Scientia Magna
Vol. 2 (2006), No. 3, 54-64

Perfect powers in Smarandache n-Expressions

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Abstract In [1] I studied the concept of Smarandache n-expressions, for example I proposed formulas, found solutions, proposed open questions, and conjectured, but all for the fixed 3, and 2 numbers, but what will happen if these equations have different fixed numbers such as 7? This paper will answer this question.

Keywords Perfect powers in Smarandache N-Expressions, perfect squares, identity.

§1. Introduction

In [4] M. Perez and E. Burton, documented that J. Castillo [5], asked how many primes are there in the Smarandache n-expressions:

$$x_1^{x_2} + x_2^{x_3} + \cdots + x_n^{x_1},$$

where $n > 1$, $x_1, x_2, \cdots, x_n > 1$, and $(x_1, x_2, \cdots x_n) = 1$.

In this paper, with only slight modification of the above equation we got the following equation namely:

$$a^{x_1} + a^{x_2} + \cdots + a^{x_n},$$

where $a > 1$, $x_1, x_2, \cdots, x_n > 1$, and $(a, x_1, x_2, \cdots, x_n) = 1$.

§2.Main results and proofs

I will study the following cases of above equation.

Case 1. The solution of equation (1) is given by

$$7^p + 7^q + 7^r + 7^s = k^2, \tag{1}$$

where $p = 2m$, $q = 2m + 1$, $r = 2m + 2$, $s = 2m + 3$, and $k = 20 \cdot 7^m$.

Proof. Assume $k = 20 \cdot 7^m$, then $k^2 = 400 \cdot 7^2m$, i.e.

$$k^2 = 400 \cdot 7^{2m} = (1 + 7 + 49 + 343)7^{2m} = 7^{2m} + 7^{2m+1} + 7^{2m+2} + 7^{2m+3}.$$

Hence $p = 2m$, $q = 2m + 1$, $r = 2m + 2$, and $s = 2m + 3$.

The first 11th solution of (1) is given in Table 1 below:

Table 1

m	$7^p + 7^q + 7^r + 7^s$	k^2
0	$7^0 + 7^1 + 7^2 + 7^3$	20^2
1	$7^2 + 7^3 + 7^4 + 7^5$	140^2
2	$7^4 + 7^5 + 7^6 + 7^7$	980^2
3	$7^6 + 7^7 + 7^8 + 7^9$	6860^2
4	$7^8 + 7^9 + 7^{10} + 7^{11}$	48020^2
5	$7^{10} + 7^{11} + 7^{12} + 7^{13}$	336140^2
6	$7^{12} + 7^{13} + 7^{14} + 7^{15}$	2352980^2
7	$7^{14} + 7^{15} + 7^{16} + 7^{17}$	16470860^2
8	$7^{16} + 7^{17} + 7^{18} + 7^{19}$	115296020^2
9	$7^{18} + 7^{19} + 7^{20} + 7^{21}$	807072140^2
10	$7^{20} + 7^{21} + 7^{22} + 7^{23}$	564950498^2

The first terms and the m-th terms of the sequence (the last column on Table 1 are:)

$$400, \quad 9600, \quad 960400, \quad 4705900, \quad \cdots, 20 \cdot 2 \cdot 7^{2m}, \cdots. \tag{2}$$

$$20, \quad 140, \quad 980, \quad 68600, \quad \cdots, 20 \cdot 7^{2m}, \cdots \tag{3}$$

I have noticed there is no prime numbers in geometric series (3), (excluding the prime 7). The

$$\sum_{i}^{m} 20 \cdot 7^{2m} = \frac{10(7^m - 1)}{3},$$

and there is no limit, since $\frac{10(7^m-1)}{3}$ becomes large as m approach infinity. The sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.

Equation (1) has the following important properties, i.e.

$$\begin{aligned} 7^p + 7^q + 7^r + 7^s &= a^2 - b^2 = c^2 - d^2 = e^2 - f^2 = g^2 - h^2 = i^2 - k^2 \\ &= j^2 - l^2 = o^2 - t^2 = u^2 - v^2 = w^2 - x^2. \end{aligned} \tag{4}$$

The solution of (4) are

$$a = 25 \cdot 7^{n-1}, \quad b = 15 \cdot 7^{n-1}, \quad c = 100 \cdot 7^{2n-2} + 1,$$

$$d = 100 \cdot 7^{2n-2} - 1, \quad e = 50 \cdot 7^{2n-2} + 2, \quad f = 50 \cdot 7^{2n-2} - 2,$$

$$g = 4 \cdot 7^{2n-2} + 25, \quad h = 4 \cdot 7^{2n-2} - 25, \quad i = 10 \cdot 7^{2n-1} + 10,$$

$$k = 10 \cdot 7^{2n-1} - 10, \quad j = 50 \cdot 7^{2n-1} + 20, \quad l = 50 \cdot 7^{2n-1} - 20,$$

$$o = 7^{2n-1} + 100, \quad t = 7^{2n-1} - 100, \quad u = 2 \cdot 7^{2n-1} + 50,$$

$$v = 2 \cdot 7^{2n-1} - 50, \quad w = 25 \cdot 7^{2n-1} + 4, \quad x = 25 \cdot 7^{2n-1} - 4.$$

For example, let $n = 2$ ($m = 1$), then we always have $n = m + 1$.

$$\begin{aligned} 175^2 - 105^2 &= 4901^2 - 4899^2 = 2452^2 - 2448^2 = 221^2 - 171^2 = 500^2 - 480^2 \\ &= 265^2 - 225^2 = 149^2 - 51^2 = 148^2 - 48^2 = 1229^2 - 1221^2 \\ &= 140^2 = 7^2 + 7^3 + 7^4 + 7^5. \end{aligned}$$

Conjecture 1. If p, q, r, s are distinct prime numbers, then the equation

$$7^p + 7^q + 7^r + 7^s = k^2,$$

will has no solution, (otherwise we have solutions in prime numbers, such as

$$7^2 + 7^2 + 7^2 + 7^2 = 14^2,$$

and

$$7^2 + 7^2 + 7^3 + 7^3 = 28^2.)$$

Case 2. The solution of equation (5) is given by

$$7^p + 7^q + 7^r + 7^s = k^2, \tag{5}$$

where $p = q = 2m$, $r = s = 2m + 1$, and $k = 4 \cdot 7^m$.

Proof. Assume $k = 4 \cdot 7^m$, then $k^2 = 16 \cdot 7^2 m$, i.e.

$$k^2 = 16 \cdot 7^{2m} = (1 + 1 + 7 + 7)7^{2m} = 7^{2m} + 7^{2m} + 7^{2m+1} + 7^{2m+1}.$$

Hence $p = q = 2m$, $r = s = 2m + 1$.

The first 11th solution of (5) is given in Table 2 below:

Table 2

m	$7^p + 7^q + 7^r + 7^s$	k^2
0	$7^0 + 7^0 + 7^1 + 7^1$	4^2
1	$7^2 + 7^2 + 7^3 + 7^3$	28^2
2	$7^4 + 7^4 + 7^5 + 7^5$	196^2
3	$7^6 + 7^6 + 7^7 + 7^7$	1372^2
4	$7^8 + 7^8 + 7^9 + 7^9$	9604^2
5	$7^{10} + 7^{10} + 7^{11} + 7^{11}$	67228^2
6	$7^{12} + 7^{12} + 7^{13} + 7^{13}$	470596^2
7	$7^{14} + 7^{14} + 7^{15} + 7^{15}$	3294172^2
8	$7^{16} + 7^{16} + 7^{17} + 7^{17}$	23059204^2
9	$7^{18} + 7^{18} + 7^{19} + 7^{19}$	161414428^2
10	$7^{20} + 7^{20} + 7^{21} + 7^{21}$	1129900996^2

Equation (5) has the following important properties, i.e.

$$7^p + 7^q + 7^r + 7^s = a^2 - b^2 = c^2 - d^2 = e^2 - f^2 = g^2 - h^2, \qquad (6)$$

where the solution of (6) are

$$a = 5 \cdot 7^{n-1}, \quad b = 3 \cdot 7^n, \quad c = 4 \cdot 7^{2n-2} + 1, \quad d = 4 \cdot 7^{2n-2} - 1,$$

$$e = 2 \cdot 7^{2n-2} + 2, \; f = 2 \cdot 7^{2n-2} - 2, \; g = 7^{2n-2} + 4, \; h = 7^{2n-2} - 4.$$

For example, let $n = 2$ $(m = 1)$, then we have

$$\begin{aligned} 35^2 - 212 &= 197^2 - 195^2 = 100^2 - 96^2 = 53^2 - 45^2 \\ &= 28^2 = 7^2 + 7^2 + 7^3 + 7^3. \end{aligned}$$

Case 3. The solution of equation (7) is given by

$$7^p + 7^q + 7^r + 7^s = k^2, \qquad (7)$$

where $p = q = 2m$, $r = s = 2m + 2$, and $k = 10 \cdot 7^m$.

Proof. Assume $k = 10 \cdot 7^m$, then $k^2 = 100 \cdot 7^2 m$, i.e.

$$k^2 = 100 \cdot 7^{2m} = (1 + 1 + 49 + 49)7^{2m} = 7^{2m} + 7^{2m} + 7^{2m+2} + 7^{2m+2}.$$

Hence $p = q = 2m$, $r = s = 2m + 2$.

The first 11th solution of (7) is given in Table 3 below:

Table 3

m	$7^p + 7^q + 7^r + 7^s$	k^2
0	$7^0 + 7^0 + 7^2 + 7^2$	10^2
1	$7^2 + 7^2 + 7^4 + 7^4$	70^2
2	$7^4 + 7^4 + 7^6 + 7^6$	490^2
3	$7^6 + 7^6 + 7^8 + 7^8$	3430^2
4	$7^8 + 7^8 + 7^{10} + 7^{10}$	24010^2
5	$7^{10} + 7^{10} + 7^{12} + 7^{12}$	168070^2
6	$7^{12} + 7^{12} + 7^{14} + 7^{14}$	1176490^2
7	$7^{14} + 7^{14} + 7^{16} + 7^{16}$	8235430^2
8	$7^{16} + 7^{16} + 7^{18} + 7^{18}$	57648010^2
9	$7^{18} + 7^{18} + 7^{20} + 7^{20}$	403536070^2
10	$7^{20} + 7^{20} + 7^{22} + 7^{22}$	2824752490^2

Equation (7) has the following important properties, i.e.

$$7^p + 7^q + 7^r + 7^s = k^2 = a^2 + b^2, \qquad (8)$$

where the solution of (8) are

$$a = 8 \cdot 7^{n-1}, \quad b = 6 \cdot 7^{n-1}.$$

For example, let $n = 2$ ($m = 1$), then we have

$$7^2 + 7^2 + 7^4 + 7^4 = 56^2 + 42^2 = 70^2.$$

Case 4. The solution of equation (9) is given by

$$7^p + 7^q + 7^r + 7^s = k^2, \tag{9}$$

where $p = q = r = s = 2m$, and $k = 2 \cdot 7^m$.

Proof. Assume $k = 2 \cdot 7^m$, then $k^2 = 4 \cdot 7^2 m$, i.e.

$$k^2 = 4 \cdot 7^{2m} = (1 + 1 + 1 + 1)7^{2m} = 7^{2m} + 7^{2m} + 7^{2m+2} + 7^{2m+2}.$$

Hence $p = q = r = s = 2m$.

The first 11th solution of (9) is given in Table 4 below:

Table 4

m	$7^p + 7^q + 7^r + 7^s$	k^2
0	$7^0 + 7^0 + 7^0 + 7^0$	2^2
1	$7^2 + 7^2 + 7^2 + 7^2$	14^2
2	$7^4 + 7^4 + 7^4 + 7^4$	98^2
3	$7^6 + 7^6 + 7^6 + 7^6$	686^2
4	$7^8 + 7^8 + 7^8 + 7^8$	4802^2
5	$7^{10} + 7^{10} + 7^{10} + 7^{10}$	33614^2
6	$7^{12} + 7^{12} + 7^{12} + 7^{12}$	235298^2
7	$7^{14} + 7^{14} + 7^{14} + 7^{14}$	1647068^2
8	$7^{16} + 7^{16} + 7^{16} + 7^{16}$	11529602^2
9	$7^{18} + 7^{18} + 7^{18} + 7^{18}$	80707214^2
10	$7^{20} + 7^{20} + 7^{20} + 7^{20}$	564950498^2

Equation (7) has the following important properties, i.e.

$$7^p + 7^q + 7^r + 7^s = k_i^2 + k_{i+1}^2 = a^2 + a^2. \tag{10}$$

where $k_i = 2 \cdot 7^m$ and $a = 10 \cdot 7^m$.

Examples of equation (10):

1) $2^2 + 4^2 = 10^2 + 10^2$, (divided both sides by 2 given $1^2 + 7^2 = 5^2 + 5^2$)

2) $14^2 + 98^2 = 70^2 + 70^2$, (divided both sides by 14 given $1^2 + 7^2 = 5^2 + 5^2$)

3) $98^2 + 686^2 = 490^2 + 490^2$. (divided both sides by 98 given $1^2 + 7^2 = 5^2 + 5^2$)

These examples suggested a formula that gives three perfect squares which are in the arithmetic progression. So for the positive m and n with $m > n$, put

$$x = 2mn - m^2 + n^2,$$

$$y = m^2 + n^2,$$

$$z = 2mn + m^2 - n^2,$$

such as

$$y^2 + y^2 = z^2 + x^2.$$

So if $m = 2$, $n = 1$, then we will have

$$1^2 + 7^2 = 5^2 + 5^2.$$

Case 5. The solution of equation (11) is given by

$$7^p + 7^q + 7^r + 7^s = 2^4 \cdot 5^2 \cdot 7^p, \tag{11}$$

where $p = 2m + 1$, $q = 2m + 2$, $r = 2m + 3$, and $s = 2m + 4$.

Proof. Assume $2^4 \cdot 5^2 \cdot 7^{2m+1}$, is the sum of equation (11), then

$$2^4 \cdot 5^2 \cdot 7^{2m+1} = (1 + 7 + 49 + 343)7^{2m+1} = 7^{2m+1} + 7^{2m+2} + 7^{2m+3} + 7^{2m+4}.$$

Hence $p = 2m + 1$, $q = 2m + 2$, $r = 2m + 3$, $s = 2m + 4$.

The first 7th solution is given in Table 5 below:

Table 5

m	$7^p + 7^q + 7^r + 7^s$	$2^4 \cdot 5^2 \cdot 7^p$
0	$7^1 + 7^2 + 7^3 + 7^4$	$2^4 \cdot 5^2 \cdot 7^1$
1	$7^3 + 7^4 + 7^5 + 7^6$	$2^4 \cdot 5^2 \cdot 7^3$
2	$7^5 + 7^6 + 7^7 + 7^8$	$2^4 \cdot 5^2 \cdot 7^5$
3	$7^7 + 7^8 + 7^9 + 7^{10}$	$2^4 \cdot 5^2 \cdot 7^7$
4	$7^9 + 7^{10} + 7^{11} + 7^{12}$	$2^4 \cdot 5^2 \cdot 7^9$
5	$7^{11} + 7^{12} + 7^{13} + 7^{14}$	$2^4 \cdot 5^2 \cdot 7^{11}$
6	$7^{13} + 7^{14} + 7^{15} + 7^{16}$	$2^4 \cdot 5^2 \cdot 7^{13}$

The first terms and the m-th terms of the sequence (the last column on Table 5 are:)

$$2^4 \cdot 5^2 \cdot 7^1, \quad 2^4 \cdot 5^2 \cdot 7^3, \quad 2^4 \cdot 5^2 \cdot 7^5, \quad \cdots, \quad 2^4 \cdot 5^2 \cdot 7^{2m+1}, \cdots, \tag{12}$$

where the square roots are

$$20 \cdot 7^{1/2}, \quad 20 \cdot 7^{3/2}, \quad 20 \cdot 7^{5/2}, \quad 20 \cdot 7^{7/2}, \quad \cdots, \quad 20 \cdot 7^{2m+1/2}, \cdots. \tag{13}$$

Notice that there is no prime numbers in geometric series (13).

The

$$\sum_{i}^{m} 20 \cdot 7^{2i+1/2} = \frac{10 \cdot 7^{1/2} \cdot 7^{2m+1} - 1}{3},$$

and there is no limit, since $\frac{10 \cdot 7^{1/2} . 7^{2m+1} - 1}{3}$ becomes large as m approach infinity. The sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.

Conjecture 2. If p, q, r, s are distinct prime numbers, then the equation

$$7^p + 7^q + 7^r + 7^s = 2^4 \cdot 5^2 \cdot 7^p,$$

has no solution.

Case 6. The solution of equation (11) is given by

$$7^p + 7^q + 7^r + 7^s = 2^4 \cdot 7^p, \tag{14}$$

where $p = q = 2m+1$, $r = s = 2m+2$.

Proof. Assume $2^4 \cdot 7^{2m+1}$ is the sum of equation (14), then

$$2^4 \cdot 7^{2m+1} = (1+1+7+7)7^{2m+1} = 7^{2m+1} + 7^{2m+1} + 7^{2m+2} + 7^{2m+2}.$$

Hence $p = q = 2m+1$, $r = s = 2m+2$.

The first 11th solution is given in Table 6 below:

Table 6

m	$7^p + 7^q + 7^r + 7^s$	$2^4 \cdot 7^p$
0	$7^1 + 7^1 + 7^2 + 7^2$	$2^4 \cdot 7^1$
1	$7^3 + 7^3 + 7^4 + 7^4$	$2^4 \cdot 7^3$
2	$7^5 + 7^5 + 7^6 + 7^6$	$2^4 \cdot 7^5$
3	$7^7 + 7^7 + 7^8 + 7^8$	$2^4 \cdot 7^7$
4	$7^9 + 7^9 + 7^{10} + 7^{10}$	$2^4 \cdot 7^9$
5	$7^{11} + 7^{11} + 7^{12} + 7^{12}$	$2^4 \cdot 7^{11}$
6	$7^{13} + 7^{13} + 7^{14} + 7^{14}$	$2^4 \cdot 7^{13}$
7	$7^{15} + 7^{15} + 7^{16} + 7^{16}$	$2^4 \cdot 7^{15}$
8	$7^{17} + 7^{17} + 7^{18} + 7^{18}$	$2^4 \cdot 7^{17}$
9	$7^{19} + 7^{19} + 7^{20} + 7^{20}$	$2^4 \cdot 7^{19}$
10	$7^{21} + 7^{21} + 7^{22} + 7^{22}$	$2^4 \cdot 7^{21}$

If we look deeply in equation (11), (14), we can find the following relation

$$20^4 - 2^4 = 97^2 - 95^2 = 50^2 - 46^2 = 28^2 - 20^2.$$

Case 7. The solution of equation (15) is given by

$$7^p + 7^q + 7^r + 7^s = 2^2 \cdot 5^2 \cdot 7^p, \tag{15}$$

where $p = q = 2m+1$, $r = s = 2m+3$.

Proof. Assume $2^2 \cdot 5^2 \cdot 7^{2m+1}$, is the sum of equation (15), then

$$2^2 \cdot 5^2 \cdot 7^{2m+1} = (1+1+49+49)7^{2m+1} = 7^{2m+1} + 7^{2m+1} + 7^{2m+3} + 7^{2m+3}.$$

Hence $p = q = 2m+1$, $r = s = 2m+2$.

The first 11th solution of (15) is given in Table 7 below:

Table 7

m	$7^p + 7^q + 7^r + 7^s$	$2^2 \cdot 5^2 \cdot 7^p$
0	$7^1 + 7^1 + 7^3 + 7^3$	$2^2 \cdot 5^2 \cdot 7^1$
1	$7^3 + 7^3 + 7^5 + 7^5$	$2^2 \cdot 5^2 \cdot 7^3$
2	$7^5 + 7^5 + 7^7 + 7^7$	$2^2 \cdot 5^2 \cdot 7^5$
3	$7^7 + 7^7 + 7^9 + 7^9$	$2^2 \cdot 5^2 \cdot 7^7$
4	$7^9 + 7^9 + 7^{11} + 7^{11}$	$2^2 \cdot 5^2 \cdot 7^9$
5	$7^{11} + 7^{11} + 7^{13} + 7^{13}$	$2^2 \cdot 5^2 \cdot 7^{11}$
6	$7^{13} + 7^{13} + 7^{15} + 7^{15}$	$2^2 \cdot 5^2 \cdot 7^{13}$
7	$7^{15} + 7^{15} + 7^{17} + 7^{17}$	$2^2 \cdot 5^2 \cdot 7^{15}$
8	$7^{17} + 7^{17} + 7^{19} + 7^{19}$	$2^2 \cdot 5^2 \cdot 7^{17}$
9	$7^{19} + 7^{19} + 7^{21} + 7^{21}$	$2^2 \cdot 5^2 \cdot 7^{19}$
10	$7^{21} + 7^{21} + 7^{23} + 7^{23}$	$2^2 \cdot 5^2 \cdot 7^{21}$

case 8. The solution of equation (16) is given by

$$7^p + 7^q + 7^r + 7^s = 1201 \cdot 2^2 \cdot 5^2 \cdot 7^p, \tag{16}$$

where $p = 2m$, $q = 2m + 2$, $r = 2m + 4$, $s = 2m + 6$.

Proof. Assume $1201 \cdot 2^2 \cdot 5^2 \cdot 7^{2m+1}$, is the sum of equation (16), then

$$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{2m+1} = (1 + 49 + 2401 + 117649)7^{2m} = 7^{2m} + 7^{2m+2} + 7^{2m+4} + 7^{2m+6}.$$

Hence $p = 2m$, $q = 2m + 2$, $r = 2m + 4$, $s = 2m + 6$.

The first 11th solution of (16) is given in Table 8 below:

Table 8

m	$7^p + 7^q + 7^r + 7^s$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^p$
0	$7^0 + 7^2 + 7^4 + 7^6$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^0$
1	$7^2 + 7^4 + 7^6 + 7^8$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^2$
2	$7^4 + 7^6 + 7^8 + 7^{10}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^4$
3	$7^6 + 7^8 + 7^{10} + 7^{12}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^6$
4	$7^8 + 7^{10} + 7^{12} + 7^{14}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^8$
5	$7^{10} + 7^{12} + 7^{14} + 7^{16}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{10}$
6	$7^{12} + 7^{14} + 7^{16} + 7^{18}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{12}$
7	$7^{14} + 7^{16} + 7^{18} + 7^{20}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{14}$
8	$7^{16} + 7^{18} + 7^{20} + 7^{22}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{16}$
9	$7^{18} + 7^{20} + 7^{22} + 7^{24}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{18}$
10	$7^{20} + 7^{22} + 7^{24} + 7^{26}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{20}$

Case 9. The solution of equation (17) is given by

$$7^p + 7^q + 7^r + 7^s = 1201 \cdot 2^2 \cdot 5^2 \cdot 7^p, \tag{17}$$

where $p = 2m + 1$, $q = 2m + 3$, $r = 2m + 5$, $s = 2m + 7$.

Proof. Assume $1201 \cdot 2^2 \cdot 5^2 \cdot 7^{2m+1}$, is the sum of equation (17), then

$$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{2m+1} = (1 + 49 + 2401 + 117649)7^{2m+1} = 7^{2m+1} + 7^{2m+3} + 7^{2m+5} + 7^{2m+7}.$$

Hence $p = 2m + 1$, $q = 2m + 3$, $r = 2m + 5$, $s = 2m + 7$.

The first 11th solution of (17) is given in Table 9 below:

Table 9

m	$7^p + 7^q + 7^r + 7^s$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^p$
0	$7^1 + 7^3 + 7^5 + 7^7$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^1$
1	$7^3 + 7^5 + 7^7 + 7^9$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^3$
2	$7^5 + 7^7 + 7^9 + 7^{11}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^5$
3	$7^7 + 7^9 + 7^{11} + 7^{13}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^7$
4	$7^9 + 7^{11} + 7^{13} + 7^{15}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^9$
5	$7^{11} + 7^{13} + 7^{15} + 7^{17}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{11}$
6	$7^{13} + 7^{15} + 7^{17} + 7^{19}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{13}$
7	$7^{15} + 7^{17} + 7^{19} + 7^{21}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{15}$
8	$7^{17} + 7^{19} + 7^{21} + 7^{23}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{17}$
9	$7^{19} + 7^{21} + 7^{23} + 7^{25}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{19}$
10	$7^{21} + 7^{23} + 7^{25} + 7^{27}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{21}$

Case 10. The solution of equation (18) is given by

$$7^p + 7^q + 7^r + 7^s + 7^t + 7^u + 7^v + 7^w + 7^z = 1201 \cdot 2^5 \cdot 5^2 \cdot 7^p, \tag{18}$$

where $p = m$, $q = m + 1$, $r = m + 2$, $s = m + 3$, $t = m + 4$, $u = m + 5$, $w = m + 6$, $z = m + 7$.

Proof. Assume $1201 \cdot 2^2 \cdot 5^2 \cdot 7^{2m+1}$, is the sum of equation (18), then

$$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{2m+1} = (1 + 49 + 2401 + 117649)7^{2m+1} = 7^{2m+1} + 7^{2m+3} + 7^{2m+5} + 7^{2m+7}.$$

Hence $p = m$, $q = m + 1$, $r = m + 2$, $s = m + 3$, $t = m + 4$, $u = m + 5$, $w = m + 6$, $z = m + 7$.

The first 11th solution of (18) is given in Table 10 below:

Table 10

m	$7^p + 7^q + 7^r + 7^s + 7^t + 7^u + 7^w + 7^z$	$1201 \cdot 2^5 \cdot 5^2 \cdot 7^p$
0	$7^0 + 7^1 + 7^2 + 7^3 + 7^4 + 7^5 + 7^6 + 7^7$	$1201 \cdot 2^5 \cdot 5^2 \cdot 7^0$
1	$7^1 + 7^2 + 7^3 + 7^4 + 7^5 + 7^6 + 7^7 + 7^8$	$1201 \cdot 2^5 \cdot 5^2 \cdot 7^1$
2	$7^2 + 7^3 + 7^4 + 7^5 + 7^6 + 7^7 + 7^8 + 7^9$	$1201 \cdot 2^5 \cdot 5^2 \cdot 7^2$
3	$7^3 + 7^4 + 7^5 + 7^6 + 7^7 + 7^8 + 7^9 + 7^{10}$	$1201 \cdot 2^5 \cdot 5^2 \cdot 7^3$
4	$7^4 + 7^5 + 7^6 + 7^7 + 7^8 + 7^9 + 7^{10} + 7^{11}$	$1201 \cdot 2^5 \cdot 5^2 \cdot 7^4$
5	$7^5 + 7^6 + 7^7 + 7^8 + 7^9 + 7^{10} + 7^{11} + 7^{12}$	$1201 \cdot 2^5 \cdot 5^2 \cdot 7^5$
6	$7^6 + 7^7 + 7^8 + 7^9 + 7^{10} + 7^{11} + 7^{12} + 7^{13}$	$1201 \cdot 2^5 \cdot 5^2 \cdot 7^6$
7	$7^7 + 7^8 + 7^9 + 7^{10} + 7^{11} + 7^{12} + 7^{13} + 7^{14}$	$1201 \cdot 2^5 \cdot 5^2 \cdot 7^7$
8	$7^8 + 7^9 + 7^{10} + 7^{11} + 7^{12} + 7^{13} + 7^{14} + 7^{15}$	$1201 \cdot 2^5 \cdot 5^2 \cdot 7^8$
9	$7^9 + 7^{10} + 7^{11} + 7^{12} + 7^{13} + 7^{14} + 7^{15} + 7^{16}$	$1201 \cdot 2^5 \cdot 5^2 \cdot 7^9$
10	$7^{10} + 7^{11} + 7^{12} + 7^{13} + 7^{14} + 7^{15} + 7^{16} + 7^{17}$	$1201 \cdot 2^5 \cdot 5^2 \cdot 7^{10}$

Case 11. The solution of equation (19) is given by

$$7^p + 7^q + 7^r + 7^s + 7^t + 7^u + 7^v + 7^w + 7^z = k^3, \tag{19}$$

where $p = q = r = s = t = u = w = z = 3m$.

Proof. Assume $k = 2 \cdot 7^m$, then $k^3 = 2^3 \cdot 7^{3m}$, i.e.

$$\begin{aligned} k^2 &= 8 \cdot 7^{3m} = (1+1+1+1+1+1+1+1)7^{3m} \\ &= 7^{3m} + 7^{3m} + 7^{3m} + 7^{3m} + 7^{3m} + 7^{3m} + 7^{3m} + 7^{3m}. \end{aligned}$$

Hence $p = q = r = s = t = u = w = z = 3m$.

The first 11th solution of (19) is given in Table 11 below:

Table 11

m	$7^p + 7^q + 7^r + 7^s + 7^t + 7^u + 7^w + 7^z$	$k^3 = 2^3 \cdot 7^{3m}$
0	$7^0 + 7^0 + 7^0 + 7^0 + 7^0 + 7^0 + 7^0 + 7^0$	2^3
1	$7^3 + 7^3 + 7^3 + 7^3 + 7^3 + 7^3 + 7^3 + 7^3$	14^3
2	$7^6 + 7^6 + 7^6 + 7^6 + 7^6 + 7^6 + 7^6 + 7^6$	98^3
3	$7^9 + 7^9 + 7^9 + 7^9 + 7^9 + 7^9 + 7^9 + 7^9$	686^3
4	$7^{12} + 7^{12} + 7^{12} + 7^{12} + 7^{12} + 7^{12} + 7^{12} + 7^{12}$	4802^3
5	$7^{15} + 7^{15} + 7^{15} + 7^{15} + 7^{15} + 7^{15} + 7^{15} + 7^{15}$	33614^3
6	$7^{18} + 7^{18} + 7^{18} + 7^{18} + 7^{18} + 7^{18} + 7^{18} + 7^{18}$	235298^3
7	$7^{21} + 7^{21} + 7^{21} + 7^{21} + 7^{21} + 7^{21} + 7^{21} + 7^{21}$	1647068^3
8	$7^{24} + 7^{24} + 7^{24} + 7^{24} + 7^{24} + 7^{24} + 7^{24} + 7^{24}$	11529602^3
9	$7^{27} + 7^{27} + 7^{27} + 7^{27} + 7^{27} + 7^{27} + 7^{27} + 7^{27}$	80707214^3
10	$7^{30} + 7^{30} + 7^{30} + 7^{30} + 7^{30} + 7^{30} + 7^{30} + 7^{30}$	564950498^3

I find interesting identity like this:

1) $14^3 - 2^3 = 685^2 - 683^2 = 344^2 - 340^2 = 175^2 - 167^2$,

2) $98^3 - 14^3 = 234613^2 - 234611^2 = 117308^2 - 117304^2 = 58657^2 - 58649^2$, and so on.

§3. Open questions

The following equations have infinitely many solutions, find them.

1) $-7^p + 7^q - 7^r + 7^s = 3 \cdot 2^2 \cdot 5^2 \cdot 7^p$ (has two different solutions),

2) $7^p + 7^q = 2^3 \cdot 7^p$,

3) $7^p + 7^q = 2 \cdot 5^2 \cdot 7^p$.

References

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