SMARANDACHE LOOPS

W. B. Vasantha Kandasamy

Abstract

In this paper we study the notion of Smarandache loops. We obtain some interesting results about them. The notion of Smarandache semigroups homomorphism is studied as well in this paper. Using the definition of homomorphism of Smarandache semigroups we give the classical theorem of Cayley for Smarandache semigroups. We also analyze the Smarandache loop homomorphism. We pose the problem of finding a Cayley theorem for Smarandache loops. Finally we show that all Smarandache loops $L_n(m)$ for $n > 3$, n odd, varying n and appropriate m have isomorphic subgroups.

Keywords :

Loops, Bruck Loop, Bol loop, Moufang loop, Smarandache loop, power associative loop, right or left alternative loop, Smarandache semigroup homomorphism, Smarandache loop homomorphism.

Definition [1, Bruck]:

A non-empty set L is said to form a loop if on L is defined a binary operation called product denoted by '$\bullet$' such that

1. For $a, b \in L$, we have $a \bullet b \in L$
2. There exists an element $e \in L$ such that $a \bullet e = e \bullet a = a$ for all $a \in L$ (e called identity element of L)
3. For every ordered pair $(a,b) \in L \times L$ there exists a unique pair $(x, y) \in L \times L$ such that $a \bullet x = b$ and $y \bullet a = b$.

By a loop, we mean only a finite loop and the operation '$\bullet$' need not always be associative for a loop. A loop is said to be a Moufang Loop if it satisfies any one of the following identity.

1. $(xy)(zx) = (x(yz))x$
2. $((xy)z)y = x(y(zy))$
3. $x(y(xz)) = ((xy)x)z$

for all $x, y, z \in L$.

A loop L is said to be Bruck Loop if $x(yx)z = x(y(xz))$ and $(xy)^{-1} = x^{-1}y^{-1}$ for all x, y, z ∈ L. L is a Bol Loop if $((xy)z)y = x((yz)y)$ for all x, y, z ∈ L. L is a right alternative loop if $(xy)y = x(yy)$ for all x, y ∈ L and left alternative if $(xx)y = x(xy)$. L is said to be an alternative loop f it is both a right and a left alternative loop. A loop L is said to be power associative if every element generates a subgroup. L is said to be di-associative if every 2 elements of L generates a subgroup. Let $L_n(m)$ = (e, 1, 2, 3, ... , n) be a set where n > 3, n is odd and m is a positive integer such that (m,n) = 1 and (m-1 ,n) = 1 with m < n. Define on $L_n(m)$ a binary operation '•' as follows .

1. e • i = i • e = i for all i ∈ $L_n(m)$
2. i • i = for all i ∈ $L_n(m)$
3. i • j = t where t = (mj-(m-1) i) (mod n) for all i • j ∈ $L_n(m)$ i ≠ j , i ≠ e and j ≠ e. $L_n(m)$ is a loop.

   We call this a new class of loops.

For more about loops and its properties please refer to [1] , [5] , [6] , [7] , [8] , [9] , [10], [11], [12] and [13] .

Definition 1:

The *Smarandache Loop* is defined to be a loop L such that a proper subset A of L is a subgroup (with respect to the same induced operation). That is φ ≠ A ⊂ L.

Example 1

Let L be a loop given by the following table

•	e	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_2	e	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_5	a_1	e	a_5	a_2	a_6	a_3	a_7	a_4
e	a_2	a_5	e	a_6	a_3	a_7	a_4	a_1
a_6	a_3	a_2	a_6	e	a_7	a_4	a_1	a_5
a_3	a_4	a_6	a_3	a_7	e	a_1	a_5	a_2
a_7	a_5	a_3	a_7	a_4	a_1	e	a_2	a_6
a_4	a_6	a_7	a_4	a_1	a_5	a_2	e	a_3
a_1	a_7	a_4	a_1	a_5	a_2	a_6	a_3	e

L is a Smarandache loop. For the pair (e, a2) is a subgroup of L.

Theorem 2

Every power associative loop is a Smarandache loop.

Proof

By definition of a power associative loop every element in L generates a subgroup in L. Hence the proof.

Theorem 3

Every di-associative loop is a Smarandache loop.

Proof

Since in a di-associative loop L every two elements of L generate a subgroup in L. So every di - associative loop has subgroups, hence L is a Smarandache loop.

Theorem 4

Every loop $L_n(m)$ for n >3, n an odd integer. (n,m) = (n, m-1) = 1 with m < n is a Smarandache loop.

Proof

Since $L_n(m)$ is power associative we have for every a in $L_n(m)$ is such that $a^2 = e$, {a,e} forms a subgroup for every a in $L_n(m)$. Hence the claim. Thus it is interesting to note that for every odd integer n there exists a class of Smarandache loops of order n+ 1. For a given n > 3, n odd we can have more than one integer m, m < n such that (m,n) = (m-1, n) = 1. For instance when n = 5 we have only 3 Smarandache loops given by $L_5(2)$, $L_5(3)$ and $L_5(4)$.

Definition 5

The *Smarandache Bol loop* is defined to be a loop L such that a proper subset A of L is a Bol loop (with respect to the same induced operation). That is $\phi \neq A \subset S$.

Note 1 - Similarly is defined *Smarandache Bruck loop*, *Smarandache Moufang loop* and *Smarandache right (left) alternative loop.*

Note 2- In definition 5 we insist that A should be a subloop of L and not a subgroup of L. For every subgroup is a subloop but a subloop in general is not a subgroup. Further every subgroup will certainly be a Moufang loop, Bol loop, Bruck loop and right(left) alternative loop, since in a group the operation is associative. Hence only we make the definition little rigid so that automatically we will not have all Smarandache loops to be Smarandache Bol loop, Smarandache Bruck loop, Smarandache Moufang loop and Smarandache right (left) alternative loop.

Theorem 6

Every Bol loop is a Smarandache Bol loop but every Smarandache Bol loop is not a Bol loop.

Proof

Clearly every Bol loop is a Smarandache Bol loop as every subloop of a Bol loop is a Bol loop. But a Smarandache Bol loop L is one which has a proper subset A which is a Bol loop. Hence L need not in general be a Bol loop.

Definition 7

Let S and S′ be two Smarandache semigroups. A map ϕ from S to S′ is said to be a *Smarandache semigroup homomorphism* if ϕ restricted to a subgroup $A \subset S$ and $A' \subset S'$ is a group homomorphism, that is $\phi : A \subset S \to A' \subset S'$ is a group homomorphism. The Smarandache semigroup homomorphism is an isomorphism if $\phi : A \to A'$ is one to one and onto.
Similarly, one can define *Smarandache semigroup automorphism* on S.

Theorem 8

Let N be any set finite or infinite. S(N) denote the set of all mappings of N to itself. S(N) is a semigroup under the composition of mappings. S(N), for every N, is a Smarandache semigroup.

Proof

S(N) is a semigroup under the composition of mappings. Now let A(N) denote the set of all one to one mappings of N. Clearly $\phi \neq A(N) \subset S(N)$ and A(N) is a subgroup of S(N) under the operation of composition of mappings, that is A(N) is the permutation group of degree N. Hence S(N) is a Smarandache semigroup for all N> 1.

<u>*Example 2*</u>

Let S = {set of all maps from the set (1, 2, 3, 4) to itself} and S′ = {set of all map from the set (1, 2, 3, 4, 5, 6) to itself}. Clearly S and S′ are Smarandache semigroups. For $A = S_4$ is the permutation subgroup of S and $A' = S_6$ is also the permutation subgroup of S′. Define the map $\phi : S \to S'$, $\phi(A) = B'$ = {set of all permutations of (1, 2, 3, 4) keeping the positions of 5 and 6 fixed} $\subseteq A'$. Clearly ϕ is a Smarandache semigroup homomorphism.

From the definition of Smarandache semigroup homomorphism one can give the modified form of the classical Cayley′s theorem for groups to Smarandache semigroups.

Theorem 9 (Cayley's Theorem for Smarandache semigroups)

Every Smarandache semigroup is isomorphic to a Smarandache semigroup of mappings of a set N to itself, for some appropriate set N.

Proof

Let S be a Smarandache semigroup. That is there exists a set A, which is a proper subset of S, such that A is a group (under the operations of S), that is $\phi \neq A \subset S$. Now let N be any set and S(N) denotes the set of all mappings from N to N. Clearly S(N) is a Smarandache semigroup. Now using the classical Cayley' s theorem for groups we can always have an isomorphism from A to a subgroup of S(N) for an appropriate N. Hence the theorem.

Thus by defining the notion of Smarandache semigroups one is able to extend the classical theorem of Cayley. Now we are interested to finding the appropriate formulation of Cayley's theorem for loops.

It is important to mention here that loops do not satisfy Cayley's theorem but for Smarandache loops the notion of Cayley's theorem unlike Smarandache semigroups is an open problem.

Definition 10

Let L and L′ be two Smarandache loops with A and A′ its subgroups respectively. A map ϕ from L to L′ is called *Smarandache loop homomorphism* if ϕ restricted to A is mapped to a subgroup A′ of L′, that is $\phi : A \to A'$ is a group homomorphism. The concept of *Smarandache loop homomorphism and automorphism* are defined in a similar way.

Problem 1 Prove or disprove that every Smarandache loop L is isomorphic with a Smarandache Loop L′ or equivalently

Problem 2 Can a loop L′ be constructed having a proper appropriate subset A′ of L′ such that A′ is a desired subgroup $\phi \neq A' \subset L'$?

Problem 3 Characterize all Smarandache loops which have isomorphic subgroups ?

<u>*Example 3*</u>

Let $L_5(3)$ be a Smarandache loop given by the following table

•	e	1	2	3	4	5
e	e	1	2	3	4	5
1	1	e	4	2	5	3
2	2	4	e	5	3	1
3	3	2	5	e	1	4
4	4	5	3	1	e	2
5	5	3	1	4	2	e

and $L_7(3)$ is another Smarandache loop given by the following table

•	e	1	2	3	4	5	6	7
e	e	1	2	3	4	5	6	7
1	1	e	4	7	3	6	2	5
2	2	6	e	5	1	4	7	3
3	3	4	7	e	6	2	5	1
4	4	2	5	1	e	7	3	6
5	5	7	3	6	2	e	1	4
6	6	5	1	4	7	3	e	2
7	7	3	6	2	5	1	4	e

These two loops have isomorphic subgroup, for $L_7(3)$ and $L_5(3)$ have subgroups of order 2.

Theorem 11

All Smarandache loops $L_n(m)$, where $n > 3$, n odd, for varying n and appropriate m, have isomorphic subgroups.

Proof

All Smarandache loops $L_n(m)$ have subgroups of order 2. Hence they have isomorphic subgroups.

Note- *This does not mean $L_n(m)$ cannot have subgroups of order other than two. the main concern is that all loops $L_n(m)$ have subgroups of order 2.*

References

[1] Bruck R.H., *A Survey of Binary Systems,* Springer Verlag, 1958.

[2] Castillo J., *The Smarandache Semigroup,* International Conference on Combinatorial Methods in Mathematics, II Meeting of the project 'Algebra, Geometria e Combinatoria', Faculdade de Ciencias da Universidade do Porto, Portugal, 9-11 July 1998.

[3] Padilla, Raul. *Smarandache Algebraic Structures,* Bulletin of Pure and Applied Sciences, Delhi, Vol. 17 E., No. 1, 119-121, (1998), http://www.gallup.unm.edu/~smarandache/ALG-S-TXT.TXT

[4] Smarandache, Florentin, *Special Algebraic Structures*, in "Collected Papers", Vol. III, Abaddaba, Oradea, 2000.

[5] W. B. Vasantha Kandasamy, *Zero Divisors in Loop Algebras of Power Associative and di-associative loops,* Acta Polytechnica Prace, CVUT, Praze, 16(IV, 3), 21-26, 1991.

[6] W. B. Vasantha Kandasamy, *Loops over Near-rings*, J. Shangai Univ. of Science and Technology, Vol. 15, No. 1, 41-43, 1992.

[7] W. B. Vasantha Kandasamy, *On Normal Elements in Loop Rings,* Ultra Scien. Phys. Sci., Vol. 4, 210-212, 1992.

[8] W. B. Vasantha Kandasamy, *Idempotents in Rational loop Algebras,* Ganit. J. Bangladesh Math. Soc., Vol. 12, 53-56, 1992.

[9] W. B. Vasantha Kandasamy, *Loop Boolean Algebras,* Zeszeyty Naukowe Pol. Vol.18, 95-98, 1993.

[10] W. B. Vasantha Kandasamy, *On Strictly Right Loop Rings,* J. Harbin Univ. Sci. & Tech., Vol. 18, 116-118, 1994.

[11] W. B. Vasantha Kandasamy, *Loop Algebras of Moufang Loops,* Mathematical Forum, Vol. XI, 21-25, 1997.

[12] W. B. Vasantha Kandasamy, *Group-Loop-Rings,* Acta Ciencia Indica, Vol. XXIVM, No. 1, 13-14, 1998.

[13] W. B. Vasantha Kandasamy, *On a new class of Jordan Loops and their loop rings*, Vol. 19, 71-75, 1999.

Current Address :

Dr. Mrs. Vasantha Kandasamy,
Associate Professor,
Department of Mathematics
Indian Institute of Technology, Madras
Chennai - 600 036, India.

E- mail :

vasantak@md3.vsnl.net.in