On an additive analogue of the function S

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The function S, and its dual S_* are defined by

$$S(n) = \min\{m \in \mathbb{N}: \ n|m!\};$$

$$S_*(n) = \max\{m \in \mathbb{N}: \ m!|n\} \quad \text{(see e.g. [1])}$$

We now define the following "additive analogue", which is defined on a subset of real numbers.

Let

$$S(x) = \min\{m \in \mathbb{N}: \ x \le m!\}, \quad x \in (1, \infty) \tag{1}$$

as well as, its dual

$$S_*(x) = \max\{m \in \mathbb{N}: \ m! \le x\}, \quad x \in [1, \infty). \tag{2}$$

Clearly, $S(x) = m$ if $x \in ((m-1)!, m!]$ for $m \ge 2$ (for $m = 1$ it is not defined, as $0! = 1! = 1!$), therefore this function is defined for $x > 1$.

In the same manner, $S_*(x) = m$ if $x \in [m!, (m+1)!)$ for $m \ge 1$, i.e. $S_* : [1, \infty) \to \mathbb{N}$ (while $S : (1, \infty) \to \mathbb{N}$).

It is immediate that

$$S(x) = \begin{cases} S_*(x) + 1, & \text{if } \ x \in (k!, (k+1)!) \quad (k \ge 1) \\ S_*(x), & \text{if } \ x = (k+1)! \qquad (k \ge 1) \end{cases} \tag{3}$$

Therefore, $S_*(x) + 1 \geq S(x) \geq S_*(x)$, and it will be sufficient to study the function $S_*(x)$.

The following simple properties of S_* are immediate:

1° S_* is surjective and an increasing function

2° S_* is continuous for all $x \in [1, \infty) \setminus A$, where $A = \{k!,\ k \geq 2\}$, and since $\lim\limits_{x \nearrow k!} S_*(x) = k - 1$, $\lim\limits_{x \searrow k!} S_*(x) = k$ $(k \geq 2)$, S_* is continuous from the right in $x = k!$ $(k \geq 2)$, but it is not continuous from the left.

3° S_* is differentiable on $(1, \infty) \setminus A$, and since $\lim\limits_{x \searrow k!} \dfrac{S_*(x) - S_*(k!)}{x - k!} = 0$, it has a right-derivative in $A \cup \{1\}$.

4° S_* is Riemann integrable in $[a, b] \subset \mathbb{R}$ for all $a < b$.

a) If $[a, b] \subset [k!, (k+1)!)$ $(k \geq 1)$, then clearly

$$\int_a^b S_*(x)dx = k(b - a) \tag{4}$$

b) On the other hand, since

$$\int_{k!}^{l!} = \int_{k!}^{(k+1)!} + \int_{(k+1)!}^{(k+2)!} + \ldots + \int_{(k+l-k-1)!}^{(k+l-k)!}$$

(where $l > k$ are positive integers), and by

$$\int_{k!}^{(k+1)!} S_*(x)dx = k[(k+1)! - k!] = k^2 \cdot k!, \tag{5}$$

we get

$$\int_{k!}^{l!} S_*(x)dx = k^2 \cdot k! + (k+1)^2(k+1)! + \ldots + [k + (l-k-1)]^2[k + (l-k-1)!] \tag{6}$$

c) Now, if $a \in [k!, (k+1)!]$, $b \in [l!, (l+1)!)$, by

$$\int_a^b = \int_a^{(k+1)!} + \int_{(k+1)!}^{l!} + \int_{l!}^{k}$$

and (4), (5), (6), we get:

$$\int_a^b S_*(x)dx = k[(k+1)! - a] + (k+1)^2(k+1)! + \ldots +$$

$$+[k+1+(l-k-2)]^2[k+1+(l-k-2)!] + l(b-l!) \tag{7}$$

We now prove the following

Theorem 1.

$$S_*(x) \sim \frac{\log x}{\log\log x} \quad (x \to \infty) \tag{8}$$

Proof. We need the following

Lemma. *Let $x_n > 0$, $y_n > 0$, $\frac{x_n}{y_n} \to a > 0$ (finite) as $n \to \infty$, where $x_n, y_n \to \infty$ $(n \to \infty)$. Then*

$$\frac{\log x_n}{\log y_n} \to 1 \quad (n \to \infty). \tag{9}$$

Proof. $\log \frac{x_n}{y_n} \to \log a$, i.e. $\log x_n - \log y = \log a + \varepsilon(n)$, with $\varepsilon(n) \to 0$ $(n \to \infty)$. So

$$\frac{\log x_n}{\log y_n} - 1 = \frac{\log a}{\log y_n} + \frac{\varepsilon(n)}{\log y_n} \to 0 + 0 \cdot 0 = 0.$$

Lemma 2. a) $\frac{n \log\log n!}{\log n!} \to 1$;

b) $\frac{\log n!}{\log(n+1)!} \to 1$;

c) $\frac{\log\log n!}{\log\log(n+1)!} \to 1$ as $n \to \infty$ (10)

Proof. a) Since $n! \sim Ce^{-n}n^{n+1/2}$ (Stirling's formula), clearly $\log n! \sim n \log n$, so b) follows by $\frac{\log n}{\log(n+1)} \sim 1$ ((9), since $\frac{n}{n+1} \sim 1$). Now c) is a consequence of b) by the Lemma. Again by the Lemma, and $\log n! \sim n \log n$ we get

$$\log\log n! \sim \log(n \log n) = \log n + \log\log n \sim \log n$$

and a) follows.

Now, from the proof of (8), remark that

$$\frac{n \log\log n!}{\log(n+1)!} < \frac{S_*(x)\log\log x}{\log x} < \frac{n\log\log(n+1)!}{\log n!}$$

and the result follows by (10).

Theorem 2. *The series* $\sum_{n=1}^{\infty} \frac{1}{n(S_*(n))^\alpha}$ *is convergent for* $\alpha > 1$ *and divergent for* $a \le 1$.

Proof. By Theorem 1,

$$A\frac{\log n}{\log\log n} < S_*(n) < B\frac{\log n}{\log\log n}$$

$(A, B > 0)$ for $n \ge n_0 > 1$, therefore it will be sufficient to study the convergence of $\sum_{n\ge n_0}^{\infty} \frac{(\log\log n)^\alpha}{n(\log n)^\alpha}$.

The function $f(x) = (\log\log x)^\alpha / x(\log x)^\alpha$ has a derivative given by

$$x^2(\log x)^{2\alpha} f'(x) = (\log\log x)^{\alpha-1}(\log x)^{\alpha-1}[1 - (\log\log x)(\log x + \alpha)]$$

implying that $f'(x) < 0$ for all sufficiently large x and all $\alpha \in \mathbb{R}$. Thus f is strictly decreasing for $x \ge x_0$. By the Cauchy condensation criterion ([2]) we know that $\sum a_n \leftrightarrow \sum 2^n a_{2^n}$ (where $\leftrightarrow$ means that the two series have the same type of convergence) for (a_n) strictly decreasing, $a_n > 0$. Now, with $a_n = (\log\log n)^\alpha / n(\log n)^\alpha$ we have to study $\sum \frac{2^n(\log\log 2^n)^\alpha}{2^n(\log 2^n)^\alpha} \leftrightarrow \sum \left(\frac{\log n + a}{n+b}\right)^\alpha$, where a, b are constants ($a = \log\log 2$, $b = \log 2$). Arguing as above, (b_n) defined by $b_n = \left(\frac{\log n + a}{n+b}\right)^\alpha$ is a strictly positive, strictly decreasing sequence, so again by Cauchy's criterion

$$\sum_{n\ge m_0} b_n \leftrightarrow \sum_{n\ge m_0} \frac{2^n(\log 2^n + a)^\alpha}{(2^n+b)^\alpha} = \sum_{n\ge m_0} \frac{2^n(nb+a)^\alpha}{(2^n+b)^\alpha} = \sum_{n\ge m_0} c_n.$$

Now, $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \frac{1}{2^{\alpha-1}}$, by an easy computation, so D'Alembert's criterion proves the theorem for $\alpha \ne 1$. But for $\alpha = 1$ we get the series $\sum \frac{2^n(nb+a)}{2^n+b}$, which is clearly divergent.

References

[1] J. Sándor, *On certain generalizations of the Smarandache function*, Notes Numb. Th. Discr. Math. 5(1999), No.2, 41-51.

[2] W. Rudin, *Principles of Mathematical Analysis*, Second ed., Mc Graw-Hill Company, New York, 1964.