ON A PROBLEM OF F. SMARANDACHE*

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ABSTRACT. Let $d_s(n)$ denotes the sum of the base 10 digits of $n \in N$. For natural $x \geq 2$ and arbitrary fixed exponent $m \in N$, let $A_m(x) = \sum_{n<x} d_s^m(n)$. The main purpose of this paper is to give two exact calculating formulas for $A_1(x)$ and $A_2(x)$.

1. INTRODUCTION

For any positive integer n, let $d_s(n)$ denotes the sum of the base 10 digits of n. For example, $d_s(0) = 0$, $d_s(1) = 1$, $d_s(2) = 2, \cdots$, $d_s(11) = 2$, $d_s(12) = 3, \cdots\cdots$. In problem 21 of book [1], Professor F.Smaradache ask us to study the properties of sequence $\{d_s(n)\}$. For natural number $x \geq 2$ and arbitrary fixed exponent $m \in N$, let

$$A_m(x) = \sum_{n<x} d_s^m(n). \tag{1}$$

The main purpose of this paper is to study the calculating problem of $A_m(x)$, and use elementary methods to deduce two exact calculating formulas for $A_1(x)$ and $A_2(x)$. That is, we shall prove the following:

Theorem. *For any positive integer x, let $x = a_1 10^{k_1} + a_2 10^{k_2} + \cdots + a_s 10^{k_s}$ with $k_1 > k_2 > \cdots > k_s \geq 0$ and $1 \leq a_i \leq 9$, $i = 2, 3, \cdots, s$. Then we have the calculating formulas*

$$A_1(x) = \sum_{i=1}^{s} a_i \cdot \left(\frac{9}{2} k_i + \sum_{j=1}^{i} a_j - \frac{a_i + 1}{2} \right) \cdot 10^{k_i};$$

$$A_2(x) = \sum_{i=1}^{s} a_i \cdot \left[\frac{k_i(81k_i + 33)}{4} + \frac{9k_i}{2}(a_i - 1) + \sum_{j=1}^{i} a_j^2 - \frac{(4a_i - 1)(a_i + 1)}{6} \right] \cdot 10^{k_i}$$
$$+ \sum_{i=2}^{s} a_i \cdot \left[(9k_i - a_i - 1)10^{k_i} + 2\sum_{j=i}^{s} a_j 10^{k_j} \right] \cdot \left(\sum_{j=1}^{i-1} a_j \right).$$

For general integer $m \geq 3$, using our methods we can also give an exact calculating formula for $A_m(x)$. But in these cases, the computations are more complex.

Key words and phrases. F.Smarandache problem; Sum of base 10 digits; Calculating formula.
* This work is supported by the N.S.F. and the P.S.F. of P.R.China.

2. Proof of the Theorem

In this section, we complete the proof of the Theorem. First we need following two simple Lemmas.

Lemma 1. *For any integer $k \geq 0$, we have the identities*

a) $A_1(10^k) = \frac{9}{2} \cdot k \cdot 10^k$;

b) $A_1(a \cdot 10^k) = \left(\frac{9}{2}k + \frac{a-1}{2}\right) \cdot a \cdot 10^k, \quad 1 \leq a \leq 9.$

Proof. We first prove a) of Lemma 1 by induction. For $k = 0$ and 1, we have $A_1(10^0) = A_1(1) = 0$, $A_1(10^1) = A_1(10) = 45$. So that the identity

$$A_1(10^k) = \sum_{n<10^k} d_s(n) = \frac{9}{2} \cdot k \cdot 10^k \tag{2}$$

holds for $k = 0$ and 1. Assume (2) is true for $k = m - 1$. Then by the inductive assumption we have

$$\begin{aligned}
A_1(10^m) &= \sum_{n<9\cdot 10^{m-1}} d_s(n) + \sum_{9\cdot 10^{m-1}\leq n<10^m} d_s(n) \\
&= A_1(9\cdot 10^{m-1}) + \sum_{0\leq n<10^{m-1}} d_s(n + 9\cdot 10^{m-1}) \\
&= A_1(9\cdot 10^{m-1}) + \sum_{0\leq n<10^{m-1}} (d_s(n) + 9) \\
&= A_1(9\cdot 10^{m-1}) + 9\cdot 10^{m-1} + \sum_{n<10^{m-1}} d_s(n) \\
&= A_1(9\cdot 10^{m-1}) + 9\cdot 10^{m-1} + A_1(10^{m-1}) \\
&= A_1(8\cdot 10^{m-1}) + (8+9)\cdot 10^{m-1} + 2A_1(10^{m-1}) \\
&= \cdots\cdots\cdots \\
&= (1+2+3+4+5+6+7+8+9)\cdot 10^{m-1} + 10A_1(10^{m-1}) \\
&= \frac{9}{2}\cdot 10^m + 10\cdot\frac{9}{2}\cdot(m-1)\cdot 10^{m-1} \\
&= \frac{9}{2}\cdot m\cdot 10^m.
\end{aligned}$$

That is, (2) is true for $k = m$. This proves the first part of Lemma 1.

The second part b) follows from a) of Lemma 1 and the recurrence formula

$$\begin{aligned}
A_1(a\cdot 10^k) &= \sum_{n<(a-1)\cdot 10^k} d_s(n) + \sum_{(a-1)\cdot 10^k\leq n<a\cdot 10^k} d_s(n) \\
&= \sum_{n<(a-1)\cdot 10^k} d_s(n) + \sum_{0\leq n<10^k} d_s(n + (a-1)\cdot 10^k) \\
&= \sum_{n<(a-1)\cdot 10^k} a(n) + (a-1)\cdot 10^k + \sum_{n<10^k} d_s(n) \\
&= A_1((a-1)\cdot 10^k) + (a-1)\cdot 10^k + A_1(10^k).
\end{aligned}$$

This completes the proof of Lemma 1.

Lemma 2. *For any integer $k \geq 0$ and $1 \leq a \leq 9$, we have the identities*

c) $A_2(10^k) = \frac{81k+33}{4} \cdot k \cdot 10^k;$

d) $A_2(a \cdot 10^k) = \left[\frac{k(81k+33)}{4} + \frac{9k}{2}(a-1) + \frac{(a-1)(2a-1)}{6}\right] \cdot a \cdot 10^k.$

Proof. These results can be deduced by Lemma 1, induction and the recurrence formula

$$
\begin{aligned}
A_2(10^{k+1}) &= \sum_{n<9\cdot 10^k} d_s^2(n) + \sum_{9\cdot 10^k \leq n < 10^{k+1}} d_s^2(n) \\
&= \sum_{n<9\cdot 10^k} d_s^2(n) + \sum_{0\leq n<10^k} d_s^2(n+9\cdot 10^k) \\
&= \sum_{n<9\cdot 10^k} d_s^2(n) + \sum_{0\leq n<10^k} (d_s(n)+9)^2 \\
&= A_2(9\cdot 10^k) + 9^2\cdot 10^k + 18A_1(10^k) + A_2(10^k) \\
&= \cdots\cdots\cdots \\
&= 10A_2(10^k) + (1^2+2^2+\cdots+9^2)\cdot 10^k + 2\cdot(1+2+\cdots+9)A_1(10^k) \\
&= 10A_2(10^k) + \frac{57}{2}\cdot 10^{k+1} + 90\cdot\frac{9}{2}\cdot k\cdot 10^k \\
&= 10A_2(10^k) + \frac{57}{2}\cdot 10^{k+1} + \frac{81}{2}\cdot k\cdot 10^{k+1}.
\end{aligned}
$$

This completes the proof of Lemma 2.

Now we use Lemma 1 and Lemma 2 to complete the proof of the Theorem. For any positive integer x, let $x = a_1 \cdot 10^{k_1} + a_2 \cdot 10^{k_2} + \cdots + a_s \cdot 10^{k_s}$ with $k_1 > k_2 > \cdots > k_s \geq 0$ under the base 10. Then applying Lemma 1 repeatedly we have

$$
\begin{aligned}
A_1(x) &= \sum_{n<a_1\cdot 10^{k_1}} d_s(n) + \sum_{a_1\cdot 10^{k_1}\leq n<x} d_s(n) \\
&= A_1(a_1\cdot 10^{k_1}) + \sum_{0\leq n<x-a_1\cdot 10^{k_1}} d_s(n+a_1\cdot 10^{k_1}) \\
&= A_1(a_1\cdot 10^{k_1}) + \sum_{0\leq n<x-a_1\cdot 10^{k_1}} (d_s(n)+a_1) \\
&= A_1(a_1\cdot 10^{k_1}) + a_1(x-a_1\cdot 10^{k_1}) + \sum_{0\leq n<x-a_1\cdot 10^{k_1}} d_s(n) \\
&= A_1(a_1\cdot 10^{k_1}) + a_1(x-a_1\cdot 10^{k_1}) + A_1(x-a_1\cdot 10^{k_1}) \\
&= A_1(a_1\cdot 10^{k_1}) + A_1(a_2\cdot 10^{k_2}) + a_1(x-a_1\cdot 10^{k_1}) \\
&\quad + a_2(x-a_1\cdot 10^{k_1}-a_2\cdot 10^{k_2}) + A_1(x-a_1\cdot 10^{k_1}-a_2\cdot 10^{k_2}) \\
&= \cdots\cdots\cdots
\end{aligned}
$$

$$= \sum_{i=1}^{s} A_1(a_i \cdot 10^{k_i}) + \sum_{i=1}^{s} a_i(x - \sum_{j=1}^{i} a_j \cdot 10^{k_j})$$

$$= \sum_{i=1}^{s} \left(\frac{9}{2} \cdot k_i + \frac{a_i - 1}{2}\right) \cdot a_i \cdot 10^{k_i} + \sum_{i=2}^{s} a_i \cdot 10^{k_i} \left(\sum_{j=1}^{i-1} a_j\right)$$

$$= \sum_{i=1}^{s} \left(\frac{9}{2} k_i + \sum_{j=1}^{i} a_j - \frac{a_i + 1}{2}\right) \cdot a_i \cdot 10^{k_i}.$$

This proves the first part of the Theorem.

Applying Lemma 2 and the first part of the Theorem repeatedly we have

$$A_2(x) = \sum_{n < a_1 \cdot 10^{k_1}} d_s^2(n) + \sum_{a_1 \cdot 10^{k_1} \le n < x} d_s^2(n)$$

$$= A_2(a_1 \cdot 10^{k_1}) + \sum_{0 \le n < x - a_1 \cdot 10^{k_1}} d_s^2(n + a_1 \cdot 10^{k_1})$$

$$= A_2(a_1 \cdot 10^{k_1}) + \sum_{0 \le n < x - a_1 \cdot 10^{k_1}} (d_s(n) + a_1)^2$$

$$= A_2(a_1 \cdot 10^{k_1}) + \sum_{0 \le n < x - a_1 \cdot 10^{k_1}} (d_s^2(n) + 2a_1 \cdot d_s(n) + a_1^2)$$

$$= A_2(a_1 \cdot 10^{k_1}) + a_1^2 \cdot (x - a_1 \cdot 10^{k_1})$$
$$+ 2a_1 A_1(x - a_1 \cdot 10^{k_1}) + A_2(x - a_1 \cdot 10^{k_1})$$

$$= \cdots\cdots\cdots$$

$$= \sum_{i=1}^{s} A_2(a_i \cdot 10^{k_i}) + \sum_{i=1}^{s} a_i^2(x - \sum_{j=1}^{i} a_j \cdot 10^{k_j}) + \sum_{i=1}^{s} 2a_i A_1(x - \sum_{j=1}^{i} a_j \cdot 10^{k_j})$$

$$= \sum_{i=1}^{s} \left[\frac{k_i(81k_i + 33)}{4} + \frac{9k_i}{2}(a_i - 1) + \frac{(a_i - 1)(2a_i - 1)}{6}\right] \cdot a_i \cdot 10^{k_i}$$

$$+ \sum_{i=2}^{s} a_i \cdot 10^{k_i} \cdot \left(\sum_{j=1}^{i-1} a_j^2\right) + \sum_{i=2}^{s} (9k_i + a_i - 1) \cdot a_i \cdot 10^{k_i} \cdot \left(\sum_{j=1}^{i-1} a_j\right)$$

$$+ 2\sum_{i=2}^{s} \left(\sum_{j=1}^{i-1} a_j\right) \cdot a_i \cdot \left(x - \sum_{j=1}^{i} a_j \cdot 10^{k_j}\right)$$

$$= \sum_{i=1}^{s} \left[\frac{k_i(81k_i + 33)}{4} + \frac{9k_i}{2}(a_i - 1) + \sum_{j=1}^{i} a_j^2 - \frac{(4a_i - 1)(a_i + 1)}{6}\right] \cdot a_i \cdot 10^{k_i}$$

$$+ \sum_{i=2}^{s} a_i \cdot \left[(9k_i - a_i - 1)10^{k_i} + 2\sum_{j=i}^{s} a_j 10^{k_j}\right] \cdot \left(\sum_{j=1}^{i-1} a_j\right).$$

This completes the proof of the second part of the Theorem.

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