On Additive Analogues of Certain Arithmetic Functions

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1. The Smarandache, Pseudo-Smarandache, resp. Smarandache-simple functions are defined as ([7], [6])

$$S(n) = \min\{m \in \mathbb{N} : \ n|m!\}, \tag{1}$$

$$Z(n) = \min\left\{m \in \mathbb{N} : \ n|\frac{m(m+1)}{2}\right\}, \tag{2}$$

$$S_p(n) = \min\{m \in \mathbb{N} : \ p^n|m!\} \text{ for fixed primes } p. \tag{3}$$

The duals of S and Z have been studied e.g. in [2], [5], [6]:

$$S_*(n) = \max\{m \in \mathbb{N} : \ m!|n\}, \tag{4}$$

$$Z_*(n) = \max\left\{m \in \mathbb{N} : \ \frac{m(m+1)}{2}|n\right\}. \tag{5}$$

We note here that the dual of the Smarandache simple function can be defined in a similar manner, namely by

$$S_{p*}(n) = \max\{m \in \mathbb{N} : \ m!|p^n\} \tag{6}$$

This dual will be studied in a separate paper (in preparation).

2. The additive analogues of the functions S and S_* are real variable functions, and have been defined and studied in paper [3]. (See also our book [6], pp. 171-174). These functions have been recently further extended, by the use of Euler's gamma function, in place of the factorial (see [1]). We note that in what follows, we could define also the additive analogues functions by the use of Euler's gamma function. However, we shall apply the more transparent notation of a factorial of a positive integer.

The additive analogues of S and S_* from (1) and (4) have been introduced in [3] as follows:

$$S(x) = \min\{m \in \mathbb{N} : \ x \leq m!\}, \quad S : (1, \infty) \to \mathbb{R}, \tag{7}$$

resp.

$$S_*(x) = \max\{m \in \mathbb{N} : \ m! \leq x\}, \quad S_* : [1, \infty) \to \mathbb{R} \tag{8}$$

Besides of properties relating to continuity, differentiability, or Riemann integrability of these functions, we have proved the following results:

Theorem 1.

$$S_*(x) \sim \frac{\log x}{\log \log x} \quad (x \to \infty) \tag{9}$$

(the same for $S(x)$).

Theorem 2. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n(S_*(n))^{\alpha}} \tag{10}$$

is convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$ (the same for $S_*(n)$ replaced by $S(n)$).

3. The additive analogues of Z and Z_* from (2), resp. (4) will be defined as

$$Z(x) = \min\left\{m \in \mathbb{N} : \ x \leq \frac{m(m+1)}{2}\right\}, \tag{11}$$

$$Z_*(x) = \max\left\{m \in \mathbb{N} : \ \frac{m(m+1)}{2} \leq x\right\} \tag{12}$$

In (11) we will assume $x \in (0, +\infty)$, while in (12) $x \in [1, +\infty)$.

The two additive variants of $S_p(n)$ of (3) will be defined as

$$P(x) = S_p(x) = \min\{m \in \mathbb{N} : \; p^x \leq m!\}; \tag{13}$$

(where in this case $p > 1$ is an arbitrary fixed real number)

$$P_*(x) = S_{p*}(x) = \max\{m \in \mathbb{N} : \; m! \leq p^x\} \tag{14}$$

From the definitions follow at once that

$$Z(x) = k \;\Leftrightarrow\; x \in \left(\frac{(k-1)k}{2}, \frac{k(k+1)}{2} \right] \text{ for } k \geq 1 \tag{15}$$

$$Z_*(x) = k \;\Leftrightarrow\; x \in \left[\frac{k(k+1)}{2}, \frac{(k+1)(k+2)}{2} \right) \tag{16}$$

For $x \geq 1$ it is immediate that

$$Z_*(x) + 1 \geq Z(x) \geq Z_*(x) \tag{17}$$

Therefore, it is sufficient to study the function $Z_*(x)$.

The following theorems are easy consequences of the given definitions:

Theorem 3.

$$Z_*(x) \sim \frac{1}{2}\sqrt{8x+1} \quad (x \to \infty) \tag{18}$$

Theorem 4.

$$\sum_{n=1}^{\infty} \frac{1}{(Z_*(n))^\alpha} \textit{ is convergent for } \alpha > 2 \tag{19}$$

and divergent for $\alpha \leq 2$. *The series* $\sum_{n=1}^{\infty} \frac{1}{n(Z_*(n))^\alpha}$ *is convergent for all* $\alpha > 0$.

Proof. By (16) one can write $\frac{k(k+1)}{2} \leq x < \frac{(k+1)(k+2)}{2}$, so $k^2 + k - 2x \leq 0$ and $k^2 + 3k + 2 - 2x > 0$. Since the solutions of these quadratic equations are $k_{1,2} = \frac{-1 \pm \sqrt{8x+1}}{2}$, resp. $k_{3,4} = \frac{-3 \pm \sqrt{8x+1}}{2}$, and remarking that $\frac{\sqrt{8x+1}-3}{2} \geq$

$1 \Leftrightarrow x \geq 3$, we obtain that the solution of the above system of inequalities is:

$$\begin{cases} k \in \left[1, \dfrac{\sqrt{1+8x}-1}{2}\right] & \text{if} \quad x \in [1,3); \\ \\ k \in \left(\dfrac{\sqrt{1+8x}-3}{2}, \dfrac{\sqrt{1+8x}-1}{2}\right] & \text{if} \quad x \in [3,+\infty) \end{cases} \tag{20}$$

So, for $x \geq 3$

$$\frac{\sqrt{1+8x}-3}{2} < Z_*(x) \leq \frac{\sqrt{1+8x}-1}{2} \tag{21}$$

implying relation (18).

Theorem 4 now follows by (18) and the known fact that the generalized harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^\theta}$ is convergent only for $\theta > 1$.

The things are slightly more complicated in the case of functions P and P_*. Here it is sufficient to consider P_*, too.

First remark that

$$P_*(x) = m \Leftrightarrow x \in \left[\frac{\log m!}{\log p}, \frac{\log(m+1)!}{\log p}\right). \tag{22}$$

The following asymptotic results have been proved in [3] (Lemma 2) (see also [6], p. 172)

$$\log m! \sim m \log m, \quad \frac{m \log\log m!}{\log m!} \sim 1, \quad \frac{\log\log m!}{\log\log(m+1)!} \sim 1 \quad (m \to \infty) \tag{23}$$

By (22) one can write

$$\frac{m \log\log m!}{\log m!} - \frac{m}{\log m!}\log\log p \leq \frac{m \log x}{\log m!} \leq \frac{m \log\log(m+1)!}{\log m!} - (\log\log p)\frac{m}{\log m!},$$

giving $\frac{m \log x}{\log m!} \to 1$ $(m \to \infty)$, and by (23) one gets $\log x \sim \log m$. This means that:

Theorem 5.

$$\log P_*(x) \sim \log x \quad (x \to \infty) \tag{24}$$

The following theorem is a consequence of (24), and a convergence theorem established in [3]:

Theorem 6. *The series* $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\log\log n}{\log P_*(n)}\right)^{\alpha}$ *is convergent for* $\alpha > 1$ *and divergent for* $\alpha \leq 1$.

Indeed, by (24) it is sufficient to study the series $\sum_{n \geq n_0}^{\infty} \frac{1}{n}\left(\frac{\log\log n}{\log n}\right)^{\alpha}$ (where $n_0 \in \mathbb{N}$ is a fixed positive integer). This series has been proved to be convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$ (see [6], p. 174).

References

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