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Ruled surfaces generated by some special curves in Euclidean 3-Space

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Abstract In this paper, a family of ruled surfaces generated by some special curves using a Frenet frame of that curves in Euclidean 3-space is investigated. Some important results are obtained in the case of general helices as well as slant helices. Moreover, as an application, circular general helices, spherical general helices, Salkowski curves and circular slant helices, which illustrate the results, are provided and graphed.

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1. Introduction

The study of some classes of surfaces with special properties in $\mathbf{E}^3$ such as developable, minimal, II-minimal, and II-flat is one of the principal aims of the classical differential geometry. There are many important kinds of surfaces such as cyclic, revolution, helicoid, rotational, canal, ruled surfaces and so on. This kind of surfaces has an important role and many applications in different fields, such as Physics, Computer Aided Geometric Design and the study of design problems in spatial mechanism, etc [1,2]. There are many studies that interested with many properties of these surfaces in Euclidean space and some characterizations [3,4]. Furthermore, many geometers have studied some of the differential geometric concepts of the ruled surfaces in Minkowski space [5–8].

A *helix* (circular helix) is a geometric curve with non-vanishing constant curvature κ and non-vanishing constant torsion τ. It is a special case of a general helix [9–11]. The general helix is the curve such that the tangent makes a constant angle with a fixed straight line which is called the axis of the general helix. A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 says that: *A necessary and sufficient condition that a curve be a general helix is that the ratio*

$$\frac{\tau}{\kappa}$$

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is constant along the curve, where κ and τ denote the curvature and the torsion, respectively [12].

The *slant helix* is the curve such that the normal line makes a constant angle with a fixed straight line which is called the axis of the slant helix [13]. Izumiya and Takeuchi [13] proved that: *A curve is a slant helix if and only if the geodesic curvature of the principal image of the principal normal indicatrix*

$$\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'$$

is constant along the curve.

The determining of the position vector of some different curves according to the intrinsic equations $\kappa = \kappa(s)$ and $\tau = \tau(s)$ (where κ and τ are the curvature and torsion of the curve) is considered as a one of important subjects. Recently, the parametric representation of general helices and slant helices as an important special curves in Euclidean space $\mathbf{E}^3$ are deduced by Ali [14,15].

Ruled surfaces are surfaces which are generated by moving a straight line continuously in the space and are one of the most important topics of differential geometry [16]. In this paper, we investigate a family of ruled surfaces generated by some special curves in Euclidean 3-space $\mathbf{E}^3$ and we obtained some important results in the case of general helices and slant helices as a base curve of this ruled surfaces.

2. Basic concepts

Let $\mathbf{E}^3$ be a 3-dimensional Euclidean space provided with the metric given by

$$\langle,\rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbf{E}^3$. Let $\mathbf{c} = \mathbf{c}(s) : I \subset \mathbb{R} \to \mathbf{E}^3$ is an arbitrary curve of arc-length parameter s. Let $\{\mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s)\}$ be the moving Frenet frame along $\mathbf{c}$, then the Frenet formulae is given by [12]

$$\begin{bmatrix}\mathbf{e}_1'(s)\\ \mathbf{e}_2'(s)\\ \mathbf{e}_3'(s)\end{bmatrix} = \begin{bmatrix}0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{bmatrix}\begin{bmatrix}\mathbf{e}_1(s)\\ \mathbf{e}_2(s)\\ \mathbf{e}_3(s)\end{bmatrix}. \tag{1}$$

where the functions $\kappa(s)$ and $\tau(s)$ are the curvature and the torsion of the curve $\mathbf{c}$, respectively.

A ruled surface is generated by a one-parameter family of straight lines and it possesses a parametric representation

$$\Psi(s,v) = \mathbf{c}(s) + v\, X(s), \tag{2}$$

where $\mathbf{c}(s)$ is called the base curve and $X(s)$ is the unit represents a space curve which representing the direction of straight line [17].

If there exists a common perpendicular to two constructive rulings in the ruled surface, then the foot of the common perpendicular on the main rulings is called a central point. The locus of the central point is called striction curve [4]. The parametrization of the striction curve on the ruled surface (2) is given by

$$\tilde{\mathbf{c}}(s) = \mathbf{c}(s) - \frac{\langle \mathbf{c}'(s), X'(s)\rangle}{\|X'(s)\|^2} X(s). \tag{3}$$

If $\|X'(s)\| = 0$, then the ruled surface does not have any striction curve. In this case the ruled surface is cylindrical. Thus the base curve can take as a striction curve.

The standard unit normal vector field $\mathbf{U}$ on a surface Ψ can be defined by:

$$\mathbf{U} = \frac{\Psi_s \wedge \Psi_v}{\|\Psi_s \wedge \Psi_v\|}, \tag{4}$$

where $\Psi_s = \frac{\partial\Psi(s,v)}{\partial s}$ and $\Psi_v = \frac{\partial\Psi(s,v)}{\partial v}$. The first I and second II fundamental forms of the surface Ψ are given by, respectively

$$I = E\,ds^2 + 2F\,ds\,dv + G\,dv^2, \tag{5}$$

$$II = e\,ds^2 + 2f\,ds\,dv + g\,dv^2, \tag{6}$$

where

$$E = \langle \Psi_s, \Psi_s\rangle, \quad F = \langle \Psi_s, \Psi_v\rangle, \quad G = \langle \Psi_v, \Psi_v\rangle, \quad e = \langle \Psi_{ss}, U\rangle, \quad f = \langle \Psi_{sv}, U\rangle, \quad g = \langle \Psi_{vv}, U\rangle.$$

On the other hand, the Gaussian curvature K, the mean curvature H and the distribution parameter λ are given by, respectively [18]

$$K = \frac{eg - f^2}{EG - F^2}, \tag{7}$$

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}, \tag{8}$$

$$\lambda = \frac{\det(\mathbf{c}', X, X')}{\|X'\|^2}. \tag{9}$$

From Brioschi's formula in a Euclidean 3-space, we are able to compute K_{II} of a surface by replacing the components of the first fundamental form E, F and G by the components of the second fundamental form e, f and g respectively. Consequently, the second Gaussian curvature K_{II} of a surface is defined by [19]:

$$\mathbf{K}_{II} = \frac{1}{(eg-f^2)^2}\left\{\begin{vmatrix} -\frac{1}{2}e_{vv} + f_{sv} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_v\\ f_v - \frac{1}{2}g_s & e & f\\ \frac{1}{2}g_v & f & g\end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_v & \frac{1}{2}g_s\\ \frac{1}{2}e_v & e & f\\ \frac{1}{2}g_s & f & g\end{vmatrix}\right\}. \tag{10}$$

Having in mind the usual technique for computing the second mean curvature H_{II} by using the normal variation of the area functional for the surfaces in $\mathbf{E}^3$ one gets [20]:

$$H_{II} = H + \frac{1}{4}\Delta_{II}\ln(K)$$

where H and K denote the mean, respectively Gaussian curvatures of surface and Δ_{II} is the Laplacian for functions computed with respect to the second fundamental form II as metric. The second mean curvature H_{II} can be equivalently expressed as

$$H_{II} = H + \frac{1}{2\sqrt{\det(II)}}\sum_{i,j}\frac{\partial}{\partial u^i}\left[\sqrt{\det(II)}h^{ij}\frac{\partial}{\partial u^j}(\ln\sqrt{K})\right], \tag{11}$$

where (h_{ij}) denotes the associated matrix with its inverse (h^{ij}), the indices i, j belong to $\{1, 2\}$ and the parameters u^1, u^2 are s, v respectively.

The geodesic curvature, the normal curvature and the geodesic torsion which associate the curve $\mathbf{c}(s)$ on the surface Ψ can be computed as follows:

$$\kappa_g = \langle \mathbf{U}\wedge\mathbf{e}_1, \mathbf{e}_1'\rangle, \quad \kappa_n = \langle \mathbf{c}'', \mathbf{U}\rangle, \quad \tau_g = \langle \mathbf{U}\wedge\mathbf{U}', \mathbf{e}_1'\rangle. \tag{12}$$

Now, we can write the following important definitions:

Definition 2.1 [21]. For a curve $\mathbf{c}(s)$ lying on a surface, the following are well-known:

(1) $\mathbf{c}(s)$ is a geodesic curve if and only if the geodesic curvature κ_g vanishes.
(2) $\mathbf{c}(s)$ is an asymptotic line if and only if the normal curvature κ_n vanishes.
(3) $\mathbf{c}(s)$ is a principal line if and only if the geodesic torsion τ_g vanishes.

Definition 2.2 [22].

(1) A regular surface is flat (developable) if and only if its Gaussian curvature vanishes identically.
(2) A regular surface for which the mean curvature vanishes identically is called a minimal surface.
(3) A surface is called II-flat if the second Gaussian curvature vanishes identically.
(4) A surface is called II-minimal if the second mean curvature vanishes identically.

It is worth noting that the ruled surfaces (2) is developable if and only if the distribution parameter λ of the surface Ψ vanishes identically [23].

3. Some characterizations of ruled surfaces in general form

For our study, we consider the following generated surface using a curve $c(s)$ as a base curve:

$$S:\Psi(s,v)=\mathbf{c}(s)+v\ X(s),\quad X'(s)\neq 0,\quad v\in\mathbb{R}, \tag{13}$$

where

$$X(s)=\sum_{i=1}^{3}x_i\ \mathbf{e}_i(s), \tag{14}$$

is a unit vector with fixed components, i.e., $x_1^2+x_2^2+x_3^2=1$. The natural frame $\{\Psi_s,\Psi_v\}$ of (13)is given by:

$$\begin{cases}\Psi_s=(1-vx_2\kappa)\mathbf{e}_1+v(x_1\kappa-x_3\tau)\mathbf{e}_2+(vx_2\tau)\mathbf{e}_3,\\ \Psi_v=x_1\mathbf{e}_1+x_2\mathbf{e}_2+x_3\mathbf{e}_3.\end{cases} \tag{15}$$

From the above equation, we can obtain the components of the first and second fundamental forms of Ψ, respectively, as the following:

$$\begin{cases}E=(1-vx_2\kappa)^2+v^2(x_1\kappa-x_3\tau)^2+(vx_2\tau)^2,\\ F=x_1,\\ G=1,\end{cases} \tag{16}$$

$$\begin{cases}e=\frac{1}{A}\Big[\big[x_2(\tau\kappa'-\kappa\tau')-x_1\left(x_2^2+x_3^2\right)\tau^3-x_3(1-3x_1^2)\kappa\tau^2-x_1(1-3x_3^2)\kappa^2\tau\\ \qquad -x_3\left(x_1^2+x_2^2\right)\kappa^3\big]v^2+\left[2x_2(x_3\kappa+x_1\tau)\kappa-x_1x_3\kappa'+(x_2+x_3)^2\tau'\right]v-x_3\kappa\Big],\\ f=\frac{1}{A}\left[\left(x_2^2+x_3^2\right)\tau-x_1x_3\kappa\right],\\ g=0,\end{cases} \tag{17}$$

where

$$A^2=\left[\left(x_1^2+x_2^2\right)\kappa^2-2x_1x_3\kappa\tau+\left(x_2^2+x_3^2\right)\tau^2\right]v^2-2x_2\kappa v+x_2^2+x_3^2.$$

Making use of the data described above, the Gaussian curvature K, the mean curvature H and the distribution parameter λ are given respectively, by

$$K=-\frac{f^2}{E-F^2}, \tag{18}$$

$$H=\frac{e-2Ff}{2(E-F^2)}, \tag{19}$$

$$\lambda=\frac{\tau\left(x_2^2+x_3^2\right)-\kappa x_1x_3}{x_2^2(\kappa^2+\tau^2)+(x_1\kappa-x_3\tau)^2}. \tag{20}$$

Also, from (10) the second Gaussian curvature of Ψ is given as follows:

$$K_{II}=\frac{f(e_{vv}-2f_{sv})-(e_v-2f_s)f_v}{2f^3}=\frac{1}{2f}\frac{\partial}{\partial v}\left(\frac{e_v-2f_s}{f}\right). \tag{21}$$

From (18)–(21) and (11), at the point $(s,0)$, we have the following results respectively

$$K=-\left[\left(\frac{x_1x_3}{x_2^2+x_3^2}\right)\kappa-\tau\right]^2, \tag{22}$$

$$H=-\frac{x_3(1-2x_1^2)\kappa+2x_1\left(x_2^2+x_3^2\right)\tau}{2\left(x_2^2+x_3^2\right)^{3/2}}, \tag{23}$$

$$\begin{aligned}H_{II}=&\frac{1}{2\sqrt{x_2^2+x_3^2}\left[x_1x_3\kappa-\left(x_2^2+x_3^2\right)^2\tau\right]^2}\\ &\times\left[2x_1\left(x_2^2+x_3^2\right)\left[3x_1x_3\kappa-\left(x_2^2+x_3^2\right)\tau\right]\tau^2+\kappa^2\right.\\ &\times\left(x_3\left[x_2^2\left(x_2^2+x_3^2\right)^2-x_1^4\left(x_2^2-2x_3^2\right)\right]\kappa\right.\\ &\quad\left.+x_2\left(x_2^2+x_3^2\right)\left(\frac{\tau}{\kappa}\right)'\right)+\left(x_2^2+x_3^2\right)\left(2x_1\left[x_1^2\left(x_2^2-3x_3^2\right)\right.\right.\\ &\quad\left.+x_2^2\left(x_2+x_3^2\right)\right]\kappa^2-x_2\left(x_2^2+x_3^2\right)\kappa'\big)\tau\\ &\quad\left.+x_1x_2x_3\left(x_2^2+x_3^2\right)\kappa\kappa'\right],\end{aligned} \tag{24}$$

$$\begin{aligned}K_{II}=&\frac{1}{2\left(x_2^2+x_3^2\right)^{3/2}\left[x_1x_3\kappa-\left(x_2^2+x_3^2\right)^2\tau\right]^2}\\ &\times\left[x_1^3x_3\left(x_2^2+x_3^2\right)\kappa[x_2\kappa'-4x_3\kappa\tau]\right.\\ &+x_1^4x_3\left(x_3^2-x_2^2\right)\kappa^3-x_1\left(x_2^2+x_3^2\right)^2\left(2x_2^2\tau^3+2x_3^2\tau(\tau^2-\kappa^2)\right.\\ &\left.-x_2x_3\kappa\kappa'\right)-x_1^2\left(x_2^2+x_3^2\right)\kappa\left(x_3\left[\left(2x_2^2+x_3^2\right)\kappa^2-5\left(x_2^2+x_3^2\right)\tau^2\right]\right.\\ &\left.+x_2\left(x_2^2+x_3^2\right)\tau'\right)-\left(x_2^2+x_3^2\right)^2\kappa\left(x_3\left[x_3^2\tau^2+x_2^2(\kappa^2+\tau^2)\right]\right.\\ &\left.\left.+x_2\left(x_2^2+x_3^2\right)\tau'\right)\right].\end{aligned} \tag{25}$$

Furthermore, we will use (12) to get the geodesic curvature, the normal curvature and the geodesic torsion which associate the curve $c(s)$ on the surface Ψ as the following forms, respectively:

$$\kappa_g=\frac{\kappa}{A}\left[x_2-\left[\left(x_1^2+x_2^2\right)\kappa-x_1x_3\tau\right]v\right], \tag{26}$$

$$\kappa_n=\frac{\kappa}{A}\left[x_3-x_2(x_3\kappa+x_1\tau)v\right], \tag{27}$$

$$\begin{aligned}\tau_g=&\frac{1}{A^2}\left[x_2x_3\kappa^2-v\left(x_3(x_1^2+2x_2^2)\kappa^3+x_1\left(x_2^2-2x_3^2\right)\kappa^2\tau\right.\right.\\ &+x_3\left(x_2^2+x_3^2\right)\kappa\tau^2+x_2\left(x_2^2+x_3^2\right)\kappa^2\left(\frac{\tau}{\kappa}\right)'\\ &\left.+x_2\left[\left(x_2^2+x_3^2\right)\tau-x_1x_3\kappa\right]\kappa'\kappa'\right)+x_2v^2\big((x_3\kappa+x_1\tau)\\ &\left.\times\kappa\left[x_2^2(\kappa^2+\tau^2)+(x_1\kappa-x_3\tau)^2\right]+x_2\kappa^3\left(\frac{\tau}{\kappa}\right)'\big)\right].\end{aligned} \tag{28}$$

At the point (*s*; 0), above equations take the simple form:

$$\kappa_g = \frac{x_2\kappa}{\sqrt{x_2^2+x_3^2}}, \quad \kappa_n = \frac{x_3\kappa}{\sqrt{x_2^2+x_3^2}}, \quad \tau_g = \frac{x_2x_3\kappa^2}{x_2^2+x_3^2}. \tag{29}$$

Then we have the following properties:

$$\kappa_g\kappa_n = \tau_g, \quad \kappa_g^2+\kappa_n^2 = \kappa^2. \tag{30}$$

From (14) and (1), it is easy to see that the parametrization of the striction curve on the ruled surface (13) is defined by:

$$\tilde{\mathbf{c}}(s) = \mathbf{c}(s) + \frac{x_2\kappa}{\|X'(s)\|^2}X(s). \tag{31}$$

From the above study, one can formulate the following corollaries:

Corollary 3.1. *At the point $(s, 0)$, the ruled surface (13) is a flat surface if and only if the curve $c(s)$ is a general helix with* $\frac{\tau(s)}{\kappa(s)} = \frac{x_1x_3}{x_2^2+x_3^2}$.

Corollary 3.2. *At the point $(s, 0)$, the ruled surface (13) is a minimal surface if and only if the curve $c(s)$ is a general helix with* $\frac{\tau(s)}{\kappa(s)} = \frac{x_3(2x_1^2-1)}{2x_1\left(x_2^2+x_3^2\right)}$.

In the following we will compute the Gaussian curvature K, the mean curvature H, the second Gaussian curvature K_{II}, the second mean curvature H_{II} as well as the geodesic curvature κ_g, the normal curvature κ_n, and the geodesic torsion τ_g in a special cases, respectively.

Case 3.1. At $x_1 = 0$, the ruled surface (13) has the following:

$$\begin{aligned} &K = -\tau^2, \quad H = -\frac{x_3\kappa}{2},\\ &K_{II} = -\frac{\kappa}{2\tau^2}\left[x_3\left(x_2^2\kappa^2+\tau^2\right)+x_2\tau'\right],\\ &H_{II} = \frac{1}{2\tau^2}\left[2x_2(2\tau\kappa'-\kappa\tau')-x_3\kappa(2x_2^2\kappa^2+3\tau^2)\right],\\ &\kappa_g = x_2\kappa, \quad \kappa_n = x_3\kappa, \quad \tau_g = x_2x_3\kappa^2. \end{aligned} \tag{32}$$

Corollary 3.3. *At the point $(s, 0)$, the ruled surface (13) with $x_1 = 0$ is:*

1. *Flat surface if the base curve is a plane curve.*
2. *Minimal surface if the base curve is straight line.*
3. *II-minimal surface if the base curve has the following characterization*

$$2x_2(2\tau\kappa'-\kappa\tau')-x_3\kappa\left(2x_2^2\kappa^2+3\tau^2\right)=0.$$

4. *II-Flat surface if the base curve has the following characterization*

$$\tau' = \frac{x_3}{x_2}\left(x_2^2\kappa^2+\tau^2\right).$$

Case 3.2. At $x_2 = 0$, the ruled surface (13) has the following:

$$\begin{aligned} &K = -\left(\frac{x_1\kappa}{x_3}-\tau\right)^2, \quad H = K_{II} = \frac{1}{2}\left(\frac{x_1^2}{x_3^2}-1\right)\kappa - \left(\frac{x_1}{x_3}\right)\tau,\\ &H_{II} = -\left[\frac{1}{2}\left(\frac{x_1^2}{x_3^2}+3\right)\kappa+\frac{x_1\tau}{x_3}\right], \quad \kappa_g = 0, \quad \kappa_n = \kappa, \quad \tau_g = 0. \end{aligned} \tag{33}$$

Corollary 3.4. *At the point $(s, 0)$, in the ruled surface (13) with $x_2 = 0$ the following are satisfied:*

1. *The ruled surface is a flat surface if the base curve is general helix with $\tau(s) = \left(\frac{x_1}{x_3}\right)\kappa(s)$.*
2. *The ruled surface is II-minimal surface if the base curve is general helix with $\tau(s) = -\frac{1}{2}\left(\frac{x_1}{x_3}+\frac{3x_3}{x_1}\right)\kappa(s)$.*

Corollary 3.5. *At the point $(s, 0)$, in the ruled surface (13) with $x_2 = 0$ the following statements are equivalent:*

1. *The ruled surface is a minimal surface.*
2. *The ruled surface is II-flat surface.*
3. *The base curve is general helix with $\tau(s) = \frac{1}{2}\left(\frac{x_1}{x_3}-\frac{x_3}{x_1}\right)\kappa(s)$.*

Case 3.3. At $x_3 = 0$, the ruled surface (13) has the following:

$$\begin{aligned} &K=-\tau^2, \quad H=-\left(\frac{x_1\tau}{x_2}\right),\\ &K_{II}=-\left(\frac{2x_1x_2\tau^3+\kappa\tau'}{2x_2^2\tau^2}\right), \quad H_{II}=\frac{x_2(2\tau\kappa'-\kappa\tau')-x_1(2\kappa^2+x_2^2\tau^2)\tau}{x_2^3\tau^2},\\ &\kappa_g=\kappa, \quad \kappa_n=0, \quad \tau_g=0. \end{aligned} \tag{34}$$

Corollary 3.6. *At the point $(s, 0)$, the ruled surface (13) with $x_3 = 0$ is:*

1. *Flat surface if the base curve is a plane curve.*
2. *Minimal surface if the base curve is a plane curve.*
3. *II-flat surface if the base curve has the intrinsic equations*

$$\kappa = \kappa(s) \quad \text{and} \quad \tau = \frac{1}{\sqrt{c_1 - 4x_1x_2\int\frac{ds}{\kappa(s)}}},$$

where c_1 is an arbitrary constant.

4. *II-minimal surface if the base curve has the intrinsic equations*

$$\kappa = \kappa(s) \quad \text{and} \quad \tau = \frac{\kappa^2(s)\, e^{-\frac{2x_1}{x_2}\int\kappa(s)ds}}{\sqrt{c_2+2x_1x_2\int\kappa^3(s)\, e^{-\frac{4x_1}{x_2}\int\kappa(s)ds}ds}},$$

where c_2 is an arbitrary constant.

Case 3.4. At $x_1 = x_2 = 0$ and $x_3 = 1$, the ruled surface (13) at the point (*s*,0), has the following:

$$\begin{aligned} &K = -\tau^2, \quad H_{II} = 3H = 3K_{II} = -\left(\frac{3\kappa}{2}\right),\\ &\kappa_g = 0, \quad \kappa_n = \kappa, \quad \tau_g = 0. \end{aligned} \tag{35}$$

Corollary 3.7. *At the point $(s, 0)$, the ruled surface (13) with $x_1 = x_2 = 0$ and $x_3 = 1$ is flat if the base curve is a plane curve.*

Corollary 3.8. *At the point $(s, 0)$, the ruled surface (13) with $x_1 = x_2 = 0$ and $x_3 = 1$, the following statements are equivalent:*

(1) *The ruled surface is minimal surface.*
(2) *The ruled surface is II-minimal surface.*
(3) *The ruled surface is II-flat surface.*
(4) *The base curve is a straight line.*

Case 3.5. At $x_1 = x_3 = 0$ and $x_2 = 1$, the ruled surface (13) has the following:

$$K = -\tau^2, \quad H = 0,$$
$$K_{II} = -\left(\frac{\kappa\tau'}{2\tau^2}\right), \quad H_{II} = \frac{2\tau\kappa' - \kappa\tau'}{2\tau^2}, \tag{36}$$
$$\kappa_g = \kappa, \quad \kappa_n = 0, \quad \tau_g = 0.$$

Corollary 3.9. *At the point* $(s, 0)$*, the ruled surface* (13) *with* $x_1 = x_3 = 0$ *and* $x_2 = 1$ *is flat if the base curve is a plane curve.*

Corollary 3.10. *At the point* $(s, 0)$*, the ruled surface* (13) *with* $x_1 = x_3 = 0$ *and* $x_2 = 1$ *is minimal surface.*

Corollary 3.11. *At the point* $(s, 0)$*, the ruled surface* (13) *with* $x_1 = x_3 = 0$ *and* $x_2 = 1$ *is II-flat surface if the base curve has a constant torsion.*

Corollary 3.12. *At the point* $(s, 0)$*, the ruled surface* (13) *with* $x_1 = x_3 = 0$ *and* $x_2 = 1$ *is II-minimal surface if the intrinsic equations of the base curve are*:

$$\kappa = \kappa(s) \quad \text{and} \quad \tau = c_3\ \kappa^2(s),$$

where c_3 *is an arbitrary constant.*

Case 3.6. At $x_2 = x_3 = 0$ and $x_1 = 1$, the ruled surface (13) has the following:

$$K = 0, \quad H = -\left(\frac{\tau}{2v\kappa}\right),$$
$$\kappa_g = -\kappa, \quad \kappa_n = 0, \quad \tau_g = 0. \tag{37}$$

Corollary 3.13. *The ruled surface* (13) *with* $x_2 = x_3 = 0$ *and* $x_1 = 1$ *is a flat (developable) surface.*

Corollary 3.14. *The ruled surface* (13) *with* $x_2 = x_3 = 0$ *and* $x_1 = 1$ *is minimal if the base curve is a plane curve.*

It is worth noting that the second mean curvature and second Gaussian curvature are defined only on the non-developable surfaces.

Remark 3.15. On the ruled surface (13) with $x_2 = x_3 = 0$ and $x_1 = 1$ we have $\Psi_s \wedge \Psi_v = -v\kappa\mathbf{e}_3$. The normal vector on this surface is $\mathbf{U} = \mathbf{e}_3$. While, at the point $(s, 0)$, the normal vector is not defined because $\Psi_s \wedge \Psi_v = 0$. Therefore, all curvatures K, H, H_{II}, K_{II}, κ_g, κ_n and τ_g are not defined at the point $(s, 0)$.

4. Ruled surfaces generated by some special curves

In this section, we consider ruled surfaces generated by some important special curves such as general helices and slant helices.

4.1. Ruled surfaces generated by general helices

Theorem 4.1. [14]:*The position vector* $\mathbf{c}$ *of general helix is expressed in the natural representation form as follows*:

$$\mathbf{c}(s) = \sqrt{1-n^2} \times \int \left(\cos\left[\sqrt{1+m^2}\int \kappa(s)ds\right], \sin\left[\sqrt{1+m^2}\int \kappa(s)ds\right], m\right) ds, \tag{38}$$

where $m = \frac{n}{\sqrt{1-n^2}}$, $n = \cos[\phi]$ *and* ϕ *is the angle between the fixed straight line* $\mathbf{e}_3$ *(axis of a general helix) and the tangent vector of the curve* $\mathbf{c}$.

From the above theorem we have

$$\begin{cases} \mathbf{e}_1(s) = \sqrt{1-n^2}\left(\cos\left[\sqrt{1+m^2}\int\kappa(s)ds\right], \sin\left[\sqrt{1+m^2}\int\kappa(s)ds\right], m\right), \\ \mathbf{e}_2(s) = \left(-\sin\left[\sqrt{1+m^2}\int\kappa(s)ds\right], \cos\left[\sqrt{1+m^2}\int\kappa(s)ds\right], 0\right), \\ \mathbf{e}_3(s) = \left(-n\cos\left[\sqrt{1+m^2}\int\kappa(s)ds\right], -n\sin\left[\sqrt{1+m^2}\int\kappa(s)ds\right], \sqrt{1-n^2}\right). \end{cases} \tag{39}$$

Then the position vector $\Psi(s, v) = (\Psi_1, \Psi_2, \Psi_3)$ of the ruled surfaces (13) generated by the general helix takes the following form:

$$\begin{cases} \Psi_1 = \frac{1}{\sqrt{1+m^2}}\left[\int\cos[\Theta]ds + v\left[(x_1 - mx_3)\cos[\Theta] - \sqrt{1+m^2}x_2\sin[\Theta]\right]\right], \\ \Psi_2 = \frac{1}{\sqrt{1+m^2}}\left[\int\sin[\Theta]ds + v\left[(x_1 - mx_3)\sin[\Theta] + \sqrt{1+m^2}x_2\cos[\Theta]\right]\right], \\ \Psi_3 = \frac{1}{\sqrt{1+m^2}}[ms + v(mx_1 + x_3)], \end{cases} \tag{40}$$

where $\Theta = \sqrt{1+m^2}\int\kappa(s)\,ds$.

Here, we introduced the position vector of ruled surfaces generated by some special cases of general helices:

Case (1) In this case we take a circular helix (the curvature and torsion are constants) with the intrinsic equations

$$\kappa(s) = \kappa \quad \text{and} \quad \tau(s) = m\ \kappa.$$

Then the components of the position vector of the ruled surfaces generated by circular helix are:

$$\begin{cases} \Psi_1 = \frac{1}{\kappa(1+m^2)}\left[[1-(1+m^2)x_2\kappa v]\sin\left[\sqrt{1+m^2}\kappa s\right] + \sqrt{1+m^2}(x_1 - mx_3)\kappa v\cos\left[\sqrt{1+m^2}\kappa s\right]\right], \\ \Psi_2 = \frac{1}{\kappa(1+m^2)}\left[[(1+m^2)x_2\kappa v - 1]\cos\left[\sqrt{1+m^2}\kappa s\right] + \sqrt{1+m^2}(x_1 - mx_3)\kappa v\sin\left[\sqrt{1+m^2}\kappa s\right]\right], \\ \Psi_3 = \frac{1}{\sqrt{1+m^2}}[ms + v(mx_1 + x_3)]. \end{cases} \tag{41}$$

Case (2) In this case we take a general helix with the intrinsic equations given by

$$\kappa(s) = \frac{a}{s} \quad \text{and} \quad \tau(s) = \frac{m\ a}{s},$$

where a is an arbitrary constant. Then the components of the position vector of the ruled surface take the form:

$$\begin{cases}\Psi_1=\frac{1}{\sqrt{1+m^2}}\left[\left(\frac{s}{1+b^2}+(x_1-mx_3)v\right)\cos[\Theta]+\left(\frac{as}{1+b^2}-x_2v\right)\sin[\Theta]\right],\\ \Psi_2=\frac{1}{\sqrt{1+m^2}}\left[\left(\frac{s}{1+b^2}+(x_1-mx_3)v\right)\sin[\Theta]-\left(\frac{as}{1+b^2}-x_2v\right)\cos[\Theta]\right],\\ \Psi_3=\frac{1}{\sqrt{1+m^2}}[ms+v(mx_1+x_3)],\end{cases} \tag{42}$$

where $b=a\sqrt{1+m^2}$ and $\Theta = b\,\mathrm{Log}[s]$.

Case (3) In this case we take a spherical general helix with the intrinsic equations are [24,25]:

$$\kappa(s)=\frac{a}{\sqrt{1-m^2s^2}}\quad\text{and}\quad\tau(s)=\frac{m\ a}{\sqrt{1-m^2s^2}},$$

where a is an arbitrary constant. The components of the position vector of the ruled surface can be written as:

$$\begin{cases}\Psi_1=\frac{n}{m}\left[(x_1-mx_3)v-\frac{m^2s}{a^2(1+m^2)-m^2}\right]\cos[\Theta]+\left[\frac{a\sqrt{1-m^2s^2}}{a^2(1+m^2)-m^2}-x_2v\right]\sin[\Theta],\\ \Psi_2=\frac{n}{m}\left[(x_1-mx_3)v-\frac{m^2s}{a^2(1+m^2)-m^2}\right]\sin[\Theta]-\left[\frac{a\sqrt{1-m^2s^2}}{a^2(1+m^2)-m^2}-x_2v\right]\cos[\Theta],\\ \Psi_3=\frac{n}{m}[ms+v(mx_1+x_3)],\end{cases} \tag{43}$$

where $\Theta=\frac{a}{n}\sin^{-1}[ms]$.

4.2. Ruled surfaces generated by slant helices

Theorem 4.2. [15]:*The position vector* $\mathbf{c}=(c_1(s),c_2(s),c_3(s))$ *of a slant helix is computed in the natural representation form*:

$$\begin{cases}c_1(s)=\frac{n}{m}\int\left[\int\kappa(s)\cos\left[\frac{1}{n}\arcsin\left(m\int\kappa(s)ds\right)\right]ds\right]ds,\\ c_2(s)=\frac{n}{m}\int\left[\int\kappa(s)\sin\left[\frac{1}{n}\arcsin\left(m\int\kappa(s)ds\right)\right]ds\right]ds,\\ c_3(s)=n\int\left[\int\kappa(s)ds\right]ds,\end{cases} \tag{44}$$

where $m=\frac{n}{\sqrt{1-n^2}}$, $n=\cos[\phi]$ *and* ϕ *is the angle between the fixed straight line (axis of a slant helix) and the principal normal vector of the curve* $\mathbf{c}$.

From the above theorem we can compute the tangent $\mathbf{e}_1=(e_{11}(s),e_{12}(s),e_{13}(s))$, the normal $\mathbf{e}_2=(e_{21}(s),e_{22}(s),e_{23}(s))$ and the binormal $\mathbf{e}_3=(e_{31}(s),e_{32}(s),e_{33}(s))$ as the following:

$$\begin{cases}e_{11}(s)=\frac{n}{m}\int\kappa(s)\cos\left[\frac{1}{n}\arcsin(m\int\kappa(s)ds)\right]ds,\\ e_{12}(s)=\frac{n}{m}\int\kappa(s)\sin\left[\frac{1}{n}\arcsin(m\int\kappa(s)ds)\right]ds,\\ e_{13}(s)=n[\int\kappa(s)ds],\end{cases} \tag{45}$$

$$\begin{cases}e_{21}(s)=\frac{n}{m}\cos\left[\frac{1}{n}\arcsin(m\int\kappa(s)ds)\right],\\ e_{22}(s)=\frac{n}{m}\sin\left[\frac{1}{n}\arcsin(m\int\kappa(s)ds)\right],\\ e_{23}(s)=n,\end{cases} \tag{46}$$

$$\begin{cases}e_{31}(s)=\frac{n^2}{m}\left[\int\kappa(s)\sin\left[\frac{1}{n}\arcsin(m\int\kappa(s)ds)\right]ds\right.\\ \qquad\left.-\left(\int\kappa(s)ds\right)\sin\left[\frac{1}{n}\arcsin\left(m\int\kappa(s)ds\right)\right]\right],\\ e_{32}(s)=\frac{n^2}{m}\left[\left(\int\kappa(s)ds\right)\cos\left[\frac{1}{n}\arcsin\left(m\int\kappa(s)ds\right)\right]\right.\\ \qquad\left.-\int\kappa(s)\cos\left[\frac{1}{n}\arcsin(m\int\kappa(s)ds)\right]ds\right]\\ e_{23}(s)=\frac{n}{m}\sqrt{1-m^2\left(\int\kappa(s)ds\right)^2}.\end{cases} \tag{47}$$

Then the position vector $\Psi(s,v)=(\Psi_1,\Psi_2,\Psi_3)$ of the ruled surface (13) generated by the slant helix takes the following form:

$$\begin{cases}\Psi_1=\frac{n}{m}\left[\int[\int\kappa(s)\cos[\Phi]ds]ds+v\left((x_2-x_1\Theta-mx_3\sqrt{1-\Theta^2})\cos[\Phi]\right.\right.\\ \qquad\left.\left.+\sqrt{1+m^2}\left(x_1\sqrt{1-\Theta^2}-x_3\Theta\right)\sin[\Phi]\right)\right],\\ \Psi_2=\frac{n}{m}\left[\int[\int\kappa(s)\sin[\Phi]ds]ds+v\left((x_2-mx_1\Theta-mx_3\sqrt{1-\Theta^2})\sin[\Phi]\right.\right.\\ \qquad\left.\left.-\sqrt{1+m^2}\left(x_1\sqrt{1-\Theta^2}-x_3\Theta\right)\cos[\Phi]\right)\right],\\ \Psi_3=\frac{n}{m}\left[\int\Theta ds+v(x_1\Theta+mx_2+x_3\sqrt{1-\Theta^2})\right],\end{cases} \tag{48}$$

where $\Theta=m\int\kappa(s)ds$ and $\Phi=\frac{1}{n}\arcsin[\Theta]$.

In what follows, we presented the position vector of some important slant helices such as Salkowski, antiSalkowski, spherical slant helix.

Case (1) In this case, we take a Salkowski curve [26,27] whose intrinsic equations are:

$$\kappa=1,\quad\tau=\frac{m\ s}{\sqrt{1-m^2\ s^2}}. \tag{49}$$

The explicit parametric representation of such curve can be written as follows:

$$\begin{cases}\psi_1(t)=\frac{n}{4m}\left[\frac{n-1}{2n+1}\cos[(2n+1)t]+\frac{n+1}{2n-1}\cos[(2n-1)t]-2\cos[t]\right],\\ \psi_2(t)=\frac{n}{4m}\left[\frac{n-1}{2n+1}\sin[(2n+1)t]-\frac{n+1}{2n-1}\sin[(2n-1)t]-2\sin[t]\right],\\ \psi_3(t)=-\frac{n}{4m^2}\cos[2nt],\end{cases} \tag{50}$$

where $t=\frac{1}{n}\arcsin(ms)$.

Case (2) In this case, we take an anti-Salkowski curve [26,27] with its intrinsic equations are:

$$\kappa=\frac{m\ s}{\sqrt{1-m^2\ s^2}},\quad\tau=1. \tag{51}$$

This curve has the following explicit parametric representation:

$$\begin{cases}\psi_1(t)=\frac{n}{4m}\left[\frac{n-1}{2n+1}\sin[(2n+1)t]+\frac{n+1}{2n-1}\sin[(2n-1)t]-2n\sin[t]\right],\\ \psi_2(t)=\frac{n}{4m}\left[\frac{1-n}{1+2n}\cos[(1+2n)t]-\frac{1+n}{1-2n}\cos[(1-2n)t]+2n\cos[t]\right],\\ \psi_3(t)=\frac{n}{4m^2}(2nt-\sin[2nt]),\end{cases} \tag{52}$$

where $t=\frac{1}{n}\arcsin(m\theta)$ and $\theta=\frac{\sqrt{1-m^2s^2}}{m}$.

Case (3) In this case, we take a circular slant helix [24] which has intrinsic equations are:

$$\kappa=\frac{\mu}{m}\cos[\mu\ s],\quad\tau=\frac{\mu}{m}\sin[\mu\ s], \tag{53}$$

The natural representation of such curve is in the following form:

$$\begin{cases}\psi_1(s)=-\frac{m^2}{n\ \mu}\left[(1+n^2)\cos[\mu\ s]\cos[\frac{\mu\ s}{n}]+2n\sin[\mu\ s]\sin[\frac{\mu\ s}{n}]\right],\\ \psi_2(s)=-\frac{m^2}{n\ \mu}\left[(1+n^2)\cos[\mu\ s]\sin[\frac{\mu\ s}{n}]-2n\sin[\mu\ s]\cos[\frac{\mu\ s}{n}]\right],\\ \psi_3(s)=-\frac{n}{m\ \mu}\ \cos[\mu\ s].\end{cases} \tag{54}$$

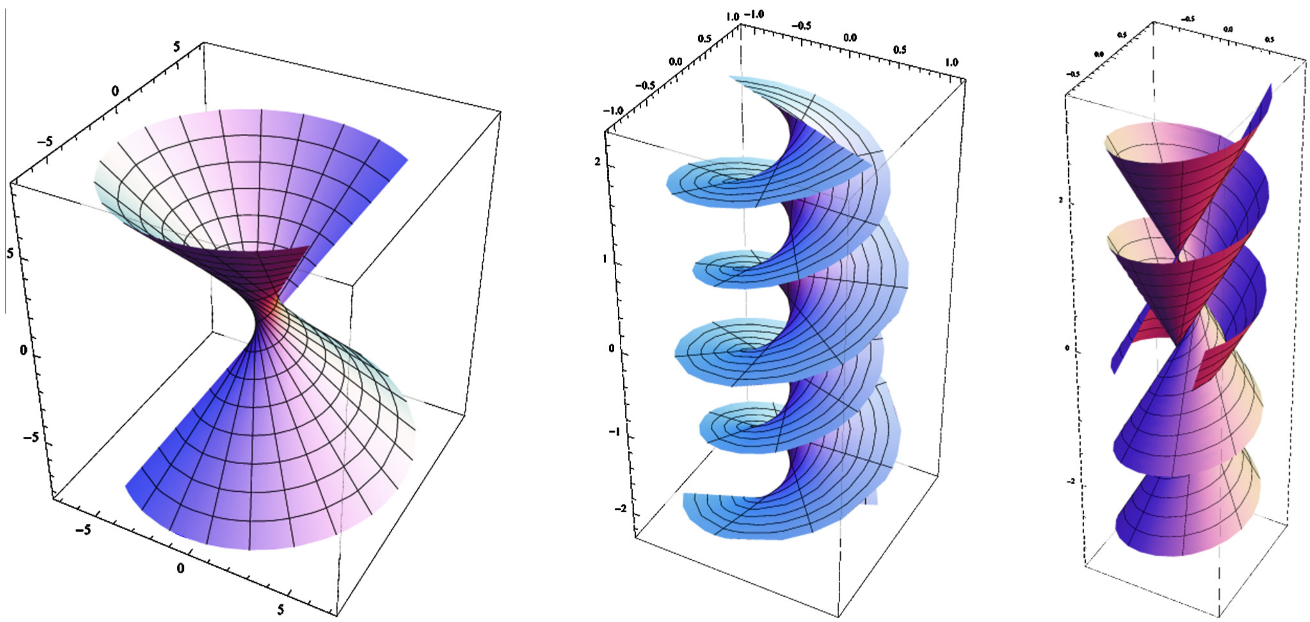

Figure 1 Some ruled surfaces generated by circular helices.

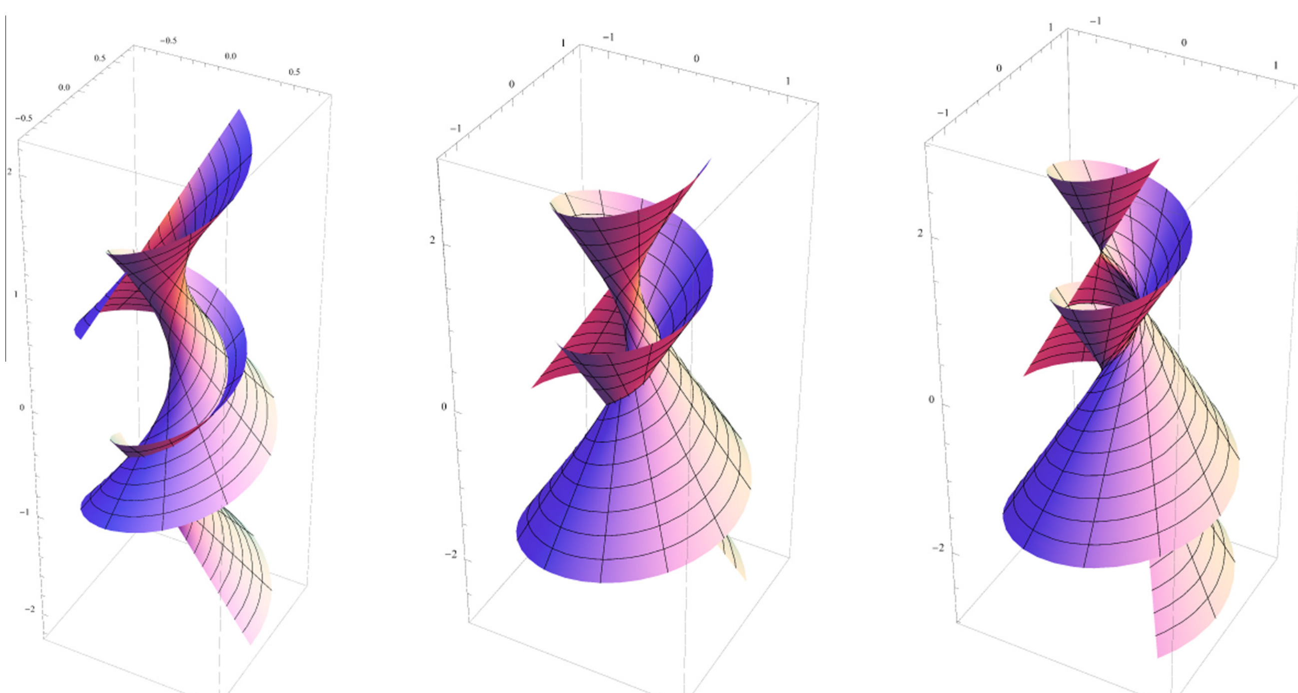

Figure 2 Some ruled surfaces generated by circular helices.

The above curve is a geodesic of the tangent developable of a general helix [13].

In the following remarks, we will illustrate in what values the graph plotted.

Remark 4.3. It is worth noting that:

(1) The ruled surfaces generated by circular general helices are illustrated by graph in Figs. 1 and 2.
(2) The ruled surfaces generated by spherical general helices are illustrated by graph in Figs. 3 and 4.
(3) The ruled surfaces generated by Salkowski curves are illustrated by graph in Figs. 5 and 6.
(4) The ruled surfaces generated by circular slant helix is illustrated by graph in Figs. 7 and 8.

Remark 4.4. We will take the symbols (L, M and R) that means (Left, Middle and Right) in the graph, respectively.

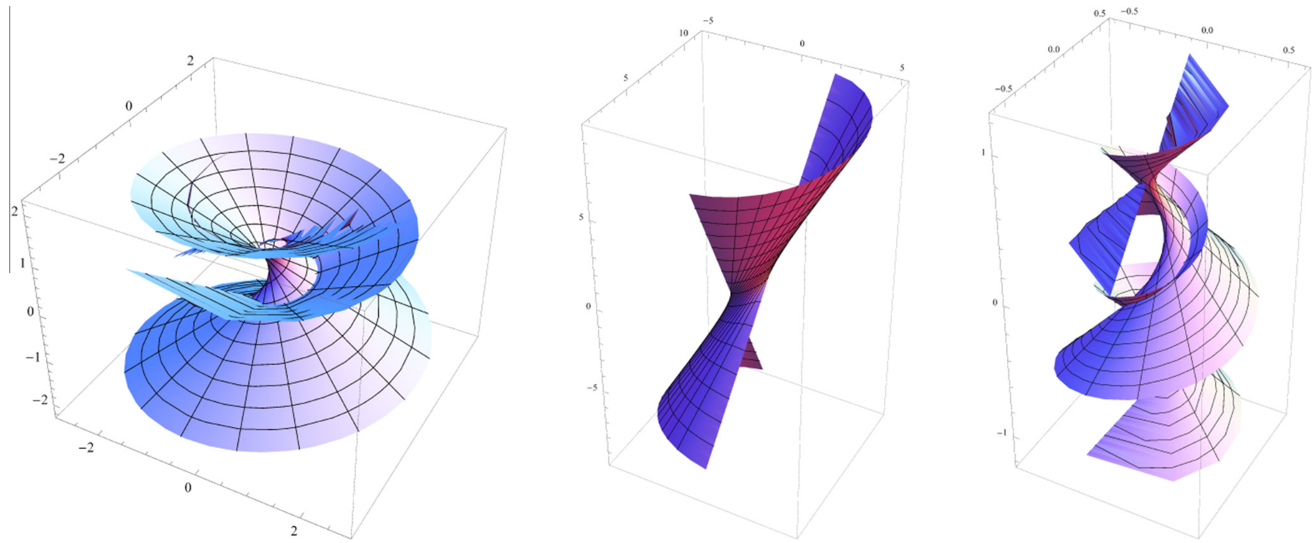

Figure 3 Some ruled surfaces generated by spherical general helices.

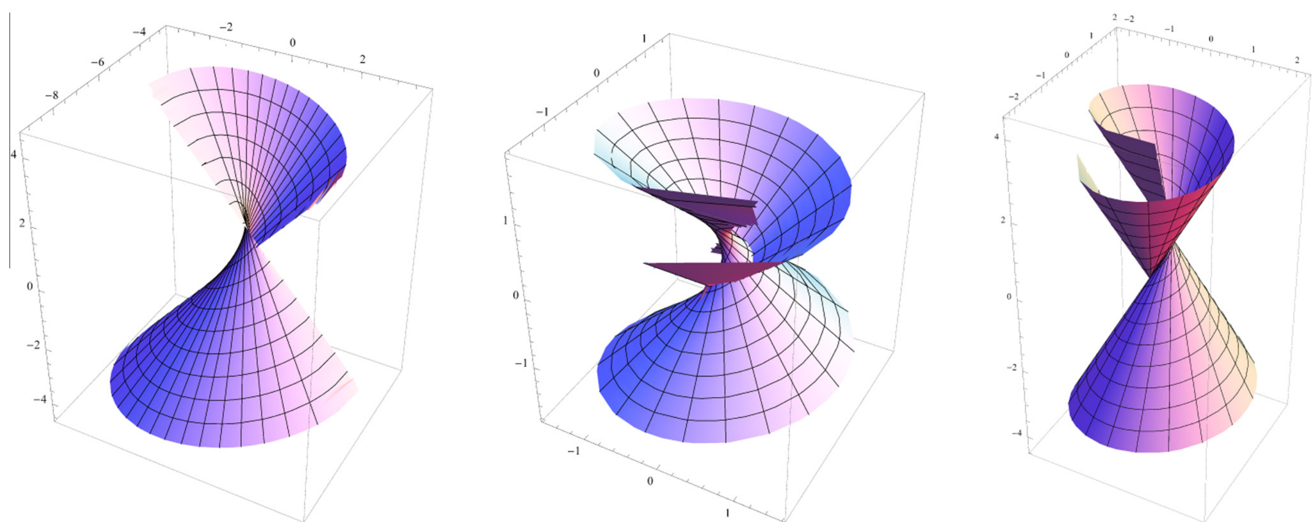

Figure 4 Some ruled surfaces generated by spherical general helices.

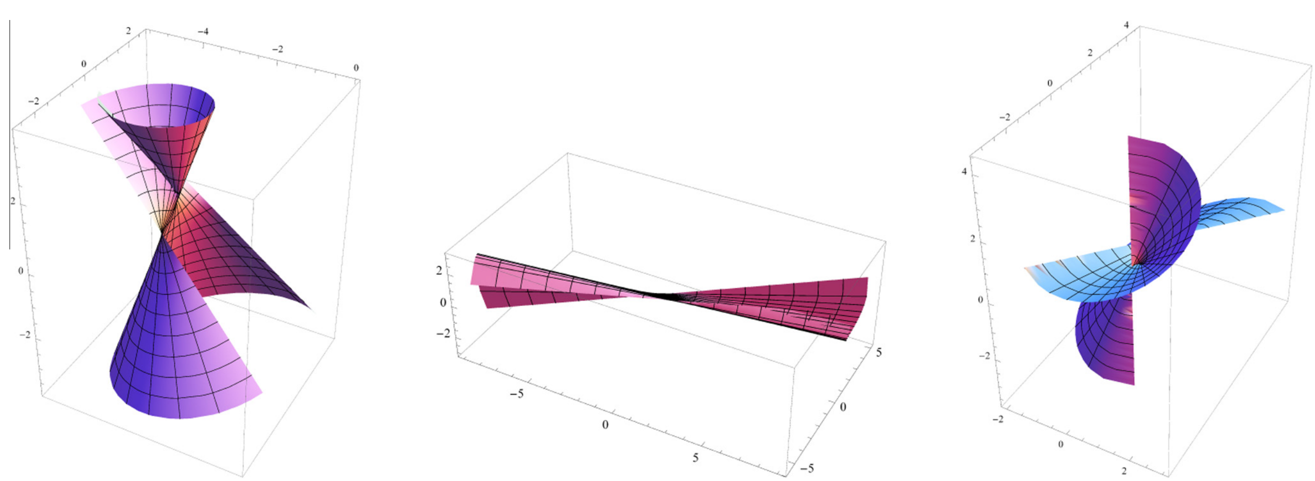

Figure 5 Some ruled surfaces generated by Salkowski curves.

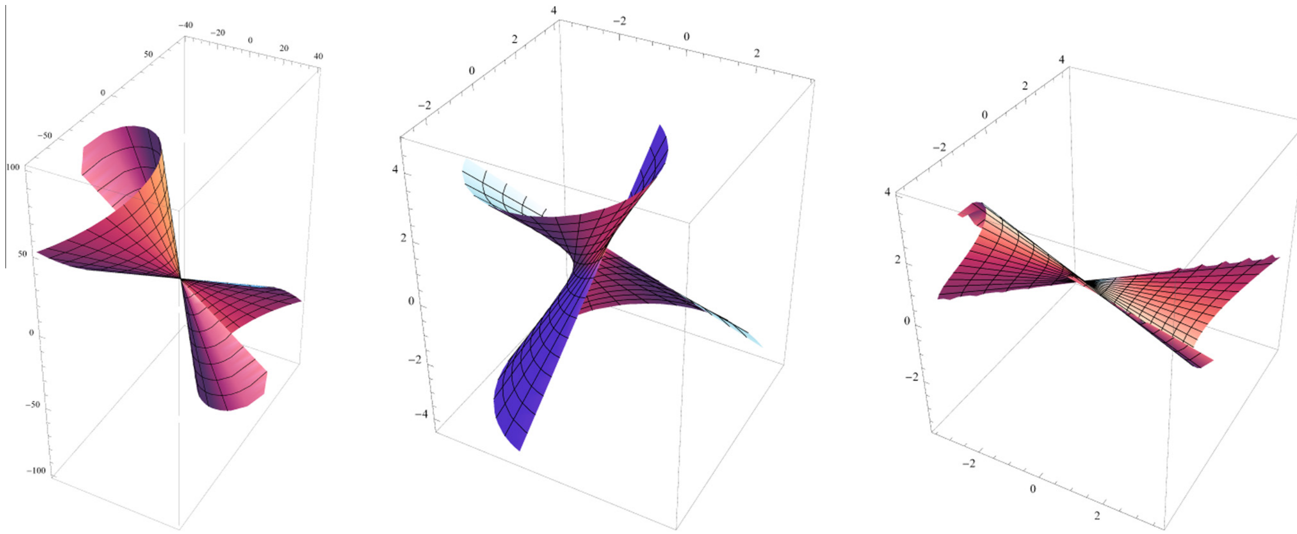

Figure 6 Some ruled surfaces generated by Salkowski curves.

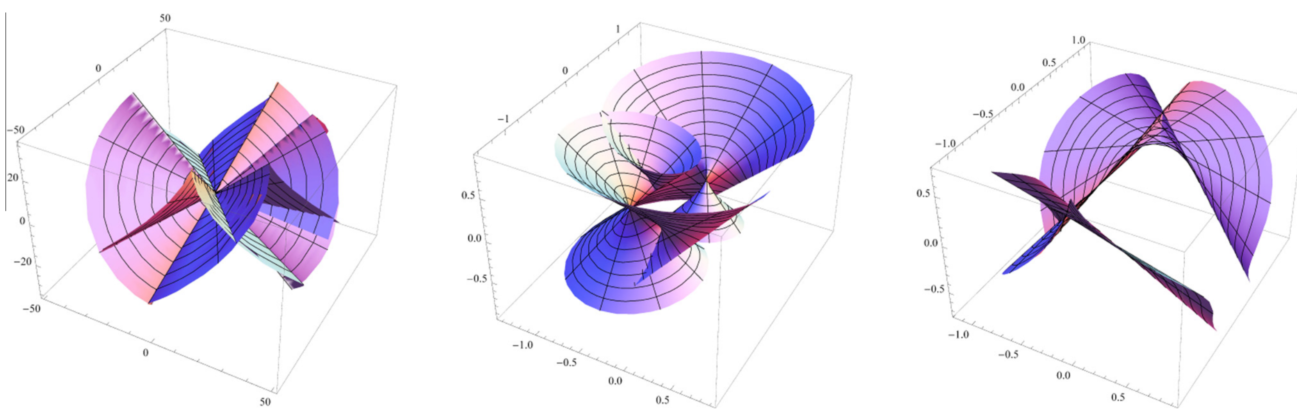

Figure 7 Some ruled surfaces generated by circular slant helices.

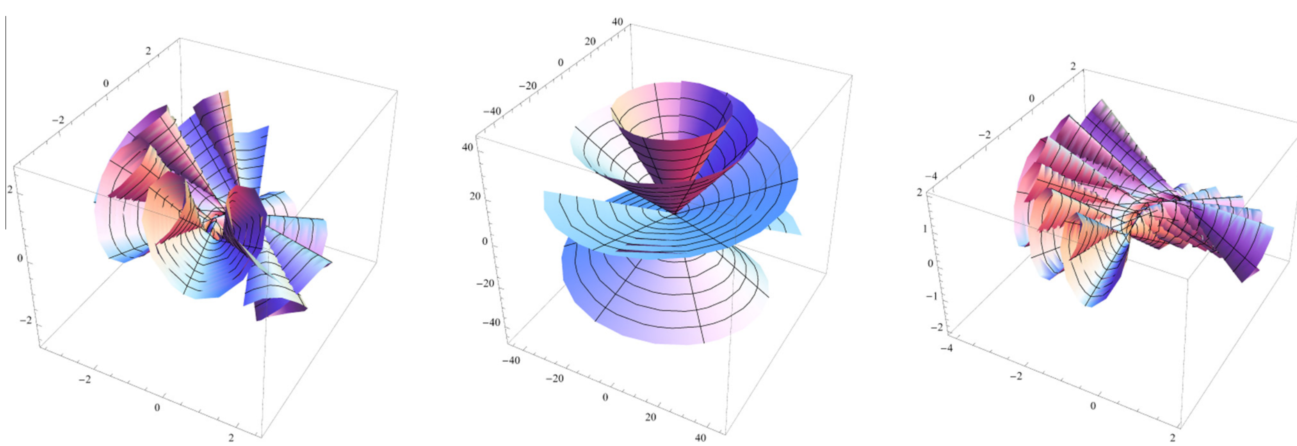

Figure 8 Some ruled surfaces generated by circular slant helices.

Fig. 1: L: ($\kappa = m = 1, x_1 = x_2 = 0, x_3 = 1$), M: ($\kappa = 1$, $m = 3, x_1 = x_3 = 0, x_2 = 1$), R: ($\kappa = \sqrt{3}, m = 2, x_2 = x_3 = 0, x_1 = 1$).

Fig. 2: L: ($\kappa = \frac{3}{2}, m = 2, x_1 = x_2 = x_3 = \frac{1}{\sqrt{3}}$), M: ($\kappa = 2, m = 1, x_1 = x_2 = \frac{1}{2}, x_3 = \frac{1}{\sqrt{2}}$), R: ($\kappa = 2, m = \frac{2}{3}, x_1 = \frac{1}{\sqrt{2}}, x_2 = \frac{1}{\sqrt{6}}, x_3 = \frac{1}{\sqrt{3}}$).

Fig. 3: L: $(a=2, m=1, x_1=0, x_2=x_3=\frac{1}{\sqrt{2}})$, M: $(a=\frac{1}{2}, m=\frac{2}{3}, x_2=0, x_1=\frac{2}{\sqrt{5}}, x_3=\frac{1}{\sqrt{5}})$, R: $(a=3, m=1, x_3=0, x_1=\frac{\sqrt{3}}{2}, x_2=\frac{1}{2})$.
Fig. 4: L: $(a=\frac{3}{4}, m=1, x_1=\frac{1}{\sqrt{3}}, x_2=\frac{\sqrt{5}}{2\sqrt{2}}, x_3=\frac{1}{2})$, M: $(a=m=2, x_1=x_3=\frac{1}{2}, x_2=\frac{1}{\sqrt{2}})$, R: $(a=\frac{3}{2}, m=\frac{3}{4}, x_1=\frac{1}{\sqrt{2}}, x_2=\frac{1}{\sqrt{6}}, x_3=\frac{1}{\sqrt{3}})$.
Fig. 5: L: $(m=\frac{1}{2}, x_1=0, x_2=x_3=\frac{1}{\sqrt{2}})$, M: $(m=3, x_2=0, x_1=\frac{1}{2}, x_3=\frac{\sqrt{3}}{2})$, R: $(m=1, x_3=0, x_1=\frac{2}{3}, x_2=\frac{\sqrt{5}}{3})$.
Fig. 6: L: $(m=1, x_1=x_2=x_3=\frac{1}{\sqrt{3}})$, M: $(m=\frac{1}{5}, x_1=x_2=\frac{1}{2}, x_3=\frac{1}{\sqrt{2}})$, R: $(m=2, x_1=\frac{1}{\sqrt{2}}, x_2=\frac{1}{\sqrt{6}}, x_3=\frac{1}{\sqrt{3}})$.
Fig. 7: L: $(\mu=5, m=1, x_1=x_2=0, x_3=1)$, M: $(\mu=3, m=1, x_1=x_3=0, x_2=1)$, R: $(\mu=3, m=1, x_2=x_3=0, x_1=1)$.
Fig. 8: L: $(\mu=10, m=2, x_1=x_2=x_3=\frac{1}{\sqrt{3}})$, M: $(\mu=\frac{1}{2}, m=\frac{1}{10}, x_1=x_2=\frac{1}{2}, x_3=\frac{1}{\sqrt{2}})$, R: $(\mu=12, m=3, x_1=\frac{1}{\sqrt{2}}, x_2=\frac{1}{\sqrt{6}}, x_3=\frac{1}{\sqrt{3}})$.

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