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# The Number of Minimum Dominating Sets in $P_n \times P_2$

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**Abstract**: A set $S$ of vertices in a graph $G$ is said to be a *Smarandachely $k$-dominating set* if each vertex of $G$ is dominated by at least $k$ vertices of $S$. The *Smarandachely $k$-domination number* $\gamma_k(G)$ of $G$ is the minimum cardinality of Smarandachely $k$-dominating sets of $G$. Particularly, if $k = 1$, a Smarandachely $k$-dominating set is called a *dominating set* of $G$ and $\gamma_k(G)$ is abbreviated to $\gamma(G)$. In this paper, we get the Smarandachely 1-dominating number, i.e., the dominating number of $P_n \times P_2$.

**Key Words**: Smarandachely $k$-dominating set, Smarandachely $k$-domination number, dominating sets, dominating number.

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## §1. Introduction

We considered finite, undirected, simple graphs $G = (V, E)$ with vertex set $V(G)$ and edge set $E(G)$. The order of $G$ is given by $n = |V(G)|$. A set $S \subseteq V$ of vertices in a graph $G$ is called a dominating set if every vertex $v \in V$ is either an element of $S$ or is adjacent to an element of $S$. A dominating set $S$ is a minimum dominating set if no proper subset is a dominating set. The domination number $\gamma(G)$ of a graph $G$ is the minimum cardinality of a dominating set in $G$. A set of vertices $S$ in a graph $G$ is said to be a *Smarandachely $k$-dominating set* if each vertex of $G$ is dominated by at least $k$ vertices of $S$. Particularly, if $k = 1$, such a set is called a dominating set of $G$. The *Smarandachely $k$-domination number* $\gamma_k(G)$ of $G$ is the minimum cardinality of a Smarandachely $k$-dominating set of $G$.

As known, a fundamental unsolved problem concerning the bounds on the domination number of product graphs is to settle Vizing's conjecture. Another basic problem is to find the domination number or bound on the domination number of specific Cartesian products, for example the $j \times k$ grid graph $P_j \times P_k$ . This too seems to be a difficult problem. It is

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known that dominating set remains NP- complete when restricted to arbitrary sub graphs of [2,12]. However, Hare, Hare and Hedetniemi [8,9] developed a linear time algorithm to solve this problem on $j \times k$ grid graph for any fixed $j$. Moreover, the domination number of $P_j \times P_k$ has been determined for small values of $j$. Jacobson and Kinch [10] established it for $j = 1, 2, 3, 4$ and all $k$. Hare [8] developed algorithm which she used to conjecture simple formulae for $\gamma(P_j \times P_k)$ for $1 \leq j \leq 10$. Chang and Clark [4] proved Hare's formulae for the domination number of $P_5 \times P_k$ and $P_6 \times P_k$ . The domination numbers for $P_j \times P_k$ $1 \leq j \leq 6$ are listed below:

1. $\gamma(P_1 \times P_k) = \lfloor \frac{k+2}{3} \rfloor$, $k \geq 1$
2. $\gamma(P_2 \times P_k) = \lfloor \frac{k+2}{2} \rfloor$, $k \geq 1$
3. $\gamma(P_3 \times P_k) = \lfloor \frac{3k+4}{4} \rfloor$, $k \geq 1$
4. $\gamma(P_3 \times P_k) = \begin{cases} k+1, & k = 1, 2, 3, 5, 6, 9; \\ k, & \text{otherwise.} \end{cases}$
5. $\gamma(P_3 \times P_k) = \begin{cases} \frac{6k+6}{5}, & k = 2, 3, 7; \\ \frac{6k+8}{5}, & \text{otherwise.} \end{cases}$
6. $\gamma(P_3 \times P_k) = \begin{cases} \frac{10k+10}{7}, & k \geq 6k \equiv 1 mod 7; \\ \frac{10k+12}{7}, & \text{otherwise} if k \geq 4. \end{cases}$

It is well known that the concept of domination is originated from the game of chess board. The problem of finding the minimum number of stones is one aspect and the number of ways of placing the minimum number of stones is another aspect. Though the first aspect has not been resolved as mentioned earlier, we consider the second aspect of the problem, that is, finding the number of ways of placing the minimum number of stones. In this paper, we consider the second aspect of the problem for $P_n \times P_2$. That is, equivalently finding the minimum number of dominating sets in $P_n \times P_2$.

$P_7 \times P_2$ :

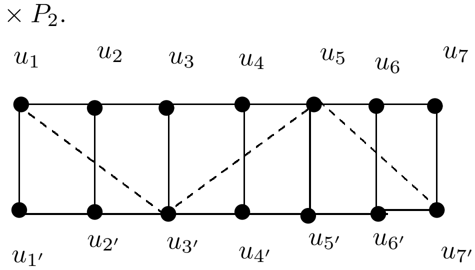


Figure 1: $P_7 \times P_2$ with dominating vertices

The minimum dominating sets of Figure 1 are $\{u_1, u_{3'}, u_5, u_{7'}\}$ and $\{u_{1'}, u_3, u_{5'}, u_7\}$.

Similarly, the minimum dominating sets of Figure 2 are: $\{u_1, u_{3'}, u_5, u_{6'}\}$ $\{u_{1'}, u_3, u_{5'}, u_6\}$, $\{u_1, u_{3'}, u_5, u_6\}$, $\{u_{1'}, u_3, u_{5'}, u_{6'}\}$, $\{u_1, u_{3'}, u_4, u_{6'}\}$, $\{u_{1'}, u_3, u_{4'}, u_6\}$, $\{u_1, u_{2'}, u_4, u_{6'}\}$, $\{u_{1'}, u_2,$

$P_6 \times P_2$ :

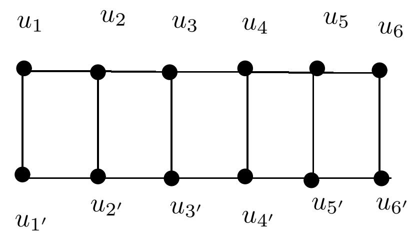


Figure 2: $P_6 \times P_2$ with dominating vertices

$u_{4'}, u_6\}$, $\{u_{1'}, u_3, u_4, u_{6'}\}$, $\{u_1, u_{3'}, u_{4'}, u_6\}$, $\{u_1, u_2, u_{4'}, u_6\}$, $\{u_2, u_{2'}, u_4, u_{6'}\}$, $\{u_{2'}, u_2, u_{4'}, u_6\}$, $\{u_{1'}, u_3, u_{5'}, u_5\}$, $\{u_1, u_{3'}, u_{5'}, u_5\}$, $\{u_2, u_{2'}, u_5, u_{5'}\}$, $\{u_{1'}, u_{2'}, u_4, u_{6'}\}$.

As such the domination number of $P_n \times P_2$ is, $\gamma\left(P_n \times P_2\right) = \left\lfloor \frac{n+2}{2} \right\rfloor$. Using this value we consider the minimum number of dominating sets $\gamma_D\left(P_n \times P_2\right)$ for the values $n = 2k + 1$ and $n = 2k$.

## §2. Results

To prove our results, we need some lemmas proved below.

**Lemma** 2.1 *Let vertices of first and second rows in $P_{2k+1} \times P_2$ are labeled with $v_1, v_2 \ldots, v_{2k-2}$, $v_{2k-1}, v_{2k}, v_{2k+1}$ and $u_1, u_2, \ldots, u_{2k-2}, u_{2k-1}, u_{2k}, u_{2k+1}$, then there is no md-set containing both the vertices $v_{2k}$ and $u_{2k}$.*

*Proof* On the contrary, assume that there is an md-set say $D$ in $P_{2k+1} \times P_2$ containing both the vertices $v_{2k}$and $u_{2k}$. Clearly, $D - \{v_{2k}u_{2k}\}$ dominating set in $P_{2k-2} \times P_2$, for otherwise there exists a vertex $v_i$ (or $u_i$ ) of $P_{2k-2} \times P_2$ which is not either in $D - \{v_{2k}u_{2k}\}$ or not adjacent to any vertex of $D - \{v_{2k}u_{2k}\}$ then this vertex $v_i$ (or $u_i$ ) is not in $D$ or is not adjacent to any vertex of $D$ in $P_{2k+1} \times P_2$ and hence $D$ is not a dominating set in $P_{2k+1} \times P_2$, a contradiction to the assumption.

Therefore,$K = \gamma\left(P_{2k-2} \times P_2\right) \leq |D - \{v_{2k}u_{2k}\}| = |D| - 2 = k + 1 - 2 = k - 1$ a contradiction, which proves the Lemma. □

**Lemma** 2.2 *There is no md-set containing both $v_{2k+1}$ and $u_{2k+1}$ , where the vertices of $P_{2k+1} \times P_2$ are labelled as in the above Lemma 2.1.*

*Proof* The proof is similar to that of Lemma 2.1 with a slight change, that is by considering $D - \{v_{2k+1}u_{2k+1}\}$ which is the dominating set in $P_{2k-1} \times P_2$ with $D$ being a md - set containing both $v_{2k+1}$and $u_{2k+1}$ in $P_{2k-1} \times P_2$. Thus, $K = \gamma\left(P_{2k-1} \times P_2\right) \leq |D - \{v_{2k+1}u_{2k+1}\}| = |D| - 2 = k + 1 - 2 = k - 1$ a contradiction, which proves that $D$ is not an md - set. □

**Corollary** 2.3 *Every md - set in $P_{2k+1} \times P_2$ contains either $v_{2k+1}$ or $u_{2k+1}$.*

**Theorem** 2.4 $\gamma(P_{2k+1} \times P_2) = \begin{cases} 3, & if\ k = 1; \\ 2, & if\ k \geq 2. \end{cases}$

**Lemma** 2.5 *There exists exactly two md - sets containing both $v_{2k-1}$ and $u_{2k-1}$ in $P_{2k} \times P_2$.*

*Proof* In $P_{2k} \times P_2$ , clearly the vertices $v_{2k-1}$ and $u_{2k-1}$ can cover $v_{2k-2}$, $v_{2k}$ and $u_{2k-2}$, $u_{2k}$ respectively. We claim that any md - set $D$ containing either $v_{2k-3}$ or $u_{2k-3}$ but not both, (follows from the Corollary 2.3)union $\{v_{2k-1}, u_{2k-1}\}$ is an md - set in $P_{2k} \times P_2$. Since $k + 1 = \gamma(P_{2k} \times P_2) \leq |D \cup \{v_{2k-1}, v_{2k-2}\}| = \gamma(P_{2k-3} \times P_2) + 2 = k - 1 + 2 = k + 1$. Hence the claim. Again by Theorem 2.4 and Corollary 2.3, there are exactly two md-sets viz $D_1$ containing $v_{2k-3}$ and $D_2$ containing $u_{2k-3}$ in $P_{2k-3} \times P_2$. Hence $D_1 \cup \{v_{2k-1}, u_{2k-1}\}$ and $D_2 \cup \{v_{2k-1}, u_{2k-1}\}$ are md-sets in $P_{2k} \times P_2$. □

**Lemma** 2.6 *There is no md-set containing both $v_{2k}$ and $u_{2k}$ in $P_{2k} \times P_2$.*

*Proof* On the contrary, assume that there is a md - set in $P_{2k} \times P_2$ containing both $v_{2k}$ and $u_{2k}$. Then, clearly,

$D - \{v_{2k}, u_{2k}\}$ is a dominating set in $P_{2k} \times P_2$. Thus, $k = \gamma(P_{2k-2} \times P_2) \leq |D - \{v_{2k}, u_{2k}\}| \leq |D| - 2 = k + 1 - 2 = k - 1$ a contradiction, which proves this lemma. □

**Theorem** 2.7 *For any $k \geq 3$, $\gamma_D\,(P_{2k} \times P_2) = \gamma(P_{2k-2} \times P_2) + 4$*

*Proof* We prove this theorem by four steps following.

**Step 1.** Let $D_1, D_2, \cdots, D_t$ be md-sets containing $u_{2k-2}$ in $P_{2k-2} \times P_2$, then, $D_i \cup \{u_{2k}\}$ and $D_i \cup \{v_{2k}\}$ are dominating sets in $P_{2k-2} \times P_2$ for $i = 1, 2, \cdots, t$ But, $k + 1 = \gamma(P_{2k} \times P_2) \leq |D_i| \cup \{u_{2k}\} = |D_i| + 1 = \gamma(P_{2k-2} \times P_2) + 1 = k + 1$. Hence, $D_i \cup \{u_{2k}\}$ is a md-set in $P_{2k} \times P_2$. And for the same reason, $D_i \cup \{v_{2k}\}$ is a md-set in $P_{2k} \times P_2$.

**Step 2.** By the Lemma 2.5, Let $D_1$ and $D_2$ be two md - sets containing both $v_{2k-3}$ and $u_{2k-3}$ in $P_{2k-2} \times P_2$. But, by the Lemma, there exists exactly two md - sets say $D_1'$ and $D_2'$ containing $v_{2k-3}$ and $u_{2k-3}$ respectively in $P_{2k-2} \times P_2$. So, $D_1$ must be obtained from $D_1' \cup \{v_{2k-3}, u_{2k-3}\}$ and $D_2$ must be obtained from $D_2' \cup \{v_{2k-3}, u_{2k-3}\}$. Thus it is not difficult to see that $(D_1 - v_{2k-3}) \cup \{v_{2k-1}, u_{2k}\}$ and $(D_1 - u_{2k-3}) \cup \{u_{2k-1}, v_{2k}\}$ are md- sets in $P_{2k} \times P_2$.

**Step 3.** For md-sets $D_1$ and $D_2$ of $P_{2k-2} \times P_2$ the sets $(D_1 - \{v_{2k-3}\}) \cup \{v_{2k-1}, v_{2k}\}$ and $(\{D_2 - u_{2k-3}\}) \cup \{u_{2k-1}, u_{2k},\}$ are md- sets in $P_{2k} \times P_2$.

**Step 4.** For md-sets $D_1$ and $D_2$ of $P_{2k-2} \times P_2$ the sets $(D_1 - \{v_{2k-3}\}) \cup \{v_{2k-1}, u_{2k-1}\}$ and $(\{D_2 - u_{2k-3}\}) \cup \{v_{2k-1}, u_{2k-1}\}$ are md- sets in $P_{2k} \times P_2$.

Thus $\gamma_D\,(P_{2k} \times P_2) = 2t + 2 + 2 + 2 = 2t + 2 + 4 = \gamma_D\,(P_{2k-2} \times P_2) + 4$ by steps 1, 2, 3, 4.□

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