follows let X denote a $PSRU$-algebra unless otherwise specified.

Theorem 1. *Let X be a $PSRU$-algebra and $n \in \mathbf{Z}^+$. If $y \in X$ such that $y * z = 1$ for all $z \in X$, then $U^n(x, y) = X = U^n(y, x)$ for all $x \in X$.*

Proof. Let $a \in X$. Then $x^n * (y * a) = x^n * 1 = 1$ and $y^n * (x * a) = y^{n-1} * (y * (x * a)) = y^{n-1} * 1 = 1$. Hence $U^n(x, y) = X = U^n(y, x)$.

By Theorem 1, we have the following.

Corollary 1. *Let X be a $PSRU$-algebra. If $y \in X$ such that $y * z = 1$ for all $z \in X$, then $U(x, y) = X = U(y, x)$ for all $x \in X$.*

Proof. Similar to the proof of Theorem 1.

Proposition 1. *Let X be a $PSRU$-algebra and $n \in \mathbf{Z}^+$. If F is a filter of X, then $U^n(x, y) \subseteq F$ for all $x, y \in F$.*

Proof. Suppose that F is a filter of X and $x, y \in F$. Let $a \in U^n(x, y)$. Then $x^n * (y * a) = 1 \in F$. Since $x, y \in F$, we have $a \in F$. Hence $U^n(x, y) \subseteq F$.

By induction we easily obtain.

Corollary 2. *Let X be a $PSRU$-algebra. If F is a filter of X, then $U(x, y) \subseteq F$ for all $x, y \in F$.*

Proof. Similar to the proof of Proposition 1.

We give the definition of a self distributive $PSRU$-algebra. A $PSRU$-algebra $(X; *, 1)$ is said to be **self distributive** if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. An element 1 of X is a **left identity** of X if $1 * x = x$ for all $x \in X$.

In the following we show that any self distributive $PSRU$-algebra with left identity need not be a self distributive BE-algebra.

Example 4. In Example 3(1), it is easy to know that $(X; *, 1)$ is a self distributive $PSRU$-algebra with left identity. Notice that $x * x = x \neq 1$. Therefore $(X; *, 1)$ is not a BE-algebra.

Let $(X; *, 1)$ be a $PSRU$-algebra. We introduce a relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 1$. Clearly, $x \leq 1$ for all $x \in X$.

https://doi.org/10.1016/j.heliyon.2018.e00863

Lemma 1. *Let X be a self distributive $PSRU$-algebra. Then the following statements hold.*

1. *For all $x, y, z \in X$ if $x \leq y$, then $z * x \leq z * y$.*
2. *For all $x, y, z \in X$ if $z \leq x$, then $(y * z) \leq (z * x) * (y * x)$.*

Proof. 1. Let $x, y, z \in X$ such that $x \leq y$. Then $x * y = 1$. Since $(z * x) * (z * y) = z * (x * y) = z * 1 = 1$, we have $z * x \leq z * y$.

2. Let $x, y, z \in X$ such that $z \leq x$. Then $z * x = 1$. Consider,

$$\begin{aligned}(y * z) * [(z * x) * (y * x)] &= (z * x) * [(y * z) * (y * x)] \\ &= (z * x) * [y * (z * x)] \\ &= (z * x) * (y * 1) \\ &= 1.\end{aligned} \tag{11}$$

By Equation (11), $(y * z) \leq (z * x) * (y * x)$.

By Lemma 1, we have the following.

Proposition 2. *Let X be a self distributive $PSRU$-algebra with left identity. For all $a, x, y, x_i, y_i \in X$ if*

$$x \leq y * a, \tag{12}$$

$$x_n \leq x_{n-1} * (* \ldots * x_2 * (x_1 * x) \ldots), \tag{13}$$

$$y_n \leq y_{n-1} * (* \ldots * y_2 * (y_1 * y) \ldots), \tag{14}$$

then $y_n \leq y_{n-1} * (* \ldots * y_2 * (y_1 * (x_n * (* \ldots * x_2 * (x_1 * a) \ldots))) \ldots)$.

Proof. Let $a, x, y, x_i, y_i \in X$ such that $x \leq y * a$. By Lemma 1(1), $x_1 * x \leq x_1 * (y * a)$. Repeating the process we have $x_n * (* \ldots * x_2 * (x_1 * x) \ldots) \leq x_n * (* \ldots * x_2 * (x_1 * (y * a)) \ldots)$. By Equations (12), (13) and (14),

$$\begin{aligned}x_n * (* \ldots * x_2 * (x_1 * (y * a)) \ldots) &= 1 * \left[x_n * (* \ldots * x_2 * (x_1 * (y * a)) \ldots)\right] \\ &= \left[x_n * (* \ldots * x_2 * (x_1 * x) \ldots)\right] * \\ &\quad \left[x_n * (* \ldots * x_2 * (x_1 * (y * a)) \ldots)\right] \\ &= 1.\end{aligned} \tag{15}$$

By Equation (15), $x_n \leq (* \ldots * x_2 * (x_1 * (y * a)) \ldots)$. Since X is a $PSRU$-algebra, we have $y * (x_n * (* \ldots * x_2 * (x_1 * a) \ldots)) = 1$. Thus $y \leq x_n * (* \ldots * x_2 * (x_1 * a) \ldots)$. Then by Lemma 1(1), $y_1 * y \leq y_1 * (x_n * (* \ldots * x_2 * (x_1 * a) \ldots))$. Repeating the process we have $y_n * (* \ldots * y_2 * (y_1 * y) \ldots) \leq y_n * (* \ldots * y_2 * (y_1 * (x_n * (* \ldots * x_2 * (x_1 * a) \ldots))) \ldots)$. Clearly, $y_n * (* \ldots *$

https://doi.org/10.1016/j.heliyon.2018.e00863

$y_2 * (y_1 * (x_n * (* \ldots * x_2 * (x_1 * a) \ldots))) \ldots) = 1$. Hence $y_n \leq y_{n-1} * (* \ldots * y_2 * (y_1 * (x_n * (* \ldots * x_2 * (x_1 * a) \ldots))) \ldots)$.

The following three theorems give the self distributive $PSRU$-algebra properties of filters.

Theorem 2. *Let X be a self distributive $PSRU$-algebra with left identity and $n \in \mathbf{Z}^+$. Then $U^n(x, y)$ is a filter of X for all $x, y \in X$.*

Proof. Clearly, $1 \in U^n(x, y)$ for all $x, y \in X$. Let $a, a * b \in U^n(x, y)$. Then $x^n * (y * a) = 1$ and $x^n * (y * (a * b)) = 1$. By assumption,

$$\begin{aligned} x^n * (y * b) &= 1 * \left[x^n * (y * b)\right] \\ &= \left[x^n * (y * a)\right] * \left[x^n * (y * b)\right] \\ &= x^n * [(y * a) * (y * b)] \\ &= x^n * [y * (a * b)] \\ &= 1. \end{aligned} \tag{16}$$

By Equation (16), $b \in U^n(x, y)$ and hence $U^n(x, y)$ is a filter of X.

By induction we easily obtain.

Corollary 3. *Let X be a self distributive $PSRU$-algebra with left identity. Then $U(x, y)$ is a filter of X for all $x, y \in X$.*

We give the definition of an initial section (generalized initial section) of $x \in X$ in a $PSRU$-algebra.

Definition 4. Let X be a $PSRU$-algebra and let $x \in X$. Define

$$U(x) := \{y \in X : x * y = 1\} \tag{17}$$

$$(U^n(x) := \{z \in X : x^n * y = 1\}). \tag{18}$$

We call $U(x)\,(U^n(x))$ an **initial section (generalized initial section)** of x.

Remark 3. Let X be a $PSRU$-algebra and let $x \in X$. By Equations (17) and (18), respectively, it is easy to see that $1 \in U(x)$ and $1 \in U^n(x)$.

Proposition 3. *Let X be a $PSRU$-algebra and $n \in \mathbf{Z}^+$. If F is a filter of X, then $U^n(x) \subseteq F$ for all $x \in F$.*

 https://doi.org/10.1016/j.heliyon.2018.e00863

Proof. Suppose that F is a filter of X and $x \in F$. Let $a \in U^n(x)$. Then $x^n * a = 1 \in F$. Since $x \in F$, we have $a \in F$. Hence $U^n(x) \subseteq F$.

By Proposition 3, we have the following.

Corollary 4. *Let X be a $PSRU$-algebra. If F is a filter of X, then $U(x) \subseteq F$ for all $x \in F$.*

Proof. Similar to the proof of Proposition 3.

Theorem 3. *Let X be a self distributive $PSRU$-algebra with left identity and $n \in \mathbf{Z}^+$. Then $U^n(x)$ is a filter of X for all $x \in X$.*

Proof. Clearly, $1 \in U^n(x)$ for all $x \in X$. Let $a, a * b \in U^n(x)$. Then $x^n * a = 1$ and $x^n * (a * b) = 1$. By assumption,

$$\begin{aligned} x^n * b &= 1 * (x^n * b) \\ &= (x^n * a) * (x^n * b) \\ &= x^n * (a * b) \\ &= 1. \end{aligned} \tag{19}$$

By Equation (19), $b \in U^n(x)$ and hence $U^n(x)$ is a filter of X.

By Theorem 3, we have the following.

Corollary 5. *Let X be a self distributive $PSRU$-algebra with left identity. Then $U(x)$ is a filter of X for all $x \in X$.*

Proof. The proof is straightforward.

Theorem 4. *Let F be a filter of a self distributive $PSRU$-algebra with left identity X and $x \in X$. If $F_x := \{y \in X : x * y \in F\}$, then F_x is a filter of X containing F.*

Proof. Since $x * 1 = 1 \in F$, we have $1 \in F_x$. Let $a, a * b \in F_x$. Then $x * a, x * (a * b) \in F$. By assumption, $x * (a * b) = (x * a) * (x * b)$. Thus $x * b \in F$ and hence $b \in F_x$. This implies that F is a filter of X.

There are no hidden difficulties to prove it and hence we omit its proof.

Theorem 5. *Let X be a $PSRU$-algebra. If F_α is a filter of X for all $\alpha \in \beta$, then $\bigcap_{\alpha \in \beta} F_\alpha$ is a filter of X.*

https://doi.org/10.1016/j.heliyon.2018.e00863

2.2. Ideals of $PSRU$-algebras

In this section, the notion of ideals is introduced in $PSRU$-algebras. Some properties of these ideals are studied. Especially we discuss relations between left ideals and filters.

Definition 5. Let $(X; *, 1)$ be a $PSRU$-algebra. A non empty subset I of X is called a **left ideal (right ideal)** if $x * a \in I$ for all $x \in X$ and $a \in I$ ($(a * (b * x)) * x \in I$ for all $x \in X$ and $a, b \in I$). A non empty subset I of X is called an **ideal** if it is both a left and a right ideal of X.

Example 5. In Example 1, $\{1, b\}$ is an ideal of a $PSRU$-algebra $(X; *, 1)$, but $\{1, c\}$ is not an ideal of X, since $c * c = d \notin \{1, c\}$.

Lemma 2. *Every right ideal of a $PSRU$-algebra $(X; *, 1)$ contains* 1.

Proof. Let I be a right ideal of X. Since $I \neq \emptyset$, there exists $a \in I$. Clearly, $1 = (a * (a * 1)) * 1 \in I$.

By Lemma 2, we have the following.

Lemma 3. *Let X be a $PSRU$-algebra with left identity. If $x \in X$ and $a \in I$, then $(a * x) * x \in I$.*

Proof. Let $x \in X$ and $a \in I$. By Lemma 2, $(a * x) * x = (1 * (a * x)) * x \in I$.

By Lemma 3, we have the following.

Lemma 4. *Let X be a $PSRU$-algebra with left identity. If $x \in X, a \in I$ such that $a \leq x$, then $x \in I$.*

Proof. Let $x \in X$ and $a \in I$ such that $a \leq x$. Then $a * x = 1$. By Lemma 3, $x = 1 * x = (a * x) * x \in I$.

Proposition 4. *Let I be an ideal of a $PSRU$-algebra with left identity X such that $x * (y * z) \in I$ and $y \in I$ imply $x * z \in I$ for all $x, y, z \in X$. If $a \in I$ such that $a \leq x$, then $x \in I$.*

Proof. Let $x \in X$ and $a \in I$ such that $a \leq x$. Then $a * x = 1$. Clearly, $1 * (a * x) = 1 * 1 = 1 \in I$. By assumption, $1 = 1 * x \in I$.

 https://doi.org/10.1016/j.heliyon.2018.e00863
2405-8440/

Proposition 5. *Let X be a $PSRU$-algebra with left identity. If I is a right ideal of X, then $U(x, y) \subseteq I$ for all $x, y \in X$.*

Proof. Suppose that I is a right ideal of X and $x, y \in X$. Let $a \in U(x, y)$. Then $x * (y * a) = 1$. Clearly, $a = 1 * a = [x * (y * a)] * a \in I$. Hence $U(x, y) \subseteq I$.

The following three theorems give the self distributive $PSRU$-algebra properties of left ideals.

Theorem 6. *Let X be a self distributive $PSRU$-algebra and $n \in \mathbf{Z}^+$. Then $U^n(x, y)$ is a left ideal of X for all $x, y \in X$.*

Proof. Let $r \in X$ and $a \in U^n(x, y)$. Then $x^n * (y * a) = 1$. By assumption,

$$\begin{aligned} x^n * [y * (r * a)] &= x^n * [(y * r) * (y * a)] \\ &= x^{n-1} * (x * [(y * r) * (y * a)]) \\ &= x^{n-1} * [(x * (y * r)) * (x * (y * a))] \\ &\ \ \vdots \\ &= \left[x^n * (y * r)\right] * \left[x^n * (y * a)\right] \\ &= \left[x^n * (y * r)\right] * 1 \\ &= 1. \end{aligned} \tag{20}$$

By Equation (20), $U^n(x, y)$ is a left ideal of X.

By Theorem 6, we have the following.

Corollary 6. *Let X be a self distributive $PSRU$-algebra. Then $U(x, y)$ is a left ideal of X for all $x, y \in X$.*

Proof. The proof is straightforward.

Theorem 7. *Let X be a self distributive $PSRU$-algebra and $n \in \mathbf{Z}^+$. Then $U^n(x)$ is a left ideal of X for all $x \in X$.*

Proof. Let $r \in X$ and $a \in U^n(x)$. Then $x^n * a = 1$. By assumption,

$$\begin{aligned} x^n * (r * a) &= x^{n-1} * [x * (r * a)] \\ &= x^{n-1} * [(x * r) * (x * a)] \\ &\ \ \vdots \\ &= (x^n * r) * (x^n * a) \end{aligned} \tag{21}$$

https://doi.org/10.1016/j.heliyon.2018.e00863

$$= (x^n * r) * 1$$
$$= 1.$$

By Equation (21), $U^n(x)$ is a left ideal of X.

By Theorem 7, we have the following.

Corollary 7. *Let X be a self distributive $PSRU$-algebra. Then $U(x)$ is a left ideal of X for all $x \in X$.*

Proof. The proof is straightforward.

Let $(X; *, 1)$ be a $PSRU$-algebra. An element 1 of X is a **zero element** of X if $1 * x = 1$ for all $x \in X$. In Example 1, 1 is a zero element of X.

Lemma 5. *Every left ideal of a $PSRU$-algebra with zero $(X; *, 1)$ contains 1.*

Proof. Let I be a left ideal of X. Since $I \neq \emptyset$, there exists $a \in I$. Clearly, $1 = 1 * a \in I$.

By Lemma 5, we have the following.

Theorem 8. *Every filter is an ideal of a $PSRU$-algebra with zero X.*

Proof. Let F be a filter of X and $a, b \in F, r \in X$. Then by Lemma 5, $1 * (r * x) = 1, 1 * [(a * (b * r)) * r] = 1 \in F$. By assumption, $r * x, (a * (b * r)) * r \in F$. Hence F is an ideal of X.

We conclude this section with the following theorem.

Theorem 9. *Let I be a left ideal of a $PSRU$-algebra X and $x \in X$. If $I_x := \{y \in X : x * y \in I\}$, then I_x is a left ideal of X.*

Proof. Let $x \in X$ and $a \in I$. By Definition 5, $x * a \in I$ then our hypothesis implies $a \in I_x$. Hence $I \subseteq I_x$. To show that I_x is a left ideal of X. Let $r \in X$ and $a \in I_x$. Then $x * a \in I$. By assumption, $x * (r * a) = r * (x * a) \in I$. This implies that $r * a \in I_x$. Thus I_x is a left ideal of X.

 https://doi.org/10.1016/j.heliyon.2018.e00863

2.3. Fuzzy filters of $PSRU$-algebras

In 2013, Dymek and Walendziak [18] introduced concepts of fuzzy filters of BE-algebras. In this section we discuss these concepts of $PSRU$-algebras and next study some of its elementary properties.

Definition 6. Let $(X; *, 1)$ be a $PSRU$-algebra. A fuzzy subset μ of X is called a **fuzzy filter** if it satisfies:

$$\mu(1) \geq \mu(x), \tag{22}$$

$$\mu(x) \geq \mu(y * x) \wedge \mu(y), \tag{23}$$

for all $x, y \in X$.

Example 6. 1. Let X be the $PSRU$-algebra from Example 3(1). Define a fuzzy subset μ of X by

$$\mu(x) = \begin{cases} 0.9 & ; x \in \{1\} \\ 0.2 & ; \text{otherwise.} \end{cases} \tag{24}$$

By Equations (22), (23) and (24), it is easily checked that μ is a fuzzy filter of X.

2. Let X be the $PSRU$-algebra from Example 1. Define a fuzzy subset μ of X by

$$\mu(x) = \begin{cases} 0.9 & ; x \in \{1\} \\ 0.2 & ; \text{otherwise.} \end{cases} \tag{25}$$

By Equations (22), (23) and (25), μ is not a fuzzy filter of X, since $\mu(c) = 0.2 \ngeq 0.9 = \mu(1) \wedge \mu(1) = \mu(1 * c) \wedge \mu(1)$.

Lemma 6. *Let μ be a fuzzy filter of $PSRU$-algebra X. Then the following statements hold.*

1. *For any $x, y \in X$ if $x \leq y$, then $\mu(x) \leq \mu(y)$.*
2. *For any $x, y \in X$ if $\mu(x * y) = \mu(1)$, then $\mu(x) \leq \mu(y)$.*

Proof. 1. Let $x, y \in X$. Since $x \leq y$, we have $x * y = 1$. By assumption, $\mu(y) \geq \mu(x * y) \wedge \mu(x) = \mu(1) \wedge \mu(x) = \mu(x)$.

2. Let $x, y \in X$. By assumption, $\mu(y) \geq \mu(x * y) \wedge \mu(x) = \mu(1) \wedge \mu(x) = \mu(x)$ This completes the proof.

Theorem 10. *Let X be a $PSRU$-algebra and let μ be a fuzzy filter of X. For any $x, y, z \in X$ if $x \leq y * z$, then $\mu(z) \geq \mu(x) \wedge \mu(y)$.*

https://doi.org/10.1016/j.heliyon.2018.e00863

Proof. Let $x, y, z \in X$ such that $x \leq y * z$. Then $x * (y * z) = 1$. By assumption,

$$\begin{aligned}\mu(y * z) &\geq \mu(x * (y * z)) \wedge \mu(x) \\ &= \mu(1) \wedge \mu(x) \\ &= \mu(x).\end{aligned} \tag{26}$$

By Equation (26), $\mu(z) \geq \mu(y * z) \wedge \mu(y) \geq \mu(x) \wedge \mu(y)$.

By Theorem 10, we have the following.

Corollary 8. *Let X be a $PSRU$-algebra and let μ be a fuzzy filter of X. For any $x, x_i \in X$ if $x_n \leq x_{n-1} * (* \ldots * x_2 * (x_1 * x) \ldots)$, then $\mu(x) \geq \bigwedge_{i=1}^{n} \mu(x_i)$.*

Proof. Similar to the proof of Theorem 10.

Theorem 11. *Let X be a $PSRU$-algebra and let μ be a fuzzy filter of X. For any $x, y, z \in X$ if $x \leq y * z$, then $\mu(z) \geq \mu(x) \wedge \mu(y)$.*

Proof. Let $x, y, z \in X$ such that $x \leq y * z$. Then $x * (y * z) = 1$. By assumption,

$$\begin{aligned}\mu(y * z) &\geq \mu(x * (y * z)) \wedge \mu(x) \\ &= \mu(1) \wedge \mu(x) \\ &= \mu(x).\end{aligned} \tag{27}$$

By Equation (27), $\mu(z) \geq \mu(y * z) \wedge \mu(y) \geq \mu(x) \wedge \mu(y)$.

Now, we characterize fuzzy filters of $PSRU$-algebras.

Theorem 12. *Let X be a $PSRU$-algebra with left identity. Then the following conditions are equivalent.*

1. *μ is a fuzzy filter of X.*
2. *For all $x, y, z \in X$,*

$$\mu(1) \geq \mu(x), \tag{28}$$

$$\mu(y * z) \geq \mu(x * (y * z)) \wedge \mu(y). \tag{29}$$

Proof. $(1 \Rightarrow 2)$ Obvious.

$(2 \Rightarrow 1)$ Let $x, y \in X$. By Equations (28) and (29), $\mu(z) = \mu(1 * z) \geq \mu(x * (1 * z)) \wedge \mu(y) = \mu(x * z) \wedge \mu(y)$. Hence μ is a fuzzy filter of X.

 https://doi.org/10.1016/j.heliyon.2018.e00863

Let μ be a fuzzy subset in a $PSRU$-algebra X. For any $t \in [0,1]$, a set $\mathcal{U}(\mu,t) =: \{x \in X : \mu(x) \geq t\}$ is called a **level subset** of μ.

Theorem 13. *Let X be a $PSRU$-algebra. Then the following conditions are equivalent.*

1. *μ is a fuzzy filter of X.*
2. *A non empty level subset $\mathcal{U}(\mu,t)$ is a filter of X for all $t \in [0,1]$.*

Proof. $(1 \Rightarrow 2)$ Let $x \in X$ and $t \in [0,1]$. Clearly, $x * (x * 1) = 1$. Then $x \leq x * 1$. By Theorem 11, $\mu(1) \geq \mu(x) \wedge \mu(1) = \mu(x)$. Since $\mathcal{U}(\mu,t) \neq \emptyset$, there exists an element $y \in \mathcal{U}(\mu,t)$. This implies that $\mu(1) \geq \mu(y) \geq t$. Thus $1 \in \mathcal{U}(\mu,t)$. For any $a, a * b \in \mathcal{U}(\mu,t)$, $\mu(a) \geq t$ and $\mu(a * b) \geq t$. Since μ is a fuzzy filter of X, we have $\mu(b) \geq \mu(a * b) \wedge \mu(a) \geq t$. Therefore $b \in \mathcal{U}(\mu,t)$ and hence $\mathcal{U}(\mu,t)$ is a filter of X.

$(2 \Rightarrow 1)$ Clearly, $1 \in \mathcal{U}(\mu,\mu(x))$ for all $x \in X$. Then $\mu(1) \geq \mu(x)$ for all $x \in X$. Suppose that μ is not a fuzzy filter of X. This implies that there exists an element $a, b \in X$ such that $\mu(b) \leq \mu(a * b) \wedge \mu(a)$. It is easy to see that $0 \leq \frac{1}{2}\left[\mu(b) + \mu(a * b) \wedge \mu(a)\right] \leq 1$. Consider,

$$\begin{aligned} \frac{1}{2}\left[\mu(b) + \mu(a * b) \wedge \mu(a)\right] &\leq \frac{1}{2}\left[\mu(a * b) \wedge \mu(a) + \mu(a * b) \wedge \mu(a)\right] \\ &= \mu(a * b) \wedge \mu(a) \\ &\leq \mu(a * b) \end{aligned} \tag{30}$$

and $\frac{1}{2}\left[\mu(b) + \mu(a * b) \wedge \mu(a)\right] \leq \mu(a)$. This implies that

$$a, a * b \in \mathcal{U}\left(\mu, \frac{1}{2}\left[\mu(b) + \mu(a * b) \wedge \mu(a)\right]\right). \tag{31}$$

Since $\mathcal{U}\left(\mu, \frac{1}{2}\left[\mu(b) + \mu(a * b) \wedge \mu(a)\right]\right)$ is a filter of X, we have

$$b \in \mathcal{U}\left(\mu, \frac{1}{2}\left[\mu(b) + \mu(a * b) \wedge \mu(a)\right]\right). \tag{32}$$

Thus by Equations (30), (31) and (32), $\mu(b) \geq \frac{1}{2}\left[\mu(b) + \mu(a * b) \wedge \mu(a)\right]$. This is a contradiction. Hence μ is a fuzzy filter of X.

By Theorem 13, we have the following.

Corollary 9. *If μ is a fuzzy filter of a $PSRU$-algebra X, then*

$$A_\mu := \{x \in X : \mu(x) = \mu(1)\} \tag{33}$$

is a filter of X.

 https://doi.org/10.1016/j.heliyon.2018.e00863
2405-8440/

Proof. By Equation (33), similar to the proof of Theorem 13.

We conclude this section with the following theorem.

Theorem 14. *Let $F_1 \subseteq F_2 \subseteq \dots$ be a strictly ascending sequence of filters of a $PSRU$-algebra X and let (a_n) be a strictly decreasing sequence in $(0,1)$. If μ is a fuzzy subset of X defined by*

$$\mu(x) = \begin{cases} 0 & ; x \notin F_n \text{ for all } n \in \mathbf{Z}^+ \\ a_n & ; x \in F_n - F_{n-1} \text{ for all } n \in \mathbf{Z}^+ \end{cases} \tag{34}$$

where $F_0 = \emptyset$, then μ is a fuzzy filter of X.

Proof. Since $F_1 \subseteq F_2 \subseteq \dots$ is a strictly ascending sequence of filters of X, we have $\bigcup_{i \in \mathbf{Z}^+} F_i$ is a filter of X. By Equation (34), $\mu(1) = a_1 \geq \mu(x)$ for all $x \in X$. Let $x, y \in X$. If $y \notin \bigcup_{i \in \mathbf{Z}^+} F_i$, then $x * y \notin \bigcup_{i \in \mathbf{Z}^+} F_i$ or $x \notin \bigcup_{i \in \mathbf{Z}^+} F_i$, since $\bigcup_{i \in \mathbf{Z}^+} F_i$ is a filter. Thus $\mu(y) = 0 = \mu(x * y) \wedge \mu(x)$. Assume that $y \in F_n - F_{n-1}$ for some $n \in \mathbf{Z}^+$. By assumption, $x * y \notin F_{n-1}$ or $x \notin F_{n-1}$. Then $\mu(x * y) \leq a_n$ or $\mu(x) \leq a_n$. Therefore $\mu(y) = a_n \geq \mu(x * y) \wedge \mu(x)$ and hence μ is a fuzzy filter of X.

3. Conclusion

$PSRU$-algebra is a type of logical algebra like BE-algebras. A $PSRU$-algebra is a another generalization of BE-algebras. In this note, we have introduced the concept of filters, left ideals (right ideal, ideal) and fuzzy filters in $PSRU$-algebras. Moreover, we investigate relationships between left ideals and filters in $PSRU$-algebras. Furthermore, we characterize filters in terms of fuzzy filters. It is shown that if I be a left ideal of a $PSRU$-algebra X, then I_x is a left ideal of X for all $x \in X$. In future we will study the following topics: We will get more results in $PSRU$-algebras and application.

Declarations

Author contribution statement

Pairote Yiarayong: Conceived and designed the analysis; Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper.

Piyada Wachirawongsakorn: Conceived and designed the analysis; Wrote the paper.

 https://doi.org/10.1016/j.heliyon.2018.e00863

Funding statement

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Competing interest statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

References

[1] H.S. Kim, Y.H. Kim, On BE-algebras, Sci. Math. Jpn. 66 (1) (2007) 113–116.

[2] S.S. Ahn, K.S. So, On ideals and upper sets in BE-algebras, Sci. Math. Jpn. 68 (2) (2008) 279–285.

[3] A. Walendziak, On commutative BE-algebras, Sci. Math. Jpn. 69 (2) (2009) 281–284.

[4] S.S. Ahn, K.K. So, On generalized upper sets in BE-algebras, Bull. Korean Math. Soc. 46 (2) (2009) 281–287.

[5] K.H. Kim, A note on BE-algebras, Sci. Math. Jpn. 72 (2) (2010) 127–132.

[6] S.S. Ahn, Y.H. Kim, J.M. Ko, Filters in commutative BE-algebras, Commun. Korean Math. Soc. 27 (2) (2012) 233–242.

[7] A. Rezaei, A.B. Saeid, Commutative ideals in BE-algebras, Kyungpook Math. J. 52 (2012) 483–494.

[8] A.B. Saeid, Smarandache weak BE-algebras, Commun. Korean Math. Soc. 27 (3) (2012) 489–496.

[9] Y.B. Jun, M.S. Kang, Ideal theory of BE-algebras based on $\mathcal{N}$-structures, Hacet. J. Math. Stat. 41 (4) (2012) 435–447.

[10] A. Rezaei, A.B. Saeid, Some results in BE-algebras, An. Univ. Oradea, Fasc. Mat. XIX (1) (2012) 33–44.

[11] A.B. Saeid, A. Rezaei, R.A. Borzooei, Some types of filters in BE-algebras, Math. Comput. Sci. 7 (2013) 341–352.

[12] A. Radfar, A. Rezaei, A.B. Saeid, Hyper BE-algebras, Novi Sad J. Math. 44 (2) (2014) 137–147.

[13] R. Borzooei, A.B. Saeid, R. Ameri, A. Rezaei, Involutory BE-algebras, J. Math. Appl. 37 (2014) 13–26.

[14] Y.B. Jun, S.S. Ahn, Energetic subsets of BE-algebras, Honam Math. J. 39 (4) (2017) 569–574.

[15] M. Hamidi, A.B. Saeid, Derivations of BE-algebra form hyper BE-algebras, Kragujev. J. Math. 42 (3) (2018) 399–417.

[16] S.Z. Song, Y.B. Jun, K.J. Lee, Fuzzy ideals in BE-algebras, Bull. Malays. Math. Sci. Soc. (2)3 (1) (2010) 147–153.

[17] A. Rezaei, A.B. Saeid, On fuzzy subalgebras of BE-algebras, Afr. Math. 22 (2011) 115–127.

[18] G. Dymek, A. Walendziak, Fuzzy filters of BE-algebras, Math. Slovaca 63 (5) (2013) 935–946.

[19] S. Abdullaha, A.F. Alib, Applications of $\mathcal{N}$-structures in implicative filters of BE-algebras, J. Intell. Fuzzy Syst. 29 (2015) 517–524.

[20] S.S. Ahn, C. Kim, Rough set theory applied to fuzzy filters in BE-algebras, Commun. Korean Math. Soc. 31 (3) (2016) 451–460.

[21] R.K. Bandaru, N. Rafi, B. Davvaz, Fuzzy implicative filters of BE-algebras based on the theory of falling shadows, Facta Univ., Ser. Math. Inform. 33 (1) (2018) 31–40.

 https://doi.org/10.1016/j.heliyon.2018.e00863