

# A NEW SEQUENCE RELATED SMARANDACHE SEQUENCES AND ITS MEAN VALUE FORMULA\*

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ABSTRACT. Let  $n$  be any positive integer,  $a(n)$  denotes the product of all non-zero digits in base 10. For natural  $x \geq 2$  and arbitrary fixed exponent  $m \in N$ , let  $A_m(x) = \sum_{n < x} a^m(n)$ . The main purpose of this paper is to give two exact calculating formulas for  $A_1(x)$  and  $A_2(x)$ .

## 1. INTRODUCTION

For any positive integer  $n$ , let  $b(n)$  denotes the product of base 10 digits of  $n$ . For example,  $b(1) = 1$ ,  $b(2)=2, \dots$ ,  $b(10) = 0$ ,  $b(11) = 1, \dots$ . In problem 22 of book [1], Professor F.Smaradache ask us to study the properties of sequence  $\{b(n)\}$ . About this problem, it appears that no one had studied it yet, at least, we have not seen such a paper before. The problem is interesting because it can help us to find some new distribution properties of the base 10 digits. In this paper, we consider another sequence  $\{a(n)\}$ , which related to Smarandache sequences. Let  $a(n)$  denotes the product of all non-zero digits in base 10 of  $n$ . For example,  $a(1) = 1$ ,  $a(2) = 2$ ,  $a(12) = 2$ ,  $\dots$ ,  $a(28) = 16$ ,  $a(1023) = 6, \dots$ . For natural number  $x \geq 2$  and arbitrary fixed exponent  $m \in N$ , let

$$A_m(x) = \sum_{n < x} a^m(n). \quad (1)$$

The main purpose of this paper is to study the calculating problem of  $A_m(x)$ , and use elementary methods to deduce two exact calculating formulas for  $A_1(x)$  and  $A_2(x)$ . That is, we shall prove the following:

**Theorem.** *For any positive integer  $x$ , let  $x = a_1 10^{k_1} + a_2 10^{k_2} + \dots + a_s 10^{k_s}$  with  $k_1 > k_2 > \dots > k_s \geq 0$  and  $1 \leq a_i \leq 9$ ,  $i = 2, 3, \dots, s$ . Then we have the calculating formulas*

$$A_1(x) = \frac{a_1 a_2 \dots a_s}{2} \sum_{i=1}^s \frac{a_i^2 - a_i + 2}{\prod_{j=i}^s a_j} \left( 45 + \left[ \frac{1}{k_i + 1} \right] \right) \cdot 46^{k_i - 1};$$

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$$A_2(x) = \frac{a_1^2 a_2^2 \cdots a_s^2}{6} \sum_{i=1}^s \frac{2a_i^3 - 3a_i^2 + a_i + 6}{\prod_{j=i}^s a_j^2} \left( 285 + \left[ \frac{1}{k_i + 1} \right] \right) \cdot 286^{k_i - 1},$$

where  $[x]$  denotes the greatest integer not exceeding  $x$ .

For general integer  $m \geq 3$ , using our methods we can also give an exact calculating formula for  $A_m(x)$ . That is, we have the calculating formula

$$A_m(x) = a_1^m a_2^m \cdots a_s^m \sum_{i=1}^s \frac{1 + B_m(a_i)}{\prod_{j=i}^s a_j^m} \left( \left[ \frac{1}{k_i + 1} \right] + B_m(10) \right) \cdot (1 + B_m(10))^{k_i - 1},$$

where  $a_i$  as the definition as in the above Theorem, and  $B_m(N) = \sum_{1 \leq n < N} n^m$ .

## 2. PROOF OF THE THEOREM

In this section, we complete the proof of the Theorem. First we need following two simple Lemmas.

**Lemma 1.** For any integer  $k \geq 1$  and  $1 \leq c \leq 9$ , we have the identities

- a)  $A_1(10^k) = 45 \cdot 46^{k-1}$ ;
- b)  $A_1(c \cdot 10^k) = 45 \cdot \left( 1 + \frac{(c-1)c}{2} \right) \cdot 46^{k-1}$ .

*Proof.* We first prove a) of Lemma 1 by induction. For  $k = 1$ , we have  $A_1(10^1) = A_1(10) = 1 + 2 + \cdots + 9 = 45$ . So that the identity

$$A_1(10^k) = \sum_{n < 10^k} a(n) = 45 \cdot 46^{k-1} \tag{2}$$

holds for  $k = 1$ . Assume (2) is true for  $k = m \geq 1$ . Then by the inductive assumption we have

$$\begin{aligned} A_1(10^{m+1}) &= \sum_{n < 9 \cdot 10^m} a(n) + \sum_{9 \cdot 10^m \leq n < 10^{m+1}} a(n) \\ &= A_1(9 \cdot 10^m) + \sum_{0 \leq n < 10^m} a(n + 9 \cdot 10^m) \\ &= A_1(9 \cdot 10^m) + 9 \cdot \sum_{0 \leq n < 10^m} a(n) \\ &= A_1(9 \cdot 10^m) + 9 \cdot \sum_{n < 10^m} a(n) \\ &= A_1(9 \cdot 10^m) + 9 \cdot A_1(10^m) \\ &= A_1(8 \cdot 10^m) + 9 \cdot A_1(10^m) + 8 \cdot A_1(10^m) \\ &= \dots \dots \dots \\ &= (1 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) \cdot A_1(10^m) \\ &= 46 \cdot A_1(10^m) \\ &= 45 \cdot 46^m. \end{aligned}$$

That is, (2) is true for  $k = m + 1$ . This proves the first part of Lemma 1.

The second part b) follows from a) of Lemma 1 and the recurrence formula

$$\begin{aligned}
 A_1(c \cdot 10^k) &= \sum_{n < (c-1) \cdot 10^k} a(n) + \sum_{(c-1) \cdot 10^k \leq n < c \cdot 10^k} a(n) \\
 &= \sum_{n < (c-1) \cdot 10^k} a(n) + \sum_{0 \leq n < 10^k} a(n + (c-1) \cdot 10^k) \\
 &= \sum_{n < (c-1) \cdot 10^k} a(n) + (c-1) \cdot \sum_{n < 10^k} a(n) \\
 &= A_1((c-1) \cdot 10^k) + (c-1) \cdot A_1(10^k).
 \end{aligned}$$

This completes the proof of Lemma 1.

**Lemma 2.** For any integer  $k \geq 1$  and  $1 \leq c \leq 9$ , we have the identities

$$\begin{aligned}
 \text{c) } A_2(10^k) &= 285 \cdot 286^{k-1}; \\
 \text{d) } A_2(a \cdot 10^k) &= 285 \cdot \left[ 1 + \frac{(a-1)a(2a-1)}{6} \right] \cdot 286^{k-1}.
 \end{aligned}$$

*Proof.* Note that  $A_2(10) = 285$ . The Lemma 2 can be deduced by Lemma 1, induction and the recurrence formula

$$\begin{aligned}
 A_2(10^{k+1}) &= \sum_{n < 9 \cdot 10^k} a^2(n) + \sum_{9 \cdot 10^k \leq n < 10^{k+1}} a^2(n) \\
 &= \sum_{n < 9 \cdot 10^k} a^2(n) + \sum_{0 \leq n < 10^k} a^2(n + 9 \cdot 10^k) \\
 &= \sum_{n < 9 \cdot 10^k} a^2(n) + 9^2 \cdot \sum_{0 \leq n < 10^k} a^2(n) \\
 &= A_2(9 \cdot 10^k) + 9^2 \cdot A_2(10^k) \\
 &= \dots \\
 &= (1 + 1^2 + 2^2 + \dots + 9^2) \cdot A_2(10^k) \\
 &= 286 \cdot A_2(10^k).
 \end{aligned}$$

This completes the proof of Lemma 2.

Now we use Lemma 1 and Lemma 2 to complete the proof of the Theorem. For any positive integer  $x$ , let  $x = a_1 \cdot 10^{k_1} + a_2 \cdot 10^{k_2} + \dots + a_s \cdot 10^{k_s}$  with  $k_1 > k_2 > \dots > k_s \geq 0$  under the base 10. Then applying Lemma 1 repeatedly we have

$$\begin{aligned}
 A_1(x) &= \sum_{n < a_1 \cdot 10^{k_1}} a(n) + \sum_{a_1 \cdot 10^{k_1} \leq n < x} a(n) \\
 &= A_1(a_1 \cdot 10^{k_1}) + \sum_{0 \leq n < x - a_1 \cdot 10^{k_1}} a(n + a_1 \cdot 10^{k_1}) \\
 &= A_1(a_1 \cdot 10^{k_1}) + a_1 \cdot \sum_{0 \leq n < x - a_1 \cdot 10^{k_1}} a(n)
 \end{aligned}$$

$$\begin{aligned}
&= A_1(a_1 \cdot 10^{k_1}) + a_1 \cdot A_1(x - a_1 \cdot 10^{k_1}) \\
&= A_1(a_1 \cdot 10^{k_1}) + a_1 \cdot A_1(a_2 \cdot 10^{k_2}) + a_1 a_2 \cdot A_1(x - a_1 \cdot 10^{k_1} - a_2 \cdot 10^{k_2}) \\
&= \dots \\
&= \sum_{i=1}^s \frac{a_1 a_2 \cdots a_s}{a_i a_{i+1} \cdots a_s} A_1(a_i \cdot 10^{k_i}) \\
&= a_1 a_2 \cdots a_s \sum_{i=1}^s \frac{(1 + \frac{(a_i-1)a_i}{2})}{\prod_{j=i}^s a_j} \left( 45 + \left[ \frac{1}{k_i + 1} \right] \right) 46^{k_i-1}.
\end{aligned}$$

This proves the first part of the Theorem.

Applying Lemma 2 and the first part of the Theorem repeatedly we have

$$\begin{aligned}
A_2(x) &= \sum_{n < a_1 \cdot 10^{k_1}} a^2(n) + \sum_{a_1 \cdot 10^{k_1} \leq n < x} a^2(n) \\
&= A_2(a_1 \cdot 10^{k_1}) + \sum_{0 \leq n < x - a_1 \cdot 10^{k_1}} a^2(n + a_1 \cdot 10^{k_1}) \\
&= A_2(a_1 \cdot 10^{k_1}) + a_1^2 \cdot \sum_{0 \leq n < x - a_1 \cdot 10^{k_1}} a^2(n) \\
&= A_2(a_1 \cdot 10^{k_1}) + a_1^2 \cdot A_2(x - a_1 \cdot 10^{k_1}) \\
&= A_2(a_1 \cdot 10^{k_1}) + a_1^2 \cdot A_2(a_2 \cdot 10^{k_2}) + a_1^2 a_2^2 \cdot A_2(x - a_1 \cdot 10^{k_1} - a_2 \cdot 10^{k_2}) \\
&= \dots \\
&= \sum_{i=1}^s \frac{a_1^2 a_2^2 \cdots a_s^2}{\prod_{j=i}^s a_j^2} A_2(a_i \cdot 10^{k_i}) \\
&= \frac{a_1^2 a_2^2 \cdots a_s^2}{6} \sum_{i=1}^s \frac{2a_i^3 - 3a_i^2 + a_i + 6}{\prod_{j=i}^s a_j^2} \left( 285 + \left[ \frac{1}{k_i + 1} \right] \right) \cdot 286^{k_i-1}.
\end{aligned}$$

This completes the proof of the second part of the Theorem.

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