ON M-TH POWER FREE PART OF AN INTEGER

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Abstract In this paper, we using the elementary method to study the convergent property of one class Dirichlet series involving a special sequences, and give an interesting identity for it.

Keywords: m-th power free part; Infinity series; Identity.

§1. Introduction and results

For any positive integer n and $m \geq 2$, we define $C_m(n)$ as the m-th power free part of n. That is,

$$C_m(n) = \min\{n/d^m : d^m \mid n, \quad d \in N\}.$$

If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ is the prime powers decomposition of n, then we have: $C_m(n_1^m n_2) = C_m(n_2)$, and

$$C_m(n) = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}, \quad if \quad \alpha_i \leq m-1.$$

Now for any positive integer k, we also define arithmetic function $\delta_k(n)$ as follows:

$$\delta_k(n) = \begin{cases} \max\{d \in N \quad | \quad d|n, (d,k) = 1\}, & if \quad n \neq 0, \\ 0, & if \quad n = 0. \end{cases}$$

Let $\mathcal{A}$ denotes the set of all positive integers n satisfy the equation $C_m(n) = \delta_k(n)$. That is, $\mathcal{A} = \{n \in N, C_m(n) = \delta_k(n)\}$. In this paper, we using the elementary method to study the convergent property of the Dirichlet series involving the set $\mathcal{A}$, and give an interesting identity for it. That is, we shall prove the following conclusion:

Theorem. *Let $m \geq 2$ be a fixed positive integer. Then for any real number $s > 1$, we have the identity*

$$\sum_{\substack{n=1 \\ n \in \mathcal{A}}}^{\infty} \frac{1}{n^s} = \frac{\zeta(s)}{\zeta(ms)} \prod_{p|k} \frac{1 - \frac{1}{p^s}}{(1 - \frac{1}{p^{ms}})^2},$$

where $\zeta(s)$ is the Riemann zeta-function, and $\prod_p$ denotes the product over all primes..

Note that $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$ and $\zeta(6) = \pi^6/945$, from our Theorem we may immediately deduce the following:

Corollary. *Let $\mathcal{B} = \{n \in N, C_2(n) = \delta_k(n)\}$ and $\mathcal{C} = \{n \in N, C_3(n) = \delta_k(n)\}$, then we have the identities:*

$$\sum_{\substack{n=1\\ n\in\mathcal{B}}}^{\infty} \frac{1}{n^2} = \frac{15}{\pi^2} \prod_{p|k} \frac{p^6}{(p^2+1)(p^4-1)}$$

and

$$\sum_{\substack{n=1\\ n\in\mathcal{C}}}^{\infty} \frac{1}{n^2} = \frac{305}{2\pi^4} \prod_{p|k} \frac{p^{10}}{(p^4+p^2+1)(p^6-1)}.$$

§2. Proof of the theorem

In this section, we will complete the proof of the theorem. First, we define the arithmetical function $a(n)$ as follows:

$$a(n) = \begin{cases} 1, & if \quad n \in \mathcal{A}, \\ 0, & otherwise. \end{cases}$$

For any real number $s > 0$, it is clear that

$$\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} < \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is convergent if $s > 1$, thus $\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty} \frac{1}{n^s}$ is also convergent if $s > 1$. Now we find the set $\mathcal{A}$. From the definition of $C_m(n)$ and $\delta_k(n)$ we know that $C_m(n)$ and $\delta_k(n)$ both are multiplicative functions. So in order to find all solutions of the equation $C_m(n) = \delta_k(n)$, we only discuss the case $n = p^\alpha$. If $n = p^\alpha$, $(p, k) = 1$, then the equation $C_m(p^\alpha) = \delta(p^\alpha)$ has solution if and only if $1 \le \alpha \le m-1$. If $n = p^\alpha$, $p \mid k$, then the equation $C_m(p^\alpha) = \delta(p^\alpha)$ have solutions if and only if $m \mid \alpha$. Thus, by the Euler product formula (see [1]), we have

$$\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty} \frac{1}{n^s} = \prod_p \left(1 + \frac{a(p)}{p^s} + \frac{a(p^2)}{p^{2s}} + \cdots + \frac{a(p^{m-1})}{p^{(m-1)s}} + \cdots\right)$$

$$
\begin{aligned}
&= \prod_{p\nmid k}\left(1+\frac{a(p)}{p^s}+\frac{a(p^2)}{p^{2s}}+\cdots+\frac{a(p^{m-1})}{p^{(m-1)s}}\right)\\
&\quad\times\prod_{p|k}\left(1+\frac{a(p)}{p^{ms}}+\frac{a(p^2)}{p^{2ms}}+\frac{a(p^3)}{p^{3ms}}+\cdots\right)\\
&= \prod_{p\nmid k}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots+\frac{1}{p^{(m-1)s}}\right)\\
&\quad\times\prod_{p|k}\left(1+\frac{1}{p^{ms}}+\frac{1}{p^{2ms}}+\frac{1}{p^{3ms}}+\cdots\right)\\
&= \frac{\zeta(s)}{\zeta(ms)}\prod_{p|k}\frac{1-\frac{1}{p^s}}{(1-\frac{1}{p^{ms}})^2},
\end{aligned}
$$

where $\zeta(s)$ is the Riemann zeta-function, and $\prod\limits_{p}$ denotes the product over all primes.

This completes the proof of Theorem.

References

[1] Tom M.Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.