THE SMARANDACHE FACTORIAL SEQUENCE *

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Abstract The main purpose of this paper is using the elementary method to study the asymptotic properties of the Smarandache factorial sequence, and give an interesting asymptotic formula.

Keywords: Factorial part; Mean value; Asymptotic formula.

§1. Introduction and result

According to reference [1], for any positive integer n, let $F(n)$ denotes the inferior factorial part of n. That is, $F(n)$ denotes the largest factorial less than or equal to n. For example, $F(1)$=1,$F(2)$=2,$F(3)$=2,$F(4)$=2,$F(5)$=2,$F(6)$=3, $\cdots$. On the other hand, $f(n)$ is called the superior factorial part of n if $f(n)$ is the smallest factorial greater than or equal to n. For example, $f(1) = 1, f(2) = 2, f(3) = 3, f(4) = 3, f(5) = 3, f(6) = 3, \cdots$ are all superior factorial part. In reference [1], Professor F. Smarandache asked us to study the properties of the factorial part. About this problem, it seems that none had studied it, at least we have not seen such a paper before. In this paper, we use the elementary method to study the mean value properties of the factorial part, and give an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. Let $x \geq 1$, $\{a(n)\}$ denotes the set of $F(n)$, Then we have the asymptotic formula

$$\sum_{\substack{n=1 \\ a(n)\leq x}}^{\infty} \frac{1}{a(n)} = \frac{\ln^2 x}{2(\ln\ln x)^2} + O\left(\frac{\ln^2 x \ln\ln\ln x}{(\ln\ln x)^3}\right).$$

§2. A Lemma

To complete the proof of the theorem, we need the lemma

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Lemma. For any $x \geq 1$ and any fixed positive integer $n > 2$, let $n! \leq x < (n+1)!$, then we have the asymptotic formula

$$n = \frac{\ln x}{\ln \ln x} + O\left(\frac{\ln x \ln \ln \ln x}{(\ln \ln x)^2}\right).$$

Proof. First noting the formula $n! \leq x < (n+1)!$, we take logarithm on both sides, then we have

$$\sum_{t=1}^{n} \ln t \leq \ln x < \sum_{t=1}^{n+1} \ln t.$$

Taking $f(t) = \ln t$ in Euler's summation formula [2] we obtain:

$$n \ln n - n + O(1) \leq \ln x \leq n \ln n - n + \ln n + O(1).$$

That is,

$$\ln x = n \ln n - n + O(\ln n). \tag{1}$$

From (1), we have $n = \frac{\ln x}{\ln n} + \frac{n}{\ln n} + O(1)$ and take logarithm on both sides, we easily get the main term of $\ln n$, that is: $\ln n = \ln \ln x + O(\ln \ln \ln x)$. So we get the asymptotic formula

$$n = \frac{\ln x}{\ln n} + O\left(\frac{n \ln n}{\ln^2 n}\right) = \frac{\ln x}{\ln \ln x} + O\left(\frac{\ln x \ln \ln \ln x}{(\ln \ln x)^2}\right).$$

This completes the proof of the lemma.

§3. Proof of the theorem

In this section, we complete the proof of the theorem. Let $a(n)$ denotes the set of all the inferior factorial part, from the above lemma, we may have

$$\begin{aligned}
\sum_{\substack{n=1 \\ a(n) \leq x}}^{\infty} \frac{1}{a(n)} &= \sum_{n \leq m} \frac{nn!}{n!} = \sum_{n \leq m} n = \frac{m(m+1)}{2} \\
&= \frac{1}{2}\left(\frac{\ln x}{\ln \ln x} + O\left(\frac{\ln x \ln \ln \ln x}{(\ln \ln x)^2}\right)\right)^2 + O\left(\frac{\ln x}{\ln \ln x}\right) \\
&= \frac{\ln^2 x}{2(\ln \ln x)^2} + O\left(\frac{\ln^2 x \ln \ln \ln x}{(\ln \ln x)^3}\right).
\end{aligned}$$

And then, we can use the same method to get the same result on the superior factorial part. This completes the proof of the theorem.

References

[1]F. Smarandache, Only Problems, Not Solutions, Xiquan Publishing House, Chicago, 1993.

[2]Tom M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.