Scientia Magna
Vol. 3 (2007), No. 4, 41-44

On the mean value of the Smarandache LCM function

Lin Cheng

Department of Mathematics, Northwest University
Xi'an, Shaanxi, P.R.China

Abstract For any positive integer n, the F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1, 2, \cdots, k]$, where $[1, 2, \cdots, k]$ denotes the least common multiple of $1, 2, \cdots, k$, and let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be the factorization of n into prime powers, then $\overline{\Omega}(n) = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_s p_s$. The main purpose of this paper is using the elementary methods to study the mean value properties of $\overline{\Omega}(n) SL(n)$, and give a sharper asymptotic formula for it.

Keywords F.Smarandache LCM function, $\overline{\Omega}(n)$ function, mean value, asymptotic formula.

§1. Introduction and Results

For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of $1,\ 2,\ \cdots,\ k$. For example, the first few values of $SL(n)$ are $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 4$, $SL(9) = 6$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $SL(13) = 13$, $SL(14) = 7$, $SL(15) = 5$, $\cdots$. About the elementary properties of $SL(n)$, some authors had studied it, and obtained some interesting results, see reference [2] and [3]. For example, Lv Zhongtian [4] studied the mean value properties of $SL(n)$, and proved that for any fixed positive integer k and any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \le x} SL(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{b_i \cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where b_i $(i = 2, 3, \cdots, k)$ are computable constants.

On the other hand, Chen Jianbin [5] studied the value distribution properties of $SL(n)$, and proved that for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \le x} (SL(n) - P(n))^2 = \frac{2}{5} \cdot \zeta\left(\frac{5}{2}\right) \cdot \frac{x^{\frac{5}{2}}}{\ln x} + O\left(\frac{x^{\frac{5}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, and $P(n)$ denotes the largest prime divisor of n.

Now we define a new arithmetical function $\overline{\Omega}(n)$ as follows: $\overline{\Omega}(1) = 0$; for $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be the factorization of n into prime powers, then $\overline{\Omega}(n) = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_s p_s$.

Obviously, for any positive integers m and n, we have $\overline{\Omega}(mn) = \overline{\Omega}(m) + \overline{\Omega}(n)$. That is, $\overline{\Omega}(n)$ is the additive function. The main purpose of this paper is using the elementary methods to study the mean value properties of $\overline{\Omega}(n)SL(n)$, and give a sharper asymptotic formula for it. That is, we shall prove the following conclusion:

Theorem. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\le x}\overline{\Omega}(n)SL(n) = \sum_{i=1}^{k}\frac{d_i x^3}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where d_i $(i = 1,\ 2,\ \cdots,\ k)$ are computable constants.

§2. Proof of the theorem

In this section, we shall use the elementary methods to complete the proof of the theorem.

In fact, for any positive integer $n > 1$, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ be the factorization of n into prime powers, then from [2] we know that

$$SL(n) = \max\{p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_s^{\alpha_s}\}, \tag{1}$$

and we easily to know that

$$\overline{\Omega}(n) = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_s p_s. \tag{2}$$

Now we consider the summation

$$\sum_{n\le x}\overline{\Omega}(n)SL(n). \tag{3}$$

We separate all integer n in the interval $[1, x]$ into four subsets A, B, C and D as follows:

A: $p \ge \sqrt{n}$ and $n = m\cdot p$;

B: $n^{\frac{1}{3}} < p_1 < p_2 \le \sqrt{n}$ and $n = m\cdot p_1\cdot p_2$, where p_i $(i = 1, 2)$ are primes;

C: $n^{\frac{1}{3}} < p \le \sqrt{n}$ and $n = m\cdot p^2$;

D: otherwise.

It is clear that if $n \in A$, then from (1) we know that $SL(n) = p$, and from (2) we know that $\overline{\Omega}(n) = \overline{\Omega}(m) + p$. Therefore, by the Abel's summation formula (See Theorem 4.2 of [6]) and the Prime Theorem (See Theorem 3.2 of [7]):

$$\pi(x) = \sum_{i=1}^{k}\frac{a_i\cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where a_i $(i = 1, 2, \ldots, k)$ are computable constants and $a_1 = 1$.

We have

$$\begin{aligned}
\sum_{n\in A}\overline{\Omega}(n)SL(n) &= \sum_{\substack{mp\le x\\ m<p}}\left(\overline{\Omega}(m)+p\right)p=\sum_{\substack{mp\le x\\ m<p}}p^2+\sum_{\substack{mp\le x\\ m<p}}\left(\overline{\Omega}(m)p\right)\\
&= \sum_{m\le\sqrt{x}}\sum_{m<p\le\frac{x}{m}}p^2+O(x^2)\\
&= \sum_{m\le\sqrt{x}}\left[\pi\left(\frac{x}{m}\right)\frac{x^2}{m^2}-\pi(m)m^2-2\int_m^{\frac{x}{m}}\pi(t)tdt\right]+O(x^2)\\
&= \frac{1}{3}\zeta(3)\frac{x^3}{\ln x}+\sum_{i=2}^{k}\frac{b_ix^3}{\ln^i x}+O\left(\frac{x^3}{\ln^{k+1}x}\right), \qquad (4)
\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function, and b_i $(i=2,3,\cdots,k)$ are computable constants.

Similarly, if $n\in B$, then we have $SL(n)=p_2$ and $\overline{\Omega}(n)=\overline{\Omega}(m)+p_1+p_2$. So

$$\begin{aligned}
\sum_{n\in B}\overline{\Omega}(n)SL(n) &= \sum_{\substack{mp_1p_2\le x\\ m<p_1<p_2}}\left(\overline{\Omega}(m)+p_1+p_2\right)p_2=\sum_{\substack{mp_1p_2\le x\\ m<p_1<p_2}}p_2^2+O\left(\sum_{\substack{mp_1p_2\le x\\ m<p_1<p_2}}p_1p_2\right)\\
&= \sum_{m\le x^{\frac{1}{3}}}\sum_{m<p_1\le\sqrt{\frac{x}{m}}}\sum_{p_1<p_2\le\frac{x}{p_1m}}p_2^2+O(x^2)\\
&= \sum_{m\le x^{\frac{1}{3}}}\sum_{m<p_1\le\sqrt{\frac{x}{m}}}\left[\pi\left(\frac{x}{p_1m}\right)\frac{x^2}{p_1^2m^2}-\pi(p_1)p_1^2-2\int_{p_1}^{\frac{x}{p_1m}}\pi(t)tdt\right]\\
&\quad +O(x^2)\\
&= \sum_{i=1}^{k}\frac{c_ix^3}{\ln^i x}+O\left(\frac{x^3}{\ln^{k+1}x}\right), \qquad (5)
\end{aligned}$$

where c_i $(i=1,2,\cdots,k)$ are computable constants.

Now we estimate the error terms in set C. Using the same method of proving (4), we have $SL(n)=p^2$ and $\overline{\Omega}(n)=\overline{\Omega}(m)+2p$, so

$$\begin{aligned}
\sum_{n\in C}\overline{\Omega}(n)SL(n) &= \sum_{\substack{mp^2\le x\\ m<p}}\left(\overline{\Omega}(m)+2p\right)p^2=2\sum_{\substack{mp^2\le x\\ m<p}}p^3+\sum_{\substack{mp^2\le x\\ m<p}}\left(\overline{\Omega}(m)p^2\right)\\
&= 2\sum_{m\le x^{\frac{1}{3}}}\sum_{m<p\le\sqrt{\frac{x}{m}}}p^3+O(x^{\frac{3}{2}})\\
&= O(x^2). \qquad (6)
\end{aligned}$$

Finally, we estimate the error terms in set D. For any integer $n\in D$, if $SL(n)=p$ then

$p \leq \sqrt{n}$; if $SL(n) = p^2$, then $p \leq n^{\frac{1}{3}}$; or $SL(n) = p^\alpha, \alpha \geq 3$. So we have

$$\begin{aligned}\sum_{n\in D}\overline{\Omega}(n)SL(n) &\ll \sum_{\substack{mp\leq x\\ p\leq m}}(\overline{\Omega}(m)+p)p+\sum_{\substack{mp^2\leq x\\ p\leq m}}(\overline{\Omega}(m)+2p)p\\ &+ \sum_{\substack{mp^\alpha\leq x\\ p\leq x^{\frac{1}{3}},\alpha\geq 3}}(\overline{\Omega}(m)\alpha p)p^\alpha \ll \frac{x^2}{\ln x}. \qquad (7)\end{aligned}$$

Combining (4), (5), (6) and (7) we may immediately obtain the asymptotic formula

$$\begin{aligned}\sum_{n\leq x}\overline{\Omega}(n)SL(n) &= \sum_{n\in A}\overline{\Omega}(n)SL(n)+\sum_{n\in B}\overline{\Omega}(n)SL(n)\\ &+\sum_{n\in C}\overline{\Omega}(n)SL(n)+\sum_{n\in D}\overline{\Omega}(n)SL(n)\\ &= \sum_{i=1}^{k}\frac{d_i x^3}{\ln^i x}+O\left(\frac{x^3}{\ln^{k+1}x}\right),\end{aligned}$$

where d_i $(i = 1, 2, \cdots, k)$ are computable constants.

This completes the proof of Theorem.

References

[1] F. Smarandache, Only Problems, not solutions, Chicago, Xiquan Publ. House, 1993.

[2] A. Murthy, Some notions on least common multiples, Smarandache Notions Journal, **12**(2001), 307-309.

[3] Le Maohua, An equation concerning the Smarandache LCM function, Smarandache Notions Journal, **14**(2004), 186-188.

[4] Lv Zhongtian, On the F.Smarandache LCM function and its mean value, Scientia Magna, **3**(2007), No.1, 22-25.

[5] Jianbin Chen, Value distribution of the F.Smarandache LCM function, Scientia Magna, **3**(2007), No.2, 15-18.

[6] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

[7] Pan Chengdong and Pan Chengbiao, The elementary proof of the prime theorem, Shanghai Science and Technology Press, Shanghai, 1988.

[8] Xu Zhefeng, On the value distribution of Smarandache function, ACTA MATHEMATICA, Chinese Series, Sept., **49**(2006), No. 5, 961-1200.