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On the primitive numbers of power p

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Abstract For any positive integer n and prime p, let $S_p(n)$ denotes the smallest positive integer m such that $m!$ is divisible by p^n. The main purpose of this paper is using the elementary method to study the properties of the summation $\sum_{d|n} S_p(d)$, and give an exact calculating formula for it.

Keywords Primitive number of power p, summation, calculating formula.

§1. Introduction and Results

Let p be a prime and n be any positive integer. Then we define the primitive number function $S_p(n)$ of power p as the smallest positive integer m such that $m!$ is divisible by p^n. For example, $S_3(1) = 3$, $S_3(2) = 6$, $S_3(3) = S_3(4) = 9$, $\cdots\cdots$. In problem 49 of book [1], Professor F.Smarandache asked us to study the properties of the sequence $\{S_p(n)\}$. About this problem, Zhang Wenpeng and Liu Duansen [3] had studied the asymptotic properties of $S_p(n)$, and obtained an interesting asymptotic formula for it. That is, for any fixed prime p and any positive integer n, they proved that

$$S_p(n) = (p-1)n + O\left(\frac{p}{\ln p}\ln n\right).$$

Yi Yuan [4] had studied the asymptotic property of $S_p(n)$ in the form

$$\frac{1}{p}\sum_{n\le x}|S_p(n+1) - S_p(n)|,$$

and obtained the following conclusion: For any real number $x \ge 2$, we have

$$\frac{1}{p}\sum_{n\le x}|S_p(n+1) - S_p(n)| = x\left(1-\frac{1}{p}\right) + O\left(\frac{\ln x}{\ln p}\right).$$

Xu Zhefeng [5] had studied the relationship between the Riemann zeta-function and an infinite series involving $S_p(n)$, and obtained some interesting identities and asymptotic formulae for $S_p(n)$. That is, for any prime p and complex number s with $\mathrm{Re}s > 1$, we have the identity:

$$\sum_{n=1}^{\infty}\frac{1}{S_p^s(n)} = \frac{\zeta(s)}{p^s - 1},$$

where $\zeta(s)$ is the Riemann zeta-function.

And, let p be a fixed prime, then for any real number $x \geq 1$,

$$\sum_{\substack{n=1 \\ S_p(n)\leq x}}^{\infty} \frac{1}{S_p(n)} = \frac{1}{p-1}\left(\ln x + \gamma + \frac{p\ln p}{p-1}\right) + O(x^{-\frac{1}{2}+\varepsilon}),$$

where γ is the Euler constant, ε denotes any fixed positive number.

Chen Guohui [7] had studied the calculating problem of the special value of the Smarandache function $S(n) = \min\{m:\ m \in N,\ n|m!\}$. That is, let p be a prime and k an integer with $1 \leq k < p$. Then for polynomials $f(x) = x^{n_k} + x^{n_{k-1}} + \cdots + x^{n_1}$ with $n_k > n_{k-1} > \cdots > n_1 \geq 1$, we have:

$$S(p^{f(p)}) = (p-1)f(p) + pf(1).$$

And, let p be a prime and k an integer with $1 \leq k < p$. Then for any positive integer n, we have:

$$S\left(p^{kp^n}\right) = k\left(\phi(p^n) + \frac{1}{k}\right)p,$$

where $\phi(n)$ is the Euler function. All these conclusions also hold for primitive function $S_p(n)$ of power p.

In this paper, we shall use the elementary method to study the calculating problem of the summation

$$\sum_{d|n} S_p(d),$$

and give some interesting calculating formulas for it. That is, we shall prove the following conclusions:

Theorem. For any prime p, we have the calculating formulas

(1) $\sum\limits_{d|n} S_p(d) = p\sigma(n)$, if $1 \leq n \leq p$;

(2) $\sum\limits_{d|n} S_p(d) = p\sigma(n) - (n-1)p$, if $p < n \leq 2p$, where $\sigma(n)$ denotes the summation over all divisors of n.

For general positive integer $n > 2p$, whether there exists a calculating formula for $\sum\limits_{d|n} S_p(d)$ is an open problem.

§2. Proof of Theorem

To complete the proof of the theorem, we need a simple lemma which stated as following:

Lemma. For any prime p, we have:

(1) $S_p(d) = dp$, if $1 \leq d \leq p$;

(2) $S_p(d) = (d-k+1)p$, if $(k-1)p + k - 2 < d \leq kp$.

Proof. First we prove the case (1). From the definition of $S_p(n) = \min\{m : p^n|m!\}$ we know that to prove the case (1) of Lemma, we only to prove that $p^d \| (dp)!$. That is, $p^d|(dp)!$ and

$p^{d+1} \dagger (dp)!$. According to Theorem 1.7.2 of reference [6], we only to prove that $\sum_{j=1}^{\infty}\left[\frac{dp}{p^j}\right] = d$. In fact, if $1 \leq d < p$, note that $\left[\frac{d}{p^j}\right] = 0$ $(j = 1,\ 2,\ \cdots\cdots)$, we have

$$\sum_{j=1}^{\infty}\left[\frac{dp}{p^j}\right] = d + \left[\frac{d}{p}\right] + \left[\frac{d}{p^2}\right] + \cdots = d.$$

That is means $S_p(d) = dp$. If $d = p$, then $\sum_{d|n}\left[\frac{dp}{p^j}\right] = d + 1$, but $p^p \dagger (p^2 - 1)!$ and $p^p | p^2!$. So from the definition of $S_p(n)$ we have $S_p(p) = p^2 = dp$. This proves the case (1) of Lemma.

Now we prove the case (2) of Lemma. Using the same method of proving the case (1) of Lemma we can also deduce that if $p < d \leq 2p$, then $\left[\frac{d-1}{p^j}\right] = 1$, $\left[\frac{d-1}{p^j}\right] = 0, (j = 2,\ 3,\ \cdots\cdots)$, we have

$$\sum_{j=1}^{\infty}\left[\frac{(d-1)p}{p^j}\right] = d - 1 + \left[\frac{d-1}{p}\right] + \left[\frac{d-1}{p^2}\right] + \cdots = d.$$

That is means that $S_p(d) = (d-1)p$. From Theorem 1.7.2 of reference [6] we know that if $p < d \leq 2p$, then $p^d \| ((d-1)p)!$. That is, $S_p(d) = (d-1)p$. This proves the lemma.

Now we use this Lemma to complete the proof of the theorem.

First we prove the case (1) of Theorem. From the case (1) of Lemma we know that if $1 \leq n \leq p$, then

$$\sum_{d|n} S_p(d) = \sum_{d|n} dp = p\sum_{d|n} d = p\sigma(n).$$

Now we prove the case (2) of Theorem. We find that if $p < n \leq 2p$ and $d|n$, then there is only one divisor $d(> p)$ of n, i.e. $d = n$, the reason is that if $d > p$ and $d|n$, then $n = d,\ 2d,\ 3d,\ \cdots$, but $2d > 2p,\ \cdots$. So we may immediately deduce the following conclusion: if $p < n \leq 2p$, then

$$\sum_{d|n} S_p(d) = \sum_{\substack{d|n\\ 1\leq d\leq p}} S_p(d) + \sum_{\substack{d|n\\ p<d\leq 2p}} S_p(d) = p\sum_{\substack{d|n\\ 1\leq d\leq p}} d + S_p(n) = p\sigma(n) + (n-1)p.$$

This completes the proof of Theorem.

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