ON THE M-TH POWER RESIDUE OF N *

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Abstract For any positive integer n, let $a_m(n)$ denote the m-th power residue of n. In this paper, we use the elementary method to study the asymptotic properties of $\log(a_m(n!))$, and give an interesting asymptotic formula for it.

Keywords: m-th power residue of n; Chebyshev's function; Asymptotic formula.

§1. Introduction

Let $m > 2$ be a fixed integer. For any positive integer n, we define $a_m(n)$ as the m-th power residue of n (See reference [1]). That is, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ denotes the factorization of n into prime powers, then $a_m(n) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r}$, where $\beta_i = \min(\alpha_i, m-1)$. Let p be a prime, and for any real number $x > 1$, $\theta(x) = \sum\limits_{p \le x} \log p$ denotes the Chebyshev's function of x. In this paper, we will use the elementary methods to study the asymptotic properties of $\log(a_m(n!))$, and give an interesting asymptotic formula for it. That is, we shall prove the following conclusion:

Theorem. *Let $m > 1$ be a fixed positive integer. Then for any positive integer n, we have the asymptotic formula:*

$$\log(a_m(n!)) = n\left(\sum_{a=1}^{m-1} \frac{1}{a}\right) + O\left(n \exp\left(\frac{-A\log^{\frac{3}{5}} n}{(\log\log n)^{\frac{1}{5}}}\right)\right),$$

where A is a fixed positive constant.

§2. Proof of the theorem

Before the proof of Theorem, a lemma will be useful.

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Lemma. *Let p be a prime. Then for any real number $x \geq 2$, we have the asymptotic formula:*

$$\theta(x) = x + O\left(x \exp\left(\frac{-A \log^{\frac{3}{5}} x}{(\log\log x)^{\frac{1}{5}}}\right)\right),$$

where A is a positive constant.

Proof. See reference [2] or [3].

Now we use this Lemma to complete the proof of Theorem. In fact, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ denotes the factorization of n into prime powers. Suppose that $m \ll n$, if $(m-1)p \leq n < mp$, then $p^{m-1} \parallel n!$. From the definition of $a_m(n)$, we can write

$$a_m(n!) = \prod_{\frac{n}{2} < p \leq n} p \prod_{\frac{n}{3} < p \leq \frac{n}{2}} p^2 \cdots \prod_{\frac{n}{m-1} < p \leq \frac{n}{m-2}} p^{m-2} \prod_{p \leq \frac{n}{m-1}} p^{m-1}.$$

By taking the logistic computation in the two sides, we have

$$\begin{aligned}
& \log(a_m(n!)) \\
= & \log\left(\prod_{\frac{n}{2} < p \leq n} p \prod_{\frac{n}{3} < p \leq \frac{n}{2}} p^2 \cdots \prod_{\frac{n}{m-1} < p \leq \frac{n}{m-2}} p^{m-2} \prod_{p \leq \frac{n}{m-1}} p^{m-1}\right) \\
= & \theta(n) - \theta\left(\frac{n}{2}\right) + 2\left(\theta\left(\frac{n}{2}\right) - \theta\left(\frac{n}{3}\right)\right) + \cdots \\
& +(m-2)\left(\theta\left(\frac{n}{m-2}\right) - \theta\left(\frac{n}{m-1}\right)\right) + (m-1)\theta\left(\frac{n}{m-1}\right) \\
= & \theta(n) + \theta\left(\frac{n}{2}\right) + \cdots + \theta\left(\frac{n}{m-1}\right).
\end{aligned}$$

Then, combining Lemma, we can get the asymptotic formula:

$$\begin{aligned}
\log(a_m(n!)) &= n + \frac{n}{2} + \cdots + \frac{n}{m-1} + O\left(n \exp\left(\frac{-A \log^{\frac{3}{5}} n}{(\log\log n)^{\frac{1}{5}}}\right)\right) \\
&= n\left(1 + \frac{1}{2} + \cdots + \frac{1}{m-1}\right) + O\left(n \exp\left(\frac{-A \log^{\frac{3}{5}} n}{(\log\log n)^{\frac{1}{5}}}\right)\right) \\
&= n\left(\sum_{a=1}^{m-1} \frac{1}{a}\right) + O\left(n \exp\left(\frac{-A \log^{\frac{3}{5}} n}{(\log\log n)^{\frac{1}{5}}}\right)\right).
\end{aligned}$$

This completes the proof of Theorem.

References

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