

Some notes on the paper “The mean value of a new arithmetical function”

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Abstract In reference [2], we used the elementary method to study the mean value properties of a new arithmetical function, and obtained two mean value formulae for it, but there exist some errors in that paper. The main purpose of this paper is to correct the errors in reference [2], and give two correct conclusions.

Keywords Smarandache multiplicative function, mean value, asymptotic formula.

§1. Introduction

For any positive integer n , we call an arithmetical function $f(n)$ as the Smarandache multiplicative function if for any positive integers m and n with $(m, n) = 1$, we have $f(mn) = \max\{f(m), f(n)\}$. For example, the Smarandache function $S(n)$ and the Smarandache LCM function $SL(n)$ both are Smarandache multiplicative functions. In reference [2], we defined a new Smarandache multiplicative function $f(n)$ as follows: $f(1) = 1$; If $n > 1$, then $f(n) = \max_{1 \leq i \leq k} \left\{ \frac{1}{\alpha_i + 1} \right\}$, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the factorization of n into prime powers. Then we studied the mean value properties of $f(n)$, and proved two asymptotic formulae:

$$\sum_{n \leq x} f(n) = \frac{1}{2} \cdot x \cdot \ln \ln x + \lambda \cdot x + O\left(\frac{x}{\ln x}\right), \quad (1)$$

where λ is a computable constant.

$$\sum_{n \leq x} \left(f(n) - \frac{1}{2} \right)^2 = \frac{1}{36} \cdot \frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)} \cdot \sqrt{x} \cdot \ln \ln x + d \cdot \sqrt{x} + O\left(x^{\frac{1}{3}}\right), \quad (2)$$

where $\zeta(s)$ is the Riemann zeta-function, and d is a computable constant.

But now, we found that the methods and results in reference [2] are wrong, so the formulae (1) and (2) are not correct. In this paper, we shall improve the errors in reference [2], and obtain two correct conclusions. That is, we shall prove the following:

Theorem 1. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} f(n) = \frac{1}{2} \cdot x + O\left(x^{\frac{1}{2}}\right).$$

Theorem 2. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} \left(f(n) - \frac{1}{2} \right)^2 = \frac{1}{36} \cdot \frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)} \cdot \sqrt{x} + O\left(x^{\frac{1}{3}}\right),$$

where $\zeta(n)$ is the Riemann zeta-function.

§2. Proof of the theorems

In this section, we shall using the elementary and the analytic methods to prove our Theorems. First we give following two simple Lemmas:

Lemma 1. Let A denotes the set of all square-full numbers. Then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{\substack{n \leq x \\ n \in A}} 1 = \frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)} \cdot x^{\frac{1}{2}} + \frac{\zeta\left(\frac{2}{3}\right)}{\zeta(2)} \cdot x^{\frac{1}{3}} + O\left(x^{\frac{1}{6}}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

Lemma 2. Let B denotes the set of all cubic-full numbers. Then for any real number $x > 1$, we have

$$\sum_{\substack{n \leq x \\ n \in B}} 1 = N \cdot x^{\frac{1}{3}} + O\left(x^{\frac{1}{4}}\right),$$

where N is a computable constant.

Proof. The proof of these two Lemmas can be found in reference [3].

Now we use these two simple Lemmas to complete the proof of our Theorems. In fact, for any positive integer $n > 1$, from the definition of $f(n)$ we have

$$\sum_{n \leq x} f(n) = f(1) + \sum_{\substack{n \leq x \\ n \in A}} f(n) + \sum_{\substack{n \leq x \\ n \in B}} f(n), \quad (3)$$

where A denotes the set of all square-full numbers. That is, $n > 1$, and for any prime p , if $p \mid n$, then $p^2 \mid n$. B denotes the set of all positive integers $n > 1$ with $n \notin A$. Note that $f(n) \ll 1$, from the definition of A and Lemma 1 we have

$$\sum_{\substack{n \leq x \\ n \in A}} f(n) = O\left(x^{\frac{1}{2}}\right). \quad (4)$$

$$\begin{aligned} \sum_{\substack{n \leq x \\ n \in B}} f(n) &= \sum_{\substack{n \leq x \\ n \in B}} \frac{1}{2} = \sum_{n \leq x} \frac{1}{2} - \sum_{\substack{n \leq x \\ n \in A}} \frac{1}{2} \\ &= \frac{1}{2} \cdot x + O\left(x^{\frac{1}{2}}\right). \end{aligned} \quad (5)$$

Now combining (3), (4) and (5) we may immediately get

$$\begin{aligned}\sum_{n \leq x} f(n) &= 1 + \sum_{\substack{n \leq x \\ n \in A}} f(n) + \sum_{\substack{n \leq x \\ n \in B}} f(n) \\ &= \frac{1}{2} \cdot x + O\left(x^{\frac{1}{2}}\right).\end{aligned}$$

This proves Theorem 1.

Now we prove Theorem 2. From the definition of $f(n)$ and the properties of square-full numbers we have

$$\begin{aligned}\sum_{\substack{n \leq x \\ n \in A}} \left(f(n) - \frac{1}{2}\right)^2 &= \frac{1}{4} + \sum_{\substack{n \leq x \\ n \in A}} \left(f(n) - \frac{1}{2}\right)^2 + \sum_{\substack{n \leq x \\ n \notin A}} \left(f(n) - \frac{1}{2}\right)^2 \\ &= \frac{1}{4} + \sum_{\substack{n \leq x \\ n \in A}} \left(f(n) - \frac{1}{2}\right)^2.\end{aligned}\tag{6}$$

where A also denotes the set of all square-full numbers. Let C denotes the set of all cubic-full numbers. Then from the properties of square-full numbers, Lemma 1 and Lemma 2 we have

$$\begin{aligned}\sum_{\substack{n \leq x \\ n \in A}} \left(f(n) - \frac{1}{2}\right)^2 &= \sum_{\substack{n \leq x \\ n \in A, f(n) = \frac{1}{3}}} \left(\frac{1}{3} - \frac{1}{2}\right)^2 + \sum_{\substack{n \leq x \\ n \in C}} \left(f(n) - \frac{1}{2}\right)^2 \\ &= \sum_{\substack{n \leq x \\ n \in A}} \left(\frac{1}{3} - \frac{1}{2}\right)^2 - \sum_{\substack{n \leq x \\ n \in C}} \left(\frac{1}{3} - \frac{1}{2}\right)^2 + O\left(\sum_{\substack{n \leq x \\ n \in C}} 1\right) \\ &= \frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)} \cdot x^{\frac{1}{2}} + O\left(x^{\frac{1}{3}}\right).\end{aligned}\tag{7}$$

where $\zeta(s)$ is the Riemann zeta-function.

Now combining (6) and (7) we have the asymptotic formula

$$\sum_{n \leq x} \left(f(n) - \frac{1}{2}\right)^2 = \frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)} \cdot x^{\frac{1}{2}} + O\left(x^{\frac{1}{3}}\right).$$

This completes the proof of Theorem 2.

References

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On the Smarandache multiplicative sequence

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Abstract In this paper, we use the elementary method to study the properties of the Smarandache multiplicative sequence, and proved that some infinite series involving the Smarandache multiplicative sequence are convergent.

Keywords Smarandache multiplicative sequence, infinite series, convergent properties.

§1. Introduction and result

For any positive integer $m \geq 2$, let $1 < d_1 < d_2 < \cdots < d_m$ are m positive integers, then we define the Smarandache multiplicative sequence A_m as: If d_1, d_2, \cdots, d_m are the first m terms of the sequence A_m , then $d_k > d_{k-1}$, for $k \geq m + 1$, is the smallest number equal to $d_1^{\alpha_1} \cdot d_2^{\alpha_2} \cdots d_m^{\alpha_m}$, where $\alpha_i \geq 1$ for all $i = 1, 2, \cdots, m$. For example, the Smarandache multiplicative sequence A_2 (generated by digits 2, 3) is:

2, 3, 6, 12, 18, 24, 36, 48, 54, 72, 96, 108, 144, 162, 192, 216, \cdots

The Smarandache multiplicative sequence A_3 (generated by digits 2, 3, 7) is:

2, 3, 7, 42, 84, 126, 168, 252, 294, 336, 378, 504, 588, 672, \cdots

The Smarandache multiplicative sequence A_4 (generated by digits 2, 3, 5, 7) is:

2, 3, 5, 7, 210, 420, 630, 840, 1050, 1260, 1470, 1680, 1890, 2100, \cdots

In the book "Sequences of Numbers Involved Unsolved Problems", Professor F.Smarandache introduced many sequences, functions and unsolved problems, one of them is the Smarandache multiplicative sequence, and he also asked us to study the properties of this sequence. About this problem, it seems that none had studied it yet, at least we have not seen any related papers before. The problem is interesting, because there are close relationship between the Smarandache multiplicative sequence and the geometric series. In this paper, we shall use the elementary method to study the convergent properties of some infinite series involving the Smarandache multiplicative sequence, and get some interesting results. For convenience, we use the symbol $a_m(n)$ denotes the n -th term of the Smarandache multiplicative sequence A_m . The main purpose of this paper is to study the convergent properties of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{a_m^s(n)}, \quad (1)$$

and prove the following:

Theorem. For any positive integer $m \geq 2$, let $1 < d_1 < d_2 < \dots < d_m$ are m positive integers, and A_m denotes the Smarandache multiplicative sequence generated by d_1, d_2, \dots, d_m . Then for any real number $s \leq 0$, the infinite series (1) is divergent; For any real number $s > 0$, the series (1) is convergent, and

$$\sum_{n=1}^{\infty} \frac{1}{a_m^s(n)} = \prod_{i=1}^m \frac{1}{d_i^s - 1} + \sum_{i=1}^m \frac{1}{d_i^s}.$$

From our Theorem we may immediately deduce the following two corollaries:

Corollary 1. Let A_2 be the Smarandache multiplicative sequence generated by 2 and 3, then we have the identity

$$\sum_{n=1}^{\infty} \frac{1}{a_2(n)} = \frac{4}{3}.$$

Corollary 2. Let A_3 be the Smarandache multiplicative sequence generated by 3, 4 and 5, then we have the identity

$$\sum_{n=1}^{\infty} \frac{1}{a_3(n)} = \frac{13}{20}.$$

Similarly, we can also introduce another sequence called the Smarandache additive sequence as follows: Let $1 \leq d_1 < d_2 < \dots < d_m$ are m positive integers, then we define the Smarandache additive sequence D_m as: If d_1, d_2, \dots, d_m are the first m terms of the sequence D_m , then $d_k > d_{k-1}$, for $k \geq m + 1$, is the smallest number equal to $\alpha_1 \cdot d_1 + \alpha_2 \cdot d_2 + \dots + \alpha_m \cdot d_m$, where $\alpha_i \geq 1$ for all $i = 1, 2, \dots, m$. It is clear that this sequence has the close relationship with the coefficients of the power series ($x^{d_1} + x^{d_2} + \dots + x^{d_m} < 1$)

$$\sum_{n=1}^{\infty} (x^{d_1} + x^{d_2} + \dots + x^{d_m})^n = \frac{x^{d_1} + x^{d_2} + \dots + x^{d_m}}{1 - x^{d_1} - x^{d_2} - \dots - x^{d_m}}.$$

For example, the Smarandache additive sequence D_2 (generated by digits 3, 5) is:

3, 5, 8, 11, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28,

It is an interesting problem to study the properties of the Smarandache additive sequence.

§2. Proof of the theorem

In this section, we shall prove our Theorem directly. First note that for any positive integer $k > m$, we have

$$a_m(k) = d_1^{\alpha_1} \cdot d_2^{\alpha_2} \cdot \dots \cdot d_m^{\alpha_m},$$

where $\alpha_i \geq 1, i = 1, 2, \dots, m$. So for any real number $s > 1$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{a_m^s(n)} &= \sum_{i=1}^m \frac{1}{a_m^s(i)} + \sum_{\alpha_1=1}^{\infty} \sum_{\alpha_2=1}^{\infty} \dots \sum_{\alpha_m=1}^{\infty} \frac{1}{(d_1^{\alpha_1} \cdot d_2^{\alpha_2} \cdot \dots \cdot d_m^{\alpha_m})^s} \\ &= \sum_{i=1}^m \frac{1}{a_m^s(i)} + \left(\sum_{\alpha_1=1}^{\infty} \frac{1}{d_1^{\alpha_1 s}} \right) \cdot \left(\sum_{\alpha_2=1}^{\infty} \frac{1}{d_2^{\alpha_2 s}} \right) \dots \left(\sum_{\alpha_m=1}^{\infty} \frac{1}{d_m^{\alpha_m s}} \right). \end{aligned} \tag{2}$$

It is clear that for any real number $s \leq 0$, the series $\sum_{\alpha_i=1}^{\infty} \frac{1}{d_i^{\alpha_i s}}$ is divergent, and for any real number $s > 0$, the series $\sum_{\alpha_i=1}^{\infty} \frac{1}{d_i^{\alpha_i s}}$ is convergent, and more

$$\sum_{\alpha_i=1}^{\infty} \frac{1}{d_i^{\alpha_i s}} = \frac{1}{d_i^s - 1}.$$

So from (2) we know that the series $\sum_{n=1}^{\infty} \frac{1}{a_m^s(n)}$ is also convergent, and

$$\sum_{n=1}^{\infty} \frac{1}{a_m^s(n)} = \prod_{i=1}^m \frac{1}{d_i^s - 1} + \sum_{i=1}^m \frac{1}{d_i^s}.$$

This completes the proof of our Theorem.

References

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