Scientia Magna
Vol. 3 (2007), No. 2, 41-43

On a conjecture involving the function $SL^*(n)$

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Received February 4, 2007

Abstract In this paper, we define a new arithmetical function $SL^*(n)$, which is related with the famous F.Smarandache LCM function $SL(n)$. Then we studied the properties of $SL^*(n)$, and solved a conjecture involving function $SL^*(n)$.

Keywords F.Smarandache LCM function, $SL^*(n)$ function, conjecture.

§1. Introduction and result

For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of all positive integers from 1 to k. For example, the first few values of $SL(n)$ are $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 8$, $SL(9) = 9$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $SL(13) = 13$, $SL(14) = 7$, $SL(15) = 5$, $SL(16) = 16$, $\cdots\cdots$. From the definition of $SL(n)$ we can easily deduce that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of n into primes powers, then

$$SL(n) = \max\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_r^{\alpha_r}\}.$$

About the elementary properties of $SL(n)$, many people had studied it, and obtained some interesting results, see references [2], [4] and [5]. For example, Murthy [2] porved that if n be a prime, then $SL(n) = S(n)$, where $S(n)$ be the F.Smarandache function. That is, $S(n) = \min\{m :\ n|m!,\ m \in N\}$. Simultaneously, Murthy [2] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n\ ? \tag{1}$$

Le Maohua [4] solved this problem completely, and proved the following conclusion:

Every positive integer n satisfying (1) can be expressed as

$$n = 12 \ \texttt{ or } \ n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where $p_1,\ p_2,\ \cdots,\ p_r,\ p$ are distinct primes and $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_r$ are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1,\ 2,\ \cdots,\ r$.

Zhongtian Lv [5] proved that for any real number $x > 1$ and fixed positive integer k, we have the asymptotic formula

$$\sum_{n\leq x} SL(n) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 2,\ 3,\ \cdots,\ k)$ are computable constants.

Now, we define another function $SL^*(n)$ as follows: $SL^*(1) = 1$, and if $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ be the factorization of n into primes powers, then

$$SL^*(n) = \min\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_r^{\alpha_r}\},$$

where $p_1 < p_2 < \ \cdots < p_r$ are primes.

About the elementary properties of function $SL^*(n)$, it seems that none has studied it yet, at least we have not seen such a paper before. It is clear that function $SL^*(n)$ is the dual function of $SL(n)$. So it has close relations with $SL(n)$. In this paper, we use the elementary method to study the following problem: For any positive integer n, whether the summation

$$\sum_{d|n}\frac{1}{SL^*(n)}, \tag{2}$$

is a positive integer? where $\sum\limits_{d|n}$ denotes the summation over all positive divisors of n.

We conjecture that there is no any positive integer $n > 1$ such that (2) is an integer. In this paper, we solved this conjecture, and proved the following:

Theorem. There is no any positive integer $n > 1$ such that (2) is an positive integer.

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem directly. For any positive integer $n > 1$, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ be the factorization of n into primes powers, from the definition of $SL^*(n)$ we know that

$$SL^*(n) = \min\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_r^{\alpha_r}\}. \tag{3}$$

Now if $SL(n) = p_k^{\alpha_k}$ (where $1 \leq k \leq r$) and n satisfy

$$\sum_{d|n}\frac{1}{SL^*(d)} = N, \quad \text{a positive integer},$$

then let $n = m\cdot p_k^{\alpha_k}$ with $(m, p_k) = 1$, note that for any $d|m$ with $d > 1$, $SL^*\left(p_k^i\cdot d\right) \mid m\cdot p_k^{\alpha_k - 1}$, where $i = 0,\ 1,\ 2,\ \cdots,\ \alpha_k$. We have

$$\begin{aligned} N &= \sum_{d|n}\frac{1}{SL^*(d)} = \sum_{i=0}^{\alpha_k}\sum_{d|m}\frac{1}{SL^*(d\cdot p_k^i)} = \sum_{i=0}^{\alpha_k}\frac{1}{SL^*(p_k^i)} + \sum_{i=0}^{\alpha_k}\sum_{\substack{d|m\\ d>1}}\frac{1}{SL^*(d\cdot p_k^i)} \\ &= 1 + \frac{1}{p_k} + \cdots + \frac{1}{p_k^{\alpha_k}} + \sum_{i=0}^{\alpha_k}\sum_{\substack{d|m\\ d>1}}\frac{1}{SL^*(d\cdot p_k^i)}, \end{aligned}$$

or

$$m \cdot p_k^{\alpha_k - 1} \cdot N = \sum_{i=0}^{\alpha_k} \sum_{\substack{d|m \\ d>1}} \frac{m \cdot p_k^{\alpha_k-1}}{SL^*(d \cdot p_k^i)} + m \cdot p_k^{\alpha_k-1} \cdot \left(1 + \frac{1}{p_k} + \cdots + \frac{1}{p_k^{\alpha_k-1}}\right) + \frac{m}{p_k}. \tag{4}$$

It is clear that for any $d|m$ with $d > 1$,

$$\sum_{i=0}^{\alpha_k} \sum_{\substack{d|m \\ d>1}} \frac{m \cdot p_k^{\alpha_k-1}}{SL^*(d \cdot p_k^i)} \quad \text{and} \quad m \cdot p_k^{\alpha_k-1} \cdot \left(1 + \frac{1}{p_k} + \cdots + \frac{1}{p_k^{\alpha_k-1}}\right),$$

are integers, but $\frac{m}{p_k}$ is not an integer. This contradicts with (4). So the theorem is true. This completes the proof of the theorem.

Open problem. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of n into primes powers, whether there exists an integer $n \geq 2$ such that $\sum_{d|n} \frac{1}{SL(n)}$ is an integer?

References

[1] Pan Chengdong and Pan Chengbiao, The elementary proof of the prime theorem, Shanghai Science and Technology Press, Shanghai, 1988.

[2] A. Murthy, Some notions on least common multiples, Smarandache Notions Journal, **12**(2001), 307-309.

[3] I. Balacenoiu and V. Seleacu, History of the Smarandache function, Smarandache Notions Journal, **10**(1999), 192-201.

[4] Le Maohua, An equation concerning the Smarandache LCM function, Smarandache Notions Journal, **14**(2004),186-188.

[5] Zhongtian Lv, On the F. Smarandache LCM function and its mean value, Scientia Magna, **3**(2007), No. 1, 22-25.

[6] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

[7] F. Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.