NNTDM 9 (2003) 2, 42-45

ON 15-TH SMARANDACHE'S PROBLEM

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Introduction

The 15-th Smarandache's problem from [1] is the following: "**Smarandache's simple numbers:**

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 25, 26, 27, 29, 31, 33, \ldots$$

A number n is called "Smarandache's simple number" if the product of its proper divisors is less than or equal to n. Generally speaking, n has the form $n = p$, or $n = p^2$, or $n = p^3$, or $n = pq$, where p and q are distinct primes".

Let us denote: by S - the sequence of all Smarandache's simple numbers and by s_n - the n-th term of S; by $\mathcal{P}$ - the sequence of all primes and by p_n - the n-th term of $\mathcal{P}$; by $\mathcal{P}^2$ - the sequence $\{p_n^2\}_{n=1}^{\infty}$; by $\mathcal{P}^3$ - the sequence $\{p_n^3\}_{n=1}^{\infty}$; by $\mathcal{PQ}$ - the sequence $\{p.q\}_{p,q\in\mathcal{P}}$, where $p < q$.

For an abitrary increasing sequence of natural numbers $C \equiv \{c_n\}_{n=1}^{\infty}$ we denote by $\pi_C(n)$ the number of terms of C, which are not greater that n. When $n < c_1$ we must put $\pi_C(n) = 0$.

In the present paper we find $\pi_S(n)$ in an explicit form and using this, we find the n-th term of S in explicit form, too.

1. $\pi_S(n)$-representation

First, we must note that instead of $\pi_{\mathcal{P}}(n)$ we shall use the well known denotation $\pi(n)$. Hence

$$\pi_{\mathcal{P}^2}(n) = \pi(\sqrt{n}), \quad \pi_{\mathcal{P}^3}(n) = \pi(\sqrt[3]{n}).$$

Thus, using the definition of S, we get

$$\pi_S(n) = \pi(n) + \pi(\sqrt{n}) + \pi(\sqrt[3]{n}) + \pi_{\mathcal{PQ}}(n). \tag{1}$$

Our first aim is to express $\pi_S(n)$ in an explicit form. For $\pi(n)$ some explicit formulae are proposed in [2]. Other explicit formulae for $\pi(n)$ are contained in [3]. One of them is known as Mináč's formula. It is given below

$$\pi(n) = \sum_{k=2}^{n}[\frac{(k-1)!+1}{k} - [\frac{(k-1)!}{k}]], \tag{2}$$

where $[.]$ denotes the function integer part. Therefore, the question about explicit formulae for functions $\pi(n), \pi(\sqrt{n}), \pi(\sqrt[3]{n})$ is solved successfully. It remains only to express $\pi_{\mathcal{PQ}}(n)$ in an explicit form.

Let $k \in \{1, 2, \ldots, \pi(\sqrt{n})\}$ be fixed. We consider all numbers of the kind $p_k.q$, where $q \in \mathcal{P}, q > p_k$ for which $p_k.q \leq n$. The number of these numbers is $\pi(\frac{n}{p_k}) - \pi(p_k)$, or which is the same

$$\pi(\frac{n}{p_k}) - k. \tag{3}$$

When $k = 1, 2, \ldots, \pi(\sqrt{n})$, numbers $p_k.q$, that were defined above, describe all numbers of the kind $p.q$, where $p, q \in \mathcal{P}, p < q, p.q \leq n$. But the number of the last numbers is equal to $\pi_{\mathcal{PQ}}(n)$. Hence

$$\pi_{\mathcal{PQ}}(n) = \sum_{k=1}^{\pi(\sqrt{n})}(\pi(\frac{n}{p_k}) - k), \tag{4}$$

because of (3). The equality (4), after a simple computation yields the formula

$$\pi_{\mathcal{PQ}}(n) = \sum_{k=1}^{\pi(\sqrt{n})}\pi(\frac{n}{p_k}) - \frac{\pi(\sqrt{n}).(\pi(\sqrt{n})+1)}{2}. \tag{5}$$

In [4] the identity

$$\sum_{k=1}^{\pi(b)}\pi(\frac{n}{p_k}) = \pi(\frac{n}{b}).\pi(b) + \sum_{k=1}^{\pi(\frac{n}{2})-\pi(\frac{n}{b})}\pi(\frac{n}{p_{\pi(\frac{n}{b})+k}}) \tag{6}$$

is proved, under the condition $b \geq 2$ (b is a real number). When $\pi(\frac{n}{2}) = \pi(\frac{n}{b})$, the right hand-side of (6) reduces to $\pi(\frac{n}{b}).\pi(b)$. In the case $b = \sqrt{n}$ and $n \geq 4$ equality (6) yields

$$\sum_{k=1}^{\pi(\sqrt{n})}\pi(\frac{n}{p_k}) = (\pi(\sqrt{n}))^2 + \sum_{k=1}^{\pi(\frac{n}{2})-\pi(\sqrt{n})}\pi(\frac{n}{p_{\pi(\sqrt{n})+k}}). \tag{7}$$

If we compare (5) with (7) we obtain for $n \geq 4$

$$\pi_{\mathcal{PQ}}(n) = \frac{\pi(\sqrt{n}).(\pi(\sqrt{n})-1)}{2} + \sum_{k=1}^{\pi(\frac{n}{2})-\pi(\sqrt{n})}\pi(\frac{n}{p_{\pi(\sqrt{n})+k}}). \tag{8}$$

Thus, we have two different explicit representations for $\pi_{\mathcal{PQ}}(n)$. These are formulae (5) and (8). We must note that the right hand-side of (8) reduces to $\frac{\pi(\sqrt{n}).(\pi(\sqrt{n})-1)}{2}$, when $\pi(\frac{n}{2}) = \pi(\sqrt{n})$.

Finally, we observe that (1) gives an explicit representation for $\pi_S(n)$, since we may use formula (2) for $\pi(n)$ (or other explicit formulae for $\pi(n)$) and (5), or (8) for $\pi_{\mathcal{PQ}}(n)$.

2. Explicit formulae for s_n

The following assertion decides the question about explicit representation of s_n.

Theorem: The n-th term s_n of S admits the following three different explicit representations:

$$s_n = \sum_{k=0}^{\theta(n)} [\frac{1}{1+[\frac{\pi_S(k)}{n}]}]; \tag{9}$$

$$s_n = -2\sum_{k=0}^{\theta(n)} \theta(-2[\frac{\pi_S(k)}{n}]); \tag{10}$$

$$s_n = \sum_{k=0}^{\theta(n)} \frac{1}{\Gamma(1-[\frac{\pi_S(k)}{n}])}, \tag{11}$$

where

$$\theta(n) \equiv [\frac{n^2+3n+4}{4}], \ n = 1,2,..., \tag{12}$$

ζ is Riemann's function zeta and Γ is Euler's function gamma.

Remark. We must note that in (9)-(11) $\pi_S(k)$ is given by (1), $\pi(k)$ is given by (2) (or by others formulae like (2)) and $\pi_{FQ}(n)$ is given by (5), or by (8). Therefore, formulae (9)-(11) are explicit.

Proof of the Theorem. In [2] the following three universal formulae are proposed, using $\pi_C(k)$ $(k = 0, 1, ...)$, which one could apply to represent c_n. They are the following

$$c_n = \sum_{k=0}^{\infty} [\frac{1}{1+[\frac{\pi_C(k)}{n}]}]; \tag{13}$$

$$c_n = -2\sum_{k=0}^{\infty} \zeta(-2[\frac{\pi_C(k)}{n}]); \tag{14}$$

$$c_n = \sum_{k=0}^{\infty} \frac{1}{\Gamma(1-[\frac{\pi_C(k)}{n}])}. \tag{15}$$

In [5] is shown that the inequality

$$p_n \le \theta(n), \ n = 1, 2, ..., \tag{16}$$

holds. Hence

$$s_n = \theta(n), n = 1, 2, ..., \tag{17}$$

since we have obviously

$$s_n \le p_n, \ n = 1, 2, ... \tag{18}$$

Then to prove the Theorem it remains only to apply (13)-(15) in the case $C = S$, i.e., for $c_n = s_n$, putting there $\pi_S(k)$ instead of $\pi_C(k)$ and $\theta(n)$ instead of ∞.

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