A NOTE ON EXPONENTIAL DIVISORS AND RELATED ARITHMETIC FUNCTIONS

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§1. Introduction

Let $n > 1$ be a positive integer, and $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ its prime factorization. A number $d \mid n$ is called an Exponential divisor (or e-divisor, for short) of n if $d = p_1^{b_1} \cdots p_r^{b_r}$ with $b_i \mid a_i (i = \overline{1, r})$. This notion has been introduced by E.G. Straus and M.V. Subbarao[1]. Let $\sigma_e(n)$, resp. $d_e(n)$ denote the sum, resp. number of e-divisors of n, and let $\sigma_e(1) = d_e(1) = 1$, by convention. A number n is called e-perfect, if $\sigma_e(n) = 2n$. For results and References involving e-perfect numbers, and the arithmetical functions $\sigma_e(n)$ and $d_e(n)$, see [4]. For example, it is well-known that $d_e(n)$ is multiplicative, and

$$d_e(n) = d(a_1) \cdots d(a_r), \tag{1}$$

where $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ is the canonical form of n, and $d(a)$ denotes the number of (ordinary) divisors of a.
The e-totient function $\varphi_e(n)$, introduced and studied in [4] is multiplicative, and one has

$$\varphi_e(n) = \varphi(a_1) \cdots \varphi(a_r), \tag{2}$$

where φ is the classical Euler totient function.
Let $\sigma(a)$ denote the sum of (ordinary) divisors of a. The product of e-divisors of n, denoted by $T_e(n)$ has the following expression (see [9]):

$$T_e(n) = p_1^{\sigma(a_1)d(a_2)\cdots d(a_r)} \cdots p_r^{\sigma(a_r)d(a_1)\cdots d(a_r-1)} \tag{3}$$

A number n is called multiplicatively e-perfect if $T_e(n) = n^2$. Based on (3), in [9] we have proved that n is multiplicatively e-perfect iff n can be written as $n = p^m$, where $\sigma(m) = 2m$, and p is a prime. Two notions of exponentially-harmonic numbers have been recently introduced by the author in [11]. Finally, we note that for a given arithmetic function $f : N^* \to N^*$, in [5], [6] we have introduced the minimun function of f by

$$F_f(n) = \min\{k \geq 1 : n \mid f(k)\} \tag{4}$$

Various particular cases, including $f(k) = \varphi(k)$, $f(k) = \sigma(k)$, $f(k) = d(k), f(k) = S(k)$ (Smarandache function), $f(k) = T(k)$ (product of ordinary divisors), have been studiedrecently by the present author. He also studied the duals of these functions (when these have sense) defined by

$$F_f^*(n) = \max\{k \geq 1 : f(k) \mid n\} \tag{5}$$

See e.g. [10] and the References therein.

§2. Main notions and Results

The aim of this note is to introduce certain new arithmetic functions, related to the above considered notions.
Since for the product of ordinary divisors of n one can write

$$T(n) = n^{d(n)/2}, \tag{6}$$

trying to obtain a similar expression for $T_e(n)$ of the product of e-divisors of n, by (3) the following can be written:

Theorem 1.

$$T_e(n) = (t(n))^{d_e(n)/2}, \tag{7}$$

where $d_e(n)$ is the number of exponential divisors of n, given by (1); while the arithmetical function $t(n)$ is given by $t(1) = 1$

$$t(n) = p_1^{2^{\frac{\sigma(a_1)}{d(a_1)}}} \cdots p_r^{2^{\frac{\sigma(a_r)}{d(a_r)}}} \tag{8}$$

$n = p_1^{a_1} \cdots p_r^{a_r}$ *being the prime factorization of* $n > 1$.

Proof. This follows easily by relation (3), and the definition of $t(n)$ given by (8).

Remark For multiplicatively perfect numbers given by $T(n) = n^2$, see [7]. For multiplicatively deficient numbers, see [8].

Remark that

$$d_e(n) \leq d(n) \tag{9}$$

for all n, with equality only for $n = 1$. Indeed, by $d(a) < a + 1$ for $a \geq 2$, via (1) this is trivial.

On the other hand, the inequality

$$t(n) \leq n \tag{10}$$

is not generally valid. Let e.g. $n = p_1^{q_1} \cdots p_r^{q_r}$, where all q_i $(i = \overline{1, r})$ are primes. Then, by (8) $t(n) = p_1^{q_1+1} \cdots p_r^{q_r+1} = (p_1 \cdots p_r)n > n$. However, there is a particular case, when (10) is always true, namely suppose that $\omega(a_i) \geq 2$ for all $i = \overline{1, r}$ (where $\omega(a)$ denotes the number of distinct prime

factors of a). In [3] it is proved that if $\omega(a) \geq 2$, then $\frac{\sigma(a)}{d(a)} < \frac{a}{2}$. This gives (10) with strict inequality, if the above conditions are valid.
Without any condition one can prove:

Theorem 2. *For all $n \geq 1$,*

$$T_e(n) \leq T(n), \tag{11}$$

with equality only for $n = 1$ and $n = prime$.

Proof. The inequality to be proved becomes

$$\left(p_1^{\frac{\sigma(a_1)}{d(a_1)}} \cdots p_r^{\frac{\sigma(a_r)}{d(a_r)}}\right)^{d(a_1)\cdots d(a_r)} \leq (p_1^{a_1} \cdots p_r^{a_r})^{(a_1+1)\cdots(a_r+1)/2} \tag{12}$$

We will prove that

$$\frac{\sigma(a_1)}{d(a_1)} d(a_1)\cdots d(a_r) \leq \frac{a_1(a_1+1)\cdots(a_r+1)}{2}$$

with equality only if $r = 1$ and $a_1 = 1$. Indeed, it is known that (see [2]) $\frac{\sigma(a_1)}{d(a_1)} \leq \frac{a_1+1}{2}$, with equality only for $a_1 = 1$ and $a_1 = prime$. On the other hand, $d(a_1)\cdots d(a_r) \leq a_1(a_2+1)\cdots(a_r+1)$ is trivial by $d(a_1) \leq a_1$, $d(a_2) < a_2+1, \cdots, d(a_r) < a_r+1$, with equality only for $a_1 = 1$ and $r = 1$. Thus (12) follows, with equality for $r = 1$, $a_1 = 1$, so $n = p_1 = prime$ for $n > 1$.
Remark In [4] it is proved that

$$\varphi_e(n) d_e(n) \geq a_1 \cdots a_r \tag{13}$$

Now, by (2), $d_e(n) \geq \frac{a_1}{\varphi(a_1)} \cdots \frac{a_r}{\varphi(a_r)} \geq 2^r$ if all a_i $(i = \overline{1,r})$ are even, since it is well-known that $\varphi(a) \leq \frac{a}{2}$ for a =even. Since $d(n) = (a_1+1)\cdots(a_r+1) \leq 2^{a_1}\cdots 2^{a_r} = 2^{a_1+\cdots+a_r} = 2^{\Omega(n)}$ (where $\Omega(n)$ denotes the total number of prime divisors of n), by (9) one can write:

$$2^{\omega(n)} \leq d_e(n) \leq 2^{\Omega(n)} \tag{14}$$

if all a_i are even, i.e. when n is a perfect square (right side always).
Similarly, in [4] it is proved that

$$\varphi_e(n) d_e(n) \geq \sigma(a_1)\cdots\sigma(a_r) \tag{15}$$

when all a_i $(i = \overline{1,r})$ are odd. Let all $a_i \geq 3$ be odd. Then, since $\sigma(a_i) \geq a_i + 1$ (with equality only if $a_i = prime$), (15) implies

$$\varphi_e(n) d_e(n) \geq d(n), \tag{16}$$

which is a converse to inequality (9).
Let now introduce the arithmetical function $t_1(n) = p_1^{2\sqrt{a_1}} \cdots p_r^{2\sqrt{a_r}}$, $t_1(1) = 1$ and let $\gamma(n) = p_1 \cdots p_r$ denote the "core" of n (see [2]). Then:

Theorem 3.

$$t_1(n) \geq t(n) \geq n\gamma(n) \quad for \;\; all \;\; n \geq 1. \tag{17}$$

Proof. This follows at once by the known double-inequality

$$\sqrt{a} \leq \frac{\sigma(a)}{d(a)} \leq \frac{a+1}{2}, \tag{18}$$

with equality for $a = 1$ on the left side, and for $a = 1$ and $a = prime$ on the right side. Therefore, in (17) one has equality when n is squarefree, while on the right side if n is squarefree, or $n = p_1^{q_1} \cdots p_r^{q_r}$ with all q_i $(i = \overline{1,r})$ primes. Clearly, the functions $t_1(n)$, $t(n)$ and $\gamma(n)$ are all multiplicative.
Finally, we introduce the minimun exponential totient function by (4) for $f(k) = \varphi_e(k)$:

$$E_e(n) = \min\{k \geq 1 : n \mid \varphi_e(k)\}, \tag{19}$$

where $\varphi_e(k)$ is the e-totient function given by (2). Let

$$E(n) = \min\{k \geq 1 : n \mid \varphi(k)\} \tag{20}$$

be the Euler minimum function (see [10]). The following result is true:

Theorem 4.

$$E_e(n) = 2^{E(n)} \quad for \;\; n > 1. \tag{21}$$

Proof. Let $k = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$. Then $k \geq 2^{\alpha_1 + \cdots + \alpha_s} \geq 2^s$. Let s be the least integer with $n \mid \varphi(s)$ (i.e. $s = E(n)$ by (20)). Clearly $\varphi_e(2^s) = \varphi(s)$, so $k = 2^s$ is the least $k \geq 1$ with property $n \mid \varphi_e(k)$. This finishes the proof of (21). For properties of $E(n)$, see [10].
Remark It is interesting to note that the "maximum e-totient", i.e.

$$E_e^*(n) = \max\{k \geq 1 : \varphi_e(k) \mid n\} \tag{22}$$

is not well defined. Indeed, e.g. for all primes p one has $\varphi_e(p) = 1 \mid n$, and $E_e^*(p) = +\infty$, so $E_e^*(n)$ given by (22) is not an arithmetic function.

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