ON THE M-POWER RESIDUES NUMBERS SEQUENCE

Ma Yuankui
Department of Mathematics and Physics, Xi'an Institute of Technology
Xi'an, Shaanxi, P.R.China
beijiguang2008@eyou.com

Zhang Tianping
1. Department of Mathematics, Northwest University
Xi'an, Shaanxi, P.R.China
2. College of Mathematics and Information Science, Shannxi Normal University
Xi'an, Shaanxi, P.R.China
tianpzhang@eyou.com

Abstract The main purpose of this paper is to study the distribution properties of m-power residues numbers, and give two interesting asymptotic formulae.

Keywords: m-power residues numbers; Mean value; Asymptotic formula.

§1. Introduction and results

For any given natural number $m \geq 2$, and any positive integer $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, we call $a_m(n) = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdot \cdots \cdot p_r^{\beta^r}$ a m-power residue number, where $\beta_i = \min(m-1, \alpha_i), 1 \leq i \leq r$. In reference [1], Professor F. Smarandache asked us to study the properties of the m-power residue numbers sequence. Yet we still know very little about it.

Now we define two new number-theoretic functions $U(n)$ and $V(n)$ as following,

$$U(1) = 1, U(n) = \prod_{p|n} p,$$

$$V(1) = 1, V(n) = V\left(p_1^{\alpha_1}\right) \cdots U\left(p_r^{\alpha_r}\right) = (p^{\alpha_1} - 1) \cdots (p^{\alpha_r} - 1),$$

where n is any natural number with the form $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. Obviously they are both multiplicative functions. In this paper, we shall use the analytic method to study the distribution properties of this sequence, and obtain two interesting asymptotic formulae. That is, we have the following two theorems:

Theorem 1. *Let $\mathcal{A}$ denotes the set of all m-power residues numbers, then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in\mathcal{A}\\ n\leq x}} U(n) = \frac{3x^2}{\pi^2}\prod_p\left(1+\frac{1}{p^3+p^2-p-1}\right)+O\left(x^{\frac{3}{2}+\varepsilon}\right),$$

where ε denotes any fixed positive number.

Theorem 2. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in\mathcal{A}\\ n\leq x}} V(n) = \frac{x^2}{2}\prod_p\left(1-\frac{1}{p^m}+\frac{1-p^m}{p^{m+2}+p^{m+1}}\right)+O\left(x^{\frac{3}{2}+\varepsilon}\right).$$

§2. Proof of the theorems

In this section, we shall complete the proof of the theorems. First we prove Theorem 1, let

$$f(s) = \sum_{\substack{n\in\mathcal{A}\\ n\leq x}} \frac{U(n)}{n^s}.$$

From the Euler product formula [2] and the definition of $U(n)$ we have

$$\begin{aligned}
f(s) &= \prod_p\left(1+\frac{U(p)}{p^s}+\frac{U(p^2)}{p^{2s}}+\cdots+\frac{U(p^{m-1})}{p^{(m-1)s}}+\frac{U(p^m)}{p^{ms}}+\frac{U(p^{m+1})}{p^{(m+1)s}}\cdots\right)\\
&= \prod_p\left(1+\frac{1}{p^{s-1}}+\frac{1}{p^{2s-1}}+\cdots+\frac{1}{p^{(m-1)s-1}}+\frac{1}{p^{ms-1}}+\frac{1}{p^{(m+1)s-1}}+\cdots\right)\\
&= \prod_p\left(1+\frac{1}{p^{s-1}}+\frac{1}{p^{2s-1}(1-\frac{1}{p^s})}\right)\\
&= \frac{\zeta(s-1)}{\zeta(2(s-1))}\prod_p\left(1+\frac{p^s}{(p^s-1)(p^{2s-1}+p^s)}\right),
\end{aligned}$$

where $\zeta(s)$ is the Riemann-zeta function. Obviously, we have inequality

$$|U(n)| \leq n, \qquad \left|\sum_{n=1}^{\infty}\frac{U(n)}{n^\sigma}\right| < \frac{1}{\sigma-2},$$

where $\sigma > 2$ is the real part of s. So by Perron formula [3]

$$\begin{aligned}
\sum_{n\leq x}\frac{U(n)}{n^{s_0}} &= \frac{1}{2i\pi}\int_{b-iT}^{b+iT} f(s+s_0)\frac{x^s}{s}ds+O\left(\frac{x^bB(b+\sigma_0)}{T}\right)\\
&\quad +O\left(x^{1-\sigma_0}H(2x)\min(1,\frac{\log x}{T})\right)+O\left(x^{-\sigma_0}H(N)\min(1,\frac{x}{||x||})\right),
\end{aligned}$$

where N is the nearest integer to x, $\|x\| = |x - N|$. Taking $s_0 = 0$, $b = 3$, $T = x^{\frac{3}{2}}$, $H(x) = x$, $B(\sigma) = \frac{1}{\sigma - 2}$, we have

$$\sum_{n \le x} U(n) = \frac{1}{2i\pi} \int_{3-iT}^{3+iT} \frac{\zeta(s-1)}{\zeta(2(s-1))} R(s) \frac{x^s}{s} ds + O(x^{\frac{3}{2}+\varepsilon}),$$

where

$$R(s) = \prod_p \left(1 + \frac{1}{p^3 + p^2 - p - 1}\right).$$

To estimate the main term

$$\frac{1}{2i\pi} \int_{3-iT}^{3+iT} \frac{\zeta(s-1)}{\zeta(2(s-1))} R(s) \frac{x^s}{s} ds,$$

we move the integral line from $s = 3 \pm iT$ to $s = \frac{3}{2} \pm iT$. This time, the function

$$f(s) = \frac{\zeta(s-1)x^s}{\zeta(2(s-1))s} R(s)$$

has a simple pole point at $s = 2$ with residue $\frac{x^2}{2\zeta(2)} R(2)$. So we have

$$\frac{1}{2i\pi} \left(\int_{3-iT}^{3+iT} + \int_{3+iT}^{\frac{3}{2}+iT} + \int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT} + \int_{\frac{3}{2}-iT}^{3-iT} \right) \frac{\zeta(s-1)x^s}{\zeta(2(s-1))s} R(s) ds$$
$$= \frac{x^2}{2\zeta(2)} \prod_p \left(1 + \frac{1}{p^3 + p^2 - p - 1}\right).$$

We can easily get the estimate

$$\left| \frac{1}{2\pi i} \left(\int_{3+iT}^{\frac{3}{2}+iT} + \int_{\frac{3}{2}-iT}^{3-iT} \right) \frac{\zeta(s-1)x^s}{\zeta(2(s-1))s} R(s) ds \right|$$
$$\ll \int_{\frac{3}{2}}^{3} \left| \frac{\zeta(\sigma - 1 + iT)}{\zeta(2(\sigma - 1 + iT))} R(s) \frac{x^3}{T} \right| d\sigma \ll \frac{x^3}{T} = x^{\frac{3}{2}}$$

and

$$\left| \frac{1}{2\pi i} \int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT} \frac{\zeta(s-1)x^s}{\zeta(2(s-2))s} R(s) ds \right| \ll \int_0^T \left| \frac{\zeta(1/2 + it)}{\zeta(1 + 2it)} \frac{x^{\frac{3}{2}}}{t} \right| dt \ll x^{\frac{3}{2}+\varepsilon}.$$

Note that $\zeta(2) = \frac{\pi^2}{6}$, from the above we have

$$\sum_{\substack{n \in \mathcal{A} \\ n \le x}} U(n) = \frac{3x^2}{\pi^2} \prod_p \left(1 + \frac{1}{p^3 + p^2 - p - 1}\right) + O\left(x^{\frac{3}{2}+\varepsilon}\right).$$

This completes the proof of Theorem 1.

Now we come to prove Theorem 2. Let

$$g(s) = \sum_{\substack{n \in \mathcal{A} \\ n \leq x}} \frac{V(n)}{n^s}.$$

From the Euler product formula [2] and the definition of $V(n)$, we also have

$$\begin{aligned}
& g(s) \\
= & \prod_p \left(1 + \frac{V(p)}{p^s} + \frac{V(p^2)}{p^{2s}} + \cdots + \frac{V(p^{m-1})}{p^{(m-1)s}} + \frac{V(p^m)}{p^{ms}} + \frac{V(p^{m+1})}{p^{(m+1)s}} + \cdots\right) \\
= & \prod_p \left(1 + \frac{p-1}{p^s} + \frac{p^2-1}{p^{2s}} + \cdots + \frac{p^{m-1}-1}{p^{(m-1)s}} + \frac{p^m-1}{p^{ms}} + \frac{p^{m+1}-1}{p^{(m+1)s}} + \cdots\right) \\
= & \prod_p \left(\frac{1 - \frac{1}{p^{m(s-1)}}}{1 - \frac{1}{p^{s-1}}} - \left(\frac{1 - \frac{1}{p^{s-1}}}{1 - \frac{1}{p^s}}\right)\left(\frac{p^{m-1}}{p^{ms}} - \frac{1}{p^s}\right)\right) \\
= & \zeta(s-1) \prod_p \left(1 - \frac{1}{p^{m(s-1)}} + \frac{(p^{m-1} - p^{(m-1)s})(p^s - p)}{p^{ms}(p^s - 1)}\right).
\end{aligned}$$

By Perron formula [3], and the method of proving Theorem 1, we can also obtain the result of Theorem 2.

This completes the proof of the theorems.

References

[1] F. Smarandache, Only Problems, Not Solutions, Xiquan Publishing House, Chicago, 1993.

[2] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

[3] Pan Chengdong and Pan Chengbiao, Foundation of Analytic Number Theory, Beijing, Science Press, 1997.